From the holdings of the Department of Special Collections , Princeton University Library, Princeton, NJ

These documents are either in the public domain, or can only be used for educational and research purposes (Fair use) as per U.S. Copyright law (text below). By accessing this file, all users agree that their use falls in the public domain or within fair use as defined by the copyrightlaw. We do not charge any permission or usage fees for the publication of images of material inour collections, including those provided by Princeton University Library, via our website, catalog, or directly from staff.For more information regarding copyright, credit, and citation please review the information onthe Copyright, Credit and Citation Guidelines page of our website.
https://library.princeton.edu/special-collections/policies/copyright-credit-and-citation-guidelines. If you have any questions, including how to order higher-quality images from the collection, contact Special Collections Public Services via our Ask Us! Form. https://library.princeton.edu/special-collections/ask-us

---

Doctoral Dissertations, 2011-2026 Mathematics

2023

Singular Structures and Generic Regularity for Minimal Hypersurfaces

Wang, Zhihan

Princeton, NJ : Princeton University

---

http://arks.princeton.edu/ark:/88435/dsp01wm117s24c

U.S. Copyright law: The copyright law of the United States (Title 17, §108, United States Code) governs the making of photocopies or other reproductions of copyrighted material. Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be used for any purpose other than private study, scholarship, or research. If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of fair use that user may be liable for copyright infringement. This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve a violation of copyright law. https://www.copyright.gov/title17/title17.pdf

Singular Structures and Generic Regularity for Minimal Hypersurfaces

Zhihan Wang

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Fernando Codá Marques

May 2023

© Copyright by Zhihan Wang, 2023.

All Rights Reserved

Abstract

Regularity theory of minimal hypersurfaces has long been studied since last century. A satisfactory regularity result for area-minimizing and stable minimal hypersurfaces in a manifold of dimension ≤ 7 has been established over 40 years ago. In higher dimensional spaces, area-minimizing or stable minimal hypersurfaces are known to potentially have singular sets, and the wild behavior of these singularities obstructs us from further exploration of global geometric behaviors of minimal hypersurfaces as well as applications in other fields.

In this thesis, we will discuss how singular sets affect the global behavior of minimal hypersurfaces.

First, we generalize a Theorem of Hardt-Simon [HS85] to approximate any minimizing (resp. viscosity mean convex) hypercone by complete smooth minimizing (resp. strictly mean convex) hypersurfaces. This suggests that singularity models for minimizing hypersurfaces may all be smoothable.

Second, we develop the spectral analysis of the Jacobi operator on a closed locally stable minimal hypersurface with strongly isolated singularities. This in particular applies to all min-max minimal hypersurfaces in an eight manifold. We include primary application of this linear theory to obtain a local exitence and a local minimizing property of minimal hypersurface under reasonable spectral assumptions.

Finally, in an ambient manifold of dimension 8, by further exploiting the linear analysis of the Jacobi operators, joint with Yangyang Li, we prove that under generic metric, every minimal hypersurface constructed by min-max approach are entirely smooth.

Acknowledgements

First and foremost, I would like to express my great gratitude to my advisor Fernando Codá Marques, for all of his helpful guidance and constant encouragement during these years in Princeton. As a super-nice teacher, Fernando leads me into a broad area of mathematics. Besides, his efforts of organizing a special year at IAS in my first year and conducting seminar courses every year have not only been instrumental in shaping my research flavor, but also stimulated much of my present work.

I owe a special gratitude to Guoyi Xu, who introduced me to the world of geometric measure theory and regularity of minimal surfaces, that I have been working on during my whole graduate study.

I am also grateful to Alice Chang and Camillo De Lellis, not only for serving in my FPO committee, but also for their constant support during my PhD study. I have learned a lot from their lectures and reading courses, and I have also benefited a lot from many helpful discussions with them. I would also like to express my gratitude to Luca Spolaor, for his interest in my research and kind seminar and conference invitations, for the inspiring discussions and for his serving as a reader of my thesis. Sincere gratitude also goes to André Neves, Leon Simon, Gábor Székelyhidi and Xin Zhou, for their invaluable supports and encouragements for my academic life.

For their interests in my work, for the helpful discussions and supports, and/or for their kind seminar invitations, I would also like to thank Xinliang An, Sigurd Angenent, Otis Chodosh, Jianchun Chu, Qi Ding, Zeno Huang, Daniel Ketover, Yi Lai, Chao Li, Yevgeny Liokumovich, Jiewon Park, Joaquim Serra, Antoine Song, Daniel Stern, Ao Sun, Liming Sun, Song Sun, Lu Wang, Yi Wang, Brian White, Paul Yang, Ruobing Zhang and others.

I would also like to all thank my friends and colleagues in math, who have influenced and support me a lot during my PhD period, including Shaoyun Bai, Eric Chen,

Letian Chen, Akashdeep Dey, Vikram Giri, Dong Yeong Ko, Shilin Lai, Sanghoon Lee, Yangyang Li, Zhenhua Liu, Ben Lowe, Zilu Ma, Xiao Ma, Paul Minter, Samuel Pérez-Ayala, Lorenzo Sarnataro, Érico Silva, Anna Skorobogatova, Doug Stryker, Ryan Unger, Siyi Zhang, Fan Zheng, Jingze Zhu, Jonathan Zhu and others.

Special thanks go to the Mathematics Department at Princeton University and all staff members, especially to Jill LeClair, for their assistance in providing the necessary resources and support throughout my research life.

Above all, I want to thank my family. I am deeply indebt to my parents Lanqin Zhang and Liyun Wang, for their unconditional love and encouragements. Their unwavering faith in my abilities kept me motivated during the most challenging times of my research journey.

To my parents.

Contents

Abstract iii

Acknowledgements iv

1 Introduction **1**

1.1 Existence Theory 1

1.2 Partial Regularity and Generic Regularity 4

1.3 Results from this Thesis and Organization 5

1.3.1 Regular foliation for general hypercones 5

1.3.2 Obstacle problem for min-max minimal hypersurfaces 7

1.3.3 Deformation theory for singular minimal hypersurfaces 9

1.3.4 Full generic regularity in eight manifolds 11

2 Preliminary **14**

2.1 Notations and Conventions 14

2.2 Basics in Geometric Measure Theory 19

2.3 Locally Stable Minimal Hypersurfaces 23

2.4 Minimal Hypercone 26

3 Analysis on Singular Minimal Hypersurfaces, I **28**

3.1 Introduction 28

3.2 Harnack Inequality for Minimal Hypersurfaces 29

3.3 Growth Rate Lower Bound for Positive Jacobi Fields 37

3.3.1 Proof of Growth Lemma 3.3.2 41
3.4 Mean Convex Smoothing of Mean Convex Hypersurfaces 48
3.4.1 Mean convex approximation 48
3.4.2 Self-expanders 55
3.4.3 Weak set flow and level set flow 60
3.4.4 Proof of Theorem 1.3.1 and 1.3.2 63

4 Min-max Minimal Hypersurafces with Obstacle 71
4.1 Introduction 71
4.2 Preliminaries 73
4.2.1 Constrained stationary varifolds 75
4.2.2 Constrained embedded hypersurfaces 79
4.2.3 Obstacle problem 81
4.3 A Structure Theorem near the Boundary 84
4.4 Min-max with Obstacle 94
4.4.1 Almost minimizer and regularity 94
4.4.2 Min-max construction 97
4.5 A Local Rigidity for Constrained Embedded Minimal Hypersurfaces . 101

5 Analysis on Singular Minimal Hypersurfaces, II 106
5.1 Introduction 106
5.2 Geometry of Regular Stable Minimal Hypercone 111
5.2.1 Asymptotic rate of Jacobi-field-type functions 114
5.2.2 Asymptotics of minimal hypersurfaces near infinity 120
5.2.3 A general local growth dichotomy 130
5.3 Spectral Analysis on LSMH 134
5.3.1 Linear analysis in large 135
5.3.2 L^2-nonconcentration for strongly isolated singularities 145

5.3.3 Asymptotic rates on LSMH 152
5.3.4 Green's functions 157
5.4 Local Minimization of Strictly Stable Minimal Hypersurfaces 162
5.5 Local Existence of LSMH under Metric Perturbation 176
5.5.1 Strong min-max property for singular minimal hypersurfaces . 179
5.5.2 A rigidity result for constrained minimal hypersurfaces 186
5.5.3 Failure of local existence 191

6 Analysis on Singular Minimal Hypersurfaces, III 194
6.1 Introduction 194
6.2 Preliminaries 196
6.2.1 Jacobi fields of slower growth 198
6.2.2 Singular capacity 199
6.2.3 Canonical pseudo neighborhoods 201
6.2.4 G-invariant minimal hypersurfaces 205
6.3 Proof of Generic Regularity 207
6.4 Applications 213
6.5 Singular Capacity 217
6.6 Almost Cone Decomposition 228
6.6.1 Cone decompositions à la Edelen 228
6.6.2 Tree representation and first decomposition 233
6.6.3 Compactness and second decomposition 244
6.7 Induced Jacobi Fields 246
6.7.1 Local growth estimate of graphical functions 249
6.7.2 Induced Jacobi field in general 256
6.7.3 Induced Jacobi field for pairs 261
6.8 Generic Semi-nondegeneracy 267

A Minimal Surface Equation for Graph **279**

B Improved Regularity for Obstacle Problems **286**

C Jacobi Fields on Minimal Hypercone **294**

D Quantitative Uniqueness of Tangent Cone **305**

Chapter 1

Introduction

In a Riemannian manifold, an n-dimensional hypersurface Σ is called **minimal** if it's a critical point of n-dimensional area functional, namely the first variation vanishes,

$$\delta \operatorname{Area}[\Sigma] = 0.$$

The theory of minimal surfaces dates back to the investigations of Euler and Lagrange, in connection with the calculus of variations, during the second half of the 18th century, and still remains an active field of research to this day. The study of minimal submanifolds also has applications in various fields. For example, in general relativity, minimal surfaces appear as models for the apparent horizon of black holes, and are used to settle the well-known positive mass theorem [SY79,SY81] and Penrose inequality [HI01, Bra01]

1.1 Existence Theory

The general existence theory of minimal hypersurfaces goes back to Douglas [Dou31] and Radó [Rad30] in the 1930's, who solved the *Plateau's problem* of finding minimal surfaces in $\mathbb{R}^3$ with prescribed boundary curve. Later in 1960s, De Giorgi [DG61],

Federer-Fleming [FF60], Reifenberg [Rei60] developed the geometric measure theory and obtained the general existence of minimizing *hypersurfaces* in $\mathbb{R}^{n+1}$ with prescribed boundary and closed minimizing *hypersurfaces* in a nontrivial homology class in general Riemannian manifolds (M^{n+1}, g). Their idea was to minimize area and take limit in a larger family of weak notions of hypersurfaces (e.g. *Caccioppoli sets*, *integral currents*). Combining the geometric identity of Simons [Sim68], these minimizing hypersurfaces were proved by Federer [Fed70] to be smooth outside a closed subset of Hausdorff dimension $\leq n-7$.

The work of Bombieri-De Giorgi-Giusti [BDGG69] indicates that the singularities could exist in general. For example, the well-known Simons' cone

$$\mathbf{C}^{3,3} := \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\},$$

is a minimizing hypersurface in $\mathbb{R}^8$ with isolated singularity at $\mathbf{0}$, and $\mathbf{C} \cap \mathbb{B}_1$ is the unique solution to Plateau's problem with boundary $\mathbf{C} \cap \mathbb{S}^7$.

In general, minimizing hypercones are singularity models of an arbitrary minimizing hypersurface Σ. More precisely, near each singularity p, consider the rescaled hypersurfaces $\Sigma_{p,r} := (\Sigma - p)/r$. Then after passing to a subsequence of $r \to 0$, $\Sigma_{p,r}$ converges to some minimizing hypercone $\mathbf{C} \subset \mathbb{R}^{n+1}$, called a *tangent cone* of Σ at p.

Without homological non-triviality, minimizers of area functional would be trivial. In search of saddle points instead of global minimizers, Almgren [Alm62, Alm65] and Pitts [Pit81] proposed a Morse-theoretic approach to find minimal hypersurfaces in an arbitrary closed manifold (M^{n+1}, g). In particular, they shown that when $2 \leq n \leq 5$, (M^{n+1}, g) admits a nontrivial closed embedded smooth minimal hypersurface Σ. Such minimal hypersurface is no longer minimizing, but they are locally **stable**, i.e. every $p \in M$ has a neighborhood $U_p \subset M$ such that the second variation with respect to

vector fields X supported in U_p is non-negative,

$$\delta^2 \operatorname{Area}_\Sigma(X, X) \geq 0, \quad \forall X \in \mathscr{X}_c(U_p).$$

Based on an ϵ-regularity Theorem, Schoen-Simon [SS81] extended this result to arbitrary dimension, where the minimal hypersurfaces Σ through min-max construction are still locally stable, smooth away from a closed subset of dimension $\leq n-7$ and having tangent cones at each singular point to be stable minimal hypercones in $\mathbb{R}^{n+1}$. We shall use the abbreviation **LSMH** to denote locally stable minimal hypersurfaces throughout the thesis.

The past decades witness the profound development of Almgren-Pitts min-max theory led by F. Codá Marques and A. Neves, which promotes the solution to various well-known open problems, such as Willmore conjecture [MN14] and Yau's conjecture on existence of infinitely many minimal hypersurfaces [MN17, IMN18, Son18]. Liokumovich-Marques-Neves [LMN18] derived the Weyl Law for volume spectrum, which led to the generic denseness [IMN18] and generic equi-distribution [MNS19] of minimal hypersurfaces. Marques-Neves [MN16] [MN21] also proposed the refined Morse theory for area functional, estimating the index and multiplicity of min-max minimal hypersurfaces; X. Zhou [Zho20] confirmed the Multiplicity 1 Conjecture, showing that for *bumpy metric* (a generic condition on Riemannian metric which roughly makes the area functional Morse), min-max minimal hypersurfaces can be chosen of multiplicity 1. Marques-Montezuma-Neves [MMN20] derived a Morse inequality for area functional with bumpy metric.

Most of these results rely on the smoothness of minimal hypersurfaces and hence is restricted to dimension $3 \leq n+1 \leq 7$.

1.2 Partial Regularity and Generic Regularity

As is mentioned above, LSMH in an $n+1$-dimensional manifold obtained by minimization or min-max approach can have closed singular set of Hausdorff codimension ≥ 7. Simon [Sim93] and Naber-Valtorta [NV20] shown that the singular set is $n-7$ rectifiable, i.e. off an $\mathscr{H}^{n-7}$-measure-zero set, $\mathrm{Sing}(\Sigma)$ is contained in a countable union of $n-7$ dimensional C^1-submanifolds. On the other hand, Smale [Sma99, Sma00] constructed homologically area-minimizing hypersurfaces under certain metric, whose singular set is a non-empty codimension 7 submanifold; Simon [Sim21b] recently constructed examples of stable minimal hypersurfaces in $\mathbb{R}^{n+1\geq 9}$ with metrics arbitrarily C^∞-close to the Euclidean metric, whose singular set can be an arbitrarily prescribed closed subset in $\mathbb{R}^{n-7}$. Székelyhidi [Szé21] gave examples of minimizing hypersurfaces in the unit ball of $\mathbb{R}^{n+1\geq 9}$ under Euclidean metric which has an isolated singularity at $\mathbf{0}$ but has tangent cone at $\mathbf{0}$ to be $\mathbf{C}^{p,q}\times\mathbb{R}$, namely, a cylindrical cone whose singular set is $\mathbb{R}$. These examples suggest that the singular set may not be avoidable and can be bad in general.

On the other hand, Yau conjectured that,

Conjecture (Yau [Yau82])**.** *A perturbation of the metric can eliminate the singularities of a given area minimizing hypersurface in its homology class.*

In their ground-breaking work, Hardt-Simon [HS85] shown that in the simplest model case, i.e. for a minimizing hypercone $\mathbf{C}\subset\mathbb{R}^{n+1}$ with isolated singularity at $\mathbf{0}$, there exists a unique (up to scaling) minimizing hypersurface $S\subset\mathbb{R}^{n+1}$ lying on one side of $\mathbf{C}$ and asymptotic to $\mathbf{C}$ near infinity. Moreover, such S has no singularity (See Theorem 2.4.1 for a precise statement). Based on this, Smale [Sma93] shown that in an 8-manifold with a generic metric, any homologically area-minimizing hypersurface is smooth. Here *generic* is in the Baire category sense. This in particular confirmed Yau's conjecture in dimension 8. Recently, a breakthrough was made by

Chodosh-Mantoulidis-Schulze [CMS23] in confirming Yau's conjecture in manifolds of dimension 9 and 10.

Similar conjecture of Yau can be asked for min-max minimal hypersurfaces. In this direction, Chodosh-Liokumovich-Spolaor [CLS20] proved that an 8-dimensional manifold M^8 with a generic metric of positive Ricci curvature admits at least one smooth minimal hypersurface. The major difficulty in the min-max setting is that tangent cones of LSMH are only stable but may not be minimizing a priori. Hence Hardt-Simon's local deformation doesn't apply. What's even worse, there's no entire minimal hypersurface lying on one-side of a stable-but-not-minimizing hypercone, making it impossible to only do the perturbation locally.

1.3 Results from this Thesis and Organization

The primary goal of this thesis is to investigate how singularities affects the global behavior of minimal hypersurfaces.

1.3.1 Regular foliation for general hypercones

In [Wan22a], we generalize the existence result by Hardt and Simon to arbitrary minimizing hypercones, not necessarily with only isolated singularity.

Theorem 1.3.1 ([Wan22a], c.f. Theorem 3.4.14)**.** *Let $E \subset \mathbb{R}^{n+1}$ be a closed subset such that $\mathbf{C} := \partial E$ is a minimizing hypercone in $\mathbb{R}^{n+1}$. Then there exists $S \subset \mathrm{Int}(E)$ a properly embedded smooth minimizing hypersurface in $\mathbb{R}^{n+1}$ with distant 1 to the origin and $\mathbf{C}$ to be the tangent cone at infinity. Moreover, S is a radial graph over $\mathrm{Int}(E) \cap \mathbb{S}^n$.*

We might expect this theorem to be a first step towards the full generic regularity problem for minimizing hypersurfaces in general dimension. However, we assert that the uniqueness (up to rescaling) of such one-sided smooth perturbation is still open,

even in the splitting case, i.e. when $\mathbf{C} = \mathbf{C}_0 \times \mathbb{R}^k$ for some regular minimizing hypercone $\mathbf{C}_0$. In [Sim21a], uniqueness in the splitting case is proved under further strict stability assumption. In [ES23], Edelen-Székelyhidi obtained the uniqueness when $\mathbf{C}_0$ is strictly minimizing and strictly stable. We also mention that [Loh18] claimed the existence and uniqueness of such one-sided minimizing perturbation, with a seemingly incomplete proof.

When $\mathbf{C}$ is not minimizing in E, by a barrier argument (see Lemma 3.4.13), such complete minimal hypersurface lying in the interior of E should not exist. [Lin87a] shown that for $n+1 \leq 8$, for a regular minimal hypercone $\mathbf{C} = \partial E \subset \mathbb{R}^{n+1}$, there exists a smooth mean convex hypersurface asymptotic to $\mathbf{C}$ near infinity. Ding [Din20] shown that more generally, if the cone $\mathbf{C}$ is of arbitrary dimension, $C^{3,\alpha}$-regular and mean convex, but not minimizing in E, then there exists a unique smooth strictly mean convex self-expander sitting inside $\mathrm{Int}(E)$ and properly embedded in $\mathbb{R}^{n+1}$. Here a *self-expander* is a hypersurface satisfying the equation,

$$\vec{H} - \frac{X^{\perp}}{2} = 0,$$

where $\vec{H}$ is the mean curvature vector and $X^{\perp}$ is the projection of position vector X onto the normal direction. Self-expander was introduced in [EH89] as model of long time behavior for mean curvature flow, and also studied by [Ilm98] as model for mean curvature flow coming out of a singularity. See Section 3.4.2 for more on geometry of self-expanders.

We also generalize Ding's result to arbitrary mean convex hypercone, not necessarily regular.

Theorem 1.3.2 ([Wan22a], c.f. Theorem 3.4.15)**.** *Let $E \subset \mathbb{R}^{n+1}$ be a closed subset such that $\mathbf{C} := \partial E$ is a mean convex hypercone in $\mathbb{R}^{n+1}$ and not minimizing in E. Then there exists $S \subset \mathrm{Int}(E)$ a smooth strictly mean convex self-expander properly*

embedded in $\mathbb{R}^{n+1}$ *with* $\mathbf{C}$ *to be the tangent cone at infinity. In particular,* S *is a radial graph over* $\mathrm{Int}(E) \cap \mathbb{S}^n$.

Here, mean convexity assumption for E is in viscosity sense, see Definition 3.4.1. In particular, Theorem 1.3.2 applies when ∂E is a stationary minimal hypercone, possibly with singularities, which is not minimizing in E. On the other hand, uniqueness is again unclear in this non-smooth case.

Combining Theorem 1.3.1 and 1.3.2, we confirm a conjecture of Lawson [Bro86, Problem 5.7]:

Theorem 1.3.3 ([Wan22a])**.** *Given a stable minimal hypercone* $\mathbf{C} \subset \mathbb{R}^{n+1}$ *and every* $\epsilon > 0$*, there exists a properly embedded smooth hypersurface with positive mean curvature in* $\mathbb{B}_1^{n+1}(\mathbf{0})$ *within Hausdorff distant* $\leq \epsilon$ *from* $\mathbf{C} \cap \mathbb{B}_1^{n+1}(\mathbf{0})$.

1.3.2 Obstacle problem for min-max minimal hypersurfaces

Given a compact manifold M with non-empty boundary, it's also natural to ask for the existence theory of *minimal* hypersurface.

Question. Can we find a *minimal* hypersurface constrained in a (M, g)?

Clearly, the answer is no in general if one only focus on closed surfaces with vanishing mean curvature. For example, any bounded domain in $\mathbb{R}^{n+1}$ contains no closed minimal hypersurface. Allowing an extension of subjects we would like to find, two distinct aspects of this problem are frequently studied.

• **Free boundary problem.** A minimal hypersurface with free boundary in M is a hypersurface with boundary in ∂M, perpendicular to ∂M and having vanishing mean curvature. The study of free boundary minimal surfaces has a long history, we refer the readers to [LZ21] for a beautiful survey of the development.

• **Obstacle problem.** Let (M, g) be a bounded smooth domain in $(\mathbb{R}^{n+1}, |dx|^2)$, $E_0 \subset \mathbb{R}^{n+1}$ be a bounded Caccioppoli set; Consider a minimizer $\hat{E}$ of the perimeter of *Caccioppoli sets* (See the definition in Section 2.2) among

$$\{E \subset \mathbb{R}^{n+1} : E\Delta E_0 \subset M\}.$$

We assert that such a $\hat{E}$ is no longer a critical point of area functional, but instead satisfies a variational inequality, and is known to be a solution to *obstacle problem*, which has been widely studied in elliptic PDE settings. The regularity of boundary of $\hat{E}$ is well studied in [Mir71, BM82, Tam82, Lin85], and a non-parametric version is studied in [CK80, BKL74, LS71, GP71] under weaker assumptions on M. We emphasize here that in general, the optimal regularity for $\partial\hat{E}$ is $C^{1,1}$ instead of C^2 where $\partial\hat{E}$ touch ∂M, and the mean curvature of $\partial\hat{E}$ does not necessarily vanish. We refer the readers to [Giu11] for a brief introduction and generalizations when M is not necessarily smooth. See also Section 4.2.3 for a more concrete discussion.

The min-max analogue of obstacle problem for area functional and other parametric elliptic integral is also studied recent years under further convexity assumption on M, and hence the stationarity of solutions valid by maximum principle. See [Son18, BW20b, DLR18, Mon20].

In [Wan22b], we obtained the existence and regularity of min-max minimal hypersurface with obstacle. More precisely, let $m \geq 1$, $n \geq 2$, $\mathcal{Z}_n(M)$ be the space of *integral cycles* (see its definition in Section 2.2) supported in M, $I^m := [0,1]^m$ be the m-dimensional unit cube; Let $\Phi_0 : I^m \to \mathcal{Z}_n(M)$ be a continuous map. Let Π be the family of sequences of continuous maps $S = \{\Phi_i : I^m \to \mathcal{Z}_n(M)\}_{i\geq 1}$ which C^0-tends to $\Phi_0|_{\partial I^m}$ restricted on ∂I^m. Let

$$\mathbf{L}(\Pi) := \inf_{S=\{\Phi_i\}_{i\geq 1}\in\Pi} \limsup_{i\to\infty} \sup_{x\in I^m} \mathbf{M}(\Phi_i(x)); \qquad \mathbf{L}_{\Phi_0,\partial I^m} := \sup_{x\in\partial I^m} \mathbf{M}(\Phi_0(x)).$$

Theorem 1.3.4 ([Wan22b], c.f. Theorem 4.4.3)**.** *If* $\boldsymbol{L}(\Pi) > \boldsymbol{L}_{\Phi_0,\partial I^m}$*, then there exists an immersed* $C^{1,1}$ *hypersurface* $\Sigma \hookrightarrow M$ *such that*

- *Area*$(\Sigma) = \boldsymbol{L}(\Pi)$
- Σ *is minimal, locally stable and embedded with multiplicity and codimension* ≥ 7 *singular set in the interior of* M*;*
- Σ *is multiple* $C^{1,1}$ *graphs near* ∂M*, and the mean curvature vector of* Σ *on* $\Sigma \cap \partial M$ *points outward, i.e.*

$$\vec{H}_\Sigma \cdot \nu_{\partial M} \leq 0,$$

for the inward normal field $\nu_{\partial M}$ *of* ∂M*.*

One application of this Theorem is the construction of the perturbation of singular minimal hypersurface under small perturbation of metric. See Theorem 5.1.1.

1.3.3 Deformation theory for singular minimal hypersurfaces

Given a smooth closed manifold M^{n+1}, White [Whi91] studied the space of metric-minimal-hypersurface pairs

$$\mathcal{M}(M) := \{(g, \Sigma) : g \text{ is a } C^{k,\alpha} \text{ metric on } M, \Sigma \subset (M, g) \text{ smooth minimal hypersurface}\}.$$

He shown that $\mathcal{M}(M)$ is a Banach manifold and the projection on to the first variable

$$\Pi : \mathcal{M}(M) \to \Gamma(M) := \{C^{k,\alpha} \text{ metrics on } M\},$$

is a Fredholm map of index 0. This in particular leads to the *non-degeneracy* of every closed smooth minimal hypersurface under generic metric. Here a minimal

hypersurface Σ being *nondegenerate* means the Jacobi operator of Σ,

$$L_\Sigma := \Delta_\Sigma + |A_\Sigma|^2 + \mathrm{Ric}_M(\nu, \nu),$$

has trivial kernel in $C^2(\Sigma)$. This fact is crucial in various recent results, for example, the generic denseness [IMN18], generic equi-distribution [MNS19] of minimal hypersurfaces, and generic multiplicity 1 of Almgren-Pitts min-max minimal hypersurfaces [Zho20] when $3 \leq n+1 \leq 7$.

A basic question in regularity theory is to understand what's going on when Σ are allowed to have singularities. Since any nearby minimal hypersurface is modeled on solutions of Jacobi field equation on Σ, the starting point may be to study the spectral theory of the Jacobi operator L_Σ. When $\mathrm{Sing}(\Sigma) \neq \emptyset$, the spectral theory of L_Σ becomes subtle and, to the best of the author's knowledge, was only known under further *strict stability* assumptions on the tangent cones, see for example [MS94, Sma99, SY17].

The simplest case is when every singularity p is *strongly isolated*, i.e. the tangent cone of Σ at p has only isolated singularity at the origin (Simon [Sim83a] shown that if one tangent cone is of multiplicity 1 and isolated singularity, then it's independent of choice of blow up radius). In particular, every LSMH in an 8-dimensional Riemannian manifold has only strongly isolated singularities. In [Wan20], we develop the spectral theory for the Jacobi operator on a GENERAL locally stable minimal hypersurface (LSMH) with only strongly isolated singularities.

Theorem 1.3.5 (cf. Proposition 5.3.5 & Lemma 5.3.9)**.** *Let $\Sigma \subset (M^{n+1}, g)$ be a closed two-sided LSMH with only strongly isolated singularities, then*

(a) There exists a constant $C_\Sigma > 1$ such that for every $\phi \in C_c^\infty(\Sigma)$, the quadratic form

$$\|\phi\|^2_{\mathscr{B}(\Sigma)} := \int_\Sigma -\phi \cdot L_\Sigma \phi + C_\Sigma \phi^2 \geq \int_\Sigma \phi^2,$$

Hence this defines a norm $\|\cdot\|_{\mathscr{B}(\Sigma)}$.

(b) The natural embedding $\mathscr{B}(\Sigma) := \overline{C_c^\infty(\Sigma)}^{\|\cdot\|_{\mathscr{B}}} \hookrightarrow L^2(\Sigma)$ *is a compact embedding.*

In particular, we have the spectral decomposition of $-L_\Sigma$ *on* $\mathscr{B}(\Sigma)$. *There exist* $\lambda_1 < \lambda_2 < \lambda_3 < \cdots \nearrow +\infty$ *and finite dimensional pairwise* L^2*-orthogonal linear subspaces* $\{E_j\}_{j\geq 1}$ *of* $\mathscr{B}(\Sigma) \cap C^\infty(\Sigma)$ *such that*

$$-L_\Sigma \phi = \lambda_j \phi, \qquad \forall \phi \in E_j;$$

and that

$$L^2(\Sigma) = \overline{\bigoplus_{j\geq 1} E_j}^{L^2}.$$

The novelty of this Theorem is that we find the correct substitute of the Sobolev space $W^{1,2}(\Sigma)$ in studying Jacobi operator L_Σ. In particular, we can define Σ to be **non-degenerate** if the spectrum $\{\lambda_j\}_{j\geq 1}$ doesn't contain 0. Note that the nondegeneracy assumption is a *generic* assumption, see Proposition 5.3.5 (6).

This spectral theory allows us to develop the deformation theory of minimal hypersurfaces near Σ, see Theorem 5.1.1 and 5.1.3. It's also fundamental in deriving full generic regularity for minimal hypersurfaces in an eight manifold, which is the joint work of the author with Yangyang Li [LW20, LW22], and will be carried out in detail in Chapter 6.

1.3.4 Full generic regularity in eight manifolds

In view of Yau's conjecture [Yau82] on instability of singularities to area minimizing hypersurfaces, a more optimistic conjecture is the generic regularity as following.

Conjecture (Full generic regularity I)**.** *For a closed smooth manifold* M^{n+1} *endowed with a (Baire)* C^∞*-generic Riemannian metric* g*, every* g*-area homology minimizer is smooth.*

Since the min-max construction doesn't produce area minimizers generally, it is natural to extend this conjecture to min-max minimal hypersurfaces, i.e., those generated from the min-max theory.

Conjecture (Full generic regularity II)**.** *For a closed smooth manifold M^{n+1} endowed with a (Baire) C^∞-generic Riemannian metric g, every min-max g-minimal hypersurface is smooth.*

In [LW20, LW22], joint with Yangyang Li, we prove the following theorem.

Theorem 1.3.6 (Full Generic Regularity for 8 Manifolds)**.** *Let M be a closed smooth manifold of dimension 8, $k \geq 4$ or $k = \infty$, $\alpha \in (0, 1)$. Then for $C^{k,\alpha}$ generic metric g on M, every embedded locally stable minimal hypersurface is regular and nondegenerate.*

Remark 1.3.7. This can be viewed as a weak generalization of B. White's structure theorem. In fact, we only need to verify the case where k is finite. As in [Whi17], we can use similar arguments therein or [ACS17, Section 7.1] to extend our result to the case $k = \infty$.

Note that every minimal hypersurfaces obtained by Almgren-Pitts min-max theory (or Allen-Cahn min-max theory) is locally stable. Therefore, we confirm the full generic regularity conjecture II in dimension eight.

Corollary 1.3.8. *Let M be a closed smooth manifold of dimension 8, $k \geq 4$ or $k = \infty$. Then for $C^{k,\alpha}$ generic metric g on M, every minimal hypersurface generated from Almgren-Pitts min-max theory (or Allen-Cahn min-max theory) is regular and nondegenerate.*

Organization of the Thesis.

Chapter 3 contains the author's work [Wan22a] on studying general minimal hypercones $\mathbf{C}$ in higher dimension. In particular, we prove Theorem 1.3.1 and 1.3.2.

We also prove a Harnack Inequality, Corollary 3.2.5, for stable minimal hypersurfaces away from varifolds of higher multiplicity, which extends a previous work of Bombieri-Giusti [BG72] for minimizing hypersurfaces. This enables us to obtain an optimal growth estimate, Proposition 3.3.1, for positive Jacobi fields on a stable minimal hypersurface, extending a result of Simon [Sim08], which plays a key role in the proof of Theorem 1.3.1 and 1.3.2.

Chapter 4 contains the author's work [Wan22b] on min-max minimal hypersurfaces with obstacle. In particular, we prove Theorem 1.3.4 based on the idea of min-max construction by Pitts [Pit81] and the regularity result of Schoen-Simon [SS81]. One novelty of this proof is an analogous Schoen-Simon ϵ-Regularity Theorem [SS81, Theorem 1] for obstacle problem, see Theorem 4.1.1. We also include a local rigidity result, discussing when solutions to the obstacle problem are genuine minimal hypersurfaces, see Theorem 4.1.2.

Chapter 5 is based on the author's work [Wan20] on Deformation Theory of singular minimal hypersurfaces. We develop the linear analysis of Jacobi operator on a minimal hypersurface with strongly isolated singularities, and then prove the spectral Decomposition Theorem 1.3.5. We then discuss two primary applications of them. Under technical assumptions on the tangent cones, first we prove a local existence Theorem of minimal hypersurface near a given non-degenerate singular minimal hypersurface; We also derive a local minimizing property for a strictly stable minimal hypersurface. See Theorem 5.1.1 and 5.1.3 for the precise statement.

In Chapter 6, based on the linear analysis and spectral theory developed in Chapter 5, as well as the joint work with Yangyang Li [LW20, LW22], we prove the full generic regularity Theorem 1.3.6 for eight manifolds. As an application, we generalize many generic geometric properties of Almgren-Pitts min-max minimal hypersurfaces, previously only known in dimension between 3 and 7, to dimension 8.

Chapter 2

Preliminary

2.1 Notations and Conventions

Throughout this thesis, let $\mathbb{R}^{n+1}$ be the Euclidean space of dimension $n+1$, $n \geq 1$; Let

- $\mathbb{B}_r^{n+1}(x)$: the open ball of radius r in $\mathbb{R}^{n+1}$ centered at x; We may omit the superscript $n+1$ if there's no confusion about dimension; We may write $\mathbb{B}_r := \mathbb{B}_r(\mathbf{0})$ to be the ball centered at the origin $\mathbf{0}$;
- $\mathbb{S}^n := \partial\mathbb{B}_1^{n+1}$: the unit sphere;
- $\mathbb{A}^{n+1}(x; r, s)$: the open annuli $\mathbb{B}_r^{n+1}(x) \setminus \mathrm{Clos}(\mathbb{B}_s^{n+1}(x))$ centered at x;
- $\mathbf{C}(\mathcal{E}) := \{r\omega : \omega \in \mathcal{E} \subset \mathbb{S}^n, r \geq 0\} \subset \mathbb{R}^{n+1}$ be the cone generated by subset $\mathcal{E} \subset \mathbb{S}^n$;
- $\eta_{x,r}$: the map between $\mathbb{R}^{n+1}$, maps y to $r(y-x)$; We may omit subscript x if $x = \mathbf{0}$; For $E \subset \mathbb{R}^{n+1}$, we denote for simplicity $r \cdot (E - x) := \eta_{x,r}(E)$;
- $\mathscr{H}^k$: the k-dimensional Hausdorff measure;
- ω_k : the volume of k-dimensional unit sphere $\mathbb{B}^k$;

- g_{euc} : the Euclidean metric on $\mathbb{R}^{n+1}$.

For any smooth oriented Riemannian manifold (M, g) of dimension $n+1$, write

- $\text{Int}(A)$ be the interior of a subset $A \subset M$;
- $\text{Clos}(A)$ be the closure of a subset $A \subset M$;
- $\partial A := \text{Clos}(A) \setminus \text{Int}(A)$ be the topological boundary of A;
- $dist_g$ be the distant function over (M, g);
- $B_r^g(A) = B_r^M(A) = B^g(A, r) := \{x \in M : dist_g(x, A) < r\}$ be the open r-neighborhood of subset $A \subset M$;
- ∇_g (or ∇^g) be the Levi-Civita connection with respect to g;
- $\exp_x^g$ be the exponential map of (M, g) on the tangent space T_xM;
- $\text{injrad}(x; M, g)$ be the injectivity radius of (M, g) at x; $\text{injrad}(M, g)$ be the infimum of $\text{injrad}(x; M, g)$ over all $x \in M$;
- Ric_g be the Ricci curvature tensor of (M, g);
- $\mathscr{X}_c(U)$ be the space of compactly supported smooth vector field on an open subset $U \subset M$;
- e^{tX} be the one-parameter family of diffeomorphism generated by $X \in \mathscr{X}_c(U)$.

We may omit the super or subscript g if there's no confusion. Also for two subsets $U, V \subset M$, write $U \subset\subset V$ if $\text{Clos}(U)$ is a compact subset of V.

For a sequence of closed subset $\{E_j\}_{1 \le j \le \infty}$ of M, we call E_j converges to E_∞ **locally in the Hausdorff distant sense**, if for every compact subset $K \subset M$ and every $\epsilon > 0$, there exists $j(\epsilon, K) >> 1$ such that $\forall j \ge j(\epsilon, K)$,

$$E_j \cap K \subset B_\epsilon(E_\infty), \qquad E_\infty \cap K \subset B_\epsilon(E_j).$$

Let $\Sigma \subset (M^{n+1}, g)$ be a C^2 hypersurface (need not be two-sided). Throughout this thesis, unless otherwise stated, every such hypersurface Σ is assumed to have $\mathscr{H}^n(\Sigma) < +\infty$ and $\mathscr{H}^{n-2}(\mathrm{Clos}(\Sigma) \setminus \Sigma) = 0$. Let

- $\mathrm{Reg}(\Sigma) := \{x \in \mathrm{Clos}(\Sigma) : \mathrm{Clos}(\Sigma)$ is an embedded C^2 hypersurface near $x\}$;
- $\mathrm{Sing}(\Sigma) := \mathrm{Clos}(\Sigma) \setminus \mathrm{Reg}(\Sigma)$.

By adding points to Σ, we may identify without loss of generality that $\Sigma = \mathrm{Reg}(\Sigma)$. Call Σ **regular** if $\mathrm{Sing}(\Sigma) = \emptyset$.

For an open subset $U \subset M$, Σ is called **two-sided** in U if there exists a globally defined unit normal field ν on $\Sigma \cap U$; if not, then it's called **one-sided** in U. If Σ is portion of the boundary of some specified domain, then it's always two-sided. Unless otherwise mentioned, we use the convention that ν is chosen to be the outward pointed normal field in this case. Also note that since $\mathscr{H}^{n-2}(\mathrm{Sing}(\Sigma)) = 0$, when U is simply connected, Σ is always two-sided in U.

When Σ is connected and one-sided in M, there's a unique double cover $\pi : \hat{M} \to M$ such that $\hat{\Sigma} := \pi^{-1}(\Sigma) \subset (\hat{M}, \hat{g} := \pi^* g)$ is a connected two-sided $\mathbb{Z}_2$-invariant LSMH, where $\mathbb{Z}_2$ acts on $\hat{M}$ by deck transformation. This also induces a natural $\mathbb{Z}_2$-action on $C^0_{loc}(\hat{\Sigma})$ by $(-\mathbf{1}, f) \mapsto \check{f}$, where $-\mathbf{1} \in \mathbb{Z}_2$ is the nontrivial element, $\check{f}(x) := -f(-x)$. For any subset $\mathscr{F} \subset C^0_{loc}(\hat{\Sigma})$, we denote by

$$\mathscr{F}^{\mathbb{Z}_2} := \{f \in \mathscr{F} : \check{f} = f\}. \tag{2.1.1}$$

The Riemannian metric on Σ is induced from (M, g). Since we may deal with families of minimal hypersurfaces and metrics, we may use subscript $\nabla_{\Sigma,g}$, $\Delta_{\Sigma,g}$ etc. to emphasis that they are operators with respect to Σ, g, and may omit the subscript if there's no confusion. Let

- $(\vec{A}_{\Sigma,g})_{ij} := (e_i \cdot \nabla^g e_j)^\perp$ be the second fundamental form of Σ, where $\{e_i\}$ is an orthonormal frame of Σ; $(A_{\Sigma,g})_{ij} := -(\vec{A}_{\Sigma,g})_{ij} \cdot \nu$ be the scalar second fundamental

form if a normal field ν is specified;

- $\vec{H}_{\Sigma,g} := (e_i \cdot \nabla^g e_i)^\perp$ be the mean curvature vector of Σ; $H_{\Sigma,g} := tr_{\Sigma,g} A_{\Sigma,g} = -\vec{H}_{\Sigma,g} \cdot \nu$ be the scalar mean curvature if a normal field ν is specified.

Recall that Σ is called **minimal** in (M, g) if and only if $H_{\Sigma,g} = 0$. Also note that under this convention, the mean curvature vector does not depend on the choice of normal field and points into the area-decreasing direction; And the scalar mean curvature for unit sphere with respect to the outward pointed normal field is positive.

We call a hypersurface $\mathbf{C} \subset \mathbb{R}^{n+1}$ **hypercone** if it's invariant under dilation η_λ for every $\lambda > 0$. Call such $\mathbf{C}$ a **regular** hypercone, if further $\mathrm{Sing}(\mathbf{C}) \subset \{\mathbf{0}\}$.

If Σ admits a global unit normal field ν, for any $E \subset \Sigma$ and function $u : E \to \mathbb{R}$, let

$$\mathrm{graph}_{\Sigma,g}(u) := \{\exp^g_x(u(x) \cdot \nu(x)) : x \in E\},$$

be the graph of u over Σ, u will be called the graphical function. In Appendix A, we carefully analyze the geometry of $\mathrm{graph}_{\Sigma,g}(u)$ in terms of u.

The following concept of *regularity scale* was introduced in [CN13].

Definition 2.1.1. For every $x \in \Sigma$, we define the **regularity scale** $r_{\mathcal{S}}(x) := r_{\mathcal{S}}(x; M, g, \Sigma)$ of Σ at x to be the supremum among all $r \in (0, \mathrm{injrad}(x; M, g)/2)$ such that,

- $r^2 \| \mathrm{Rm}_g \|_{C^0, B^g_r(x)} + r^3 \| \nabla \mathrm{Rm}_g \|_{C^0, B^g_r(x)} \leq 1/10$;

- After pulling back by $\exp^g_x$, in $T_x M$,

$$\frac{1}{r}(\exp^g_x)^{-1}(\Sigma) \cap \mathbb{B}_1 = \mathrm{graph}_L u \cap \mathbb{B}_1,$$

for some linear hyperplane $L \subset T_x M$ and $u \in C^3(L)$ with $\|u\|_{C^3} \leq 1/10$.

It's easy to check by definition that,

(i) If $(M, g, \Sigma) \neq (\mathbb{R}^{n+1}, g_{\text{euc}}, \mathbb{R}^n)$, then $r_{\mathcal{S}} < +\infty$;

(ii) $r_{\mathcal{S}}$ is locally Lipschitz on Σ and $r_{\mathcal{S}}(x) \to 0$ when $x \to x_0 \in \overline{\Sigma} \setminus \Sigma$;

(iii) $|A_\Sigma|^2 + |\operatorname{Rm}_g| \leq C_n \cdot r_{\mathcal{S}}^{-2}$ on Σ;

(iv) For a minimal hypersurface with only *strongly isolated singularities*, we can bound the regularity scale from above and below by the distant function to the singular set. See Corollary 2.3.2 for the precise statement.

We shall work with the following function spaces on Σ. For every open subset $\mathcal{U} \subset \Sigma$ and every integer $k \geq 0$, let

- $L^p(\mathcal{U})$ measurable functions f with $\int_{\mathcal{U}} |f|^p < +\infty$;
- $W^{1,2}_{loc}(\mathcal{U})$ measurable functions which restricts to $W^{1,2}$-function on each compact smooth sub-domain in $\mathcal{U}$;
- $C^k_{loc}(\mathcal{U})$ functions on Σ which admit up to k-th order continuous derivatives, possibly unbounded on $\mathcal{U}$;

Also let $L^2(\mathcal{U})$, $W^{1,2}(\mathcal{U})$ and $C^k(\mathcal{U})$ be the space of functions with finite L^2-, $W^{1,2}$- and C^k- norms respectively.

For $\phi \in C^{k,\alpha}_{loc}(\Sigma)$, $k \in \mathbb{Z}_{\geq 0}$, and $\alpha \in [0, 1]$, we define

$$\begin{aligned} \|\phi\|_{C^{k,\alpha}_*;\mathcal{U},g} := \sup_{x \in \mathcal{U}} \sum_{j=0}^{k} r_{\mathcal{S}}(x)^{j-1} |\nabla^j_{\Sigma,g} \phi|(x) \\ + \sup_{\substack{x \neq y \in \mathcal{U}; \\ dist_g(x,y) \leq r_{\mathcal{S}}(x)}} r_{\mathcal{S}}(x)^{k-1+\alpha} \cdot \frac{|\nabla^k_{\Sigma,g}\phi(x) - \nabla^k_{\Sigma,g}\phi(y)|}{dist_g(x,y)^\alpha}. \end{aligned} \tag{2.1.2}$$

And we denote for simplicity $\|\phi\|_{C^k_*;\mathcal{U},g} = \|\phi\|_{C^{k,0}_*;\mathcal{U},g}$ When $\mathcal{U} = \Sigma$, we may omit the subscript $\mathcal{U}$ and if there's no ambiguity about choice of metric, then we may also omit the subscript g. The reason we use this norm is that it's invariant under rescalings,

i.e. for every $\lambda > 0$, we have that $\mathrm{graph}_{\Sigma,\lambda^2 g}(\lambda\phi) = \mathrm{graph}_{\Sigma,g}(\phi)$ and that,

$$\|\lambda\phi\|_{C^k_*;\mathcal{U},\lambda^2 g} = \|\phi\|_{C^k_*;\mathcal{U},g}\,.$$

2.2 Basics in Geometric Measure Theory

Let (M^{n+1}, g) be a Riemannian manifold (not necessarily complete, isometrically embedded in some $\mathbb{R}^L$ if necessary). We first list some notations in geometric measure theory. Interested readers may refer to [Sim83b].

- $\mathbf{I}_n(M;\mathbb{Z}_2)$: the space of n-dimensional mod 2 flat chains in M endowed with $\mathcal{F}$ metric;
- $\mathcal{Z}_n(M;\mathbb{Z}_2)$: the space of n-dimensional mod 2 flat boundaries in M endowed with $\mathcal{F}$ metric;
- $\mathbf{M}_U$: mass in an open subset $U \subset M$ of flat chains in $\mathbf{I}_n(M;\mathbb{Z}_2)$. We omit U if $U = M$ hereafter;
- $\mathcal{F}_U$: flat metric in an open subset $U \subset M$ on $\mathbf{I}_n(M;\mathbb{Z}_2)$,

$$\mathcal{F}_U(S,T) := \inf\{\mathbf{M}_U(Q) + \mathbf{M}_U(R) : S - T = Q + \partial R,\ \mathrm{spt}(R) \subset U\}\,;$$

- $\mathcal{V}_n(M)$: the space of n-dimensional varifolds in M;
- $\mathcal{IV}_n(M)$: the space of n-dimensional integral varifolds in M;
- $\mathbf{F}_U$: varifold distant in an open subset $U \subset M$ on $\mathcal{V}_n(M)$;
- $\mathrm{spt}(V)$ be the support of a varifold $V \in \mathcal{V}_k(M)$;
- $f_\sharp$ be the push forward of currents or varifolds associated to a proper Lipschitz map f.

- $|T|$: the associated integral varifold for $T \in \mathbf{I}_n(M;\mathbb{Z}_2)$;
- $\|T\|$: the associated Radon measure for $T \in \mathbf{I}_n(M;\mathbb{Z}_2)$;
- $\|V\|$: the associated Radon measure for $V \in \mathcal{IV}_n(M)$;
- $\theta^n_V(a,r) = \theta^n(a,r;\|V\|_g)$: the n-dimensional volume ratio of V in $B^g(a,r)$;

For a hypersurface $\Sigma \subset (M,g)$, let $|\Sigma|_g \in \mathcal{IV}(M)$ be the integral varifold associated to Σ; If further Σ is oriented, denote $[\Sigma] \in \mathbf{I}_n(M)$ to be the associated integral current. More generally, for $T \in \mathbf{I}_n(M)$, denote $|T|_g$, $\|T\|_g$ to be the associated integral n-varifold and Radon measure on M correspondingly. The metric subscript g above will be omit if there's no confusion.

For a smooth vector field $X \in \mathscr{X}_c(U)$ and $V \in \mathcal{IV}_n(U)$, the first variation of the area of V with respect to X is

$$\delta V(X) = \frac{d}{dt}\Big|_{t=0} \|V\|(U) = \int div^{\pi} X \; dV(x,\pi),$$

Recall V is called **stationary** in U if and only if $\delta V(X) = 0$, $\forall X \in \mathscr{X}_c(U)$. By constructing appropriate test vector fields, whenever V is stationary in M, for every $x \in M$, the following

$$r \mapsto \tilde{\theta}^n_V(x,r) := \frac{e^{n\lambda r}}{\omega_n r^n}\|V\|(B_r(x)) = e^{n\lambda r} \cdot \theta^n_V(x,r)$$

is monotone non-decreasing in $r \in (0, \mathrm{injrad}(x;M,g))$, where

$$\lambda = \lambda(x,r;g) := \inf\{C \geq 0 : \nabla^2_g(\rho_x^2) \geq 2(1 - C\rho_x^2) \text{ as a quadratic form, on } B_r(x)\};$$

where $\rho_x := dist_g(\cdot, x)$ is the distant function to x. And when $g = g_{\mathrm{euc}}$, $\tilde{\theta}_g(x,r;\|V\|)$ is constant in r if and only if V is a cone. Also, denote $\theta(x;\|V\|) := \lim_{r \searrow 0} \tilde{\theta}_g(x,r;\|V\|)$

to be the density of V at x.

Following [DG61, Sim83b], we call a Lebesgue measurable subset $P \subset M$ a **Caccioppoli set**, if for every open subset $W \subset\subset M$ and every smooth vector field $X \in \mathscr{X}_c(W)$, there exists some constant $C(P, W)$ such that

$$|\int_P div(X)\ d\mathscr{H}^{n+1}| \leq C(P, W)\|X\|_{C^0}.$$

By [DG61, Sim83b], a Caccioppoli set P is naturally an integral $n+1$-current $[P] \in \mathbf{I}_{n+1}(M)$. $\|\partial[P]\|(W)$ is called the **perimeter** of P in W, and is usually denoted as $\mathcal{P}(P; W)$. Also, we define the **F**-metric among Caccioppoli sets by

$$\mathbf{F}(P_1, P_2) := \mathcal{F}(\partial[P_1], \partial[P_2]) + \mathbf{F}(\|\partial[P_1]\|, \|\partial[P_2]\|).$$

When $E \subset M$ is a measurable subset, we call a Caccioppoli set $P \subset M$ (homologically) **perimeter minimizing** in E if $P \subset E$ and for every open subset $W' \subset\subset W \subset M$ and every Caccioppoli set $Q \subset M$ with $P \Delta Q \subset E \cap W'$, we have

$$\mathcal{P}(P; W) \leq \mathcal{P}(Q; W).$$

If $E \subset M$ is a closed subset, $P \subset E$ is minimizing in E and $|\partial[P]|$ is stationary in M, then the regular part of $\partial[P]$ is a multiplicity one *stable hypersurface* (see Section 2.3 for the notion of stability); Being minimizing in E guarantees that there's no singularity of $\partial[P]$ near which $\mathrm{spt}(\partial[P])$ is a union of embedded $C^{1,\alpha}$ hypersurface-with-boundary meeting along their common boundary. Hence, by [Wic14, Theorem 18.1], $\mathrm{Reg}(\partial[P])$ is a stable minimal hypersurface in M with *optimal regularity* (See Section 2.3 for the notion of optimal regularity)

The compactness of perimeter minimizing Caccioppoli sets in closed subsets is a

little bit subtle, due to the lack of direct cut-pasting argument as in [Sim83b, Theorem 34.5].

Lemma 2.2.1 (Compactness)**.** *Let $\{E_j\}_{1\leq j\leq\infty}$ be a sequence of closed subsets in M such that $E_j \to E_\infty$ and $\partial E_j \to \partial E_\infty$ both in locally Hausdorff distant sense in M. Let $\{g_j\}_{1\leq j\leq\infty}$ be a sequence of smooth metrics on M such that g_j smoothly converges to g_∞. For $1 \leq j < \infty$, let $P_j \subset E_j$ be a Caccioppoli set in M minimizing perimeter in E_j, with $|\partial[P_j]|$ stationary in M.*

Then there exists some Caccioppoli set $P_\infty \subset E_\infty$ with $\mathrm{Reg}(\partial[P_\infty])$ an optimally regular stable minimal hypersurface in (M, g_∞), such that up to a subsequence of $j \to \infty$, $\mathbf{F}(P_j, P_\infty) \to 0$.

Moreover, if $E_j \equiv E_\infty$, then P_∞ is also perimeter minimizing in E_∞.

Proof. For each smooth domain $W \subset\subset M$, by comparing the perimeter of P_j with $P_j \setminus W$, we know that the $\limsup \|\partial[P_j]\|(W) < +\infty$. Hence by Fleming-Federer compactness Theorem, $\partial[P_j]$ subconverges in flat topology to some boundary of Caccioppoli set $\partial[P_\infty]$; And by [SS81], the stable minimal hypersurfaces $|\partial[P_j]|$ subconverges to some stationary integral varifold V_∞ supported on a stable minimal hypersurace in E_∞.

Moreover, V_∞ must be of multiplicity 1, since otherwise, by [SS81, Theorem 1], near some regular point x of higher multiplicity, $\|\partial[P_j]\|$ are multiple graphs over some hyperplane through $x \in \mathrm{spt}(V_\infty) \subset E_\infty$. When $x \in \mathrm{Int}(E_\infty)$, since ∂E_j converges to ∂E_∞ locally in the Hausdorff distant sense, this is impossible by the local perimeter-minimizing property of P_j near x; When $x \in \partial E_\infty$, we know that for $j >> 1$, there must be two adjacent graphs bounding a slab in P_j near x. This violates that P_j is perimeter-minimizing in E_j by subtracting a small ball from this slab.

Using Schoen-Simon's Epsilon Regularity Theorem [SS81, Theorem 1], V_∞ being of multiplicity 1 implies that $V_\infty = |\partial[P_\infty]|$, hence P_j subconverges to P_∞ in $\mathbf{F}$-metric.

When $E_j \equiv E_\infty$, the cut-pasting argument in [Sim83b, Theorem 34.5] works here to show that P_∞ is perimeter minimizing in E_∞. □

2.3 Locally Stable Minimal Hypersurfaces

Following [Sim68,SS81], a minimal hypersurface $\Sigma \subset (M^{n+1}, g)$ is called **stable** in an open subset $U \subset M$, if for every C^2 family of diffeomorphism $\{\phi_t : U \to U\}_{t\in(-1,1)}$ such that $\phi_t = id_U$ near ∂U, we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathscr{H}^n(\phi_t(\Sigma)) \geq 0.$$

By [SS81], the singular set of a stable minimal hypersurfaces Σ in an $n+1$ dimensional domain U satisfies

$$\mathrm{Sing}(\Sigma) = \begin{cases} \emptyset, & \text{if } 3 \leq n+1 \leq 7; \\ \text{isolated points}, & \text{if } n+1 = 8; \\ \text{a closed subset of } U \text{ with Hausdorff dimension } \leq n-7, & \text{if } n+1 \geq 9. \end{cases}$$

We say a hypersurface has **optimal regularity**, if its singular set satisfying this estimate.

If Σ has a global unit normal field $\nu = \nu_{\Sigma,g}$ in U, then Σ being stable in U is equivalent to that

$$Q_{\Sigma,g}(\varphi,\varphi) := \int_\Sigma |\nabla_{\Sigma,g}\varphi|^2 - (|A_{\Sigma,g}|^2 + \mathrm{Ric}_g(\nu,\nu))\varphi^2 \geq 0, \quad \forall \varphi \in C_c^2(\Sigma \cap U). \tag{2.3.1}$$

We call Σ a **locally stable minimal hypersurface (LSMH)** if for every $p \in M$,

there exists some neighborhood $U_p \ni p$ in which Σ is stable. Let

$$L_{\Sigma,g} := \Delta_{\Sigma,g} + |A_{\Sigma,g}|^2 + \mathrm{Ric}_g(\nu,\nu), \tag{2.3.2}$$

be the E-L operator associated to $Q_{\Sigma,g}$, called **Jacobi operator**. Any solution $u \in C^2_{loc}(\Sigma)$ to $L_{\Sigma,g}u = 0$ is called a **Jacobi field**.

Following [MN21,Dey19], the index of a two-sided LSMH $\Sigma \subset (M^{n+1}, g)$ is defined by

$$\begin{aligned}\mathrm{index}(\Sigma) := \sup\Big\{ \dim \mathcal{L} : \mathcal{L} \subset \mathscr{X}_c(M) \text{ is a linear subsapce such that} \\ \frac{d^2}{dt^2}\Big|_{t=0} \mathscr{H}^n(e^{tX}(\Sigma)) < 0,\ \forall \mathbf{0} \neq X \in \mathcal{L}\Big\}.\end{aligned} \tag{2.3.3}$$

When $n+1 = 8$, it is proved in [Wan20] that a two-sided LSMH in a closed manifold always has finite index, and coincides with the index of its Jacobi operator, see Section 5.3.1 for detail. Also, we define the index of a one-sided LSMH to be the index of its two-sided double cover.

It is known from [SS81,Ilm96] that near each singular point p, a LSMH $\Sigma \subset M^{n+1}$ is modeled on stable minimal hypercones. More precisely, for every $r_j \searrow 0$, the blow up sequence $r_j^{-1}(\Sigma - p)$ **F**-subconverges to some stable minimal hyercone $\mathbf{C} \subset \mathbb{R}^{n+1}$ with multiplicity 1, which is called a **tangent cone** of Σ at p.

We assert that a priori, the tangent cone $\mathbf{C}$ might depend on the choice of blow-up radii $\{r_j\}$. A well-known conjecture in geometric measure theory, usually referred to as the *uniqueness of tangent cone conjecture*, asserts that the tangent cones at a fixed point p for any choice of $r_j \searrow 0$ are the same. When one of the tangent cone $\mathbf{C}$ at p is of multiplicity one and regular (this is always the case when $n+1 = 8$ and Σ is locally stable), in other words $\mathrm{Sing}(\mathbf{C}) \subset \{\mathbf{0}\}$, then the uniqueness of tangent cone is proved by Allard and Almgren [AAJ81] under further integrability assumption, and then by L. Simon [Sim83a] in general. This allows us to denote this tangent

cone of Σ at p by $\mathbf{C}_p\Sigma$ in this thesis. Moreover, by adapting the proof in [Sim83a], Edelen [Ede21, Theorem 6.3] proved a quantitative version of uniqueness of tangent cone. For sake of completeness, we include in Appendix D a brief discussion of this.

We call a stationary varifold $V \in \mathcal{IV}_n(M)$ have **strongly isolated singularity** at $p \in M$ if the tangent cone of V at p is a multiplicity one regular minimal hypercone. The following Lemma, which is an easy consequence of Allard Regularity Theorem [All72], guarantees that we can locally parametrize LSMH using their regular tangent cone.

Lemma 2.3.1. *Let $V \in \mathcal{IV}_n(U)$ be a stationary integral varifold under metric g, $p \in \mathrm{spt}(V) \cap U$. Suppose the tangent cone $\mathbf{C} \subset T_pM \approx \mathbb{R}^{n+1}$ of V at p is regular and of multiplicity 1. We let $\mathbf{B}_r := \mathbf{C} \cap \mathbb{B}^{n+1}_r$. Then there exists $0 < \tau_0(\Sigma, U, g, \mathbf{C}) < \mathrm{injrad}(p; U, g)/2$ and a function $u \in C^3_{loc}(\mathbf{B}_{\tau_0})$ such that*

$$\|u\|_{C^3_*(\mathbf{B}_s)} \leq \epsilon(s), \quad \forall s \in (0, \tau_0]\,,$$

for some decreasing function $\epsilon \in C^0([0, \tau_0])$ with $\epsilon(0) = 0$; And after pulled back to M by $\exp^g_p$,

$$V \llcorner B^g_{\tau_0}(p) = |\,\mathrm{graph}_{\mathbf{C},g}(u)| \llcorner B^g_{\tau_0}(p)\,.$$

A direct consequence of this Lemma is that we can estimate the regularity scale using the distant to the singular set. The proof is left to readers.

Corollary 2.3.2. *Let Σ be a LSMH in a closed manifold (M^{n+1}, g) with only strongly isolated singularities. Then there exists constant $C(M, g, \Sigma) > 1$ such that*

$$C^{-1} r_{\mathcal{S}} \leq dist_g(\cdot, \mathrm{Sing}(\Sigma)) \leq C r_{\mathcal{S}}\,.$$

2.4 Minimal Hypercone

For every $n+1 \geq 8$ and $\Lambda, \Lambda' > 0$, we define,

$$\mathscr{C}^n_\Lambda := \{\text{stable minimal hypercone } \mathbf{C} \subset \mathbb{R}^{n+1} : \|\mathbf{C}\|(\mathbb{B}_1) \leq \Lambda\}, \tag{2.4.1}$$

equipped with $\mathbf{F}$-metric in $\mathbb{B}^{n+1}_1$. Note that by Schoen-Simon's Compactness Theorem [SS81] and Lemma 3.2.2 later, for each $\Lambda > 0$, $\mathscr{C}^n_\Lambda$ is compact under $\mathbf{F}$-metric.

Since $\mathbb{R}^{n+1}$ is simply connected, every $\mathbf{C} \in \mathscr{C}^n_\Lambda$ is two-sided. And by Ilmanen's strong maximum principle [Ilm96], $\mathbb{R}^{n+1} \setminus \mathbf{C}$ has exactly 2 connected component, denoted by $E_\pm$. We can choose a unit normal field ν on $\mathbf{C}$ pointing in E_+.

Now suppose that $(\mathbf{C}, \nu) \subset \mathbb{R}^{n+1}$ is a regular area-minimizing hypercone, i.e. $\mathrm{Sing}(\mathbf{C}) \subset \{\mathbf{0}\}$ and $[\mathbf{C}]$ is an area-minimizing current in $\mathbb{R}^{n+1}$. Hardt and Simon [HS85] obtained the following groundbreaking result in smoothing minimizing hypercones.

Theorem 2.4.1 ([HS85, Theorem 2.1]). *Let $\mathbf{C} \neq \mathbb{R}^n$ be a nontrivial regular minimizing hypercone as above; E be one of the components $E_\pm$. Then there exists an area-minimizing hypersurface $\Sigma \subset E$ with $\mathrm{Sing}(\Sigma) = \emptyset$, $dist_{\mathbb{R}^{n+1}}(\mathbf{0}, \Sigma) = 1$ and $[\Sigma] = \partial[P]$ for some closed Caccioppoli set $P \subset E$.*

Furthermore, there exist $h \in C^\infty(\mathbf{C})$ and $c \neq 0$ such that $\Sigma = \mathrm{graph}_{\mathbf{C}}(h)$ outside some large ball, and when $R \to +\infty$, either

$$h(R,\omega) = \begin{cases} cR^{\gamma_1^+} w_1(\omega) + O(R^{\gamma_1^+ - \epsilon}), & \text{if } \gamma_1^+ > \gamma_1^-; \\ cR^{\gamma_1^+} \log R \; w_1(\omega) + O(R^{\gamma_1^+ - \epsilon}), & \text{if } \gamma_1^+ = \gamma_1^-; \end{cases} \tag{2.4.2}$$

or

$$h(R,\omega) = cR^{\gamma_1^-} w_1(\omega) + O(R^{\gamma_1^- - \epsilon}), \tag{2.4.3}$$

where $\gamma_1^{\pm}$ is defined in (C.0.1).

Moreover, Σ foliates E by rescaling (and hence is usually refered to as a leaf of ***Hardt-Simon foliation****). And any other area-minimizing integral n-cycle ∂Q with* $\mathrm{spt}(Q) \subset E$ *is a rescaling of* $[\Sigma]$.

Following Hardt-Simon [HS85], we call a regular minimizing hypercone $\mathbf{C} \subset \mathbb{R}^{n+1}$ **strictly minimizing**, if (2.4.2) holds for every of its Hardt-Simon foliation. Otherwise, $\mathbf{C}$ is called **non-strictly minimizing**. Similarly, we call $\mathbf{C}$ **strictly minimizing in** E if its Hardt-Simon foliation in E satisfies (2.4.2).

We generalize the existence part of Hardt-Simon's Theorem in Chapter 3 to abitrary minimizing or stable hypercones, not necessarily with only isolated singularity. See Theorem 1.3.1 and 1.3.2.

We also prove an analogue of this Regularity Theorem for regular stable minimal hypercone in Lemma 5.2.13, which plays an important role in deriving full generic regularity for 8-dimensional manifolds.

More discussions of regular stable minimal hypercones are presented in Section 5.2 and Appendix C.

Chapter 3

Analysis on Singular Minimal Hypersurfaces, I

3.1 Introduction

In this chapter, we present the work in [Wan22a] to smooth out minimizing or mean convex hypercones without isolated-singularity assumption. More precisely, we prove Theorem 1.3.1 and 1.3.2. A key ingredient of the proof of both Theorem is the following divergence of positive super-solution to Jacobi field equation near singularities.

Theorem 3.1.1 (c.f. Corollary 3.3.3)**.** *Let (M, g) be a smooth Riemannian manifold (not necessarily complete), $\Lambda > 0$. Let $\Sigma \subset M$ be a two-sided stable minimal hypersurface with singular set of codimension ≥ 7. Let $u \in C^2_{loc}(\Sigma)$ be a positive function such that*

$$\Delta_\Sigma u + |A_\Sigma|^2 u - \Lambda u \leq 0,$$

on Σ. Then for every $x \in \mathrm{Sing}(\Sigma)$,

$$\liminf_{y \to x} u(y) = +\infty.$$

Such behavior of u has been established by [Sim08] for minimizing hypersurfaces, or more generally, for submanifolds belonging to a *regular multiplicity 1 class*. A phenomenon of similar spirit was also exploited in [SY17] for minimal slicings and by [Wan20, Lemma 2.14] to study stable minimal hypersurfaces lying on one side of a regular cone near infinity. In this chapter, we prove it for general stable minimal hypersurfaces, following the strategy of [Sim08] but more directly by first proving a Harnack inequality for stable minimal hypersurfaces.

Organization of the Chapter.

In Section 3.2, we prove a multiplicity 1 result for stable minimal hypersurfaces, which enable us to derive a Neumann-Sobolev inequality and a Harnack inequality for stable minimal hypersurfaces following the same argument as [BG72] for minimizing boundaries. Using this, in Section 3.3 we prove a more precise asymptotic lower bound for super-solution of Jacobi field equations on a stable minimal hypersurface, and derive Theorem 3.1.1 as a corollary. Finally, we state and prove a more concrete version of Theorem 1.3.1, 1.3.2 and finish the proof of Theorem 1.3.3 in Section 3.4.

3.2 Harnack Inequality for Minimal Hypersurfaces

By [SS81], the convergence of stable minimal hypersurfaces might result in multiplicity in the limit. The first goal of this section is to show that if a stable minimal hypersurface is away from higher multiplicity at scale 1, then so is it in every smaller scales.

Let $n \geq 3$. Denote $\mathscr{M}^{\geq 2}$ to be the space of hypersurfaces with at least one piece

of higher multiplicity , more precisely,

$$\mathscr{M}^{\geq 2} := \Big\{ V \in \mathcal{IV}_n(\mathbb{B}_4^{n+1}) : V = \sum_{i\geq 1} m_i |\Sigma_i| \text{ for some disjoint minimal hypersurfaces } \Sigma_i \subset (\mathbb{B}_4, g) \text{ and } m_i \in \mathbb{N} \text{ such that } \|g - g_{\mathrm{euc}}\|_{C^{3,1/2}} \leq 1/10, m_1 \geq 2 \text{ and } \Sigma_1 \cap \mathrm{Clos}(\mathbb{B}_2) \neq \emptyset \Big\}.$$

Proposition 3.2.1. *Let $\Lambda > 0$, $\epsilon > 0$. There exists $\delta_1(n, \Lambda) > 0$ and $\Psi(\epsilon|n, \Lambda) \in (0, 1)$ with the following property. If $\Sigma \subset (\mathbb{B}_4^{n+1}, g)$ is a stable minimal hypersurface with*

$$\|g - g_{\mathrm{euc}}\|_{C^4} \leq \delta_1, \qquad \mathbf{M}(\Sigma) \leq \Lambda, \qquad \mathbf{F}(|\Sigma|, \mathscr{M}^{\geq 2}) \geq \epsilon. \tag{3.2.1}$$

Then for every $r \in (0, 1/4)$ and $x \in \mathrm{spt}(\Sigma) \cap \mathbb{B}_1$, we have

$$\mathbf{F}_{\mathbb{B}_4}(|\eta_{x,r^{-1}}(\Sigma)|, \mathscr{M}^{\geq 2}) \geq \Psi(\epsilon|n, \Lambda).$$

Here, $\Psi(\epsilon|n, \Lambda)$ denotes a function in ϵ, n, Λ, which tends to 0 when n, Λ are fixed and $\epsilon \to 0$.

To prove Proposition 3.2.1, we need the following preparations.

Lemma 3.2.2. *For every $\Lambda > 0$, there exists $\delta_2(n, \Lambda) > 0$ such that for every stable minimal hypercone $\mathbf{C} \subset \mathbb{R}^{n+1}$ with $\|\mathbf{C}\|(\mathbb{B}_1) \leq \Lambda$, we have*

$$\mathbf{F}_{\mathbb{B}_4}(|\mathbf{C}|, \mathscr{M}^{\geq 2}) \geq 10\delta_2.$$

Proof. Suppose otherwise, there exist stable minimal hypercones $\mathbf{C}_j$ with uniformly bounded density at origin such that $\mathbf{F}_{\mathbb{B}_4}(|\mathbf{C}_j|, \mathscr{M}^{\geq 2}) \to 0$. Then by [SS81] and definition of $\mathscr{M}^{\geq 2}$ we have, $|\mathbf{C}_j| \to m|\mathbf{C}_\infty|$ for some stable minimal hypercone $\mathbf{C}_\infty$ and integer $m \geq 2$. Also by the same argument as [Sha17, Section 4, Claim 5 & 6] we see

that $\mathbf{C}_j$ induces a positive homogeneous degree 1 Jacobi field $u \in C^\infty_{loc}(\mathbf{C}_\infty)$.

The goal next is to show that such Jacobi field do not exist. Let $\Gamma := \mathbf{C}_\infty \cap \mathbb{S}^n$, and since u is homogeneous degree 1, we can write $u(r\omega) = r \cdot v(\omega)$, $r > 0$, $\omega \in \Gamma$. By rewriting the Jacobi field equation in polar coordinates we derive

$$\begin{aligned} 0 = (\Delta_{C_\infty} + |A_{C_\infty}|^2)u &= [\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}(\Delta_\Gamma + |A_\Gamma|^2)](r \cdot v) \\ &= r^{-1} \cdot [\Delta_\Gamma v + (|A_\Gamma|^2 + n - 1)v], \end{aligned}$$

where $\Delta_\Gamma v + |A_\Gamma|^2 + n - 1$ is the Jacobi operator of $\Gamma \subset \mathbb{S}^n$. Since $v > 0$, this implies $\Gamma \subset \mathbb{S}^n$ is a stable minimal hypersurface, which is impossible. □

The following Almost Cone Rigidity Lemma follows directly from Schoen-Simon's Compactness Theorem [SS81, Theorem 2] and monotonicity of area.

Lemma 3.2.3 (Almost Cone Rigidity)**.** *For every $\epsilon > 0$ and $\Lambda > 0$, there exists $\delta_3(\epsilon, \Lambda, n) \in (0, 1)$ such that if $\Sigma \subset (\mathbb{B}_4^{n+1}, g)$ is a stable minimal hypersurface with $\Sigma \cap \mathbb{B}_1 \neq \emptyset$ and*

$$\|g - g_{\text{euc}}\|_{C^4} \leq \delta_3, \qquad \|\Sigma\|(\mathbb{B}_4) \leq \Lambda, \qquad \theta_g(\mathbf{0}, 4; \|\Sigma\|) - \theta_g(\mathbf{0}, 1; \|\Sigma\|) \leq \delta_3.$$

Then there exists a stable minimal hypercone $\mathbf{C} \subset \mathbb{R}^{n+1}$ and $m \in \mathbb{N}$ such that

$$\mathbf{F}_{\mathbb{B}_2}(|\Sigma|, m|\mathbf{C}|) \leq \epsilon\,.$$

Proof of Proposition 3.2.1. Take $\delta_1 = \delta_3(\delta_2, n, \Lambda)/10$, given by Lemma 3.2.2 and 3.2.3.

Suppose for contradiction, there exists some $\epsilon > 0$ and stable minimal hypersurfaces $\Sigma_j \subset (\mathbb{B}_4, g_j)$, $x_j \in \text{spt}(\Sigma_j) \cap \mathbb{B}_1$ and $r_j \in (0, 1/4)$ such that $\|g_j - g\|_{C^4} \leq \delta_1$, $\mathbf{F}_{\mathbb{B}_4}(|\Sigma_j|, \mathscr{M}^{\geq 2}) \geq \epsilon$ but $\mathbf{F}_{\mathbb{B}_4}(|\eta_{x_j, r_j^{-1}}(\Sigma_j)|, \mathscr{M}^{\geq 2}) \to 0$ when $j \to \infty$. By unique con-

tinuation for stable minimal hypersurfaces, $r_j \to 0$ as $j \to \infty$. By possibly replace ϵ by a smaller ϵ' and translate-rescale Σ, WLOG $x_j \equiv \mathbf{0}$.

For each $j \geq 1$, let

$$I_j := \{\alpha \in \mathbb{N} : \theta_{g_j}(\mathbf{0}, 4^{-\alpha}; \|\Sigma_j\|) - \theta_{g_j}(\mathbf{0}, 4^{-\alpha-1}; \|\Sigma_j\|) \geq \delta_1\} \cup \{+\infty, 0\}$$

Since $\theta_{g_j}(\mathbf{0}, r; \|\Sigma\|)$ is non-decreasing in r, by the mass upper bound on Σ_j, we have $\sharp I_j \leq C(\Lambda, n)$. Write

$$I_j =: \{0 = \alpha_j^{(0)} < \alpha_j^{(1)} < ... < \alpha_j^{(\sharp I_j - 1)} = +\infty\}; \qquad \alpha_j^k := +\infty, \ \text{ for } k \geq \sharp I_j.$$

Consider

$$i_0 := \sup\Big\{ i \in \mathbb{Z}_{\geq 0} : \ \text{for every sequence } \{0 < s_j \in [4^{-\alpha_j^{(i)}-1}, 1/2)\}_{j\geq 1}, \\ \liminf_{j\to\infty} \mathbf{F}_{\mathbb{B}_4}(|\eta_{\mathbf{0},s_j^{-1}}(\Sigma_j)|, \mathscr{M}^{\geq 2}) > 0 \Big\}.$$

By our contradiction assumption, $0 \leq i_0 < C(\Lambda, n)$ and for infinitely many j, $\alpha_j^{(i_0)} < +\infty$. Also by passing to a subsequence in j, there exists $0 < s_j \in [4^{-\alpha_j^{(i_0+1)}-1}, 4^{-\alpha_j^{(i_0)}-1})$ such that $\mathbf{F}_{\mathbb{B}_4}(|\eta_{s_j^{-1}}(\Sigma_j)|, \mathscr{M}^{\geq 2}) \to 0$ when $j \to \infty$.

Denote for simplicity $\hat{\Sigma}_j := \eta_{4^{\alpha_j^{(i_0)}+1}}(\Sigma_j)$, $\hat{g}_j := 4^{2\alpha_j^{(i_0)}+2} \cdot \eta^*_{4^{-\alpha_j^{(i_0)}-1}} g_j$ be the corresponding rescaled metric, by definition of i_0 and Schoen-Simon's Compactness Theorem [SS81], we have

$$|\hat{\Sigma}_j| \to |\hat{\Sigma}_\infty| \quad \text{ in } \mathbb{B}_2, \tag{3.2.2}$$

for some multiplicity 1 stable minimal hypersurface $\hat{\Sigma}_\infty$ in $\mathbb{B}_2$; Also, $\hat{s}_j := s_j \cdot 4^{\alpha_j^{(i_0)}+1} \in$

$(0,1)$ satisfies

$$\mathbf{F}(|\eta_{\hat{s}_j^{-1}}(\hat{\Sigma}_j)|, \mathscr{M}^{\geq 2}) \to 0 \quad \text{as } j \to \infty. \tag{3.2.3}$$

And by definition of I_j, for every integer $l \in [0, -\log_4(\hat{s}_j) - 2)$, we have

$$\theta_{\hat{g}_j}(\mathbf{0}, 4^{-l}; \|\hat{\Sigma}_j\|) - \theta_{\hat{g}_j}(\mathbf{0}, 4^{-l-1}; \|\hat{\Sigma}_j\|) \leq \delta_1. \tag{3.2.4}$$

By (3.2.2) and (3.2.3), $\hat{s}_j \searrow 0$; By (3.2.4) and Lemma 3.2.3, for every $j \geq 1$ and $l \in [1, -\log_4(\hat{s}_j) - 1)$, there exists some stable minimal hypercone $\mathbf{C}_j^{(l)}$ and integer $m_j^{(l)} \in \mathbb{N}$ such that

$$\mathbf{F}_{\mathbb{B}_2}(|\eta_{4^l}(\hat{\Sigma}_j)|, m_j^{(l)}|\mathbf{C}_j^{(l)}|) \leq \delta_2. \tag{3.2.5}$$

On the other hand, recall by [Ilm96], each tangent cone of a stable minimal hypersurface is a multiplicity 1 stable hypercone. Hence, by Lemma 3.2.2, there exists some integer $l_0 = l_0(\hat{\Sigma}_\infty) >> 1$ such that

$$\mathbf{F}_{\mathbb{B}_4}(|\eta_{4^{l_0}}(\hat{\Sigma}_\infty)|, \mathscr{M}^{\geq 2}) \geq 9\delta_2.$$

And since $|\hat{\Sigma}_j| \to |\hat{\Sigma}_\infty|$, we can choose $j_0 >> 1$ such that for every $j \geq j_0$, $\hat{s}_j < 4^{-l_0-3}$ and

$$\mathbf{F}_{\mathbb{B}_4}(|\eta_{4^{l_0}}(\hat{\Sigma}_j)|, \mathscr{M}^{\geq 2}) \geq 8\delta_2$$

Combine this with (3.2.5), we derive $\mathbf{F}_{\mathbb{B}_2}(m_j^{(l_0)}|\mathbf{C}_j^{(l_0)}|, \mathscr{M}^{\geq 2}) \geq 7\delta_2$, and hence $m_j^{(l_0)} = 1$. Moreover, recall for every two integral varifold $V_1, V_2 \in \mathcal{IV}_n(\mathbb{B}_2^{n+1})$ and every $\lambda > 1$, we have

$$\mathbf{F}_{\mathbb{B}_2}((\eta_\lambda)_\sharp V_1, (\eta_\lambda)_\sharp V_2) \leq \lambda \cdot \mathbf{F}_{\mathbb{B}_2}(V_1, V_2).$$

Thus combined with (3.2.5) we see,

$$\begin{aligned}&\mathbf{F}_{\mathbb{B}_2}(m_j^{(l)}|\mathbf{C}_j^{(l)}|, m_j^{(l+1)}|\mathbf{C}_j^{(l+1)}|)\\ \leq\ &\mathbf{F}_{\mathbb{B}_2}(m_j^{(l)}|\mathbf{C}_j^{(l)}|, |\eta_{4^{l+1}}(\hat{\Sigma}_j)|) + \mathbf{F}_{\mathbb{B}_2}(|\eta_{4^{l+1}}(\hat{\Sigma}_j)|, m_j^{(l+1)}|\mathbf{C}_j^{(l+1)}|) \leq 4\delta_2 + \delta_2 = 5\delta_2.\end{aligned}$$

Therefore, using Lemma 3.2.2 and starting at l_0, we can prove inductively that $m_j^{(l)} = 1$ for all integers $l \in [l_0, -\log_4(\hat{s}_j) - 1)$ and again by Lemma 3.2.2 and (3.2.5), we have $\limsup_{j\to\infty} \mathbf{F}_{\mathbb{B}_4}(|\eta_{\hat{s}_j^{-1}}(\hat{\Sigma}_j)|, \mathscr{M}^{\geq 2}) \geq \delta_2 > 0$. This contradicts to (3.2.3). □

Following the argument in [BG72], an immediate corollary of Proposition 3.2.1 is the Neumann-Sobolev type inequality for stable minimal hypersurfaces,

Corollary 3.2.4 (Neumann-Sobolev Inequality)**.** *Let $\epsilon, \Lambda > 0$; δ_1 be in Proposition 3.2.1. Then there exists $\beta(n, \epsilon, \Lambda) \in (0, 1/2)$, $C(n, \epsilon, \Lambda) > 0$ with the following property. If Σ is a stable minimal hypersurface in $(\mathbb{B}_4^{n+1}, g)$ satisfying (3.2.1). Then for every $f \in W^{1,1}(\Sigma)$, we have*

$$\inf_{k\in\mathbb{R}} \|f - k\|_{L^{\frac{n}{n-1}}(\mathbb{B}_\beta; \|\Sigma\|)} \leq C \cdot \int_{\mathbb{B}_4} |\nabla_\Sigma f|\ d\|\Sigma\|.$$

Proof. We shall first show the following,

Claim. There exists $\beta(n, \epsilon, \Lambda) \in (0, 1/2)$, $\tilde{C}(n, \epsilon, \Lambda) > 0$ with the following property. If Σ is a stable minimal hypersurface in $(\mathbb{B}_4^{n+1}, g)$ satisfying (3.2.1); $T_1, T_2 \in \mathbf{I}^n(\mathbb{B}_4)$ are integral currents such that

$$[\Sigma] = T_1 + T_2; \qquad\qquad \|\Sigma\| = \|T_1\| + \|T_2\|. \tag{3.2.6}$$

Then we have for every $x \in \mathrm{spt}(\Sigma) \cap \mathbb{B}_1$,

$$\mathbf{M}_{\mathbb{B}_4}(\partial T_1) = \mathbf{M}_{\mathbb{B}_4}(\partial T_2) \geq \tilde{C}^{-1} \cdot min\{\mathbf{M}_{\mathbb{B}_\beta(x)}(T_1), \mathbf{M}_{\mathbb{B}_\beta(x)}(T_2)\}^{1-1/n}. \tag{3.2.7}$$

Proof of Claim. By a translation and rescaling of Σ and considering a smaller ϵ, WLOG $x = \mathbf{0}$. Suppose the Claim fails, then there are stable minimal hypersurfaces Σ^j satisfying (3.2.1) and integral currents $T_1^j, T_2^j \in \mathbf{I}_n(\mathbb{B}_4)$ such that $[\Sigma^j] = T_1^j + T_2^j$, $\|\Sigma^j\| = \|T_1^j\| + \|T_2^j\|$ and that

$$\mathbf{M}_{\mathbb{B}_4}(\partial T_1^j) = \mathbf{M}_{\mathbb{B}_4}(\partial T_2^j) < \frac{1}{j} \cdot min\{\mathbf{M}_{\mathbb{B}_{j^{-1}}}(T_1^j), \mathbf{M}_{\mathbb{B}_{j^{-1}}}(T_2^j)\}^{1-1/n}. \tag{3.2.8}$$

Also by slicing Theorem [Sim83b], for each $T = T_i^j$ ($j \geq 1$, $i = 1, 2$) and a.e. $s \in (j^{-1}, 4)$,

$$\mathbf{M}_{\mathbb{B}_4}(\partial(T \llcorner \mathbb{B}_s)) \leq \mathbf{M}_{\mathbb{B}_s}(\partial T) + \frac{d}{ds}\mathbf{M}_{\mathbb{B}_s}(T).$$

Combined with (3.2.8) and Micheal-Simon inequality [Sim83b, Theorem 18.6], this implies

$$\mathbf{M}_{\mathbb{B}_s}(T)^{\frac{n-1}{n}} \leq \frac{S(n)}{j}\mathbf{M}_{\mathbb{B}_s}(T)^{\frac{n-1}{n}} + \frac{d}{ds}\mathbf{M}_{\mathbb{B}_s}(T), \quad \text{for a.e. } s \in (j^{-1}, 1). \tag{3.2.9}$$

Take $j \geq 10S(n)$ in (3.2.9) and integrate over (j^{-1}, t), we derive

$$\mathbf{M}_{\mathbb{B}_t}(T_i^j) \geq \delta(n) \cdot (t - j^{-1})^n, \quad \forall t \in (j^{-1}, 1). \tag{3.2.10}$$

By Proposition 3.2.1 and Schoen-Simon compactness, when $j \to \infty$, $|\eta_{\mathbf{0},j}(\Sigma^j)| \to |\Sigma^\infty|$ in $\mathcal{IV}_n(\mathbb{R}^n)$ for some multiplicity 1 stable minimal hypersurface Σ^∞. Moreover, since the tangent cone $\mathbf{C}$ of Σ^∞ at infinity is also some rescaled limit of Σ^j, we must have by Proposition 3.2.1 that $\mathbf{C}$ is of multiplicity 1 and hence Σ^∞ is connected.

By (3.2.8), Fleming-Federer Compactness Theorem and area monotonicity formula for minimal hypersurfaces, we have

$$\mathbf{F}((\eta_{\mathbf{0},j})_\sharp T_i^j, T_i^\infty) \to 0,$$

for some $T_i^\infty \in \mathbf{I}_n(\mathbb{R}^{n+1})$ $(i = 1, 2)$ such that

$$\partial T_1^\infty = \partial T_2^\infty = 0; \qquad T_1^\infty + T_2^\infty = [\Sigma^\infty], \qquad \|T_1^\infty\| + \|T_2^\infty\| = \|\Sigma^\infty\|.$$

And by (3.2.10), $\mathrm{spt}(T_1^\infty) \cap \mathbb{B}_2 \neq \emptyset$, $\mathrm{spt}(T_2^\infty) \cap \mathbb{B}_2 \neq \emptyset$. But by constancy Theorem [Sim83b, Theorem 26.27], this is impossible for a connected stable minimal hypersurface Σ^∞. Hence the Claim is proved. □

Now to prove Corollary 3.2.4, suppose WLOG $f \in C^1(\Sigma)$, and denote for simplicity $p := n/(n-1)$. Choose $k \in \mathbb{R}$ such that

$$\mathbf{M}_{\mathbb{B}_\beta}(\{f > k\}),\ \mathbf{M}_{\mathbb{B}_\beta}(\{f < k\}) \leq \frac{1}{2}\mathbf{M}_{\mathbb{B}_\beta}(\Sigma). \tag{3.2.11}$$

Let $f^\pm := (f-k)^\pm = max\{\pm(f-k), 0\}$; $A^\pm(t) := [\Sigma]\llcorner\{f^\pm > t\}$. Then by co-area formula, we have

$$\begin{aligned}
\int_{\mathbb{B}_4 \cap \Sigma} |\nabla_\Sigma f^+| \ d\|\Sigma\| &= \int_0^\infty \mathbf{M}_{\mathbb{B}_4}(\partial A^+(t)) \ dt \\
&\geq \tilde{C}^{-1} \int_0^\infty \mathbf{M}_{\mathbb{B}_\beta}(A^+(t))^{1/p} \ dt \\
&\geq \tilde{C}^{-1} \left(p \cdot \int_0^\infty \mathbf{M}_{\mathbb{B}_\beta}(A^+(t)) t^{p-1} \ dt \right)^{1/p} \\
&= C^{-1} \|f^+\|_{L^p(\mathbb{B}_\beta; \|\Sigma\|)},
\end{aligned}$$

where the first inequality follows from the Claim and choice of k; the second inequality follows from the following Hardy-Littlewood-Polya Inequality: If $p > 1$ and $\eta \geq 0$ is a non-increasing function on $(0, +\infty)$, then

$$\left(\int_0^\infty \eta^{1/p} \ dt \right)^p \geq p \int_0^\infty \eta(t) t^{p-1} \ dt.$$

Similarly one derive $\int_{\mathbb{B}_4} |\nabla_\Sigma f^-| \ d\|\Sigma\| \geq C^{-1} \|f^-\|_{L^p(\mathbb{B}_\beta; \|\Sigma\|)}$. Hence Corollary 3.2.4 is

proved. □

By De Giorgi-Moser iteration process [GT01] and the abstract John-Nirenberg Inequality [BG72, Section 4], Corollary 3.2.4 implies the following Harnack inequality for stable minimal hypersurfaces.

Corollary 3.2.5 (Harnack Inequality)**.** *Let Σ be in Corollary 3.2.4. There exists $\eta(\epsilon, n, \Lambda) \in (0,1)$ such that if $0 < u \in W^{1,2}_{loc}(\Sigma)$ satisfies $\Delta_\Sigma u - \Lambda u \leq 0$ on Σ in the distribution sense, then for every $p \in (0, \frac{n}{n-2})$, every $x \in \mathrm{spt}(\Sigma) \cap \mathbb{B}_1$ and every $r \in (0, \eta)$, we have*

$$\left(\frac{1}{\|\Sigma\|(\mathbb{B}_r(x))} \int_{\Sigma \cap \mathbb{B}_r(x)} |u|^p \, d\|\Sigma\| \right)^{1/p} \leq C(p, \epsilon, \Lambda, n) \cdot \inf_{\Sigma \cap \mathbb{B}_r(x)} u.$$

Remark 3.2.6. By a slight modification, a similar Neumann-Sobolev Inequality and Harnack inequality still holds for minimal hypersurfaces with bounded index, where the constant will also depend on the index upper bound.

On the other hand, the assumption in (3.2.1) that Σ is away from varifolds with multiplicity ≥ 2 at scale 1 CANNOT be dropped. A typical counterexample is a sequence of pairs of parallel hyperplanes $V_j := |L_j^{(1)}| + |L_j^{(2)}|$ converging to a multiplicity 2 hyperplane, and a sequence of functions on $\mathrm{spt}(V_j)$ which is 1 on $L_j^{(1)}$ and $1/j$ on $L_j^{(2)}$. Counterexample in unstable situation could be a blow down sequence of catenoids converging to a multiplicity 2 hyperplane and a positive harmonic function which approaches 1 on one end and 0 on the other end.

3.3 Growth Rate Lower Bound for Positive Jacobi Fields

The goal of this section is to prove the following.

Proposition 3.3.1. *Let $\epsilon \in (0,1)$, $\Lambda > 0$, $n \geq 7$;*

$$\gamma_n := -\frac{n-2}{2} + \sqrt{(\frac{n-2}{2})^2 - n + 1}$$

For every $\gamma > \gamma_n$, there exists $C = C(\epsilon, \Lambda, n, \gamma) > 0$ and $\tau_0(\epsilon, \Lambda, n, \gamma) \in (0, 1/2)$ such that the following hold.

Let $\Sigma \subset (\mathbb{B}_4^{n+1}, g)$ be a two-sided stable minimal hypersurface satisfying (3.2.1); $u \in W^{1,2}_{loc}(\Sigma)$ be a positive super-solution of $(\Delta_\Sigma + |A_\Sigma|^2 - \Lambda)u = 0$, i.e.

$$\int_\Sigma \nabla_\Sigma u \cdot \nabla_\Sigma \xi + (\Lambda - |A_\Sigma|^2) u\xi \ \geq 0, \quad \forall \xi \in C^1_c(\Sigma, \mathbb{R}_{\geq 0}).$$

Then for every $p \in \mathrm{Sing}(\Sigma) \cap \mathbb{B}_1$ and every $r \in (0, \tau_0)$, we have

$$\inf_{\Sigma \cap \mathbb{B}_r(p)} u \geq C \cdot \|u\|_{L^1(\Sigma \cap \mathbb{B}_{\tau_0}(p))} \cdot r^\gamma. \tag{3.3.1}$$

Note that the exponent γ_n is optimal and can be realized by a homogeneous positive Jacobi field on Simons cone.

The key for the proof is the following Growth Lemma. Let

$$\mathscr{C}_n^{=1}(\Lambda) := \{|\mathbf{C}| : \mathbf{C} \subset \mathbb{R}^{n+1} \text{ stable minimal hypercone with } \|\mathbf{C}\|(\mathbb{B}_4) \leq \Lambda\} \backslash \{\text{hyperplanes}\},$$

be the space of multiplicity 1 varifolds associated to nontrivial stable minimal hypercones. For a function ϕ over Σ, $p > 0$ and a domain $\Omega \subset \Sigma$, let

$$\|\phi\|_{L^p_*(\Omega; \|\Sigma\|)} := \big(\|\Sigma\|(\Omega)^{-1} \cdot \int_\Omega |\phi|^p \ d\|\Sigma\|\big)^{1/p}.$$

Lemma 3.3.2 (Growth Lemma)**.** *Let $\Lambda > 0$, $\gamma > \gamma_n$ be fixed as in Prop 3.3.1. There exists $\delta(n, \Lambda, \gamma) \in (0,1)$ and $r_0(n, \Lambda, \gamma) \in (0, 1/2)$ with the following property.*

Let $\Sigma \subset (\mathbb{B}_{4r_0^{-1}}, g)$ be a two-sided stable minimal hypersurface with normal field ν,

satisfying

$$\mathrm{Sing}(\Sigma) \ni \mathbf{0}, \qquad \|g - g_{\mathrm{euc}}\|_{C^4} \leq \delta, \qquad \mathbf{F}_{\mathbb{B}_4}(|\Sigma|, \mathscr{C}_n^{=1}(\Lambda)) \leq \delta. \tag{3.3.2}$$

Let $0 < u \in W^{1,2}_{loc}(\Sigma)$ *be a positive supersolution of* $\Delta_\Sigma u + (|A_\Sigma|^2 - \delta)u = 0$ *in the distribution sense on* Σ*, in other words,*

$$\int \nabla_\Sigma u \cdot \nabla_\Sigma \xi + (\delta - |A_\Sigma|^2) u\xi \; d\|\Sigma\| \geq 0, \quad \forall \xi \in C^1_c(\Sigma, \mathbb{R}_{\geq 0}). \tag{3.3.3}$$

Then $u \in L^1(\mathbb{B}_1; \|\Sigma\|)$ *and*

$$\|u\|_{L^1_*(\mathbb{B}_r; \|\Sigma\|)} \geq \|u\|_{L^1_*(\mathbb{B}_1; \|\Sigma\|)} \cdot r^\gamma, \quad \forall r \in [r_0/2, r_0].$$

In [Sim08, Theorem 1], a general Growth Theorem is established for multiplicity 1 class of minimal submanifolds. It follows directly from Proposition 3.2.1 that minimal hypersurfaces in Euclidean balls together with their blow-up limits forms a multiplicity 1 class of hypersurfaces. Hence Simon's result applies here to obtain Lemma 3.3.2. For sake of completeness, we include in Section 3.3.1 a direct proof using Harnack Inequality in Corollary 3.2.5.

Proof of Proposition 3.3.1. WLOG $p = \mathbf{0}$. Let $\tau := min\{r_0^{N+1} : r_0^N \geq \eta(\epsilon, \Lambda, n)/4\}$ be given by Lemma 3.3.2 and Corollary 3.2.5; $\tau_0 := \tau^2$. By definition, τ, τ_0 only depend on $\epsilon, \Lambda, n, \gamma$. Let

$$J := \{j \in \mathbb{Z}_{\geq 2} : \eta_{\tau^j}^{-1}(\Sigma) \cap \mathbb{B}_{4r_0^{-1}} \subset (\mathbb{B}_{4r_0^{-1}}, \tau^{-2j}\eta_{\tau^j}^* g) \text{ satisfies (3.3.2)}, u \circ \eta_{\tau^j} \text{ satisfies (3.3.3)}\}.$$

By area monotonicity, Lemma 3.2.3 and Proposition 3.2.1, we see that

$$\sharp(\mathbb{N} \setminus J) \leq N(n, \Lambda, \epsilon, \gamma) < +\infty. \tag{3.3.4}$$

Denote for simplicity $I_u(r) := \|u\|_{L^1_*(\mathbb{B}_r;\|\Sigma\|)}$. Note that by Corollary 3.2.5, for every $j \geq 1$,

$$I_u(\tau^{j+1}) \geq \frac{\|\Sigma\|(\mathbb{B}_{\tau^j})}{\|\Sigma\|(\mathbb{B}_{\tau^{j+1}})} \inf_{\mathbb{B}_{\tau^j}} u \geq c(n,\Lambda,\epsilon,\gamma) I_u(\tau^j). \tag{3.3.5}$$

And for every $j \in J$, by Lemma 3.3.2, we have better estimate,

$$I_u(\tau^{j+1}) \geq I_u(\tau^j) \cdot \tau^\gamma. \tag{3.3.6}$$

Now for every $r \in (0,\tau_0)$, let $l \in \mathbb{Z}_{\geq 2}$ be such that $\tau^{l+1} \leq r < \tau^l$. Combining Corollary 3.2.5, (3.3.4), (3.3.5) and (3.3.6) we have,

$$\begin{aligned}
\inf_{\Sigma\cap\mathbb{B}_r} u &\geq c(n,\Lambda,\epsilon)\|u\|_{L^1_*(\mathbb{B}_{\tau^l};\|\Sigma\|)} \\
&= c(n,\Lambda,\epsilon)\left(\prod_{j=2}^{l-1}\frac{I_u(\tau^{j+1})}{I_u(\tau^j)}\right)\cdot\|u\|_{L^1_*(\mathbb{B}_{\tau_0};\|\Sigma\|)} \\
&= c(n,\Lambda,\epsilon)\left(\prod_{2\leq j\leq l-1;j\in J}\frac{I_u(\tau^{j+1})}{I_u(\tau^j)}\right)\cdot\left(\prod_{2\leq j\leq l-1;j\notin J}\frac{I_u(\tau^{j+1})}{I_u(\tau^j)}\right)\cdot\|u\|_{L^1_*(\mathbb{B}_{\tau_0};\|\Sigma\|)} \\
&\geq c(n,\Lambda,\epsilon)\cdot(\tau^\gamma)^{l-\sharp(\mathbb{N}\setminus J)}\cdot c(n,\Lambda,\epsilon,\gamma)^{\sharp(\mathbb{N}\setminus J)}\cdot\|u\|_{L^1_*(\mathbb{B}_{\tau_0};\|\Sigma\|)} \\
&\geq c(n,\Lambda,\epsilon,\gamma)\|u\|_{L^1_*(\mathbb{B}_{\tau_0};\|\Sigma\|)}\cdot r^\gamma.
\end{aligned}$$

This finishes the proof of the Proposition. □

Since $\gamma_n < -1 < 0$, a direct corollary of Proposition 3.3.1 is,

Corollary 3.3.3. *Let (M,g) be a smooth Riemannian manifold (not necessarily complete), $\Lambda > 0$. Let $\Sigma \subset M$ be a two-sided stable minimal hypersurface with optimal regularity. Let $u \in C^2_{loc}(\Sigma)$ be a positive function such that*

$$\Delta_\Sigma u + |A_\Sigma|^2 u - \Lambda u \leq 0,$$

on Σ. *Then for every* $x \in \mathrm{Sing}(\Sigma)$,

$$\liminf_{y \to x} u(y) = +\infty.$$

3.3.1 Proof of Growth Lemma 3.3.2

The goal of this section is to prove Lemma 3.3.2. We begin with the following uniform control of first eigenfunction for Jacobi operator of cross section. For every $\mathbf{C} \in \mathscr{C}_n^{=1}(\Lambda)$, let $\Gamma = \Gamma(\mathbf{C}) := \mathbf{C} \cap \mathbb{S}^n$ be the cross section of $\mathbf{C}$, and for each domain $\Omega \subset \Gamma$, let

$$\mu_1(\Omega) := \inf\{\int_\Gamma |\nabla_\Gamma \phi|^2 - |A_\Gamma|^2 \phi^2 : \phi \in C_c^1(\Omega), \|\phi\|_{L^2(\Gamma)} = 1\}.$$

Denote for simplicity $\mu_1(\mathbf{C}) := \mu_1(\Gamma(\mathbf{C}))$. Recall by [Law69, Per02, Zhu18], $\mu_1(\mathbf{C}) \leq -(n-1)$, and equality holds if and only if $\mathbf{C}$ is a quadratic cone.

For each $x \in \Gamma(\mathbf{C})$, following [CN13], let

$$r_\Gamma(x) := \sup\{r \in (0,1) : |A_\Gamma| \leq r^{-1} \text{ on } \mathbb{B}_r(x) \cap \Gamma\},$$

be the regularity scale. Clearly, r_Γ is continuous in x and in smooth convergence of Γ. For each $\delta > 0$, denote $\Omega_\delta(\mathbf{C}) := \{x \in \Gamma : r_\Gamma(x) > \delta\}$.

Lemma 3.3.4. *There exists* $\rho_0(n, \Lambda) > 0$ *such that* $\mu_1(\Omega_{\rho_0}(\mathbf{C})) \leq -(n-1)$ *for every* $\mathbf{C} \in \mathscr{C}_n^{=1}(\Lambda)$.

Proof. Suppose otherwise, there are $\mathbf{C}_j \in \mathscr{C}_n^{=1}(\Lambda)$ such that $\mu_1(\Omega_{1/j}(\mathbf{C}_j)) > -(n-1)$.

Suppose $\mathbf{C}_j \to \mathbf{C}_\infty \in \mathscr{C}_n^{=1}(\Lambda)$ and write $\Gamma_j := \mathbf{C}_j \cap \mathbb{S}^n$, $1 \leq j \leq \infty$. Then we must have $\mu_1(\mathbf{C}_\infty) = -(n-1)$. Since otherwise by [Zhu18], $\mu_1(\mathbf{C}_\infty) < -(n-1)$ and

thus there exists $\psi \in C_c^1(\Gamma(\mathbf{C}_\infty))$ such that

$$\int_{\Gamma_\infty} |\nabla_{\Gamma_\infty}\psi|^2 - |A_{\Gamma_\infty}|^2\psi^2 < -(n-1)\int_{\Gamma_\infty}\psi^2.$$

Since the convergence of Γ_j is smooth near $\mathrm{spt}(\psi)$ and r_{Γ_j} is continuous in j and x near $\mathrm{spt}(\psi)$, this is a contradiction to $\mu_1(\Omega_{1/j}(\mathbf{C}_j)) > -(n-1)$.

Now that $\mu_1(\mathbf{C}_\infty) = -(n-1)$. By [Zhu18], $\mathbf{C}_\infty$ is a quadratic cone, which has smooth cross section; Then by Allard Regularity, for $j >> 1$, $\mu_1(\Omega_{1/j}(\mathbf{C}_j)) = \mu_1(\mathbf{C}_j) \leq -(n-1)$, that's a contradiction. □

Lemma 3.3.5. *For every $\Lambda > 1$, there exists a constant $C(n,\Lambda) > 1$ with the following property. Let $\delta_2(n,\Lambda) > 0$ be in Lemma 3.2.2 and $\eta(n,\Lambda) = \eta(\delta_2/2, n, \Lambda) \in (0,1)$ be in Corollary 3.2.5; Also let γ_n be in Proposition 3.3.1.*

If $\mathbf{C} \in \mathscr{C}_n^{=1}(\Lambda)$ is a stable minimal hypercone; $0 < u \in C^\infty_{loc}(\mathbf{C} \cap \mathbb{B}_{4\eta^{-1}})$ is a Jacobi field, i.e. $\Delta_{\mathbf{C}} u + |A_{\mathbf{C}}|^2 u = 0$. Then,

$$\|u\|_{L^1_*(\mathbb{B}_r;\|\mathbf{C}\|)} \geq C(n,\Lambda)^{-1} r^{\gamma_n} \cdot \|u\|_{L^1_*(\mathbb{B}_1;\|\mathbf{C}\|)}, \qquad \text{for every } r \in (0,1).$$

Proof. Let $\Gamma := \mathbf{C} \cap \mathbb{S}^n$ be the cross section of $\mathbf{C}$ in Lemma 3.3.5; ρ_0 be in Lemma 3.3.4. Let Ω be a smooth domain in Γ such that $\Omega_{\rho_0}(\mathbf{C}) \subset \Omega \subset \Omega_{\rho_0/2}(\mathbf{C})$, and $0 < \varphi_\Omega$ be a first Dirichlet eigenfunction of $-\Delta_\Gamma - |A_\Gamma|^2$, i.e.

$$\begin{cases} -(\Delta_\Gamma + |A_\Gamma|^2)\varphi_\Omega = \mu_1(\Omega)\varphi_\Omega, & \text{on } \Omega; \\ \varphi_\Omega = 0, & \text{on } \partial\Omega \text{ if } \partial\Omega \neq \emptyset. \end{cases} \tag{3.3.7}$$

with eigenvalue $\mu_1(\Omega) \leq \mu_1(\Omega_{\rho_0}(\mathbf{C})) \leq -(n-1)$ by Lemma 3.3.4. By definition of Ω, $|A_\Gamma| \leq 2\rho_0^{-1}$ on $\mathbb{B}_{\rho_0/2}(\Omega)$. Hence by covering Ω with ball of radius ρ_0 and applying

classical elliptic estimates [GT01] we see,

$$\sup_{\Omega} |\varphi_\Omega| \leq C(n, \Lambda) \|\varphi_\Omega\|_{L^1}. \tag{3.3.8}$$

Now let $U(r) := \int_\Omega \varphi_\Omega(\omega) u(r\omega)\, d\omega$. Since

$$0 = (\Delta_{\mathbf{C}} + |A_{\mathbf{C}}|^2)u = \partial_r^2 u + \frac{n-1}{r}\partial_r u + \frac{1}{r^2}(\Delta_\Gamma + |A_\Gamma|^2)u.$$

We then have for $r \in (0, 1]$,

$$\begin{aligned} U''(r) + \frac{n-1}{r}U'(r) &= -\frac{1}{r^2}\int_\Omega \varphi_\Omega(\omega)(\Delta_\Gamma + |A_\Gamma|^2)u(r\omega) \\ &= -\frac{1}{r^2}\int_\Omega (\Delta_\Gamma + |A_\Gamma|^2)\varphi_\Omega(\omega)\cdot u(r\omega) + \frac{1}{r^2}\int_{\partial\Omega} \partial_\zeta \varphi_\Omega \cdot u \\ &\leq \frac{\mu_1(\Omega)}{r^2}\int_\Omega \varphi_\Omega(\omega) u(r\omega) \\ &\leq -\frac{n-1}{r^2}U(r), \end{aligned} \tag{3.3.9}$$

where the first inequality follows from (3.3.7) as well as positivity of u and the inward normal derivative $-\partial_\zeta \varphi$ once $\partial\Omega \neq \emptyset$; the second inequality follows from $\mu_1(\Omega) \leq -(n-1)$ by Lemma 3.3.4. Let

$$\gamma_n = -\frac{n-2}{2} + \sqrt{(\frac{n-2}{2})^2 - (n-1)}, \qquad \beta_n := 2\sqrt{(\frac{n-2}{2})^2 - (n-1)}.$$

And set $W(s) := U(s^{-1/\beta_n}) \cdot s^{\gamma_n/\beta_n}$, $s \geq 1$, then (3.3.9) is equivalent to $W''(s) \leq 0$ on $s \geq 1$. Since $u > 0$ on $\mathbf{C}$ and then $W > 0$ on $[1, +\infty)$, we must have $W' \geq 0$ on $[1, +\infty)$. Thus,

$$U(r) \geq U(1) \cdot r^{\gamma_n}, \quad \forall r \in (0, 1). \tag{3.3.10}$$

We then have for every $r \in (0,1)$,

$$\begin{aligned}
\|u\|_{L^1(\mathbb{B}_r;\|\mathbf{C}\|)} \sup_{\Omega} \varphi_\Omega &\geq \int_{\mathbb{B}_r} \varphi_\Omega(\frac{x}{|x|}) u(x)\ d\|\mathbf{C}\|(x) \\
&= \int_0^r s^{n-1} U(s)\ ds \geq \int_0^r s^{n-1+\gamma_n} U(1)\ ds \\
&= c_n r^{n+\gamma_n} \cdot \int_\Omega \varphi_\Omega(\omega) u(\omega)\ d\omega \\
&\geq c_n r^{n+\gamma_n} \cdot \|\varphi_\Omega\|_{L^1} \inf_{\mathbb{B}_1} u \\
&\geq C(n,\Lambda)^{-1} r^{n+\gamma_n} \cdot \sup_{\Omega} \varphi_\Omega \cdot \|u\|_{L^1(\mathbb{B}_1;\|\mathbf{C}\|)},
\end{aligned}$$

where the second inequality follows from (3.3.10), the last inequality follows from (3.3.8) and Corollary 3.2.5. This finish the proof. □

Lemma 3.3.6. *Let $\Sigma \subset (\mathbb{B}_4, g)$ be a two-sided minimal hypersurface under metric g such that, the Hessian of distant function square satisfies $\nabla_g^2(dist_g(\mathbf{0},\cdot)^2) \geq 1$ on $\mathbb{B}_2$. (Note that this is automatically true when g is C^2 close to g_{euc}.)*

Let $u \in W^{1,2}_{loc}(\Sigma)$ be a positive super-solution of $\Delta_\Sigma u + (|A_\Sigma|^2 - \delta)u = 0$ in distribution sense, i.e.

$$\int \nabla_\Sigma u \cdot \nabla_\Sigma \xi + (\delta - |A_\Sigma|^2) u\xi\ d\|\Sigma\| \geq 0, \quad \forall \xi \in C^1_c(\Sigma, \mathbb{R}_{\geq 0}), \tag{3.3.11}$$

where $\delta \leq n/2$. Then there exists $v \in C^2_{loc}(\Sigma \cap \mathbb{B}_1)$ solving $\Delta_\Sigma v + (|A_\Sigma|^2 - \delta)v = 0$ on $\Sigma \cap \mathbb{B}_1$ such that

$$\frac{1}{2} \inf_{\mathbb{B}_1} u \leq v \leq u$$

Proof. **Claim.** With a positive $u \in W^{1,2}_{loc}(\Sigma)$ satisfying (3.3.11), we have

$$\int |\nabla \phi|^2 + (\delta - |A_\Sigma|^2)\phi^2\ d\|\Sigma\| \geq 0, \quad \forall \phi \in C^1_c(\Sigma),$$

with equality if and only if $\phi \equiv 0$.

Proof of the Claim. For every $\phi \in C_c^1(\Sigma)$, let $\psi := u^{-1}\phi \in W_0^{1,2}(\Sigma)$. Take $\xi = u\psi^2 = \phi\psi$ in (3.3.11), we have

$$0 \leq \int \nabla u \cdot \nabla(\phi\psi) + (\delta - |A_\Sigma|^2)\phi^2 \, d\|\Sigma\| = \int |\nabla\phi|^2 + (\delta - |A_\Sigma|^2)\phi^2 - u^2|\nabla\psi|^2 \, d\|\Sigma\|.$$

This finishes the proof of Claim. □

Now we turn to the proof of Lemma 3.3.6. By a renormalization, suppose WLOG that $\inf_{\mathbb{B}_1} u = 1$. Consider for every smooth domain $\Omega \subset\subset \Sigma \cap \mathbb{B}_1$, minimize

$$J_\Omega[\phi] := \int |\nabla\phi|^2 + (-|A_\Sigma|^2 + \delta)\phi^2 \, d\|\Sigma\|,$$

among $\{\phi : \phi - 1 \in W_0^{1,2}(\Omega)\}$ to get a minimizer $v_\Omega \in C^2(\mathrm{Clos}(\Omega))$ solving $\Delta_\Sigma v_\Omega + (|A_\Sigma|^2 - \delta)v_\Omega = 0$ in Ω. Since $u \geq 1$ on $\Sigma \cap \mathbb{B}_1$, we know that $(v_\Omega - u)^+ \in W_0^{1,2}(\Sigma)$ is supported in $\mathrm{Clos}(\Omega)$, thus by taking $\xi = (v_\Omega - u)^+$ in (3.3.11) we have,

$$\begin{aligned} 0 &\geq \int -\nabla u \cdot \nabla(v_\Omega - u)^+ - (\delta - |A_\Sigma|^2)u(v_\Omega - u)^+ \, d\|\Sigma\| \\ &= \int \nabla(v_\Omega - u) \cdot \nabla(v_\Omega - u)^+ + (\delta - |A_\Sigma|^2)(v_\Omega - u) \cdot (v_\Omega - u)^+ \, d\|\Sigma\| \\ &= \int |\nabla\xi|^2 + (\delta - |A_\Sigma|^2)\xi^2 \, d\|\Sigma\| \ \geq 0, \end{aligned}$$

where the first equality follows by integration by parts and the equation satisfied by v_Ω; the last inequality follows from the Claim. Hence the Claim guarantees that $(v_\Omega - u)^+ = 0$, in other words $v_\Omega \leq u$ in Ω. On the other hands, let $w := (1 + dist_g(\cdot, \mathbf{0}))/2 \leq 1$ on $\Sigma \cap \mathbb{B}_1$, a simple computation shows that

$$\Delta_\Sigma w + (|A_\Sigma|^2 - \delta)w \geq 0 \ \text{ on } \Sigma \cap \mathbb{B}_1.$$

This provides a sub-solution to the equation. Repeat the same process above gives $v_\Omega \geq w \geq 1/2$ on Ω.

Now take Ω_j be an increasing family of smooth domain approximating $\Sigma \cap \mathbb{B}_1$, $1/2 \leq v_{\Omega_j} \leq u$ be the solution constructed above relative to Ω_j. By classical Harnack inequality for Elliptic equations [GT01], v_{Ω_j} subconverges to some solution $v \in C^2_{loc}(\Sigma \cap \mathbb{B}_1)$ of $\Delta_\Sigma v + (|A_\Sigma|^2 - \delta)v = 0$ which satisfies $1/2 \leq v \leq u$, thus finishes proof of the Lemma. □

Proof of Lemma 3.3.2. Take $r_0 = r_0(n, \Lambda, \gamma) \in (0, \eta(\delta_2, \Lambda, n)/8)$ TBD, where $\delta_2 = \delta_2(n, \Lambda)$ be given by Lemma 3.2.2 and η is given by the Harnack inequality Corollary 3.2.5.

Suppose for contradiction that there exists $\gamma > \gamma_n$ and stable minimal hypersurfaces $\Sigma_j \subset (\mathbb{B}_4, g_j)$ such that $\mathbf{0} \in \mathrm{Sing}(\Sigma_j)$, $g_j \to g_{\mathrm{euc}}$ in C^4 and $\mathbf{F}_{\mathbb{B}_{4r_0^{-1}}}(|\Sigma_j|, |\mathbf{C}|) \to 0$ for some stable minimal hypercone $\mathbf{C} \subset \mathbb{R}^{n+1}$; But there are $u_j \in W^{1,2}_{loc}(\Sigma_j)$ weak supersolutions to $\Delta_{\Sigma_j} u_j + (|A_{\Sigma_j}|^2 - 1/j)u_j = 0$ and $r_j \in [r_0/2, r_0]$, $r_j \to r_\infty \in [r_0/2, r_0]$ satisfying

$$\|u_j\|_{L^1_*(\mathbb{B}_{r_j}; \|\Sigma\|)} < \|u_j\|_{L^1_*(\mathbb{B}_1; \|\Sigma\|)} \cdot r_j^\gamma \tag{3.3.12}$$

Note that by Lemma 3.2.2 and $\mathbf{F}_{\mathbb{B}_{4r_0^{-1}}}(|\Sigma_j|, |\mathbf{C}|) \to 0$, we have for sufficiently large j, $\mathbf{F}_{\mathbb{B}_{4r_0^{-1}}}(|\Sigma_j|, \mathscr{M}^{\geq 2}) \geq \delta_2(n, 2\Lambda)$. Hence the Harnack inequality Corollary 3.2.5 applies for functions on Σ_j.

Also by Lemma 3.3.6, we can take $v_j \in C^2_{loc}(\Sigma_j \cap \mathbb{B}_1)$ such that on $\Sigma_j \cap \mathbb{B}_1$,

$$0 < \frac{1}{2} \inf_{\Sigma_j \cap \mathbb{B}_1} u_j \leq v_j \leq u_j; \qquad \Delta_{\Sigma_j} v_j + (|A_{\Sigma_j}|^2 - 1/j)v_j = 0. \tag{3.3.13}$$

Thus by applying Harnack inequality Corollary 3.2.5 on u_j and (3.3.12), (3.3.13), we

have

$$\|v_j\|_{L^1_*(\mathbb{B}_{r_j})} \leq C(n,\Lambda) \inf_{\Sigma_j \cap \mathbb{B}_1} u_j \cdot r_j^\gamma \leq C(n,\Lambda) \inf_{\Sigma_j \cap \mathbb{B}_1} v_j \cdot r_j^\gamma. \tag{3.3.14}$$

On the other hand, fix a regular point $x_\infty \in \mathbf{C} \cap \mathbb{B}_{\eta/4}$ and suppose $\Sigma_j \ni x_j \to x_\infty$. Renormalize v_j such that $v_j(x_j) = 1$. Then by classical Harnack inequality [GT01], after passing to a subsequence, $v_j \to v_\infty$ in C^2_{loc} away from $\mathrm{Sing}(\mathbf{C})$ for some positive Jacobi field $v_\infty \in C^\infty_{loc}(\mathbf{C} \cap \mathbb{B}_1)$, i.e. $(\Delta_{\mathbf{C}} + |A_{\mathbf{C}}|^2)v_\infty = 0$. Again by Harnack inequality Corollary 3.2.5, for every $p \in (1, n/(n-2))$ we have,

$$\|v_j\|_{L^p}(\mathbb{B}_{\eta/4}; \|\Sigma_j\|) \leq C(p,n,\Lambda) v_j(x_j) = C(p,n,\Lambda).$$

Hence by Hölder inequality, for every small neighborhood $\mathcal{N} \subset \mathbb{B}_{\eta/4}$ of $\mathrm{Sing}(\mathbf{C}) \cap \mathbb{B}_{\eta/8}$, we have

$$\|v_j\|_{L^1(\mathcal{N};\|\Sigma_j\|)} \leq \|v_j\|_{L^p(\mathcal{N};\|\Sigma_j\|)} \cdot \|\Sigma_j\|(\mathcal{N})^{(p-1)/p} \leq C(p,n,\Lambda) \cdot \|\Sigma_j\|(\mathcal{N})^{(p-1)/p}.$$

This guarantees that v_j are not L^1-concentrating near $\mathrm{Sing}(\Sigma_j) \cap \mathbb{B}_{\eta/8}$. Hence v_j subconverges to v_∞ also in $L^1(\mathbb{B}_{\eta/8})$. By (3.3.14) and Lemma 3.3.5, we derive,

$$\begin{aligned}
\|v_\infty\|_{L^1_*(\mathbb{B}_{r_\infty};\|\mathbf{C}\|)} &\leq C(n,\Lambda) \inf_{\mathbf{C}\cap\mathbb{B}_{\eta/8}} v_\infty \cdot r_\infty^\gamma \\
&\leq C(n,\Lambda) \|v_\infty\|_{L^1_*(\mathbb{B}_{\eta/8};\|\mathbf{C}\|)} \cdot r_\infty^\gamma \\
&\leq \bar{C}(n,\Lambda) \|v_\infty\|_{L^1_*(\mathbb{B}_{r_\infty};\|\mathbf{C}\|)} \cdot (\frac{\eta}{8r_\infty})^{\gamma_n} \cdot r_\infty^\gamma \,,
\end{aligned}$$

which then implies

$$\bar{C}(n,\Lambda) \cdot (\frac{\eta}{8})^{\gamma_n} \geq r_\infty^{\gamma_n - \gamma} \geq r_0^{\gamma_n - \gamma}.$$

This will be a contradiction if we take $r_0 < \left(\bar{C}(n,\Lambda) \cdot (\eta/8)^{\gamma_n}\right)^{1/(\gamma_n - \gamma)}$. □

3.4 Mean Convex Smoothing of Mean Convex Hypersurfaces

We shall prove in this section Theorem 1.3.1 and 1.3.2. We start with the mean convex approximation Theorem in Section 3.4.1 and some preliminaries on self-expanders and weak mean curvature flow in Section 3.4.2 and 3.4.3. It's worth noting that the preliminaries on self-expanders and weak mean curvature flow are only used in proving Theorem 1.3.2. Readers interested in Theorem 1.3.1 can skip Section 3.4.2 and 3.4.3.

3.4.1 Mean convex approximation

Definition 3.4.1. We call a closed subset $E \subset M$ **viscosity mean convex**, if $E = \text{Clos}(\text{Int}(E))$ and for every smooth open domain $\Omega \subset E$ with $x \in \partial\Omega \cap \partial E$, the mean curvature with respect to the outward pointed normal field of $\partial\Omega$ at x is $H_{\partial\Omega}(x) \geq 0$.

We call a closed viscosity mean convex subset $E \subset M^{n+1}$ $C^{1,1}$**-optimally regular**, if there exists a closed subset $\mathcal{S} \subset \partial E$ with Hausdorff dimension $\leq n-7$, such that $\partial E \setminus S$ is a $C^{1,1}$ embedded hypersurface and $\forall x \in \mathcal{S}$, ∂E is a stable minimal hypersurface with optimal regularity near x. Call the smallest such $\mathcal{S} \subset \partial E$ the **singular set** of ∂E.

Without assuming $E = \text{Clos}(\text{Int}(E))$, the notion of viscosity mean convex was introduced and studied in [ISZ98, Definition 3.1], where it was called *barrier for minimal surface equation*.

Remark 3.4.2. (1) By [SW89], the support of every Caccioppoli set in M with stationary boundary is viscosity mean convex. In particular, every stable minimal hypercone in $\mathbb{R}^{n+1}$ bounds a viscosity mean convex subset.

(2) It follows directly from the definition that if $E \subset M$ is a viscosity mean convex closed subset, then for every connected component U of $\mathrm{Int}(E)$, $\mathrm{Clos}(U)$ is also viscosity mean convex. Hence, in the following discussion, we may additionally assume that a viscosity mean convex subset has connected interior.

(3) When a closed viscosity mean convex set $E \subset M$ is $C^{1,1}$-optimally regular, on its regular part, we can define the mean curvature vector $\vec{H}_{\partial E} \in L^\infty(\partial E)$ almost everywhere. And E being viscosity mean convex guarantees that $\langle \vec{H}_{\partial E}, \nu \rangle \leq 0$, where ν is the outward pointed normal field.

The rest of this subsection is devoted to proving that any compact viscosity mean convex subset can be approximated by $C^{1,1}$-optimally regular mean convex subset (see Corollary 3.4.5). We start with a minimization process with obstacle.

Theorem 3.4.3. *Let E be a compact viscosity mean convex subset in a Riemannian manifold (M, g) of dimension $n+1$; Let $Z \subset \mathrm{Int}(E)$ be a closed C^2 domain. Then there exists a closed subset P with the following properties,*

(i) $Z \subset P \subset E$;

(ii) The topological boundary ∂P is a stable minimal hypersurface with optimally regularity in the portion $M \setminus Z$ and is $C^{1,1}$ near each point of $\partial P \cap \partial Z$;

(iii) For every Caccioppoli set Q such that $Z \subset Q \subset\subset \mathrm{Int}(E)$, we have $\mathcal{P}(P) \leq \mathcal{P}(Q)$;

(iv) If $(M, g) \cong (\mathbb{S}^{n+1}, g_{round})$ is isometric to the round sphere, then one can choose so that $P \subset \mathrm{Int}(E)$.

In particular, P is a $C^{1,1}$-optimally regular mean convex subset.

In the Euclidean space, such existence-with-barrier result was established by [ISZ98, Theorem 3.6]. Here we shall use a similar argument. The only difficulty is

that in a general Riemannian manifold, there's usually no isometric translations. So instead of minimizing area, we shall first minimize an $\mathcal{A}^\epsilon$ functional to find prescribed mean curvature approximations. Such $\mathcal{A}^\epsilon$ functional was introduced and carefully studied by [ZZ20].

Proof. Suppose WLOG that $\text{injrad}(M, g) > 1$ and every ball of radius < 1 in (M, g) is strictly convex. Let $d(x) := dist_g(x, E^c)$ be the distant function defined in E; Then there exists $j_0 >> 1$ such that $Z \subset \{d > 1/j_0\}$; Let $\eta \in C^\infty(M; [0, 1])$ be a cut-off function such that $\eta = 0$ in a neighborhood of Z and $\eta = 1$ on $\{d \leq 1/(2j_0)\}$.

Let $\epsilon \in (0, 1)$, consider the the following $\mathcal{A}^\epsilon$-functional [ZZ20] for a Caccioppoli set Q,

$$\mathcal{A}^\epsilon(Q) := \mathcal{P}(Q) + \int_Q \epsilon \cdot \eta(x) \, d\mathscr{H}^{n+1}(x).$$

By first variation [ZZ20], a stationary Caccioppoli set for $\mathcal{A}^\epsilon$ has boundary mean curvature equals to $\epsilon \cdot \eta$.

Claim 1. There exists $j_\epsilon \geq 4j_0 + 1/\epsilon$ such that for every $j > j_\epsilon$, there exists a closed Caccioppoli set $Z \subset Q_j \subset \{d > 1/j\}$ which minimize $\mathcal{A}^\epsilon$ among

$$\mathscr{T}_j := \{Q : Z \subset Q \subset \{d \geq 1/j\}\}. \tag{3.4.1}$$

In particular, ∂Q_j is optimally regular and stable in $M \setminus Z$ and $C^{1,1}$ near each point on ∂Z.

With this Claim 1, choose $Q^\epsilon := Q_{j_\epsilon+1}$ and send $\epsilon \searrow 0$, by spirit of Lemma 2.2.1 and [Lin85, Chapter 2], Q^ϵ **F**-subconverges to some closed Caccioppoli set $P = \text{spt}[P] \subset E$, with ∂P stable minimal hypersuraces in $M \setminus Z$ and $C^{1,1}$ near ∂Z. This proves (i) and (ii). Also since P is the **F**-limit of $\mathcal{A}^\epsilon$-minimizer Q^ϵ, (iii) also holds for P. When $(M, g) \cong (\mathbb{S}^{n+1}, g_{\text{round}})$, since ∂Q^ϵ is $\mathcal{A}^\epsilon$-stationary and thus has mean curvature > 0 on $\{d < 1/(2j_0)\}$, by a rotation of $\mathbb{S}^{n+1}$ and viscosity mean convexity

of E, we must have $d \geq 1/(2j_0)$ along Q^ϵ and then along P. This proves (iv), and finish the proof of Theorem 3.4.3.

Proof of Claim 1. Let $j_\epsilon = j(\epsilon, g) \geq 4j_0 + 1/\epsilon$ to be determined later.

For $j > j_\epsilon$, let $\bar{Q}_j \in \mathscr{T}_j$ be a general $\mathcal{A}^\epsilon$-minimizer among $\mathscr{T}_j$, where $\mathscr{T}_j$ is defined in (3.4.1). For every $x \in \mathrm{spt}[\bar{Q}_j] \cap \{d = 1/j\}$, consider minimize $\mathcal{A}^\epsilon$ among

$$\{R \subset M : R \Delta \bar{Q}_j \subset B_{1/j}(x)\},$$

to find a Caccioppoli set $\tilde{Q}_j$.

Claim 2. By taking j_ϵ large, for every $y \in \mathrm{spt}[\tilde{Q}_j] \cap B_{1/j}(x)$, we have $d(y) > 1/j$.

With this Claim 2, $\tilde{Q}_j \in \mathscr{T}_j$ is also an $\mathcal{A}^\epsilon$-minimizer among $\mathscr{T}_j$, and whenever $\bar{Q}_j$ is $\mathcal{A}^\epsilon$-stationary and optimally regular in an open subset $W \subset \{d > 1/j\}$, by unique continuation of CMC hypersurfaces [ZZ20], $\tilde{Q}_j$ must be $\mathcal{A}^\epsilon$-stationary and optimally regular in $W \cup B_{1/j}(x)$. Hence by covering $\{d = 1/j\}$ with finitely many ball of radius $1/j$ and repeat the replacement process, we can find Q_j satisfying the assertion of Claim 1. □

Proof of Claim 2. Let $j_\epsilon = j(\epsilon, g) \geq 4j_0 + 1/\epsilon$ TBD. By definition of $\tilde{Q}_j$ and strict convexity of $B_{1/j}(x)$, we must have

$$\mathrm{spt}[\tilde{Q}_j] \setminus B_{1/j}(x) \subset \mathrm{spt}[\bar{Q}_j] \setminus B_{1/j}(x) \subset \{d \geq 1/j\}.$$

If Claim 2 fails, then there exists $y \in \mathrm{spt}(\partial[\tilde{Q}_j]) \cap B_{1/j}(x)$ realizing $\inf\{d(z) : z \in \mathrm{spt}(\tilde{Q}_j) \cap B_{1/j}(x)\}$. Let $y' \in \partial E$ and $0 < d_0 \leq 1/j$ be such that $dist_g(y, y') = d(y) = d_0$. Since $B_{d_0}(y') \cap \mathrm{spt}(\tilde{Q}_j) = \emptyset$, the tangent cone of $\partial_* \tilde{Q}_j$ at y must be a plane, and hence by $\mathcal{A}^\epsilon$-minimizing of $\tilde{Q}_j$ in $B_{1/j}(x)$, we know that y is a regular point of $\partial[\tilde{Q}_j]$. Moreover, by comparing the second fundamental form $\tilde{A}_j$ of $\partial[\tilde{Q}_j]$ and second fundamental form of $\partial B_{d_0}(y')$, together with the fact that the mean curvature of $\partial[\tilde{Q}_j]$

at y is ϵ by first variation of $\mathcal{A}^\epsilon$, we know that,

$$|\tilde{A}_j(y)| \leq C(n,g) \cdot d_0^{-1}. \tag{3.4.2}$$

On the other hand, let L_y be the image of $B_\epsilon(\mathbf{0}) \cap \tilde{\nu}_y^\perp \subset T_yM$ under exponential map under metric g centered at y, where $\tilde{\nu}_y$ is the unit normal vector of $\partial[\tilde{Q}_j]$ at y pointing towards y'. We shall work under Fermi coordinates

$$\Phi : L_y \times (-\epsilon, \epsilon) \to M, \quad (z,t) \mapsto \exp_z^g(t \cdot \nu^L(z)),$$

where ν^L be the unit normal field of L_y such that $\nu^L(y) = \tilde{\nu}_y$.

Consider $R := \Phi(\{(z, t+d_0) : \Phi(z,t) \in \mathrm{spt}(\tilde{Q}_j) \cap B_{1/j}(x)\})$. Since $dist_g(\Phi(z,t), \Phi(z,t+d_0)) \leq d_0$, we have $R \subset E$ and $y' = \Phi(y, d_0) \in R$ is a regular point of ∂R. Moreover, by (3.4.2) and the following Lemma 3.4.4, the mean curvature $H_{\partial R}(y')$ of ∂R at y' satisfies,

$$H_{\partial R}(y') \leq \tilde{H}_j(y) + \bar{C}(n,g) \cdot d_0 \leq -\epsilon + \bar{C}(n,g) \cdot j_\epsilon^{-1},$$

where recall the mean curvature of $\partial[\tilde{Q}_j]$ at y is $\tilde{H}_j(y) = -\epsilon$. Choosing $j_\epsilon > \bar{C}(n,g) \cdot \epsilon^{-1}$, we have $H_{\partial R}(y') < 0$, violates the viscosity mean convexity of E and hence is a contradiction. This finish the proof of Claim 2. □

□

Lemma 3.4.4. *Let $\{g^t\}_{t\in(-1,1)}$ be a smooth family of Riemannian metric on $L := \mathbb{B}_1^n(\mathbf{0})$ (not necessarily complete); Let $\{A^t := \frac{1}{2}\frac{d}{dt}(g^t)\}_{t\in(-1,1)}$ be the family of symmetric 2-tensors of derivative of g^t. Assume that $A^0(\mathbf{0}) = 0$.*

Consider on the cylinder $L \times (-1,1)$, the metric $g := g^t \oplus dt^2$. Assume that the Riemannian curvature tensor of g satisfies $|\,\mathrm{Rm}_g\,| \leq 1/2n$ on $L \times (-1,1)$.

Let u be a C^2-function on some small ball $\mathbb{B}_r(\mathbf{0}) \subset L$ taking value in $(-1,1)$ such that $u(\mathbf{0}) =: a$ and $du(\mathbf{0}) = 0$. Let $\Gamma_u := \{(z, u(z)) : z \in \mathbb{B}_r\} \subset L \times (-1,1)$ be a C^2

hypersurface. Let $\{z^i\}_{1\leq i\leq n}$ *be a normal coordinates of* g^0 *defined near* $\mathbf{0}$ *on* L*. Then the mean curvature of* Γ_u *at* $(\mathbf{0}, a)$ *with respect to the upper-pointed normal field is given by*

$$H_{\Gamma_u}(\mathbf{0}, a) = -(\nabla^2_{g_0} u(\mathbf{0}) - A^a(\mathbf{0}))_{ij} \cdot g_a(\mathbf{0})^{ij},$$

In particular, the difference between mean curvature of Γ_u *at* $(\mathbf{0}, a)$ *and* Γ_{u-a} *at* $(\mathbf{0}, 0)$ *has estimate*

$$|H_{\Gamma_u}(\mathbf{0}, a) - H_{\Gamma_{u-a}}(\mathbf{0}, 0)| \leq C(n)(a^2 |A_{\Gamma_{u-a}}|(\mathbf{0}, 0) + |a|).$$

Proof. First note that $\{z^1, ..., z^n, t\}$ is a local coordinates of $L \times (-1, 1)$ with the coordinate vector fields denoted by $\{\partial_1, ..., \partial_n, \partial_t\}$. By [Pet06], under this,

$$\begin{aligned} 2A^t(\partial_i, \partial_j) &= \frac{d}{dt} g^t(\partial_i, \partial_j) = \nabla^g_{\partial_t}(g(\partial_i, \partial_j)) = 2g(\nabla^g_{\partial_i} \partial_t, \partial_j); \\ \frac{d}{dt} A^t(\partial_i, \partial_j) &= \frac{1}{2} g(\nabla^g_{\partial_t} \nabla^g_{\partial_i} \partial_t, \partial_j) + \frac{1}{2} g(\nabla^g_{\partial_t} \nabla^g_{\partial_j} \partial_t, \partial_i) + g(\nabla^g_{\partial_i} \partial_t, \nabla^g_{\partial_j} \partial_t) \\ &= \mathrm{Rm}_g(\partial_t, \partial_i, \partial_t, \partial_j) + A^t(\partial_i, \partial_k) A^t(\partial_j, \partial_l) g_t^{kl}, \end{aligned}$$

where $[g_t^{kl}]$ is the inverse of matrix $[g_{ij}^t]$. In particular, by a bootstrap argument, we have

$$|A^t(\partial_i, \partial_j)| \leq \frac{t}{2}, \qquad\qquad |g_{ij}^t - \delta_{ij}| \leq \frac{t^2}{2}. \tag{3.4.3}$$

Let $\Phi_u : z \mapsto (z, u(z))$ be a parametrization of Γ_u. Then the upward pointed normal field of Γ_u at $(\mathbf{0}, a)$ is

$$\nu^u(\mathbf{0}, a) = \frac{-\partial_i u \cdot g_u^{ij} \partial_j + \partial_t}{\sqrt{1 + \partial_i u \partial_j u \cdot g_u^{ij}}} = \partial_t;$$

The induced metric of Γ_u under this parametrization is

$$g(\partial_i \Phi_u, \partial_j \Phi_u) = g(\partial_i + \partial_i u \cdot \partial_t, \partial_j + \partial_j u \cdot \partial_t);$$

And the second fundamental form of Γ_u at $(\mathbf{0}, a)$ is

$$A_{\Gamma_u}(\partial_i \Phi_u, \partial_j \Phi_u)|_{(\mathbf{0},a)} = \langle \nabla^g_{\partial_i + \partial_i u \cdot \partial_t}(\partial_j + \partial_j u \cdot \partial_t), \nu^u \rangle|_{(\mathbf{0},a)} = -A^a(\mathbf{0}) + \nabla^2_{g^0} u(\mathbf{0})(\partial_i, \partial_j).$$

Hence the mean curvature of Γ_u at $(\mathbf{0}, a)$ is

$$H_{\Gamma_u}(\mathbf{0}, a) = -(\nabla^2_{g^0} u(\mathbf{0}) - A^a(\mathbf{0}))_{ij} \cdot g_a(\mathbf{0})^{ij};$$

The estimate on the difference of mean curvature follows from the estimate (3.4.3). □

Corollary 3.4.5. *Let $\mathcal{E} \subset (\mathbb{S}^{n+1}, g_{round})$ be a closed viscosity mean convex subset with connected interior. Then there exists an increasing sequence of closed connected $C^{1,1}$ optimally regular mean convex subsets $\{\mathcal{E}_j \subset \mathrm{Int}(\mathcal{E})\}_{j \geq 1}$ such that when $j \to \infty$, $\mathcal{E}_j \to \mathcal{E}$, $\partial \mathcal{E}_j \to \partial \mathcal{E}$ both in Hausdorff distant sense. Furthermore, if the topological boundary $\partial \mathcal{E}$ is n-rectifiable (In particular $\mathcal{E}$ is a Caccioppoli set), then we can choose so that*

$$\lim_{j \to \infty} \mathcal{P}(\mathcal{E}_j) = \mathcal{P}(\mathcal{E}).$$

Proof. Let $d_{\mathcal{E}} := dist_{\mathbb{S}^{n+1}}(\cdot, \mathcal{E}^c)$ be the distant function to the boundary defined in $\mathcal{E}$. $\tilde{d}_{\mathcal{E}}$ be a smooth function on $\mathrm{Int}(\mathcal{E})$ such that $d_{\mathcal{E}} \leq \tilde{d}_{\mathcal{E}} \leq 2 d_{\mathcal{E}}$. Let $\epsilon_j \searrow 0$ be a sequence of regular value of $\tilde{d}_{\mathcal{E}}$, and $Z_j := \{\tilde{d}_{\mathcal{E}} \geq \epsilon_j\} \subset \mathrm{Int}(\mathcal{E})$ are compact smooth domains.

By Theorem 3.4.3, there exists $C^{1,1}$-optimally regular closed mean convex subset $Z_j \subset P_j \subset \mathrm{Int}(\mathcal{E})$. Clearly by definition, $P_j \to \mathcal{E}$ and $\partial P_j \to \partial \mathcal{E}$ both in Hausdorff distant sense. Also, let $j_0 >> 1$ such that $Z_{j_0} \neq \emptyset$ and fix $\omega_0 \in Z_{j_0}$; Take $\mathcal{E}_j$ to be the connect component of P_j containing ω_0, we know that since $\mathrm{Int}(\mathcal{E})$ is connected,

$\mathcal{E}_j \to \mathcal{E}$, $\partial\mathcal{E}_j \to \partial\mathcal{E}$ also in Hausdorff distant sense.

If further $\partial\mathcal{E}$ is n-rectifiable, by [Fed69, Theorem 3.2.39] and co-area formula, there exists $R_j := \{d_{\mathcal{E}} \geq \delta_j\} \subset \mathrm{Int}(\mathcal{E})$ such that $\lim_{j\to\infty} \mathcal{P}(R_j) = \mathcal{P}(\mathcal{E})$. Then by Theorem 3.4.3 (iii) and upper semi-continuity of perimeter under convergence, we have $\lim_{j\to\infty} \mathcal{P}(\mathcal{E}_j) = \mathcal{P}(\mathcal{E})$. □

3.4.2 Self-expanders

For a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ and a bounded open subset $W \subset \mathbb{R}^{n+1}$, define the **E**-functional of Σ in W to be,

$$\mathbf{E}[\Sigma; W] := \int_W e^{|x|^2/4} \, d\|\Sigma\|(x).$$

Such **E**-functional is introduced in [Ilm98, Lecture 2 C] in which it's named **K**-functional, and frequently studied by Bernstein-Wang [BW18, BW19a, BW19b, BW20b, BW21a, BW21b] and by Ding [Din20]. Here we keep the notations with Bernstein-Wang. Critical points Σ of **E** are known to be **self-expanders**, i.e. it satisfies the following equation

$$\vec{H}_\Sigma - \frac{X^\perp}{2} = 0,$$

where $X^\perp$ denotes the projection of position vector onto normal direction of Σ. Equivalently, by [EH89], $\{\eta_{\sqrt{t}}(\Sigma)\}_{t>0}$ is a family of hypersurfaces flowing by mean curvature.

Clearly, the **E**-functional can be extended to any Radon measure μ on $\mathbb{R}^{n+1}$,

$$\mathbf{E}[\mu; W] := \int_W e^{|x|^2/4} \, d\mu(x).$$

We also denote for simplicity that for a Caccioppoli sets $P \subset \mathbb{R}^{n+1}$, $\mathbf{E}[P; W] := \mathbf{E}[\|\partial[P]\|; W]$ and for an n-varifold V, $\mathbf{E}[V; W] := \mathbf{E}[\|V\|; W]$.

For a bounded open subset $W \subset \mathbb{R}^{n+1}$, we call a Caccioppoli set $P \subset \mathbb{R}^{n+1}$ **E-minimizing** in W, if for every Caccioppoli set $Q\Delta P \subset\subset W' \subset\subset W$, we have

$$\mathbf{E}[Q; W'] \geq \mathbf{E}[P; W'].$$

And we call an integral n-varifold $V \in \mathcal{IV}_n(W)$ **E-stationary** in W, if for every $X \in \mathscr{X}_c(W)$,

$$\delta\mathbf{E}[V; W](X) := \frac{d}{dt}\Big|_{t=0} \mathbf{E}[e^{tX}{}_\sharp V; W] = 0;$$

Call V **E-stable** if it's **E**-stationary and for every $X \in \mathscr{X}_c(W)$,

$$\delta^2\mathbf{E}[V; W](X, X) := \frac{d^2}{dt^2}\Big|_{t=0} \mathbf{E}[e^{tX}{}_\sharp V; W] \geq 0.$$

Note that for an n-varifold, **E**-functional is the n-th area-functional under the metric $\hat{g} := e^{|x^2|/2n} \cdot g_{\text{euc}}$ on $\mathbb{R}^{n+1}$. Hence, being **E**-stationary (resp. **E**-stable, **E**-minimizing) is equivalent to being stationary (resp. stable, minimizing) with respect to the area functional under $\hat{g}$.

For a 2-sided hypersurface $\Sigma \subset W$, by [Din20, (3.16)], being **E**-stable is equivalent to that for every $\phi \in C_c^1(\Sigma \cap W)$,

$$\int_W [|\nabla\phi|^2 - (|A_\Sigma|^2 - \frac{1}{2})\phi^2] \cdot e^{|x|^2/4} d\|\Sigma\|(x) \geq 0. \tag{3.4.4}$$

The Euler-Lagrangian operator of the quadratic form on the left hand side of (3.4.4) is,

$$\mathcal{L}_\Sigma := \Delta_\Sigma + \frac{X}{2} \cdot \nabla_\Sigma + |A_\Sigma|^2 - \frac{1}{2}. \tag{3.4.5}$$

Moreover, if denote $\hat{\Delta}_\Sigma, \hat{A}_\Sigma$ to be the laplacian and second fundamental form of Σ

under metric $\hat{g}$, then a simple computation shows that for every function u on Σ,

$$\mathcal{L}_\Sigma u = e^{\frac{|x|^2}{4n}} \cdot \big(\hat{\Delta}_\Sigma + |\hat{A}_\Sigma|^2 - e^{-\frac{|x|^2}{2n}}(1 + \frac{n-1}{4n^2}|X^\Sigma|^2)\big)(e^{\frac{|x|^2}{4n}} u), \tag{3.4.6}$$

where X^Σ is the projection of the position vector X onto $T\Sigma$.

Let $\mathcal{E} \subset \mathbb{S}^n$ be a closed subset; Let $E = \mathbf{C}(\mathcal{E})$ be the closed cone over $\mathcal{E}$ in $\mathbb{R}^{n+1}$. We call a closed set $P \subset \mathbb{R}^{n+1}$ **distant asymptotic to** E near infinity if locally in the Hausdorff distant sense, when $R \to +\infty$, $\eta_{1/R}(P) \to E$ and $\eta_{1/R}(\partial P) \to \partial E$.

For a unit vector $v \in \mathbb{S}^n$ and $\alpha \in (0, \pi)$, define

$$\mathcal{C}(v; \alpha) := \{x \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} : \langle x/|x|, v\rangle > \cos\alpha\},$$

be the open cone in v direction with open angle α. $O(n)$-invariant self-expanders distant asymptotic to $\mathrm{Clos}(\mathcal{C}(v;\alpha))$ are constructed in [AIC95, Lemma 2], and also studied using variational approach in [Din20, Section 4]. We summarize the results here. By a rotation, WLOG $v = e_{n+1}$.

Theorem 3.4.6 ([AIC95, Din20])**.** *For each $\alpha \in (0, \pi/2)$, there exists a unique $O(n)$-symmetric convex closed smooth domain $F(e_{n+1}; \alpha) \subset \mathcal{C}(e_{n+1}, \alpha)$ minimizing* **E***-functional in $\mathbb{R}^{n+1}$ and distant asymptotic to $\mathrm{Clos}(\mathcal{C}(e_{n+1}; \alpha))$ near infinity.*

Moreover, $F(e_{n+1}, \alpha)$ is an $O(n)$-symmetric super graph, i.e. there exists $u_\alpha \in C^\infty([0, +\infty))$ such that

$$F(e_{n+1}, \alpha) = \{(x, z) : x \in \mathbb{R}^n, z \geq u_\alpha(|x|)\},$$

and u_α satisfies

(i) $u_\alpha(0) > 0$, $u'_\alpha(0) = 0$, $u'_\alpha(r), u''_\alpha(r) > 0$ for $r > 0$ and $u(r)/r \to \cot\alpha$ when $r \to +\infty$;

(ii) u_α is continuous and strictly monotone in α and for every fixed $r \geq 0$,

$$u_\alpha(r) \to +\infty \quad \textit{when } \alpha \to 0, \qquad u_\alpha(r) \to 0 \quad \textit{when } \alpha \to \pi/2.$$

In particular, $\{\partial F(e_{n+1}, \alpha)\}_{\alpha \in (0,\pi/2)}$ foliates $\mathbb{R}^n \times (0, +\infty)$.

Proof. For each $\alpha \in (0, \pi/2)$, in [AIC95] and [Din20], a unique $O(n)$-invariant entire convex graph $S_\alpha \subset \mathcal{C}(e_{n+1}, \alpha)$ solving self-expanding equation and asymptotic to $\partial\mathcal{C}(e_{n+1}; \alpha)$ is constructed, where the graphical function u_α satisfies (i). Moreover, the uniqueness of S_α guarantees they varies continuously in α. Let $F(e_{n+1}, \alpha)$ be the closed domain above S_α.

As an entire graph, by [Din20, Lemma 3.1], S_α are **E**-minimizing in $\mathbb{R}^{n+1}$. Since when $\alpha \searrow 0$, $\mathrm{Clos}(\mathcal{C}(e_{n+1}, \alpha)) \to \{r \cdot e_{n+1} : r \geq 0\}$ locally in the Hausdorff distant sense, by compactness of minimizing boundary we must have, $dist(\mathbf{0}, S_\alpha) \to +\infty$ when $\alpha \searrow 0$, hence for each $r \geq 0$, $u_\alpha(r) \to +\infty$ when $\alpha \searrow 0$. This also implies that for each $\alpha \in (0, \pi/2)$, when α' sufficiently close to 0, $S_{\alpha'} \subset F(e_{n+1}, \alpha)$. Let $\bar{\alpha} := \sup\{\beta \in (0, \alpha] : S_t \subset F(e_{n+1}, \alpha) \text{ for every } t \in (0, \beta)\}$. By strong maximum principle for **E**-functional, $\bar{\alpha} = \alpha$ and $S_t \subset \mathrm{Int}\, F(e_{n+1}, \alpha)$ whenever $t \in (0, \alpha)$. In other words, u_α is strictly monotone in α.

Note that (i) in the Theorem also implies that $0 \leq u'_\alpha < \cot\alpha$ on $[0, +\infty)$ and is monotone increasing. Hence when $\alpha \nearrow \pi/2$, u_α converges to a constant function whose graph is also a self-expander, hence must be 0. This finishes the proof of (ii). □

For every unit vector $v \in \mathbb{S}^n$ and $\alpha \in (0, \pi/2)$, let $T \in O(n+1)$ be an orthogonal transform mapping e_{n+1} to v, and denote $F(v; \alpha) := T(F(e_{n+1}, \alpha)) \subset \mathcal{C}(v; \alpha)$ to be the domain with **E**-minimizing boundary and distant asymptotic to $\mathcal{C}(v; \alpha)$ near infinity, where $F(e_{n+1}, \alpha)$ is given by Theorem 3.4.6. This family of self-expanders are good barriers for general asymptotic conic self-expanders.

Corollary 3.4.7. *Let $\mathcal{E} \subset \mathbb{S}^n$ be a closed subset such that $\mathrm{Clos}(\mathrm{Int}(\mathcal{E})) = \mathcal{E}$, let $0 < R \leq +\infty$. Suppose $F = \mathrm{spt}[F] \subset \mathbb{R}^{n+1}$ is a closed subset distant asymptotic to $\mathbf{C}(\mathcal{E})$ near infinity with $|\partial[F]|$ being $\mathbf{E}$-stationary in $(\mathbb{R}^{n+1} \setminus \partial\mathbf{C}(\mathcal{E})) \cup \mathbb{B}_R$ and coincide with $\mathbf{C}(\mathcal{E})$ outside $\mathbb{B}_R$, i.e. $F \setminus \mathbb{B}_R = \mathbf{C}(\mathcal{E}) \setminus \mathbb{B}_R$. (We use the convention that $\mathbb{B}_R = \mathbb{R}^{n+1}$ if $R = +\infty$.) Then,*

$$\begin{cases} F(v,\alpha) \subset F, & \text{if } \mathcal{C}(v;\alpha) \cap \mathbb{S}^n \subset \mathcal{E}; \\ F(v,\alpha) \cap F = \emptyset, & \text{if } \mathrm{Clos}(\mathcal{C}(v;\alpha) \cap \mathbb{S}^n) \cap \mathcal{E} = \emptyset. \end{cases}$$

Proof. We prove for $R < +\infty$; The case when $R = +\infty$ will be similar. Since $F \setminus \mathbb{B}_R = \mathbf{C}(\mathcal{E}) \setminus \mathbb{B}_R$, by Theorem 3.4.6 (ii), for sufficiently small $\beta \in (0,\alpha)$, $F(v,\beta) \subset F$. Consider

$$\bar{\alpha} := \sup\{\beta \in (0,\alpha) : F(v,\beta) \subset F\},$$

If $\bar{\alpha} < \alpha$, then by continuity of $F(v,\beta)$ in β from Theorem 3.4.6, $F(v,\bar{\alpha}) \subset F$ and $\partial F(v,\bar{\alpha}) \cap \mathrm{spt}(\|\partial[F]\|) \neq \emptyset$. Since $|\partial[F]|$ and $|\partial F(v,\bar{\alpha})|$ are both $\mathbf{E}$-stationary in $(\mathbb{R}^{n+1} \setminus \partial\mathbf{C}(\mathcal{E})) \cup \mathbb{B}_R$, by strong maximum principle [SW89], $\partial F(v,\bar{\alpha}) \subset \mathrm{spt}(\partial[F])$, contradicts to that $\mathrm{Clos}(\mathcal{C}(v;\bar{\alpha})) \setminus \mathbb{B}_R \subset \mathrm{Int}(F) \setminus \mathbb{B}_R$. Hence $\bar{\alpha} = \alpha$ and $F(v,\alpha) \subset F$.

Similarly, when $\mathrm{Clos}(\mathcal{C}(v;\alpha) \cap \mathbb{S}^n) \cap \mathcal{E} = \emptyset$, the argument above provides,

$$\sup\{\beta \in (0,\pi/2) : F(v,\beta) \cap F = \emptyset\} > \alpha.$$

Hence by Theorem 3.4.6, $F(v,\alpha) \cap F = \emptyset$. □

Note that using Corollary 3.4.7, one can also prove the existence of $\mathbf{E}$-minimizing closed subset distant asymptotic to $\mathbf{C}(\mathcal{E})$ near infinity, as is sketched in [Ilm98, Lecture 2 F] and proved for C^2-domains $\mathcal{E}$ in [Din20, Theorem 6.3].

In Section 3.4.4, we need the following existence result with constraint.

Lemma 3.4.8. *Let $E \subset \mathbb{R}^{n+1}$ be a closed $C^{1,1}$-optimally regular mean convex cone,*

which is not perimeter minimizing itself in E. Then there exists a closed Caccioppoli set $F \subset \mathrm{Int}(E)$ such that $F = \mathrm{spt}[F]$ is distant asymptotic to E near ∞, and that $\partial[F]$ minimizes **E***-functional among*

$$\{\partial[Q] : Q\Delta F \subset E\}.$$

In particular, $\partial[F]$ is a self-expanding integral cycle with optimal regularity.

Proof. For $R \geq 1$, minimize **E**-functional in $\mathbb{B}_{2R}$ among Caccioppoli sets

$$\{Q \subset \mathbb{R}^{n+1} : E \setminus \mathbb{B}_R \subset Q \subset E\}$$

to find a Caccioppoli set $F_R \subset E$. Since $\partial\mathbb{B}_R$ and $\mathrm{Reg}(\partial E)$ are both expander mean convex (i.e. $\vec{H} - X^{\perp}/2$ either vanishes or points inward), by [Lin85, Chapter 2], $|\partial[F_R]|$ is **E**-stationary in $(\mathbb{R}^{n+1} \setminus \partial E) \cup \mathbb{B}_R$. WLOG $F_R = \mathrm{spt}(F_R)$. When $R \to +\infty$, by Lemma 2.2.1 and Corollary 3.4.7, F_R subconverges in **F**-metric to some closed Caccioppoli set $F \subset E$ which is distant asymptotic to E near infinity, **E**-minimizing in E and **E**-stationary in $\mathbb{R}^{n+1}$. In particular, ∂F is optimally regular.

By strong maximum principle [SW89, Ilm96], since $F \neq E$ (Otherwise E is **E**-minimizing in E, and by blowing up E at $\mathbf{0}$, E is perimeter minimizing in E, which is a contradiction), we must have $F \subset \mathrm{Int}(E)$. □

3.4.3 Weak set flow and level set flow

Level set flow in the Euclidean space is introduced by [ES91] and independently by [CGG91], and is later generalized to defined on manifolds by [Ilm92, Ilm93]. Here, we shall use the notion of weak (set) flow and level set flow possibly with boundary, developed by [Whi95].

Definition 3.4.9 ([Whi95])**.** Let M be a compact n-manifold, possibly with bound-

ary, $f : M \times [a, b] \to \mathbb{R}^{n+1}$ be a continuous map. Call $\mathcal{M} := \{(f(t, x), t) : t \in [a, b], x \in M\} \subset \mathbb{R}^{n+1} \times \mathbb{R}$ a **classical flow** if,

- f is smooth on $\mathrm{Int}(M) \times (a, b]$;
- $f(\cdot, t)$ is one-to-one on M for each $t \in [a, b]$ and is a smooth embedding on $\mathrm{Int}(M)$ for each $t \in (a, b]$;
- For each $(x, t) \in \mathrm{Int}(M) \times (a, b]$, we have

$$(\frac{\partial}{\partial t} f(x, t))^{\perp} = \vec{H}(x, t),$$

  where $^{\perp}$ means projection onto normal direction of $f(M, t)$, and $\vec{H}(x, t)$ denotes the mean curvature vector of $f(M, t)$ at $f(x, t)$.

We call

$$\partial_h \mathcal{M} := \{(f(x, t), t) : \text{either } t = a \text{ and } x \in M, \text{ or } t \in [a, b] \text{ and } x \in \partial M\},$$

the **heat boundary** of $\mathcal{M}$.

Given a closed subset $\Gamma \subset \mathbb{R}^{n+1} \times \mathbb{R}_+$, we call a closed subset $\mathcal{M} \subset \mathbb{R}^{n+1} \times \mathbb{R}_+$ a **weak (set) flow** generated by Γ at $t = 0$, if

- $\mathcal{M}(0) = \Gamma(0)$;
- For every classical flow $\mathcal{M}'$ with $\partial_h \mathcal{M}' \cap \mathcal{M} = \emptyset$ and $\mathcal{M}' \cap \Gamma = \emptyset$, we always have $\mathcal{M}' \cap \mathcal{M} = \emptyset$.

In this chapter, we need the following

Theorem 3.4.10 ([Whi95, Theorem 4.1]). *Let $i = 1, 2$, $\mathcal{M}_i$ be weak flow generated by Γ_i at $t = 0$. Suppose for each $\tau \geq 0$, $\{(t, x) \in \mathcal{M}_a : t \leq \tau\}$ is compact. Suppose*

also that at some $T > 0$*,*

$$dist_T(\mathcal{M}_1, \mathcal{M}_2) < min_{i \neq j} dist_T(\mathcal{M}_i, \Gamma_j).$$

Then $dist_t(\mathcal{M}_1, \mathcal{M}_2) \geq dist_T(\mathcal{M}_1, \mathcal{M}_2)$ *for* t *on some interval* $[T, T + \epsilon]$*.*

Here $dist_t(\mathcal{M}_1, \mathcal{M}_2) := dist_{\mathbb{R}^{n+1}}(\mathcal{M}_1(t), \mathcal{M}_2(t))$, where $\mathcal{M}_i(t) := \{x \in \mathbb{R}^{n+1} : (x, t) \in \mathcal{M}_i\}$. In particular, this Theorem implies that whenever $\mathcal{M}_1 \cap \Gamma_2 = \mathcal{M}_2 \cap \Gamma_1 = \emptyset$ in Theorem 3.4.10, we always have $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$.

[Whi95] also provide the construction of level set flow under this definition.

Proposition 3.4.11 ([Whi95, Proposition 5.1])**.** *Given a closed subset* $\Gamma \subset \mathbb{R}^{n+1} \times \mathbb{R}_+$*, there's a unique weak flow* $\mathcal{M}$ *generated by* Γ *at* $t = 0$ *(known as the* ***level set flow****) such that it contains every weak flow generated by* Γ *at* $t = 0$*.*

We end up this section with an example, which is used in the proof of Theorem 3.4.15.

Lemma 3.4.12. *Let* $W \subset \mathbb{R}^{n+1}$ *be an open subset;* $\Sigma \subset W$ *be an* **E***-stable hypersurface with optimal regularity. Then for every compact* $W' \subset W$ *and every* $\lambda \geq 0$*,*

$$\mathcal{M} := \bigsqcup_{t \geq \lambda} \eta_{\sqrt{t}}(W' \cap \mathrm{Clos}(\Sigma)) \times \{t - \lambda\},$$

is a weak (set) flow generated at $t = 0$ *by*

$$\Gamma = \eta_{\sqrt{\lambda}}(W' \cap \mathrm{Clos}(\Sigma)) \times \{0\} \cup \bigsqcup_{t \geq \lambda} \eta_{\sqrt{t}}(\partial W' \cap \mathrm{Clos}(\Sigma)) \times \{t - \lambda\}.$$

Proof. Since the restriction of weak flow onto a sub-interval is also a weak flow, we can assume WLOG that $\lambda = 0$. Suppose for contradiction that there exists a classical flow $\mathcal{M}'$ over $[a, b] \subset (0, +\infty)$ such that $\mathcal{M}' \cap \Gamma = \partial_h \mathcal{M}' \cap \mathcal{M} = \emptyset$ but $\mathcal{M}' \cap \mathcal{M} \neq \emptyset$.

By definition,

$$\mathbf{d}(t) := dist_{\mathbb{R}^{n+1}}(\mathcal{M}(t), \mathcal{M}'(t)),$$

is a continuous function in $t \in [a, b]$ and vanishes at some point in $(a, b]$; And

$$\mathbf{d}_\partial(t) := min\{dist_{\mathbb{R}^{n+1}}(\mathcal{M}(t), \partial_h\mathcal{M}'\ (t)), dist_{\mathbb{R}^{n+1}}(\Gamma(t), \mathcal{M}'(t))\},$$

has infimum > 0, and $\mathbf{d}(a) = \mathbf{d}_\partial(a)$. Let

$$T := \inf\{s \in [a, b] : \mathbf{d}(s) = 0\} \in (a, b],$$

Then since $\mathbf{d}_\partial(T) > 0$ and $\mathrm{Int}(\mathcal{M}'(t))$ is smooth, we know that $\mathcal{R} := \mathcal{M}(T) \cap \mathcal{M}'(T) \neq \emptyset$ is a closed subset contained in $\mathrm{Reg}(\mathcal{M}(T)) \setminus \Gamma(T) = \eta_{\sqrt{T}}(\mathrm{Int}(W') \cap \Sigma)$. Take a smooth closed neighborhood $\mathcal{N} \supset \eta_{1/\sqrt{T}}(\mathcal{R})$ in $\Sigma \cap \mathrm{Int}(W')$, we know that $\sqcup_{t\in[a,T]}\eta_{\sqrt{t}}(\mathcal{N}) \times \{t\}$ and restriction of $\mathcal{M}'$ on $[a, T]$ are two classical flows over $[a, T]$ which intersects along $t = T$ but not intersects along the heat boundary. This violates the maximum principle for classical flows [Whi95, Lemma 3.1]. □

Note that a more general result can be proved for integral Brakke motions with codimension 1 restricted to a spacetime closed subset, using the same argument as [Ilm94, 10.5]. But we don't need it in this thesis.

3.4.4 Proof of Theorem 1.3.1 and 1.3.2

Throughout this subsection, let $E \subset \mathbb{R}^{n+1}$ be a closed viscosity mean convex cone with connected nonempty interior. Then $\mathcal{E} := E \cap \mathbb{S}^n$ is a closed viscosity mean convex domain in $\mathbb{S}^n$. Let's start with the following regularity Lemma.

Lemma 3.4.13. *Suppose $\partial[E]$ is not minimizing in E (This is automatically true if E is not a Caccioppoli set). Then there exists a closed subset $P \subset E$ such that,*

(i) $P \supset E \setminus \mathbb{B}_1$ *and* $P \cap \mathbb{B}_1 \subset \mathrm{Int}(E)$*;*

(ii) $\partial P \cap \mathbb{B}_1$ *is a smooth radial graph over* $\mathrm{Int}(E) \cap \mathbb{S}^n$*, i.e., there exists* $v \in C^\infty(\mathrm{Int}(\mathcal{E}); (0,1))$ *such that*

$$P \cap \mathbb{B}_1 = \{r\omega : \omega \in \mathrm{Int}(\mathcal{E}), v(\omega) \leq r < 1\};$$

(iii) $\partial[P] \llcorner \mathbb{B}_1$ *is area-minimizing in* $\mathbb{B}_1 \cap \mathrm{Int}(E)$*.*

Proof. Let's first further assume that $\partial E \cap \mathbb{S}^n$ is $C^{1,1}$ optimally regular mean convex in $\mathbb{S}^n$. Consider minimize perimeter in $\mathbb{B}_2$ among family of Caccioppoli sets

$$\{Q : E \setminus \mathbb{B}_1 \subset Q \subset E\},$$

to get a Caccioppoli set $P \subset E$. Since $\partial[E]$ is not minimizing in E, we must have $P \neq E$. By [Lin85, Chapter 2], since E and $\mathrm{Clos}(\mathbb{B}_1)$ are $C^{1,1}$ mean convex away from a low dimensional subset, we know that $\|\partial[P]\|$ is a stationary integral varifold in $(\mathbb{R}^{n+1} \setminus \partial E) \cup \mathbb{B}_1$. Then by Solomon-White's strong maximum principle [SW89] and Ilmanen's strong-maximum principle [Ilm96], either $E = P$ (which is impossible) or $\mathrm{spt}(P) \cap \partial E \cap \mathbb{B}_1 \subset \{\mathbf{0}\}$. If $\mathbf{0} \in \mathrm{spt}(P)$, then by Lemma 2.2.1, any tangent cone P' of P at $\mathbf{0}$ is also perimeter-minimizing in E, hence cannot coincide with E, this also violates the strong maximum principle for cross sections of E and P' in $\mathbb{S}^n$.

Now we have shown that $\mathrm{spt}(P) \cap \mathbb{B}_1 \subset \mathrm{Int}(E)$, since P minimize perimeter in $E \cap \mathbb{B}_1$, we have $\dim \mathrm{Sing}(\partial[P]) \leq n-7$. In particular, WLOG $P = \mathrm{spt}[P]$. Consider the rescalings of P and

$$r := \sup\{s > 0 : \eta_t(P) \supset P, \forall t \in (0,s)\} \in (0,1]. \tag{3.4.7}$$

If $r < 1$, then there exists $y \in \mathbb{B}_1 \cap \partial P \cap \eta_r(\partial P)$ being a touching point of $P \subset \eta_r(P)$

on the boundary, which violates strong maximum principle [Ilm96]. Hence $r = 1$.

Let ν_P be the inward pointed normal field defined on $\mathrm{Reg}(\partial P)$. Consider the Jacobi field $\phi := X \cdot \nu_P$ on $\mathrm{Reg}(\partial P)$ induced by rescalings, where X is the position vector. Since $r = 1$ in (3.4.7), we know that $\phi \geq 0$ on $\mathrm{Reg}(\partial P \cap \mathbb{B}_1)$. And since ϕ satisfies the Jacobi field equation

$$\Delta\phi + |A_{\partial P}|^2\phi = 0,$$

by strong maximum principle, either $\phi \equiv 0$ (which then implies P being a cone, contradicts to that $\mathbf{0} \notin P$), or $\phi > 0$ on regular part of ∂P. Then by Corollary 3.3.3 and that $\phi \leq 1$, we know that $\mathrm{Sing}(\partial P \cap \mathbb{B}_1) = \emptyset$. And thus, by $\phi > 0$ and $r = 1$ in (3.4.7), we know that the projection $x \mapsto x/|x|$ is injective restricting to $\partial P \cap \mathbb{B}_1$ and onto $\mathcal{E} \subset \mathbb{S}^n$, which indicates that $\partial P \cap \mathbb{B}_1$ is a radial graph.

Now for a general E, by Corollary 3.4.5, we can approximate $\mathcal{E}$ from interior by a family of connected $C^{1,1}$-optimally regular mean convex closed subset $\mathcal{E}_j \subset \mathbb{S}^n$. Let $P_j \subset \mathbf{C}(\mathcal{E}_j)$ be the corresponding closed subset given above, which is perimeter minimizing in $\mathbf{C}(\mathcal{E}_j) \cap \mathbb{B}_1$ and has stationary boundary in $(\mathbb{R}^{n+1} \setminus \partial\mathbf{C}(\mathcal{E}_j)) \cup \mathbb{B}_1$, with inward pointed normal vector ν_j on $\partial P_j \cap \mathbb{B}_1$, and $\phi_j := r\partial_r \cdot \nu_j$ be the Jacobi field on $\partial P_j \cap \mathbb{B}_1$ induced by rescaling. Let $j \to \infty$, by Lemma 2.2.1, after passing to a subsequence, P_j converges to some closed subset $P \subset E$ locally in the Hausdorff distant sense, where $P \supset E \setminus \mathbb{B}_1$ and $|\partial P|$ is a stable minimal hypersurface in $(\mathbb{R}^{n+1} \setminus \partial E) \cup \mathbb{B}_1$. Thus by mean convexity of $\mathrm{Clos}(\mathbb{B}_1)$, we have $\mathrm{Clos}(P \cap \mathbb{B}_1) \cap \mathbb{S}^n = \mathcal{E}$. Also, on $\mathrm{Reg}(\partial P \cap \mathbb{B}_1)$, ϕ_j subconverges to $\phi = X \cdot \nu_{\partial P} \geq 0$, which is a Jacobi field on $\partial P \cap \mathbb{B}_1$. By strong maximum principle, either $\phi \equiv 0$ (which implies that P is a cone, and then $E = P$ with boundary being a stable minimal hypercone, reduces to the previous situation), or $\phi > 0$ everywhere. Repeat the argument above using Corollary 3.3.3, we know that P is also a smooth radial graph over $\mathcal{E} \subset \mathbb{S}^n$. $\square$

Now we are ready to prove the following stronger version of Theorem 1.3.1.

Theorem 3.4.14. *Let $E \subset \mathbb{R}^{n+1}$ be closed and invariant under scaling. Suppose $\mathbf{C} = \partial[E]$ is a stationary integral cone and is area-minimizing in E. Then there exists a smooth hypersurface $S \subset \mathrm{Int}(E)$ minimizing area in E and foliating $\mathrm{Int}(E)$ by rescaling.*

Proof. By Corollary 3.4.5, there exists an increasing sequence of connected closed $C^{1,1}$ optimally regular mean convex domain $\mathcal{E}_j \subset \mathrm{Int}(\mathcal{E})$ approximating $\mathcal{E}$ such that $\limsup_j \|\partial\mathcal{E}_j\|(\mathbb{S}^n) = \|\partial\mathcal{E}\|(\mathbb{S}^n)$. And since $\mathcal{E}_j \subset \mathrm{Int}(\mathcal{E})$, by Frankel property of minimal hypersurfaces in $\mathbb{S}^n$, for $j >> 1$, $E_j := \mathbf{C}(\mathcal{E}_j)$ do not have stationary boundary, hence are not perimeter minimizing in E_j. By Lemma 3.4.13, there exist closed subsets $P_j \subset \mathbf{C}(\mathcal{E}_j)$ such that $P_j \cap \mathbb{B}_1$ are smooth minimizing radial graphs in $\mathrm{Int}(\mathbf{C}(\mathcal{E}_j)) \cap \mathbb{B}_1$. When $j \to \infty$, by Lemma 2.2.1, P_j **F**-subconverges to some Caccioppoli set $P_\infty \subset E$ with ∂P_∞ a stable minimal hypersurface in $(\mathbb{R}^{n+1} \setminus \partial E) \cup \mathbb{B}_1$ and P_∞ coincide with E on $\mathbb{B}_1^c$. Moreover, as a minimizing boundary in $\mathbf{C}(\mathcal{E}_j)$, $\partial[P_j]$ satisfies

$$\|\partial P_j\|(\mathbb{B}_2) \leq \|\partial[E_j]\|(\mathbb{B}_2).$$

Hence by taking $j \to \infty$ we have,

$$\|\partial[P_\infty]\|(\mathbb{B}_2) \leq \|\partial[E]\|(\mathbb{B}_2). \tag{3.4.8}$$

Since E is perimeter minimizing in E, (3.4.8) implies that P_∞ also minimizes perimeter in $\mathbb{B}_2 \cap E$. Hence by unique continuation of stable minimal hypersurfaces, $P_\infty = E$. Therefore, when $j \to \infty$,

$$d_j := dist(\mathbf{0}, P_j) \searrow 0.$$

By Lemma 2.2.1, when $j \to \infty$, $\hat{P}_j := \eta_{1/d_j}(P_j)$ are perimeter-minimizing in $\mathbb{B}_{1/d_j} \cap E_j$ and hence **F**-subconverges to some $\hat{P}_\infty \subset E$, minimizing perimeter in

$\mathrm{Int}(E)$ and has $\partial\hat{P}_\infty$ a stable minimal hypersurface in $\mathbb{R}^{n+1}$. Since

$$dist(\mathbf{0}, \mathrm{spt}(\hat{P}_\infty)) = \lim_{j\to\infty} dist(\mathbf{0}, \hat{P}_j) = 1,$$

we know that $\partial\hat{P}_\infty$ is not supported on ∂E. Hence by strong maximum principle [SW89, Ilm96], $\mathrm{spt}(\hat{P}_\infty) \subset \mathrm{Int}(E)$ and $\hat{P}_\infty$ also minimize perimeter in E.

Let $\hat{\nu}_j$ be the inward pointed normal field of $\hat{P}_j$ on $\partial\hat{P}_j \cap \mathbb{B}_{1/d_j}$, $1 \leq j \leq \infty$; $\psi_j := X \cdot \hat{\nu}_j$ be the Jacobi field induced by rescaling defined on $\mathrm{Reg}(\partial\hat{P}_j) \cap \mathbb{B}_{1/d_j}$. Since $\hat{P}_j$ are also radial graphs, when $j < \infty$, $\psi_j > 0$. When $j \to \infty$, ψ_j smoothly subconverges to ψ_∞ on $\mathrm{Reg}(\partial\hat{P}_\infty)$. Hence $\psi_\infty \geq 0$. Again by strong maximum principle for Jacobi field equation, either $\psi_\infty \equiv 0$ (which implies $\hat{P}_\infty$ is a cone and violates that $dist(\mathbf{0}, \hat{P}_\infty) = 1$), or $\psi_\infty > 0$ everywhere. Then by Corollary 3.3.3, $\mathrm{Sing}(\partial\hat{P}_\infty) = \emptyset$ and $S = \partial\hat{P}_\infty \subset \mathrm{Int}(E)$ is area-minimizing in E. Furthermore, $\psi_\infty > 0$ everywhere implies, by the same argument as Lemma 3.4.13, that S is a smooth radial graph over $\mathcal{E}$, hence foliates $\mathrm{Int}(E)$ by rescalings. □

Now we turn to the non-minimizing case,

Theorem 3.4.15. *Let $E \subset \mathbb{R}^{n+1}$ be a closed, viscosity mean convex cone with connected interior. Suppose $\mathbf{C} = \partial[E]$ is NOT minimizing in E (This is the case when E is not a Caccioppoli set).*

Let $\mathcal{D} := \bigcup_{t\geq 0} D_t \times \{t\} \subset \mathbb{R}^{n+1} \times \mathbb{R}$ be the level set flow of $D_0 := \mathbb{R}^{n+1} \setminus \mathrm{Int}(E)$. Then $D_t = \eta_{\sqrt{t}}(D_1)$, $\forall t > 0$; D_1 is a smooth domain and ∂D_1 is a self-expander supported in $\mathrm{Int}(E)$ and minimizing $\mathbf{E}$-functional in $\mathrm{Int}(E)$.

Moreover, ∂D_1 is a radial graph over $\mathcal{E}$, i.e. there exists $v \in C^\infty(\mathcal{E}; (0, +\infty))$ which diverges near $\partial\mathcal{E}$ such that

$$D_1^c = \{r\omega : \omega \in \mathcal{E}, r > v(\omega)\}.$$

In particular, the mean curvature of ∂D_1 with respect to outward pointed normal field is positive along ∂D_1.

Proof. That $D_t = \eta_{\sqrt{t}}(D_1)$ follows from scaling invariance of D_0 and the uniqueness of level set flow, see Proposition 3.4.11;

Now let $\mathcal{E}_j \subset \mathrm{Int}(\mathcal{E})$ be the $C^{1,1}$-optimally regular closed connected mean convex approximation given by Corollary 3.4.5; Thus $E_j := \mathbf{C}(\mathcal{E}_j)$ satisfies the assumption in Lemma 3.4.8. Let $D_0^j := \mathbb{R}^{n+1} \setminus \mathrm{Int}(E_j)$. We know that $\mathrm{Int}(D_0^j) \supset D_0 \setminus \{\mathbf{0}\}$ and $\partial D_0^j \to \partial D_0$ locally in Hausdorff distant sense as $j \to \infty$.

Fix $j \geq 1$, by Lemma 3.4.8, there exists a closed Caccioppoli set $F^j \subset (D_0^j)^c = \mathrm{Int}(E_j)$ minimizing $\mathbf{E}$-functional in E_j and distant asymptotic to E_j near infinity. In particular, ∂F^j is a self-expander in $\mathbb{R}^{n+1}$ with optimal regularity. We now Claim that

$$\eta_\lambda(F^j) \cap \mathrm{Int}(D_1) = \emptyset; \quad \eta_\lambda(\mathrm{Int}(F^j)) \cap D_1 = \emptyset, \quad \forall \lambda \geq 1. \tag{3.4.9}$$

To see this, recall that by avoidance principle of level set flow, we have, locally in the Hausdorff distant sense,

$$\lim_{\lambda \searrow 0} D_\lambda = \lim_{\lambda \searrow 0} \eta_\lambda(D_1) \subset D_0 \subset \mathrm{Int}(D_0^j) \cup \{\mathbf{0}\}.$$

Hence there exists $\epsilon_j > 0$ and $\lambda_j \in (0,1)$ such that for every $0 < \lambda \leq \lambda_j$ and every $\mu \in (0,1]$,

$$dist(D_{\lambda\mu} \cap \mathcal{A}, \eta_{\sqrt{\lambda}}(F^j) \cap \mathcal{A}) \geq \frac{1}{2} dist(D_0 \cap \mathcal{A}, (D_0^j)^c \cap \mathcal{A}) \geq \epsilon_j, \tag{3.4.10}$$

where $\mathcal{A} := \mathbb{A}(\mathbf{0}; 1/2, 1)$. In particular, (3.4.10) implies that for every $\mu \in (0,1]$,

$$F^j \cap D_\mu \setminus \mathbb{B}_{1/\sqrt{\lambda_j}} = \emptyset.$$

Hence (3.4.9) follows from Theorem 3.4.10 and Lemma 3.4.12 by comparing $\mathcal{D}$ with the weak set flow

$$t \mapsto \eta_{\sqrt{t+\epsilon}}\left(\mathrm{spt}(\partial[F^j]) \cap \mathrm{Clos}(\mathbb{B}_{2/\sqrt{\lambda_j}})\right),$$

for every $\epsilon > 0$ and then send $\epsilon \searrow 0$.

Now let $j \to \infty$ in (3.4.9), by **E**-minimizing property of F^j, Corollary 3.4.7 and Lemma 2.2.1, we have $F^j \to F^\infty$ in **F**-metric, where $F^\infty = \mathrm{spt}[F^\infty] \subset E$ is a closed Caccioppoli set distant asymptotic to E near infinity, and has $S^\infty := \partial F^\infty$ an **E**-stable hypersurface with optimal regularity, minimizing **E**-functional in $\mathrm{Int}(E)$. By (3.4.9), we have

$$\eta_\lambda(F^\infty) \cap \mathrm{Int}(D_1) = \emptyset,\ \eta_\lambda(\mathrm{Int}(F^\infty)) \cap D_1 = \emptyset, \quad \forall \lambda \geq 1. \tag{3.4.11}$$

Also, we assert that F^∞ is not a cone. In fact otherwise, $E = F^\infty$ whose boundary is a stable minimal hypercone. In particular, $\mathbf{0} \in F^\infty$. Hence when $j \to \infty$, $d_j := dist_{\mathbb{R}^{n+1}}(\mathbf{0}, F^j) \to 0$, and by Lemma 2.2.1, $\eta_{1/j}(F^j)$ **F**-subconverges to some Caccioppoli set $\hat{F}^\infty = \mathrm{spt}[\hat{F}^\infty] \subset E$ with distant 1 to the origin and an optimally regular stable minimal boundary. By maximum principle [SW89, Ilm96], $\hat{F}^\infty \subset \mathrm{Int}(E)$. But since E is not minimizing, by comparing rescalings of $\hat{F}^\infty$ with the Caccioppoli set constructed in Lemma 3.4.13 and using maximum principle [SW89, Ilm96], we get a contradiction.

On the other hand, by Lemma 3.4.12, $t \mapsto \mathbb{R}^{n+1} \setminus \eta_{\sqrt{t}}(\mathrm{Int}(F^\infty))$ is a weak-set flow from D_0. Hence by Proposition 3.4.11, we have $D_1 \supset \mathbb{R}^{n+1} \setminus \mathrm{Int}(F^\infty)$. Combined with (3.4.11) when $\lambda = 1$, we have $\partial D_1 = \partial F^\infty$ and $\mathrm{Int}(D_1) = \mathbb{R}^{n+1} \setminus F^\infty$.

Finally by (3.4.11), we have $\eta_\lambda(F^\infty) \subset F^\infty$ for $\lambda \geq 1$. Hence $\phi := 2H^{S^\infty} = X \cdot \nu_{S^\infty} \geq 0$ on regular part of S^∞ and is not identically 0 since S^∞ is not a cone. Recall that by self-expanding equation [Din20, (3.9)], we have $(\mathcal{L}_{S^\infty} + 1)\phi = 0$, where $\mathcal{L}_{S^\infty}$ is the Jacobi operator for **E**-functional defined in (3.4.5). By (3.4.6), this is also

a Jacobi-type equation. Hence by strong maximum principle, $\phi > 0$ and is bounded on regular part of S^∞ in each ball. And therefore by Corollary 3.3.3, $\mathrm{Sing}(S^\infty) = \emptyset$ and $\mathrm{spt}(S^\infty)$ is a radial graph over $\mathrm{Int}(\mathcal{E})$. □

Proof of Theorem 1.3.3. Let $E \subset \mathbb{R}^{n+1}$ be a closed cone bounded by $\mathbf{C}$. Recall by Remark 3.4.2 (1), E is viscosity mean convex. If E is not perimeter minimizing in E, then Theorem 3.4.15 provides entirely smooth, strictly mean convex self-expander $S \subset \mathrm{Int}(E)$ and distant asymptotic to E near infinity. Thus the blow-down of S restricted to $\mathbb{B}_1$ will be strictly mean convex hypersurface approximating $\mathbf{C} \cap \mathbb{B}_1$ in the Hausdorff distant sense.

If E minimizes perimeter in E, Theorem 3.4.14 provides a smooth minimizing hypersurface $S \subset \mathrm{Int}(E)$ with $\mathbf{C}$ to be the tangent cone at infinity. Blow-down $\eta_\lambda(S)$ ($\lambda \searrow 0$) also converges to $\mathbf{C}$ locally in the Hausdorff distant sense, and restricting to $\mathbb{B}_1$, $\eta_\lambda(S)$ are strictly stable. Let $\phi_1 > 0$ be the first eigenfunction of Jacobi operator $L_{\eta_\lambda(S)}$ on the domain $\eta_\lambda(S) \cap \mathbb{B}_1$, then for sufficiently small ε,

$$\{x - \varepsilon\phi_1(x) \cdot \nu_{\eta_\lambda(S)} : x \in \eta_\lambda(S) \cap \mathbb{B}_1\},$$

is a properly embedded smooth hypersurface in $\mathbb{B}_1$ with positive mean curvature, and still approximate $\mathbf{C} \cap \mathbb{B}_1$ in Hausdorff distant sense. □

Chapter 4

Min-max Minimal Hypersurafces with Obstacle

The work described in this chapter was first presented in [Wan22b].

4.1 Introduction

We consider in this Chapter min-max problem in a manifold with boundary.

The first goal of this Chapter is to prove the existence Theorem 1.3.4 for min-max minimal hypersurface with obstacle. For the precise statement, see Theorem 4.4.3 and the set-up in Section 4.4.

Our proof is based on the idea of min-max construction in [Pit81] and the regularity result in [SS81]. Two major difficulties are the lack of Schoen-Simon-type regularity for stable varifolds near ∂M, and the invalidity of unique continuation result for minimal hypersurface with obstacle. To fix this, we propose a general structure theorem for a certain class of varifolds V near ∂M, which may be viewed as generalization of ϵ-regularity results in [Mir71] to the stable case analogous to [SS81].

Theorem 4.1.1 (c.f. Theorem 4.3.2.)**.** *Let U be a relatively open subset in M with*

$U \cap \partial M \neq \emptyset$. Suppose Σ is an integral varifold in U such that

- *Restricted to the interior of M, Σ is a stable minimal hypersurface in U with multiplicities;*
- *Σ satisfies the following variational inequality, the first variation of area*

$$\delta \operatorname{Area}[\Sigma](X) \geq 0,$$

for every vector field $X \in \mathscr{X}_c(M)$ which points inward or tangent to the boundary along ∂M.

Then Σ is the sum of finitely many C^1 graphs over ∂M near each point $p \in \partial M \cap U$.

The improved regularity of min-max solution then follows from a standard replacement argument together with a uniform $C^{1,1}$ estimate for graphs satisfying a variational inequality. Based on this, the compactness is also proved for a family of constrained embedded stable minimal hypersurfaces, see Corollary 4.3.3.

A natural question is when the hypersurface we construct above has vanishing mean curvature. By strong maximum principle [Whi10], this is the case when M is weakly mean convex. Without mean convexity, in Section 4.5, we prove the following rigidity theorem,

Theorem 4.1.2. *Let $\Sigma \subset N^{n+1}$ be a two-sided, non-degenerate smooth closed minimal hypersurface; $\epsilon \in (0,1)$. Then there's $s_0 = s_0(\Sigma, N, \epsilon) > 0$ such that if $\delta \in (0, s_0)$, then any constrained embedded minimal hypersurface in the δ-neighborhood $\operatorname{Clos}(B_\delta(\Sigma))$ of Σ with the mass between $\mathscr{H}^n(\Sigma)$ and $(2-\epsilon)\mathscr{H}^n(\Sigma)$ is Σ itself.*

In view of the local min-max property by [Whi94], we can see from Theorem 4.4.3 and Theorem 4.1.2 that a non-degenerate two-sided smooth minimal hypersurface could be reconstruct from min-max in a sufficiently small neighborhood. In Chapter

5, we generalize the approach above to construct minimal hypersurfaces lying nearby certain singular minimal hypersurface under any sufficiently small perturbation of the metric. See Section 5.5 for the precise statement.

Organization of the Chapter.

In Section 4.2, we introduce some notations and preliminaries for constrained stationary varifold, an object we shall deal with throughout this chapter. We also include a brief survey on obstacle problem in the minimizing setting in Section 4.2.3 and two $C^{1,1}$-regularity theorems we shall use later, the proof of which are in Appendix B; Section 4.3 is devoted to the proof of main structure theorem for constrained stationary varifolds that are stable in the interior. In Section 4.4, we discuss the min-max constructions and prove a precise version of Theorem 1.3.4. Section 4.5 discusses Theorem 4.1.2.

4.2 Preliminaries

Throughout this chapter, let (M, g) be a compact oriented $n+1$-dimensional smooth manifold with boundary $\partial M \neq \emptyset$, $n \geq 2$; For the sake of simplicity, we can always extend (M, g) to a closed $n+1$-manifold $(\tilde{M}, \tilde{g})$ isometrically embedded in $\mathbb{R}^L$. We always work under the intrinsic topology of $\tilde{M}$. Write

- $\mathcal{B}_r(p)$ the open geodesic ball in ∂M of radius r centered at $p \in \partial M$;
- $\nu_{\partial M}$ the inward unit normal field of ∂M with respect to M, extended to a neighborhood of ∂M by $\nabla_{\nu_{\partial M}} \nu_{\partial M} = 0$; The scalar mean curvature $H_{\partial M}$ and second fundamental form $A_{\partial M}$ of ∂M in this chapter are with respect to this inward normal field;
- $T_p^+ M := \{v \in T_p M : v \cdot \nu_{\partial M} \geq 0\}$, the inward pointing vectors in $T_p M$;

- $\mathscr{X}_c(U)$ the space of compactly supported smooth vector fields in a relative open subset $U \subset M$;
- $\mathscr{X}_c^+(U) := \{X \in \mathscr{X}_c(U) : X_p \in T_p^+M,\ \forall p \in \partial M\}$;
- $\mathscr{X}_c^{tan}(U) := \{X \in \mathscr{X}_c(U) : X_p \in T_p\partial M,\ \forall p \in \partial M\}$.

Let r_M be the relative injectivity radius of ∂M in $\tilde{M}$, i.e.,

$$r_M := min\Big\{ \text{injrad}(\partial M), \sup\{r : (x,s) \mapsto \exp_x^M(s \cdot \nu_{\partial M}(x)) \text{ is a diffeomorphism on } \partial M \times (-r,r)\}\Big\}. \tag{4.2.1}$$

We shall work under Fermi coordinates in $B_{r_M}(\partial M)$,

$$\Phi : \partial M \times (-r_M, r_M) \to B_{r_M}(\partial M), \quad (x,s) \mapsto \exp_x^M(s \cdot \nu_{\partial M}(x)),$$

and shall simply write a point $p = \Phi(x,s) \in B_{r_M}(\partial M)$ as (x,s). Under this notation, for $u \in C^1(\mathcal{U}, (-r_M, r_M))$ on some domain $\mathcal{U} \subset \partial M$, we have

$$\text{graph}_{\partial M}(u) := \{(x, u(x)) : x \in \mathcal{U}\}.$$

For $x \in \partial M$ and $r \in (0, r_M)$ write

- $C_r(x) := \{(y,s) : dist_{\partial M}(y,x) < r, |s| < r\}$ the open cylinder in $\tilde{M}$;
- $C_r^+(x) := \{(y,s) : dist_{\partial M}(y,x) < r, 0 < s < r\}$ the open half cylinder in M;
- $\partial^+ C_r(x) := \{(y,r) : dist_{\partial M}(y,x) < r\}$ the top boundary of $C_r(x)$;
- $\partial^s C_r(x) := \{(y,s) : dist_{\partial M}(y,x) = r, |s| < r\}$ the side boundary of $C_r(x)$.

4.2.1 Constrained stationary varifolds

Throughout this subsection, let M be a compact $n+1$-manifold with boundary as above, $1 \leq k \leq n$, $U \subset M$ be a bounded relatively open subset. We note that the following definitions also generalize directly to properly embedded $M \subset \mathbb{R}^L$, not necessarily compact.

Definition 4.2.1. Call $V \in \mathcal{V}_k(M)$ stationary in U if

$$\delta V(X) = 0, \quad \forall X \in \mathscr{X}_c(U);$$

Call $V \in \mathcal{V}_k(M)$ **constrained stationary** along ∂M in U if

$$\delta V(X) \geq 0, \quad \forall X \in \mathscr{X}_c^+(U);$$

Call V **stationary with free boundary** along ∂M in U if

$$\delta V(X) = 0, \quad \forall X \in \mathscr{X}_c^{tan}(U).$$

We shall omit the description *along* ∂M in both cases if there's no ambiguity.

The same condition of a constrained stationary varifold is studied in [Whi10], where it is called *varifold minimizes area to first order*. The notion of stationary varifold with free boundary was introduced in [GcLZ20] for the study of free boundary minimal surfaces. Since $\mathscr{X}_c^{tan}(U) \subset \mathscr{X}_c^+(U)$ and $X \in \mathscr{X}_c^{tan}(U)$ iff $-X \in \mathscr{X}_c^{tan}(U)$ by definition, every constrained stationary varifold in U is stationary with free boundary.

We list some basic results about constrained stationary varifolds, which will be used in the next few sections.

Lemma 4.2.2 (Compactness)**.** *Let $(M_j, g_j) \subset \mathbb{R}^L$ be a family of manifolds with boundary, converging to (M, g) in C^2_{loc}, $\mathscr{U} \subset \mathbb{R}^L$ be an open subset such that $U =$*

$\mathscr{U} \cap M$; Let $V_j \in \mathcal{V}_k(M_j)$ be constrained stationary in $\mathscr{U} \cap M_j$ and $V_j \overset{*}{\rightharpoonup} V \in \mathcal{V}_k(\mathbb{R}^L)$. Then, $V \in \mathcal{V}_k(M)$ is constrained stationary in U.

Proof. By the continuity of $\delta V(X)$ in V w.r.t. varifold topology and in X w.r.t. to C^1 topology. □

Lemma 4.2.3 (Boundary Monotonicity)**.** *Let $\mathcal{K} \subset U \cap \partial M$ be a compact subset. Then there exists $R_0 = R_0(M, U, \mathcal{K}) > 0$, $\Lambda = \Lambda(M) > 0$ such that if $V \in \mathcal{V}_k(M)$ is a stationary varifold with free boundary along ∂M in U, $p \in \mathcal{K}$, then the function*

$$r \mapsto \frac{e^{\Lambda r}}{r^k} \|V\|(\mathbb{B}^L_r(p))$$

is non-decreasing in $r \in (0, R_0)$. Moreover, when $(U, M, g) = (\mathbb{R}^{n+1}_+, \mathbb{R}^{n+1}_+, |dx|^2)$ and $p = \mathbf{0}$, if $\|V\|(B_r(\mathbf{0}))/r^k$ is constant in $r \in (0, +\infty)$, then V is a cone, i.e.

$$(\eta_{\mathbf{0},r})_\sharp V = V, \quad \forall r \in (0, +\infty).$$

This is a corollary of [GcLZ20, Theorem 3.4], where $N = \partial M$ and an explicit monotonicity formula is proved.

Corollary 4.2.4. *Suppose $V \in \mathcal{IV}_k(M)$ is a stationary integral varifold with free boundary along ∂M in U; $x \in \mathrm{spt}(V) \cap U \cap \partial M$, then*

$$\theta^k(x, \|V\|) \geq 2^{-k}.$$

Proof. Since V is integer multiplicity k-rectifiable, there exist (under Fermi coordinates) $p_j = (x_j, t_j) \to x$ such that $\theta^k(p_j, \|V\|) \geq 1$. Let $r_j := |x - x_j|$. Suppose WLOG that either $t_j \equiv 0$ or $t_j > 0$ for all $j \geq 1$. For every $0 < r << 1$, take $j >> 1$

such that $r - r_j > 2t_j$. In the first case, by Lemma 4.2.3

$$\frac{e^{\Lambda r}}{\omega_k r^k}\|V\|(\mathbb{B}^L_r(x)) \geq \limsup_{j\to\infty} \frac{e^{\Lambda(r-r_j)}}{\omega_k (r-r_j)^k}\|V\|(\mathbb{B}^L_{r-r_j}(x_j)) \geq 1$$

While in the second case,

$$\begin{aligned}
\frac{e^{\Lambda r}}{\omega_k r^k}\|V\|(\mathbb{B}^L_r(x)) &\geq \limsup_{j\to\infty} \frac{e^{\Lambda r}}{\omega_k (r-r_j)^k}\|V\|(\mathbb{B}^L_{r-r_j}(x_j)) \\
&\geq \limsup_{j\to\infty} \frac{e^{\Lambda 2t_j}}{\omega_k (2t_j)^k}\|V\|(\mathbb{B}^L_{2t_j}(x_j)) \\
&\geq \limsup_{j\to\infty} \frac{e^{\Lambda 2t_j}}{2^k\cdot\omega_k t_j^k}\|V\|(\mathbb{B}^L_{t_j}(p_j)) \geq 2^{-k}.
\end{aligned}$$

where the second inequality follows from Lemma 4.2.3, and the last inequality follows from interior monotonicity formula for stationary varifolds. Let $r \to 0$, the corollary is proved. □

Lemma 4.2.5 (Strong maximum principle)**.** *Suppose $\mathcal{W} := U \cap \partial M$ is connected and mean convex, i.e. $\vec{H}_{\partial M}\cdot\nu_{\partial M}|_{\mathcal{W}} \geq 0$; $V \in \mathcal{IV}_n(M)$ be a constrained stationary integral varifold along ∂M in U. Suppose* $\mathrm{spt}(V)\cap\mathcal{W} \neq \emptyset$*, then*

(1) $\mathcal{W} \subset \mathrm{spt}(V)$ and $\vec{H}_{\partial M} = 0$ on $\mathcal{W}$.

(2) Moreover, $V \llcorner U = V_b + V_I$ where $V_b = m|\mathcal{W}|$ for some $m \in \mathbb{N}$ and $\mathrm{spt}(V_I)\cap W = \emptyset$*.*

Proof. (1) is proved in [Whi10, Theorem 4], so is (2) under the additional assumption that V is stationary. The only place stationarity is used is that $x \in \mathrm{spt}(V)\cap\mathcal{W}$ implies $\theta^n(x, \|V\|) \geq 1$, which also holds for constrained stationary integral varifolds in mean convex domain by combining (1) and the proof of Corollary 4.2.4. □

The following lemma guarantees that constrained stationary varifolds have locally bounded first variation when viewed as a varifold in $\tilde{M}$. Recall r_M is defined in (4.2.1).

Lemma 4.2.6. *Let $V \in \mathcal{V}_k(M)$ be constrained stationary in U with $\|V\|(U) \leq \Lambda$; $U' \subset\subset U$. Then $\exists\, C = C(M, U, U') > 0$ s.t.*

$$|\delta V(X)| \leq C\|V\|(U) \cdot \|X\|_{C^0}, \quad \forall X \in \mathscr{X}_c(U').$$

Proof. We shall work under Fermi coordinate in $B_{r_M}(\partial M)$, where $\nu_{\partial M}$ is extended by $\nabla_{\nu_{\partial M}}(\nu_{\partial M}) = 0$. Let $\eta \in C^1_c(B_{r_M}(\partial M) \cap U), [0,1])$ and equal to 1 near $\partial M \cap U'$. For every $X \in \mathscr{X}_c(U')$ with $\|X\|_{C^0} \leq 1$, let $X_0 := X - \eta \cdot \langle X, \nu_{\partial M}\rangle \nu_{\partial M}$. Notice that $\pm X_0 \in \mathscr{X}_c^+(U)$ and $1 \pm \langle X, \nu_{\partial M}\rangle \geq 0$, hence

$$\begin{aligned}
\delta V(X) &= \int div^S X_0 + div^S(\eta \cdot \langle X, \nu_{\partial M}\rangle \nu_{\partial M})\ dV(x,S) \\
&= \int div^S(\eta \cdot (\langle X, \nu_{\partial M}\rangle - 1)\nu_{\partial M})\ dV(x,S) + \int div^S(\eta \cdot \nu_{\partial M})\ dV(x,S) \\
&\leq \int div^S(\eta \cdot \nu_{\partial M})\ dV(x,S) \leq C(M,U,U')\|V\|(U).
\end{aligned}$$

where the first inequality follows from that $\eta \cdot (1 - \langle X, \nu_{\partial M}\rangle)\nu_{\partial M} \in \mathscr{X}_c^+(U)$. Similarly,

$$\begin{aligned}
\delta V(X) &= \int div^S(\eta \cdot (\langle X, \nu_{\partial M}\rangle + 1)\nu_{\partial M})\ dV(x,S) - \int div^S(\eta \cdot \nu_{\partial M})\ dV(x,S) \\
&\geq -\int div^S(\eta \cdot \nu_{\partial M})\ dV(x,S) \geq -C(M,U,U')\|V\|(U).
\end{aligned}$$

where $C(M,U,U') := \|\eta \cdot \nu_{\partial M}\|_{C^1}$. □

Corollary 4.2.7. *Let $V \in \mathcal{IV}_n(M)$ be a constrained stationary integral varifold in U. Then $\forall x \in \mathrm{spt}(V) \cap U \cap \partial M$, the tangent varifold of V at x exists and equals to $m|T_x\partial M|$ for some $m \in \mathbb{N}$.*

Proof. By Lemma 4.2.2, 4.2.3, 4.2.6 and Allard Compactness Theorem [Sim83b, Chapter 8, 42.7, 42.8], up to a subsequence of $r_j \to 0$, $(\eta_{x,r_j})_\sharp V \overset{*}{\rightharpoonup} \mathbf{C} \in \mathcal{IV}(T_x^+(M))$. Moreover, by the rigidity part of Lemma 4.2.3, $\mathbf{C}$ is a constrained stationary cone along $T_x(\partial M)$ centered at 0. Since $0 \in \mathrm{spt}(\mathbf{C})$, by Lemma 4.2.5, $\mathbf{C} = m|T_x(\partial M)| + \mathbf{C}_I$

for some stationary varifold $\mathbf{C}_I$ supported in $T_x^+M\backslash T_x(\partial M)$ and $m \in \mathbb{N}$. But since $\mathbf{C}_I$ is also a cone, $\mathbf{C}_I = 0$ and $\mathbf{C} = m|T_x(\partial M)|$ and hence is unique since $m = \theta^n(x, \|V\|)$ is independent of the converging sequences. □

4.2.2 Constrained embedded hypersurfaces

A major problem in the regularity theory of constrained stationary varifolds is that one should not expect embeddedness near ∂M. Consider for example, $M = \mathbb{R}^{n+1} \setminus \mathbb{B}_1^{n+1}(0)$, $V = |\mathbb{S}_1^n(0)| + |\{x_{n+1} = 1\}| \in \mathcal{IV}_n(M)$ is constrained stationary in M but is supported on an immersed submanifold with self-touching points. We introduce in this Subsection the following notion of constrained embedded hypersurfaces in order to describe the optimal regularity for constrained stationary varifolds. Suppose $U \subset M$ is a relatively open subset. Recall $\mathrm{Int}(U) = U \setminus \partial M$.

Definition 4.2.8. Call $V \in \mathcal{IV}_n(U)$ a **constrained embedded hypersurface** with optimal regularity if $\forall p \in U$, there's a neighborhood U_p of p in U, $m \in \mathbb{N} \cup \{0\}$ and $C^{1,1}$ embedded hypersurfaces $\Sigma_j \subset U_p$ without boundary, $1 \leq j \leq m$ such that

(1) $V \llcorner U_p = \sum_{j=1}^m |\Sigma_j|$,

(2) $\partial M \cap U_p \cap \mathrm{Clos}(\Sigma_j) \setminus \Sigma_j = \emptyset$ and $\mathscr{H}^{n-2}(U_p \cap \mathrm{Clos}(\Sigma_j) \setminus \Sigma_j) = 0$,

(3) If $q \in U_p \cap \mathrm{Clos}(\Sigma_i) \cap \mathrm{Clos}(\Sigma_j)$, then either $q \in \partial M$ or $\mathrm{Clos}(\Sigma_i) \cap \mathrm{Int}(U_p)$ coincide with $\mathrm{Clos}(\Sigma_j) \cap \mathrm{Int}(U_p)$ on the connected component containing q.

Let

$$\mathrm{Sing}(V) := \{p \in \mathrm{spt}(V) \cap \mathrm{Int}(U) : p \text{ is not a regular point of hypersurfaces above in } U_p\}.$$

Here a hypersurface $\Sigma \subset M$ is called $C^{1,\alpha}$ if locally it's the graph of some $C^{1,\alpha}$ function over some n-plane.

Remark 4.2.9. (1) When a constrained embedded V is constrained stationary along ∂M, it's clear that $\mathrm{spt}(V) \cap \mathrm{Int}(U)$ is a minimal hypersurface in $\mathrm{Int}(U)$; Also for every $x \in \partial M \cap U$, suppose for some neighborhood U_x of x that $V \llcorner U_x = \sum_j |\operatorname{graph}_{\partial M}(u_j)|$, since u_j are $C^{1,1}$ and by Lemma 4.2.5, $\{u_j = 0\} \subset \{y \in \partial M : H_{\partial M} \geq 0\}$. Hence $\nabla_{\partial M} u_j(x) = 0$, $\Delta_{\partial M} u_j(x) = 0$ for a.e. $x \in \{u_j = 0\}$. And as a $C^{1,1}$ graph, $\operatorname{graph}_{\partial M}(u_j)$ has mean curvature of form

$$H = \mathbf{a}(x, u_j, \nabla_{\partial M} u_j) \cdot \nabla^2_{\partial M} u_j + b(x, u_j, \nabla_{\partial M} u_j),$$

whenever u_j is differentiable up to second order. In particular, $|\operatorname{graph}_{\partial M}(u_j)|$ has L^∞-mean curvature, which vanishes in $\mathrm{Int}(M)$ and equals to $H_{\partial M} - \Delta_{\partial M} u_j = H_{\partial M}$ for a.e $x \in \{u_j = 0\}$. And thus, $|\operatorname{graph}_{\partial M} u_j|$ is constrained stationary in U along ∂M.

(2) Also note that a constrained embedded hypersurface V could be viewed as the associated varifold of some immersed hypersurface. More precisely, consider $\Sigma \subset M \times \mathbb{N}$ defined by

$$\Sigma := \{(p, j) : p \in \mathrm{spt}(V) \setminus \mathrm{Sing}(V), 1 \leq j \leq \theta^n(p, \|V\|)\},$$

and endow topology on Σ by specifying neighborhoods of points in Σ:

- If $(p, j) \in \mathrm{Int}(M) \times \mathbb{N} \cap \Sigma$, $\mathcal{O} \subset \Sigma$ is a neighborhood of (p, j) iff $\exists U \ni p$ an open subset of M such that $U \times \{j\} \subset \mathcal{O}$;

- If $(p, j) \in \partial M \times \mathbb{N} \cap \Sigma$, let U_p be a neighborhood of p in M such that $V \llcorner U_p = \sum_{i=1}^{m_p} |\operatorname{graph}_{\partial M}(u_i)|$, where $0 \leq u_1 \leq u_2 \leq ... \leq u_{m_p}$ are $C^{1,1}$ functions over $U_p \cap \partial M$; And such that for every $x \in U_p \cap \partial M$, if $u_i(x) = u_j(x)$, then either they both vanish or $u_i = u_j$ on the connected component of $\{u_i > 0\}$ containing x. By definition of Σ, $m_p \geq j$; for each $x \in U_p \cap \partial M$, define $l_j(x) := \inf\{l \in \mathbb{N} :$

$u_l(x) = u_j(x)\}$.

Now, $\mathcal{O} \subset \Sigma$ is a neighborhood of (p, j) iff $\exists \mathcal{U} \subset \partial M \cap U$ an open neighborhood of p in ∂M such that

$$\{(x, u_j(x), j - l_j(x) + 1) : x \in \mathcal{U}\} \subset \mathcal{O},$$

where recall that $(x, u_j(x))$ represent a point in $\mathrm{spt}(V)$ under Fermi coordinate near ∂M.

We leave it to readers to verify that such Σ admits a natural $C^{1,1}$ structure making $\Sigma \to M$ by projection onto first factor a $C^{1,1}$ immersion, whose associated varifold is V.

Definition 4.2.10. Let $\{V_j\}, V$ be constrained embedded hypersurfaces with optimal regularity in U. Call V_j C^1**-converges** to V if $\mathbf{F}(V, V_j) \to 0$ and $\forall p \in \mathrm{spt}(V) \setminus \mathrm{Sing}(V)$, there's some neighborhood $U_p \cap \mathrm{Sing}(V) = \emptyset$ of p such that

(1) If $p \in \mathrm{Int}(U)$, then $U_p \cap \partial M = \emptyset$, $V \llcorner U_p = m|\Sigma|$ for some $m \in \mathbb{N}$, $\Sigma \subset U_p$ properly embedded hypersurface and $V_j \llcorner U_p = \sum_{i=1}^m |\operatorname{graph}_\Sigma(v_j^{(i)})|$ for some $v_j^{(i)}$ with $|v_j^{(i)}|_{C^1(\Sigma)} \to 0$ as $j \to \infty$.

(2) If $p \in \partial M$, then $V \llcorner U_p = \sum_{i=1}^m |\operatorname{graph}_{\partial M}(u^{(i)})|$, $V_j \llcorner U_p = \sum_{i=1}^m |\operatorname{graph}_{\partial M}(u_j^{(i)})|$ for some $m \in \mathbb{N}$, $u^{(i)}, u_j^{(i)} \in C^{1,1}(\partial M \cap U_p)$ and $|u^{(i)} - u_j^{(i)}|_{C^1} \to 0$.

4.2.3 Obstacle problem

Here we only focus on minimal hypersurfaces with smooth obstacle. Problems in two different settings are well studied, mainly for hypersurfaces in the Euclidean spaces.

• **Non-parametric obstacle problem.** Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded convex domain. $\psi \in C^2(\bar{\Omega})$, $\varphi \in C^2(\partial\Omega)$. Consider the problem

(N) Minimize

$$\mathscr{A}(u) := \int_\Omega \sqrt{1+|\nabla u|^2}\, dx$$

among all Lipschitz functions u such that $u = \varphi$ on $\partial\Omega$ and $u \geq \psi$ in Ω.

The existence and uniqueness of solutions to problem (N) was studied by [LS71] and [GP71] under weaker assumptions on ψ, φ and Ω. See also [Giu11].

The regularity of minimizer $\bar{u}$ in problem (N) was also studied in [LS71] and [GP71]. Assuming $\psi \in H^{2,p}$, they are able to prove $\bar{u} \in H^{2,p}$, $\forall n < p < +\infty$. Using an approximation argument, [BKL74] shown that $\bar{u}$ is $C^{1,1}$ provided $\psi \in C^2(\bar{\Omega})$.

A linearization of problem (N) is studied in [CK80], where they considered the minimizer of enegry functional $\mathscr{E}(u) := \int_\Omega |\nabla u|^2\, dx$ instead of area functional $\mathscr{A}$, and obtained $C^{1,\alpha}$-regularity for minimizers by a Harnack-type inequality only assuming $\psi \in C^{1,\alpha}(\bar{\Omega})$, $0 \leq \alpha \leq 1$. Their approach was generalized in [Lin85] to deal with the non-parametric obstacle problem for area functional.

The regularity result we shall use in the rest of this chapter is

Theorem 4.2.11 (Interior Regularity)**.** *Let $M \subset \mathbb{R}^L$ be as in Section 4.2, $\alpha \in (0,1)$. Then there exists $r_0 = r_0(M) \in (0, r_M/2)$ such that, if $r < r_0$, $y \in \partial M$, $u \in C^1(\mathcal{B}_r(y), [0,r)) \cap C^{1,\alpha}_{loc}(\mathcal{B}_r(y) \setminus \{y\})$ with $|\nabla u| \leq 1$ and $|\operatorname{graph}_{\partial M}(u)|$ constrained stationary in $C_r(y)$ along ∂M, then $u \in C^{1,1}_{loc}(\mathcal{B}_r(y))$ and for every compact subset $\mathcal{K} \subset \mathcal{B}_r(y)$,*

$$|\nabla u(x) - \nabla u(x')| \leq C(M, \mathcal{K}, r_0)|x - x'|, \qquad \text{for every } x, x' \in \mathcal{K}.$$

For later applications, we include the case with isolated singularity. The proof essentially follows [Lin85]. For some technical reason, we have to assume a-priori $C^{1,\alpha}$ regularity instead of only C^1 regularity. But the point is we don't need a-priori a uniform $C^{1,\alpha}$ bound. For sake of completeness, we include in Appendix B the proof of Theorem 4.2.11 and the following Theorem 4.2.12.

• **Parametric obstacle problem.** Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain, $E_0 \subset \mathbb{R}^{n+1}$ be a set of finite perimeter. Consider the following obstacle problem for minimizing boundary

(P) Minimize $\mathbf{M}(\partial[E])$ among all Cacciappoli sets $E \subset \mathbb{R}^{n+1}$ with $\mathrm{spt}([E]-[E_0]) \subset \bar{\Omega}$.

Regularity of solution $\bar{E}$ to Problem (P) was first studied by [Mir71], in which he shown $\partial_* \bar{E}$ is of class C^1 near $\partial\Omega$ provided $\partial\Omega$ is C^1 (Call a hypersurface *of class* $C^{k,\alpha}$ if locally it's the graph of some $C^{k,\alpha}$ function). Combined with the result in non-parametric problem, $\partial_* \bar{E}$ is $C^{1,\alpha}$ if $\partial\Omega$ is $C^{1,\alpha}$, $0 \leq \alpha \leq 1$. Different approaches were exploited in [BM82] and [Tam82] where $C^{1,\alpha}$ regularity ($\alpha \in (0,1)$) of $\partial_* \bar{E}$ is proved assuming $H^{2,p}$ or $C^{1,\alpha}$ regularity of $\partial\Omega$ correspondingly. We assert that [Tam82] proves a general regularity result on the reduced boundary with good excess estimate (see also [Alm75, Bom82]), hence directly generalizes to minimizing boundary with obstacle in a manifold. The following regularity for minimizing currents with obstacle in a compact manifold with boundary will be used in Section 4.4.

Theorem 4.2.12. *Let $M \subset \mathbb{R}^L$ be as in Section 4.2, $U \subset M$ be a relative open subset with non-empty intersection with ∂M. Suppose $T \in \mathcal{Z}_n(U)$ minimizes mass in $\{S \in \mathcal{Z}_n(U) : \mathrm{spt}(T-S) \subset\subset U\}$. Then $\forall y \in U \cap \partial M$, there's a neighborhood U_y of y and $C^{1,1}$ functions $u_1 \geq u_2 \geq ... \geq u_q \geq 0$, $|\nabla u_j| \leq 1$ such that*

(1) $\pm T \llcorner U_y = \sum_{j=1}^{q} [\mathrm{graph}_{\partial M}(u_j)] \llcorner U_y$.

(2) For every $x \in U_y \cap \partial M$, if $u_i(x) = u_j(x)$, then either $u_i(x) = u_j(x) = 0$, or $u_i \equiv u_j$ on the connected component of $\{u_i > 0\}$ containing x.

4.3 A Structure Theorem near the Boundary

The goal of this Section is to prove the following structure theorem for a class of constrained stationary varifold near ∂M. Throughout the section, $U \subset M$ be a relatively open subset with nonempty intersection with ∂M, $\mathcal{K} \subset \partial M \cap U$ be a compact subset. Keep working under Fermi coordinates near ∂M. Use x, y, z to denote points on ∂M and $p = (x, t), q = (y, s)$ to denote points in $\mathrm{Int}(M)$. We shall not distinguish between x and $(x, 0)$.

Definition 4.3.1. Call $V \in \mathcal{V}_n(\mathrm{Int}(U))$ a **stable minimal hypersurface with multiplicity**, if there exists a countable family of disjoint properly embedded stable minimal hypersurfaces $\{\Sigma_j\}_{j\geq 1}$ with optimal regularity in $\mathrm{Int}(U)$ (i.e. $\mathrm{Sing}(\Sigma_j) \cap \mathrm{Int}(U)$ are of Hausdorff dimension $\leq n-7$.) and $m_j \in \mathbb{N}$ such that

- $\bigcup_{j\geq 1} \Sigma_j$ has no accumulation point in $\mathrm{Int}(U)$, i.e. $\mathrm{Clos}(\bigcup_{j\geq 1} \Sigma_j) \cap \mathrm{Int}(U) = \bigcup_{j\geq 1} \Sigma_j$;
- $V = \sum_j m_j |\Sigma_j|$.

Call points in $\bigcup_{j\geq 1} Reg(\Sigma_j)$ **regular points** of V in $\mathrm{Int}(U)$.

Call $V \in \mathcal{IV}(U)$ **constrained stable varifold** if V is constrained stationary in U and a stable minimal hypersurface with multiplicity in $\mathrm{Int}(U)$. Call it **constrained embedded stable minimal hypersurface** if in addition it's a constrained embedded hypersurface.

Theorem 4.3.2. *Suppose $V \in \mathcal{IV}_n(M)$ be a constrained stable varifold in U along ∂M with $\|V\|(U) \leq \Lambda$.*

Then $\exists\ r_0 = r_0(M, U, \mathcal{K}, \Lambda) \in (0, dist(\mathcal{K}, U^c)/4)$, $m_0 = m_0(M, U, \mathcal{K}, \Lambda) < +\infty$ such that, for every $y \in \mathcal{K}$,

$$V \llcorner C_{r_0}(y) = V_0 + V_+,$$

where V_+ is a stable minimal hypersurfaces with multiplicity in $C_{r_0}(y)$ and $\mathrm{spt}(V_+) \cap B_{r_0/2}(\partial M) = \emptyset$*; $V_0 = \sum_{j=1}^{m} |\,\mathrm{graph}_{\partial M}(u_j)|$ for some $0 \le m \le m_0$ and C^1 functions $u_j : \mathcal{B}_{r_0}(y) \to [0, 3r_0/4)$ satisfying*

(1) $\mathrm{spt}(V_0) \cap \mathrm{spt}(V_+) = \emptyset$*;*

(2) $u_1 \ge u_2 \ge ... \ge u_m$*,* $|\nabla u_j| \le 1/2$*;*

(3) If $y' \in \mathcal{B}_{r_0}(y)$ such that $u_i(y') = u_j(y')$, then either $u_i(y') = 0$ or $u_i \equiv u_j$ on the connected component of $\{u_i > 0\}$ containing y'.

If further, $u_j \in C^{1,1}(\mathcal{B}_{r_0}(y))$, then $|\,\mathrm{graph}_{\partial M}(u_j)|$ is a constrained stationary varifold in $C_{r_0}(y)$.

An interesting analytic question is whether one can directly conclude that $u_j \in C^{1,1}$.

In view of Theorem 4.2.11, a direct corollary of Theorem 4.3.2 is the compactness of constrained embedded stable minimal hypersurfaces. The proof is left to readers.

Corollary 4.3.3. *Let $\{V_j\}_{j\ge1}$ be a sequence of constrained embedded stable minimal hypersurfaces in U along ∂M, with $\|V_j\|(U) \le \Lambda$. Then up to a subsequence, there exists a constrained embedded stable minimal hypersurface $V_\infty \in \mathcal{IV}_n(U)$ such that V_j converges to V in varifold sense and in C^1 sense (see definition 4.2.10).*

The rest of this Section is devoted to the proof of Theorem 4.3.2. We start with a tilt estimate of constrained stable varifolds near ∂M.

Lemma 4.3.4. *Let V be in Theorem 4.3.2. There exists a continuous function $\rho : (0, 1/2) \to (0, dist(\mathcal{K}, U^c)/2)$ depending only on $M, U, \mathcal{K}, \Lambda$ such that, for every $\epsilon \in (0, 1/2)$, if $p = (y, t) \in \mathcal{K} \times (0, \rho(\epsilon)) \cap \mathrm{spt}(V)$, then V is regular near p and $1 - |\nu_V \cdot \nu_{\partial M}|(p)^2 < \epsilon$, where ν_V denotes the unit normal of V.*

Proof. We shall argue by contradiction. If the statement is not true, then there's some $\epsilon > 0$, $V_i \in \mathcal{IV}_n(M)$ constrained stationary in U, stable minimal hypersurfaces in $\mathrm{Int}(U)$ with $\|V_i\|(U) \leq \Lambda$ and $p_i = (y_i, t_i) \in \mathrm{spt}(V_i) \cap \mathrm{Int}(U)$ with $t_i \to 0$, $y_i \in \mathcal{K}$ such that one of the following holds,

(a) p_i is a singular point of $V_i \llcorner \mathrm{Int}(U)$;

(b) p_i is a regular point but $1 - |\nu_{V_i} \cdot \nu_{\partial M}|(p)^2 \geq \epsilon$.

Suppose $y_i \to y_0 \in \mathcal{K}$. By Lemma 4.2.6, 4.2.3, 4.2.2 and Allard compactness [Sim83b], up to a subsequence,

$$(\eta_{y_i,t_i})_\sharp V_i \overset{*}{\rightharpoonup} P,$$

for some $P \in \mathcal{IV}_n(T^+_{y_0}M)$ constrained stationary along $T_{y_0}\partial M$. Thus by Lemma 4.2.5, $P = k|T_{y_0}\partial M| + P_I$ for some $k \in \{0\} \cup \mathbb{N}$ and stationary integral varifold P_I in $T_{y_0}M$ with $\mathrm{spt}(P_I) \subset \mathrm{Int}(T^+_{y_0}M)$.

On the other hand, since $V_i \llcorner \mathrm{Int}(U)$ are stable minimal hypersurfaces, by Schoen-Simon compactness [SS81], P_I is a stable minimal hypersurface in $T_{y_0}M$. Since $\mathrm{spt}(P_I) \subset \mathrm{Int}(T^+_{y_0}M)$, any tangent cone C of P_I at infinity is also supported in the half space $T^+_{y_0}M$. Since $\mathbf{0} \in \mathrm{spt}(C)$ and C is a cone, by the strong maximum principle [Ilm96], C is an integer multiple of $|T_{y_0}\partial M|$. Then by Schoen-Simon's ϵ-regularity Theorem [SS81, Theorem 1], $P_I = \sum_j |L_j|$ for some hyperplane $L_j \subset T_{y_0}M$ parallel to $T_{y_0}\partial M$.

Now that, consider (up to a subsequence) $\eta_{y_i,t_i}(p_i) \to p_0 \in \mathrm{Int}(T^+_{y_0}M)$. Since p_0 is a smooth point of P_I with tangent plane parallel to $T_{y_0}\partial M$, by [SS81, Theorem 1], neither of (a) (b) is true. □

From now on, let ρ be the function in Lemma 4.3.4 with respect to some $\mathcal{K}' \subset U \cap \partial M$ such that $\mathcal{K} \subset \mathrm{Int}(\mathcal{K}')$. We shall also assume, WLOG, that $\rho < min\{1, dist(\mathcal{K}, (\mathcal{K}')^c)\}$. The following is a direct corollary of Lemma 4.3.4, and the proof is left to the readers.

Corollary 4.3.5. *For every $k \in \mathbb{N}$ and $p = (y,t) \in \mathcal{K} \times (0, \rho(1/(10k^2))/2) \cap \mathrm{spt}(V)$, the connected component of $\mathrm{spt}(V) \cap C_{kt}(y)$ containing p is the graph of some function $u \in C^\infty(\mathcal{B}_{kt}(y))$ with*

$$0 < |u| < 2t, \quad |\nabla u| < 1/(2k).$$

Now we proceed with the proof of Theorem 4.3.2. For $y \in \mathcal{K}$, let $r_0 = \rho(1/4000)/2$. Suppose $V \llcorner C^+_{3r_0}(y) = \sum_{j \geq 1} m_j |\Sigma_j|$ where Σ_j are connected, disjoint regular stable minimal hypersurfaces. Let $J := \{j \geq 1 : \Sigma_j \cap B_{r_0/2}(\partial M) = \emptyset\}$; $V_+ := \sum_{j \in J} m_j |\Sigma_j|$ and $V_0 = V \llcorner C_{3r_0}(y) - V_+$. By Corollary 4.3.5, $\mathrm{spt}(V_0) \subset B_{3r_0/4}(\partial M)$, $\mathrm{spt}(V_0) \cap \mathrm{spt}(V_+) = \emptyset$ and V_0 is also constrained stationary in $C_{3r_0}(y) \cap M$ along ∂M.

Define $m : \mathcal{B}_{r_0}(y) \to \mathbb{Z} \cup \{+\infty\}$ by

$$m(x) := \begin{cases} 0, & \text{if } V_0 = 0; \\ \displaystyle\sum_{p=(x,t) \in \mathrm{spt}(V_0)} \theta^n(p, \|V_0\|), & \text{if } V_0 \neq 0. \end{cases} \tag{4.3.1}$$

Note that by Corollary 4.2.7 and Lemma 4.3.4, $\theta^n(p, \|V\|) \in \mathbb{Z}$ for $p \in \mathrm{spt}(V_0) \cap C_{r_0}(y)$, hence $m(x) \in \mathbb{Z} \cup \{+\infty\}$. We shall show that m is constant on $\mathcal{B}_{r_0}(y_0)$, bounded uniformly from above.

Lemma 4.3.6. *$m(x) \leq m_0$ for some $m_0 = m_0(M, U, \mathcal{K}, \Lambda, r_0)$ and every $x \in \mathcal{B}_{r_0}(y)$.*

Proof. For every $x \in \mathcal{B}_{r_0}(y)$ fixed, let $\{t > 0 : (x,t) \in \mathrm{spt}(V_0)\} = \{t_1 > t_2 > ...(> 0)\}$, $t_1 < 3r_0/4$; and let $n_i := \theta^n((x,t_i), \|V_0\|) \in \mathbb{N}$. By Corollary 4.3.5, for each $i \geq 1$, the connected component of $\mathrm{spt}(V_0 \llcorner C_{2t_i}(x))$ containing (x,t_i), denoted by S_i, is a smooth minimal graph over $\mathcal{B}_{2t_i}(x)$, and $\{S_i\}_{i \geq 1}$ are disjoint. Moreover

$$\|S_i\|(\mathbb{B}_{2t_i}(x)) \geq e^{-\Lambda_1 t_i} \cdot \omega_n t_i^n$$

for some $\Lambda_1 = \Lambda(M) \geq 0$ by interior monotonicity of area.

Also, by Lemma 4.2.3, suppose WLOG $\Lambda_1 \geq 0$ such that $t \mapsto e^{\Lambda_1 t}\|V_0\|(\mathbb{B}_t(x))/t^n$ is non-decreasing on $(0, 2r_0)$. Therefore,

$$\begin{aligned}
\frac{e^{\Lambda_1 2r_0}}{\omega_n(2r_0)^n}\|V_0\|(\mathbb{B}_{2r_0}(x)) &\geq \frac{e^{\Lambda_1\cdot(2t_1)}}{\omega_n(2t_1)^n}\|V_0\|(\mathbb{B}_{2t_1}(x)) \\
&= \frac{e^{\Lambda_1\cdot(2t_1)}}{\omega_n(2t_1)^n}\cdot n_1\|S_1\|(\mathbb{B}_{2t_1}(x)) + \frac{e^{\Lambda_1\cdot(2t_1)}}{\omega_n(2t_1)^n}\|\hat{V}_1\|(\mathbb{B}_{2t_1}(x)) \\
&\geq n_1\cdot 2^{-n} + \frac{e^{\Lambda_1\cdot(2t_2)}}{\omega_n(2t_2)^n}\|\hat{V}_1\|(\mathbb{B}_{2t_2}(x)) \\
&= n_1\cdot 2^{-n} + \frac{e^{\Lambda_1\cdot(2t_2)}}{\omega_n(2t_2)^n}\cdot n_2\|S_2\|(\mathbb{B}_{2t_2}(x)) + \frac{e^{\Lambda_1\cdot(2t_2)}}{\omega_n(2t_2)^n}\|\hat{V}_2\|(\mathbb{B}_{2t_2}(x)) \\
&\geq (n_1+n_2)\cdot 2^{-n} + \frac{e^{\Lambda_1\cdot(2t_3)}}{\omega_n(2t_3)^n}\|\hat{V}_2\|(\mathbb{B}_{2t_3}(x)) \\
&\quad \ldots \\
&\geq (\sum_{i\geq 1} n_i)\cdot 2^{-n} + \theta^n(x, \|V_0\|),
\end{aligned}$$

where $\hat{V}_i$ is a constrained stationary integral varifold in $C_{2t_i}(x)$ defined inductively by

$$\hat{V}_0 := V_0; \quad \hat{V}_i := (\hat{V}_{i-1} - n_i|S_i|)\llcorner C_{2t_i}(x).$$

Hence, the lemma follows from $\|V_0\|(U) \leq \|V\|(U) \leq \Lambda$. □

Let $\bar{m} := \sup_{x\in\mathcal{B}_{r_0}(y)} m(x)$. WLOG $\bar{m} > 0$. We are going to define inductively the functions u_i in Theorem 4.3.2. Let

- $\bar{V}_0 := V_0\llcorner C_{r_0}(y)$;
- For $1 \leq i \leq \bar{m}$, $u_i(x) := \sup\{t : (x,t) \in \mathrm{spt}(\bar{V}_{i-1})\}$, $x \in \mathcal{B}_{r_0}(y)$;
- For $1 \leq i \leq \bar{m}$, $\bar{V}_i := \bar{V}_{i-1} - |\operatorname{graph}_{\partial M}(u_i)|$;

- For $1 \leq i \leq \bar{m}$, $x \in \mathcal{B}_{r_0}(y)$,

$$m_i(x) := \begin{cases} 0, & \text{if } \bar{V}_i = 0; \\ \displaystyle\sum_{p=(x,t)\in \mathrm{spt}(\bar{V}_i)} \theta^n(p, \|\bar{V}_i\|), & \text{if } \bar{V}_i \neq 0, \end{cases}$$

where we set $\sup \emptyset = -\infty$. The following lemma guarantees they are well-defined. (We set $m_0(x) := m(x)$.)

Lemma 4.3.7. *For u_i, $\bar{V}_i$ defined above, $1 \leq i \leq \bar{m}$,*

(1) u_i are nonnegative C^1 functions on $\mathcal{B}_{r_0}(y)$, and is C^∞ in $\{x : u_i(x) > 0\}$.

(2) $\bar{V}_i$ are integral varifolds in $C_{r_0}(y)$ such that $\bar{V}_i \llcorner C^+_{r_0}(y)$ are stable minimal hypersurfaces with multiplicity supported in $\mathrm{spt}(V_0)$.

(3) $\forall x \in \mathrm{spt}(\bar{V}_i) \cap \mathcal{B}_{r_0}(y)$, the tangent varifold of $\bar{V}_i$ at x is $k|T_x\partial M|$ for some positive $k \in \mathbb{N}$.

(4) $m_i(x) = m_{i-1}(x) - 1$, $\forall x \in \mathcal{B}_{r_0}(y)$.

Remark 4.3.8. By (4) of Lemma 4.3.7 and that $m_i \geq 0$, we see $m_{\bar{m}} \equiv 0$ and $V_0 = \sum_{i=1}^{\bar{m}} |\operatorname{graph}_{\partial M}(u_i)|$. Together with (1), this proves the first part of Theorem 4.3.2.

If further, $u_j \in C^{1,1}(\mathcal{B}_{r_0}(y))$, then (denote $|\operatorname{graph}_{\partial M}(u_j)|$ by Γ for simplicity) the mean curvature of Γ is in $L^\infty(M, \|\Gamma\|)$ satisfying

$$\vec{H}_\Gamma = \vec{H}_{\partial M} \quad \mathscr{H}^n\text{-a.e. on } \{u_i = 0\}.$$

Combined with Lemma 4.2.5, we see $\vec{H}_\Gamma \cdot \nu_{\partial M} \leq 0$, hence Γ is constrained stationary in $C_{r_0}(y)$ along ∂M. This proves the second part of Theorem 4.3.2.

Proof of Lemma 4.3.7. : It's sufficient to prove (1)-(4) inductively in i assuming they hold for $1 \leq l \leq i-1$. $i = 1$ case is treated in the same way without a-priori inductive assumption.

(1) u_i is clearly upper semi-continuous by definition; While by Corollary 4.3.5, together with the assumption that $\mathrm{spt}(\bar{V}_{i-1}) \subset \mathrm{spt}(V_0)$ and $\bar{V}_{i-1} \llcorner C^+_{r_0}(y)$ is a stable minimal hypersurface with multiplicity, we know that u_i is lower semi-continuous, and then smooth, in open subset $\{u_i > 0\} \subset \mathcal{B}_{r_0}(y)$.

Note also that by assumption (4) for $1 \leq l \leq i-1 < \bar{m}$, $\mathrm{spt}(\bar{V}_{i-1}) \neq \emptyset$. Hence $\{u_i \geq 0\} \neq \emptyset$. To prove the continuity of u_i, it suffices to verify that $u_i \geq 0$. If not, then by upper semi-continuity of u_i, there exists an open ball $\mathcal{B} \subset \{u_i < 0\}$ and some point $z \in \partial\mathcal{B} \cap \{u \geq 0\}$. Moreover, by the continuity of u_i on $\{u_i > 0\}$, $u_i(z) = 0$. By definition, $z \in \mathrm{spt}(\bar{V}_{i-1})$, then by the assumption (3) for $i-1$ when $i \geq 2$ and Corollary 4.2.7 when $i = 1$, we have

$$(\eta_{z,\lambda})_\sharp \bar{V}_{i-1} \overset{*}{\rightharpoonup} k|T_z\partial M|, \qquad \text{as } \lambda \to 0,$$

for some $k \geq 1$. On the other hand, by definition, $\mathcal{B} \times (-r_0, r_0) \cap \mathrm{spt}(\bar{V}_{i-1}) = \emptyset$, hence the weak limit above should be supported in a half space. Here comes a contradiction.

That u_i is differentiable and C^1 on $\mathcal{B}_{r_0}(y)$ follows from Corollary 4.3.5 and Lemma 4.3.4.

(2) Integrality of $\bar{V}_i$ follows from assumption (3) for $i-1$ if $i \geq 2$, from Corollary 4.2.7 if $i = 1$, from that $u_i \in C^1(\mathcal{B}_{r_0}(y))$ by (1) and from that $\mathrm{graph}_{\partial M}(u_i) \subset \mathrm{spt}(\bar{V}_{i-1})$ by definition. Hence $\mathrm{spt}(\bar{V}_i) \subset \mathrm{spt}(\bar{V}_{i-1}) \subset ... \subset \mathrm{spt}(V_0)$. Since $V_0 \llcorner C^+_{r_0}(y)$ is smooth minimal hypersurface with multiplicity and $\{u_l\}_{1 \leq l \leq i}$ are C^1, $\bar{V}_i \llcorner C^+_{r_0}(y)$ is also a smooth minimal hypersurface with multiplicity.

(3) $\forall x \in \mathrm{spt}(\bar{V}_i \cap \mathcal{B}_{r_0}(y))$ fixed, by assumption (3) for $i-1$ and that $u_i \in C^1(\mathcal{B}_{r_0}(y))$,

$$(\eta_{x,\lambda})_\sharp \bar{V}_i = (\eta_{x,\lambda})_\sharp \bar{V}_{i-1} - (\eta_{x,\lambda})_\sharp |\operatorname{graph}_{\partial M}(u_i)| \overset{*}{\rightharpoonup} k|T_p \partial M|, \qquad \text{as } \lambda \to 0,$$

for some non-negative $k \in \mathbb{Z}$. Thus, to prove (3) for i, it suffices to verify that $\theta^n(x, \bar{V}_i) > 0$. The proof is similar to the one of Corollary 4.2.4. The only difference is that it's unclear whether $\bar{V}_i$ is constrained stationary in $C_{r_0}(y)$ or not, so Lemma 4.2.3 does not apply directly. In what follows, we shall establish a quasi-volume monotonicity for $\bar{V}_i$.

Claim: $\forall \epsilon > 0$, there's an $r_1 \in (0, r_0)$ depending on ϵ such that, for every $0 < \sigma < \eta < r_1$ and every $x \in \mathcal{B}_{r_0}(y)$ such that $\mathbb{B}^L_{2\eta}(x) \subset C_{r_0}(y)$, we have

$$\frac{e^{\Lambda \eta}}{\omega_n \eta^n} \|\bar{V}_i\|(\mathbb{B}^L_\eta(x)) \geq \frac{e^{\Lambda \sigma}}{\omega_n \sigma^n} \|\bar{V}_i\|(\mathbb{B}^L_\sigma(x)) - 2\bar{m}^2 \epsilon, \tag{4.3.2}$$

where Λ is in Lemma 4.2.3. Notice that by taking ϵ sufficiently small in (4.3.2), the proof of Corollary 4.2.4 generalizes directly here to prove the density lower bound at every point in $\mathrm{spt}(\bar{V}_i)$.

Proof of the claim: First notice that $\exists\, r_2 > 0$ and $k > 4$ both depend on ϵ such that, for every $0 < r < r_2$, every $\mathcal{B}_r(z) \subset \mathcal{B}_{r_0}(y)$ and an arbitrary function $v \in C^1(\mathcal{B}_{2r}(z))$ satisfying

$$\frac{1}{r}|v| + |\nabla_{\partial M} v| \leq \frac{1}{k} \quad \text{ on } \mathcal{B}_{2r}(z),$$

by area formula we always have,

$$1 - \epsilon \leq \frac{e^{\Lambda r}}{\omega_n r^n} \|\operatorname{graph}_{\partial M}(v)\|(\mathbb{B}^L_r(z)) \leq 1 + \epsilon, \tag{4.3.3}$$

Note that we don't require v to satisfy any equation at this point.

Now take $r_1 := min\{r_2, \rho(1/(40k^2)), 1/k\}/10$, where ρ is specified in Corollary

4.3.5. For every $0 < r < r_1$ and every $\mathcal{B}_r(x) \subset \mathcal{B}_{r_0}(y)$, by our choice of r_1 and Corollary 4.3.5, $\forall 1 \leq l \leq i$, either

$$\frac{1}{r}|u_l| \leq \frac{1}{2k}, \quad |\nabla_{\partial M} u_l| \leq \frac{1}{2k}, \quad \{u_l = 0\} \neq \emptyset, \quad \text{on } \mathcal{B}_{2r}(x); \tag{4.3.4}$$

or

$$u_l > 0, \quad \text{on } \mathcal{B}_{2r}(x). \tag{4.3.5}$$

Note that if u_l satisfies (4.3.5) on $\mathcal{B}_{2r}(x)$, then it satisfies (4.3.5) on every $\mathcal{B}_t(x) \subset \mathcal{B}_{2r}(x)$. Also note that once $u_l > 0$ on $\mathcal{B}_{2r}(x)$, $|\operatorname{graph}_{\partial M}(u_l)|$ is a stationary varifold in $\mathcal{B}_{2r}(x) \times (-r_0, r_0)$ and hence $V_0 - |\operatorname{graph}_{\partial M}(u_l)|$ is constrained stationary in $\mathcal{B}_{2r}(x) \times (-r_0, r_0)$.

Now that $\forall 0 < \sigma < \eta < r_1$, $\forall \mathcal{B}_{2\eta}(x) \subset \mathcal{B}_{r_0}(y)$, let

$$I_1 := \{l : 1 \leq l \leq i, u_l \text{ satisfies (4.3.4) for every } r \in [\sigma, \eta]\};$$

$$I_2 := \{l : 1 \leq l \leq i, u_l \text{ satisfies (4.3.5) with } r = \eta\};$$

$$I_3 := \{1, 2, ..., i\} \setminus (I_1 \cup I_2).$$

And for $l \in I_3$, let $t_l := \inf\{t \in [\sigma, \eta] : (4.3.5) \text{ fails with } r = t\}$. After relabeling the index, WLOG $I_3 = \{1, 2, ...q\}$ and $t_1 \leq t_2 \leq ...t_q$, $q \in \mathbb{N} \cup \{0\}$. Denote for simplicity,

$$a(x, t; V) := \frac{e^{\Lambda t}}{\omega_n t^n} \|V\|(\mathbb{B}^L_t(x)); \quad a_l(x, t) := a(x, t; |\operatorname{graph}_{\partial M}(u_l)|),$$

and $\Gamma_l := |\operatorname{graph}_{\partial M}(u_l)|$. Then by the discussion above, for every $l \in I_1$, by (4.3.3), we have

$$|a_l(x, \eta) - a_l(x, \sigma)| \leq |a_l(x, \eta) - 1| + |a_l(x, \sigma) - 1| \leq 2\epsilon. \tag{4.3.6}$$

And for every $l \in I_3$, by definition of t_l and (4.3.3), we have

$$|a_l(x,\eta) - a_l(x,t)| \le |a_l(x,\eta) - 1| + |a_l(x,t) - 1| \le 2\epsilon, \quad \forall t_l \le t \le \eta. \tag{4.3.7}$$

Also for every $1 \le q' \le q+1$, and for every $l \in I_3$ such that $q' \le l \le q$, we have $\mathrm{spt}(\Gamma_l) \cap \mathcal{B}_{2t_l} = \emptyset$. Thus we know that $V_0 - \sum_{l\in I_2}\Gamma_l - \sum_{q'\le l\le q}\Gamma_l$ is constrained stationary in $\mathcal{B}_{2t_{q'}}(x)\times(-r_0,r_0)$. By Lemma 4.2.3 we then have,

$$t \mapsto a(x,t;V_0 - \sum_{l\in I_2}\Gamma_l - \sum_{q'\le l\le q}\Gamma_l) \quad \text{is monotone nondecreasing in } t \in (0,t_{q'}). \tag{4.3.8}$$

Here we use the convention that $t_{q+1} := \eta$. Combining (4.3.6), (4.3.7) and (4.3.8), we have,

$$\begin{aligned}
a(x,\eta;\bar{V}_i) &= a(x,\eta;V_0 - \sum_{l\in I_2}\Gamma_l) - \sum_{l=1}^{q} a_l(x,\eta) - \sum_{k\in I_1} a_k(x,\eta)\\
&\ge a(x,t_q;V_0 - \sum_{l\in I_2}\Gamma_l) - \sum_{l=1}^{q}(a_l(x,t_q)+2\epsilon) - \sum_{k\in I_1}(a_k(x,\sigma)+2\epsilon)\\
&\ge a(x,t_q;V_0 - \sum_{l\in I_2}\Gamma_l - \Gamma_q) - \sum_{l=1}^{q-1} a_l(x,t_q) - \sum_{k\in I_1} a_k(x,\sigma) - 2\bar{m}\epsilon\\
&\ge a(x,t_{q-1};V_0 - \sum_{l\in I_2}\Gamma_l - \Gamma_q) - \sum_{l=1}^{q-1}(a_l(x,t_{q-1})+2\epsilon) - \sum_{k\in I_1} a_k(x,\sigma) - 2\bar{m}\epsilon\\
&\ge a(x,t_{q-1};V_0 - \sum_{l\in I_2}\Gamma_l - \Gamma_q - \Gamma_{q-1}) - \sum_{l=1}^{q-2} a_l(x,t_{q-1}) - \sum_{k\in I_1} a_k(x,\sigma) - 4\bar{m}\epsilon\\
&\quad \cdots\ \cdots\\
&\ge a(x,t_1;V_0 - \sum_{l\in I_2}\Gamma_l - \sum_{l=1}^{q}\Gamma_l) - 0 - \sum_{k\in I_1} a_k(x,\sigma) - 2q\bar{m}\epsilon\\
&\ge a(x,\sigma;V_0 - \sum_{l\in I_2\cup I_3}\Gamma_l) - \sum_{k\in I_1} a_k(x,\sigma) - 2q\bar{m}\epsilon\\
&\ge a(x,\sigma;\bar{V}_i) - 2\bar{m}^2\epsilon,
\end{aligned}$$

Here we also use the fact that $q = \#I_3 \leq i \leq \bar{m}$. This completes the proof of claim.

(4) follows immediately from (3) and Corollary 4.3.5. □

Remark 4.3.9. From the proof we can also see that $r_0 = r_0(M, U, \mathcal{K}, \Lambda)$ can be chosen to depend continuously on $(M, \partial M)$ in C^2 topology.

4.4 Min-max with Obstacle

4.4.1 Almost minimizer and regularity

The notion of almost minimizing varifolds was introduced in [Pit81] to prove the regularity of stationary varifolds obtained from min-max construction. We employ similar notions here. The only difference is that everything is supported in an ambient manifold with boundary, making the regularity result more subtle near the boundary. M will be as in Section 4.2, $U \subset M$ be a relatively open subset.

Definition 4.4.1. Call $\{T^{(l)}\}_{l=0}^{q} \subset \mathbf{I}_n(U)$ a **δ-sequence** in U starting at $T^{(0)}$ if $\forall 1 \leq l \leq q$,

- $\mathrm{spt}(T^{(l)} - T^{(0)}) \subset\subset U$;
- $\mathbf{M}_U(T^{(l)}) \leq \mathbf{M}_U(T^{(0)}) + \delta$;
- $\mathbf{M}_U(T^{(l)} - T^{(l-1)}) \leq \delta$.

Call $V \in \mathcal{V}_n(M)$ **almost minimizing** in U, if there exist $\epsilon_i \to 0$, $\delta_i \to 0$, $T_i \in \mathbf{I}_n(M)$ with $|T_i| \to V$ such that for every i and every δ_i-sequence $\{T_i^{(l)}\}_{i=0}^{q}$ in U start at T_i, we have

$$\mathbf{M}_U(T_i^{(q)}) \geq \mathbf{M}_U(T_i) - \epsilon_i.$$

Note that by definition, if V is not almost minimizing in U, then $\exists\, \epsilon > 0$ such that $\forall T \in \mathbf{I}_n(M)$ with $\mathbf{F}(|T|, V) < \epsilon$ and $\forall \delta > 0$, there exists some δ-sequence $\{T^{(l)}\}_{l=0}^{q}$ start at T with

$$\mathbf{M}_U(T^{(q)}) \leq \mathbf{M}_U(T) - \epsilon$$

Theorem 4.4.2. *Let $V \in \mathcal{V}_n(M)$ be almost minimizing in U. Then V is a constrained embedded stable minimal hypersurface in U with optimal regularity.*

Proof. First see that by [Pit81], $V \llcorner \mathrm{Int}(U)$ is a stable minimal hypersurface with optimal regularity; moreover, a similar argument in [Pit81, 3.3] yields that V is constrained stationary in U. Now we focus on the behavior of V near ∂M. The proof of integrality of V is almost the same as in [Pit81, Chapter 3], with aid of the lemma in Subsection 4.2.1. We briefly go through the proof.

• **Replacement.** $\forall U' \subset\subset U$ be a subdomain. Since V is almost minimizing in U, let $\epsilon_i \to 0$, $\delta_i \to 0$ and $|T_i| \to V$ be as in the definition. For each i, let T_i^* be a mass minimizer among

$$\begin{aligned}\mathscr{E}(T_i, \mathrm{Clos}(U')) := \{T^{(q)} : \exists\ \delta_i\text{-sequence } \{T^{(l)}\}_{l=0}^{q} \text{ start at } T_i \\ \text{such that } \mathrm{spt}(T^{(l)} - T_i) \subset \mathrm{Clos}(U')\}\end{aligned}$$

The existence of such $T_i^* \in \mathscr{E}(T_i, \mathrm{Clos}(U'))$ follows from Federer-Fleming Compactness [FF60] and Discretization-Interpolation Theorem [MN14, 13.1, 14.1].

Moreover, T_i^* is locally constrained minimizing in U' and satisfies the mass bound

$$\|T_i\|(U') - \epsilon_i \leq \|T_i^*\|(U') \leq \|T_i\|(U').$$

Hence, by Theorem 4.2.12, $|T_i^*| \llcorner U'$ is a constrained embedded stable minimal hypersurface in U'. Then by Corollary 4.3.3, when $i \to \infty$, $|T_i^*| \to V^* \in \mathcal{V}_n(U)$ satisfying

(a) V^* is almost minimizing in U and $V^* \llcorner U'$ is a constrained embedded stable

minimal hypersurface;

(b) $V^* \llcorner \mathrm{Clos}(U')^c = V \llcorner \mathrm{Clos}(U')^c$;

(c) $\|V^*\|(U) = \|V\|(U)$.

Those $V^* \in \mathcal{V}_n(U)$ satisfying (a)-(c) above will be called **replacements** of V in U'.

• **Rectifiability.** Thanks to Allard Rectifiability Theorem [Sim83b, Chapter 8, 42.4] and Lemma 4.2.6, it suffices to show that $\theta^n(x, \|V\|) > 0$ $\|V\|$-a.e. $x \in \partial M \cap U$. $\forall x \in \mathrm{spt}(V) \cap U \cap \partial M$, let $r_j \in (0, dist_M(x, M \setminus U)/2)$, $r_j \to 0$; V_j^* be a replacement of V in $A_{r_j, 2r_j}(x)$. Then by Lemma 4.2.3, $\|V_j^*\|(A_{5r_j/4, 7r_j/4}) > 0$ for $j >> 1$. By the same trick as in Corollary 4.2.4, $\|V_j^*\|(A_{r_j,2r_j}) \geq c(n, M) r_j^n$ for some $c(n, M) > 0$. Hence, $\theta^n(x, \|V\|) \geq c'(n, M) > 0$.

• **Integrality.** For V_j^* constructed above, by Lemma 4.2.2 and 4.2.3, up to a subsequence, $(\eta_{x,r_j})_\sharp V \to P$, $(\eta_{x,r_j})_\sharp V_j^* \to Q$, where $P, Q \in \mathcal{V}_n(\mathbb{R}^{n+1}_+)$ are constrained stationary along $\mathbb{R}^n \times \{0\}$. (We identify $(T_x^+ M, T_x \partial M)$ with $(\mathbb{R}^{n+1}_+, \mathbb{R}^n \times \{0\})$ for simplicity.) Moreover, by definition of replacements, $Q \llcorner \mathbb{A}^{n+1}_{1,2}(0)$ is constrained embedded stable minimal hypersurface, hence stable minimal hypersurface by Lemma 4.2.5; And $P = Q$ outside $\mathrm{Clos}(\mathbb{A}^{n+1}_{1,2}(0))$. By Lemma 4.2.3, P, Q are cones and hence $P = Q$ is a stable minimal cone with optimal regularity supported in $\mathbb{R}^{n+1}_+$. By strong maximum principle, $P = Q = m|\mathbb{R}^n \times \{0\}|$ for some $m \in \mathbb{N}$. Therefore $\theta^n(x, \|V\|) = m \in \mathbb{N}$. This proves $V \in \mathcal{IV}_n(U)$.

Now by Theorem 4.3.2, V is locally multiple C^1 graphs near ∂M. To get $C^{1,1}$ regularity of the graphical function, we use again the replacements. In what follows, we keep working under Fermi coordinates.

Let $x \in \partial M \cap U$, $r_0 = r_0(M, U, \{x\}, \|V\|(U)) > 0$ be in Theorem 4.3.2; $U_x \subset C_{r_0}(x)$ be a neighborhood of x such that $V \llcorner U_x = \sum_{i=1}^m |\,\mathrm{graph}_{\partial M}(u_i)|$, where $u_1 \geq$

$u_2 \geq ... \geq u_m \in C^1(U_x \cap \partial M, \mathbb{R}_+)$, $|\nabla u_i| \leq 1/2$ and $\{u_1 = 0\} \neq \emptyset$. For any $p = (y,t) \in \mathrm{graph}_{\partial M}(u_i) \cap \mathrm{Int}(M)$, let V^* be a replacement of V in $U_x \setminus \mathrm{Clos}(B_{t/2}(p))$. By Theorem 4.3.2, $V^* \llcorner U_x = |\mathrm{graph}_{\partial M}(u^*)| + \hat{V}^*$ for some $\hat{V}^* \in \mathcal{IV}_n(U_x)$ and $u^* = u_i$ on the connected component of $\{u_i > 0\}$ containing y by unique continuation; Since V^* is a constrained embedded stable minimal hypersurface in U_x, by Theorem 4.2.11, $u^* \in C^{1,1}(\partial M \cap U)$ and for every $\mathcal{K} \subset \partial M \cap U$ compact,

$$\|u^*\|_{C^{1,1}(\mathcal{K})} \leq C(M, r_0, \mathcal{K}).$$

By applying the replacement on every connected component of $\mathrm{spt}(V) \cap \mathrm{Int}(U_x)$, we see $u_i \in C^{1,1}_{loc}(\partial M \cap U_x)$. By the arbitrariness of i and x and second part of Theorem 4.3.2, V is a constrained embedded stable minimal hypersurface in U. □

4.4.2 Min-max construction

In this subsection, we describe the min-max construction with obstacle following the idea from [Pit81]. The notions coincide with [MN14, Section 7,8] for the cell complexes, subcomplex and homotopy relations. We also use the notions of relative homotopy classes in [Zho20].

Suppose $m \geq 1$, M be described in Section 4.2. Let (X, Z) be a pair of cubical complex in I^m with $Z \subset X$. Fix $\Phi_0 : X \to (\mathcal{Z}_n(M), \mathbf{F})$ be a continuous map (i.e. continuous in $\mathbf{F}$-metric). Let Π be the space of sequence of continuous maps $S = \{\Phi_i : X \to (\mathcal{Z}_n(M), \mathbf{F})\}_{i \geq 1}$ satisfying

$$\sup\{\mathbf{F}(\Phi_i(x), \Phi_0(x)) : x \in Z\} \to 0 \text{ as } i \to \infty$$

And define the width

$$\begin{aligned}\mathbf{L}(S) &:= \limsup_{i\to\infty}\sup_{x\in X}\mathbf{M}(\Phi_i(x)),\\ \mathbf{L}(\Pi) &:= \inf_{S\in\Pi}\mathbf{L}(S),\\ \mathbf{L}_{\Phi_0,Z} &:= \sup_{x\in Z}\mathbf{M}(\Phi_0(x)).\end{aligned}$$

The goal of this Section is to prove

Theorem 4.4.3. *Let (M,g) be a compact $n+1$ manifold with boundary, $\Phi_0 : X \to (\mathcal{Z}_n(M),\mathbf{F})$ be a continuous map as above. If $\boldsymbol{L}(\Pi) > \boldsymbol{L}_{\Phi_0,Z}$, then there's a constrained embedded minimal hypersurface $V \in \mathcal{IV}_n(M)$ with optimal regularity such that $\|V\|(M) = \boldsymbol{L}(\Pi)$.*

With aid of Theorem 4.4.2, the proof is almost the same as in [Pit81, Chapter 4]. We briefly go through the major part of the argument.

• **Minimizing sequence and pull tight**. By a diagonalization process, there exists some $S \in \Pi$ such that $\mathbf{L}(S) = \mathbf{L}(\Pi)$. Call such S a **minimizing sequence**.

For a minimizing sequence $S = \{\Phi\}_i \in \Pi$, denote

$$\mathbf{C}(S) := \{V \in \mathcal{V}_n(M) : V_n = \lim_{j\to\infty}|\Phi_{i_j}(x_j)|,\ \|V\|(M) = \mathbf{L}(S)\},$$

called the **critical set** of S. Following [Pit81, 4.3] and [ZZ20], we can construct $H : [0,1]\times\mathcal{Z}_n(M)\cap\{\|T\|(M)\le 2\mathbf{L}(\Pi)\}\to\mathcal{Z}_n(M)\cap\{\|T\|(M)\le 2\mathbf{L}(\Pi)\}$ continuous in $\mathbf{F}$-metric such that

(i) $H(0,\text{-}) = id$;

(ii) $H(t,T) = T, \forall t\in[0,1], \forall|T|\in\mathcal{V}^\infty := \{\text{constrained stationary varifolds in } M\}\cup\{|\Phi(x)| : x\in Z\}$;

(iii) There's an $L : [0, +\infty) \to [0, +\infty)$ continuous, $L(0) = 0$, $L(\tau) > 0$ if $\tau > 0$ such that

$$\|H(1, T)\|(M) \leq \|T\|(M) - \mathbf{L}(\mathbf{F}(|T|, \mathcal{V}^\infty));$$

(iv) $\forall \epsilon > 0$, $\exists\ \delta > 0$ such that if $\mathbf{F}(|T|, |\Phi_0(x)|) \leq \delta$ for some $x \in Z$, then $\mathbf{F}(H(t, T), |\Phi_0(x)|) \leq \epsilon$ for all $t \in [0, 1]$.

Hence, if $S \in \Pi$ is a minimizing sequence, then $S' := \{H(1, \Phi_i)\}_{i\geq 1} \in \Pi$ is also a minimizing sequence such that $\emptyset \neq \mathbf{C}(S') \subset \mathbf{C}(S)$ consists of constrained stationary varifolds in M. Call such $S' \in \Pi$ a **pulled tight sequence**.

• **Discretization and interpolation**. Given a pulled-tight sequence $S = \{\Phi_i\}_{i\geq 1} \in \Pi$, by the Interpolation Theorem [MN14, 13.1] and diagonalization process, there exist $k_i \nearrow \infty$, $\delta_i \to 0$ and $\varphi_i : X(k_i) \to \mathcal{Z}_n(M)$ such that

(i) $\sup\{\mathbf{F}(\varphi_i(x), \Phi_i(x)) : x \in X(k_i)_0\} \leq \delta_i$;

(ii) $\sup\{\mathbf{F}(\varphi_i(x), \varphi_i(y)) : x, y \in \alpha \cap X \text{ for some } \alpha \in I(m, k_i)_m\} \leq \delta_i$;

(iii) $\mathbf{f}(\varphi_i) \to 0$ as $i \to \infty$;

(iv) $\mathbf{L}(\{\varphi_i\}_i) := \limsup_{i\to\infty} \sup_{x\in X(k_i)_0} \|\varphi_i(x)\|(M) = \mathbf{L}(\Pi)$.

In particular,

$$\mathbf{C}(\{\varphi_i\}_{i\geq 1}) := \{V : V = \lim_{j\to\infty} |\varphi_{i_j}(x_j)| \text{ for some } i_j \to \infty \text{ and } x_j \in X(k_{i_j})_0;$$
$$\|V\|(M) = \mathbf{L}(\Pi)\} \subset \mathbf{C}(S).$$

Conversely, given a sequence of $k_i \nearrow \infty$, $\delta_i \to 0$ and $\varphi_i : X(k_i)_0 \to \mathcal{Z}_n(M)$ satisfying (ii), (iii) above and

(i') $\sup\{\mathbf{F}(\varphi_i(x), \Phi_0(x)) : x \in Z(k_i)_0\} \leq \delta_i$.

By the Interpolation Theorem in [Alm62, Section 6], (see also [MN14, 14.1]), there exists $S = \{\Phi_i\}_{i\geq 1} \in \Pi$ such that

$$\mathbf{L}(S) = \mathbf{L}(\{\varphi_i\}_{i\geq 1}).$$

• **Pitts' deformation**. Let $\{\varphi_i : X(k_i)_0 \to \mathcal{Z}_n(M)\}_{i\geq 1}$ be satisfying (i'), (ii), (iii) above and such that

$$\mathbf{L}(\{\varphi_i\}_{i\geq 1}) > \mathbf{L}_{\Phi_0,Z}, \quad \mathbf{C}(\{\varphi_i\}_{i\geq 1}) \subset \{\text{constrained stationary varifolds in } M\}$$

Assuming $\forall V \in \mathbf{C}(\{\varphi_i\}_i)$, $\exists\ x \in M$ such that $\forall r > 0$, $\exists s \in (0, r)$ with V not almost minimizing in $A_{s,r}(x)$, [Pit81] introduced a deformation process to construct $\{\varphi_i' : X(k_i')_0 \to \mathcal{Z}_n(M)\}$ satisfying (i'), (ii), (iii) and

$$\mathbf{L}(\{\varphi_i'\}_{i\geq 1}) < \mathbf{L}(\{\varphi_i\}_{i\geq 1}).$$

Applying this to the interpolation theorem above, one see that given a pulled-tight minimizing sequence $S \in \Pi$, $\exists\ V \in \mathbf{C}(S)$ such that $\forall x \in M$, $\exists r = r_x > 0$ and V being almost minimizing in $A_{s,r}(x)$, $\forall s \in (0, r)$. By Theorem 4.4.2 and 4.2.11, V is a constrained embedded minimal hypersurface. This complete the proof of Theorem 4.4.3.

Remark 4.4.4. By Pitts' combinatorial argument [Pit81, 4.9, 4.10], there's a constant $N(m) \in \mathbb{N}$ such that in Theorem 4.4.3, we can choose V such that if $p \in M$ and $\{A_j := A_{r_j,s_j}(p)\}_{j=1}^{N(m)}$ is a family of annuli in $\tilde{M}$ such that $s_j > r_j > 4s_{j+1}$ for every j, then V is almost minimizing in at lease one of $A_j \cap M$. Hence in particular V is a constrained embedded **stable** minimal hypersurface in that $A_j \cap M$.

4.5 A Local Rigidity for Constrained Embedded Minimal Hypersurfaces

We close this chapter by pointing out the following local rigidity theorem for constrained embedded minimal hypersurface. A slightly modified version is discussed in Section 5.5.2 in singular minimal hypersurface setting.

Let (N, g) be a closed $n+1$ manifold; $(\Sigma, \nu) \subset N$ be a connected, two-sided, closed smooth minimal hypersurface; r_Σ be the injective radius of Σ in N, i.e.

$$r_\Sigma := \sup\{r : \Phi : \Sigma\times(-r, r) \to N,\ (x, t) \mapsto \exp_x^N(t\nu(x)) \text{ is a diffeomorphism onto its image}\}.$$

Work under Fermi coordinates in $\Phi(\Sigma\times(-r_\Sigma, r_\Sigma))$. For $K > 1$ and $w^\pm \in C^\infty(\Sigma, (0, r_\Sigma))$, call the domain

$$\mathscr{N}_{w^\pm} := \{(x, t) \in N : x \in \Sigma, -w^- \leq t \leq w^+\}$$

a K**-uniform** neighborhood of Σ, if

(1) $\sup_{x\in\Sigma, i\in\{\pm\}} w^i(x) \leq K \inf_{x\in\Sigma, i\in\{\pm\}} w^i(x)$;

(2) The mean curvature $H_\pm$ of $\text{graph}_\Sigma(\pm w^\pm)$ satisfies

$$|H_\pm| \leq K \sup_\Sigma w^\pm.$$

Note that for a smooth minimal hypersurface as above, the δ-neighborhood $\mathscr{N}_\delta := \{p \in N : dist_N(p, \Sigma) \leq \delta\}$ and the neighborhoods $\mathscr{N}_{\delta w_1} := \{(x, t) : x \in \Sigma, |t| \leq \delta w_1(x)\}$ given by first eigenfunction w_1 of Jacobi operator of Σ are both K-uniform neighborhood of Σ for every sufficiently small δ and K independent of δ.

Theorem 4.5.1. *Let $\Sigma \subset N$ be a non-degenerate closed minimal hypersurface as*

above; $K > 1$*;* $\epsilon \in (0,1)$*. Then* $\exists\ s_0 = s_0(K, \epsilon, \Sigma, N) > 0$ *such that if* $\mathscr{N} = \mathscr{N}_{w^\pm}$ *is a* K*-uniform neighborhood of* Σ *and* $\sup_\Sigma |w^\pm| \leq s_0$*, and* V *is a constrained embedded minimal hypersurface in* $\mathscr{N}$ *with*

$$\mathscr{H}^n(\Sigma) \leq \|V\|(N) \leq (2-\epsilon)\mathscr{H}^n(\Sigma),$$

then $V = |\Sigma|$*.*

We emphasis that the mass two-side bound is crucial for such a theorem to be true. In fact, for an unstable minimal hypersurface Σ, let $w^+ = w^- = \delta w_1$, where w_1 be the first eigenfunction of Jacobi operator, $\delta << 1$, then each component of $\partial \mathscr{N}_{w^\pm}$ is constrained embedded minimal in $\mathscr{N}_{w^\pm}$, and so is their union.

Proof. We prove by contradiction. First recall some basic facts on minimal surface equation on Σ. Let $F = F(x, z, p)$, $x \in \Sigma$, $z \in (-r_\Sigma, r_\Sigma), p \in T_x\Sigma$ be the area integrand for graph over Σ, i.e.

$$\mathscr{H}^n(\mathrm{graph}_\Sigma(\phi)) = \int_\Sigma F(x, \phi(x), \nabla\phi(x))\ dx, \qquad \forall \phi \in C^1(\Sigma, (-r_\Sigma, r_\Sigma)).$$

Here are some basic properties we shall use later, see also Appendix A.

(a) $F(x,0,0) = 1$; $F_z(x,0,0) := \partial_z F(x,0,0) = 0$, $F_p := \partial_p F(x,0,0) = 0$; $F_{pp}(x,0,0)^{ij} = \delta^{ij}$, $F_{zp}(x,0,0) = F_{pz}(x,0,0) = 0$, $F_{zz}(x,0,0) = -|A_\Sigma|^2 - Ric_N(\nu_\Sigma, \nu_\Sigma)$.

(b) $\|F\|_{C^3} \leq C(\Sigma, N)$.

(c) If further $\phi \in C^1 \cap W^{2,2}(\Sigma)$, then

$$\mathscr{M}\phi := -div F_p(x, \phi, \nabla\phi) + F_z(x, \phi, \nabla\phi) = \tilde{G}(x, \phi, \nabla\phi) \cdot H,$$

where H is the mean curvature of $\mathrm{graph}_\Sigma(\phi)$ with normal field having positive inner product with ν; $\tilde{G}$ is a positive smooth function, with a positive upper and lower bound only depend on M, g, Σ.

(d) $L_\Sigma\phi := \frac{d}{ds}\big|_{s=0}\mathscr{M}(s\phi) = -(\Delta + |A_\Sigma|^2 + \mathrm{Ric}_N(\nu,\nu))\phi$ is the Jacobi operator of Σ.

Suppose otherwise, $K > 1$ and there are K-uniform neighborhood $\mathscr{N}_j = \mathscr{N}_{w_j^\pm}$ of Σ with $\sup|w_j^\pm| \to 0$ and $|\Sigma| \neq |\Gamma_j| \in \mathcal{IV}_n(\mathscr{N}_j)$ constrained embedded minimal hypersurfaces in $\mathscr{N}_j$ with $\|\Gamma_j\|(N) \in [\|\Sigma\|(N), (2-\epsilon)\|\Sigma(N)\|]$. The strategy is to show that these $\{\Gamma_j\}$ induces a nontrivial Jacobi field on Σ, which contradicts to the non-degeneracy assumption.

Recall by Remark 4.2.9 (1), Γ_j has mean curvature uniformly bounded by the mean curvature of $\partial\mathscr{N}_j$, hence tends to 0 in L^∞. By Allard Compactness Theorem [All72, Sim83b], up to a subsequence, $\Gamma_j \to \Gamma_\infty$ for some stationary integral varifold $\Gamma_\infty \in \mathcal{IV}_n(N)$ supported in Σ and having mass between $\|\Sigma\|(N)$ and $(2-\epsilon)\|\Sigma\|(N)$. Hence $\Gamma_\infty = |\Sigma|$, and by Allard Regularity Theorem [All72,Sim83b], for $j >> 1$, there exist $u_j \in C^{1,\alpha} \cap W^{2,2}(\Sigma)$ such that $\Gamma_j = |\,\mathrm{graph}_\Sigma(u_j)|$, $u_j \to 0$ in C^α, not identically 0 and

$$\begin{aligned} \mathscr{M}u_j &= 0 \quad \text{on } \{-w_j^- < u_j < w_j^+\}; \\ u_j \cdot \mathscr{M}u_j &\leq 0 \quad \text{on } \Sigma; \\ |\mathscr{M}u_j| &\leq C(M,g,\Sigma) \cdot \sup_{i\in\{\pm\}, x\in\Sigma} Kw_j^i(x) \quad \text{on } \Sigma, \end{aligned} \tag{4.5.1}$$

where the last two inequality follows from (c) and Remark 4.2.9 (1). Note that $\{u_j = \pm w_j^\pm\} \neq \emptyset$, otherwise by standard elliptic estimate, $u_j/\|u_j\|_{L^2}$ will C^∞-converge to some nontrivial Jacobi field on Σ. Thus,

$$\inf w_j^\pm \leq \|u_j\|_{C^0(\Sigma)} \leq \sup w_j^\pm. \tag{4.5.2}$$

Also by standard elliptic estimate,

$$\|u_j\|_{C^{1,\alpha}} + \|u_j\|_{W^{2,2}} \leq C(\Sigma, N)(\|u_j\|_{C^0} + |\mathscr{M} u_j|_{C^0}) \leq C(\Sigma, N, K)\|u_j\|_{C^0}.$$

Hence, let $c_j := \|u_j\|_{C^0}$, up to a subsequence, $\hat{u}_j := u_j/c_j \to \hat{u}_\infty \in C^{1,\alpha} \cap W^{2,2}(\Sigma)$ in C^1. Moreover, $\|\hat{u}_\infty\|_{C^0} = 1$ and by (4.5.1), (4.5.2) and definition of K-uniform domain,

$$\begin{aligned} L_\Sigma \hat{u}_\infty &= 0 \quad \text{on } \{|\hat{u}_\infty| < 1/K\}; \\ \hat{u}_\infty \cdot L_\Sigma \hat{u}_\infty &\leq 0 \quad \text{on } \Sigma. \end{aligned} \tag{4.5.3}$$

Now we make use of the lower bound on mass to show that $\int_\Sigma \hat{u}_\infty \cdot L_\Sigma \hat{u}_\infty \geq 0$. Combining this with (4.5.3), we see that $\hat{u}_\infty$ is a nontrivial Jacobi field on Σ, which contradicts to the non-degeneracy.

Observe that

$$\begin{aligned} 0 \leq \|\Gamma_j\|(N) - \|\Sigma\|(N) &= \int_\Sigma F(x, u_j, \nabla u_j) - 1 \, dx \\ &= \int_\Sigma (\int_0^1 F_p(x, su_j, s\nabla u_j) \, ds) \cdot \nabla u_j + (\int_0^1 F_z(x, su_j, s\nabla u_j)) \cdot u_j \, dx \\ &= \frac{1}{2} \int_\Sigma F_p(x, u_j, \nabla u_j) \cdot \nabla u_j + F_z(x, u_j, \nabla u_j) u_j \, dx \; + \; \mathscr{R}, \end{aligned} \tag{4.5.4}$$

where by (a),(b) and standard elliptic estimate, $|\mathscr{R}| \leq C(\Sigma, N, K) c_j^3$. Here we use the fact that

$$\int_0^1 f(s) \, ds - \frac{1}{2}(f(0) + f(1)) = \frac{1}{2} \int_0^1 f''(s)(s^2 - s) \, ds.$$

Multiply (4.5.4) by c_j^{-2} and let $j \to \infty$ one gets,

$$\int_\Sigma \hat{u}_\infty \cdot L_\Sigma \hat{u}_\infty = \int_\Sigma |\nabla \hat{u}_\infty|^2 - (|A_\Sigma|^2 + \mathrm{Ric}_N(\nu, \nu)) \hat{u}_\infty^2 \, dx \geq 0,$$

which completes the proof. □

Remark 4.5.2. An interesting question is when the constrained embedded minimal hypersurface obtained by min-max construction in Theorem 4.4.3 is a minimal hypersurface (i.e. has vanishing mean curvature). When ∂M is minimal, this is clearly true by strong maximum principle; In general, is this true when one assume further that the width is invariant under local perturbation of ∂M in $\tilde{M}$? Is this true under assumption above for generically perturbed boundary?

Chapter 5

Analysis on Singular Minimal Hypersurfaces, II

The work described in this chapter was first presented in [Wan20].

5.1 Introduction

In this chapter, we study how the singular set of a minimal hypersurface affects its *global behavior* in different aspects. We first develop the analysis on singular minimal hypersurfaces and prove the spectral Theorem 1.3.5. Then we discuss the following two primary applications.

• **Local Moduli.** White [Whi91,Whi17] studied the space Γ of Riemannian metrics and smooth minimal surfaces pairs (g, Σ), which admits a natural Banach manifold structure and makes the projection onto the first variable a Fredholm mapping with index 0.

When Σ is allowed to have singularities, no local structural result of Γ is known in the literature. In fact, when g is Euclidean, possible singular structures for minimal hypersurfaces in a neighborhood of Σ near singular set of Σ was studied in [CHS84,

SS86, CES17, ES19], for certain classes of singularity models. And it was shown that nearby minimal surface might have different singularity type from Σ, making it hard to study the local structure of Γ using parametric approach.

In this chapter, we shall address the following local existence problem,

Question. Given a closed singular minimal hypersurfaces Σ_0 in a Riemannian manifold (M, g_0) and a smoothly perturbed metric g of g_0, is there always at least one, possibly singular, hypersurface $\Sigma \subset M$ minimal with respect to g and staying close to Σ_0?

Our first application provide partial answers to the question above when the singular set are strongly isolated, in other words, every tangent cone of Σ is a regular cone with multiplicity 1.

Theorem 5.1.1. *Suppose $\Sigma \subset (M, g)$ is an optimally regular, locally stable, two-sided minimal hypersurface with only strongly isolated singularities. Also suppose that*

(1) Σ is nondegenerate, i.e. in the notation of Theorem 1.3.5, 0 is not an eigenvalue of $-L_\Sigma$;

(2) the tangent cone at each singularity is strictly minimizing;

(3) the tangent cone at each singularity is strictly stable.

Then for every $\epsilon > 0$, there's some $\delta > 0$ depending on Σ, M, g, ϵ, such that if g' is a smooth metric with $\|g' - g\|_{C^4} \leq \delta$, then there exists a hypersurface $\Sigma' \subset M$ minimal, locally stable under g' with optimal regularity, lying in the ϵ-neighborhood of Σ and satisfying

$$\mathbf{F}(|\Sigma'|_{g'}, |\Sigma|_g) \leq \epsilon\,.$$

The notions of *strictly minimizing* and *strict stability* are introduced in Section 2.4, 5.2 respectively. Also note that Theorem 5.1.1 fails if we drop the assumption (2), see Proposition 5.5.9.

Examples of minimal hypercones have been constructed in [HLJ71,FKM81,FK85], many of which were verified to be area-minimizing by [Law72, Sim74, Law91, WS94]. It's also worth mentioning that all examples of area-minimizing hypercones in the literature above were proved to be strictly stable and strictly minimizing, see [WS94, TZ20]. Therefore, Theorem 5.1.1 applies for these cones to be the tangent cones.

Unlike the case when $\mathrm{Sing}(\Sigma) = \emptyset$, Theorem 5.1.1 is not derived by fixed point method when singularities exist. In fact, unlike in the smooth setting, the uniqueness of such nearby minimal hypersurfaces under a fixed perturbed metric is not known (and even not expected in general).

One key step in proving Theorem 5.1.1 is to control the behavior of graphical functions of minimal hypersurfaces near a minimal hypercone. To do so, one first need to study minimal hypersurfaces asymptotic to a given cone near infinity. This subject was studied in [BDGG69,HS85,SS86,Whi89,WS94,Cha97,Maz17,ES19], most focusing on minimizing hypersurfaces and hypercones. In Section 5.2.2, we prove the following generalization of Hardt-Simon's Regularity Theorem 2.4.1 in stable case, which is not only interesting itself, but also useful in obtaining full generic regularity in eight manifolds in Chapter 6.

Theorem 5.1.2 (cf. Theorem 5.2.12)**.** *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a stable minimal hypersurface with optimal regularity asymptotic to a regular minimal hypercone $\mathbf{C} \subset \mathbb{R}^{n+1}$ near infinity. If write Σ as the graph over $\mathbf{C}$ of some function u outside a large ball, then we have*

$$\liminf_{R\to+\infty} R^{-2\gamma_1^- - n} \int_{\mathbf{A}_{R,2R}} u^2 > 0\,,$$

where $\mathbf{A}_{R,2R}$ is the annulus in $\mathbf{C}$ and $-(n-2) < \gamma_1^- \leq -(n-2)/2$ is a geometric constant depending on $\mathbf{C}$, defined in Section 5.2.

Moreover, if Σ lies on one side of $\mathbf{C}$ near infinity, then $\mathrm{Sing}(\Sigma) = \emptyset$ and Σ is contained entirely in that side and foliate the component of $\mathbb{R}^{n+1} \setminus \mathbf{C}$ containing it by

rescaling.

• **Local Minimizing Property.** A natural question by Lawlor [Law91] is that given a (strictly) minimizing hypercone $\mathbf{C} \subset \mathbb{R}^{n+1}$, is every minimal hypersurface Σ with $\mathbf{C}$ to be the tangent cone at some singularity p minimizing in some neighborhood of p? When $\mathbf{C}$ is strictly stable or the asymptotic rate of Σ to $\mathbf{C}$ near p is strictly greater than 1, [HS85, Theorem 4.4] confirmed this question.

We get a positive answer in this chapter to a more general version of the question under stability assumption.

Theorem 5.1.3 (cf. Theorem 5.4.1)**.** *Suppose $\Sigma \subset (M, g)$ is a stable minimal hypersurface with only strongly isolated singularities, satisfying (1) and (2) in Theorem 5.1.1. Then there's a small neighborhood U of $\mathrm{Clos}(\Sigma)$ which admits a mean convex foliation. More precisely, there's a family of piecewisely smooth mean convex neighborhood $\{U_t\}_{t\in(0,1)}$ of $\mathrm{Clos}(\Sigma)$ such that*

(1) $U_{t_1} \subset U_{t_2}$, if $t_1 \leq t_2$;

(2) $U = \bigcup_{t\in(0,1)} U_t$, $\mathrm{Clos}(\Sigma) = \bigcap_{t\in(0,1]} U_t$; $U_s = \bigcup_{t<s} U_t$; $\mathrm{Clos}(U_s) = \bigcap_{t>s} U_t$, $\forall s \in (0, 1)$.

In particular, Σ is homologically area-minimizing in U and is the unique stationary varifold in U up to multiplicity.

For a precise description of mean convexity of piecewise smooth domain, see Theorem 5.4.1.

Similar local minimizing property was proved by a contradiction argument in [Sma99, Section 3, Lemma 4] assuming that each tangent cone are strictly stable and strictly minimizing, and that Σ is a minimal cone in the Euclidean space near each singularity; By a calibration argument, [Zha18] obtained similar local minimizing result for singular minimal submanifolds of higher codimension with Lawlor's

cones in [Law91] to be the tangent cone, and with specifically chosen metrics away from singular points. The calibration method was recently exploited by Zhenhua Liu [Liu21, Liu22] to construct area-minimizing cycles of higher codimensions with fractal or nonsmoothable singular sets.

Organization of the Chapter.

In Section 5.2, we study asymptotic analysis of functions on a stable minimal hypercone. This has applications to the asymptotic behavior near infinity of entire minimal hypersurfaces in $\mathbb{R}^{n+1}$, as well as the asymptotic behavior near singular set of the graphical functions for a family of converging stable minimal hypersurfaces. In particular, this enables us to prove Theorem 5.1.2.

In Section 5.3, we study the analytic behavior of the linearized minimal surfaces operator, namely, the Jacobi operator L_Σ, on a singular minimal hypersurface Σ. The key property is the L^2-compactness of a family of functions with uniformly bounded L^2-norm and uniformly bounded quadratic form associated to L_Σ. Similar compactness has been proved by Schoen-Yau using a uniform L^2-nonconcentration property and coercivity in [SY17] for the operators associated to the specific quadratic form of minimal slicings and by Lohkamp using the associated skin structure in [Loh19] for the $\mathcal{S}$-adapted Schrödinger operators. In fact, their argument is sufficient to obtain the desired compactness if each tangent cone is strictly stable.

For a general Σ, the coercivity of L_Σ might fail. Hence, we establish another L^2-compactness in Section 5.3.1 and 5.3.2 when Σ has strongly isolated singularities, which allows us to still get spectral decomposition of L_Σ as stated in Theorem 1.3.5 and define the Green's function. We also derive an asymptotic estimate of Green's function near singularities in Section 5.3.4, which is used in Section 5.4 to deduce certain mean convexity of one-sided perturbations of minimal hypersurfaces.

In Section 5.4, we prove Theorem 5.1.3. The key is an existence theorem of mini-

mal graph over compact subset of regular part with weighted estimate, see Theorem 5.4.3. Note that this estimate doesn't require the tangent cone to be minimizing.

In Section 5.5, we carry out the proof of Theorem 5.1.1. To do this, inspired by [Whi94], we first prove in Section 5.5.1 an analogue of strong min-max property for Σ in Theorem 5.1.1 in a sufficiently small neighborhood U. The proof closely follows the argument in [Whi94], except that we have to estimate error terms near singularities using using a growth estimate obtained in Section 5.2.3. This allows us to construct an admissible family of sweepouts under any perturbed metric, with width close to the volume of Σ. Then by applying the min-max theory for obstacle problem in Chapter 4, for any slightly perturbed metric g', there's a constrained embedded minimal hypersurface $\Sigma' \subset (\mathrm{Clos}(U), g')$.

To conclude that Σ' doesn't intersect ∂U for certain small U and g sufficiently close to g_0, we prove in Section 5.5.2 a local rigidity result for constrained embedded minimal hypersurfaces, as an analogue of Theorem 4.5.1.

5.2 Geometry of Regular Stable Minimal Hypercone

We introduce the following space of *uniformly regular* stable minimal hypercone,

$$\mathscr{C}^n_{\Lambda,\Lambda'} := \{\mathbf{C} \in \mathscr{C}^n_\Lambda : r_{\mathcal{S}}^{-1} \leq \Lambda' \text{ on } \mathbf{C} \cap \mathbb{S}^n\}, \tag{5.2.1}$$

where recall $\mathscr{C}^n_\Lambda$ is defined in (2.4.1), $r_{\mathcal{S}}$ is the regularity scale defined in Definition 2.1.1. Note that

$$\mathscr{C}^n := \bigcup_{\Lambda,\Lambda'>1} \mathscr{C}^n_{\Lambda,\Lambda'} = \{\text{regular stable minimal hypercones in } \mathbb{R}^{n+1}\}.$$

Moreover, for each $\Lambda, \Lambda' > 0$, $\mathscr{C}^n_{\Lambda,\Lambda'}$ is a closed subset in $(\mathscr{C}^n_\Lambda, \mathbf{F})$, and it's also compact in C^∞-topology, i.e. any sequence $\{\mathbf{C}_j \in \mathscr{C}^n_{\Lambda,\Lambda'}\}_{j\geq 1}$ contains a subsequence whose cross section converges smoothly to the cross section of some $\mathbf{C}_\infty \in \mathscr{C}^n_{\Lambda,\Lambda'}$. And if $n+1 = 8$, then Schoen-Simon's Regularity Theorem [SS81] guarantees that there's a function $C(\Lambda) >> 1$ in Λ such that $\mathscr{C}^7_{\Lambda,\Lambda'} = \mathscr{C}^7_\Lambda$ whenever $\Lambda' \geq C(\Lambda)$.

Since $\mathscr{C}^n_{\Lambda,\Lambda'}$ are compact under smooth topology, by the Łojasiewicz-Simon inequality [Sim83a], the densities of stable minimal hypercones in $\mathscr{C}^n_{\Lambda,\Lambda'}$ is a finite subset of $\mathbb{R}$. In particular, when $n+1 = 8$, we can label the set of density of stable minimal hypercones in $\mathbb{R}^8$ by at most countably many real numbers:

$$\{\theta_{\mathbf{C}}(\mathbf{0}) : \mathbf{C} \in \mathscr{C}^7\} = \{1 = \theta_0 < \theta_1 < \theta_2 < \dots\}. \tag{5.2.2}$$

Let $\mathbf{C} \subset \mathbb{R}^{n+1}$ be a regular stable minimal hypercone. Thus $S := \mathbf{C} \cap \mathbb{S}^n$ is a regular minimal hypersurface in $\mathbb{S}^n$. We parametrize $\mathbf{C}$ by

$$(0, +\infty) \times S \to \mathbf{C}, \quad (r, \omega) \mapsto x = r\omega. \tag{5.2.3}$$

And denote for simplicity $\mathbf{B}_r := \mathbb{B}^{n+1}_r(\mathbf{0}) \cap \mathbf{C}$, and $\mathbf{A}(r,s) := \mathbb{A}_{r,s}(\mathbf{0}) \cap \mathbf{C}$. Recall by [Sim68], the Jacobi operator of $\mathbf{C}$ is decomposed under this parametrization as

$$L_{\mathbf{C}} = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}(\Delta_S + |A_S|^2),$$

where A_S is the second fundamental form of S in $\mathbb{S}^n$.

In Appendix C, we include a precise discussion of Jacobi fields on $\mathbf{C}$. We list some basic notations and facts here to build up intuitions. To start with, let

$$\mu_1 < \mu_2 \leq \mu_3 \leq \cdots \nearrow +\infty$$

be the eigenvalues of $-(\Delta_S + |A_S|^2)$, and $\varphi_1, \varphi_2, \dots$ be the corresponding $L^2(S)$-orthonormal eigenfunctions, where $\varphi_1 > 0$. By [Sim68], the stability of $\mathbf{C}$ is equivalent to that $\mu_1 \geq -(n-2)^2/4$. Following [CHS84], we call $\mathbf{C}$ **strictly stable**, if $\mu_1 > -(n-2)^2/4$.

Clearly, if $\{\mathbf{C}_k\}_{1 \leq k \leq \infty}$ is a family of regular cone such that $\mathbf{F}(|\mathbf{C}_k|, |\mathbf{C}_\infty|) \to 0$, then the asymptotic spectrum also converges: For every $j \geq 1$, as $k \to \infty$, $\mu_j(\mathbf{C}_k) \to \mu_j(\mathbf{C}_\infty)$.

By [CHS84], a general Jacobi field $v \in C^\infty(\mathbf{C})$ is given by a linear combination of homogeneous Jacobi fields,

$$v(r, \omega) = \sum_{j \geq 1} (v_j^+(r) + v_j^-(r)) \varphi_j(\omega), \tag{5.2.4}$$

where

$$\begin{aligned} v_j^+(r) &= c_j^+ \cdot r^{\gamma_j^+}; \\ v_j^-(r) &= \begin{cases} c_j^- \cdot r^{\gamma_j^-}, & \text{if } \mu_j > -\frac{(n-2)^2}{4}; \\ c_j^- \cdot r^{\gamma_j^-} \log r, & \text{if } \mu_j = -\frac{(n-2)^2}{4}. \end{cases} \end{aligned} \tag{5.2.5}$$

for some $c_j^\pm \in \mathbb{R}$ and

$$\gamma_j^\pm = \gamma_j^\pm(\mathbf{C}) := -\frac{n-2}{2} \pm \sqrt{\mu_j + \frac{(n-2)^2}{4}}. \tag{5.2.6}$$

Note that $\mu_j = -(n-2)^2/4$ if and only if $j = 1$ and $\gamma_1^+ = \gamma_1^- = -(n-2)/2$.

In view of (5.2.4) and (5.2.5), we define the **asymptotic spectrum** of $\mathbf{C}$ by

$$\Gamma(\mathbf{C}) := \{\gamma_j^\pm : j \geq 1\}. \tag{5.2.7}$$

Heuristically, $\Gamma(\mathbf{C})$ consists of all possible degrees of homogeneous Jacobi fields on

C. For later reference, we also denote for $\gamma \in \{\gamma_j^{\pm}\}$,

$$W_\gamma := L^2(S)\text{-eigenspace of } (\Delta_S + |A_S|^2) \text{ with eigenvalue } \mu_j\,. \tag{5.2.8}$$

Lemma 5.2.1. *For every $\Lambda, \Lambda' > 0$,*

$$\gamma^n_{\mathrm{gap};\Lambda,\Lambda'} := \inf\{\gamma_2^+(\mathbf{C}) - \gamma_1^+(\mathbf{C}) : \mathbf{C} \in \mathscr{C}^n_{\Lambda,\Lambda'}\} > 0\,;$$
$$\gamma_{\mathrm{gap},\Lambda} := \inf_{\Lambda'>1} \gamma^7_{\mathrm{gap};\Lambda,\Lambda'} > 0\,.$$

Here we use the convention that $\inf \emptyset = +\infty$.

Proof. Since $\mathscr{C}^n_{\Lambda,\Lambda'}$ is C^∞-compact and $\gamma_2^+(\mathbf{C}) - \gamma_1^+(\mathbf{C})$ is continuous w.r.t. the C^∞-metric, $\gamma^n_{\mathrm{gap},\Lambda,\Lambda'}$ (if not $+\infty$) is realized by some $\mathbf{C} \in \mathscr{C}^n_{\Lambda,\Lambda'}$, and hence is positive. And since when $n = 7$, $\mathscr{C}^7_{\Lambda,\Lambda'} = \mathscr{C}^7_\Lambda$ for $\Lambda' \geq C(\Lambda)$, we then have $\gamma_{\mathrm{gap},\Lambda} = \gamma^7_{\mathrm{gap};\Lambda,C(\Lambda)} > 0$. □

5.2.1 Asymptotic rate of Jacobi-field-type functions

The following Lemma on Growth Rate Monotonicity is proved in Appendix C ,

Lemma 5.2.2 (c.f. Lemma C.0.6)**.** *For $\sigma > 0$, $\Lambda, \Lambda' > 1$, there exists $K = K(\sigma, \Lambda, \Lambda') > 2$ with the following property. Let $\gamma \in \mathbb{R}$, $\mathbf{C} \in \mathscr{C}^n_{\Lambda,\Lambda'}$ be a stable minimal hypercone satisfying*

$$dist_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}) \cup \{-(n-2)/2\}) \geq \sigma\,.$$

Suppose that $u \in C^2_{loc}(\mathbf{A}(K^{-3}, 1)) \cap L^2(\mathbf{A}(K^{-3}, 1))$ solves $L_{\mathbf{C}} u = 0$. Then

$$J^\gamma_K(u; r) := \int_{\mathbf{A}(K^{-1}r, r)} u^2 \cdot |x|^{-2\gamma-n} d\|\mathbf{C}\|\,, \tag{5.2.9}$$

satisfies

$$J^\gamma_K(u; K^{-2}) - 2J^\gamma_K(u; K^{-1}) + J^\gamma_K(u; 1) \geq 0\,.$$

Moreover, the equality holds if and only if $u \equiv 0$.

For later reference, we need the following quantitative version of this Lemma.

Corollary 5.2.3 (Perturbed version of Lemma 5.2.2)**.** *For* $\sigma > 0$ *and* $\Lambda, \Lambda' > 1$*, let* $K(\sigma, \Lambda, \Lambda') > 2$ *be the same as in Lemma 5.2.2. Then there exists* $\varepsilon(\sigma, \Lambda, \Lambda') > 0$ *small enough satisfying the following property.*

For $\gamma \in (-\Lambda, \Lambda)$ *and any stable hypercone* $\mathbf{C} \in \mathscr{C}^n_{\Lambda,\Lambda'}$ *satifying*

$$dist_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}) \cup \{-(n-2)/2\}) \geq \sigma\,,$$

suppose that $0 \neq u \in W^{1,2}_{loc}(\mathbf{A}(K^{-3}, 1)) \cap L^2(\mathbf{A}(K^{-3}, 1))$ *is a weak solution to*

$$div_{\mathbf{C}}(\nabla_{\mathbf{C}} u + \vec{b}_0(x)) + |A_{\mathbf{C}}|^2 u + |x|^{-1} b_1(x) = 0, \tag{5.2.10}$$

where $\vec{b}_0, b_1$ *satisfy the following estimates pointwisely,*

$$|\vec{b}_0|(x) + |b_1|(x) \leq \varepsilon \left(|x|^{-1}|u|(x) + |\nabla u|(x) + \|u\|_{L^2(\mathbf{A}(K^{-3},1))} \right), \tag{5.2.11}$$

on $\mathbf{A}(K^{-3}, 1)$*. Then we have,*

$$J^{\gamma}_K(u; K^{-2}) - 2(1+\varepsilon) J^{\gamma}_K(u; K^{-1}) + J^{\gamma}_K(u; 1) > 0.$$

Remark 5.2.4. When $\Sigma := \mathrm{graph}_{\mathbf{C}}(\phi) \cap \mathbb{A}(\mathbf{0}, K^{-3}/2, 2)$ is a minimal hypersurface under the metric g parametrized by the cone $\mathbf{C}$, with

$$\|\phi\|_{C^2_*} + \|g - g_{\mathrm{euc}}\|_{C^2} \leq \epsilon,$$

then by Appendix A, any Jacobi field u on Σ, viewed as a function over $\mathbf{C}$ under this

parametrization, solves an equation of form (5.2.10) with

$$|\vec{b}_0| + |b_1| \leq \Psi(\epsilon|\mathbf{C}, K) \cdot (|u| + |\nabla u|).$$

where $\Psi(\epsilon|\mathbf{C}, K)$ is a constant tending to 0 when $\mathbf{C}, K$ is fixed and $\epsilon \to 0$. Therefore, Corollary 5.2.3 applies when $\epsilon << 1$.

Proof of Corollary 5.2.3. First note that by multiplying (5.2.10) with $u \cdot \eta^2$ and integration by parts (η is some smooth cut-off function) and using (5.2.11) with $\varepsilon < 1/2$, we have for every domain $\Omega \subset\subset \mathbf{A}(K^{-3}, 1)$,

$$\int_\Omega |\nabla_{\mathbf{C}} u|^2 d\|\mathbf{C}\| \leq C(n, K, \Omega) \int_{\mathbf{A}(K^{-3},1)} (1 + |A_{\mathbf{C}}|^2) u^2. \tag{5.2.12}$$

Now to prove the corollary, suppose for contradiction that there exist $\sigma > 0$, $\Lambda, \Lambda' > 0$, a sequence of $\mathbf{C}_j \in \mathscr{C}^n_{\Lambda,\Lambda'}$, a sequence of real number $\{\gamma_j\} \subset (-\Lambda, 1)$ with

$$dist_{\mathbb{R}}(\gamma_j, \Gamma(\mathbf{C}_j) \cup \{-(n-2)/2\}) \geq \sigma\,,$$

and a sequence of nonzero function $u_j \in W^{1,2}_{loc}(\mathbf{A}(K^{-3}, 1)) \cap L^2(\mathbf{A}(K^{-3}, 1))$ satisfying (5.2.10) and (5.2.11) with $\mathbf{C} = \mathbf{C}_j$ and $\varepsilon = 1/j$, but

$$J^{\gamma_j}_K(u_j, K^{-2}) - 2(1 + \frac{1}{j}) J^{\gamma_j}_K(u_j, K^{-1}) + J^{\gamma_j}_K(u_j, 1) \leq 0. \tag{5.2.13}$$

By a renormalization of u_j, WLOG, we can assume $J^{\gamma_j}_K(u_j, K^{-1}) = 1$. By the C^∞-compactness of $\mathscr{C}^n_{\Lambda,\Lambda'}$, we can assume by passing to a subsequence of $j \to \infty$ that $\mathbf{C}_j$ will smoothly converge to $\mathbf{C}_\infty \in \mathscr{C}^n_{\Lambda,\Lambda'}$, and $\gamma_j \to \gamma_\infty$. And then by the continuity of asymptotic spectrum under smooth convergence, we have

$$dist_{\mathbb{R}}(\gamma_\infty, \Gamma(\mathbf{C}_\infty) \cup \{-(n-2)/2\}) \geq \sigma\,.$$

By (5.2.12), (5.2.13) and Reilly compactness, we have u_j subconverges to $u_\infty \in W^{1,2}_{loc}(\mathbf{A}(K^{-3},1)) \cap L^2(\mathbf{A}(K^{-3},1))$ in $L^2_{loc}(\mathbb{A}_{K^{-3},1}(\mathbf{0}))$ with $J^{\gamma_\infty}_K(u_\infty, K^{-1}) = 1$, and

$$(\Delta_{\mathbf{C}_\infty} + |A_{\mathbf{C}_\infty}|^2)u_\infty = 0$$

on $\mathbf{A}(K^{-3},1)$, but (5.2.13) then implies

$$J^{\gamma_\infty}_K(u_\infty, K^{-2}) - 2J^{\gamma_\infty}_K(u_\infty, K^{-1}) + J^{\gamma_\infty}_K(u_\infty, 1) \leq 0.$$

This contradicts Lemma 5.2.2. □

To finish this Section, we introduce the following notion of *asymptotic rate*, which plays a crucial role in studying Jacobi fields on a general LSMH with strongly isolated singularities.

Definition 5.2.5. For a function $v \in L^2_{loc}(\mathbf{B}_r)$ defined on $\mathbf{C}$ near $\mathbf{0}$, defined the **asymptotic rate** of v at $\mathbf{0}$ to be,

$$\mathcal{AR}_{\mathbf{0}}(v) := \sup\left\{\gamma : \lim_{s\searrow 0}\int_{\mathbf{A}(s,2s)} v^2\cdot|x|^{-n-2\gamma}\, d\|\mathbf{C}\|(x) = 0\right\} \in [-\infty,+\infty],$$

Here we use the convention that $\sup\emptyset = -\infty$ and $\sup\mathbb{R} = +\infty$. Heuristically, if v grows like r^γ near $\mathbf{0}$, then $\mathcal{AR}_{\mathbf{0}}(v) = \gamma$.

We can also define the **asymptotic rate at infinity** of some function $v \in L^2_{loc}(\mathbf{C}\setminus \mathbf{B}_R)$:

$$\mathcal{AR}_{\infty}(v) := \inf\left\{\gamma : \lim_{s\nearrow+\infty}\int_{\mathbf{A}(s,2s)} v^2\cdot|x|^{-n-2\gamma}\, d\|\mathbf{C}\|(x) = 0\right\} \in [-\infty,+\infty]\,.$$

A direct application of Corollary 5.2.3 is the following quantization of asymptotic rate for solutions to Jacobi-field-type equations.

Corollary 5.2.6. *Let* $\mathbf{C}$ *be a regular stable minimal hypercone,* $0 \neq u \in W^{1,2}_{loc}(\mathbf{B}_1)$ *be solving (5.2.10) on* $\mathbf{B}_1$*, with* $\vec{b}_0$ *and* b_1 *satisfying*

$$|\vec{b}_0|(x) + |b_1|(x) \leq \varepsilon(|x|)\left(|x|^{-1}|u|(x) + |\nabla u|(x) + 1\right); \tag{5.2.14}$$

for some decreasing function $\varepsilon \in C^0([0,1])$ *such that* $\varepsilon(0) = 0$*. Also assume that*

$$\gamma := \mathcal{AR}_{\mathbf{0}}(u) \in (-\infty, 1).$$

Then $\gamma \in \Gamma(\mathbf{C})$*; And for every sequence* $t_j \searrow 0$*,* $\hat{u}_j(x) := J_2^0(u;t_j)^{-1/2}u(t_j x)$ *will subconverges in* L^2_{loc} *to some Jacob field* $\hat{u}_\infty$ *on* $\mathbf{C}$ *given by*

$$\hat{u}_\infty = r^\gamma(c + c'\log r)w, \tag{5.2.15}$$

where $0 \neq w \in W_\gamma(\mathbf{C})$ *is the corresponding eigenfunction of* γ*;* $(c, c') \neq (0,0)$ *and* $c' = 0$ *unless* $\gamma = -(n-2)/2$*.*

Similar result also holds for functions defined on $\mathbf{C} \setminus \mathbf{B}_1$*.*

Proof. We only prove the corollary for $\mathbf{B}_1$, the one for $\mathbf{C} \setminus \mathbf{B}_1$ is similar. Suppose $\Lambda, \Lambda' > 1$ such that $\mathscr{C}^n_{\Lambda,\Lambda'} \ni \mathbf{C}$; Since $-\infty < \gamma < 1$ and $\Gamma(\mathbf{C})$ is discrete, we can choose $\sigma > 0$ such that $\gamma + \sigma < 1$ and $[\gamma - \sigma, \gamma + \sigma] \cap \Gamma(\mathbf{C}) \subset \{\gamma\}$. Let $K(\Lambda, \Lambda', \sigma) > 2$ be given by Lemma 5.2.2.

By definition of $\mathcal{AR}_{\mathbf{0}}(u)$, there exists a sequence of $r_j \searrow 0$ such that

$$J_K^{\gamma+\sigma}(u, K^{-1}r_j) \geq J_K^{\gamma+\sigma}(u, r_j) \geq 1.$$

Also notice that for every $\lambda \in (0,1)$, $u^{(\lambda)}(x) := \lambda^{-1}u(\lambda x)$ and $b_j^{(\lambda)}(x) := b_j(\lambda x)$

satisfies,

$$\begin{aligned} &div_{\mathbf{C}}(\nabla u^{(\lambda)} + \vec{b}_0^{(\lambda)}) + |A_{\mathbf{C}}|^2 u^{(\lambda)} + |x|^{-1} b_1^{(\lambda)} = 0\,, \\ &|\vec{b}_0^{(\lambda)}(x)| + |b_1^{(\lambda)}(x)| \leq \varepsilon(\lambda) \cdot (|x|^{-1}|u^{(\lambda)}| + |\nabla u^{(\lambda)}| + 1)\,, \end{aligned} \tag{5.2.16}$$

on $\mathbf{A}(K^{-3}, 1)$. Also, since

$$\begin{aligned} \|u^{(\lambda)}\|^2_{L^2(\mathbf{A}(K^{-2},K^{-1}))} &= \int_{\mathbf{A}(\lambda K^{-2},\lambda K^{-1})} \lambda^{-n-2} u^2 \\ &\geq c(K,\gamma)\lambda^{2(\gamma+\sigma)-2} \cdot J_K^{\gamma+\sigma}(u, K^{-1}\lambda)\,, \end{aligned} \tag{5.2.17}$$

and $\gamma + \sigma < 1$, we can thus choose $j_0 >> 1$ such that $\|u^{(r_{j_0})}\|^2_{L^2(\mathbf{A}(K^{-2},K^{-1}))} \geq 1$ and $\varepsilon(r_{j_0})$ less than the constant $\varepsilon(\sigma, \Lambda, \Lambda')$ given by Corollary 5.2.3. Therefore, Corollary 5.2.3 applies to $u^{(r_{j_0})}$ which gives,

$$J_K^{\gamma+\sigma}(u, K^{-2} r_{j_0}) - J_K^{\gamma+\sigma}(u, K^{-1} r_{j_0}) \geq J_K^{\gamma+\sigma}(u, K^{-1} r_{j_0}) - J_K^{\gamma+\sigma}(u, r_{j_0}) \geq 0\,.$$

Repeat the process above with r_{j_0} replaced by $\lambda_l := K^{-l} r_{j_0}$, $l \geq 1$, we get

$$\begin{aligned} &|\vec{b}_0^{(\lambda_l)}(x)| + |b_1^{(\lambda_l)}(x)| \leq \varepsilon \cdot \left(|x|^{-1}|u^{(\lambda_l)}| + |\nabla u^{(\lambda_l)}| + \|u^{(\lambda_l)}\|_{L^2(\mathbf{A}(K^{-2},K^{-1}))}\right)\,, \\ &J_K^{\gamma+\sigma}(u, K^{-i}\lambda_l) \geq J_K^{\gamma+\sigma}(u, \lambda_l), \qquad \forall i \geq 0\,. \end{aligned} \tag{5.2.18}$$

On the other hand, by definition of $\gamma - \sigma < \mathcal{AR}_{\mathbf{0}}(u)$, there exists $l_0 >> 1$ such that

$$J_K^{\gamma-\sigma}(u, K^{-1}\lambda_{l_0}) \leq J_K^{\gamma-\sigma}(u, \lambda_{l_0})\,.$$

Thus by (5.2.16) and (5.2.18), Corollary 5.2.3 applies again iteratively to give for every $l \geq l_0$,

$$J_K^{\gamma-\sigma}(u, K^{-i}\lambda_l) \leq J_K^{\gamma-\sigma}(u, \lambda_l), \qquad \forall i \geq 0\,. \tag{5.2.19}$$

Now suppose $t_j \searrow 0$ be an arbitrary sequence. By first passing to a subsequence, let $\bar{t}_j \in \{\lambda_l\}_{l\geq l_0}$ be such that $\bar{t}_j \in (K^{-1}t_j, t_j]$. Let $\tilde{u}_j(x) := J_K^0(u;\bar{t}_j)^{-1/2}u(\bar{t}_j x)$. Note that estimates (5.2.18) and (5.2.19) implies that for every $R > 1$, there exists $j_R >> 1$ such that whenever $j \geq j_R$,

$$\begin{aligned} \|\tilde{u}_j\|_{L^2(\mathbf{A}(1,R))} &\leq C(\mathbf{C},K,\gamma)R^{\gamma+\sigma}\|\tilde{u}_j\|_{L^2(\mathbf{A}(K^{-1},1))} \leq C(\mathbf{C},K,\gamma)R^{\gamma+\sigma}\,, \\ \|\tilde{u}_j\|_{L^2(\mathbf{A}(R^{-1},1))} &\leq C(\mathbf{C},K,\gamma)R^{-\gamma+\sigma}\|\tilde{u}_j\|_{L^2(\mathbf{A}(K^{-1},1))} \leq C(\mathbf{C},K,\gamma)R^{-\gamma+\sigma}\,. \end{aligned} \tag{5.2.20}$$

This gives the $L^2_{loc}(\mathbf{C})$-estimate with coefficients independent of j. And by combining with (5.2.18), the equation (5.2.16) also provides uniform $W^{1,2}_{loc}$ bounds for $\tilde{u}_j$. Thus, $\tilde{u}_j$ subconverges in $L^2_{loc}(\mathbf{C})$ to some $\tilde{u}_\infty$ which is non-zero by the way of renormalization, solves $L_{\mathbf{C}}\tilde{u}_\infty = 0$ on $\mathbf{C}$ and satisfies the same growth estimates as in (5.2.20) near $\mathbf{0}$ and ∞. By our choice of σ and Lemma C.0.2, $\tilde{u}_\infty$ must be of form (5.2.15), and so is $\hat{u}_\infty$ since it's just obtained by an equivalent normalization. □

5.2.2 Asymptotics of minimal hypersurfaces near infinity

The goal of this subsection is to study minimal hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$ which asymptotic to a given regular stable minimal hypercone $\mathbf{C}$ near infinity. First notice that since $S := \mathbf{C} \cap \mathbb{S}^n$ is a smooth minimal hypersurface in $\mathbb{S}^n$, by Frankel property, S is connected and $\mathbb{R}^{n+1} \setminus \mathbf{C}$ has 2 connected components, denoted by $E_\pm$. We let ν be the unit normal field of $\mathbf{C}$ pointing into E_+.

Definition 5.2.7. We call an integral varifold $V \in \mathcal{IV}_n(\mathbb{R}^{n+1} \setminus \mathbb{B}_\Lambda)$ **asymptotic to** $\mathbf{C}$ near infinity, if there's a function $h \in C^2(\mathbf{C})$ and $R_0 > \Lambda$ such that

$$\|h\|_{C^2_*,\mathbf{A}(R,2R)} := \sup_{\mathbf{A}(R,2R)} \left(R^{-1}|h| + |\nabla_{\mathbf{C}}h| + R|\nabla^2_{\mathbf{C}}h|\right) \to 0, \quad \text{as } R \to +\infty\,. \tag{5.2.21}$$

and that

$$|\operatorname{graph}_{\mathbf{C}}(h)|\llcorner(\mathbb{R}^{n+1}\setminus\mathbb{B}_{R_0}) = V\llcorner(\mathbb{R}^{n+1}\setminus\mathbb{B}_{R_0}) \tag{5.2.22}$$

For such V, following Definition 5.2.5, we can define the **asymptotic rate** of V to $\mathbf{C}$ at infinity by,

$$\mathcal{AR}_\infty(V) = \mathcal{AR}_\infty(V;\mathbf{C}) := \mathcal{AR}_\infty(h)\,.$$

Note that by our convention, $\mathcal{AR}_\infty(V;\mathbf{C}) = -\infty$ if V coincide with $\mathbf{C}$ outside a large ball.

If $\Sigma \subset \mathbb{R}^{n+1}\setminus\mathbb{B}_\Lambda$ is a hypersurface, we denote $\mathcal{AR}_\infty(\Sigma) := \mathcal{AR}_\infty(|\Sigma|)$.

For stationary integral varifolds with finite density at infinity, by Allard regularity [All72, Sim83b], being asymptotic to $\mathbf{C}$ is equivalent to having $|\mathbf{C}|$ to be the tangent varifold at ∞.

Remark 5.2.8. By (5.2.21), it's clear that $\mathcal{AR}_\infty(V) = \mathcal{AR}_\infty(\lambda\cdot V) \le 1$, $\forall\lambda > 0$. And since the graphical function h satisfies the minimal surface equation on cone, by combining Corollary 5.2.6 and Theorem A.0.1, we know that $\mathcal{AR}_\infty(V) \in \Gamma(\mathbf{C})\cup\{-\infty\}$. In Lemma 5.2.9 below, this will be refined to a decomposition of h into degree $\mathcal{AR}_\infty(V)$ part and faster decaying part.

Lemma 5.2.9. *Suppose $\Sigma \subset \mathbb{R}^{n+1}\setminus\mathbb{B}_1$ is a minimal hypersurface asymptotic to $\mathbf{C}$ near infinity, with $\gamma := \mathcal{AR}_\infty(\Sigma) \in (-\infty, 1)$. Let h be the graphical function as in (5.2.22), then*

$$h(r,\omega) = r^\gamma(c + c'\log r)w(\omega) + O_2(r^{\gamma-\epsilon})\,,$$

for some $(c,c') \neq (0,0)$, where $c' = 0$ unless $\gamma = \gamma_1^+ = \gamma_1^-$ (in which case $\mathbf{C}$ is not strictly stable); Here $\epsilon = \epsilon(\mathbf{C},\gamma) > 0$; $w \in W_\gamma$ is a unit eigenfunction of $(\Delta_S + |A_S|^2)$; $O_2(r^{\gamma-\epsilon})$ means a function with C^2_*-norm decaying like $r^{\gamma-1-\epsilon}$ on $\mathbf{C}\setminus\mathbb{B}_1$.

This characterization is well-known, at least when $\gamma = \gamma_1^{\pm}$, in [HS85, 1.9]. For sake of completeness, we sketch the proof here. In what follows, every constant depends on the cone $\mathbf{C}$.

Proof. Since Σ is minimal, h satisfies the minimal surface equation

$$\mathscr{M}_{\mathbf{C}} h = -L_{\mathbf{C}} h + \mathscr{R}_{\mathbf{C}} h = 0$$

where by [CHS84, (2.2)],

$$\begin{aligned}
&\mathscr{R}_{\mathbf{C}} h\ (r,\omega) := N(r\omega, h/r, \nabla h)\cdot \nabla^2 h + r^{-1} P(r\omega, h/r, \nabla h)\,;\\
&|P(x,z,p)| \le C(|z|+|p|)^2\,,\\
&|N(x,z,p)| + |P_z(x,z,p)| + |P_p(x,z,p)| \le C(|z|+|p|)\,,\\
&r(|N_x(x,z,p) + P_x(x,z,p)|) + |N_z(x,z,p)| + |N_p(x,z,p)| \le C\,.
\end{aligned}$$

By elliptic estimate, for $R >> 2R_0$,

$$\|\nabla_{\mathbf{C}} h\|_{C^0(\mathbf{A}(R,2R))} + R\|\nabla^2_{\mathbf{C}} h\|_{C^0(\mathbf{A}(R,2R))} \le C R^{-1-n/2}\|h\|_{L^2(\mathbf{A}(R/2,4R))}\,.$$

Hence by definition, for any $\delta > 0$, since $R^{-\gamma-\delta-n/2}\|h\|_{L^2(\mathbf{A}(R,2R))} \to 0$ as $R \to +\infty$, we have

$$|\mathscr{R}_{\mathbf{C}} h|(R,\omega) \le C_\delta R^{\gamma-2+(\gamma-1+2\delta)}, \quad \forall R >> 2R_0\,.$$

Fix $\delta > 0$ such that $\gamma - 1 + 3\delta < 0$ and $\sigma := \gamma + (\gamma - 1 + 3\delta) \notin \Gamma(\mathbf{C})$. By Lemma C.0.4, $\exists u \in C^\infty(\mathbf{C} \setminus \mathbf{B}_{2R_0})$ solving $L_{\mathbf{C}} u = \mathscr{R}_{\mathbf{C}} h$ and

$$\|u(R,\cdot)\|_{L^2(S)} \le C R^\sigma$$

But since $L_{\mathbf{C}}(h-u) = 0$ and $\forall \delta' > 0$, $\limsup_{R\to\infty} R^{-\gamma+\delta'}\|h(R,\cdot)\|_{C^0(S)} > 0$, by

(C.0.4), $h + u = r^\gamma(c + c' \log r) + l.o.t.$ and so does h. □

Example 5.2.10. *(1) One natural example of minimal hypersurface asymptotic to a given are-minimizing cone* $\mathbf{C}$ *near infinity is the Hardt-Simon foliation [HS85] (see also Theorem 2.4.1), which lies on one side of* $\mathbf{C}$*. Notice that by (2.4.2) and (2.4.3), each of the Hardt-Simon foliation* $\Sigma \subset E_+$ *has asymptotic rate,*

$$\mathcal{AR}_\infty(\Sigma; \mathbf{C}) = \begin{cases} \gamma_1^+, & \text{if } \mathbf{C} \text{ is strictly minimizing in } E_+ ; \\ \gamma_1^-, & \text{if } \mathbf{C} \text{ is not strictly minimizing in } E_+ . \end{cases}$$

On the other hand, since the only eigenfunction of $(\Delta_S + |A_S|^2)$ *that has a sign is the first eigenfunction, Lemma 5.2.9 above guarantees that if* $V \in \mathcal{IV}_n(\mathbb{R}^{n+1} \setminus \mathbb{B}_\Lambda)$ *is a stationary integral varifold asymptotic to* $\mathbf{C}$ *near infinity. Then* $\mathcal{AR}_\infty(V; \mathbf{C}) \in \{\gamma_1^\pm\}$ *if and only if* $\mathrm{spt}(V)$ *lies on one side of* $\mathbf{C}$ *outside a sufficiently large compact subset.*

(2) Another typical example is the translation of the cone in $\mathbb{R}^{n+1}$*, which has asymptotic rate* 0 *near infinity by the following Lemma.*

Lemma 5.2.11. *Let* $\mathbf{C} \subset \mathbb{R}^{n+1}$ *be a regular minimal hypercone and* $\mathbf{x} \neq \mathbf{0}$*. Then*

$$\mathcal{AR}_\infty(\mathbf{C} + \mathbf{x}; \mathbf{C}) = 0 .$$

Proof. First by implicit function theorem, there exists $\delta(\mathbf{C}) \in (0, 1)$ such that for every $\mathbf{x} \in \mathbb{R}^{n+1}$ with $|\mathbf{x}| \leq \delta(\mathbf{C})$, the map

$$\Pi_{\mathbf{x}} : \mathbf{C} \cap \mathbb{A}(\mathbf{0}, 1/4, 4) \to \mathbf{C}, \quad p \mapsto \text{nearest point projection of } p + \mathbf{x} \text{ onto } \mathbf{C},$$

is a diffeomorphism onto its image, and its image contains $\mathbb{A}(\mathbf{0}, 1/2, 3) \cap \mathbf{C}$. Also, if

write $u_{\mathbf{x}}$ be the graphical function of $\mathbf{C}+\mathbf{x}$ over $\mathbf{C}$ in $\mathbb{A}(\mathbf{0}, 1/2, 3)$, then we know that

$$u_{\mathbf{x}}(q) = \langle \Pi_{\mathbf{x}}^{-1}(q) - q + \mathbf{x}, \nu_{\mathbf{C}}(q)\rangle, \quad \forall q \in \mathbb{A}(\mathbf{0}, 1, 2).$$

And since $\langle \Pi_{\mathbf{x}}^{-1}(q) - q, \nu_{\mathbf{C}}(q)\rangle = O(|\mathbf{x}|^2)$, we have,

$$|u_{\mathbf{x}}(q) - \langle \mathbf{x}, \nu_{\mathbf{C}}(q)\rangle| \leq C(\mathbf{C}) \cdot |\mathbf{x}|^2. \tag{5.2.23}$$

Now for a general $\mathbf{x} \in \mathbb{R}^{n+1} \setminus \mathbf{0}$, since $(\mathbf{C}+\mathbf{x})/R = \mathbf{C}+\mathbf{x}/R$, we know that $\mathbf{C}+\mathbf{x}$ is graphical over $\mathbf{C}$ outside the ball $\mathbb{B}_{|\mathbf{x}|/\delta(\mathbf{C})}$, and the graphical function satisfies $u_{\mathbf{x}}(p) = Ru_{\mathbf{x}/R}(p/R)$. Hence, for every $|p| > |\mathbf{x}|/\delta(\mathbf{C})$ we have,

$$|u_{\mathbf{x}}(p) - \langle \mathbf{x}, \nu_{\mathbf{C}}(p)\rangle| = R \cdot |u_{\mathbf{x}/R}(p/R) - \langle \mathbf{x}/R, \nu_{\mathbf{C}}(p/R)\rangle| \leq C(\mathbf{C}) \cdot |\mathbf{x}|^2/|p|, \tag{5.2.24}$$

where the last inequality follows by taking $R = |p|$ and applying (5.2.23).

Also, observe that $\langle \mathbf{x}, \nu_{\mathbf{C}}\rangle$ is a homogeneous function on $\mathbf{C}$ of degree 0, which is not identically 0 since otherwise, $\mathbf{C}$ splits in $\mathbf{x}$-direction and is not of isolated singularity. By combining this with (5.2.24), one can check that $\mathcal{AR}_\infty(u_{\mathbf{x}}; \mathbf{C}) = 0$ from the definition. □

The rest of this Subsection is devoted to proving the following asymptotic rate lower bound of minimal hypersurface towards it's tangent cone at infinity.

Theorem 5.2.12. *Suppose $V \in \mathcal{IV}_n(\mathbb{R}^{n+1})$ is a stationary integral varifold asymptotic to a regular stable minimal hypercone $\mathbf{C}$ near infinity, and $V \neq |\mathbf{C}|$. Then*

(1) If $\mathbf{C}$ is one-sided minimizing (resp. strictly minimizing), then $\mathcal{AR}_\infty(V) \geq \gamma_1^-$(resp. γ_1^+). Moreover, if $\mathbf{C}$ is area minimizing and $\mathcal{AR}_\infty(\Sigma) = \gamma_1^\pm$, then Σ has no singularity and lies on one side of $\mathbf{C}$.

(2) If $V = |\Sigma|$ for some stable minimal hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, then $\mathcal{AR}_\infty(\Sigma) \geq$

γ_1^-. *If further,* $\mathcal{AR}_\infty(\Sigma) = \gamma_1^\pm$, *then* Σ *has no singularity and lies on one-side of* $\mathbf{C}$, *inside which* Σ *and* $\mathbf{C}$ *are minimizing.*

We start with some preparation. The following lemma is an analogue of [HS85, Theorem 2.1] for stable minimal cones.

Lemma 5.2.13. *Suppose* $\Sigma \subset \mathbb{R}^{n+1}$ *be a connected stable minimal hypersurface with finite density at infinity; Also suppose that* Σ *is contained in* E_+ *outside a large ball, i.e.* $\exists\ R_0 > 1$ *such that* $\Sigma \setminus \mathbb{B}_{R_0} \subset E_+$. *Then*

(1) $\mathrm{Sing}(\Sigma) = \emptyset$, Σ *is contained entirely in* E_+ *and area-minimizing in* E_+.

(2) $\mathbf{C}$ *is area-minimizing in* $\mathrm{Clos}(E_+)$.

(3) $\{r \cdot \Sigma\}_{r>0}$ *foliates* E_+.

(4) Any stationary integral n*-varifold* $V \in \mathcal{IV}_n(\mathbb{R}^{n+1})$ *with* $\mathrm{spt}\, V \subset E_+$ *and having the same density at* ∞ *as* $\mathbf{C}$ *is a rescaling of* $|\Sigma|$.

Remark 5.2.14. (1) When $\mathbf{C}$ is a priori assumed to be area-minimizing, (1), (3), (4) of Lemma 5.2.13 is a corollary of [HS85, Theorem 2.1] by a barrier argument. However, in the proof we present here, (1), (3) are established first based on Theorem 3.1.1, and the one-sided minimizing property of $\mathbf{C}$ is part of the conclusion. With (1) and (2), the proof of (4) is similar to [HS85]. For sake of completeness, we also sketch the idea of (4).

(2) By solving Plateau problem with one-sided perturbed boundary, [HS85] shown that such Σ in Lemma 5.2.13 do exist if $\mathbf{C}$ is area-minimizing. The similar argument also works for one-sided area-minimizers and even for irregular minimizing hypercone, as is carried out in Section 3.4.4.

(3) We assert here that the stability assumption on Σ can't be dropped. In fact when

$$\mathbf{C} = \mathbf{C}^{5,1} := \{(x, y) \in \mathbb{R}^6 \times \mathbb{R}^2 : |x|^2 = 5|y|^2\} \subset \mathbb{R}^8\,,$$

is the Simons cone over $\mathbb{S}^5 \times \mathbb{S}^1$, [Maz17] constructed two distinct families of $O(6) \times O(2)$-invariant smooth minimal hypersurfaces asymptotic to $\mathbf{C}$ near infinity, each of which is closed under homothety. Recall that by [Sim74, Law91], $\mathbf{C}^{5,1}$ is minimizing only in one side, hence Lemma 5.2.13 suggests that one of the two families above is unstable. An interesting question, which we don't have space to discuss here, would be to determine its index.

Proof. To save the notation, we denote $B_R := \mathbb{B}^{n+1}_R \cap \Sigma$ in this proof. Since Σ is stable with finite density at infinity, by [SS81], any tangent varifold V_∞ of $|\Sigma|$ at infinity is a stable minimal cone. And since Σ is contained in E_+ outside a large ball, we have $\mathrm{spt}(V_\infty) \subset \mathrm{Clos}(E_+)$. Then by $\mathbf{0} \in \mathrm{spt}\, V_\infty \cap \mathbf{C}$ and strong maximum principle [SW89, Ilm96], we have $\mathrm{spt}(V_\infty) = \mathbf{C}$; Since $\mathbf{C}$ is regular, by [SS81] and [CSZ97], Σ has only one end and hence $V = |\mathbf{C}|$, i.e. Σ is asymptotic to $\mathbf{C}$ near infinity. Take $R_0 > 1$ and $h \in C^2(C)$ such that (5.2.21) and (5.2.22) holds.

Since Σ is contained in E_+ near ∞, by a blow down argument and combining Lemma C.0.3 and 5.2.9, see also [HS85, 1.9], we have when $R \to +\infty$,

$$\begin{cases} \text{either} \qquad h(R,\omega) = (c_1 + c_2 \log R) R^{\gamma_1^+} w_1(\omega) + O_2(R^{\gamma_1^+ - \epsilon}) \,, \\ \text{or} \qquad\quad h(R,\omega) = c_1 R^{\gamma_1^-} w_1(\omega) + O_2(R^{\gamma_1^- - \epsilon}) \,, \end{cases} \tag{5.2.25}$$

for some $\epsilon > 0$; $c_1 + c_2 \log R > 0$ for $R > R_0$; $c_2 = 0$ if and only if $\gamma_1^+ = \gamma_1^- = -(n-2)/2$.

Let ν_Σ be the normal field of Σ, pointing away from $\mathbf{C}$. Consider the function $\psi = (\nabla_{\mathbb{R}^{n+1}} |x|^2) \cdot \nu_\Sigma$ on Σ. Note that $|\nabla \psi|(x) \leq |x| \cdot |A_\Sigma|_x$ and $L_\Sigma \psi = 0$; And by [CN13], $|A_\Sigma| \in L^7_{weak}(B_R)$, $|\nabla A_\Sigma| \in L^{7/2}_{weak}(B_R)$, thus $\psi \in W^{1,2}(B_R)$ for each $R > 0$. By (5.2.25), $\psi > 0$ outside some large ball B_{R_1}, $R_1 > R_0$; Hence by stability of Σ and weak maximum principle, $\psi > 0$ on the whole Σ. Hence by Theorem 3.1.1, $\psi(x) \to +\infty$ as $x \to \mathrm{Sing}(\Sigma)$. But notice that by its definition, ψ is bounded on

every B_R. This means $\mathrm{Sing}(\Sigma) = \emptyset$ and $\mathbf{0} \notin \Sigma$.

To show that $\Sigma \subset E_+$, let $\Omega_+ := E_+ \cap \mathbb{S}^n$ and consider the smooth map

$$p : \Sigma \to \mathbb{S}^n \quad x \mapsto \frac{x}{|x|}$$

By $\psi > 0$, p is a diffeomorphism locally; By (5.2.25), p is diffeomorphism between $\Sigma \setminus B_{R_1}$ and a collar neighborhood of $S = \partial\Omega_+$ in Ω_+. By the following topological Lemma 5.2.15, we see that p is a glocal diffeomorphism onto Ω_+. Hence, $\Sigma \subset E_+$ and $\{r \cdot \Sigma\}_{r>0}$ foliates E_+.

To show that Σ and $\mathbf{C}$ are area-minimizing in $\mathrm{Clos}(E_+)$, it suffices to observe that $\{r \cdot \Sigma\}_{r>0}$ determines a calibration $\Theta := \iota_{\nu_+} \mathrm{dVol}_{\mathbb{R}^{n+1}}$ on $\mathrm{Clos}(E_+) \setminus \{\mathbf{0}\}$, where ν_+ be the normal vector field of $\{r \cdot \Sigma\}_{r>0}$, C^1 extended to $\mathrm{Clos}(E_+) \setminus \{\mathbf{0}\}$. Hence, for any $R > 0$, and any integral cycle $T \in \mathcal{Z}_n(\mathbb{R}^{n+1})$,

$$\mathbf{M}_{\mathbb{B}_R}(T) \geq \langle T, \Theta\rangle = \langle [\mathbf{C}], \Theta\rangle = \mathbf{M}_{\mathbb{B}_R}([\mathbf{C}]) \quad \text{if } \mathrm{spt}([\mathbf{C}] - T) \subset \mathbb{B}_R$$
$$\mathbf{M}_{\mathbb{B}_R}(T) \geq \langle T, \Theta\rangle = \langle [\Sigma], \Theta\rangle = \mathbf{M}_{\mathbb{B}_R}([\Sigma]) \quad \text{if } \mathrm{spt}([\Sigma] - T) \subset \mathbb{B}_R$$

We finally sketch the idea of (4). If V is a stationary integral n-varifold with the same density at ∞ as $\mathbf{C}$ and $\mathrm{spt}\, V \subset E_+$, then by a similar argument as the first part of this proof, $|\mathbf{C}|$ is the tangent varifold of V at infinity, and hence V could be parametrized as a graph over $\mathbf{C}$ near infinity as in (5.2.22) and (5.2.25). Let

$$r_+ := \inf\{r : \text{for any } R > r, \mathrm{spt}\, V \cap R \cdot \Sigma = \emptyset\}$$
$$r_- := \sup\{r : \text{for any } R < r, \mathrm{spt}\, V \cap R \cdot \Sigma = \emptyset\}$$

By (5.2.25) and strong maximum principle, $0 < r_- = r_+ < \infty$ and $V = |r_+ \cdot \Sigma|$. □

Lemma 5.2.15. *Let N be a connected compact n manifold with nonempty boundary, M be a closed simply connected n manifold, $p : N \to M$ be a local diffeomorphism*

onto its image, and restricted to a bijection near ∂N. If $M \setminus p(\partial N) = M_+ \sqcup M_-$ has 2 connected component, with $p(N) \cap M_+ \neq \emptyset$, then p is a diffeomorphism onto M_+.

Proof. Notice that the glued map $p \cup_{\partial N} id_{M_-} : N \cup_{\partial N} M_- \to M$ is a local homeomorphism between closed manifolds, hence a covering map. Since N is connected and M is simply connected, this is a bijection, so is p. □

Lemma 5.2.16. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a stable minimal hypersurface. Then the space of positive Jacobi field on Σ is nonempty, i.e.*

$$\mathcal{G}(\Sigma) := \{u \in C^\infty(\Sigma) : L_\Sigma u = 0,\ u > 0\} \neq \emptyset\,.$$

Moreover, if Σ is asymptotic to $\mathbf{C}$ near infinity for some regular minimal hypercone $\mathbf{C}$, assume WLOG (by a rescaling) $\mathrm{Sing}(\Sigma) \subset \mathbb{B}_1$, then for every $\epsilon > 0$,

$$u(x) \geq C(\Sigma, \epsilon, u)|x|^{\gamma_1^- - \epsilon} > 0, \qquad \forall x \in \Sigma \setminus \mathbb{B}_2\,.$$

Proof. To see $\mathcal{G}(\Sigma) \neq \emptyset$, consider an exhaustion $\{\mathcal{U}_i\}_{i\geq 1}$ of Σ and $B_i := \mathbb{B}_{r_i}(p_i) \cap \Sigma \subset\subset \mathcal{U}_i$, where $p_i \to \infty$, $r_i \in (0,1)$. Let $u_i \in W_0^{1,2}(\mathcal{U}_i)$ be the solution of $L_\Sigma u_i = 0$ on $\mathcal{U}_i \setminus B_i$ and $u_i = 1$ on B_i. By the stability of Σ and maximum principle, $u_i > 0$ on $\mathcal{U}_i$; Also let $q \in \Sigma$ be fixed. Then by Harnack inequality, up to a subsequence,

$$u_i/u_i(q) \to u_\infty \quad \text{ in } C^\infty_{loc}(\Sigma)\,,$$

for some positive Jacobi field $u_\infty \in \mathcal{G}(\Sigma)$.

To deduce the lower bound of u near infinity, suppose WLOG that $h \in C^\infty(\mathbf{C})$ and $R_0 = 1$ such that (5.2.21) and (5.2.22) holds for $V = |\Sigma|$. Parametrize $\Sigma \setminus \mathbb{B}_1$ by

$$\mathbf{C} \to \Sigma \quad x \mapsto x + h(x)\nu(x)$$

By (5.2.21), under this parametrization and the normal coordinates of $\mathbf{C}$, $L_\Sigma = a_0 \cdot \partial_i(a^{ij}\partial_j u) + |A_\Sigma|^2 u$, where

$$\sup_{\omega\in S} |a_0 - 1|(R,\omega) + |a^{ij} - \delta^{ij}|(R,\omega) + R^2\big||A_\Sigma|^2 - |A_{\mathbf{C}}|^2\big|(R,\omega) \to 0 \quad \text{as } R \to \infty\,.$$

Consider for each $R > 2$, $v_R \in C^\infty(\mathbf{C} \setminus \mathbb{B}_{2/R})$ defined by

$$v_R(r,\omega) := u(rR,\omega)/\inf_{\partial B_R} u$$

By elliptic estimate, up to subsequences, $v_R \to v_\infty$ in $C^0_{loc}(\mathbf{C})$ for some positive Jacobi fields v_∞ on $\mathbf{C}$ with $\inf_{\mathbb{S}^n\cap\mathbf{C}} v_\infty = 1$. By Lemma C.0.3,

$$v_\infty(r,\omega) \geq C(S) r^{\gamma_1^-}\,. \tag{5.2.26}$$

The lower bound of decaying rate for u then follows from (5.2.26) and an approximation argument. □

Proof of Theorem 5.2.12 (1). Let $\Sigma_+ = \partial P_+$ be the leaf of Hardt-Simon foliation in Theorem 2.4.1 lying on the side E where $\mathbf{C}$ is minimizing (resp. strictly minimizing), $P_+ \subset E$; Let h_+, h be the garphical functions of Σ_+ and V correspondingly over $\mathbf{C}$. Suppose for contradiction that $\mathcal{AR}_\infty(V) < \gamma_1^-$ (resp. γ_1^+). Then by Lemma 5.2.9, (2.4.2) and (2.4.3), $h_+ > h$ near infinity. Hence for $R >> 1$, $R\cdot P_+ \cap \mathrm{spt}(V) = \emptyset$. By considering

$$\inf\{R : s\cdot P_+ \cap \mathrm{spt}(V) = \emptyset,\ \forall s > R\}\,,$$

and using the strong maximum principle [SW89], we see that $\mathrm{spt}(V) \subset \mathbb{R}^{n+1}\setminus \mathrm{Clos}(E)$, i.e. lying on the other side of $\mathbf{C}$. Then by Lemma 5.2.13 and Theorem 2.4.1, V is a rescaling of Hardt-Simon foliation on the compliment of $\mathrm{Clos}(E)$ and thus $\mathcal{AR}_\infty(V) \geq \gamma_1^-$ (resp. $= \gamma_1^+$), from which we get a contradiction.

If $\mathcal{AR}_\infty(V) \in \{\gamma_1^\pm\}$, then by Lemma 5.2.9, $\mathrm{spt}(V)$ lies on one side of $\mathbf{C}$ near infinity. By using the Hardt-Simon foliation on the other side and strong maximum principle as above, one see that $\mathrm{spt}(V)$ lies entirely on one side of $\mathbf{C}$, hence becomes a leaf of Hardt-Simon foliation by Theorem 2.4.1. □

Proof of Theorem 5.2.12 (2). Consider $\psi := \nabla_{\mathbb{R}^{n+1}}|x|^2 \cdot \nu_\Sigma$. By Lemma 5.2.9, if $\mathcal{AR}_\infty(\Sigma) < \gamma_1^-$, then for each $\epsilon > 0$, $\exists\ R(\Sigma, \epsilon) > 1$ such that

$$|\psi(x)| \leq |x|^{\gamma_2^- + \epsilon}, \quad \forall |x| > R(\Sigma, \epsilon)\,.$$

On the other hand, let $v \in \mathcal{G}(\Sigma)$ be a positive Jacobi field on Σ. By Theorem 3.1.1, Lemma 5.2.16 and weak maximum principle, $\exists\ c > 0$ such that $cv > \psi$ on Σ. Consider the smallest one among all such c and using strong maximum principle, we conclude that $\psi = 0$ and thus $\Sigma = \mathbf{C}$, which contradicts to our assumption. This proves the lower bound on $\mathcal{AR}_\infty(\Sigma)$.

When $\mathcal{AR}_\infty(\Sigma) \in \{\gamma_1^\pm\}$, the conclusion follows from Lemma 5.2.9 and 5.2.13. □

5.2.3 A general local growth dichotomy

We close this section by the following local growth dichotomy, which plays a central role in studying induced Jacobi fields and full generic regularity in Chapter 6.

Lemma 5.2.17. *Let $\Lambda, \Lambda' > 1$, $\sigma > 0$, $\kappa > 0$; $K(\sigma, \Lambda, \Lambda') > 2$ be given by Corollary 5.2.3. Then there exists $\delta_0(\Lambda, \Lambda', \sigma, \kappa) > 0$ with the following significance.*

Let $\mathbf{C} \in \mathscr{C}^n_{\Lambda,\Lambda'}$ be a regular stable minimal hypercone; $\gamma \in (-\Lambda, 1)$ be such that

$$dist_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}) \cup \{-(n-2)/2\}) \geq \sigma.$$

Let $\{(g_j^i, V_j^i)\}_{1 \leq j \leq \infty, i=0,1}$ be a family of pairs such that

(a) g_j^i *are* C^4 *Riemannian metrics on* $\mathbb{B}_2^{n+1}$ *such that*

$$\|g_j^1 - g_j^0\|_{C^4} \to 0 \quad \text{as } j \to \infty, \qquad \|g_j^0 - g_{\text{euc}}\|_{C^4} \leq \delta_0 \text{ for every } j \geq 1\,;$$

(b) $V_j^i \in \mathcal{IV}_n(\mathbb{B}_2^{n+1})$ *are* g_j^i*-stationary such that,*

$$\mathbf{F}_{\mathbb{B}_2}(V_j^1, V_j^0) \to 0 \quad \text{as } j \to \infty; \qquad V_j^0 = |\operatorname{graph}_{\mathbf{C}, g_{\text{euc}}}(v_j^0)| \llcorner \mathbb{B}_2^{n+1}\,,$$

for some $v_j^0 \in C^4(\mathbf{C})$ *with* $\|v_j^0\|_{C^2_*, \mathbf{B}_2} \leq \delta_0$.

We further assume that $\Sigma_j := \operatorname{spt}(V_j^0)$ *is a* g_j^0*-minimal hypersurface in* $\mathbb{B}_2$ *with isolated singularity at* $\mathbf{0}$*; And let*

$$\tau_j := \inf\{t \in (0,1) : V_j^1 \llcorner \mathbb{A}(\mathbf{0}; t, 1) = |\operatorname{graph}_{\Sigma_j, g_j^0}(u_j)| \llcorner \mathbb{A}(\mathbf{0}; t, 1), \text{ with } \|u_j\|_{C^2_*} \leq \delta_0\}. \tag{5.2.27}$$

Then after passing to a subsequence, either of the following hold for $j \geq 1$*,*

(i) $\tau_j > 0$ *and there exists a linear map* $T \in \mathrm{GL}_{n+1}(\mathbb{R})$ δ_0*-close to* $id_{\mathbb{R}^{n+1}}$ *such that* $\tau_j^{-1} \cdot T_\sharp V_j^1$ $\mathbf{F}$*-converges to some* g_{euc}*-stationary* $\hat{V}_\infty \in \mathcal{IV}_n(\mathbb{R}^{n+1})$ *which doesn't equal to* $|\mathbf{C}|$ *and asymptotic to* $\mathbf{C}$ *near infinity, with asymptotic rate* $\mathcal{AR}_\infty(\hat{V}_\infty, \mathbf{C}) < \gamma$*. We call such* $\hat{V}_\infty$ *a* ***minimal blow-up limit*** *of* $\{V_j^1\}_{j \geq 1}$*;*

(ii) For every $t \in (2K^3\tau_j, 1)$*, we have*

$$max\{J_K^\gamma(u_j; K^{-1}t), \kappa\|g_j^1 - g_j^0\|_{C^4}\} \leq max\{J_K^\gamma(u_j; t), \kappa\|g_j^1 - g_j^0\|_{C^4}\}\,,$$

where u_j *is the graphical function in (5.2.27),* $J_K^\gamma = J^\gamma_{K;\Sigma_j^0, g_j^0}$ *is defined by*

$$J^\gamma_{K;\Sigma,g}(u;r)^2 := \int_{A^g(\mathbf{0}, K^{-1}r, r)} u^2(x) \cdot |dist_g(\mathbf{0}, x)|^{-2\gamma - n}\, d\|\Sigma\|_g(x)\,. \tag{5.2.28}$$

Note that by its definition we have,

$$C(K,\Lambda)^{-1} \leq \frac{J^{\gamma}_{K;\Sigma,g}(u;r)r^{\gamma+n/2}}{\|u\|_{L^2(A^g(\mathbf{0},K^{-1}r,r);\|\Sigma\|_g)}} \leq C(K,\Lambda). \tag{5.2.29}$$

Thus, Lemma 5.2.17 (ii) provides a growth rate bound between different scales.

Proof of Lemma 5.2.17. By composing with a linear map δ_0-close to $id_{\mathbb{R}^{n+1}}$, we can assume that

$$\|g^0_j|_{\mathbf{0}} - g_{\text{euc}}\| \to 0, \qquad \limsup_{r\searrow 0}\limsup_{j\to\infty}\|v^0_j\|_{C^2_*,\mathbf{B}_r} = 0\,.$$

First note that by Allard Regularity Theorem [All72], $\tau_j \to 0$ as $j \to \infty$. We parametrize Σ^0_j graphically by $\mathbf{C}$ such that $dist_{\Sigma^0_j,g^0_j}(\mathbf{0},\cdot) = dist_{\mathbf{C}}(\mathbf{0},\cdot)$; And we view the graphical function u_j given by (5.2.27) as a function on $\mathbf{C}$ by this parametrization. Let $\varepsilon = \varepsilon(\sigma,\Lambda,\Lambda')$ be given by Corollary 5.2.3. Then by Theorem A.0.1, we can choose $\delta_0(\sigma,\Lambda,\Lambda',\kappa) << 1$ such that

$$(1-\varepsilon/10) \leq \frac{J^{\gamma}_{K;\Sigma^0_j,g^0_j}(v;r)}{J^{\gamma}_{K;\mathbf{C},g_{\text{euc}}}(v;r)} \leq (1+\varepsilon/10), \quad \forall v \in L^2_{loc}(\mathbf{B}_1) \text{ and } r \in (0,1)\,. \tag{5.2.30}$$

and that every rescaling $u^{(r)}_j(x) := r^{-1}u_j(rx)$, $r \in (K^3\tau_j, 1)$ satisfies equation on $\mathbf{A}(\mathbf{0}; 2\tau_j/r, 1)$ of the following form

$$div_{\mathbf{C}}(\nabla_{\mathbf{C}}u^{(r)}_j + \vec{b}_0) + |A_{\mathbf{C}}|^2u^{(r)}_j + b_1 = 0\,, \tag{5.2.31}$$

$$|\vec{b}_0(x)| + |b_1(x)| \leq \varepsilon\left(|x|^{-1}|u^{(r)}_j| + |\nabla u^{(r)}_j| + C(K,\Lambda)\cdot\kappa\|g^1_j - g^0_j\|_{C^4(\mathbb{B}_2)}\right)\,. \tag{5.2.32}$$

Here $C(K,\Lambda)$ is given by (5.2.29).

Now suppose (ii) of Lemma 5.2.17 fails for any subsequence, namely for $j >> 1$,

there exists $t_j \in (2K^3\tau_j, 1)$ such that

$$J^{\gamma}_{K;\Sigma^0_j,g^0_j}(u_j; K^{-1}t_j) > max\{J^{\gamma}_{K;\Sigma^0_j,g^0_j}(u_j; t_j), \kappa\|g^1_j - g^0_j\|_{C^4(\mathbb{B}_2)}\}\,.$$

This means $\hat{u}_j := u_j^{(t_j)}$ satisfies

$$J^{\gamma}_{K;\mathbf{C},g_{\mathrm{euc}}}(\hat{u}_j; K^{-1}) > (1+\varepsilon)^{-1}max\{J^{\gamma}_{K;\mathbf{C},g_{\mathrm{euc}}}(\hat{u}_j; 1), t_j^{\gamma-1}\kappa\|g^1_j - g^0_j\|_{C^4(\mathbb{B}_2)}\}\,,$$

Hence by combining (5.2.29) and (5.2.32) we see that $\hat{u}_j$ satisfies (5.2.31) with pointwise error estimate,

$$|\vec{b}_0(x)| + |b_1(x)| \leq \varepsilon\left(|x|^{-1}|\hat{u}_j| + |\nabla\hat{u}_j| + \|\hat{u}_j\|_{L^2(\mathbf{A}(\mathbf{0};K^{-2},K^{-1}))}\right)\,.$$

Therefore, Corollary 5.2.3 applies to imply that

$$J^{\gamma}_{K;\mathbf{C},g_{\mathrm{euc}}}(\hat{u}_j; K^{-2}) > (1+\varepsilon)J^{\gamma}_{K;\mathbf{C},g_{\mathrm{euc}}}(\hat{u}_j; K^{-1})\,.$$

Repeat the argument above inductively for $u_j^{(t_j\cdot K^{-l})}$, $0 \leq l < \log_K(t_j/\tau_j) - 3$, and by combining (5.2.30) we deduce that

$$l \mapsto J^{\gamma}_K(u_j; t_jK^{-l}) \quad \text{is increasing in} \;\; 0 \leq l < \log_K(t_j/\tau_j) - 2\,. \tag{5.2.33}$$

Since $\gamma < 1$, by (5.2.29) and definition of τ_j, we see that $\tau_j > 0$; By Allard Compactness Theorem, $\tau_j^{-1}\cdot V^1_j$ will $\mathbf{F}$-subconverges to some g_{euc}-stationary $\hat{V}_\infty \in \mathcal{IV}_n(\mathbb{R}^{n+1})$ which is graphical over $\mathbf{C}$ outside $\mathbb{B}^{n+1}_2$ and not equal to $|\mathbf{C}|$; And by (5.2.33), the definition of asymptotic rate and Remark 5.2.8, we know that $\mathcal{AR}_\infty(\hat{V}_\infty; \mathbf{C}) < \gamma$. This means (i) holds in Lemma 5.2.17. □

Here's a direct Corollary of this growth dichotomy. For more applications, see Section 6.7.

Corollary 5.2.18. *Suppose $\Sigma \subset (\mathbb{B}_2^{n+1}, g)$ is a LSMH with $\mathbf{0} \in \mathrm{Sing}(\Sigma)$ such that the tangent cone $\mathbf{C_0}$ is strictly minimizing. Let $\{V_j\}_{j\geq 1}$ be a family of stationary integral varifold in $(\mathbb{B}_2, g)$ which converges in varifold sense to $|\Sigma|\llcorner(\mathbb{B}_2, g)$.*

Let $\sigma \in (-\infty, \gamma_1^+) \setminus (\Gamma(\mathbf{C_0}) \cup \{-(n-2)/2\})$ be fixed; Then there exist constants $K(\mathbf{C_0}, \sigma) > 2$, $\delta_1(\mathbf{C_0}, \sigma) \in (0, 1)$ and $s_0(\Sigma, \mathbf{C_0}, \sigma) \in (0, 1/2)$ such that the following holds.

Suppose $\|g - g_{\mathrm{euc}}\|_{C^3} \leq \delta_1$; $r_j \searrow 0_+$ and $\|v_j\|^*_{2,\mathcal{A}_{r_j,1}} \leq \delta_1$ is the graphical function of V_j over Σ, i.e. $V_j \llcorner A^g_{r_j,1} = |\operatorname{graph}_\Sigma(v_j)| \llcorner A^g_{r_j,1}$. Let ρ be the distant function on Σ to $\mathbf{0}$. Then for $j >> 1$, the following pointwise bound holds on $\mathcal{A}_{Kr_j, s_0/K}$,

$$|v_j|/\rho + |\nabla v_j| \leq C(\sigma, \mathbf{C_0}, \Sigma)\rho^{\sigma-1} \cdot \left(\int_{\mathcal{A}_{s_0/K, s_0}} v_j^2 \right)^{1/2}.$$

Proof. Lemma 5.2.17 applies by first rescaling Σ. And by Theorem 5.2.12, there's no non-conic g_{euc}-stationary integral varifold with asymptotic rate $< \gamma_1^+$ to a strictly minimizing hypercone. Hence the dichotomy (ii) happens in Lemma 5.2.17, allowing us to get an L^2 bound of v_j on each $\mathcal{A}_{r,Kr}$ in terms of $\int_{\mathcal{A}_{s_0/K,s_0}} v_j^2$; Then since v_j solves the minimal surface equation, classical interior Schauder estimate [GT01] yields the pointwise C^1 bound in the Corollary. □

5.3 Spectral Analysis on LSMH

Throughout this Section, let $\Sigma \subset (M, g)$ be a two-sided locally stable minimal hypersurface (LSMH) in a Riemannian manifold with a unit normal field ν; $U \subset\subset M$ will always be a smooth domain with ∂U transversely intersect with $\mathrm{Reg}(\Sigma)$; $\mathcal{U} := U \cap \Sigma$ and $\bar{\mathcal{U}} := \mathrm{Clos}(U) \cap \Sigma$. Let L_Σ, Q_Σ be the Jacobi operator and its associated quadratic form as in (2.3.2), (2.3.1), where we omit subscript g to save notation. For our ap-

plications below, we may also deal with the weighted quadratic forms

$$Q^h_\Sigma(\phi,\phi) := Q_\Sigma(\phi,\phi) + \int_\Sigma h\phi^2\,,$$

for a given $h \in L^\infty(\Sigma)$, and its associated operator $L_\Sigma - h$.

5.3.1 Linear analysis in large

Note that we do not require $\mathrm{Sing}(\Sigma)$ to be isolated in this subsection. We start with a basic observation,

Lemma 5.3.1. *There's a constant $C_0 = C_0(\Sigma, M, g, U) > 0$ such that $\forall \phi \in C^1_c(\mathcal{U})$,*

$$Q_\Sigma(\phi,\phi) + C_0\|\phi\|^2_{L^2(\Sigma)} \geq 0\,.$$

Proof. Let $\{U_j \subset M\}$ be a finite open cover of $\mathrm{Clos}(\mathcal{U})$ such that Σ is stable in each U_j; Let $\{\eta_j^2\}$ be a partition of unity on $\{U_j\}$ such that η_j is also smooth. Then $\forall \phi \in C^1_c(U \cap \Sigma)$, since $\eta_j \cdot \phi$ is support in U_j,

$$\begin{aligned}
Q_\Sigma(\phi,\phi) &= \int_\Sigma \sum_j |\nabla\phi|^2\eta_j^2 - \sum_j (|A_\Sigma|^2 + Ric_M(\nu,\nu))\phi^2\eta_j^2 \\
&= \sum_j Q_\Sigma(\phi\eta_j, \phi\eta_j) - \sum_j \int_\Sigma \left(\nabla(\phi^2)\cdot\eta_j\nabla\eta_j + \phi^2|\nabla\eta_j|^2\right) \\
&= \sum_j Q_\Sigma(\phi\eta_j, \phi\eta_j) + \sum_j \int_\Sigma \phi^2\left(div(\eta_j\nabla\eta_j) - |\nabla\eta_j|^2\right) \\
&\geq \int_\Sigma \left(\sum_j \eta_j\cdot\Delta\eta_j\right)\phi^2\,.
\end{aligned} \tag{5.3.1}$$

Since Σ is minimal,

$$|\Delta\eta_j| = |\Delta_M\eta_j - \nabla^2_M\eta_j(\nu,\nu)| \leq n|\nabla^2_M\eta_j|\,.$$

Hence, $\sum_j \eta_j \cdot \Delta\eta_j$ is bounded from below by some constant $-C_0 = -C_0(M, g, \Sigma, U)$. This proves the Lemma. □

Such C_0 in the Lemma will be fixed throughout this Section. Consider the Hilbert space given by metric completion $\mathscr{B}(\mathcal{U}) := \overline{C_c^1(\bar{\mathcal{U}})}^{\|\cdot\|_{\mathscr{B}}}$ and $\mathscr{B}_0(\mathcal{U}) := \overline{C_c^1(\mathcal{U})}^{\|\cdot\|_{\mathscr{B}}}$ where

$$\|\phi\|^2_{\mathscr{B}(\mathcal{U})} := Q_\Sigma(\phi, \phi) + (C_0 + 1)\|\phi\|^2_{L^2(\mathcal{U})}\,. \tag{5.3.2}$$

Note that by definition and Lemma 5.3.1,

$$-C_0\|\phi\|^2_{\mathscr{B}(\mathcal{U})} \leq Q_\Sigma(\phi, \phi) \leq \|\phi\|^2_{\mathscr{B}(\mathcal{U})}\,, \quad \forall \phi \in C_c^1(\bar{\mathcal{U}})\,.$$

Hence, the quadratic form Q_Σ extends to a bilinear form on $\mathscr{B}(\mathcal{U})$.

$\mathscr{B}(\mathcal{U})$ (resp. $\mathscr{B}_0(\mathcal{U})$) will play the similar role as $W^{1,2}(\mathcal{U})$ (resp. $W_0^{1,2}(\mathcal{U})$) for inverting L_Σ or finding eigenfunctions. In fact, $\mathscr{B}_0(\mathcal{U}) = W_0^{1,2}(\mathcal{U})$ provided $L = L_\Sigma$ satisfies the coercivity condition

$$\int_\Sigma |\nabla\phi|^2 \leq \delta^{-1}\Big(\int_\Sigma -\phi \cdot L\phi + \int_\Sigma \phi^2\Big), \quad \forall\ \phi \in C_c^2(\Sigma \cap U)\,. \tag{5.3.3}$$

for some $\delta > 0$. In particular, this is true when every singularity is strongly isolated and strictly stable. However, when Σ has non-strictly stable tangent cones, this will always fail, and $W_0^{1,2} \subsetneq \mathscr{B}_0$. See the following Example 5.3.3.

On the other hand, Lemma 5.3.1 asserts that $\|\phi\|_{\mathscr{B}} \geq \|\phi\|_{L^2}$, hence there's a natural continuous embedding $\mathscr{B}(\mathcal{U}) \hookrightarrow L^2(\mathcal{U})$. It's convenient to have the following characterization for $\phi \in L^2(\mathcal{U})$ to be in $\mathscr{B}(\mathcal{U})$.

Lemma 5.3.2. *Suppose $\phi \in L^2(\mathcal{U})$. Then the following are equivalent,*

(1) $\phi \in \mathscr{B}(\mathcal{U})$ (resp. $\mathscr{B}_0(\mathcal{U})$);

(2) $\phi \in W^{1,2}_{loc}(\mathcal{U})$; And for any decreasing family $\ldots V_j \supset\supset V_{j+1} \ldots$ of neighborhood

of $\mathrm{Sing}(\Sigma)$ *with* $\bigcap_j V_j = \mathrm{Sing}(\Sigma)$*, there are* $\phi_j \in W^{1,2}(\mathcal{U})$ *(resp.* $W_0^{1,2}(\mathcal{U})$*) such that* $\mathrm{spt}(\phi_j - \phi) \subset V_j$*,* $\mathrm{spt}(\phi_j) \cap V_{N_j} = \emptyset$ *for some* $N_j >> j$*, and* $\sup_j \|\phi_j\|_{\mathscr{B}} < +\infty$*.*

(3) $\exists \phi_j \in \mathscr{B}(\mathcal{U}) \cap L^2$ *(resp.* $\mathscr{B}_0(\mathcal{U}) \cap L^2$*) such that* $\phi_j \to \phi$ *a.e. and* $\limsup \|\phi_j\|_{\mathscr{B}} < +\infty$*.*

Proof. Clearly (2) implies (3); To see (3) implies (1), first observe that by weak compactness of Hilbert space and Banach-Saks Theorem, if ϕ_j is uniformly bounded in $\mathscr{B}$, then there's a subsequence ϕ_{k_j} such that $(\sum_{j=1}^m \phi_{k_j})/m \to \phi'$ in $\mathscr{B}(\mathcal{U})$ (resp. in $\mathscr{B}_0(\mathcal{U})$) as $m \to \infty$. Since $\mathscr{B} \hookrightarrow L^2$ and that $\phi_j \to \phi$ a.e., we have $\phi = \phi' \in \mathscr{B}(\mathcal{U})$ (resp. $\phi \in \mathscr{B}_0(\mathcal{U})$).

Now we shall prove (1) implies (2) beginning with $\phi \in \mathscr{B}_0(\Sigma)$; the proof when $\phi \in \mathscr{B}(\Sigma)$ is similar.

Claim. If $\Omega \subset\subset \mathcal{U}$, then for every $\varphi \in C_c^1(\mathcal{U})$,

$$\|\varphi\|_{W^{1,2}(\Omega)} \leq C(\Omega, \mathcal{U}, \Sigma, M, g)\|\varphi\|_{\mathscr{B}(\mathcal{U})}\,.$$

Proof of the Claim. Take a finite open cover $\{W_l\}_{l\geq 0}$ of $\mathrm{Clos}(\mathcal{U})$ in M such that $\mathrm{Clos}(\Omega) \subset W_0 \subset\subset M \setminus \mathrm{Sing}(\Sigma)$, and that $W_l \cap \Omega = \emptyset$ and Σ is stable in W_l for $l \geq 1$. Let $\{\eta_l^2\}_{l\geq 0}$ be a partition of unity for $\{W_l\}$ such that η_l are also smooth. By the same computation in (5.3.1) we have,

$$\begin{aligned} Q_\Sigma(\varphi,\varphi) &= Q_\Sigma(\eta_0\varphi, \eta_0\varphi) + \sum_{l\geq 1} Q_\Sigma(\eta_l\varphi, \eta_l\varphi) + \sum_{l\geq 0}\int \varphi^2(\eta_l \Delta_\Sigma \eta_l) \\ &\geq Q_\Sigma(\eta_0\varphi, \eta_0\varphi) - C(\{\eta_l\})\|\varphi\|_{L^2}^2 \geq \int_\Omega |\nabla\varphi|^2 - C(\Omega, \{\eta_l\})\|\varphi\|_{L^2}^2\,. \end{aligned}$$

Together with the fact that $\|\varphi\|_{L^2} \leq \|\varphi\|_{\mathscr{B}}$, this completes the proof of Claim. □

Now to prove (1) implies (2), let $\varphi_k \in C_c^1(\mathcal{U})$ be a Cauchy sequence in $\mathscr{B}_0(\mathcal{U})$

which converges to ϕ. Then by the claim, $\phi \in W^{1,2}_{loc}(\mathcal{U})$.

For every nested neighborhood $\{V_j\}_{j\geq 1}$ in (2), let $\xi_j \in C^\infty_c(V_j, [0,1])$ be a cut-off which equals to 1 on V_{j+1}. For each fixed j, let $\psi_k := \psi^{(j)}_k := (1-\xi_j)\phi + \xi_j\varphi_k$. Then by the similar computation as in (5.3.1) we have,

$$\begin{aligned}
&Q_\Sigma(\psi_k - \psi_{k'}, \psi_k - \psi_{k'}) \\
=& \int_{\mathcal{U}} -\xi_j \Delta \xi_j (\varphi_k - \varphi_{k'})^2 + \xi_j^2 \left(|\nabla(\varphi_k - \varphi_{k'})|^2 - (|A_\Sigma|^2 + Ric_M(\nu,\nu))(\varphi_k - \varphi_{k'})^2 \right) \\
\leq& C(\xi_j)\|\varphi_k - \varphi_{k'}\|^2_{L^2} + Q_\Sigma(\varphi_k - \varphi_{k'}, \varphi_k - \varphi_{k'}) \\
&+ \int_{\mathcal{U}\setminus V_{j+1}} |\nabla(\varphi_k - \varphi_{k'})|^2 + \left||A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu)\right| (\varphi_k - \varphi_{k'})^2 \\
\leq& C(\xi_j, V_j, V_{j+1})\|\varphi_k - \varphi_{k'}\|^2_{\mathscr{B}}\,,
\end{aligned}$$

where the last inequality follows form the Claim. Hence $\{\psi_k\}_{k\geq 1}$ is also a Cauchy sequence in $\mathscr{B}_0(\mathcal{U})$, which still converges to ϕ. Thus we can choose $k_j >> j$ such that $\|\psi^{(j)}_{k_j} - \phi\|_{\mathscr{B}} \leq 2^{-j}$, and $N_j >> j$ such that $\mathrm{spt}(\psi^j_{k_j}) \cap V_{N_j} = \emptyset$. Thus $\phi_j := \psi^{(j)}_{k_j}$ is what we want in (2). □

The following example shows that for non-strictly stable singularity model, $\mathscr{B}_0 \neq W^{1,2}_0$. It also suggests a unified behavior of the spectral theory for non-strictly stable and strictly stable minimal cones.

Example 5.3.3. *Let $\Sigma = \mathbf{C} \subset \mathbb{R}^{n+1}$ be a regular stable minimal hypercone which is not strictly stable, i.e. $\mu_1 = -(\frac{n-2}{2})^2$; Let $U = \mathbb{B}^{n+1}_1$ and $\mathcal{U} = \mathbf{B}_1 \subset \mathbf{C}$. Use the notations and parametrization as in (5.2.3). Let $u(r,\omega) := r^{-(n-2)/2} w_1(\omega)$. Then*

$$\|\nabla u\|^2_{L^2(\mathbf{B}_1(0))} = \int_0^1 r^{n-1}\, dr \int_S |\partial_r u|^2 + \frac{1}{r^2}|\nabla_S u|^2\, d\omega = +\infty\,.$$

But we can construct

$$u_j(r,\omega) := max\{r, \frac{1}{j}\}^{-(n-2)/2} \cdot w_1(\omega)\,.$$

Clearly $u_j \in W^{1,2}(\mathcal{U}) \subset \mathscr{B}(\mathcal{U})$ *and* $u_j \to u$ *pointwisely. We verify now that* $\sup_j \|u_j\|_{\mathscr{B}} < +\infty$. *In fact, notice that* $L_{\mathbf{C}}u = 0$ *on* $\mathbf{B}_1 \setminus \mathbf{B}_{1/j}$, *thus,*

$$\begin{aligned}
Q_{\mathbf{C}}(u_j,u_j) &= \int_{\mathbf{B}_1\setminus\mathbf{B}_{1/j}} |\nabla u|^2 - |A_C|^2u^2 + \int_{\mathbf{B}_{1/j}} |\nabla u_j|^2 - |A_C|^2u_j^2 \\
&= \int_{\partial\mathbf{B}_1} u\partial_r u - \int_{\partial\mathbf{B}_{1/j}} u\partial_r u - \int_{\mathbf{A}(1/j,1)} u\cdot L_C u \\
&\quad + \int_0^{1/j} r^{n-1}\,dr \int_S r^{-2}(|\nabla_S u_j|^2 - |A_S|^2u_j^2)\,d\omega \\
&= \int_{\partial\mathbf{B}_1} (-\frac{n-2}{2})\cdot r^{-n+1}w_1^2 - \int_{\partial\mathbf{B}_{1/j}} (-\frac{n-2}{2})\cdot r^{-n+1}w_1^2 \quad -0 \\
&\quad + \int_0^{1/j} r^{n-3}\,dr \int_S j^{n-2}\cdot(|\nabla_S w_1|^2 - |A_S|^2w_1^2)\,d\omega \\
&\le C(n)\,.
\end{aligned}$$

And $\|u_j\|_{L^2}^2 \le \|u\|_{L^2}^2 \le C(n)$. *Therefore, by Lemma 5.3.2,* $u \in \mathscr{B}(\mathcal{U})$.

On the other hand, let $v = -r^{-(n-2)/2}\log r \cdot w_1$ *on* B_1, *then* $v \in W^{1,2}_{loc}(\mathbf{C})$ *and* v *solves the equation* $L_{\mathbf{C}}v = 0$ *on* B_1. *However, we shall see that that* $v \notin \mathscr{B}(\mathcal{U})$. *In fact, by Lemma 5.3.2, consider* $\forall 0 < s < t < 1$, *the function* $v_{s,t}$ *that achieves,*

$$\inf\{Q_{\mathbf{C}}(\phi,\phi) : \phi \in C_c^1(\mathbf{C}),\ \phi = v \text{ on } \mathbf{C}\setminus\mathbf{B}_t,\ \mathrm{spt}(\phi)\cap\mathbf{B}_s = \emptyset\}\,.$$

By writing down the Euler-Lagrangian equation and solving an ODE, we have

$$v_{s,t}(r,\omega)=\begin{cases} v(r,\omega), & \text{if } t\le r\le 1\,;\\ \left(\dfrac{\log t\cdot\log s}{\log t-\log s}-\dfrac{\log t}{\log t-\log s}\log r\right)\cdot r^{-\frac{n-2}{2}}w_1(\omega), & \text{if } s\le r\le t\,;\\ 0, & \text{if } 0<r<s\,. \end{cases}$$

Therefore,

$$Q_{\mathbf{C}}(v_{s,t},v_{s,t})=(-\log t)+\frac{(-\log t)^2}{\log t-\log s}\ge(-\log t)\,.$$

And then $\|v_{s,t}\|_{\mathscr{B}}\to+\infty$ *when* $t\to 0$. *Hence by Lemma 5.3.2,* $v\notin\mathscr{B}(\mathcal{U})$.

As a final remark, by comparing solutions to $L_{\mathbf{C}}u=0$ *on* $\mathbf{B}_1$ *with harmonic functions on* $\mathbb{B}^n_1\setminus\{0\}$, *the calculation above suggests that we should view* $r^{-(n-2)/2}\cdot w_1$ *as a solution with finite energy, and view* $r^{-(n-2)/2}\log r\cdot w_1$ *as a Green's function.*

To study the spetral theory of L_Σ, we may wish that $\mathscr{B}\hookrightarrow L^2$ is also compact.

Definition 5.3.4. We call Σ satisfy the **L^2-nonconcentration property** in U, if for any $\epsilon>0$, there's a sufficiently small neighborhood $V_\epsilon\supset\mathrm{Sing}(\Sigma)\cap U$ such that

$$\int_{\Sigma\cap V_\epsilon}\phi^2\le\epsilon\cdot\|\phi\|^2_{\mathscr{B}},\qquad\forall\phi\in C^1_c(\mathcal{U})\,.\tag{5.3.4}$$

Proposition 5.3.5. *Suppose* Σ *satisfies the* L^2*-nonconcentration property in* U; $h\in L^\infty(\mathcal{U})$. *Then,*

(1) $\mathscr{B}_0(\mathcal{U})\hookrightarrow L^2(\mathcal{U})$ *is compact.*

(2) $\exists\ \lambda_1<\lambda_2<\lambda_3<\ldots\nearrow+\infty$ *and finite dimensional pairwise* L^2*-orthogonal linear subspaces* $\{E_j\}_{j\ge1}$ *of* $\mathscr{B}_0(\mathcal{U})\cap C^\infty(\mathcal{U})$ *such that*

$$(-L_\Sigma+h)\phi=\lambda_j\phi,\qquad\forall\phi\in E_j\,;$$

and that

$$L^2(\mathcal{U}) = \overline{\bigoplus_{j\geq 1} E_j}^{L^2} \; ; \qquad \mathscr{B}_0(\mathcal{U}) = \overline{\bigoplus_{j\geq 1} E_j}^{\mathscr{B}} \; .$$

Moreover, $\dim E_1 = 1$ *and any* $0 \neq \phi_1 \in E_1$ *is nowhere vanishing in* $\mathcal{U}$*.*

(3) If $(L_\Sigma - h)\big|_{\mathscr{B}_0}$ *is* ***nondegenerate****, i.e.* $\lambda_j \neq 0$ *for all* $j \geq 1$ *in (2). Then for each* $f \in L^2(\Sigma)$*,*

$$(-L_\Sigma + h)\phi = f, \qquad \text{on } \mathcal{U} \, ,$$

has a unique solution $\phi \in \mathscr{B}_0(\mathcal{U})$*. Moreover, we have energy-type estimate*

$$\|\phi\|_{\mathscr{B}(\mathcal{U})} \leq C(h, \Sigma, U, M, g)\|f\|_{L^2} \, .$$

(4) Suppose $-L_\Sigma + h$ *is* ***strictly positive****, i.e.* $\lambda_1 > 0$ *in (2). Suppose* $u \in \mathscr{B}(\mathcal{U})$ *satisfies* $(-L_\Sigma + h)u \geq 0$ *in the distribution sense, i.e.* $\forall \phi \in C_c^\infty(\mathcal{U}; \mathbb{R}_+)$*,*

$$\int_{\mathcal{U}} \nabla u \cdot \nabla \phi - (|A_\Sigma|^2 + \mathrm{Ric}_M(\nu, \nu) - h)u \cdot \phi \geq 0 \, .$$

Also assume that either $\partial U \cap \mathrm{Sing}(\Sigma) = \emptyset$ *and* $u|_{\partial \mathcal{U}} \geq 0$*, or* $u \in \mathscr{B}_0(\mathcal{U})$*. Then* $u \geq 0$ *on* $\mathcal{U}$*.*

(5) There exists $\vartheta \in C_c^\infty(\bar{\mathcal{U}})$ *such that* $-L_\Sigma + h + \vartheta$ *is strictly positive on* $\mathscr{B}_0(\mathcal{U})$*.*

(6) $\{f \in C_c^\infty(\mathcal{U}) : (-L_\Sigma + h + f)|_{\mathscr{B}_0}$ *is nondegenerate*$\}$ *is open and dense in* $C_c^\infty(\mathcal{U})$*.*

Remark 5.3.6. Proposition 5.3.5 (4) is usually called **weak maximum principle** for $-L_\Sigma + h$. The techinical assumption on boundary value of u, which is used to guarantee that $u^- \in \mathscr{B}_0(\mathcal{U})$, is conjectured to be simply replaced by $u|_{\partial \mathcal{U}} \geq 0$.

Proposition 5.3.5 (6) shows that $-L_\Sigma + h$ being non-degenerate is a generic property.

Proof of Proposition 5.3.5. (1) Suppose $\{\phi_k\}$ is a bounded sequence in $\mathscr{B}_0(\mathcal{U})$, WLOG

$\|\phi_k\|_{\mathscr{B}} \leq 1$. Then by the proof of Lemma 5.3.2, for every subdomain $\Omega \subset\subset \Sigma$, $\|\phi_k\|_{W^{1,2}} \leq C(\Sigma, M, g, \Omega)$. By Reilly Compactness Theorem [GT01] and Lemma 5.3.2, up to a subsequence of $k \to \infty$, $\phi_k \to \phi$ in $L^2(\Omega)$ for some $\phi \in \mathscr{B}_0(\mathcal{U})$, $\forall \Omega \subset\subset \Sigma$. We shall see that $\|\phi_k - \phi\|_{L^2} \to 0$, which proves the compactness. In fact, $\forall \epsilon > 0$, let V_ϵ be the neighborhood of $\mathrm{Sing}(\Sigma)$ in Definition 5.3.4, then

$$\begin{aligned}\limsup_{k\to\infty} \int_\Sigma (\phi_k - \phi)^2 &\leq \limsup_{k\to\infty} \int_{V_\epsilon} (\phi_k - \phi)^2 + \lim_{k\to\infty} \int_{V_\epsilon^c} (\phi_k - \phi)^2 \\ &\leq 4\epsilon \|\phi_k\|_{\mathscr{B}}^2 + 0 \leq 4\epsilon\,.\end{aligned}$$

Let $\epsilon \to 0$, we see that $\phi_k \to \phi$ in $L^2(\Sigma)$.

(2) and (3) follows directly from (1) and the standard method in spectral theory. We omit the proof here.

To prove (4), notice that in either case, there exists $u_j \in C^1(\bar{\mathcal{U}})$ converges to u in $\mathscr{B}(\mathcal{U})$ such that $u_j^- := (|u_j| - u_j)/2 \in W_0^{1,2}(\mathcal{U})$. And since $u_j^- \to u^-$ a.e. and $\|u_j^-\|_{\mathscr{B}} \leq \|u_j\|_{\mathscr{B}}$, by Lemma 5.3.2, u_j^- weakly converges to u^- in $\mathscr{B}_0(\mathcal{U})$ as $j \to \infty$. Using u^- as a text function we have,

$$\begin{aligned}0 &\leq Q_\Sigma(u, u^-) + \int_{\mathcal{U}} h \cdot u u^- \\ &= \lim_{j\to\infty} Q_\Sigma(u_j, u_j^-) + \int_{\mathcal{U}} h \cdot u_j u_j^- \\ &= \lim_{j\to\infty} -Q_\Sigma(u_j^-, u_j^-) - \int_{\mathcal{U}} h \cdot (u_j^-)^2 \quad \leq -\lambda_1 \cdot \liminf_{j\to\infty} \|u_j^-\|_{L^2(\mathcal{U})}.\end{aligned}$$

This means $u^- = 0$, i.e. $u \geq 0$.

To prove (5), first observe that since Σ satisfies the L^2-nonconcentration property in U, there exists a small neighborhood $\mathcal{V}_0 \supset \mathrm{Sing}(\Sigma)$ such that $\forall \phi \in C_c^1(\mathcal{U} \cap \mathcal{V}_0)$,

$$Q_\Sigma^h(\phi, \phi) := Q_\Sigma(\phi, \phi) + \int_\Sigma h\phi^2 \geq 0\,.$$

Let $\mathcal{V}_1 \subset\subset M\backslash \mathrm{Sing}(\Sigma)$ be an open subset such that $\mathcal{V}_0 \cup \mathcal{V}_1 \supset \mathrm{Clos}(U)$. Let $\{\eta_0^2, \eta_1^2\}$ be the partition of unity corresponding to $\mathcal{V}_0, \mathcal{V}_1$. Let $\beta(> -\infty)$ be the first eigenvalue of $-L_\Sigma + h$ on $\mathscr{B}_0(\mathcal{V}_1 \cap \Sigma)$. Then by the same argument as (5.3.1), for each $\phi \in C_c^\infty(\mathcal{U})$,

$$\begin{aligned} Q_\Sigma^h(\phi,\phi) &= Q_\Sigma^h(\phi\eta_0, \phi\eta_0) + Q_\Sigma^h(\phi\eta_1, \phi\eta_1) + \int_\Sigma \phi^2(\eta_0 \Delta \eta_0 + \eta_1 \Delta \eta_1) \\ &\geq \int_{\mathcal{V}_1} \phi^2(\beta\eta_1^2 + \eta_0\Delta\eta_0 + \eta_1\Delta\eta_1)\,. \end{aligned} \tag{5.3.5}$$

Hence take $\vartheta \in C_c^\infty(\bar{\mathcal{U}}; \mathbb{R}_+)$ such that $\vartheta \geq 1 + |\beta + \eta_0\Delta\eta_0 + \eta_1\Delta\eta_1|$ on $\mathcal{V}_1$, we see from (5.3.5) that $-L_\Sigma + h + \vartheta$ is strictly positive on $\mathscr{B}_0(\mathcal{U})$.

To prove (6), first note that openness follows directly from (2); To see denseness, denote for simplicity $L^h := L_\Sigma - h$ and $\mathcal{K} := \mathrm{Ker}(-L^h) \subset \mathscr{B}_0(\mathcal{U})$. By arbitrariness of h, it suffices to show that if $\mathcal{K} \neq 0$, then there exists $f \in C_c^\infty(\mathcal{U})$ such that $-L^h + tf$ is non-degenerate on $\mathscr{B}_0(\mathcal{U})$ for sufficiently small $|t|$.

Choose f by the following approach. Let $\{\mathcal{U}_l \subset\subset \mathcal{U}\}_{l\geq 1}$ be an increasing exhaustion of $\mathcal{U}$, and $f_l \in C_c^\infty(\mathcal{U}; [0,1])$ be a family of cut-off which restricts to 1 on $\mathcal{U}_l$. Clearly, $f_l \to 1$ on $\mathcal{U}$ weakly in L^∞, hence for sufficiently large l, the bilinear form on $\mathcal{K}\times\mathcal{K}$, $(\phi_1,\phi_2) \mapsto \int_{\mathcal{U}} f_l\phi_1\phi_2\, dx$ is nondegenerate. Choose $f = f_l$ for such l.

To show $-L^h + tf$ is non-degenerate for $|t| << 1$, suppose for contradiction that $\exists\, t_i \to 0$ and $u_i \in \mathrm{Ker}(-L^h + t_i f) \cap \mathscr{B}_0(\mathcal{U})$ with $\|u_i\|_{L^2} = 1$. Let v_i be the L^2-orthogonal projection of u_i onto $\mathcal{K}$; And let $\{E_j\}_{j\geq 1}$ be the L^2-orthogonal eigensubspaces of $-L^h$ given in (2), with corresponding eigenvalues $\lambda_1 < \lambda_2 < \ldots \nearrow +\infty$. Multiply the equation $L^h(u_i - v_i) = t_i f u_i$ with $\phi \in E_j$, integrate by parts and take sum over j we see that

$$\|u_i - v_i\|_{L^2} \leq \lambda^{-1}|t_i|\|f\|_{L^\infty}$$

where $\lambda := \inf\{|\lambda_j| : \lambda_j \neq 0\} > 0$. Let $\hat{u}_i := (u_i - v_i)/t_i$, since $\hat{u}_i$ satisfies the equation $L^h\hat{u}_i = fu_i$ and the L^2 bound above, by integration by part we have $\limsup_i \|\hat{u}_i\|_{\mathscr{B}(\mathcal{U})} \leq C < +\infty$; Also, since $v_i \in \mathcal{K}$ and $\|v_i\|_{L^2} \leq 1$, we have $\limsup_i \|v_i\|_{\mathscr{B}} <$

$+\infty$.

By compactness assumption for $\mathscr{B}_0(\mathcal{U}) \hookrightarrow L^2$ we conclude that up to a subsequence, $\hat{u}_i \to \hat{u}_\infty$, $v_i \to v_\infty$ both in $L^2(\mathcal{U})$; And they satisfy $\|v_\infty\|_{L^2} = 1$, $L^h \hat{u}_\infty = f v_\infty$. Multiply by $\phi \in \mathcal{K}$ and take integration by part, we see that

$$\int_{\mathcal{U}} f v_\infty \cdot \phi = 0 \quad \text{for every } \phi \in \mathcal{K}$$

This contradicts to the choice of f. □

We finish this Subsection by discussing a corollary of the L^2-nonconcentration property. First recall (2.3.3) the index of a singular minimal hypersurface Σ in $U \subset M$ is defined by

$$\operatorname{index}(\Sigma, U) := \sup \Big\{ \dim \mathscr{V} : \mathscr{V} \subset \mathscr{X}_c(U) \text{ be a linear subspace such that } \frac{d^2}{ds^2}\Big|_{s=0} \mathscr{H}^n(e^{sX}(\Sigma)) < 0, \quad \forall 0 \neq X \in \mathscr{V} . \Big\}$$

Corollary 5.3.7. *Suppose Σ satisfies the L^2-nonconcentration property in U. Let $\lambda_1 < \lambda_2 < \lambda_3 < ... \nearrow +\infty$ be the spectrum of L_Σ on $\mathscr{B}$ as in Proposition 5.3.5, where $h = 0$; and E_j be the L^2-eigenspaces. Then*

$$\operatorname{index}(\Sigma, U) = \sum_{\lambda_j < 0} \dim E_j .$$

In particular, Σ has finite index.

Combined with Lemma 5.3.9 in next Subsection, this partially answers a question proposed by A. Neves.

Corollary 5.3.8. *Let Σ be a closed, locally stable minimal hypersurface in an 8 dimensional manifold, then Σ has finite index.*

Proof of Corollary 5.3.7. First recall that if $X \in \mathscr{X}_c(U)$, then $\phi := X \cdot \nu \in W_0^{1,2}(\mathcal{U})$, and

$$\frac{d^2}{ds^2}\Big|_{s=0} \mathscr{H}^n(e^{sX}(\Sigma)) < 0 \quad \text{if and only if} \quad Q_\Sigma(\phi,\phi) < 0\,. \tag{5.3.6}$$

Hence, we define

$$\begin{aligned}
\operatorname{index}_c(\Sigma, U) &:= \sup\{\dim \mathscr{V}_c : \mathscr{V}_c \subset C_c^\infty(\Sigma \cap U) \text{ and } Q_\Sigma(\phi,\phi) < 0, \quad \forall 0 \neq \phi \in \mathscr{V}_c\}\,;\\
\operatorname{index}_{\mathscr{B}}(\Sigma, U) &:= \sup\{\dim \mathscr{V}_{\mathscr{B}} : \mathscr{V}_{\mathscr{B}} \subset \mathscr{B}_0(\mathcal{U}) \text{ and } Q_\Sigma(\phi,\phi) < 0, \quad \forall 0 \neq \phi \in \mathscr{V}_{\mathscr{B}}\}\,.
\end{aligned}$$

Since for each $\phi \in C_c^\infty(\mathcal{U})$, $\phi\nu$ could be extended to a smooth vector field in M supported in U, by (5.3.6) and Proposition 5.3.5, we have

$$\operatorname{index}_c(\Sigma, U) \leq \operatorname{index}(\Sigma, U) \leq \operatorname{index}_{\mathscr{B}}(\Sigma, U) = \sum_{\lambda_j < 0} \dim E_j\,.$$

On the other hand, since $C_c^\infty(\mathcal{U})$ is dense in $\mathscr{B}_0(\mathcal{U})$, we conclude that $\operatorname{index}_c(\Sigma, U) = \operatorname{index}_{\mathscr{B}}(\Sigma, U)$. □

5.3.2 L^2-nonconcentration for strongly isolated singularities

In this subsection, we keep assuming that Σ has only strongly isolated singularities. The goal of this Subsection is the following.

Lemma 5.3.9. *Suppose Σ has only strongly isolated singularities and $\partial U \cap \operatorname{Sing}(\Sigma) = \emptyset$. Then, Σ satisfies the L^2-nonconcentration property in U.*

The proof is based on a refined analysis near singularities. Since Σ has only strongly isolated singularities, by a cut-off argument similar to the proof of Lemma 5.3.1, it suffices to work near each singularity p. In the proof below, we shall view $B^g_{\tau_0}(p)$ as a open ball in T_pU by exponential map, and parametrize $\Sigma \cap B^g_{\tau_0}(p)$ by

the tangent cone $\mathbf{C}_p = \mathbf{C}_p\Sigma$ as in Lemma 2.3.1. Recall that $\mathbf{B}_r$ denote the r-ball in $\mathbf{C}_p$ centered at origin, and we shall abuse the notation to also view it as an open subset in Σ by the parametrization. We may write for simplicity $\mathbf{C} := \mathbf{C}_p$. And let $S := \mathbf{C} \cap \mathbb{S}^n$; $r_2 = r_2(\Sigma, M, g) \in (0, \tau_0/10)$ be fixed such that Σ is stable in $B^g_{3r_2}(p)$ and that for every $\phi \in C^2_{loc}(\mathbf{B}_{r_2})$,

$$|L_\Sigma \phi - L_{\mathbf{C}}\phi| \leq \left(100n^4 \|\varphi_1\|_{C^2,S}/(\inf_S \varphi_1)\right)^{-1} \left(|\nabla^2_{\mathbf{C}}\phi| + r^{-1}|\nabla_{\mathbf{C}}\phi| + r^{-2}|\phi|\right) , \quad (5.3.7)$$

where φ_1 is the first eigenfunction of the Jacobi operator of $S \subset \mathbb{S}^n$. The existence of such r_2 is guaranteed by Lemma 2.3.1 and Theorem A.0.1 (iv).

We shall prove Lemma 5.3.9 when $U \cap \Sigma = \mathbf{B}_{r_2}$. To do so, first observe since we can isometrically embed U into some large $\mathbb{R}^N$, by the Michael-Simon Inequality [Sim83b, 18.6] and Hölder inequality, we have $\forall \phi \in C^1_c(\Sigma \cap U)$ and $\forall \mathcal{W} \subset \Sigma$,

$$\begin{aligned} \int_\Sigma |\nabla_\Sigma \phi|^2 + \phi^2 \, d\|\Sigma\| &\geq c(U,g) \cdot \left(\int_\Sigma \phi^{2n/(n-2)}\right)^{(n-2)/2n} \\ &\geq c(U,g) \cdot \|\Sigma\|(\mathcal{W})^{-2/n} \cdot \int_\Sigma \phi^2 \, . \end{aligned} \quad (5.3.8)$$

As a consequence, if $\mathbf{C}_p$ is strictly stable, then by Lemma 2.3.1 and C.0.1, $\exists\, r_3(\Sigma, p) \in (0, r_2)$ and $\delta_{\mathbf{C}_p} > 0$ such that

$$\delta_{\mathbf{C}_p} \int_\Sigma |\nabla \phi|^2 \leq Q_\Sigma(\phi, \phi), \quad \forall \phi \in C^1_c(\mathbf{B}_{r_3}) \, .$$

Hence Lemma 5.3.9 holds directly by (5.3.8) and a cut-off argument. Thus, it suffices to deal with non-strictly stable tangent cones. From now on, assume $\mu_1(\mathbf{C}_p) = -(n-2)^2/4$.

Consider for each $s \in (0,1)$, the perturbed Jacobi operator

$$L^s_\Sigma := L_\Sigma - s|A_\Sigma|^2 = \Delta_\Sigma + (1-s)|A_\Sigma| + \mathrm{Ric}(\nu, \nu) \, ,$$

and its assciated quadratic form

$$Q^s_\Sigma(\phi,\phi) = Q_\Sigma(\phi,\phi) + s\int_\Sigma |A_\Sigma|^2\phi^2 = \int_\Sigma |\nabla^\Sigma\phi|^2 - \left((1-s)|A_\Sigma|^2 + \mathrm{Ric}(\nu,\nu)\right)\phi^2\,.$$

The advantage of Q^s_Σ is that it's coercive: for every $\phi \in C^1_c(\mathbf{B}_{3r_2})$,

$$\begin{aligned} Q^s_\Sigma(\phi,\phi) &\geq (1-s)Q_\Sigma(\phi,\phi) + s\int_\Sigma |\nabla\phi|^2 - s\|\,\mathrm{Ric}_M\,\|_{L^\infty}\int_\Sigma \phi^2 \\ &\geq s\int_\Sigma |\nabla\phi|^2 - s\|\,\mathrm{Ric}_M\,\|_{L^\infty}\int_\Sigma \phi^2\,, \end{aligned} \tag{5.3.9}$$

Thus by (5.3.8) and (5.3.9), there's a continuous function $1 \leq \alpha(r,\omega) \leq 1 + \log(r_1/r)$ on B_{2r_2} (independent of choice of s) such that

$$\alpha(x) \to \infty, \qquad \text{as } x \to \mathrm{Sing}(\Sigma) = \{p\}\,, \tag{5.3.10}$$

and that for each $\epsilon > 0$ and $s \in (0,1)$, there's a neighborhood $V_{\epsilon,s} \ni p$ such that $\forall \phi \in C^1_c(\mathbf{B}_{2r_2})$,

$$\int_{\Sigma\cap V_{\epsilon,s}} \phi^2\cdot\alpha \leq \epsilon\cdot\left(Q^s_\Sigma(\phi,\phi) + \int_\Sigma \phi^2\right)\,.$$

Hence, by the similar argument as Proposition 5.3.5, there's a unique $u^{(s)} \in C^0_{loc}\cap W^{1,2}_0(\mathbf{B}_{2r_2})$ with

$$u^{(s)}(r,\cdot)\begin{cases} = 0 & \text{if } r \geq r_2 \\ > 0 & \text{if } r < r_2 \end{cases}; \qquad \int_\Sigma (u^{(s)})^2\cdot\alpha = 1\,. \tag{5.3.11}$$

and

$$\lambda_s := Q^s_\Sigma(u^{(s)},u^{(s)}) = \inf\{Q^s_\Sigma(\phi,\phi) : \phi \in C^1_c(B_{r_2}),\ \int_\Sigma \phi^2\cdot\alpha = 1\}\,. \tag{5.3.12}$$

In particular, $u^{(s)}$ satisfies the equation

$$-L_\Sigma^s u^{(s)} = \lambda_s \cdot \alpha \cdot u^{(s)} \qquad \text{on } \mathbf{B}_{r_2} \tag{5.3.13}$$

Also by definition and stability of Σ in $\mathbf{B}_{2r_2}$, for any $0 < s_1 < s_2 < 1$ and any $\phi \in C_c^1(\mathbf{B}_{r_2})$, $0 \le Q_\Sigma^{s_1}(\phi,\phi) \le Q_\Sigma^{s_2}(\phi,\phi)$. Together with (5.3.11) we see, $0 \le \lambda_{s_1} < \lambda_{s_2}$.

We need a uniform lower bound on $u^{(s)}$ away from singularities. To derive it, we need the following Lemma 5.3.10. In fact, this is the only place where structures of strongly isolated singularity is essentially made use of.

Lemma 5.3.10. *There exist $s_0 = s_0(\Sigma, M, g) \in (0,1)$, $\delta = \delta(\Sigma, M, g) > 0$ such that for any $0 < s < s_0$,*

$$\sup\{u^{(s)}(r,\omega)^2 : \frac{r_2}{2} < r < r_2\} \ge \delta\,.$$

Proof. For each $0 < r < r_2$, $s \in (0,1)$, let $m^{(s)}(r) := \inf\{u^{(s)}(r,\omega) : \omega \in S\}$. By (5.3.11) and Harnack inequality, there's a $C_1 = C_1(\Sigma, M, g) > 0$ (independent of s and r) such that,

$$u^{(s)}(r,\omega) \le C_1 m^{(s)}(r), \quad \forall 0 < r < r_2/2,\ \forall s \in (0,1),\ \forall \omega \in S\,. \tag{5.3.14}$$

On the other hand, let $0 < s_0 << 1$ such that,

$$r^2 |A_\Sigma|^2_{(r,\cdot)} \le \frac{1}{10n \cdot s_0}, \quad \forall\ 0 < r < r_1\,.$$

Let $U(r) := (\frac{r}{r_2})^{-(n-1)/2} - (\frac{r}{r_2})^\gamma$, where $\gamma := -(n-3)/2 - 1/(4n-4)$. The choice of the exponents guarantees that $U(r) > 0$, $U'(r) < 0$ on $r \in (0, r_2)$, and that

$$\begin{aligned} &U''(r) + \frac{n-1+1/(4n-4)}{r} U'(r) + \frac{1}{r^2}\left((\frac{n-2}{2})^2 - \frac{1}{8}\right) U(r) = 0\,, \\ &L_{\mathbf{C}}(U(r)\varphi_1(\omega)) \ge \frac{r^{-2}}{4n-4}(r|U'(r)| + |U(r)|)\varphi_1(\omega)\,. \end{aligned} \tag{5.3.15}$$

In particular, for any $s \in (0, s_0)$ and any constant $D > 0$, we have

$$\begin{aligned}
&L^s_\Sigma(D \cdot U(r)\varphi_1(\omega)) \\
&\geq D \cdot \left(L_{\mathbf{C}}(U\varphi_1) - \frac{r^{-2}}{100n^4} \left(r^2|U''| + r|U'| + |U| \right) \varphi_1 - s|A_\Sigma|^2 U\varphi_1 \right) \\
&\geq D \cdot \left(\frac{1}{4n-4}(r|U'| + |U|) - \frac{1}{4n^2} \left(r|U'| + |U| \right) - \frac{1}{10n}|U| \right) r^{-2}\varphi_1(\omega) \\
&\geq 0 \quad \geq -\lambda_s \alpha \cdot u^{(s)} = L^s_\Sigma u^{(s)} \,,
\end{aligned}$$

on $\mathbf{B}_{r_2}$, where the first inequality and follows from the choice of r_2, the second inequality follows from (5.3.15), the third inequality is a direct calculation since $n \geq 7$ as assumed, and the last is by (5.3.11). Hence, by weak maximum principle, $\forall 0 < r_3 < r_2$, if

$$D \cdot U(r_3) \sup_S w_1 \leq m^{(s)}(r_3) \,,$$

then,

$$D \cdot U(r) \sup_S w_1 \leq m^{(s)}(r), \qquad \forall r_3 < r < r_2 \,. \tag{5.3.16}$$

Choose $D > 0$ such that,

$$D \cdot U(r_2/2) \sup_S w_1 = m^{(s)}(r_2/2) \,.$$

By (5.3.16), for any $0 < r < r_2/2$,

$$m^{(s)}(r) \leq D \cdot U(r) \sup_S w_1 = m^{(s)}(r_2/2) \cdot \frac{U(r)}{U(r_2/2)} \,. \tag{5.3.17}$$

Thus, by (5.3.11),

$$
\begin{aligned}
1 = \int_\Sigma (u^{(s)})^2 \cdot \alpha &= \int_{\mathbf{B}_{r_2}\setminus\mathbf{B}_{r_2/2}} (u^{(s)})^2 \cdot \alpha + \int_{\mathbf{B}_{r_2/2}} (u^{(s)})^2 \cdot \alpha \\
&\le C(\Sigma, M, g) \left(\sup_{\mathbf{B}_{r_2}\setminus\mathbf{B}_{r_2/2}} (u^{(s)})^2 + \int_0^{r_2/2} r^{n-1} |\log r| \int_S (u^{(s)})^2(r,\omega)\ d\omega\ dr \right) \\
&\le C(\Sigma, M, g) \left(\sup_{\mathbf{B}_{r_2}\setminus\mathbf{B}_{r_2/2}} (u^{(s)})^2 + \int_0^{r_2/2} r^{n-1} |\log r| \cdot m^{(s)}(r)^2\ dr \right) \\
&\le C(\Sigma, M, g, r_2) \left(\sup_{\mathbf{B}_{r_2}\setminus\mathbf{B}_{r_2/2}} (u^{(s)})^2 + \int_0^{r_2/2} r^{n-1} |\log r| \cdot m^{(s)}(r_2/2)^2 U(r)^2\ dr \right) \\
&\le C_2(\Sigma, M, g, r_2) \cdot \sup_{\mathbf{B}_{r_2}\setminus\mathbf{B}_{r_2/2}} (u^{(s)})^2 \,,
\end{aligned}
$$

Where the first inequality follows by definition of α and Lemma 2.3.1, the second inequality follows from (5.3.14), the third inequality is by (5.3.17) and the last inequality is by

$$\int_0^1 r^{n-1} \log r \cdot U(r)^2 < +\infty\ dr\,.$$

Hence, the Lemma is proved by choosing $\delta = C_2(\Sigma, M, g, r_2)^{-1}$. □

Proof of Lemma 5.3.9. Recall that $u^{(s)}$ satisfies the equation (5.3.13) and (5.3.11), where $\lambda_s > 0$ is decreasing in s. Let $s \to 0$. Then by classical elliptic estimates [GT01], up to a subsequence,

$$u^{(s)} \to u_0 \quad \text{in } C^2_{loc}(\mathbf{B}_{r_2}) \cap W^{1,2}_{loc} \cap C^0_{loc}(\mathbf{B}_{r_1}), \qquad \lambda_s \to \lambda_0\,. \tag{5.3.18}$$

Clearly,

$$\lambda_0 \ge \inf\{Q_\Sigma(\phi,\phi) : \phi \in C^1_c(\mathbf{B}_{r_2}),\ \int_\Sigma \phi^2 \cdot \alpha = 1\} \ge 0\,;$$

and the following equation holds in $\mathbf{B}_{r_2}$,

$$-L_\Sigma u_0 = \lambda_0 \alpha \cdot u_0\,. \tag{5.3.19}$$

Also, by Lemma 5.3.10, $u_0 \neq 0$.

Claim. $\lambda_0 > 0$.

We first complete the proof assuming this Claim to be true. For each $\epsilon > 0$, let $\mathcal{V}_\epsilon := \{x \in \mathbf{B}_{r_2} : \alpha(x) > 1/(\lambda_0 \cdot \epsilon)\}$. By (5.3.10), $\mathcal{V}_\epsilon \neq \emptyset$ is a neighborhood of $\mathrm{Sing}(\Sigma) \cap B^g_{r_1}(p) = \{p\}$. For any $\phi \in C^1_c(\mathbf{B}_{r_2})$ fixed,

$$Q_\Sigma(\phi,\phi) = \lim_{s\to 0} Q^s_\Sigma(\phi,\phi) \geq \lim_{s\to 0} \lambda_s \cdot \int_\Sigma \phi^2 \cdot \alpha = \lambda_0 \cdot \int_\Sigma \phi^2 \cdot \alpha = \frac{1}{\epsilon}\int_{\mathcal{V}_\epsilon} \phi^2 \,.$$

Hence, Σ satisfies the L^2-nonconcentration property in $\mathbf{B}_{r_2}$.

Proof of the Claim. We shall argue by contradiction. Suppose $\lambda_0 = 0$. By (5.3.11), (5.3.18) and (5.3.19) we have,

$$L_\Sigma u_0 = 0 \quad \text{on } \mathbf{B}_{r_2}; \qquad\qquad u_0(r,\cdot)\begin{cases} = 0 & \text{if } r > r_2 \\ \geq 0 & \text{if } 0 < r < r_2 \end{cases}. \tag{5.3.20}$$

Also notice that for $s \in (0, 1/2)$,

$$Q_\Sigma(u^{(s)}, u^{(s)}) \leq Q^s_\Sigma(u^{(s)}, u^{(s)}) = \lambda_s \leq \lambda_{1/2} \,.$$

Hence combined with (5.3.18), (5.3.20) and Lemma 5.3.2, $u_0 \in \mathscr{B}_0(\mathbf{B}_{2r_2})$ and $Q_\Sigma(u_0,u_0) \leq \lambda_0 = 0$.

On the other hand, recall that Σ is stable in $\mathbf{B}_{2r_2}$, i.e. $Q_\Sigma(\phi,\phi) \geq 0$, $\forall \phi \in C^1_c(\mathbf{B}_{2r_2})$. Therefore, $u_0 \neq 0$ achieves the infimum of Q_Σ in $\mathscr{B}_0(\mathbf{B}_{2r_2})$. The Euler-Lagrangian equation gives

$$L_\Sigma u_0 = 0, \quad \text{on } \mathbf{B}_{2r_2} \,.$$

However, (5.3.20) contradict to the strong maximum principle of u_0 in $\mathbf{B}_{2r_2}$. This completes the proof of the Claim. □

5.3.3 Asymptotic rates on LSMH

Let Σ be an LSMH in a closed manifold (M, g) with a strongly isolated singularity p. For a $L^2_{loc}(\Sigma)$-function v defined near p, in view of Definition 5.2.5, we can also define the **asymptotic rate** of v at p to be,

$$\mathcal{AR}_p(v) := \sup\{\gamma : \lim_{s\searrow 0} \int_{A^g(p;s,2s)} v^2 \cdot \rho^{-n-2\gamma}\, d\|\Sigma\| = 0\}, \tag{5.3.21}$$

where $\rho(x) := dist_g(x, p)$ is the distance function to the singular point, and $A^g(p; s, t)$ is the geodesic annulus in (M, g) centered at p with inner radius s and outer radius t. Recall that heuristically, if v grows like ρ^γ near p, then $\mathcal{AR}_p(v) = \gamma$. Note that by Lemma 2.3.1, by parametrizing Σ using $\mathbf{C}_p := \mathbf{C}_p\Sigma$ near p, v can be viewed as a function on $\mathbf{C}$ near the origin, and by Theorem A.0.1, the definition above is compatible with Definition 5.2.5 for function on cones. In particular, the following Lemma is a direct consequence of Lemma 2.3.1 and Corollary 5.2.6.

Lemma 5.3.11. *Let $h \in L^\infty(M)$, $L^h := L_\Sigma - h$. Suppose $u \in W^{2,2}_{loc}(\Sigma)$ satisfies $L^h u \in L^\infty(\Sigma)$. Then*

$$\mathcal{AR}_p(u) \in \{-\infty\} \cup \Gamma(\mathbf{C}_p) \cup [1, +\infty]\,.$$

Moreover, if $\gamma := \mathcal{AR}_p(u) \in (-\infty, 1)$, then by viewing u as a function on $\mathbf{C}_p$ we have, for every $t_j \searrow 0$, $u_j(x) := u(t_j x)/(\|u\|_{L^2(\mathbf{A}(t_j,2t_j))} \cdot t_j^{-n/2})$ will L^2_{loc}-subconverges to some Jacobi field $\hat{u}_\infty$ on $\mathbf{C}_p$ given by

$$\hat{u}_\infty = r^\gamma(c + c' \log r)w\,,$$

where $0 \neq w \in W_\gamma(\mathbf{C}_p)$ is the corresponding eigenfunction of γ; $(c, c') \neq (0, 0)$ and $c' = 0$ unless $\gamma = -(n-2)/2$.

We list some applications of this Lemma. In what follows, let $h \in L^\infty(M)$, $L^h := L_\Sigma - h$; $U \subset M$ be a smooth domain such that $\mathrm{Clos}(U) \cap \mathrm{Sing}(\Sigma) = \{p\}$, $\mathcal{U} := U \cap \Sigma$.

Corollary 5.3.12. *Suppose $u \in W^{2,2}_{loc}(\mathcal{U})$ such that $L^h u \in L^\infty(\mathcal{U})$. Then*

(i) If $u > 0$ and $L^h u$ vanishes near p, then $\mathcal{AR}_p(u) \in \{\gamma_1^\pm(\mathbf{C}_p\Sigma)\}$;

(ii) If $u \in \mathscr{B}(\mathcal{U})$, then $\mathcal{AR}_p(u) \geq \gamma_1^+(\mathbf{C}_p\Sigma)$;

(iii) If $\mathcal{AR}_p(u) > -(n-2)/2$, then

$$\int_{\mathcal{U}} |\nabla_\Sigma u|^2 + \rho^{-2}u^2 \; d\|\Sigma\| < +\infty\,.$$

In particular, if $u, v \in W^{2,2}_{loc}(\Sigma)$ such that $L^h u, L^h v \in L^\infty(\Sigma)$ and that $\mathcal{AR}_q(u) > -(n-2)/2$, $\mathcal{AR}_q(v) > -(n-2)/2$ for every $q \in \mathrm{Sing}(\Sigma)$, then we can do the following integration by parts,

$$\int_\Sigma u \cdot L^h v \; d\|\Sigma\| = \int_\Sigma v \cdot L^h u \; d\|\Sigma\|\,.$$

Proof. To prove (i), first notice that by classical Harnack inequality for u, we have a rough lower bound $\mathcal{AR}_p(u) \geq -C(\mathbf{C}_p)$; While by Proposition 3.3.1, $\mathcal{AR}_p(u) < 0$. Therefore, Lemma 5.3.11 and the classification Lemma C.0.3 of positive Jacobi field together implies that $\mathcal{AR}_p(u) \in \{\gamma_1^\pm(\mathbf{C}_p)\}$.

To prove (ii), first by Lemma 5.3.9, we can take another sufficiently small smooth neighborhood $U' \subset\subset U$ of p such that $\partial U'$ is transverse to Σ and that the first L^2-eigenvalue $\lambda_1 = \lambda_1(-L^h) > 1$ on $\mathscr{B}_0(\mathcal{U}')$. Also by classical elliptic estimate, $u \in L^\infty_{loc}(\mathcal{U})$. Therefore by a renormalization of u, we can assume WLOG that $|u| \leq 1$ near $\partial\mathcal{U}'$ and $|L^h u| \leq 1$ on $\mathcal{U}'$. By Proposition 5.3.5 (3), we can find $\xi \in \mathscr{B}(\mathcal{U}')$

solving,

$$\begin{cases} -L^h\xi = \xi, & \text{on } \mathcal{U}', \\ \xi = 1, & \text{on } \partial\mathcal{U}'. \end{cases}$$

And by Proposition 5.3.5 (4), we have $\inf_{\mathcal{U}'} \xi > 0$ and $|u| \leq \xi / \inf_{\mathcal{U}'} \xi$ on $\mathcal{U}'$. And by (i), $\mathcal{AR}_p(\xi) \in \{\gamma_1^{\pm}(\mathbf{C}_p)\}$. Therefore it suffices to show that if $\gamma_1^- \neq \gamma_1^+$ (i.e. $\mathbf{C}_p$ is strictly stable, and in this case $\gamma_1^- < -(n-2)/2 < \gamma_1^+$), then $\mathcal{AR}_p(\xi) \neq \gamma_1^-$. In viewing of Lemma C.0.1 and the fact that $\mathscr{B}(\mathcal{U}') = W^{1,2}(\mathcal{U}')$ in this case, this immediately follows from the fact that

$$\liminf_{s\to 0} \int_{A^g(p;s,2s)} (\rho^\sigma)^2 \cdot \rho^{-2} \, d\|\Sigma\| = +\infty, \quad \forall \sigma < -(n-2)/2 \,.$$

To prove (iii), first note that $\mathcal{AR}_q(u) > -(n-2)/2$ implies the existence of $\gamma > -(n-2)/2$ and $s_0 > 0$ satisfying that for every $s \in (0, s_0)$,

$$\int_{A^g(q,s,2s)} u^2 \, d\|\Sigma\| \leq s^{n+2\gamma}. \tag{5.3.22}$$

By taking sum of (5.3.22) over $s = 2^{-l}s_0$, $l \geq 1$ and all singularities, we get

$$\int_{B^g(\mathrm{Sing}(\Sigma),s_0)} u^2\rho^{-2} \, d\|\Sigma\| < +\infty. \tag{5.3.23}$$

Now to estimate the energy, let $\varphi \in C_c^\infty((1/2,+\infty),[0,1])$ be a non-negative cut-off function which equals to 1 on $[1,+\infty)$. Then for every $0 < r << 1$, let $\varphi_r := \varphi(\rho/r)$, and we have an energy estimate,

$$\int_\Sigma |\nabla_{\Sigma,g} u|^2 \varphi_r^2 \leq \int_\Sigma C(n,g)(|\nabla\varphi_r|^2 + 1 + |A_{\Sigma,g}|^2)u^2 + |L_{\Sigma,g}u| \cdot |u| \cdot \varphi_r^2, \tag{5.3.24}$$

Since $|\nabla\varphi_r| \leq C(\varphi) r^{-1} \cdot \chi_{\{r/2\leq\rho\leq r\}} \leq C'(\varphi)\rho^{-1}$, $|A_{\Sigma,g}| \leq C(g,\Sigma)\rho^{-1}$ and $L_{\Sigma,g}u \in$

$L^\infty(\Sigma)$, (5.3.23) implies that the right hand side is bounded as $r \to 0$. In particular, we have

$$\int_\Sigma |\nabla_{\Sigma,g} u|^2 + \rho^{-2} u^2 \, d\|\Sigma\|_g < +\infty.$$

Therefore, that we can do the integration by parts simply follows from a standard cut-off argument with the estimates above. □

The following Lemma, which plays a key role in the next chapter in globally smoothing out singularities, shows that having top asymptotic growth rate near each singularity is a generic property for solutions of $L^h u = f$.

Lemma 5.3.13. *Suppose $p \in \mathrm{Sing}(\Sigma)$. Then there exists an open subset $\mathscr{V}_p \subset L^\infty(\Sigma)$ satisfying*

(1) $f \in \mathscr{V}_p$ implies $cf \in \mathscr{V}_p$, $\forall c \in \mathbb{R} \setminus \{0\}$;

(2) If $f \in \mathscr{V}_p$ and $g \in L^\infty(\Sigma) \setminus \mathscr{V}_p$, then $f + g \in \mathscr{V}_p$;

(3) $\mathscr{V}_p \cap C_c^\infty(\Sigma)$ is dense in $C_c^\infty(\Sigma)$;

(4) If $u \in \mathscr{B}_0(\Sigma)$ is the solution of $L^h u = f$ for some $f \in \mathscr{V}_p$, then $\mathcal{AR}_p(u) = \gamma_1^+(\mathbf{C}_p)$.

In particular, for every $k \in \mathbb{Z}_{\geq 1}$ and $\alpha \in [0,1]$,

$$\mathscr{T}^{k,\alpha} := \{f \in C^{k,\alpha}(M) : \text{whenever } u \in \mathscr{B}(\Sigma) \text{ solves } L^h u = \nu_\Sigma(f), \\ \text{we have } \mathcal{AR}_p(u) = \gamma_1^+(\mathbf{C}_p) \text{ for some } p \in \mathrm{Sing}(\Sigma)\}$$

contains an open dense subset of $C^{k,\alpha}(M)$.

Proof. Let $0 < \tau_0 << 1$ be a constant such that $\{B^g_{2\tau_0}(p)\}_{p \in \mathrm{Sing}(\Sigma)}$ are pairwise disjoint, and that $-L^h$ is strictly positive on each $B^g_{2\tau_0}(p) \cap \Sigma$. For each $p \in \mathrm{Sing}(\Sigma)$,

we define $\xi_p,\ \check{\xi}_p \in \mathscr{B}(\mathcal{B}_{\tau_0}(p))$ to be the solution of

$$\begin{cases} L^h \xi_p = 0, & \text{on } \mathcal{B}_{\tau_0}(p)\,; \\ \xi_p = 1, & \text{on } \partial\mathcal{B}_{\tau_0}(p)\,; \end{cases} \qquad \begin{cases} (-L^h + 1)\check{\xi}_p = 0, & \text{on } \mathcal{B}_{\tau_0}(p)\,; \\ \check{\xi}_p = 1, & \text{on } \partial\mathcal{B}_{\tau_0}(p)\,. \end{cases}$$

Then by weak maximum principle, $\xi_p \geq \check{\xi}_p > 0$ on $\mathcal{B}_{\tau_0}(p)$ for every $p \in \mathrm{Sing}(\Sigma)$, and by Corollary 5.3.12, we have $\check{\xi}_p(x) \to +\infty$ as $x \to p$.

Now for fixed p, we define $\mathscr{V}_p$ by,

- $\mathscr{V}_p := \{f \in L^\infty(\Sigma) : u := (L^h)^{-1} f \in \mathscr{B}_0(\mathcal{U}) \text{ satisfies } \limsup_{x\to p} |u|(x)/\xi_p(x) > 0\}$, if L^h is non-degenerate;
- $\mathscr{V}_p := \{f \in L^\infty(\Sigma) : L^h u = f \text{ has no solution } u \in \mathscr{B}(\Sigma)\}$, if L^h is degenerate.

Then (1), (2) directly follows from this definition; (4) follows from Corollary 5.3.12 (i) and (ii). If (1)-(4) and openness of $\mathscr{V}_p$ are verified, then notice that

$$\mathscr{T}^{k,\alpha} \supset \bigcup_{p\in \mathrm{Sing}(\Sigma)} \{f \in C^{k,\alpha} : \nu_\Sigma(f) \in \mathscr{V}_p\}\,,$$

where the right hand side is open and dense in $C^{k,\alpha}(M)$.

Now we proceed to prove (3) and openness of $\mathscr{V}_p$ in L^∞.

Proof of Openness. When L^h is degenerate, the openness of $\mathscr{V}_p$ follows directly from that

$$L^\infty(\Sigma) \setminus \mathscr{V}_p = L^\infty(\Sigma) \cap (\mathrm{Ker}\, L^h)^\perp\,, \tag{5.3.25}$$

is a closed proper linear subspace. In case when L^h is non-degenerate, clearly, it suffices to show that there exists a $C = C(h, \Sigma, M, g) > 0$ such that for every

$\|f\|_{L^\infty(\Sigma)} \leq 1$ we have

$$\limsup_{x\to p} |(L^h)^{-1} f|(x)/\xi_p(x) \leq C \tag{5.3.26}$$

To prove this bound, denote $(L^h)^{-1} f$ by v. First recall that by Proposition 5.3.5 (3) and interior C^0-estimate, $|v| \leq C_1 = C_1(h, \Sigma, M, g, r_2)$ on $\mathcal{A}_{\tau_0/2,\tau_0}(p)$. Let $\kappa := \inf_{\mathcal{B}_{\tau_0}(p)} \check{\xi}_p (> 0)$, $C := max\{1, C_1\}/\kappa$. Then by

$$|L^h v| \leq 1 \leq -C \cdot L^h(2\xi_p - \check{\xi}_p) \text{ on } \mathcal{B}_{\tau_0}(p), \quad |v| \leq C\kappa \leq C(2\xi_p - \check{\xi}_p) \text{ on } \partial\mathcal{B}_{r_2}(p)\,.$$

and weak maximum principle, we see that $|v| \leq C(2\xi_p - \check{\xi}_p)$ on $\mathcal{B}_{r_2}(p)$. Hence (5.3.26) is proved. □

Proof of (3). By (1) and (2), it suffices to show that $\mathscr{V}_p \cap C_c^\infty(\Sigma) \neq \emptyset$. When L^h is degenerate, this follows immediately by (5.3.25), the L^2-denseness of $C_c^\infty(\Sigma)$ and that $(\operatorname{Ker} L^h)^\perp$ is a proper closed linear subspace of $L^2(\Sigma)$.

When L^h is non-degenerate, first recall that by Proposition 5.3.5 (5), there exists $\vartheta \in C_c^\infty(\Sigma; \mathbb{R}_+)$ such that $-L^{h+\vartheta}$ is strictly positive.

Let $w := (-L^{h+\vartheta})^{-1}(\vartheta) \in \mathscr{B}(\Sigma)$, then $w > 0$ on Σ by weak and strong maximum principle; Let $\tau_1 \in (0, \tau_0)$ such that $\mathcal{B}_{\tau_1}(p) \cap \operatorname{spt}(\vartheta) = \emptyset$. Then w satisfies the equation $-L^h w = \vartheta - \vartheta w = 0$ on $\mathcal{B}_{\tau_1}(p)$, and by weak maximum principle, $w \geq \varepsilon\xi_p$ for some $\varepsilon > 0$. Therefore, $\vartheta - \vartheta w \in \mathscr{V}_p \cap C_c^\infty(\Sigma)$. □

□

5.3.4 Green's functions

In this subsection, we introduce the notion of Green's function for Jacobi field type operator on a LSMH. This will be used in Section 5.4 to construct a mean convex foliation of a strictly stable minimal hypersurface with certain tangent cones. We

keep the assumptions on Σ, (M, g), U, $p \in \mathrm{Sing}(\Sigma) \cap U$ and $r_2 > 0$ as in Section 5.3.2. Also keep parametrizing $\Sigma \cap B^M_{r_2}(p)$ by the tangent cone $\mathbf{C}_p$ and using the notation that $\mathbf{B}_r := \mathbf{C}_p \cap \mathbb{B}_r(\mathbf{0})$. Recall that $-L_\Sigma$ is strictly positive on $\mathscr{B}_0(\mathbf{B}_{r_2})$. For later applications, we shall consider operators of a more general form.

$$L^h := L_\Sigma - h, \qquad\qquad Q^h(\phi, \phi) := Q_\Sigma(\phi, \phi) + \int_\Sigma h \cdot \phi^2 \,,$$

where $h \in L^\infty(\Sigma)$ be fixed throughout this subsection. By choosing an even smaller r_2, we can assume WLOG that $-L^h$ is also strictly positive on $\mathscr{B}_0(\mathbf{B}_{r_2})$.

We first slightly extend the function space we are dealing with.

Lemma 5.3.14. *Up to a normalization, there exists a unique $G \in C^\infty_{loc}(\mathrm{Clos}(\mathbf{B}_{r_2}))$ such that $L^h G = 0$ on $\mathbf{B}_{r_2}(p)$, $G > 0$ on $\mathbf{B}_{r_2}$, $G = 0$ on $\partial\mathbf{B}_{r_2}$.*

Proof. We first prove the existence. Consider $(r_2/2 >) t_j \to 0_+$ and the solution u_j of $L^h u_j = 0$ on $\mathbf{A}_{t_j, r_2}$, $u_j = 0$ on $\partial\mathbf{B}_{r_2}$, $u_j = 1$ on $\partial\mathbf{B}_{t_j}$. By weak maximum principle, $u_j > 0$ on $\mathbf{A}_{t_j, r_2}$. By Harnack inequality and standard elliptic estimate, $u_j / \sup_{\partial\mathbf{B}_{r_2/2}} u_j \to G$ in $C^\infty_{loc}(\mathrm{Clos}(\mathbf{B}_{r_2}))$ and $G > 0$ on $\mathbf{B}_{r_2}$, $G = 0$ on $\partial\mathbf{B}_{r_2}$.

To see the uniqueness, if G' is another positive smooth function vanishing on $\partial\mathbf{B}_{r_2}$ and satisfies the equation above, first note that by weak maximum principle, $\forall t \in (0, r_2)$,

$$\sup_{\mathbf{A}_{t, r_2}} G/G' = \sup_{\partial\mathbf{B}_t} G/G' \,, \qquad\qquad \inf_{\mathbf{A}_{t, r_2}} G/G' = \inf_{\partial\mathbf{B}_t} G/G' \,. \tag{5.3.27}$$

And by Harnack inequality, $\sup_{\partial\mathbf{B}_t} G/G' \le C(\Sigma) \inf_{\partial\mathbf{B}_t} G/G'$, $\forall t \in (0, r_2/2)$. Hence,

$$0 < a_- := \inf_{\mathbf{B}_{r_2}} G/G' \le a_+ := \sup_{\mathbf{B}_{r_2}} G/G' < +\infty$$

We now show that $a_- = a_+$, with which the Lemma is proved.

By (5.3.27), $a_- = \liminf_{t \to 0_+} \inf_{\partial\mathbf{B}_t} G/G'$ and $a_+ = \liminf_{t \to 0_+} \inf_{\partial\mathbf{B}_t} G/G'$. Thus,

up to a subsequence of $t \to 0_+$, $G_t(x) := G(tx)/\inf_{\partial \mathbf{B}_t} G \to G_0$, $G'_t(x) := G'(tx)/\inf_{\partial \mathbf{B}_t} G \to G'_0$ and G_0, G'_0 are positive Jacobi fields on $\mathbf{C}_p$ with $a_- = \inf_{\partial \mathbf{B}_1} G_0/G'_0$, $a_+ = \sup_{\partial \mathbf{B}_1} G_0/G'_0$. By Lemma C.0.3, $a_- = a_+$. □

We call such G in Lemma 5.3.14 a **Green's function** of L^h at p in $\mathbf{B}_{r_2}$. Since $-L^h$ is strictly positive on $\mathscr{B}_0(\mathbf{B}_{r_2})$, G doesn't belong to $\mathscr{B}(\mathbf{B}_{r_2})$. Just like Green's functions in a smooth domain, G dominates functions in $\mathscr{B}(\mathbf{B}_{r_2})$:

Lemma 5.3.15. *Let $\varphi_1 \in \mathscr{B}_0(\mathbf{B}_{r_2})$ be the first eigenfunction of $-L^h$ in $\mathscr{B}_0(\mathbf{B}_{r_2})$ with $\|\varphi_1\|_{L^2} = 1$; $\lambda_1 > 0$ to be the corresponding eigenvalue, i.e.*

$$\lambda_1 = \inf\{Q^h(\phi,\phi) : \phi \in \mathscr{B}_0(\mathbf{B}_{r_2}), \|\phi\|_{L^2} = 1\} = Q^h(\varphi_1,\varphi_1) > 0\,.$$

Then $\lim_{r\to 0} \sup_{\partial \mathbf{B}_r}(\varphi_1/G) = 0$. In particular, if $u \in \mathscr{B}(\mathbf{B}_{r_2})$ satisfies $L^h u \in L^\infty(\mathbf{B}_{r_2})$, then $\lim_{r\to 0} \sup_{\partial \mathbf{B}_r}(|u|/G) = 0$.

Proof. By weak maximum principle and Harnack inequality, for each $r \in (0, r_2)$,

$$\sup_{\partial \mathbf{B}_r} \varphi_1/G \le C(\Sigma) \inf_{\partial \mathbf{B}_r} \varphi_1/G = C(\Sigma) \inf_{\mathbf{A}_{r,r_2}} \varphi_1/G\,.$$

Hence, if assume for contradiction that $\limsup_{r\to 0} \sup_{\partial \mathbf{B}_r}(\varphi_1/G) > 0$, then by a renormalization in G, WLOG $G \le \varphi_1$ on $\mathbf{B}_{r_2}$.

For each $r < r_2$, consider the minimizer v_r of $Q^h(\phi,\phi)$ among

$$\{\phi \in \mathscr{B}_0(\mathbf{B}_{r_2}) : \phi = \varphi_1 \text{ in } \mathbf{B}_r\}\,.$$

The existence of such minimizer follows from Proposition 5.3.5 and Lemma 5.3.9. Moreover, $L^h v_r = 0$ on $\mathbf{A}_{r,r_2}$; $G \le v_r \le \varphi_1$ on $\mathbf{B}_{r_2}$ by weak maximum principle; $\|v_r\|_{\mathscr{B}(\mathbf{B}_{r_2})} \le (1 + \|h\|_{L^\infty})\|\varphi_1\|_{\mathscr{B}(\mathbf{B}_{r_2})}$ by definition. Let $r \to 0_+$, up to a subsequence, $v_r \to v_0$ in $C^\infty_{loc}(\mathrm{Clos}(\mathbf{B}_{r_2}))$ for some $v_0 \in \mathscr{B}_0(\mathbf{B}_{r_2})$ satisfying $L^h v_0 = 0$ on $\mathbf{B}_{r_2}$ and

$G \leq v_0$. But since L^h is nondegenerate in $\mathscr{B}_0(\mathbf{B}_{r_2})$, $v_0 \equiv 0$, contradicts to that $G > 0$ on $\mathbf{B}_{r_2}$.

To see that near p, G also dominates $u \in \mathscr{B}(\mathbf{B}_{r_2})$ with $L^h u \in L^\infty(\mathbf{B}_{r_2})$, first observe that by Harnack inequality, Lemma C.0.3 and a blow up argument, $\varphi_1(x) \to +\infty$ as $x \to p$. Hence there exists $C >> 1$ depending on u and φ_1 such that $|L^h u| \leq -C \cdot L^h \varphi_1$ in $\mathbf{B}_{r_2/2}$ and $|u| \leq C\varphi_1$ on $\partial \mathbf{B}_{r_2/2}$. Then by Proposition 5.3.5 (4), $|u| \leq C\varphi_1$ on $\mathbf{B}_{r_2/2}$ and the Lemma is proved. □

The study of asymptotic rate in Section 5.3.3 also has a direct implication on Green's function.

Corollary 5.3.16. *Let G be a Green's function of L^h in $\mathbf{B}_{r_2}$ at p; $\varphi_1 \in \mathscr{B}_0(\mathbf{B}_{r_2})$ be the first eigenfunction of $-L^h$ in $\mathscr{B}_0(\mathbf{B}_{r_2})$; $\xi \in \mathscr{B}(\mathbf{B}_{r_2})$ be the solution of $-L^h \xi = 0$ on $\mathbf{B}_{r_2}$ and $\xi = 1$ on $\partial \mathbf{B}_{r_2}$. Then $\mathcal{AR}_p(G) = \gamma_1^-$, $\mathcal{AR}_p(\varphi_1) = \mathcal{AR}_p(\xi) = \gamma_1^+$.*

Proof. By weak maximum principle, $\varphi_1, \xi > 0$; Also by definition, $G > 0$. By Lemma 5.3.11, Lemma C.0.3 and Harnack inequality, $\mathcal{AR}_p(G), \mathcal{AR}_p(\varphi_1), \mathcal{AR}_p(\xi) \in \{\gamma_1^\pm\}$. And to distinguish them, it suffices to assume in addition that $\mathbf{C}_p$ is strictly stable, in which case $\gamma_1^+ > -(n-2)/2 > \gamma_1^-$.

With this assumption, by Lemma C.0.1, 2.3.1 and A.0.1, $\|\cdot\|_{\mathscr{B}} \sim \|\cdot\|_{W^{1,2}}$ and in particular $\mathscr{B}_0(\mathbf{B}_{r_2}) = W_0^{1,2}(\mathbf{B}_{r_2})$. Therefore, combined with Corollary 5.3.12 (iii) and the standard elliptic estimate, we have for every $u \in \{\varphi_1, G, \xi\}$, $\mathcal{AR}_p(u) = \gamma_1^+$ if and only if $u \in W_0^{1,2}(\mathbf{B}_{r_2})$. Thus proves the corollary. □

Corollary 5.3.17. *Let G be a Green's function of L^h in $\mathbf{B}_{r_2}$ at p. Then*

$$\lim_{r \to 0_+} G(rx) / \inf_{\partial \mathbf{B}_r} G = G_0(x)\,,$$

where $G_0(r, \omega) = r^{\gamma_1^-} \varphi_1(\omega) / \inf_{\partial \mathbf{B}_1} \varphi_1$ is a homogeneous positive Jacobi field on $\mathbf{C}_p$ with fast asymptotic rate. Moreover, the convergence is in $C^\infty_{loc}(\mathbf{C}_p)$.

Proof. First note that by Harnack inequality, when $r \to 0_+$, $G_{(r)}(x) := G(rx)/\inf_{\partial \mathbf{B}_r} G$ always subconverges in $C^\infty_{loc}(\mathbf{C}_p)$ to some positive Jacobi field on $\mathbf{C}_p$ with infimum 1 on ∂B_1. To see this limit is independent of the choice of subsequence, notice that if $\mathbf{C}_p$ is not strictly stable, then by Lemma C.0.3, this is the only positive Jacobi field on $\mathbf{C}_p$. While if $\mathbf{C}_p$ is strictly stable, then by Corollary 5.3.16, $\mathcal{AR}_p(G) = \gamma_1^-$; And by Lemma 5.3.11, for every sequence $r_j \to 0_+$,

$$G_{(r_j)} \to r^{\gamma_1^-}\varphi_1/\inf\varphi_1\,.$$

This finishes the proof of the Corollary. □

Corollary 5.3.18. *Suppose $u \in W^{2,2}_{loc}(\mathbf{B}_{r_2})$ with $L^h u \in L^\infty(\mathbf{B}_{r_2})$. G be a Green's function of L^h in $\mathbf{B}_{r_2}$ at p. Then,*

(1) If $u \in \mathscr{B}(\mathbf{B}_{r_2})$, then $\mathcal{AR}_p(u) \geq \gamma_1^+$;

(2) If $\mathcal{AR}_p(u) \geq \gamma_1^-$, then there exists a unique $c \in \mathbb{R}$ such that $u - c\cdot G \in \mathscr{B}(\mathbf{B}_{r_2/2})$.

Proof. (1) follows directly from weak maximum principle and Lemma 5.3.16, which assert that $\mathcal{AR}_p(\varphi_1) = \gamma_1^+$.

To prove (2), consider $v \in \mathscr{B}(\mathbf{B}_{r_2/2})$ be the solution of $L^h v = L^h u$ on $\mathbf{B}_{r_2/2}$, $v = u$ on $\mathbf{A}_{r_2/2,r_2}$. Let $w := u - v$. By (1) and definition we see that $\mathcal{AR}_p(w) \geq \gamma_1^-$ and $L^h(u-v) = 0$ on $\mathbf{B}_{r_2/2}$, $w = 0$ outside $\mathbf{B}_{r_2/2}$.

If $\mathcal{AR}_p(w) > \gamma_1^-$, then by Corollary 5.3.16, $\limsup_{t\to 0_+}\sup_{\partial\mathbf{B}_t}|w|/G' = 0$, where G' denotes the Green's function of L^h at p in $\mathbf{B}_{r_2/2}$. Then by weak maximum principle, $w = 0$.

If $\mathcal{AR}_p(w) = \gamma_1^-$, then by Lemma 5.3.11 and Lemma C.0.2, WLOG there exists $t_j \to 0_+$ such that $w > 0$ on $\partial\mathbf{B}_{t_j}$. Then by classical weak and strong maximum principle for elliptic PDE, $w > 0$ on $\mathbf{B}_{r_2/2}$. Hence by Lemma 5.3.14, $w = c'G'$ for some $c' \in \mathbb{R}\setminus\{0\}$. To go back to a Green's function on $\mathbf{B}_{r_2}$, consider $\eta \in C^\infty_c(B^M_{r_2/4}(p))$

which equals to 1 near p and the solution $\psi \in \mathscr{B}_0(\mathbf{B}_{r_2})$ of $L^h\psi = L^h(\eta \cdot w)$ on $\mathbf{B}_{r_2}$. By Lemma 5.3.15 and weak maximum principle, $-\psi + \eta \cdot w$ is a Green's function at p in $\mathbf{B}_{r_2}$, combined with Lemma 5.3.14, $u - cG \in \mathscr{B}(\mathbf{B}_{r_2/2})$ for some $c \in \mathbb{R}$, which completes the proof. □

Now we turn to the globally defined functions on Σ. Recall that $\mathcal{U} := U \cap \Sigma$. Denote $G_{r,p}$ to be a Green's function of L^h in $\mathcal{B}_r(p)$, where $p \in \mathrm{Sing}(\Sigma)$, $r \in (0, r_2]$.

Definition 5.3.19. Define $\mathscr{B}_0(\mathcal{U}) \bigoplus \mathbb{R}_{L^h}\langle \mathrm{Sing}(\Sigma) \cap U\rangle$ to be the function space linearly generated by $\mathscr{B}_0(\mathcal{U})$ and $G_{r,p}$, $p \in \mathrm{Sing}(\Sigma) \cap U$.

The proof of Corollary 5.3.18 (2) actually guarantees that this space depends only on Σ, U and p, but not on the choice of r.

Corollary 5.3.20. *Suppose L^h be nondegenerate on $\mathscr{B}_0(\mathcal{U})$. Then, for each $p \in \mathrm{Sing}(\Sigma) \cap U$, there is a unique (up to normalization) $0 \neq G_p \in \mathscr{B}_0(\mathcal{U}) \bigoplus \mathbb{R}_{L^h}\langle \mathrm{Sing}(\Sigma) \cap U\rangle$ such that $L^h G_p = 0$ on $\mathcal{U}$, $G_p \in \mathscr{B}(\mathcal{U} \backslash \mathrm{Clos}(B^M_{r_2}(p)))$. Moreover, $\mathscr{B}_0(\mathcal{U}) \bigoplus \mathbb{R}_{L^h}\langle \mathrm{Sing}(\Sigma) \cap U\rangle$ is linearly generated by $\mathscr{B}_0(\mathcal{U})$ and $\{G_p : p \in \mathrm{Sing}(\Sigma) \cap U\}$.*

If further $-L^h$ is strictly positive on $\mathscr{B}_0(\mathcal{U})$, then G_p can be chosen positive on $\mathcal{U}$.

We call such G_p a **Green's function of L^h on $\mathcal{U}$ at p.**

Proof. The existence of G_p follows the same construction as in the proof of corollary 5.3.18 (2) from G' to G; The uniqueness of such G_p up to normalization follows directly from Lemma 5.3.14, 5.3.15 and Corollary 5.3.16, 5.3.18; The positivity of G_p when $-L^h$ is strictly positive follows from weak maximum principle. □

5.4 Local Minimization of Strictly Stable Minimal Hypersurfaces

Throughout this section, let $(\Sigma, \nu) \subset (M, g)$ be a strictly stable, closed minimal hypersurface with strongly isolated singularities, i.e. $-L_\Sigma$ is strictly positive on $\mathscr{B}_0(\Sigma)$

as in Proposition 5.3.5 (4). Here "closed" means $\mathrm{Clos}(\Sigma)$ is compact. Suppose $M \setminus \mathrm{Clos}(\Sigma)$ has 2 connected component $M_\pm$. This is always possible if we replace M by a small neighborhood of Σ. Suppose M_+ be the one which ν points in.

Let $\lambda_1 > 0$ be the first L^2-eigenvalue of $-L_\Sigma$ on $\mathscr{B}_0(\Sigma)$; $\Lambda \in (0, \lambda_1/2]$ be fixed throughout this section. Note that by definition we have,

$$Q_\Sigma(\phi, \phi) \geq 2\Lambda \int_\Sigma \phi^2, \quad \forall \phi \in C^1_c(\Sigma)\,. \tag{5.4.1}$$

The aim of this section is to prove the following Theorem.

Theorem 5.4.1. *Let Σ, M be as above. Suppose each tangent cone of Σ is strictly minimizing. Then there exist a constant $\tau = \tau(\Sigma, M, g) > 0$ and a family of piecewise smooth neighborhood $\{U_t\}_{t\in(0,1]}$ of $\mathrm{Clos}(\Sigma)$ such that*

(1) $U_{t_1} \subset\subset U_{t_2}$ for any $t_1 < t_2$;

(2) $\bigcap_{t\in(0,1]} U_t = \mathrm{Clos}(\Sigma)$; $\bigcap_{s\in(t,1]} U_s = \mathrm{Clos}(U_t)$ and $\bigcup_{s\in(0,t)} U_s = U_t$, $\forall t \in (0,1]$;

(3) There exist smooth domains $\{\Omega_t\}_{t\in(0,1]}$ such that, $\Sigma \setminus B^M_\tau(\mathrm{Sing}(\Sigma)) \subset \Omega_t \subset\subset \Sigma$ and we have decomposition

$$\partial U_t \cap M_\pm = \bar\Sigma^{\pm,ext}_t \cup \bigcup_{p\in\mathrm{Sing}(\Sigma)} \bar\Sigma^{\pm,int}_{t,p}\,,$$

where $\bar\Sigma^{\pm,ext}_t$ are smooth minimal graphs over Ω_t with smooth boundaries; $\bar\Sigma^{\pm,int}_{t,p}$ are smooth minimal hypersurfaces in $B^M_\tau(p)$ with boundary $\partial\bar\Sigma^{\pm,int}_{t,p} = \partial\bar\Sigma^{\pm,ext}_t \cap B^M_\tau(p)$.

(4) Along their intersection, $\bar\Sigma^{\pm,ext}_t$ and $\bar\Sigma^{\pm,int}_{t,p}$ forms a convex wedge.

In particular, $[\Sigma]$ is homological area-minimizing in $\mathrm{Clos}(U_1)$; And every stationary integral varifold in $\mathrm{Clos}(U_1)$ is supported on $\mathrm{Clos}(\Sigma)$.

Remark 5.4.2. Note that ∂U_t are Lipschitz hypersurfaces. Hence by [EH91], by running mean curvature flow starting from ∂U_t, we see from (3) and (4) above that U_t can be approximated by smooth mean convex domains.

We first sketch the construction of U_t. Let

$$\rho(x) := min\{dist_\Sigma(x,p) : p \in \mathrm{Sing}(\Sigma)\}\,.$$

And by Lemma 2.3.1, we can find $\tau(\Sigma, M, g) << 1$ such that $\{B^g_{4\tau}(p)\}_{p\in\mathrm{Sing}(\Sigma)}$ are disjoint, and that for each singular point p, $\Sigma \cap B^g_{4\tau}(p)$ can be parametrized by the tangent cone $\mathbf{C}_p$ of Σ. When working near a certain singularity p, we shall stick to this parametrization. We may abuse the notations and simply write any point $x \in \mathcal{B}_{2\tau}(p)$ as $x = r\omega$. Then near each $p \in \mathrm{Sing}(\Sigma)$, $\rho(r,\omega) \sim r$ if $r \leq 2\tau$.

Denote for simplicity $L = L_\Sigma$ and $L_\Lambda := L + \Lambda$. By (5.4.1), $-L_\Lambda$ is strictly positive, with the first L^2-eigenvalue $\geq \Lambda > 0$. Fixed $p_0 \in \Sigma \setminus \mathcal{B}_{4\tau}(\mathrm{Sing}(\Sigma))$. For each $p \in \mathrm{Sing}(\Sigma)$, let G_p be the Green's function of L_Λ at p constructed in Corollary 5.3.20 with $G_p(p_0) = 1$. Let $G := \sum_{p\in\mathrm{Sing}(\Sigma)} G_p$. By Corollary 5.3.20 and 5.3.16, $G > 0$ satisfies the equation $LG = -\Lambda \cdot G$ on Σ and growth estimate

$$G(x)/\rho(x)^{\gamma_1^-(\mathbf{C}_p)+\epsilon} \to +\infty, \quad \text{as } x \to p\,,$$

for every $\epsilon > 0$, where recall $\gamma_1^-(\mathbf{C}_p)$ is an asymptotic spectrum of $\mathbf{C}_p$ defined in (5.2.6). Therefore, there's a sufficiently large $T_0 > 0$ such that

$$\{x \in \Sigma : G(x)\rho(x)^{-1} > T_0\} \subset \mathcal{B}_\tau(\mathrm{Sing}(\Sigma))\,.$$

From now on in this section, every constant will depend at least on $M, g, \Sigma, \Lambda, \tau, p_0$.

For each $T > T_0$ and $p \in \mathrm{Sing}(\Sigma)$, let

$$r_p(T) := \sup\{R : G(x)\rho(x)^{-1} > T \text{ on } \mathcal{B}_R(p)\}\,; \tag{5.4.2}$$

$$\Sigma_T := \Sigma \setminus \bigcup_{p\in\mathrm{Sing}(\Sigma)} \mathcal{B}_{r_p(T)}(p)\,. \tag{5.4.3}$$

By Corollary 5.3.17, there's a $T_1 > T_0$ such that on $[T_1, +\infty)$, $r_p(T)$ varies continuously in T, Σ_T are smooth manifolds with boundary; Also by Harnack inequality, for every $p \in \mathrm{Sing}(\Sigma)$,

$$T \cdot r_p(T) = \inf_{\partial\Sigma_T\cap\mathcal{B}_\tau(p)} G \le \sup_{\partial\Sigma_T\cap\mathcal{B}_\tau(p)} G \le C(\Sigma,\Lambda,\tau)T\cdot r_p(T)\,. \tag{5.4.4}$$

Σ_T will be the spaces on which we shall build graph.

For each $T >> T_1$, we shall choose specific small positive function $\varphi^+ \in C^\infty(\partial\Sigma_T)$ and construct a graph of some positive function $u^+ \in C^\infty(\Sigma_T)$ with vanishing mean curvature and boundary $\mathrm{graph}_\Sigma \varphi^+$. Such a graph will be called an **exterior hypersurface** with boundary φ^+.

Then near each singularity p, we solve Plateau problem with boundary $\mathrm{graph}_\Sigma(\varphi^+)$ in $\mathcal{B}_\tau(p)$ to find a minimizing hypersurface, called an **interior hypersurface** with boundary $\varphi+$ near p. The union of these exterior and interior hypersurfaces, denoted by Σ_T^+, is piecewise smooth and will lie in M_+ for sufficiently large T. Repeat the same process to construct Σ_T^- lying in M_-. Then for each $t \in (0,1)$, a reparametrization $T = T(t) \ge T_1$ will be specified below such that regions U_t bounded by $\Sigma^\pm_{T(t)}$ are what we want.

The mean convexity of ∂U_t will be derived by a blow up argument letting $T \to +\infty$ and using the Hardt-Simon foliation as barriers. To make it precise, we need an existence result of exterior graphs over Σ_T with uniform growth estimate near each singularity. We assert here that this existence result does **NOT** require the tangent cones of Σ to be strictly stable or minimizing.

Theorem 5.4.3. *Let $\alpha \in (0,1)$. Then there exists $C = C(\alpha) > 1$, $T_2(\alpha) > T_1$ and $\delta_5 = \delta_5(\alpha) > 0$ sufficiently small such that, for each $T \in (T_2, +\infty)$, each $0 < \delta \leq \delta_5$ and each $\varphi \in C^{2,\alpha}(\partial\Sigma_T)$ with estimate near each singularity p,*

$$\frac{1}{r_p(T)}|\varphi| + |\nabla\varphi| + r_p(T)|\nabla^2\varphi| \leq \delta\,, \tag{5.4.5}$$

there's a unique $u \in C^{2,\alpha}(\Sigma_T)$ solving

$$\begin{cases} \mathscr{M}_\Sigma u(x) = 0, & \text{on } \Sigma_T\,; \\ u = \varphi, & \text{on } \partial\Sigma_T\,. \end{cases} \tag{5.4.6}$$

*and satisfying the **pointwise estimate**: for every $x \in \Sigma_T$,*

$$\begin{aligned} &\frac{|u(x)|}{\rho(x)} + |\nabla u(x)| + \\ &\sup\left\{\rho(x)^\alpha \cdot \frac{|\nabla u(x) - \nabla u(y)|}{dist_\Sigma(x,y)^\alpha} : y \in \Sigma_T, \frac{1}{10} \leq \frac{\rho(x)}{\rho(y)} \leq 10\right\} \leq C\delta \cdot \frac{G(x)}{T\rho(x)}\,. \end{aligned} \tag{5.4.7}$$

where $\mathscr{M}_\Sigma := \mathscr{M}^g$ is the minimal surface operator on Σ specified in Theorem A.0.1.

Remark 5.4.4. By the uniqueness of solution above and the estimate (5.4.7), we know further that u varies continuously with respect to φ in C^2-norm and in T.

We first finish the proof of Theorem 5.4.1 assuming Theorem 5.4.3 to be true.

Proof of Theorem 5.4.1. Let δ_5 be determined in Theorem 5.4.3. For each singularity p with strictly minimizing tangent cone $(\mathbf{C}_p, \nu_p)$ at p, let S_p be the cross section of $\mathbf{C}_p$. For sufficiently small $\delta_6 < \delta_5/2$, there's a leaf of Hardt-Simon foliation $\Sigma_p^+ \subset \mathbb{R}^{n+1}$ and a positive function $u_p^+ \in C^\infty(\mathbf{C}_p)$ satisfying,

$$\begin{aligned} \Sigma_p^+ \setminus \mathbb{B}^{n+1}_{1/4} &= \mathrm{graph}_{\mathbf{C}_p}(u_p^+) \setminus \mathbb{B}^{n+1}_{1/4}\,; \\ \|u_p^+(1,\cdot)\|_{C^2(S_p)} = \delta_6 &\leq C \inf_{S_p} u_p^+(1,\cdot)\,. \end{aligned} \tag{5.4.8}$$

We fix such a δ_6 from now on so that (5.4.8) is true with certain u_p^+ for every $p \in \mathrm{Sing}(\Sigma)$.

Now for each $T > T_2$, define $\varphi_T^+ \in C^2(\partial\Sigma_T)$ to be

$$\varphi_T^+(r_p(T), \omega) := r_p(T) u_p^+(1, \omega), \quad \forall \omega \in S_p \,. \tag{5.4.9}$$

By a standard compactness argument we have, for sufficiently large T, φ_T^+ satisfies (5.4.5) near each singularity p with $\delta = 2\delta_6$, and

$$\inf_{\partial\Sigma_T} \varphi_T^+ / r_p(T) \geq \delta_6 / C \,.$$

Let u_T^+ be the solution to (5.4.6) with $\varphi = \varphi_T^+$ satisfying the pointwise estimate (5.4.7). Then $\Sigma_T^{+,ext} := \mathrm{graph}_{\Sigma_T}(u_T^+)$ is a smooth minimal graph over Σ_T with boundary $\mathrm{graph}_{\Sigma}(\varphi_T^+)$.

Claim 1. $u_T^+ > 0$ for every $T \in (T_2, +\infty)$.

Proof of Claim 1. For fixed $T > T_1$ and every $s \in (0, 1]$, consider the solution $u_{T,s}^+$ of (5.4.6) on Σ_T with $\varphi = s\varphi_T^+$. Then by Theorem 5.4.3, $u_{T,s}^+$ is unique and varies continuously in s. Moreover, by (5.4.7) and the definition of φ_T^+ we have, $u_{T,s}^+ > 0$ in some neighborhood of $\partial\Sigma_T$ independent of s.

When $s \to 0_+$, by (5.4.7) and classical elliptic estimate [GT01], $u_{T,s}^+/s \to u_{T,0}^+$ in $C^2(\Sigma_T)$ for some $u_{T,0}^+$ satisfying $L\hat{u}_{T,0}^+ = 0$ on Σ_T and $\hat{u}_{T,0}^+ = \varphi_T^+ > 0$ on $\partial\Sigma_T$. Then by the strict positivity of $-L$, $\hat{u}_{T,0}^+ > 0$ on Σ_T. Hence $u_{T,s}^+ > 0$ on Σ_T for $s << 1$.

Suppose for contradiction that $\{u_T^+ \leq 0\} \neq \emptyset$. Consider $\inf\{s \in (0,1] : \{u_{T,s}^+ \leq 0\} \neq \emptyset\} =: s_0 > 0$. By continuous dependence of $u_{T,s}^+$ in s, $u_{T,s_0}^+ \geq 0$, nonzero on $\partial\Sigma_T$ and vanishes somewhere in $\mathrm{Int}(\Sigma_T)$. This contradicts to strong maximum principle since 0 is also a solution to (5.4.6) with $\varphi = 0$. □

Now for each $p \in \mathrm{Sing}(\Sigma)$, let $\Sigma_{T,p}^{+,int}$ be the minimizing hypersurface with bound-

ary $\partial\Sigma_T^{+,ext} \cap \mathcal{B}_\tau(p)$. Such hypersurface is unique and smooth for $T \in (T_3, +\infty)$ for some $T_3 > T_2$ by the uniqueness of Hardt-Simon's foliation Σ_p^+, nondegeneracy of $L_{\Sigma_p^+}$ on bounded subsets, Allard regularity theorem [All72, Sim83b] and implicit function theorem. Let

$$\Sigma_T^+ := \Sigma_T^{+,ext} \cup \left(\bigcup_{p \in \mathrm{Sing}(\Sigma)} \Sigma_{T,p}^{+,int} \right) ,$$

with normal field ν_T^+ in M pointing away from Σ, defined in the interior of each smooth component.

Claim 2. For each $p \in \mathrm{Sing}(\Sigma)$, let $\nu_T^{+,ext} := \nu_T^+|_{\Sigma_T^{+,ext}}$ be the normal field of $\Sigma_T^{+,ext}$ in M (pointing away from Σ), and $\xi_{T,p}^{+,int}$ be the inward normal field of $\partial\Sigma_{T,p}^{+,int}$ in $\Sigma_{T,p}^{+,int}$. Then there exists $T_4 > T_3$ such that for every $T \in (T_4, +\infty)$,

$$\nu_T^{+,ext} \cdot \xi_{T,p}^{+,int} < 0, \qquad \text{on } \partial\Sigma_{T,p}^{+,int} .$$

Proof of Claim 2. Identify $\Sigma_T^{+,ext}$ as a hypersurface with boundary in $T_pM \approx \mathbb{R}^{n+1}$ using $\exp_p^M$. Suppose up to a subsequence,

$$\frac{1}{r_p(T)} \Sigma_T^{+,ext} \to \Sigma_p^{+,ext}, \qquad \text{as } T \to +\infty .$$

By (5.4.7), the convergence is in C^1 sense as an exterior graph over $\mathbf{C}_p$, and $\Sigma_p^{+,ext}$ is a smooth graph of some $u_p^{+,ext} > 0$ over $\mathbf{C}_p \setminus \mathbb{B}_1^{n+1}$ with vanishing mean curvature. Moreover, by the boundary condition (5.4.8) & (5.4.9) in (5.4.6),

$$u_p^{+,ext}(1, \omega) = u_p^+(1, \omega), \qquad \text{for every } \omega \in S_p ;$$

and by (5.4.7), Corollary 5.3.17 and the definition of strictly minimizing, we have $u_p^{+,ext} < u_p^+$ near ∞. Hence, by considering $\inf\{R > 1 : R \cdot \Sigma_p^+ \text{ lies above } \Sigma_p^{+,ext}\}$ and

using strong maximum principle for minimal hypersurfaes, we see that

$$u_p^{+,ext} < u_p^+ \quad \text{on } \mathbf{C}_p \setminus \mathrm{Clos}(\mathbb{B}_1^{n+1})\,. \tag{5.4.10}$$

Since both $u_p^{+,ext}$ and u_p^+ satisfies the minimal surface equation on $\mathbf{C}_p \setminus \mathrm{Clos}(\mathbb{B}_1^{n+1})$, by (5.4.10) and Hopf boundary lemma,

$$\partial_r u_p^{+,ext}(1,\omega) < \partial_r u_p^+(1,\omega), \quad \text{for every } \omega \in S_p\,.$$

And since the convergence of $\frac{1}{r_p(T)}\Sigma_T^{+,ext}$ is in C^1 sense, the strict inequality in claim 2 is true for sufficiently large T. □

With Claim 2, (Σ_T^+, ν_T^+) is mean convex in $B_{2\tau}^M(p)$ for each $p \in \mathrm{Sing}(\Sigma)$ and all $T > T_4$ in a viscosity sense. Let U_T^+ be the open domain bounded by Σ_T^+ and $\mathrm{Clos}(\Sigma)$. Using the same blow up argument as Claim 2, one can show that there's a $T_5 > T_4$ such that for every $T \in (T_5, +\infty)$,

$$\partial\Sigma_{T+s}^{+,int} \subset U_T^+, \quad \forall s > 0\,.$$

Hence by the uniqueness of solutions to (5.4.6), the uniqueness of $\Sigma_{T,p}^{+,int}$ and a barrier argument similar to Claim 1, we see that for every $T > T' > T_5$, $\Sigma_T^+ \subset U_{T'}^+$. In particular, combined with the uniqueness of solutions to (5.4.6) and $\Sigma_{T,p}^{+,int}$, we have $\{\Sigma_{T'}^+\}_{T>T'}$ forms a foliation of $U_{T'}^+$.

Similarly, one can construct Σ_T^- on the other side of Σ satisfying the similar properties for $T > T_6$.

Let $\bar{T} = max\{T_5, T_6\}$ and U_t be the domain bounded by $\Sigma_{\bar{T}+1/t-1}^+$ and $\Sigma_{\bar{T}+1/t-1}^-$, $t \in (0,1)$. (1)-(4) in Theorem 5.4.1 are easily verified by the constructions above. □

Now we start to prove Theorem 5.4.3. We first introduce the following weighted

norms for functions u defined on $E \subset \Sigma$. Let $\alpha \in (0,1)$, $q \geq 1$, $k \in \mathbb{N}$.

$$\begin{aligned}
\|u\|_{C_G^{k,\alpha}(E)} &:= \sum_{j=0}^{k} \|\nabla^j u \cdot \frac{\rho^j}{G}\|_{C^0(E)} \\
&\quad + \sup\left\{ \frac{|\nabla^k u(x) - \nabla^k u(y)|}{dist_\Sigma(x,y)^\alpha} \cdot \frac{\rho(x)^{k+\alpha}}{G(x)} : x \neq y \in E, \frac{1}{10} < \frac{\rho(x)}{\rho(y)} < 10 \right\}; \\
\|u\|_{L_G^q(E)} &:= \sup_{t>0} \left(t^{-n} \cdot \int_{E \cap \{t \leq \rho \leq 100t\}} (u/G)^q \, d\mathscr{H}^n(x) \right)^{1/q}; \\
\|u\|_{W_G^{k,q}(E)} &:= \sum_{j=0}^{k} \| |\nabla^j u| \rho^j \|_{L_G^q(E)} .
\end{aligned} \tag{5.4.11}$$

By applying Morrey's inequality in every annuli $\mathcal{A}_{t,100t}(p) \subset \Sigma_T$, for $q \geq n/(1-\alpha)$ we have,

$$\|u\|_{C_G^{1,\alpha}(\Sigma_T)} \leq C(\alpha, q) \|u\|_{W_G^{2,q}(\Sigma_T)} , \tag{5.4.12}$$

for every u and $T > T_1$. We emphasis that by Harnack inequality for G, the constant here doesn't depend on T.

Fix $q = n/(1-\alpha)$ from now on, and define for each $T > T_1$, $\|u\|_{X_T} := T \cdot \|u\|_{W_G^{2,q}(\Sigma_T)}$. Note that (5.4.5) hold for all $p \in \mathrm{Sing}(\Sigma)$ is equivalent to that $T \cdot \|\varphi\|_{C_G^2(\partial\Sigma_T)} \leq \delta$, and (5.4.7) is equivalent to $T \cdot \|u\|_{C_G^{1,\alpha}} \leq C\delta$, hence is implied by that $\|u\|_{X_T} \leq C\delta$ by (5.4.12).

Let $\mathscr{R}$ be the error term operator given by

$$\mathscr{M}_\Sigma u(x) =: -Lu + \mathscr{R} u .$$

By Lemma 2.3.1 and Theorem A.0.1, there exists $0 < \delta_2' << 1$ such that if $\|u_1\|_{X_T}, \|u_2\|_{X_T} \leq$

$\theta \leq \delta_2'$, then the following pointwise inequality holds,

$$|\mathscr{R}u_1 - \mathscr{R}u_2| \leq C_1\theta \cdot \frac{G}{\rho^3 T} \cdot f\,, \tag{5.4.13}$$

where

$$f := \left[(|u_1 - u_2| + \rho|\nabla(u_1 - u_2)|)\,(1 + \frac{\rho^2|\nabla^2 u_2| \cdot T}{G \cdot \theta}) + \rho^2|\nabla^2(u_1 - u_2)| \right].$$

Note that by definition and (5.4.12) we have,

$$\begin{aligned} \|f\|_{L^q_G(\Sigma_T)} &\leq C\left[(\|\frac{|u_1 - u_2|}{G}\|_{C^0} + \|\frac{\rho|\nabla(u_1 - u_2)|}{G}\|_{C^0}) \cdot (1 + \|u_2\|_{W^{2,q}_G} \cdot \frac{T}{\theta}) + \|u_1 - u_2\|_{W^{2,q}_G} \right] \\ &\leq C_1'\|u_1 - u_2\|_{X_T} \cdot T^{-1}\,. \end{aligned} \tag{5.4.14}$$

The following linear estimate is the key part of the proof.

Lemma 5.4.5. *$\exists\ C_2 = C_2(q) > 1$, $T_2(q) > T_1$ such that, if $T > T_2$, $v \in W^{2,q}(\Sigma_T)$ is the solution to*

$$\begin{cases} Lv = \frac{G}{\rho^3 T} \cdot f, & \text{on } \Sigma_T\,; \\ v = \phi, & \text{on } \partial\Sigma_T\,. \end{cases} \tag{5.4.15}$$

Then

$$\|v\|_{X_T} \leq C_2 T(\|f\|_{L^q_G(\Sigma_T)} + \|\phi\|_{C^2_G(\partial\Sigma_T)})\,.$$

Proof. The key is that C_2 doesn't depend on T. Note that by (5.4.3) and (5.4.4), $G/\rho^3 T \sim \rho^{-2}$ near $\partial\Sigma_T$. Hence, we need to carefully deal with the estimate near $\partial\Sigma_T$.

Step 1 (Local estimate) We shall work near any fixed singularity p of Σ and parametrize $\Sigma \cap \mathcal{B}_{2\tau}(p)$ by the tangent cone C_p or by $(0, +\infty) \times S_p$ as in Section

5.2. By Corollary 5.3.17, $\rho(x)\cdot\partial_r G/G(x)\to\gamma_1^-(C_p)$ as $x\to p$. Hence by Harnack inequality and the definition (5.4.2) of $r_p(T)$, for every $\sigma<\gamma_1^-(C_p)<\sigma'$ and every $T>T_1$,

$$C(\Sigma,\sigma)(\frac{\rho}{r_p(T)})^{\sigma}\leq\frac{G}{r_p(T)T}\leq C(\Sigma,\sigma')(\frac{\rho}{r_p(T)})^{\sigma'}\,, \tag{5.4.16}$$

on $\Sigma_T\cap\mathcal{B}_{4\tau}(p)$. We now fix a choice of $\sigma_p\in(-\infty,\gamma_1^-(C_p))\backslash\Gamma_{C_p}$ and $\sigma_p'\in(\gamma_1^-(C_p),(1+\sigma_p)/2)$.

Let $t_p=t_p(\sigma_p,\Sigma,M)\in(0,\tau)$ such that $L=L_{C_p}+\mathscr{R}_p$ satisfies the condition in Corollary C.0.5. The existence of such t_p is guaranteed by Lemma 2.3.1 and Theorem A.0.1. Also let $T_{2,p}>T_1$ such that $r_p(T_{2,p})<t_p/10$. By Corollary C.0.5, for every $T>T_{2,p}$, there exists a solution $v_p\in W^{2,2}_{\sigma_p}(\mathcal{B}_{t_p}(p))$ to the equation $Lv_p=\chi_{\Sigma_T}fG/(\rho^3T)$ on $\mathcal{B}_{t_p}(p)$ satisfying the weighted estimate,

$$\sup_{t\in(0,t_p)}\|v_p(t,\cdot)\|_{L^2(S_p)}\cdot t^{-\sigma_p}\leq C(\sigma_p,t_p)\|\frac{G}{\rho^3T}\cdot\chi_{\Sigma_T}f\|_{L^2_{\sigma_p-2}(\mathcal{B}_{t_p}(p))}\,. \tag{5.4.17}$$

And if we denote for simplicity $\mathcal{A}_j:=\mathcal{A}_{100^{-j}t_p,100^{-j+1}t_p}(p)$, then by definition,

$$\begin{aligned}
&\|\frac{G}{\rho^3T}\cdot\chi_{\Sigma_T}f\|^2_{L^2_{\sigma_p-2}(\mathcal{B}_{t_p}(p))}\\
&=\int_{\mathcal{B}_{t_p}(p)}\chi_{\Sigma_T}\cdot(\frac{G}{\rho^3T}f)^2\rho^{-2\sigma_p+4-n}\,dx\\
&=T^2\cdot\sum_{j\geq1}\int_{\mathcal{A}_j}(\frac{f}{G})^2\rho^{-2n/q}\cdot\chi_{\Sigma_T}\cdot(\frac{G}{T})^4\rho^{-2\sigma_p-2-n+2n/q}\,dx\\
&\leq C(q)T^2\|f\|^2_{L^q_G(\Sigma_T)}\cdot\sum_{j\geq1}\big(\int_{\mathcal{A}_j}\chi_{\Sigma_T}(\frac{G}{T})^{4q'}\rho^{(-2\sigma_p-2)q'-n}\,dx\,\big)^{1/q'}\\
&\leq C(q,\sigma_p,\sigma_p',t_p)T^2\|f\|^2_{L^q_G(\Sigma_T)}\cdot\int_{\mathcal{A}_{r_p(T),t_p}(p)}r_p(T)^4(\frac{\rho}{r_p(T)})^{4\sigma_p'}\cdot\rho^{-2\sigma_p-2-n}\,dx\\
&\leq C(q,\sigma_p,\sigma_p',t_p)r_p(T)^{2-2\sigma_p}T^2\|f\|^2_{L^q_G(\Sigma_T)}\,,
\end{aligned}$$

where the first inequality is by Holder inequality with $q'=q/(q-2)$; the second

inequality is by RHS of (5.4.16) and the last inequality is true since we choose $\sigma_p' < (1+\sigma_p)/2$. Combined with (5.4.17), the interior L^q estimate for elliptic equations and the Morrey's inequality we get the pointwise estimate on $\mathcal{B}_{t_p/2}(p)$,

$$|v_p| + \rho|\nabla v_p| \leq C(q, \sigma_p, \sigma_p', t_p)(\frac{\rho}{r_p(T)})^{\sigma_p} \cdot r_p(T)T \cdot \|f\|_{L^q_G(\Sigma_T)} \leq C\|f\|_{L^q_G(\Sigma_T)} \cdot G\,, \tag{5.4.18}$$

where the last inequality above follow from LHS of (5.4.16). Since we have fixed the choice of σ_p, σ_p', t_p, from now on every constant might depend on them in this proof.

Step 2 (Global C^0 estimate) Now let $T > T_2 := \sup_{p\in \mathrm{Sing}(\Sigma)} T_{2,p}$; Let v be the solution to the equation (5.4.15). Let $\bar{v} := v - \sum_{p\in\mathrm{Sing}(\Sigma)} \eta_p v_p$, where v_p is given by Step 1 satisfying the estimate (5.4.18) in $\mathcal{B}_{t_p/2}(p)$; $\eta_p \in C_c^\infty(\mathcal{B}_{t_p/2}(p))$ is a cut-off function which equals to 1 on $\mathcal{B}_{t_p/4}(p)$ and has $|\nabla\eta_p| + |\nabla^2\eta_p| \leq C(t_p)$.

By definition of v_p we have

$$L\bar{v} = \frac{G}{\rho^3 T} \cdot f(1 - \sum_{p\in\mathrm{Sing}(\Sigma)} \eta_p) - \sum_{p\in\mathrm{Sing}(\Sigma)} (2\nabla\eta_p \cdot \nabla v_p + v_p \Delta\eta_p)\,.$$

In particular, $\mathrm{spt}(L\bar{v}) \subset \Sigma \backslash \bigcup_{p\in\mathrm{Sing}(\Sigma)} \mathcal{B}_{t_p/4}(p)$ and by (5.4.18), $\|L\bar{v}\|_{L^q(\Sigma)} \leq C\|f\|_{L^q_G(\Sigma_T)}$, where C is independent of T.

Let $\mathcal{U}_0 := \Sigma \setminus \bigcup_{p\in\mathrm{Sing}(\Sigma)} \mathcal{B}_{t_p/8}(p)$ and $\bar{v}_0 \in W_0^{1,2}(\mathcal{U}_0)$ be the solution of $L\bar{v}_0 = L\bar{v}$ on $\mathcal{U}_0$. Then by global L^q estimate and Morrey's inequality,

$$\|\bar{v}_0\|_{C^1_G(\mathcal{U}_0)} \leq C\|\bar{v}_0\|_{W^{2,q}_G(\mathcal{U}_0)} \leq C\|f\|_{L^q_G(\Sigma_T)}\,. \tag{5.4.19}$$

Let $\eta_0 \in C_c^\infty(\mathcal{U}_0)$ be a cut-off function with value 1 on $\Sigma \setminus \bigcup_{p\in\mathrm{Sing}(\Sigma)} \mathcal{B}_{t_p/4}(p)$ and $|\nabla\eta_0| + |\nabla^2\eta_0| \leq C(\mathcal{U}_0)$. Then by (5.4.18) and the boundary condition in equation

(5.4.15), we have

$$\begin{cases} |L(\bar{v} - \eta_0 \bar{v}_0)| = |2\nabla\eta_0 \cdot \nabla\bar{v}_0 + \bar{v}_0 \Delta\eta_0| \leq C\|f\|_{L^q_G(\Sigma_T)} \cdot G, & \text{on } \Sigma_T\,; \\ |\bar{v} - \eta_0\bar{v}_0| = |v - \sum_{p\in \mathrm{Sing}(\Sigma)} v_p| \leq C(\|f\|_{L^q_G(\Sigma_T)} + \|\phi\|_{C^0_G(\partial\Sigma_T)}) \cdot G, & \text{on } \partial\Sigma_T\,. \end{cases}$$

But recall that by definition, $-LG = \Lambda \cdot G$ and $-L$ is strictly positive on Σ, hence by weak maximum principle for $\bar{v} - \eta_0\bar{v}_0$ and the estimate (5.4.18) (5.4.19), we have the following weighted C^0 estimate for v,

$$|v| \leq C(\|f\|_{L^q_G(\Sigma_T)} + \|\phi\|_{C^0_G(\partial\Sigma_T)}) \cdot G \quad \text{on } \Sigma_T\,, \tag{5.4.20}$$

where the constant C is independent of f, ϕ, T.

Step 3 (Global $W^{2,q}_G$ estimate) Note that since $0 < G/\rho T \leq C(\Sigma)$ on Σ_T, we have $|Lv| \leq C(\Sigma) f/\rho^2$. Hence the desired estimate for $\|v\|_{X_T}$ follows from (5.4.20) and interior and boundary L^q estimate on every domain $\{t < \rho < 100t\} \cap \Sigma_T$, $\forall t > 0$. This completes the proof of Lemma 5.4.5. □

Proof of Theorem 5.4.3. Let $\bar{C} > 1$ and $\delta_5 \in (0, 1/2\bar{C})$ TBD subsequently. For every $T > T_1$, $\delta \in (0, \delta_5]$ and $\varphi \in C^2(\partial\Sigma_T)$ with $T \cdot \|\varphi\|_{C^2_G(\partial\Sigma_T)} \leq \delta$, define

$$X_{T,\bar{C}\delta} := \{u \in W^{2,q}(\Sigma_T) : \|u\|_{X_T} \leq \bar{C}\delta\}\,.$$

Consider the map $\mathscr{T} : X_{T,\bar{C}\delta} \to W^{2,q}(\Sigma_T)$, $\mathscr{T}u =: v$ solves

$$\begin{cases} Lv = \mathscr{R}u, & \text{on } \Sigma_T\,; \\ v = \varphi, & \text{on } \partial\Sigma_T\,. \end{cases}$$

Note that by non-degeneracy of L, the equation above has a unique solution. We shall appropriately choose $\bar{C}$ and δ_5 to make $\mathscr{T}$ a contraction map from $X_{T,\bar{C}\delta}$ to

itself.

For every $u \in X_{T,\bar{C}\delta}$, let $v := \mathscr{T}u$. By taking $u_1 = u$ and $u_2 = 0$ in (5.4.13) and applying (5.4.14) and Lemma 5.4.5,

$$\|v\|_{X_T} \leq C_2(C_1C_1'\bar{C}\delta \cdot \|u\|_{X_T} + \delta) \leq C_2(C_1C_1'\bar{C}^2\delta^2 + \delta)\,. \tag{5.4.21}$$

For every $u_1, u_2 \in X_{T,\bar{C}\delta}$, let $v_i := \mathscr{T}u_i$. Also by (5.4.13), (5.4.14) and Lemma 5.4.5,

$$\|v_1 - v_2\|_{X_T} \leq C_2(C_1C_1'\bar{C}\delta \cdot \|u_1 - u_2\|_{X_T} + 0)\,. \tag{5.4.22}$$

Hence, taking $\bar{C} := 4C_2$ and $\delta_5 := 1/(C_1C_1'\bar{C}^2)$, we see from (5.4.21) and (5.4.22) that $\mathscr{T}$ is a contraction map from $X_{T,\bar{C}\delta}$ to itself. Therefore, the Theorem follows from Banach fixed point theorem. □

Remark 5.4.6. The uniqueness of solution u in the statement of Theorem 5.4.3 seems slightly stronger since we only assume $\|u\|_{C^{1,\alpha}_G}$ bound. However, by similar argument as Step 3 in the proof of Lemma 5.4.5, for sufficiently small δ' (not depending on T), (5.4.7) hold for u with $\delta = \delta'$ implies $\|u\|_{X_T} \leq C'\delta'$. Thus reduce the uniqueness of solutions with (5.4.7) estimate to the uniqueness of solutions in $X_{T,\bar{C}\delta}$ above, which comes from Banach fixed point theorem.

Remark 5.4.7. By Theorem 5.4.1 and strong maximum principle [SW89], Σ is homologically area minimizing in $\mathrm{Clos}(U_1)$ and is the unique stationary integral varifold supported in $\mathrm{Clos}(U_1)$.

Also notice that, directly by a compactness argument, Theorem 5.1.1 is proved in case when Σ is nondegenerate stable, even without assuming (3).

Remark 5.4.8. Theorem 5.4.1 also guarantees that a locally stable minimal hypersurface Σ must be uniquely area-minimizing near the singular point p at which the

tangent cone is regular and strictly minimizing. In fact, by Proposition 5.3.5 (5), one can change the metric outside a small neighborhood of a p to make Σ strictly stable. Then Theorem 5.4.1 provides a mean convex foliation of $\mathrm{Clos}(\Sigma)$, hence contains a small neighborhood of p, inside which $[\Sigma]$ is uniquely area minimizing.

5.5 Local Existence of LSMH under Metric Perturbation

The goal of this Section is to prove Theorem 5.1.1.

Let's begin with the basic set up and assumptions.

(S1) Let $(\Sigma, \nu) \subset (M^{n+1}, g)$ be a two-sided closed LSMH with only strongly isolated singularities; $M \setminus \mathrm{Clos}(\Sigma) = M_- \sqcup M_+$ where ν points into M_+.

(S2) The Jacobi operator L_Σ is nondegenerate in $\mathscr{B}_0(\Sigma)$ with $I := \mathrm{index}(\Sigma; M) \geq 1$;

(S3) $\forall p \in \mathrm{Sing}(\Sigma)$, the tangent cone $\mathbf{C}_p$ of Σ at p is strictly minimizing and strictly stable.

(S4) $\tau \in (0, \tau_0/10)$, where τ_0 is given by Lemma 2.3.1, $K > 1$; $\{U_j\}_{j\geq 1}$ is a decreasing family of smooth domains satisfying

- $\bigcap_{j\geq 1} U_j = \mathrm{Clos}(\Sigma)$;
- ∂U_j are mean convex in $B^M_{4\tau}(\mathrm{Sing}(\Sigma))$;
- $\partial U_j \cap M_\pm$ are graphs of smooth functions $u_j^\pm \in C^2(\Sigma)$ in $M \setminus B^M_\tau(\mathrm{Sing}(\Sigma))$ such that

$$|\nabla u_j^\pm| + |\nabla^2 u_j^\pm| \leq K|u_j^\pm| \quad \text{on } \Sigma \setminus B^M_\tau(\mathrm{Sing}(\Sigma))\,;$$

$$\sup_{\Sigma\setminus B^M_\tau(\mathrm{Sing}(\Sigma))} |u_j^\pm| \leq K \cdot \inf_{\Sigma\setminus B^M_\tau(\mathrm{Sing}(\Sigma))} |u_j^\pm| \to 0_+ \quad \text{as } j \to \infty\,.$$

Assuming (S1)-(S3), the existence of such K, τ and $\{U_j\}_{j\geq 1}$ in (S4) can be seen from the following argument. By Remark 5.4.8, for each $p \in \mathrm{Sing}(\Sigma)$, let $O_p \subset M$ be a smooth convex neighborhood of p in which $[\Sigma]$ is uniquely area-minimizing; Let $\Sigma^{\pm}_{p,j} \subset M_{\pm} \cap O_p$ be a family of area-minimizing hypersurfaces in O_p which converges to $\Sigma \cap O_p$ from both side; Such $\Sigma^{\pm}_{p,j}$ exists by solving Plateau problem with boundary to be the one-sided perturbation of $\partial O_p \cap \Sigma$; For every compact subset $Z \subset O_p$, we have $\mathrm{Sing}(\Sigma^{\pm}_{p,j}) \cap Z = \emptyset$ for $j >> 1$ by Allard Regularity [All72, Sim83b], Theorem 2.4.1 and a blow-up argument similar to [Sma93]. Hence U_j in (S4) can be constructed by (passing to a subsequence if necessary) gluing these $\Sigma^{\pm}_{p,j}$ with a constant function over a neighborhood of $\Sigma \setminus \bigcup_{p\in \mathrm{Sing}(\Sigma)} O_p$ by partition of unity.

Throughout this section every constant will depend on the quantities in (S1)-(S4). The proof of Theorem 5.1.1 is devided into the following two facts.

Theorem 5.5.1 (Strong min-max property)**.** *There exists $k_1 \geq 1$, $\Phi_0 : \mathrm{Clos}(\mathbb{B}^I_1) \to \mathcal{Z}_n(\mathrm{Clos}(U_{k_1}))$ $\mathbf{F}$-continuous and $\epsilon : (0,1] \to (0,1)$ non-decreasing such that*

(i) $\Phi_0(\mathbf{0}) = [\Sigma]$*;*

(ii) $\mathbf{M}(\Phi_0(z)) \leq \mathbf{M}(\Sigma) - \epsilon(r)$ *for every* $z \in \partial \mathbb{B}^I_r$*;*

(iii) If $r \in (0,1]$ *and* $\{\Phi_i\}_{i\geq 1} : \mathrm{Clos}(\mathbb{B}^I_r) \to \mathcal{Z}_n(\mathrm{Clos}(U_{k_1}))$ *is a family of* $\mathbf{F}$*-continuous map such that*

$$\limsup_{i\to\infty} \sup\{\mathbf{F}(\Phi_0(z), \Phi_i(z)) : z \in \partial \mathbb{B}^I_r\} = 0\,.$$

Then we have

$$\liminf_{i\to\infty} \sup\{\mathbf{M}(\Phi_i(z)) : z \in \mathbb{B}^I_r\} \geq \mathbf{M}(\Sigma)\,.$$

Also recall that we have introduced the notion of constraint embedded minimal hypersurfaces in Section 4.2. The following Lemma is a slight modification of Theorem 4.5.1.

Lemma 5.5.2 (c.f. Lemma 5.5.8)**.** *Let $\epsilon \in (0,1)$. There exists $k_2 = k_2(\epsilon) \geq 1$ such that for every $j \geq k_2(\epsilon)$, if V is a constrained embedded minimal hypersurface in* $\mathrm{Clos}(U_j)$ *with*

$$\mathscr{H}^n(\Sigma) \leq \|V\|(M) \leq (2-\epsilon)\mathscr{H}^n(\Sigma)\,,$$

then $V = |\Sigma|$.

Proof of Theorem 5.1.1. First recall that by Remark 5.4.7, Theorem 5.1.1 is already proved when Σ is non-degenerate stable. We now assume $\mathrm{index}(\Sigma) \geq 1$.

Since $\Sigma \subset (M,g)$ in Theorem 5.1.1 satisfies (S1)-(S3) above, we can choose K, τ and $\{U_j\}_{j\geq 1}$ as in (S4).

Let $k_1 \geq 1$ be given in Theorem 5.5.1, $k_2 \geq 1$ be given in Lemma 5.5.2; Let $\bar{k} \geq max\{k_1, k_2(1/2)\}$ be fixed; $R_\Sigma \in (0,1)$ is also fixed such that $\mathrm{spt}(\Phi_0(x)) \subset U_{\bar{k}+1}$ for all $x \in \mathrm{Clos}(\mathbb{B}^I_{R_\Sigma})$, where Φ_0 is specified in Theorem 5.5.1. Also let $\eta \in C^\infty_c(U_{\bar{k}+1}, [0,1])$ be a fixed cut off function with $\eta|_{U_{\bar{k}+2}} = 1$.

Let Π be the family of sequences of $\mathbf{F}$-continuous maps $S = \{\Phi_i : \mathrm{Clos}(\mathbb{B}^I_{R_\Sigma}) \to \mathcal{Z}_n(\mathrm{Clos}(U_{\bar{k}}))\}_{i\geq 1}$ such that

$$\sup\{\mathbf{F}(\Phi_i(z), \Phi_0(z)) : z \in \partial\mathbb{B}^I_{R_\Sigma}\} \to 0 \quad \text{ as } i \to \infty\,.$$

Hence by Theorem 5.5.1 we have

$$\begin{aligned}\mathbf{L}(\Pi; g) := \inf_{S=\{\Phi_i\}_{i\geq 1}\in\Pi} \limsup_{i\to\infty} \sup_{z\in\mathbb{B}^I_{R_\Sigma}} \mathbf{M}^g(\Phi_i(z)) &= \mathscr{H}^n(\Sigma)\\ &> \mathbf{L}_{\Phi_0,\partial;g} := \sup_{z\in\partial\mathbb{B}^I_{R_\Sigma}} \mathbf{M}^g(\Phi_0(z))\,.\end{aligned}$$

Observe that $\mathbf{L}(\Pi; g)$ and $\mathbf{L}_{\Phi,\partial;g}$ is Lipschitz in g in C^0_{loc} sense. Therefore, $\exists\, \epsilon_1 > 0$ such that for every $\|g' - g\|_{C^4} \leq \epsilon_1$, if write $g'_\eta := \eta g' + (1-\eta)g$, then $\mathbf{L}(\Pi; g'_\eta) > \mathbf{L}_{\Phi_0,\partial;g'_\eta}$. By Theorem 4.4.3, there exists a constrained embedded minimal hypersurface $V_{g'}$

in $(\mathrm{Clos}(U_t), g'_\eta)$ with $\|V_{g'}\|(M) = \mathbf{L}(\Pi; g'_\eta)$. Moreover, by Remark 4.4.4, there's a constant N_I only depending on I such that, we can choose $V_{g'}$ satisfying the following,

*For every $p \in M$, if $\{A_l = A^M_{r_l,s_l}(p)\}_{l=1}^{N_I}$ be a family of disjoint annuli in M centered at p with $s_l > r_l > 4s_{l+1}$; Then V is a constrained embedded **stable** minimal hypersurface in at least one of $\{A_l\}$.*

From this fact, we see that if $g'_j \to g$ in C^4, then $V_{g'_j}$ will subconverges in varifold sense to some constrained stationary integral varifold V_g in $\mathrm{Clos}(U_{\bar{k}}, g)$ with the same mass as Σ, which is stable locally in every sufficiently small annuli. Hence by Theorem 4.2.11 and 4.3.2, V_g is a constrained embedded stable minimal hypersurface; And by Lemma 5.5.2, we see that $V_g = |\Sigma|$.

Also note that since $V_{g'_j}$ has uniformly bounded mean curvature (bounded by the mean curvature of $\partial U_{\bar{k}}$), $\mathrm{spt}(V_{g'_j}) \to \mathrm{spt}(V_g) = \Sigma$ in Hausdorff distant sense. Therefore, for g' sufficiently close to g, we have $\mathrm{spt}(V_{g'}) \subset U_{\bar{k}+2}$. In particular, $V_{g'}$ is a locally stable minimal hypersurface in (M, g') and can be arbitrarily close to Σ by taking $\|g' - g\|_{C^4}$ small enough. This finishes the proof of Theorem 5.1.1. □

5.5.1 Strong min-max property for singular minimal hypersurfaces

The goal of this subsection is to prove the strong min-max property, namely Theorem 5.5.1. The proof closely follows the argument in [Whi94], The only difference is the treatment of singularities, which is based on the results in Section 5.2 and 5.4.

We start with some analytic preparation. Recall (2.3.1), Q_Σ is the quadratic form of second variation of Σ.

Lemma 5.5.3. *There exists $\phi_1, \phi_2, ...\phi_I \in C^\infty_c(\Sigma)$ and $\lambda, \mu > 0$ such that*

(i) For every $u \in span_{1\le i\le I}\{\phi_i\}$,

$$Q_\Sigma(u,u) \le -\lambda \|u\|^2_{L^2(\Sigma)}\,;$$

(ii) For every $v \in C^1_c(\Sigma)$,

$$Q_\Sigma(v,v) + \mu^2 \sum_{i=1}^{I} (\int_\Sigma v\cdot\phi_i\ )^2 \ge \lambda \int_\Sigma |\nabla v|^2 + v^2/\rho^2\,;$$

(iii) The matrix $[\langle \phi_i, \phi_j\rangle_{L^2(\Sigma)}]_{(i,j)\in I\times I}$ *is nondegenerate.*

Proof. Let $\psi_1, ..., \psi_I \in \mathscr{B}_0(\Sigma)$ be the first I pairwise orthogonal unit L^2-eigenfunctions of $-L_\Sigma$. Since $C^\infty_c(\Sigma)$ is dense in $\mathscr{B}_0(\Sigma)$, we can find $\phi_i \in C^\infty_c(\Sigma)$ such that $\sup_{1\le i\le I} \|\phi_i - \psi_i\|_{\mathscr{B}(\Sigma)} \le \varepsilon$, where $\varepsilon \in (0,1)$ TBD.

To prove (i), notice that for every $a \in \mathbb{R}^I$,

$$Q_\Sigma(\sum_i a_i\psi_i, \sum_i a_i\psi_i) \le -|\lambda'| \sum_i a_i^2\,,$$

where λ' be the largest negative eigenvalue of $-L_\Sigma$; Also notice that

$$\begin{aligned} &|Q_\Sigma(\sum_i a_i\phi_i, \sum_i a_i\phi_i) - Q_\Sigma(\sum_i a_i\psi_i, \sum_i a_i\psi_i)| \\ =\ &|\sum_{1\le i,j\le I} a_ia_j(Q_\Sigma(\phi_i,\phi_j) - Q_\Sigma(\psi_i,\psi_j))| \le C(\Sigma,I)\varepsilon\cdot\sum_i a_i^2\,. \end{aligned}$$

Hence by taking $\varepsilon < |\lambda'|/2C(\Sigma,I)$, we have (i) for $\lambda \le |\lambda'|/2$.

To prove (ii), first notice that for every $v \in \mathscr{B}_0(\Sigma)$,

$$Q_\Sigma(v,v) + |\mu'| \sum_{i=1}^{I} (\int_\Sigma v\cdot\psi_i\ )^2 \ge \lambda''\|v\|^2_{L^2(\Sigma)}\,,$$

where λ'' be the smallest positive eigenvalue of $-L_\Sigma$ and μ' be the difference between

λ'' and the first eigenvalue of $-L_\Sigma$. Also note that,

$$(\int_\Sigma v\cdot\psi_i\ )^2-(\int_\Sigma v\cdot\phi_i\ )^2\le C'(\Sigma)\varepsilon\cdot\|v\|^2_{L^2}\,.$$

Hence if take $\varepsilon<\lambda''/(2I|\mu'|C'(\Sigma))$, $\mu^2\ge|\mu'|$ and $\lambda\le\lambda''/2$, then we have,

$$Q_\Sigma(v,v)+\mu^2\sum_{i=1}^I(\int_\Sigma v\cdot\phi_i\ )^2\ge\lambda\|v\|^2_{L^2(\Sigma)}\,. \tag{5.5.1}$$

Also note that since each tangent cone of Σ is strictly stable, by Lemma C.0.1 and Lemma 2.3.1, there is some small neighborhood $\mathcal{V}$ of $\mathrm{Sing}(\Sigma)$ and constant $\vartheta(\Sigma)>0$ such that,

$$Q_\Sigma(v,v)-\vartheta\int_{\mathcal{V}}v^2/\rho^2\ge0,\quad\forall v\in C^1_c(\mathcal{V})\,.$$

Hence by a similar argument as Lemma 5.3.1 we have, for some $C''(\Sigma)>0$,

$$Q_\Sigma(v,v)-\vartheta\int_\Sigma v^2/\rho^2+C''\|v\|^2_{L^2(\Sigma)}\ge0,\quad\forall v\in C^1_c(\Sigma)\,. \tag{5.5.2}$$

Also by Lemma 2.3.1, $\big||A_\Sigma|^2+\mathrm{Ric}_M(\nu,\nu)\big|\le C'''(\Sigma)/\rho^2$ on Σ, hence $\forall v\in C^1_c(\Sigma)$,

$$\begin{aligned}&(1+a)\left(Q_\Sigma(v,v)+C''\|v\|^2_{L^2(\Sigma)}\right)\\&\ge a\int_\Sigma|\nabla v|^2+\left(Q_\Sigma(v,v)-\vartheta\int_\Sigma v^2/\rho^2+C''\|v\|^2_{L^2(\Sigma)}\right)\ge a\int_\Sigma|\nabla v|^2\,,\end{aligned} \tag{5.5.3}$$

where $0<a\le\vartheta/C'''$. Hence Lemma 5.5.3 (ii) follows by combining (5.5.1), (5.5.2) and (5.5.3) and take $0<\lambda\le min\{1,\lambda''/(2C'')\}\cdot min\{\vartheta,a/(1+a)\}/2$.

To prove (iii), it suffices to notice that $|\langle\phi_i,\phi_j\rangle_{L^2(\Sigma)}-\delta_{ij}|\le10\varepsilon$ and take $\varepsilon<1/100I^2$. □

Remark 5.5.4. We see from its proof that Lemma 5.5.3 (i) and (5.5.1) holds for any Σ satisfying L^2 nonconcentration property; while the proof of (ii) essentially use strictly stability of tangent cones of Σ.

Now we take $\vec{f} = (f_1, ..., f_I) \in C_c^\infty(M \setminus \mathrm{Sing}(\Sigma); \mathbb{R}^I)$ such that $\vec{f} = 0$ on Σ and $\nu(f_i) = \mu\phi_i$ on Σ; Let $\bar{\tau} \in (0, \tau)$ be such that $B^M_{\bar{\tau}}(\mathrm{Sing}(\Sigma)) \cap \mathrm{spt}(\vec{f}) = \emptyset$.

Let $\zeta \in C^2(\mathbb{R}^I; [0, +\infty))$ such that $\zeta(\mathbf{0}) = 0$, $\nabla^2\zeta(\mathbf{0}) = id$ and $\sup_{\mathbb{R}^I} |\nabla\zeta| \cdot \sup_M |\vec{f}| < 1$. Define for $T \in \mathcal{Z}_n(M)$,

$$\mathbf{A}^*(T) := \mathbf{M}(T) + \zeta\left(\int_M \vec{f}(x)\ d\|T\|(x)\right).$$

Lemma 5.5.5. *$\boldsymbol{A}^*$ is lower semi continuous in flat topology.*

Proof. The proof is the same as [Whi94, Section 3, Claim 2]. □

For a general parametric elliptic integrand F, the regularity problem of (F, ct^α, δ)-almost minimizing surfaces is studied in [Alm75,SS82,Bom82], and is used in [Whi94] to gain better regularity of convergence of minimizers for F^* in a smaller and smaller neighborhood of the given surface stationary with respect to F. $F \equiv 1$ corresponds to the mass functional.

Now we observe that since U_j are mean convex near $\mathrm{Sing}(\Sigma)$ and $\vec{f}$ vanishes near $\mathrm{Sing}(\Sigma)$, minimizers of $\mathbf{A}^*$ in $\mathrm{Clos}(U_t)$ are minimal near each singularity. Hence the same argument in [Whi94, Section 3, Claim 1] yields the following,

Lemma 5.5.6. *For every $\alpha \in (0, 1)$, there exists some constant $c, k_2 > 0$ such that for $j \geq k_2$, every minimizer $T \in \mathcal{Z}_n(\mathrm{Clos}(U_j))$ of $\boldsymbol{A}^*$ among cycles homologous to $[\Sigma]$ in U_j is (ct^α, δ_7)-almost minimizing in M.*

In particular, if T_j is a minimizer of $\boldsymbol{A}^*$ in $\mathrm{Clos}(U_j)$ homologous to $[\Sigma]$, then $T_j \to [\Sigma]$ in flat topology as $j \to \infty$ and

(i) T_j are homologically area minimizing in $B^M_{\bar{\tau}}(\mathrm{Sing}(\Sigma)) \cap \mathrm{Clos}(U_j)$;

(ii) There are $u_j \in C^1(\Sigma)$ such that for every $U \subset\subset M \setminus \mathrm{Sing}(\Sigma)$,

$$T_j \llcorner U = [\mathrm{graph}_\Sigma(u_j)] \llcorner U, \qquad \text{for } j >> 1\,.$$

Lemma 5.5.7. *There exists $k_1 \geq k_2$ such that $[\Sigma]$ is the only $\boldsymbol{A}^*$-minimizer in its homology class in $\mathcal{Z}_n(\mathrm{Clos}(U_{k_1}))$.*

Proof. The proof closely follows [Whi94, Theorem 2]. The major extra effort is made to estimate error terms near singularities.

Suppose for contradiction that after passing to a subsequence, there are $[\Sigma] \neq T_j \in \mathcal{Z}_n(\mathrm{Clos}(U_j))$ homologous to $[\Sigma]$, i.e. $T_j - [\Sigma] = \partial P_j$ for some $P_j \in \mathbf{I}_{n+1}(M)$ supported in $\mathrm{Clos}(U_j)$, which minimize $\mathbf{A}^*$ among its homology class in $\mathcal{Z}_n(\mathrm{Clos}(U_j))$.

Let $0 \neq u_j$ be the graphical functions of T_j over Σ as described in Lemma 5.5.6 (ii). Moreover, for each $p \in \mathrm{Sing}(\Sigma)$, since each $\mathbf{C}_p$ is strictly stable, one can take $\sigma \in \big(-(n-2)/2, \inf_{p\in\mathrm{Sing}(\Sigma)} \gamma_1^+(\mathbf{C}_p)\big)$; Let

$$r_j(p) := \inf\{t > 0 : T_j \llcorner A^M_{t,2\tau}(p) = [\mathrm{graph}_\Sigma(u_j)] \llcorner A^M_{t,2\tau}(p) \text{ and } \|u_j\|^*_{2,\mathcal{A}_{t,2\tau}(p)} \leq \delta_1\},$$

where $\delta_1 = \delta_1(\mathbf{C}_p, \sigma)$ is specified in Corollary 5.2.18. Clearly, $r_j(p) \to 0_+$ as $j \to \infty$; WLOG $r_j(p) < \tau$.

By Corollary 5.2.18, there exists $\Omega_0 \subset\subset \Sigma$ such that for $j >> 1$,

$$|u_j|/\rho + |\nabla u_j| \leq C(\sigma)\rho^{\sigma-1} \cdot \Big(\int_{\Omega_0} u_j^2\Big)^{1/2} \quad \text{on } \mathcal{A}_{2r_j(p),\tau}(p) \tag{5.5.4}$$

Claim. $\liminf_{j\to\infty} r_j(p)^{2\sigma-2} \int_{\Omega_0} u_j^2 > 0$.

Proof of Claim. Otherwise, consider $\hat{V}_j := (\eta_{p,r_j(p)})_\sharp V_j$. By (5.5.4), up to a subsequence, $\hat{V}_j \to \hat{V}_\infty$ for some stationary integral varifold $\hat{V}_\infty$ equal to $|\mathbf{C}_p|$ outside the ball of radius 2. Since $\mathbf{C}_p$ is area minimizing, by a barrier argument using Hardt-Simon foliations, $\hat{V}_\infty = |\mathbf{C}_p|$ in $\mathbb{R}^{n+1}$. By Allard regularity, this contradict to the minimality of $r_j(p)$. □

Let $\tilde{\mathcal{U}}_j := \Sigma \setminus \bigcup_{p\in\mathrm{Sing}(\Sigma)} \mathcal{B}_{2r_j(p)}(p)$; $\mathcal{U}_j := \Sigma \setminus \bigcup_{p\in\mathrm{Sing}(\Sigma)} \mathcal{B}_{4r_j(p)}(p)$. Take $j >> 1$ such that $\mathcal{U}_j \supset \Omega_0 \cup (\Sigma \cap \mathrm{spt}(\vec{f}))$.

Let $\xi_j \in C_c^\infty(\tilde{\mathcal{U}}_j)$ be a cut-off function which equal to 1 on $\mathcal{U}_j$ and $|\nabla \xi_j| \leq 10/r_j(p)$ on $\mathcal{B}_\tau(p)$; Let $\tilde{u}_j := u_j \cdot \xi_j$. Note that by (5.5.4),

$$\int_{\mathcal{B}_{4r_j(p)}(p)} |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2 \leq C(\sigma) r_j(p)^{2\sigma-2+n} \cdot \int_{\Omega_0} u_j^2 \,. \tag{5.5.5}$$

Now we are ready to estimate $\mathbf{A}^*(T_j) - \mathbf{A}^*(\Sigma)$. Work under Fermi coordinate near Σ, let $F(x,z,p) = F^0(x,z,p)$ be the volume element for graphs over Σ as in Theorem A.0.1, we have

$$\begin{aligned}
0 \geq\ & \mathbf{A}^*(T_j) - \mathbf{A}^*(\Sigma) \\
\geq\ & \int_{\mathcal{U}_j} \big(F(x,u_j,\nabla u_j) - 1\big) - \mathscr{H}^n(\Sigma \setminus \mathcal{U}_j) + \zeta\big(\int_\Sigma \vec{f}(x,u_j(x)) F(x,u_j,\nabla u_j)\ d\mathscr{H}^n(x)\big) \\
\geq\ & \underbrace{\Big[\int_{\mathcal{U}_j} dx \int_0^1 \partial_p F(x,su_j,s\nabla u_j)\cdot \nabla u_j + \partial_z F(x,su_j,s\nabla u_j)\cdot u_j\ ds\Big]}_{(I)} - \underbrace{\mathscr{H}^n(\Sigma\setminus\mathcal{U}_j)}_{(II)} \\
&+ \underbrace{\zeta\big(\int_\Sigma \vec{f}(x,u_j(x)) F(x,u_j,\nabla u_j)\ d\mathscr{H}^n(x)\big)}_{(III)} \,.
\end{aligned} \tag{5.5.6}$$

To estimate (I), notice that by Theorem A.0.1,

$$|\partial_p F(x,z,p) - p| + \rho\,\big|\partial_z F(x,z,p) + (|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu))z\big| \leq C(|z|/\rho + |p|)^2 \,.$$

Hence, combined with (5.5.4) and (5.5.5) we have,

$$\begin{aligned}
(I) &\geq \frac{1}{2}\int_{\mathcal{U}_j} |\nabla u_j|^2 - \big(|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu)\big)\, u_j^2 - C(|\nabla u_j| + |u_j|/\rho)^3 \\
&\geq \frac{1}{2}\int_\Sigma |\nabla \tilde{u}_j|^2 - \big(|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu)\big)\, \tilde{u}_j^2\ d\mathscr{H}^n \\
&\quad - C(\sigma,\Omega_0)\int_\Sigma (|\nabla \tilde{u}_j| + |\tilde{u}_j/\rho| + r_j(p)^{2\sigma-2+n})\cdot(|\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2) \\
&= \frac{1}{2}\int_\Sigma |\nabla \tilde{u}_j|^2 - \big(|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu)\big)\, \tilde{u}_j^2\ d\mathscr{H}^n + o\big(\int_\Sigma |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2\big) \,.
\end{aligned} \tag{5.5.7}$$

To estimate (II), it suffices to notice that by Claim 1, $r_j(p)^n = o(\int_{\Omega_0} u_j^2)$ as $j \to \infty$. Hence,

$$(II) \leq \sum_{p \in \mathrm{Sing}(\Sigma)} \mathscr{H}^n(\mathcal{B}_{4r_j(p)}(p)) \leq C(\Sigma) \cdot \sum_p r_j(p)^n = o(\int_\Sigma |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2)\,. \quad (5.5.8)$$

To estimate (III), note that by definition, $\zeta(v) \geq |v|^2/2 - C_\zeta |v|^3$ for some $C_\zeta > 0$; and by the choice of f,

$$|\int_\Sigma \vec{f}(x, u_j(x)) F(x, u_j, \nabla u_j) - \mu \vec{\phi} \cdot \tilde{u}_j \; d\mathscr{H}^n(x)| \leq C(f) \int_\Sigma |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2\,.$$

Hence, we have

$$(III) \geq \frac{\mu^2}{2} \big| \int_\Sigma \tilde{u}_j \cdot \vec{\phi} \; d\mathscr{H}^n \big|^2 - o(\int_\Sigma |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2)\,. \quad (5.5.9)$$

Now, plug (5.5.7), (5.5.8) and (5.5.9) into (5.5.6) and apply Lemma 5.5.3 (ii), we get,

$$\begin{aligned} 0 &\geq Q_\Sigma(\tilde{u}_j, \tilde{u}_j) + \mu^2 \big| \int_\Sigma \tilde{u}_j \cdot \vec{\phi} \; d\mathscr{H}^n \big|^2 - o(\int_\Sigma |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2) \\ &\geq \lambda \int_\Sigma |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2 \; d\mathscr{H}^n - o(\int_\Sigma |\nabla \tilde{u}_j|^2 + \tilde{u}_j^2/\rho^2)\,. \end{aligned}$$

This contradicts to that $\lambda > 0$ in Lemma 5.5.3 and that $u_j \neq 0$. Thus the proof is completed. □

Proof of Theorem 5.5.1. Let k_1 be specified in Lemma 5.5.7. Define $\Phi_0 : \mathrm{Clos}(\mathbb{B}_1^I) \to \mathcal{Z}_n(\mathrm{Clos}(U_{k_1}))$ by

$$\Phi_0(z) := \left[\mathrm{graph}_\Sigma \left(\bar{t} \cdot (z_1\phi_1 + z_2\phi + ... + z_I\phi_I)\right)\right]\,,$$

where $\bar{t} \in (0,1)$ small enough such that the image of Φ_0 is contained in $\mathcal{Z}_n(\mathrm{Clos}(U_{k_1}))$

and that (ii) in Theorem 5.5.1 hold for some $\epsilon(r) > 0$. Such $\bar{t}$ and $\epsilon(r)$ exists by Lemma 5.5.3 (i).

Also by Lemma 5.5.3 (iii), we can take $\bar{t} > 0$ even smaller such that

$$\vec{f}\Phi_0 : \mathrm{Clos}(\mathbb{B}^I_1) \to \mathbb{R}^I, \quad z \mapsto \int_M \vec{f}\, d\|\Phi_0(z)\|\,.$$

has non-zero degree at $\mathbf{0} \in \mathbb{R}^I$. Fix the choice of $\bar{t}$ and then Φ_0 from now on.

Now if $\{\Phi_i\}_{i\geq 1}$ be a family of $\mathbf{F}$-continuous sweepouts described in Theorem 5.5.1 (iii), then

$$\vec{f}\Phi_i : \mathrm{Clos}(\mathbb{B}^I_1) \to \mathbb{R}^I, \quad z \mapsto \int_M \vec{f}\, d\|\Phi_i(z)\|$$

C^0-converges to Φ_0 restricted to $\partial\mathbb{B}^I_1$, and hence also has non-zero degree at $\mathbf{0}$ for $i >> 1$. In particular, $\exists\ z_i \in \mathbb{B}^I_1$ such that $\vec{f}\Phi_i(z) = 0^I$. Therefore, by Lemma 5.5.7,

$$\sup\{\mathbf{M}(\Phi_i(z)) : z \in \mathbb{B}^I_1\} \geq \mathbf{M}(\Phi(z_i)) = \mathbf{A}^*(\Phi(z_i)) \geq \mathbf{A}^*([\Sigma]) = \mathbf{M}(\Sigma)\,,$$

for $i >> 1$, thus proves (iii) of Theorem 5.5.1. □

5.5.2 A rigidity result for constrained minimal hypersurfaces

As is introduced in Section 4.2, for a smooth domain $U \subset (M, g)$, an integral varifold $V \in \mathcal{IV}_n(\mathrm{Clos}(U))$ is called a *constrained embedded minimal hypersurface* in $\mathrm{Clos}(U)$ if it's a locally stable minimal hypersurface with optimal regularity inside U, locally sum of $C^{1,1}$ graphs with multiplicity near ∂U, possibly with self-touching on ∂U, and satisfies the variational inequality

$$\delta V(X) \geq 0\,,$$

for every vector field $X \in \mathscr{X}(M)$ which is tangent to ∂U or pointing inward U along ∂U. Note that by [Wan22b, Remark 2.9, (1)], the mean curvature vectors of these $C^{1,1}$ graph comprising V all points outward U along ∂U and equals to the mean curvature of ∂U almost everywhere on $\mathrm{spt}(V) \cap \partial U$. This is the only additional fact we shall use in the proof of Lemma 5.5.8.

[Wan22b] discusses the existence of such constrained embedded minimal hypersurface through min-max construction, as well as a local rigidity result [Wan22b, Theorem 5.1] near a smooth nondegenerate minimal hypersurface. The goal of this section is to prove an analogue of this local rigidity in the singular setting. The proof is almost the same, but additional efforts will be made to deal with error terms near singularities.

We still assume (S1)-(S4) holds in this subsection. Note that by strong maximum principle and mean convexity of ∂U_j in (S4) inside $B^M_{4\tau}(\mathrm{Sing}(\Sigma))$, any constrained embedded minimal hypersurface V in $\mathrm{Clos}(U_j)$ must be stationary (i.e have vanishing mean curvature) in $B^M_{4\tau}(\mathrm{Sing}(\Sigma))$.

Lemma 5.5.8. *Let $\epsilon \in (0,1)$. There exists $k_2 = k_2(\epsilon) \geq 1$ such that for every $j \geq k_2(\epsilon)$, if V is a constrained embedded minimal hypersurface in $\mathrm{Clos}(U_j)$ with*

$$\mathscr{H}^n(\Sigma) \leq \|V\|(M) \leq (2-\epsilon)\mathscr{H}^n(\Sigma)\,,$$

then $V = |\Sigma|$.

Proof. We shall argue by contradiction as is done in [Wan22b, Theorem 5.1].

Suppose for sake of contradiction that after passing to a subsequence, there exist constrained embedded minimal hypersurfaces $|\Sigma| \neq V_j \in \mathcal{IV}_n(\mathrm{Clos}(U_j))$, $j \to \infty$. Since by the choice of $\{U_j\}_{j\geq 1}$ in (S4), the mean curvature of ∂U_j on $\mathrm{spt}(h)$ tends to 0 as $j \to \infty$, we see that the mean curvature of V_j converges to 0 in L^∞. Hence by Allard Compactness Theorem [All72, Sim83b], up to a subsequence, $V_j \to V$ for some

stationary integral varifold $V \in \mathcal{IV}_n(M)$ supported in $\mathrm{Clos}(\Sigma)$ and has $\mathscr{H}^n(\Sigma) \leq \|V\|(M) \leq (2-\epsilon)\mathscr{H}^n(\Sigma)$. This implies $V = |\Sigma|$. Thus by Allard Regularity Theorem [All72, Sim83b], this convergence is in $C^{1,\alpha}$-graphical sense on each compact subset of Σ, $\forall \alpha \in (0,1)$.

Let $\alpha \in (0,1)$ be fixed from now on; For each singularity p, since the tangent cone $\mathbf{C}_p\Sigma$ is strictly stable and strictly minimizing, we can choose $\sigma_p \in (-(n-2)/2, \gamma_1^+(\mathbf{C}_p))$. Let $\delta_1 = \delta_1(\mathbf{C}_p, \sigma) > 0$ be specified in Corollary 5.2.18, $v_j \in C^{1,\alpha}(\Sigma)$ be the graphical function of V_j over Σ, $r_j(p) \to 0_+$ be such that

$$r_j(p) = \inf\{t > 0 : V_j \llcorner A^M_{t,2\tau}(p) = |\operatorname{graph}_\Sigma(v_j)| \llcorner A^M_{t,2\tau}(p),\ \|v_j\|^*_{2;\mathcal{A}_{t,2\tau}(p)} \leq \delta_3\}\,.$$

We take $j >> 1$ such that $r_j(p) < \tau/2$. Let $\sigma := \inf_{p\in\mathrm{Sing}(\Sigma)} \sigma_p > -(n-2)/2$. By Corollary 5.2.18, there exists $\Sigma \setminus \mathcal{B}_\tau(\mathrm{Sing}(\Sigma)) \subset \Omega_0 \subset\subset \Sigma$ such that for $j >> 1$,

$$|v_j|/\rho + |\nabla v_j| \leq C(\sigma)\rho^{\sigma-1} \cdot \left(\int_{\Omega_0} v_j^2\right)^{1/2}, \quad \text{on } \mathcal{A}_{2r_j,\tau}(p)\,, \tag{5.5.10}$$

where recall that ρ is the distant function on Σ to p. Hence by the same argument as in the proof of Lemma 5.5.7, Claim, we have,

$$\liminf_{j\to\infty} r_j(p)^{2\sigma-2} \int_{\Omega_0} v_j^2 > 0\,. \tag{5.5.11}$$

Let $F = F^0(x,z,p)$ be the area integrand of graph over Σ defined in Theorem A.0.1;

$$\mathscr{M} u := \mathscr{M}^g u = -div_\Sigma(\partial_p F(x,u,\nabla u)) + \partial_z F(x,u,\nabla u)\,,$$

be the minimal surface operator. Also for each $j \geq 1$, let $\mathcal{U}_j := \Sigma \setminus \bigcup_{p\in\mathrm{Sing}(\Sigma)} \mathcal{B}_{4r_j(p)}(p)$; and let $u_j^\pm$ be C^2 functions over Ω_0 such that under Fermi coordinates of M near Σ,

$$\{(x,t) : x \in \Omega_0, u_j^-(x) < t < u_j^+(x)\} \setminus B^M_{2\tau}(\mathrm{Sing}(\Sigma)) = U_j \setminus B^M_{2\tau}(\mathrm{Sing}(\Sigma))\,.$$

Since V_j are constrained embedded minimal hypersurfaces in $\mathrm{Clos}(U_j)$, we have

$$\begin{cases} v_j \cdot \mathscr{M} v_j \leq 0, & \text{weakly on } \Omega_0; \\ \mathscr{M} v_j = 0, & \text{on } \{u_j^- < v_j < u_j^+\} \cup (\mathcal{U}_j \setminus \Omega_0); \\ |\mathscr{M} v_j| \leq C|v_j|, & \text{weakly on } \mathcal{U}_j. \end{cases} \tag{5.5.12}$$

Let $c_j^2 := \int_{\Omega_0} |\nabla v_j|^2 + v_j^2/\rho^2$, then by (5.5.10), (5.5.12), Theorem A.0.1 and standard elliptic estimate [GT01], we know that up to a subsequence, $v_j/c_j \to \hat{v}_\infty$ in $W^{1,2}_{loc} \cap C^0_{loc}(\Sigma)$ for some $0 \neq \hat{v}_\infty \in W^{1,2}_{loc}(\Sigma)$ satisfying

$$\begin{cases} -\hat{v}_\infty \cdot L_\Sigma \hat{v}_\infty \leq 0, & \text{weakly on } \Sigma; \\ -L_\Sigma \hat{v}_\infty = 0, & \text{on } (\{\hat{u}_\infty^- < \hat{v}_\infty < \hat{u}_\infty^+\} \cap \Omega_0) \cup \Sigma \setminus \Omega_0\,, \end{cases} \tag{5.5.13}$$

where $\hat{u}_\infty^\pm$ are the subsequential limit of $u_j^\pm/c_j$, which is either everywhere $\pm\infty$ or everywhere finite and nonzero by the assumption in (S4) and that $|v_j| \leq max\{|u_j^+|, |u_j^-|\}$ on Ω_0 for $j \geq 1$. Moreover by (5.5.10), $\mathcal{AR}_p(\hat{v}_\infty) \geq \sigma > \gamma_1^-(C_p)$ for every singular point p. Hence by Corollary 5.3.18 (2) and Corollary 5.3.16, we have $\hat{v}_\infty \in \mathscr{B}_0(\Sigma)$.

Claim.

$$\int_\Sigma -\hat{v}_\infty \cdot L_\Sigma \hat{v}_\infty = \int_\Sigma |\nabla \hat{v}_\infty|^2 - (|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu))\hat{v}_\infty^2 \, dx \geq 0\,.$$

Note that by combining this Claim with (5.5.13), we see that $\hat{v}_\infty$ actually satisfies the Jacobi field equation $L_\Sigma \hat{v}_\infty = 0$ on Σ. Together with that $\hat{v}_\infty \in \mathscr{B}_0(\Sigma)$, this contradicts to the non-degeneracy of L_Σ and completes the proof of Lemma 5.5.8.

Proof of the Claim. By Theorem A.0.1,

$$
\begin{aligned}
0 &\le \|V_j\|(M) - \mathscr{H}^n(\Sigma) \\
&\le \int_{\mathcal{U}_j} F(x, v_j, \nabla v_j) - 1 \; d\mathscr{H}^n(x) + \sum_{p \in \mathrm{Sing}(\Sigma)} \|V_j\|(B^M_{8r_j(p)}(p)) \\
&\le \int_{\mathcal{U}_j} dx \int_0^1 \partial_p F(x, sv_j, s\nabla v_j) \cdot \nabla v_j + \partial_z F(x, sv_j, s\nabla v_j) \cdot v_j \; ds \\
&\quad + \sum_{p \in \mathrm{Sing}(\Sigma)} \|V_j\|(B^M_{8r_j(p)}(p)) \\
&= \underbrace{\frac{1}{2} \int_{\mathcal{U}_j} \partial_p F(x, v_j, \nabla v_j) \cdot \nabla v_j + \partial_z F(x, v_j, \nabla v_j) \cdot v_j \; dx}_{(I)} + \mathscr{E} \\
&\quad + \underbrace{\sum_{p \in \mathrm{Sing}(\Sigma)} \|V_j\|(B^M_{8r_j(p)}(p))}_{(II)} ,
\end{aligned}
\tag{5.5.14}
$$

where by the fact that

$$
\int_0^1 f(s) \; ds - (f(0) + f(1))/2 = (\int_0^1 f''(s)(s^2 - s) \; ds)/2 \, ,
$$

and Theorem A.0.1, we have

$$
\begin{aligned}
|\mathscr{E}| &\le C \int_{\mathcal{U}_j} \sup_{s \in [0,1]} \Big[|\partial^3_{ppp} F(x, sv_j, s\nabla v_j)||\nabla v_j|^3 + |\partial^3_{ppz} F(x, sv_j, s\nabla v_j)||\nabla v_j|^2 |v_j| \\
&\qquad + |\partial^3_{pzz} F(x, sv_j, s\nabla v_j)||\nabla v_j||v_j|^2 + |\partial^3_{zzz} F(x, sv_j, s\nabla v_j)||v_j|^3 \Big] \; dx \\
&\le C \int_{\mathcal{U}_j} (|\nabla v_j| + |v_j|/\rho)^3 \; dx = o(c_j^2) \, .
\end{aligned}
\tag{5.5.15}
$$

Here the last equality follows from (5.5.10) and the definition of c_j; And by (5.5.11)

and volume monotonicity for stationary varifolds in M,

$$(II) \leq \sum_{p\in \mathrm{Sing}(\Sigma)} C(\tau) r_j(p)^n \cdot \|V_j\|(B^M_\tau(p)) = o(c_j^2)\,. \tag{5.5.16}$$

Now to estimate (I), first observe by

$$|\partial_p F(x,z,p) - p| + \rho\left|\partial_z F(x,z,p) + \left(|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu)\right) z\right| \leq C(\tau)(|p| + |z|/\rho(x))^2\,.$$

Hence,

$$\begin{aligned}(I) &\leq \frac{1}{2}\int_{\mathcal{U}_j} |\nabla v_j|^2 - (|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu)) v_j^2 \, dx + C(\tau)\int_{\mathcal{U}_j} (|\nabla v_j| + |v_j|/\rho)^3 \\ &\leq \frac{1}{2}\int_{\mathcal{U}_j} |\nabla v_j|^2 - (|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu)) v_j^2 \, dx + o(c_j^2)\,.\end{aligned} \tag{5.5.17}$$

Note that by (5.5.10), as $j \to \infty$,

$$\int_{\mathcal{U}_j} |\nabla(v_j/c_j)|^2 - \left(|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu)\right)(v_j/c_j)^2 \, dx \to \int_\Sigma |\nabla \hat{v}_\infty|^2 - \left(|A_\Sigma|^2 + \mathrm{Ric}_M(\nu,\nu)\right)\hat{v}_\infty \, dx\,.$$

Therefore, combined with (5.5.17), (5.5.16) and (5.5.15), by dividing (5.5.14) by c_j^2 and taking $j \to \infty$, the proof of Claim is completed. □

□

5.5.3 Failure of local existence

We point out at the end of this Section that the strict minimizing assumption in Theorem 5.1.1 can not be dropped. In fact, we have the following example.

Proposition 5.5.9. *There exist a Riemannian manifold (M, g) and a two-sided strictly stable minimal hypersurface $(\Sigma, \nu) \subset (M, g)$ with only strongly isolated singularities satisfying (1), (3) but not (2) in Theorem 5.1.1, together with a neighborhood*

U of $\mathrm{Clos}(\Sigma)$ *and a family of metrics* $g_j \to g$ *in* C^∞ *on* M*, such that there's no stationary integral varifold supported in* U *under each* g_j*.*

Recall that by [Sim68, Lin87b, Sim74, Law91], the Simons cone over $\mathbb{S}^5 \times \mathbb{S}^1$,

$$\mathbf{C}^{5,1} = \{(x, y) \in \mathbb{R}^6 \times \mathbb{R}^2 : |x|^2 = 5|y|^2\} \subset \mathbb{R}^8\,,$$

is strictly stable but only minimizing in one side. Using this cone as the singularity model and applying the following Lemma 5.5.10, one can derive Proposition 5.5.9 by the following steps, the detailed proof of which is left to readers.

(i) Consider $\Sigma := \mathbf{C}^{5,1} \times \mathbb{R} \cap \mathbb{S}^8$, a LSMH in $(\mathbb{S}^8, g_0 := g_{round})$, with two singularities and a unit normal field ν pointing into the non-minimizing direction of $\mathbf{C}^{5,1}$. By Proposition 5.3.5 (5), there's a $0 \leq \vartheta \in C^\infty_c(\Sigma)$ such that $-L_{\Sigma,g_0} + \vartheta$ is strictly positive on $\mathscr{B}(\Sigma)$. Let $f_1 \in C^\infty_c(\mathbb{S}^8 \setminus \mathrm{Sing}(\Sigma))$ be such that

$$f_1|_\Sigma = 0, \qquad \nabla f_1|_\Sigma = 0, \qquad \nabla^2_{g_0} f_1(\nu, \nu)|_\Sigma = \vartheta\,.$$

Thus Σ is minimal and strictly stable in $(\mathbb{S}^8, g := e^{f_1} g_0)$.

(ii) Using the same argument as Theorem 5.4.1, one can construct piecewise smooth mean convex hypersurfaces $\{\Sigma_t\}_{t\in(-1,1)}$ foliating a neighborhood of $\mathrm{Clos}(\Sigma)$ with $\Sigma_0 = \Sigma$, such that the mean curvature vector of Σ_t point into ν's direction for every $t \neq 0$, and that on a domain containing $\mathrm{spt}(f) \cap \Sigma$, Σ_t are graphs over Σ.

(iii) Take a non-zero $0 \leq \varrho \in C^\infty_c(\Sigma)$ supported in $\mathrm{spt}(\vartheta)$, let $f_2 \in C^\infty(\mathbb{S}^8)$ be supported in $\mathrm{spt}(f_1)$ such that $f_2|_\Sigma = 0$, $\nu(f_2)|_\Sigma = -\varrho$. Thus for $|t| \leq \delta << 1$ and every $j \geq 1$, under metric $g_j := e^{f_2/j} g$, Σ_t still all have mean curvature pointing in the same direction as ν. Moreover, none of Σ_t are entirely minimal. Therefore, by strong maximum principle, there's no stationary integral varifold under g_j supported in the region bounded by $\Sigma_{\pm\delta}$.

Lemma 5.5.10. *Let $\mathbf{C} \subset \mathbb{R}^{n+1}$ be a regular stable minimal hypercone which is minimizing only in one-side. Let E_- be the component of $\mathbb{R}^{n+1} \setminus \mathbf{C}$ in which $\mathbf{C}$ is not minimizing, $\nu_{\mathbf{C}}$ be the normal field pointing away from E_-.*

Then the area-minimizer S among all integral currents with boundary $\partial[\mathbf{C} \cap \mathbb{B}_1^{n+1}]$ is unique and is a smooth minimal hypersurface supported in $\mathrm{Clos}(E_-)$ with boundary $\mathbf{C} \cap \partial\mathbb{B}_1^{n+1}$. Moreover,

(1) $\{r \cdot S\}_{r>0}$ foliates E_-;

(2) S is a minimal graph over $\mathbf{C}$ near ∂S of some function h satisfying

$$h(r,\omega) \leq 0, \quad h(1,\omega) = 0; \quad \partial_r h(1,\omega) > 0, \quad \forall \omega \in \mathbf{C} \cap \partial\mathbb{B}_1^{n+1}. \tag{5.5.18}$$

Proof. First note that since $\mathbf{C}$ is area-minimizing in the other side, by comparing mass, $\mathrm{spt}(S) \subset \mathrm{Clos}(E_-)$. Hence by strong maximum principle [Ilm96] and that $\mathbf{C}$ isn't minimizing in E_-, we have $\mathrm{spt}(S) \cap \mathbf{C} = \mathbf{C} \cap \partial\mathbb{B}_1^{n+1}$. In particular, $dist(\mathbf{0}, \mathrm{spt}(S)) > 0$. Also, by [HS79], S is smooth near boundary. We may abuse the notation and do not distinguish between S and $\mathrm{spt}(S)$.

The fact that $\{r \cdot S\}_{r>0}$ foliates E_- follows by considering $\sup\{r \in (0,1) : r \cdot S \cap S = \emptyset\}$ and using strong maximum principle [Sim87,Ilm96]. The uniqueness of such S also follows from the foliation structure.

Consider the Jacobi field $u := \nu_S \cdot \nabla_{\mathbb{R}^{n+1}} |x|^2/2$, where ν_S is the normal field of S_- pointing away from $\mathbf{C}$. Then since $\{r \cdot S\}_{r>0}$ foliates E_-, $0 \leq u \leq 1$ and is not identically 0. By strong maximum principle for L_S, $u > 0$ on $\mathrm{Int}(S)$, hence by Theorem 3.1.1, $\mathrm{Sing}(S) = \emptyset$.

Also by weak maximum principle for Δ_S, we have $u|_{\partial S} \leq \inf_S u < 1$. Hence S is a minimal graph over $\mathbf{C}$ near ∂S. By Hopf boundary Lemma, $u > 0$ on ∂S and hence (5.5.18) holds. □

Chapter 6

Analysis on Singular Minimal Hypersurfaces, III

The work described in this chapter is a combination of [Wan20, LW20, LW22].

6.1 Introduction

In this chapter, we prove the Full Generic Regularity Theorem 1.3.6 and then discuss its applications to the min-max theory.

Outline of the Proof

The starting point is a Regular Deformation Theorem. Let $\Sigma \subset (M^8, g)$ be a LSMH with $\mathrm{Sing}(\Sigma) \neq \emptyset$, by exploiting Theorem 5.2.12, we shall prove that under a technical *semi-nondegenerate* assumption, there's a generic way to conformally perturb the metric g to g', under which every LSMH sufficiently close to Σ must be regular near some singular point of Σ. For the precise statement, see Theorem 6.2.4. Heuristically, this means we can smooth out at least one singularity of Σ by perturbing metric.

There are two difficulties in applying this Regular Deformation Theorem induc-

tively and eventually eliminate all singularities. The first one is that the technical *semi-nondegenerate* assumption may not be preserved under perturbation a priori; The second one is, although those LSMH Σ' under perturbed metric is regular near some $p \in \mathrm{Sing}(\Sigma)$, it's still possible that near some other $p' \neq p \in \mathrm{Sing}(\Sigma)$, Σ' has multiple singularities. Note that if this happens, then the TOTAL NUMBER of singular points does NOT drop under metric perturbation.

To deal with the second problem, we introduce a way to count the number of singular points weighted by their tangent cones, such that this counting is upper-semi-continuous under **F**-convergence of LSMH. We show that there exists a finite weighted singular-set-counting like this, which we call *singular capacity*. See Theorem 6.2.5.

To deal with the first problem, our goal is to first derive a Sard-type Theorem asserting that under generic metric, every LSMH is semi-nondegenerate. To achieve this, for each LSMH $\Sigma \subset (M^8, g)$, we introduce a *canonical pseudo neighborhood* of (g, Σ), which roughly speaking consists of metric-LSMH pairs (g', Σ') that are *bi-Lipschitzly close* to (g, Σ). Inside this canonical pseudo neighborhood, inspired by White's Bumpiness Theorem [Whi91, Whi17], we are able to establish a local Sard Theorem by artificially unwrapping the proof Sard-Smale Theorem for smooth Fredholm map between Banach manifolds, see Theorem 6.2.8 for its precise statement. Then to obtain the Sard-type Theorem globally, we adapt the proof in a recent work by N. Edelen [Ede21] to find countable cover of the space of all metric-LSMH pairs $(\bar{g}, \bar{\Sigma})$ by canonical pseudo neighborhoods.

After overcoming these two techical difficulties, the proof of Full Generic Regularity can be carried out by induction in the singular capacity of LSMH.

Organiztion of the Chapter

In Section 6.2, we introduce some technical notions ralated with locally stable minimal hypersurfaces, and state the Regular Deformation Theorem 6.2.4, the Existence Theorem 6.2.5 of Singular Capacity, the Local Sard Theorem 6.2.8 and the Countable Cover Theorem 6.2.9. Assuming these Theorems, the detailed proof of Theorem 1.3.6 will be carried out in Section 6.3, followed by several applications of the Full Generic Regularity Theorem to min-max Theory in 8-manifolds.

In Section 6.5, we prove the Existence Theorem 6.2.5 of Singular Capacity.

In Section 6.6, inspired by Edelen's cone decomposition, we introduce a two-step decomposition scheme to show that countably many "canonical (pseudo-)neighborhoods" could cover the whole space. In particular, we prove the Countable Cover Theorem 6.2.9.

In Section 6.7, we collected several facts on induced Jacobi fields from a converging sequence of minimal hypersurfaces or minimal hypersurfaces pairs. This allows us to establish the Regular Deformation Theorem 6.2.4, which serves as an induction step in proving full generic regularity.

In Section 6.8, based on the knowleadge of certain induced Jacobi fields, we unwrap the Sard-Smale type arguments in White's Structure Theorem to derive the Local Sard Theorem 6.2.8.

6.2 Preliminaries

Let Σ be an LSMH in a closed manifold (M^8, g) with a singularity p. For a L^2_{loc}-function v defined near p, recall that we have defined and discussed the **asymptotic rate** of v at p in Section 5.3.3 to be,

$$\mathcal{AR}_p(v) := \sup\{\gamma : \lim_{s\searrow 0} \int_{A^g(p;s,2s)} v^2 \cdot \rho^{-n-2\gamma}\, d\|\Sigma\| = 0\},$$

where $\rho(x) := dist_g(x, \mathrm{Sing}(\Sigma))$ is the distance function to the singular set, and $A^g(p; s, t)$ is the geodesic annulus in (M, g) centered at p with inner radius s and outer radius t. Heuristically, if v grows like ρ^γ near p, then $\mathcal{AR}_p(v) = \gamma$. Note that by Lemma 2.3.1, by parametrizing Σ using $\mathbf{C}_p\Sigma$ near p, v can be viewed as a function on $\mathbf{C}$ near the origin, and by Theorem A.0.1, the definition above coincide with Definition 5.2.5 for function on cones.

The advantage of this asymptotic rate is that it relates the behavior of Jacobi fields on Σ near p with the asymptotic spectrum of $\mathbf{C}_p\Sigma$. We collect our analysis in Section 5.2.1 by the following Lemma.

Lemma 6.2.1. *Suppose that a nonzero function $u \in W^{2,2}_{loc}(\Sigma)$ satisfies $L_{\Sigma,g}u \in L^\infty(\Sigma)$. Then $\mathcal{AR}_p(u) \in \{-\infty\} \cup \Gamma(\mathbf{C}_p\Sigma) \cup [1, +\infty]$. In addition,*

(i) If $u \in W^{1,2}(\Sigma)$, then $\mathcal{AR}_p(u) \geq \gamma_1^+(\mathbf{C}_p\Sigma)$;

(ii) If $u > 0$ and $L_{\Sigma,g}u$ vanishes near p, then $\mathcal{AR}_p(u) \in \{\gamma_1^\pm(\mathbf{C}_p\Sigma)\}$;

(iii) If $\mathcal{AR}_q(u) > -(n-2)/2$ for every $q \in \mathrm{Sing}(\Sigma)$, then

$$\int_\Sigma |\nabla_{\Sigma,g}u|^2 + \rho^{-2}u^2 \; d\|\Sigma\| < +\infty.$$

In particular, if $u, v \in W^{2,2}_{loc}(\Sigma)$ such that $L_{\Sigma,g}u, L_{\Sigma,g}v \in L^\infty(\Sigma)$ and that $\mathcal{AR}_q(u) > -(n-2)/2$, $\mathcal{AR}_q(v) > -(n-2)/2$ for every $q \in \mathrm{Sing}(\Sigma)$, then we can do the following integration by parts,

$$\int_\Sigma u \cdot L_{\Sigma,g}v \; d\|\Sigma\| = \int_\Sigma v \cdot L_{\Sigma,g}u \; d\|\Sigma\|.$$

Proof. Combine Lemma 5.3.11 and Corollary 5.3.12. □

6.2.1 Jacobi fields of slower growth

Definition 6.2.2 (Slower growth)**.** A function $u \in L^2_{loc}(\Sigma)$ is a **function of slower growth**, if $\forall p \in \mathrm{Sing}(\Sigma)$,

$$\mathcal{AR}_p(u) \geq \gamma_2^+(\mathbf{C}_p\Sigma)\,.$$

In particular, if $\mathrm{Sing}(\Sigma) = \emptyset$, the condition is always satisfied for any L^2 function.

Similarly, when Σ is two-sided, a Jacobi field on Σ is a **Jacobi field of slower growth** if it is a function of slower growth. We denote by $\mathrm{Ker}^+ L_{\Sigma,g}$ the space of all the Jacobi fields of slower growth on $\Sigma \subset (M, g)$.

A two-sided LSMH Σ is called **semi-nondegenerate**, if

$$\mathrm{Ker}^+ L_{\Sigma,g} = \{0\}\ .$$

Following the notation in (2.1.1), a one-sided connected LSMH Σ is called **semi-nondegenerate**, if its connected two-sided double cover $\hat{\Sigma}$ has

$$(\mathrm{Ker}^+ L_{\hat{\Sigma},\hat{g}})^{\mathbb{Z}_2} = \{0\}\ .$$

Remark 6.2.3. By definition, if $\mathrm{Sing}(\Sigma) = \emptyset$, then Σ is semi-nondegenerate if and only if Σ is nondegenerate.

By Lemma 5.2.1, for every stable minimal hypercone in $\mathbb{R}^8$, we have $\gamma_2^+ > \gamma_1^+ \geq -(n-2)/2$. Thus by Lemma 6.2.1, a Jacobi field of slower growth has finite energy.

The following Regular Deformation Theorem will be the induction step to proving full generic regularity. The proof is based on the analysis of induced Jacobi field from a converging sequence of LSMH, which will be discussed in Section 6.7.

Theorem 6.2.4. *Let $k \geq 4$ be an integer, $\alpha \in (0,1)$, and $\Sigma \subset (M^8, g)$ be a semi-nondegenerate LSMH with* $\mathrm{Sing}(\Sigma) \neq \emptyset$*. Then there exists an open dense subset*

$\mathscr{F}^{k,\alpha} \subset C^{k,\alpha}(M)$ depending on Σ, M, g with the following property. For any $f \in \mathscr{F}^{k,\alpha}$, and $j \geq 1$, Suppose that

(a) $g_j = (1 + t_j f_j)g$ is a family of metrics, where either $\mathbb{R} \ni t_j \searrow 0$ and $f_j \to f$ in $C^4(M)$ as $j \to \infty$, or $t_j \equiv 0$ for all j.

(b) $\Sigma_j \subset (M, g_j)$ is an LSMH with

$$\mathrm{index}(\Sigma_j) = \mathrm{index}(\Sigma)\,; \qquad \mathbf{F}(|\Sigma_j|_{g_j}, |\Sigma|_g) \to 0 \quad \text{as } j \to \infty\,.$$

Then there exists $p \in \mathrm{Sing}(\Sigma)$ and some neighborhood $U_p \subset M$ both depending on the sequence $\{(g_j, \Sigma_j)\}$ such that for infinitely many $j >> 1$,

$$\mathrm{Sing}(\Sigma_j) \cap U_p = \emptyset.$$

6.2.2 Singular capacity

Let $\mathscr{M}$ be the space of triples (Σ, N, g), where (N, g) is a Riemannian manifold of dimension 8, not necessarily complete; $\Sigma \subset (N, g)$ be a connected LSMH with finitely many singularities. Let $\mathscr{M}^{2\text{-sided}}$ be the space of triples $(\Sigma, N, g) \in \mathscr{M}$ such that Σ is two-sided. As before, we endow topology on $\mathscr{M}$ and $\mathscr{M}^{2\text{-sided}}$ by C^4_{loc} convergence in (N, g) and $\mathbf{F}$ convergence in Σ with multiplicity 1.

Singular capacity, first introduced in [LW20, Section 4] as a funcational defined on $\mathscr{M}^{2\text{-sided}}$, serves as a way to count the number of singularities with weight such that it is upper semi-continuous under convergence.

In this chapter, we shall need a modified singular capacity, which is still upper semi-continuous under convergence with multiplicities.

Theorem 6.2.5. *There exists a functional $\mathbf{SCap}^* : \mathscr{M} \to \mathbb{N}$, which we call **singular capacity**, with the following properties.*

(i) For every nontrivial stable minimal hypercone $\mathbf{C} \subset \mathbb{R}^8$*, we have*

$$1 \leq \mathbf{SCap}^*(\mathbf{C}, \mathbb{R}^8, g_{\mathrm{euc}}) < +\infty.$$

We denote for simplicity $\mathbf{SCap}^*(\mathbf{C}) := \mathbf{SCap}^*(\mathbf{C}, \mathbb{R}^8, g_{\mathrm{euc}})$*;*

(ii) For every triple $(\Sigma, N, g) \in \mathscr{M}$*,*

$$\mathbf{SCap}^*(\Sigma, N, g) := \begin{cases} \sum_{p \in \mathrm{Sing}(\Sigma)} \mathbf{SCap}^*(\mathbf{C}_p\Sigma) & \text{if } \Sigma \text{ is two-sided;} \\ 2 \sum_{p \in \mathrm{Sing}(\Sigma)} \mathbf{SCap}^*(\mathbf{C}_p\Sigma) & \text{if } \Sigma \text{ is one-sided.} \end{cases} \tag{6.2.1}$$

Conventionally, $\mathbf{SCap}^*(\Sigma, N, g) = 0$ *if* $\mathrm{Sing}(\Sigma) = \emptyset$*;*

(iii) Given an integer $I \geq 0$*, Suppose that* $(\Sigma_j, N, g_j) \to (\Sigma_\infty, N, g_\infty)$ *in* $\mathscr{M}$*, where* $\forall 1 \leq j \leq +\infty$*,* $\mathrm{index}(\Sigma_j) \equiv I$*, and either all two-sided or all one-sided.*

Then for every open subset $U \subset N$ *with* $\partial U \cap \mathrm{Sing}(\Sigma) = \emptyset$*, we have*

$$\limsup_{j \to \infty} \mathbf{SCap}^*(\Sigma_j, U, g_j) \leq \mathbf{SCap}^*(\Sigma_\infty, U, g_\infty). \tag{6.2.2}$$

(iv) Suppose N *is a closed* 8*-manifold. Let* $\{g_j\}$ *be a family of* C^4 *metrics on* N*, and* $\Sigma_j \subset (N, g_j)$ *be connected stable minimal hypersurfaces,* $1 \leq j \leq \infty$*, such that when* $j \to \infty$*,* $g_j \to g_\infty$ *in* C^4*,* $|\Sigma_j|$ $\mathbf{F}$*-converges to* $m|\Sigma_\infty|$ *for some integer* $m \geq 2$*.*

If $\mathrm{Sing}(\Sigma_\infty) \neq \emptyset$*, then we have,*

$$\limsup_{j \to \infty} \mathbf{SCap}^*(\Sigma_j, N, g_j) \leq \mathbf{SCap}^*(\Sigma_\infty, N, g_\infty) - 1. \tag{6.2.3}$$

The construction of such a functional is based on the density quantization (5.2.2)

of stable minimal hypercones. It also uses a fact about the multiplicity of limiting connected stable minimal hypersurface. See Section 6.5 for details of the proof.

6.2.3 Canonical pseudo neighborhoods

Throughout this section, let M be a closed smooth manifold of dimension 8, $k \geq 4$ be an integer, and $\alpha \in (0,1)$. Let

- $\mathcal{G}^{k,\alpha}(M)$ be the space of $C^{k,\alpha}$-Riemannian metrics on M, which is naturally a Banach manifold;
- $\mathcal{M}^{k,\alpha}(M)$ be the space of pairs (g, Σ), where $g \in \mathcal{G}^{k,\alpha}(M)$ and Σ is a connected LSMH, two-sided or one-sided, in (M, g); Endow topology on $\mathcal{M}^{k,\alpha}$ by $C^{k,\alpha}$-convergence in g and multiplicity 1 $\mathbf{F}$-convergence in Σ;
- $\Pi : \mathcal{M}^{k,\alpha}(M) \to \mathcal{G}^{k,\alpha}(M)$ be the projection onto the first factor.

Definition 6.2.6. Let g be a $C^{k,\alpha}$-metric on M, $\Sigma \subset (M^8, g)$ be a LSMH, $\Lambda \geq 1$, and $\delta > 0$. We denote

$$\text{injrad}(g, \Sigma) := \min\{\text{injrad}(M, g), \min\{dist_g(p, p')/2 : p \neq p' \in \text{Sing}(\Sigma)\}\}.$$

We define $\mathcal{L}^{k,\alpha}(\Sigma, g; \Lambda, \delta)$ to be the space of all pairs $(g', \Sigma') \in \mathcal{M}^{k,\alpha}(M)$ such that,

- g' is a $C^{k,\alpha}$ metric on M with $\|g'\|_{C^{k,\alpha}} \leq \Lambda$ and $\|g' - g\|_{C^{k-1,\alpha}} \leq \delta$;
- Σ' is a LSMH in (M, g') satisfying

$$\mathbf{F}(|\Sigma|_g, |\Sigma'|_{g'}) \leq \delta, \quad \text{index}(\Sigma'; g') \leq \Lambda\,;$$

- For every $p \in \text{Sing}(\Sigma)$, there exists $p' \in \text{Sing}(\Sigma') \cap B^g(p, \text{injrad}(g, \Sigma))$ such that

$$\theta_{|\Sigma'|}(p') = \theta_{|\Sigma|}(p)\,.$$

Here, the norms of tensors are all measured under metric g.

$\mathcal{L}^{k,\alpha}(\Sigma, g; \Lambda, \delta)$ is called a **canonical (pseudo-)neighborhood** of (g, Σ) in $\mathcal{M}^{k,\alpha}(M)$, and its topology is induced from $\mathcal{M}^{k,\alpha}(M)$.

Clearly, when g, Σ is fixed, $\mathcal{L}^{k,\alpha}(\Sigma, g; \Lambda, \delta)$ is monotonely increasing in Λ, δ. Also note that in general, $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ is NOT a genuine *neighborhood* of (g, Σ) in $\mathcal{M}^{k,\alpha}(M)$. In other words, there might be $(g', \Sigma') \in \mathcal{M}^{k,\alpha}(M)$ arbitrarily close to (g, Σ) but not contained in $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$.

Lemma 6.2.7 (Compactness of $\mathcal{L}^{k,\alpha}$)**.** *For every $(g, \Sigma) \in \mathcal{M}^{k,\alpha}(M)$ and every $\Lambda > 1$, there exists $\delta_0(g, \Sigma, \Lambda, k, \alpha) \in (0, 1)$ with the following property. For every $\delta \in (0, \delta_0)$, $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ is compact under $C^{k-1,\alpha}$-norm in metric and $\mathbf{F}$-metric in minimal hypersurfaces.*

Proof. Suppose otherwise, there exists $\delta_j \searrow 0$ and for each $j \geq 1$, a sequence of pairs $\{(g_j^q, \Sigma_j^q)\}_{q \geq 1}$ in $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_j)$ with no limit point in $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_j)$. By compact embedding of $C^{k,\alpha}$ into $C^{k-1,\alpha}$ and Schoen-Simon's Compactness Theorem [SS81], after passing to a subsequence, when $q \to \infty$,

$$g_j^q \to \bar{g}_j \ \text{ in } C^{k-1,\alpha}, \quad \Sigma_j^q \to \bar{V}_j \ \text{ in } \mathbf{F}\text{-metric of varifolds},$$

where $\bar{V}_j$ is a LSMH with multiplicity. Moreover, $\mathbf{F}(|\Sigma|_g, \bar{V}_j) \leq \delta_j \to 0$ as $j \to \infty$. Hence by connectivity of Σ and upper semi-continuity of density, for $j >> 1$, $\bar{V}_j = |\Gamma_j|_{\bar{g}_j}$ for some LSMH $\Gamma_j \subset (M, \bar{g}_j)$ with index $\leq \Lambda$.

By the contradiction assumption, $(\bar{g}_j, \Gamma_j) \notin \mathcal{L}^k(g, \Sigma; \Lambda, \delta_j)$. Therefore, by Allard's Regularity Theorem [All72] and the upper semi-continuity property of density under the convergence $|\Sigma_j^q| \to |\Gamma_j|$, after passing to a subsequence, there exists $p \in \mathrm{Sing}(\Sigma)$, $p_j \in \mathrm{Sing}(\Gamma_j)$ such that

$$\theta(\mathbf{C}_{p_j}\Gamma_j) > \theta(\mathbf{C}_p\Sigma_j^q) = \theta(\mathbf{C}_p\Sigma)\,, \quad p_j \to p\,,$$

as $j \to \infty$. Since by (5.2.2), the density of stable minimal hypercones form a discrete subset of $\mathbb{R}$, we then have

$$\limsup_{j\to\infty} \theta(\mathbf{C}_{p_j}\Gamma_j) > \theta(\mathbf{C}_p\Sigma),$$

which violates the upper semi-continuity of density under convergence $|\Gamma_j| \to |\Sigma|$. $\square$

The main reason to consider canonical neighborhoods is that the following Local Sard-Smale-type Theorem holds.

Theorem 6.2.8. *For every $(g,\Sigma) \in \mathcal{M}^{k,\alpha}(M)$ and every $\Lambda > 0$, there exists $\kappa_0 = \kappa_0(g,\Sigma,\Lambda) > 0$ such that*

$$\mathcal{G}^{k,\alpha}(g,\Sigma;\Lambda,\kappa_0) := \{g' \in \mathcal{G}^{k,\alpha}(M) : \text{ either } g' \notin \Pi(\mathcal{L}^{k,\alpha}(g,\Sigma;\Lambda,\kappa_0)), \text{ or every } \Sigma' \text{ with } (g',\Sigma') \in \mathcal{L}^{k,\alpha}(g,\Sigma;\Lambda,\kappa_0) \text{ is semi-nondegenerate}\}.$$

is an open dense subset of $\mathcal{G}^{k,\alpha}(M)$.

Heuristically, one can view $\mathcal{G}^{k,\alpha}(g,\Sigma;\Lambda,\kappa_0)$ as the set of regular values of $\Pi|_{\mathcal{L}^{k,\alpha}(g,\Sigma;\Lambda,\kappa_0)}$. When we only focus on regular minimal hypersurfaces, such a Sard-Smale Theorem was proved in [Whi91, Whi17] by showing that $\mathcal{M}^{k,\alpha}$ is a Banach manifold, and Π is a Fredholm map with Fredholm index 0. Here, however, it's hard to expect any Banach manifold structure on $\mathcal{L}^k(g,\Sigma;\Lambda,\delta)$. The way we prove Theorem 6.2.8 is by unwrapping the proof of infinite dimensional Sard-Smale Theorem by hand. More precisely, we need

(i) A controlled behavior of Jacobi fields of slower growth under convergence in $\mathcal{L}^{k,\alpha}(g,\Sigma;\Lambda,\kappa_0)$;

(ii) A good way to slice $\mathcal{G}^{k,\alpha}(M)$ into union of finite dimensional subspaces $\{\mathcal{F}_{g',\Sigma'}\}$ such that each $\Pi^{-1}(\mathcal{F}_{g',\Sigma'}) \cap \mathcal{L}^{k,\alpha}(g,\Sigma;\Lambda,\kappa_0)$ is bi-Lipschitz to a compact subset in the finite dimensional vector space $\mathrm{Ker}^+ L_{\Sigma',g'}$.

Both of these rely on a good control on growth rate of certain functions (Jacobi fields or graphical functions of nearby LSMH) on $(g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_0)$, where we make full use of the technical definition of $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ and Jacobi fields of slower growth. The proof of Theorem 6.2.8 is carried out in Section 6.8.

To derive a Global Sard-Smale type Theorem on $\mathcal{M}^{k,\alpha}(M)$, it suffices to show that it can be covered by countably many canonical neighborhoods.

Theorem 6.2.9. *Let $\kappa : \mathcal{M}^{k,\alpha}(M) \times \mathbb{R}_+ \to (0, +\infty)$ be a positive function, not necessarily continuous. Then there exists a countable number of triples $\{(g_j, \Sigma_j; \Lambda_j)\} \in \mathcal{M}^{k,\alpha}(M) \times \mathbb{R}_+$ such that*

$$\mathcal{M}^{k,\alpha}(M) = \bigcup_{j \geq 1} \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j),$$

where $\kappa_j := \kappa(g_j, \Sigma_j; \Lambda_j)$.

Theorem 6.2.9 will be proved in Section 6.6, based on decomposition arguments inspired by the work [Ede21].

A direct corollary of Theorem 6.2.8 and 6.2.9 is the genericity of semi-nondegeneracy.

Corollary 6.2.10. *Given an integer $k \geq 4$, $\alpha \in (0, 1)$ and a eight-dimensional closed smooth manifold M^8, let $\mathcal{G}_0^{k,\alpha}(M)$ be the space of metrics $g \in \mathcal{G}^{k,\alpha}(M)$ such that*

- *every regular connected minimal hypersurface Σ under g is **strongly nondegenerate**, i.e. either Σ is two-sided and nondegenerate, or Σ is one-sided and has a non-degenerate double cover;*
- *every connected LSMH under g is semi-nondegenerate.*

Then $\mathcal{G}_0^{k,\alpha}(M)$ is a generic subset of $\mathcal{G}^{k,\alpha}(M)$ in the Baire Category sense, i.e. $\mathcal{G}_0^{k,\alpha}(M)$ contains a countable intersection of open dense subsets in $\mathcal{G}^{k,\alpha}(M)$.

Proof. In Theorem 6.2.9, by choosing κ to be κ_0 in Theorem 6.2.8, we have

$$\mathcal{M}^{k,\alpha}(M) = \bigcup_{j\geq 1} \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j),$$

where $\kappa_j = \kappa_0(g_j, \Sigma_j; \Lambda_j)$. For each $\mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$, Theorem 6.2.8 generates an open dense subset $\mathcal{G}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$, and we can define

$$\tilde{\mathcal{G}}^{k,\alpha}(M) := \bigcap_{j=1}^{\infty} \mathcal{G}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j),$$

which by definition is clearly generic.

By White's Structure Theorem [Whi91, Whi17], there exists a generic subset $\mathcal{G}^{k,\alpha}_{\text{bumpy}}(M)$ of $\mathcal{G}^{k,\alpha}(M)$, such that every regular connected minimal hypersurface Σ under g is strongly nondegenerate.

It suffices to show that $\tilde{\mathcal{G}}^{k,\alpha}(M) \cap \mathcal{G}^{k,\alpha}_{\text{bumpy}}(M) \subset \mathcal{G}^{k,\alpha}_0(M)$. Indeed, for any $g \in \tilde{\mathcal{G}}^{k,\alpha}(M) \cap \mathcal{G}^{k,\alpha}_{\text{bumpy}}(M) \subset \mathcal{G}^{k,\alpha}_0(M)$, and any LSMH $\Sigma \subset (M, g)$, there exists a $j \in \mathbb{N}^+$ such that $(g, \Sigma) \in \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$. Hence, by the fact that $g \in \tilde{\mathcal{G}}^{k,\alpha}(M) \subset \mathcal{G}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$, we have Σ is semi-nondegenerate. If further Σ is regular, then by the definition of $\mathcal{G}^{k,\alpha}_{\text{bumpy}}(M)$, it is also strongly non-degenerate. In summary, the inclusion holds and $\mathcal{G}^{k,\alpha}_0(M)$ is also generic. □

6.2.4 G-invariant minimal hypersurfaces

To deal with one-sided and two-sided cases simultaneously, we also slightly extend the notions of canonical neighborhoods $\mathcal{L}^{k,\alpha}$ and Jacobi fields of slower growth to manifolds with symmetry.

Definition 6.2.11. Let $G \subset \text{Diff}(M)$ be a discrete subgroup of the diffeomorphism group of M. We call a two-sided LSMH $\Sigma \subset (M, g)$ **G-invariant** if for every $\gamma \in G$,

we have

$$\gamma^* g = g, \quad \gamma(\Sigma) = \Sigma.$$

Given a G-invariant two-sided LSMH $\Sigma \subset (M, g)$ with a unit normal field ν, there is an induced G-representation on $C^0_{loc}(\Sigma)$ given by

$$\begin{aligned}(\gamma \cdot f)(x) &:= \langle \frac{d}{dt}\Big|_{t=0} \gamma(\exp^g_y(t \cdot f(y)\nu(y))), \nu(\gamma(y))\rangle_g \Big|_{y=\gamma^{-1}(x)} \\ &= f(\gamma^{-1}(x)) \cdot \langle \gamma_* \nu(\gamma^{-1}(x)), \nu(x)\rangle_g,\end{aligned}$$

for every $\gamma \in G$, $f \in C^0_{loc}(\Sigma)$, $x \in \Sigma$. Under this representation, for every subset $\mathcal{K} \subset C^0_{loc}(\Sigma)$ (resp. $C^0(M)$), let $\mathcal{K}^G$ be the set of all G-invariant elements in $\mathcal{K}$, i.e.

$$\mathcal{K}^G := \{f \in \mathcal{K} : \gamma \cdot f = f \text{ (resp. } f \circ \gamma = f), \ \forall \gamma \in G\}.$$

We call a G-invariant two-sided LSMH $\Sigma \subset (M, g)$ **G-semi-nondegenerate** if $(\mathrm{Ker}^+ L_{\Sigma,g})^G = \{\mathbf{0}\}$. We also use the notation

$$\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)^G := \{(g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta) : \gamma^* g' = g', \gamma(\Sigma') = \Sigma', \forall \gamma \in G\}.$$

Remark 6.2.12. Throughout this section, unless otherwise stated, $G < \mathrm{Diff}(M)$ is fixed and $\Sigma \subset (M, g)$ is a connected G-invariant two-sided LSMH.

(i) Under the induced G-representation on $C^0_{loc}(\Sigma)$, we have, for every $\gamma \in G$ and every $f \in C^0_{loc}(\Sigma)$,

$$\gamma(\mathrm{graph}_\Sigma(f)) = \mathrm{graph}_\Sigma(\gamma \cdot f).$$

(ii) If $\Sigma_0 \subset (M_0, g_0)$ be a connected one-sided LSMH, let $\pi : \hat{M}_0 \to M_0$ be the double cover given by Section 2.1 such that $\hat{\Sigma}_0 := \pi^{-1}(\Sigma_0) \subset (\hat{M}_0, \hat{g}_0 := \pi^* g_0)$ is a connected two-sided LSMH. This makes $\hat{\Sigma}_0 \subset (\hat{M}_0, \hat{g}_0)$ a $\mathbb{Z}_2$-invariant LSMH,

where $\mathbb{Z}_2$ acts on $\hat{M}_0$ by deck transformation. It's not hard to verify that

$$\begin{aligned}\mathcal{L}^{k,\alpha}(\hat{g}_0, \hat{\Sigma}_0; \Lambda, \delta)^{\mathbb{Z}_2} &\subset \{(\pi^* g', \pi^{-1}(\Sigma')) : (g', \Sigma') \in \mathcal{L}^{k,\alpha}(g_0, \Sigma_0; \Lambda, \delta)\} \\ &\subset \mathcal{L}^{k,\alpha}(\hat{g}_0, \hat{\Sigma}_0; 2\Lambda, 2\delta)^{\mathbb{Z}_2}\end{aligned}$$

Note that functions of slower growth are preserved under convergence in $(\mathcal{L}^{k,\alpha})^G$.

6.3 Proof of Generic Regularity

The goal of this section is to prove Theorem 1.3.6 assuming Theorem 6.2.4, 6.2.5, 6.2.8 and 6.2.9 . Recall that $\mathcal{G}_0^{k,\alpha}(M)$ defined in Corollary 6.2.10 is shown to be a generic subset of $\mathcal{G}^{k,\alpha}(M)$.

Now for $\Lambda > 0$ and integer $I \geq 1$, we can define

$$\begin{aligned}\mathcal{G}_{\Lambda,I}^{k,\alpha}(M) := \big\{ g \in \mathcal{G}_0^{k,\alpha}(M) : &\text{ every } \Sigma \text{ with } (g, \Sigma) \in \mathcal{M}^{k,\alpha}(M), \text{index}(\Sigma) \leq I - 1 \\ &\text{and } \|\Sigma\|_g(M) \leq \Lambda \text{ is regular} \big\}.\end{aligned} \tag{6.3.1}$$

For simplicity, we may omit the superscript k, α. For any $\Lambda > 0$, we also write

$$\mathcal{G}_{\Lambda,0}(M) := \mathcal{G}_0^{k,\alpha}(M). \tag{6.3.2}$$

Lemma 6.3.1. *For every $\Lambda > 0$ and integer $I \geq 0$, $\mathcal{G}_{\Lambda,I}(M)$ is relatively open and dense in $\mathcal{G}_0^{k,\alpha}(M)$.*

Theorem 1.3.6 will be a direct consequence of Lemma 6.3.1 by taking countable cover in Λ and I. To prove Lemma 6.3.1, we start with some preparation. Given

$g \in \mathcal{G}_0^{k,\alpha}(M)$, $\Lambda > 0$ and integer $I \geq 0$, let

$$\mathcal{M}(g;\Lambda,I) := \{(g,\Sigma) \in \mathcal{M}^{k,\alpha}(M) : \|\Sigma\|_g(M) \leq \Lambda, \mathrm{index}(\Sigma) = I\};$$

$$\mathbf{SCap}^*(g;\Lambda,I) := \begin{cases} \sup\{\mathbf{SCap}^*(\Sigma,M,g) : (g,\Sigma) \in \mathcal{M}(g;\Lambda,I)\}, & \text{if } \mathcal{M}(g;\Lambda,I) \neq \emptyset; \\ 0, & \text{if } \mathcal{M}(g;\Lambda,I) = \emptyset; \end{cases}$$

$$\mathcal{M}^{max}(g;\Lambda,I) := \{(g,\Sigma) \in \mathcal{M}(g;\Lambda,I) : \mathbf{SCap}^*(\Sigma,M,g) = \mathbf{SCap}^*(g;\Lambda,I)\}.$$

Note that by definition, for every $g \in \mathcal{G}_{\Lambda,I}(M)$,

$$\mathbf{SCap}^*(g;\Lambda,I) = 0 \text{ if and only if } g \in \mathcal{G}_{\Lambda,I+1}(M). \tag{6.3.3}$$

Lemma 6.3.2. *Let $\Lambda > 0$, integer $I \geq 0$ and $g \in \mathcal{G}_{\Lambda,I}(M)$. If $\{(g,\Sigma_j)\}_{j\geq 1}$ is a sequence in $\mathcal{M}(g;\Lambda,I)$, then $|\Sigma_j|$ subconverges to $m|\Sigma_\infty|$, where*

(i) $(g,\Sigma_\infty) \in \mathcal{M}(g;\Lambda,I)$;

(ii) Either $m = 1$, or $m \geq 2$ and $I = 0$.

In particular, $\mathbf{SCap}^*(g;\Lambda,I) < +\infty$.

Proof. By [SS81] and connectedness of Σ_j, $|\Sigma_j|$ **F**-subconverges to $m|\Sigma_\infty|$ for some connected LSMH $\Sigma_\infty \subset (M,g)$ and integer $m \geq 1$ with

$$m\|\Sigma_\infty\|_g(M) \leq \Lambda, \quad m \cdot \mathrm{index}(\Sigma_\infty) \leq I\,.$$

To see (i), suppose for contradiction $\mathrm{index}(\Sigma_\infty) \leq I - 1$, and hence $I \geq 1$. Since $g \in \mathcal{G}_{\Lambda,I}(M)$, we know from definition that Σ_∞ is regular and strongly nondegenerate. Hence by [Sha17], $m = 1$ and $\Sigma_j \equiv \Sigma_\infty$ for $j >> 1$. This contradicts to the fact that $\mathrm{index}(\Sigma_j) = I$. Therefore,

$$\mathrm{index}(\Sigma_\infty) = I\,.$$

(ii) also follows from [Sha17].

By Theorem 6.2.5, for any $\{(g, \Sigma_j)\}_{j\geq 1} \subset \mathcal{M}(g; \Lambda, I)$, up to a subsequence,

$$\limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, M, g) \leq \mathbf{SCap}^*(\Sigma_\infty, M, g) < +\infty\,.$$

Hence, we have $\mathbf{SCap}^*(g; \Lambda, I) < +\infty$. □

Lemma 6.3.3. *For every $\Lambda > 0$, integer $I \geq 0$ and every $g \in \mathcal{G}_{\Lambda,I}(M)$, $\mathcal{M}^{max}(g; \Lambda, I)$ is a finite set.*

Proof. Suppose otherwise, there exists a pairwise distinct sequence

$$\{(g, \Sigma_j)\}_{j\geq 1} \subset \mathcal{M}^{max}(g; \Lambda, I)\,.$$

By [SS81] and Lemma 6.3.2, $|\Sigma_j|$ $\mathbf{F}$-subconverges to $m|\Sigma_\infty|$ for some $(g, \Sigma_\infty) \in \mathcal{M}(g; \Lambda, I)$. And by the definition of $\mathcal{M}^{max}$, we have

$$\mathbf{SCap}^*(\Sigma_\infty, M, g) \leq \mathbf{SCap}^*(g; \Lambda, I) = \mathbf{SCap}^*(\Sigma_j, M, g). \tag{6.3.4}$$

Then by Theorem 6.2.5, either $m = 1$ or $\mathrm{Sing}(\Sigma_\infty) = \emptyset$. Since $g \in \mathcal{G}_{\Lambda,I}(M) \subset \mathcal{G}_0^{k,\alpha}(M)$, the latter case can't happen by the strong nondegeneracy of Σ_∞. In other words, we have $m = 1$.

Applying Theorem 6.2.4 to $\Sigma_j \to \Sigma_\infty$, we know that $\exists\ p \in \mathrm{Sing}(\Sigma_\infty)$ near which $\mathrm{Sing}(\Sigma_j)$ has no singularity for infinitely many $j >> 1$. Then by Theorem 6.2.5,

$$\liminf_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, M, g) \leq \mathbf{SCap}^*(\Sigma_\infty, M, g) - 1,$$

which contradicts to (6.3.4). □

Lemma 6.3.4. *For every $\Lambda > 0$, integer $I \geq 0$ and every $g \in \mathcal{G}_{\Lambda,I}(M) \cap \mathrm{Int}(\mathrm{Clos}(\mathcal{G}_{\Lambda,I}(M)))$ such that $\mathbf{SCap}^*(g; \Lambda, I) \geq 1$, there exists a sequence of metrics $g_j \in \mathcal{G}_{\Lambda,I}(M) \cap$*

$\mathrm{Int}(\mathrm{Clos}(\mathcal{G}_{\Lambda,I}(M)))$, *$j \geq 1$, such that $g_j \to g$ in $C^{k,\alpha}$ and*

$$\mathbf{SCap}^*(g_j;\Lambda,I) \leq \mathbf{SCap}^*(g;\Lambda,I) - 1. \tag{6.3.5}$$

Proof. By Lemma 6.3.3, we can write

$$\mathcal{M}^{max}(g;\Lambda,I) =: \{(g,\Sigma^{(1)}),(g,\Sigma^{(2)}),\ldots,(g,\Sigma^{(q)})\}\,.$$

For each $1 \leq l \leq q$, let $\mathscr{F}^{(l)} \subset C^{k,\alpha}(M)$ be the open dense subset associated to $(g,\Sigma^{(l)})$ given by Theorem 6.2.4. Thus $\mathscr{F} := \bigcap_{l=1}^{q} \mathscr{F}^{(l)} \neq \emptyset$.

Take $f \in \mathscr{F}$, and there exists a sequence of metrics

$$g_j = (1+f_j/j)g \in \mathcal{G}_{\Lambda,I} \cap \mathrm{Int}(\mathrm{Clos}(\mathcal{G}_{\Lambda,I}(M)))\,,$$

such that $f_j \to f$ in $C^{k,\alpha}$ when $j \to \infty$. We now claim that by passing to a subsequence, g_j satisfies (6.3.5).

To see this, suppose for contradiction that when $j >> 1$, there exists $(g_j,\Sigma_j) \in \mathcal{M}(g_j;\Lambda,I)$ such that

$$\mathbf{SCap}^*(\Sigma_j,M,g_j) \geq \mathbf{SCap}^*(g;\Lambda,I) \geq 1\,. \tag{6.3.6}$$

By [SS81] and the connectedness of Σ_j, $|\Sigma_j|$ **F**-subconverges to $m|\Sigma_\infty|$ for some connected LSMH $\Sigma_\infty \subset (M,g)$ and integer $m \geq 1$ with $m\|\Sigma_\infty\|_g(M) \leq \Lambda$ and $m\,\mathrm{index}(\Sigma_\infty) \leq I$.

We shall deal with the following cases separately.

Case I index$(\Sigma_\infty) = I$ and $m = 1$. Then $(g, \Sigma_\infty) \in \mathcal{M}(g; \Lambda, I)$, and by Theorem 6.2.5 (iii) and (6.3.6),

$$\mathbf{SCap}^*(\Sigma_\infty, M, g) \geq \limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, M, g_j) \geq \mathbf{SCap}^*(g; \Lambda, I).$$

This means $(g, \Sigma_\infty) \in \mathcal{M}^{max}(g; \Lambda, I)$. WLOG $\Sigma_\infty = \Sigma^{(1)}$. But since $f \in \mathscr{F} \subset \mathscr{F}^{(1)}$, by the choice of g_j and Theorem 6.2.4 we have,

$$\begin{aligned}\mathbf{SCap}^*(\Sigma_\infty, M, g) - 1 &\geq \limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, M, g_j) \\ &\geq \mathbf{SCap}^*(g; \Lambda, I) = \mathbf{SCap}^*(\Sigma_\infty, M, g)\,,\end{aligned}$$

which is a contradiction.

Case II index$(\Sigma_\infty) = I$ and $m \geq 2$. Then $I = 0$ and $(g, \Sigma_\infty) \in \mathcal{M}(g; \Lambda, I)$.

- *Case II (1)*. Sing$(\Sigma_\infty) = \emptyset$. Since Σ_j's are stable minimal hypersurfaces, by [SS81], for large enough j, Σ_j is a multi-graph over Σ_∞, and thus,

  $$\mathrm{Sing}(\Sigma_j) = \emptyset\,,$$

  contradicting (6.3.6).

- *Case II (2)*. Sing$(\Sigma_\infty) \neq \emptyset$. By Theorem 6.2.5 (iv) and (6.3.6) we have,

  $$\begin{aligned}\mathbf{SCap}^*(\Sigma_\infty, M, g) &\geq \limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, M, g_j) + 1 \\ &\geq \mathbf{SCap}^*(g; \Lambda, I) + 1 \geq \mathbf{SCap}^*(\Sigma_\infty, M, g) + 1\,,\end{aligned}$$

  which is a contradiction.

Case III index$(\Sigma_\infty) \leq I-1$ and $m=1$. Then $I \geq 1$ and by the definition of $g \in \mathcal{G}_{\Lambda,I}(M)$, Σ_∞ is both regular and strongly nondegenerate. By Allard's Regularity Theorem [All72], Sing$(\Sigma_j) = \emptyset$ for $j >> 1$, contradicting (6.3.6).

Case IV index$(\Sigma_\infty, g) \leq I-1$ and $m \geq 2$. Then again $I \geq 1$ and Σ_∞ is regular and strongly nondegenerate. By Sharp's Compactness Theorem [Sha17], either Σ_∞ is two-sided and $\{\Sigma_j\}$ induces a bounded positive Jacobi field on Σ_∞, or Σ_∞ is one-sided and after passing to double covers, $\{\Sigma_j\}$ induces a bounded positive Jacobi field on the two-sided double cover $\hat{\Sigma}_\infty$ of Σ_∞. Both violate the strong nondegeneracy of Σ_∞. □

Proof of Lemma 6.3.1. To prove the relative openness, let's consider an arbitrary family of metrics $\{g_j\}_j \subset \mathcal{G}_0^{k,\alpha}(M) \setminus \mathcal{G}_{\Lambda,I}(M)$ such that

$$g_j \to g_\infty \in \mathcal{G}_0^{k,\alpha}(M) \text{ in } C^{k,\alpha}\,.$$

The choice of g_j guarantees that there exists an LSMH $\Sigma_j \subset (M, g_j)$ with index$(\Sigma_j) \leq I-1$, vol$(\Sigma_j) \leq \Lambda$ and Sing$(\Sigma_j) \neq \emptyset$. By Sharp's compactness again, $|\Sigma_j|$ **F**-subconverges to some LSMH with multiplicity $m|\Sigma_\infty|$ with area $\leq \Lambda$ and index $\leq I-1$. We now assert that $g_\infty \notin \mathcal{G}_{\Lambda,I}(M)$, to conclude the proof of relative openness. If $g_\infty \in \mathcal{G}_{\Lambda,I}(M)$, by definition, Σ_∞ is both regular and strongly nondegenerate. Hence $m=1$ and then by Allard's Regularity Theorem [All72], for sufficiently large j, Sing$(\Sigma_j) = \emptyset$, which contradicts the choice of Σ_j.

To prove denseness, we shall do the induction in $I \geq 0$.

- Case $I=0$ follows directly from (6.3.2).
- Now suppose $I \geq 0$ such that $\mathcal{G}_{\Lambda,I}(M)$ is dense in $\mathcal{G}_0^{k,\alpha}(M)$, and then by Corollary 6.2.10, it's also dense in $\mathcal{G}^{k,\alpha}(M)$. To prove the denseness of $\mathcal{G}_{\Lambda,I+1}(M)$, it

suffices to show that every metric $g \in \mathcal{G}_{\Lambda,I}(M)$ can be approximated by metrics in $\mathcal{G}_{\Lambda,I+1}(M)$.

Note that by inductive assumption, $\mathrm{Clos}(\mathcal{G}_{\Lambda,I}(M)) = \mathcal{G}^{k,\alpha}(M)$. By Lemma 6.3.2, $\mathbf{SCap}^*(g;\Lambda,I) < \infty$. For any $\epsilon > 0$, we can apply Lemma 6.3.4 finitely many times to find a metric $\bar{g}_\epsilon \in \mathcal{G}_{\Lambda,I}(M)$ such that $\|g - g_j\|_{C^{k,\alpha}} < \epsilon$ and $\mathbf{SCap}(\bar{g}_j;\Lambda,I) = 0$. By (6.3.3), $\bar{g}_\epsilon \in \mathcal{G}_{\Lambda,I+1}(M)$. In conclusion, the arbitrariness of ϵ implies that $\mathcal{G}_{\Lambda,I+1}(M)$ is dense in $\mathcal{G}_{\Lambda,I}(M)$ and thus dense in $\mathcal{G}_0^{k,\alpha}(M)$.

□

Proof of Theorem 1.3.6. By Corollary 6.2.10 and Lemma 6.3.1, we know that

$$\mathcal{G}_\infty^{k,\alpha}(M) := \bigcap_{\text{integer } \Lambda>0, I\geq 0} \mathcal{G}_{\Lambda,I}(M),$$

is a generic subset of $\mathcal{G}^{k,\alpha}(M)$ in the Baire category sense. By (6.3.1), (6.3.2) and the definition of $\mathcal{G}_0^{k,\alpha}(M)$ in Corollary 6.2.10, for every $g \in \mathcal{G}_\infty^{k,\alpha}(M)$, every LSMH $\Sigma \subset (M,g)$ is regular and strongly non-degenerate. □

6.4 Applications

In this section, we shall sketch proofs of various properties of min-max minimal hypersurfaces in an 8-dimensional closed manifold, as applications of our main generic smoothness result. In what follows, we will use $\mathcal{G}_{\mathrm{reg}}(M)$ to denote the set of all metrics where every embedded locally stable minimal hypersurface is smooth, and $\mathcal{G}_{\mathrm{reg},\Lambda,I}$ to denote the set of all metrics where every embedded locally stable minimal hypersurface with both Morse index upper bound I and area upper bound Λ is smooth.

• **Generic Denseness.** In 2017, using the Weyl law for volume spectra proved by Liokumovich-Marques-Neves [LMN18], Irie-Marques-Neves [IMN18] showed that for a generic metric g on a closed Riemannian manifold with dimension between 3 and 7, the union of all closed, smooth, embedded minimal hypersurfaces is a dense subset in the ambient manifold. In 2019, Y. Li [Li19] improved the density result and proved that the union of all min-max minimal hypersurfaces is also dense under the same dimensional conditions.

The denseness result can be extended to dimension 8 with the aid of generic smoothness.

Theorem 6.4.1 (Density of min-max minimal hypersurfaces)**.** *For a generic metric on a closed manifold M^8, the union of all (smooth) minimal hypersurfaces generated from Almgren-Pitts min-max theory is dense in M.*

Sketch of proof. Near the end of [Li19], it is noted that the generic density of all min-max minimal hypersurfaces with optimal regularity $\mathcal{APR}(M, g)$ follows from Conjecture 4.2 therein.

Theorem 1.3.6 together with Sharp's compactness theorem [Sha17] implies that for generic metric g on M^8, there are only countably many embedded locally stable minimal hypersurfaces, i.e., countably many singular minimal hypersurfaces with optimal regularity. This confirms Conjecture 4.2, and thus, the generic density result holds. □

• **Multiplicity One.** In [MN16], F. C. Marques and A. Neves conjectured that for bumpy metrics, min-max minimal hypersurfaces should have multiplicity exactly one, and they proved this for the first width minimal hypersurface. This multiplicity one conjecture was confirmed by X. Zhou [Zho20] for dimensions $3 \leq n+1 \leq 7$, where he applied the min-max construction for hypersurfaces with prescribed mean curvature developed by J. Zhu and himself [ZZ20].

Let's denote by $\mathcal{G}_{\text{bumpy}}(M)$ all the metrics where every immersed smooth minimal hypersurface is non-degenerate. B. White has proved that it is also generic in the Baire sense [Whi91, Whi17].

Theorem 6.4.2 (Multiplicity one)**.** *Given a closed manifold M^8, for any metric $g \in \mathcal{G}_{\text{reg}}(M) \cap \mathcal{G}_{\text{bumpy}}(M)$ and any $k \in \mathbb{N}$, there exists a disjoint collection of smooth, connected, closed, embedded, two-sided minimal hypersurfaces $\left\{\Sigma_i^k : i = 1, \cdots, l_k\right\}$, such that*

$$\omega_k(M, g) = \sum_{i=1}^{l_k} \text{vol}_g(\Sigma_i^k)\,, \quad \sum_{i=1}^{l_k} \text{index}(\Sigma_i^k) \leq k\,.$$

where $\omega_k(M, g)$ is the volume spectrum.

Sketch of proof. This essentially follows the proof of [Zho20, Theorem 5.2], and it suffices to show that we can construct suitable min-max PMC hypersurfaces to approximate min-max minimal hypersurfaces as in the proof of [Zho20, Theorem 4.1].

First, since every minimal hypersurface in (M, g) is smooth and nondegenerate, we can also find good $h \in \mathcal{S}(g)$ as in Lemma 4.2 therein.

Then, the existence and regularity of h-PMC hypersurfaces with optimal regularity follows from [ZZ20] together with the regularity theory by Bellettini-Wickramasekera [BW20a].

Finally, we can obtain the Morse index upper bound from the constrained min-max theory in [Li20a]. This resolves the difficulty of the lack of good pairs for singular PMC hypersurfaces as in Section 3.3 therein. □

• **Morse Index.** In [MN21], Marques-Neves shows that for bumpy metrics on $M^{n+1}(3 \leq n+1 \leq 7)$, the Morse index of a p-width min-max minimal hypersurface of multiplicity one exactly equals to p. This together with X. Zhou's affirmation of multiplicity one conjecture gives a fine geometric description of min-max minimal hypersurfaces.

Theorem 6.4.3 (Morse index estimates)**.** *Given a closed manifold M^8, for any metric $g \in \mathcal{G}_{\mathrm{reg}}(M) \cap \mathcal{G}_{\mathrm{bumpy}}(M)$ and any $k \in \mathbb{N}$, there exists a disjoint collection of smooth, connected, closed, embedded, two-sided minimal hypersurfaces $\left\{\Sigma_i^k : i = 1, \cdots, l_k\right\}$, such that*

$$\omega_k(M, g) = \sum_{i=1}^{l_k} \mathrm{vol}_g(\Sigma_i^k)\,, \quad \sum_{i=1}^{l_k} \mathrm{index}(\Sigma_i^k) = k\,.$$

where $\omega_k(M, g)$ is the volume spectrum.

Sketch of proof. Most of the proofs in [MN21] can also work in our setting, and it suffices to prove Proposition 8.5 therein for M^8.

Indeed, one only needs to replace $\mathcal{U}_{p,\alpha}$ by $\mathcal{U}_{p,\alpha} \cap \mathcal{G}_{\mathrm{reg},p,\alpha}(M)$. Then the denseness follows from the fact that both $\mathcal{U}_{p,\alpha}$ and $\mathcal{G}_{\mathrm{reg},p,\alpha}(M)$ are residual. It is also easy to prove the openness of $\mathcal{U}_{p,\alpha} \cap \mathcal{G}_{\mathrm{reg},p,\alpha}(M)$ with the aid of Allard's Regularity Theorem [All72]. Hence, Propostion 8.5 therein still holds for M^8. □

• **Geometric Complication.** In [CM19], Chodosh-Mantoulidis shows that for bumpy metrics $M^{n+1}(3 \leq n+1 \leq 7)$, there exists a sequence of connected minimal hypersurfaces whose areas tend to infinity. Later in [Li20b], Y. Li improve their results and show that the sequence can have unbounded Morse indices simultaneously. With the generic full regularity in dimension 8, we can also extend the result to dimension 8 via the same arguments.

Theorem 6.4.4 (Main Theorem)**.** *Given any closed Riemannian manifold M^8 with a generic metric $g \in \mathcal{G}_{\mathrm{reg}}(M) \cap \mathcal{G}_{\mathrm{bumpy}}(M)$, there exists a sequence of connected closed embedded two-sided minimal hypersurfaces $\{\Gamma_i\}$ such that*

$$\lim_{i\to\infty} \mathrm{Area}(\Gamma_i) = +\infty\,, \quad \lim_{i\to\infty} \mathrm{index}(\Gamma_i) = +\infty\,. \tag{6.4.1}$$

6.5 Singular Capacity

The goal of this section is to prove Theorem 6.2.5. To deal with multiplicity, we first derive a singular sheeting theorem for stable minimal hypersurfaces in dimension eight, extending the well-known smooth sheeting theorem [SS81, Theorem 1]. This is equivalent to the affirmative answer to a question asked by Ilmanen in dimension eight.

Question ([Ilm96, Question 4])**.** Show that if a stable hypersurface M lies within an ε-neighborhood of another stable hypersurface N, then on a smaller ball, M decomposes into disjoint components, each of which is weakly close to N as a Radon measure (that is, constitutes a single layer over N).

Let's start with the following multiplicity one Lemma.

Lemma 6.5.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a connected stable minimal hypersurface with optimal regularity and a varifold tangent cone $m|\mathbf{C}|$ near infinity, i.e.*

$$\lim_{r \searrow 0} \mathbf{F}(|\eta_{1/r}(\Sigma)|, m|\mathbf{C}|) = 0\,.$$

If $\mathbf{C} \subset \mathbb{R}^{n+1}$ is a regular cone, then $m = 1$.

Proof. This is a direct consequence of [SS81] and [CSZ97], which combine to show that Σ has only one end and thus, $m = 1$. □

Proposition 6.5.2. *Let $\{\Sigma_j \subset (\mathbb{B}_2^8, g_j)\}_j$ be a family of two-sided stable minimal hypersurfaces, with*

$$g_j \to g_{\mathrm{euc}} \text{ in } C^4_{loc}, \quad |\Sigma_j| \ \mathbf{F}\text{-converges to } m|\mathbf{C}| \text{ in } \mathbb{B}_2\,,$$

where $\mathbf{C} \subset \mathbb{R}^8$ is a stable minimal hypercone, and $m \geq 1$ is an integer. If all Σ_j's are connected in $\mathbb{B}_1$, then $m = 1$.

Proof. Recall in (5.2.2), the densities of stable minimal hypercones in $\mathbb{R}^8$ form a discrete set, denoted by

$$\{\theta_{\mathbf{C}}(\mathbf{0}) : \mathbf{C} \subset \mathbb{R}^8 \text{ stable minimal hypercone}\} = \{1 = \theta_0 < \theta_1 < \theta_2 < \cdots \nearrow +\infty\}.$$

We shall prove the lemma by induction on θ_l.

Note that when $\theta_{\mathbf{C}}(\mathbf{0}) = 1$, by volume monotonicity, $\mathbf{C}$ is a hyperplane. Hence by [SS81, Theorem 1], Σ_j is a multi-graph over $\mathbf{C}$ in $\mathbb{B}_1$, and the connectedness of $\Sigma_j \cap \mathbb{B}_1$ implie that this should be a graph, so $m = 1$.

Now fix $l \geq 0$ and suppose the lemma holds for stable minimal hypercones with density $\leq \theta_l$. Consider $\mathbf{C}$ with $\theta_{\mathbf{C}}(\mathbf{0}) = \theta_{l+1}$ and suppose for contradiction that $m \geq 2$. By [SS81, Theorem 1] again, when $j >> 1$, $\Sigma_j \cap A_{g_j}(\mathbf{0}, 1/2, 1)$ is a disjoint union of m graphs over $\mathbf{C}$.

Fix an $\epsilon \in (0, 1)$ (To be determined later) and let

$$r_j := \inf\{s \in (0, 1/16) : \exists p \in B^{g_j}(\mathbf{0}, 1/16) \text{ s.t. } \theta_{|\Sigma_j|}(p, 1) - \theta_{|\Sigma_j|}(p, s) \leq \epsilon\}.$$

And the multi-graphical property mentioned above implies that $r_j \to 0$ when $j \to \infty$. By Lemma D.0.1, we can take $\epsilon = \epsilon(\theta_{l+1}, \theta_l, \mathbf{C}) << 1$, such that there exists $p_j \in B^{g_j}(\mathbf{0}, 1/4)$ and $\Sigma_j \cap A^{g_j}(p_j, 2r_j, 1)$ is a disjoint union of m graphs over translated cone $\mathbf{C} + p_j$ with graphical functions of small C^2_*-norm. Note that $p_j \to \mathbf{0}$.

Since $m > 1$, by Ilmanen's strong maximum principle [Ilm96, Theorem A], we have $r_j > 0$. Note that the rescaled minimal hypersurface $\hat{\Sigma}_j := \eta_{p_j, 4r_j}(\Sigma_j) \hookrightarrow (\mathbb{B}_{1/(2r_j)}, (4r_j)^{-2}(\eta^{-1}_{p_j,4r_j})^* g_j)$ is still connected in $\mathbb{B}_R$ for every $1 \leq R \leq 1/2r_j$. When $j \to \infty$, $|\hat{\Sigma}_j|$ $\mathbf{F}$-subconverges to some $m'|\hat{\Sigma}_\infty|$, where $\hat{\Sigma}_\infty \subset (\mathbb{R}^8, g_{\text{euc}})$ is a connected stable minimal hypersurface, and a multi-graph over $\mathbf{C}$ outside $\mathbb{B}_{1/2}$. By Lemma 6.5.1, $\hat{\Sigma}_\infty$ is a single graph over $\mathbf{C}$ outside $\mathbb{B}_{1/2}$. Therefore, $m' = m \geq 2$ and by the choice of r_j, $\hat{\Sigma}_\infty$ is not a translation of a cone.

Now either

- $\mathrm{Sing}(\hat{\Sigma}_\infty) = \emptyset$, or
- by volume monotonicity and the fact that $\mathbf{C}$ is the tangent cone of $\hat{\Sigma}$ at infinity, every $\hat{p} \in \mathrm{Sing}(\hat{\Sigma}_\infty)$ has density strictly less than $\theta_{|\hat{\Sigma}_\infty|}(\infty) = \theta_{l+1}$. Hence, $\forall \hat{p} \in \mathrm{Sing}(\hat{\Sigma}_\infty),\ \theta_{|\hat{\Sigma}_\infty|}(\hat{p}) \leq \theta_l$.

Moreover, since $\hat{\Sigma}_\infty$ is a single graph over $\mathbf{C}$ outside $\mathbb{B}_{1/2}$, $\mathrm{Sing}(\hat{\Sigma}_\infty) \subset \mathbb{B}_1$. We are going to obtain contradictions in either case.

- If $\mathrm{Sing}(\hat{\Sigma}_\infty) = \emptyset$, then $\hat{\Sigma}_j$ is a disjoint union of m graphs over $\hat{\Sigma}_\infty$, which contradicts the connectedness of $\hat{\Sigma}_j \cap \mathbb{B}_1$.
- If $\mathrm{Sing}(\hat{\Sigma}_\infty) \neq \emptyset$, then there should be only finitely many points, i.e.

$$\mathrm{Sing}(\hat{\Sigma}_\infty) = \{p_i\}_{i=1}^L \subset \mathbb{B}_1 \,.$$

Hence, there exists a strictly increasing sequence $\{j_k\} \subset \mathbb{N}$ satisfying the following property.

For every k, there exists a radius $r_k \in (0, 1/k)$, such that

(1) $\hat{\Sigma}_{j_k} \cap \mathbb{B}_1$ is a disjoint union of m graphs over $\hat{\Sigma}_\infty \cap \mathbb{B}_1$ outside $\mathbb{B}(\mathrm{Sing}(\hat{\Sigma}_\infty), r_k)$;

(2) For each $i \in \{1, 2, \cdots, L\}$, $\hat{\Sigma}_{j_k} \cap \partial\mathbb{B}(p_i, r_k)$ is a disjoint union of m connected hypersurfaces;

(3) For each $i \in \{1, 2, \cdots, L\}$, $\mathbf{F}_{\mathbb{B}_1}(|\eta_{p_i,r_k}(\hat{\Sigma}_{j_k})|, m|\eta_{p_i,r_k}(\hat{\Sigma}_\infty)|) < 1/k$;

(4) There exists an $i_k \in \{1, 2, \cdots, L\}$, such that $\hat{\Sigma}_{j_k} \cap \mathbb{B}(p_{i_k}, r_k)$ has a connected component $\hat{\Gamma}_{j_k}$ with

$$\mathbf{F}_{\mathbb{B}_1}(|\eta_{p_{i_k},r_k}(\hat{\Gamma}_{j_k})|, m_k|\eta_{p_{i_k},r_k}(\hat{\Sigma}_\infty)|) < 1/k \,,$$

for some $m_k \in \{2, \cdots, m\}$. In particular, since L and m are finite, by taking a subsequence, we may replace i_k and m_k by some fixed i_0 and $m_0 \geq 2$ independent of k.

The last property follows from the connectedness of $\hat{\Sigma}_j \cap \mathbb{B}_1$ and the other three properties. Moreover, $\eta_{p_{i_0},r_k}(\hat{\Sigma}_\infty)$ would $\mathbf{F}$-converge to its tangent cone $\mathbf{C}_{p_{i_0}}(\hat{\Sigma}_\infty)$, and thus,

$$\mathbf{F}_{\mathbb{B}_1}(|\eta_{p_{i_0},r_k}(\hat{\Gamma}_{j_k})|, m_0|\mathbf{C}_{p_{i_0}}(\hat{\Sigma}_\infty)|) \to 0\,.$$

This violates our inductive hypothesis as $\theta_{\mathbf{C}_{p_{i_0}}(\hat{\Sigma}_\infty)}(\mathbf{0}) \leq \theta_l$.

In summary, we obtain that $m = 1$ for any stable hypercone in $\mathbb{R}^8$. □

Corollary 6.5.3. *Let g_j be a family of C^4 metrics on a closed manifold M^8, $\Sigma_j \subset (M, g_j)$ be connected stable minimal hypersurfaces, $1 \leq j \leq \infty$. Suppose that when $j \to \infty$,*

$$g_j \to g_\infty \text{ in } C^4, \quad |\Sigma_j| \ \mathbf{F}\text{-converges to } m|\Sigma_\infty|\,,$$

for some integer $m \geq 1$. Then

(i) If Σ_∞ is two-sided, then so are Σ_j when $j >> 1$, and $m = 1$.

(ii) If Σ_∞ is one-sided, then either $m = 1$ and Σ_j are one-sided, or $m = 2$ and Σ_j are two-sided, when $j >> 1$.

Proof.

(i) By [SS81], for j sufficiently large, away from $\mathrm{Sing}(\Sigma_\infty)$, every Σ_j is a disjoint union of m graphs over Σ_∞. As the part in the proof of Proposition 6.5.2 dealing with $\hat{\Sigma}_\infty$, the multiplicty m should be 1. Moreover, since every LSMH in a contractible space is two-sided, $\Sigma_j \cap B^{g_\infty}(\mathrm{Sing}(\Sigma_\infty), r_0)$ is two-sided for small enough r_0. When

$j >> 1$, a unit normal field over Σ_∞ induces a unit normal field on Σ_j away from $\mathrm{Sing}(\Sigma_\infty)$, and by the previous statement, can extend globally to Σ_j. This proves that Σ_j is two-sided for $j >> 1$.

(ii) Let $\pi : \hat{M} \to M$ be the double cover associated to Σ_∞ such that $\pi^{-1}(\Sigma_\infty)$ is connected and two-sided in $\hat{M}$. Then $\pi^{-1}(\Sigma_j)$ are either connected or have two connected components, which corresponding to Σ_j being one-sided or two-sided. Since $|\pi^{-1}(\Sigma_j)|$ $\mathbf{F}$-converges to $m|\hat{\Sigma}_\infty|$, by (i), we have $m = 1$ or 2. And since Σ_j is connected and π is a double cover, for sufficiently large j, we must have that Σ_j is two-sided provided $m = 2$, and one-sided provided $m = 1$. □

Theorem 6.5.4 (Singular sheeting theorem)**.** *Given a closed Riemannian manifold (M^8, g) and a connected stable two-sided minimal hypersurface Σ, for any $\Lambda > 0$ and $\delta > 0$, there exists a constant $\varepsilon = \varepsilon(M, g, \Sigma, \Lambda, \delta) > 0$ with the following property. For any stable minimal hypersurface Σ' in (M, g) with*

$$\mathscr{H}^7(\Sigma') \leq \Lambda, \quad dist_H(\Sigma', \Sigma) \leq \varepsilon\,,$$

then Σ' can be decomposed into finitely many disjoint minimal hypersurfaces $\{\Sigma'_i\}$ with

$$\mathbf{F}(|\Sigma|, |\Sigma'_i|) < \delta\,.$$

Proof. We take $\{\Sigma'_i\}$ to be the disjoint union of connected components of Σ', each of which is obviously a stable minimal hypersurface. Since $\mathscr{H}^7(\Sigma') \leq \Lambda$, the collection is finite.

Let's suppose for contradiction that there exists a sequence of connected stable minimal hypersurfaces $\left\{\Sigma''_j\right\}_{j=1}^{\infty}$ with

$$\Sigma''_j \subset B^g(\Sigma, 1/j), \quad \mathscr{H}^7(\Sigma''_j) \leq \Lambda\,,$$

but

$$\mathbf{F}(|\Sigma|, |\Sigma_j''|) \geq \delta\,.$$

By [SS81] again, we know that $|\Sigma_j''|$ $\mathbf{F}$-subconverges to some $m|\Sigma_\infty''|$ whose support $\overline{\Sigma}_\infty''$ is a stable minimal hypersurface. Moreover, it follows from $\Sigma_j'' \subset B^g(\Sigma, 1/j)$ that

$$\overline{\Sigma}_\infty'' \subset \overline{\Sigma}\,,$$

which, by the constancy theorem, indicates that $\Sigma_\infty'' = \Sigma$.

By Corollary 6.5.3 (i), $m = 1$, which contradicts our assumption that $\mathbf{F}(|\Sigma|, |\Sigma_j''|) \geq \delta$. □

Lemma 6.5.5. *Let $\mathbf{C} \subset \mathbb{R}^8$ be a stable minimal hypercone, and $\{g_j\}_{j\geq 1}$ be a sequence of Riemannian metrics on $\mathbb{B}_2$ which C^4_{loc}-converges to g_{euc}. For each $j \geq 1$, let $\Sigma_j^{(1)}, \Sigma_j^{(2)}$ be two disjoint stable minimal hypersurfaces in $(\mathbb{B}_2, g_j)$, and such that when $j \to \infty$, $|\Sigma_j^{(i)}|$ $\mathbf{F}$-converge to $|\mathbf{C}|$, $i = 1, 2$.*

Then for $j >> 1$, at least one of the following holds.

(a) At least one of the $\Sigma_j^{(i)}$'s has no singularity in $\mathbb{B}_1$;

(b) For every $i = 1, 2$ and every $p \in \text{Sing}(\Sigma_j^{(i)}) \cap \mathbb{B}_1$, we have

$$\theta_{|\Sigma_j^{(i)}|}(p) < \theta_{\mathbf{C}}(\mathbf{0}).$$

Proof. By passing to a subsequence, suppose WLOG that (b) doesn't hold for $j \geq 1$. By the upper-semi continuity of the density function at singular points under $\mathbf{F}$-convergence, suppose WLOG that $p_j \in \text{Sing}(\Sigma_j^{(1)}) \cap \mathbb{B}_1$ has $\theta_{|\Sigma_j^{(1)}|}(p_j) = \theta_{\mathbf{C}}(\mathbf{0})$. The goal is to show that under these assumptions, $\Sigma_j^{(2)}$ is smooth for $j >> 1$.

First note that by the volume monotonicity,

$$\limsup_{j\to\infty}\left(\theta_{|\Sigma_j^{(1)}|}(p_j,1)-\theta_{|\Sigma_j^{(1)}|}(p_j)\right)=\theta_{\mathbf{C}}(\mathbf{0},1)-\theta_{\mathbf{C}}(\mathbf{0})=0.$$

Hence by Lemma D.0.1, for every sequence $\{s_j\searrow 0\}$, $|\eta_{p_j,s_j}(\Sigma_j^{(1)})|$ $\mathbf{F}$-converges to $|\mathbf{C}|$ as $j\to\infty$.

Let r_j be the infimum among all $s\in(0,1)$ such that

$$\Sigma_j^{(2)}\cap A^{g_j}(p_j,s,1)=\mathrm{graph}_{\Sigma_j^{(1)}}(u_j)\cap A^{g_j}(p_j,s,1),$$

for some $u_j\in C^2(\Sigma_j^{(1)})$ satisfying $\|u_j\|_{C^2_*}\leq 1$. Since $\Sigma_j^{(i)}$ are disjoint, and so are their closures by Ilmanen's strong maximum principle [Ilm96], we know that $r_j>0$. Also by Allard's Regularity Theorem [All72], $r_j\to 0$ when $j\to\infty$. And by the definition of r_j, for $j>>1$.

$$\mathrm{Sing}(\Sigma_j^{(2)})\cap A^{g_j}(p_j,2r_j,3/2)=\emptyset. \tag{6.5.1}$$

Consider the blow-up sequence $\hat{\Sigma}_j^{(i)}:=\eta_{p_j,r_j}(\Sigma_j^{(i)})$. By the discussion above, when $j\to\infty$, $|\hat{\Sigma}_j^{(1)}|$ $\mathbf{F}$-converges to $|\mathbf{C}|$, and $\hat{\Sigma}_j^{(2)}$ $\mathbf{F}$-subconverges to some stable minimal hypersurface $\hat{\Sigma}_\infty\subset(\mathbb{R}^8,g_{\mathrm{euc}})$, which is different from $\mathbf{C}$ by the choice of r_j. Therefore, $\hat{\Sigma}_\infty$ is disjoint from $\mathbf{C}$ again by the strong maximum principle. Finally, by Lemma 5.2.13 (see also [HS85]), $\mathrm{Sing}(\hat{\Sigma}_\infty)=\emptyset$, and thus, by Allard's Regularity Theorem [All72] again,

$$\mathrm{Sing}(\Sigma_j^{(2)})\cap B^{g_j}(p_j,3r_j)=\emptyset\,,$$

for $j>>1$. Together with (6.5.1), we obtain the smoothness of $\Sigma_j^{(2)}$ for j sufficiently large. □

Before proceed the proof of Theorem 6.2.5, we also need the following uniform

bound on the number the singularities of stable minimal hypersurfaces near 7-dimensional minimal hypercones.

Lemma 6.5.6. *For every $\Lambda > 1$, there exists $N(\Lambda) \geq 1$ such that, for every $\mathbf{C} \in \mathscr{C}^7_\Lambda$ and any sequence of stable minimal hypersurfaces $\Sigma_j \subset (\mathbb{B}^8_6, g_j)$ satisfying $(\Sigma_j; \mathbb{B}_6, g_j) \to (\mathbf{C}; \mathbb{B}_6, g_{\mathrm{euc}})$ in $\mathscr{M}$, we have*

$$\limsup_{j\to\infty} \#(\mathrm{Sing}(\Sigma_j) \cap \mathbb{B}_4) \leq N(\Lambda)\,. \tag{6.5.2}$$

Proof. The lemma essentially follows from [CN13] in dimension 8. For completeness, here we give a simpler and more self-contained proof by induction on quantized densities $\{\theta_k\}_{k\geq 0}$ in (5.2.2). More precisely, we shall prove that (6.5.2) holds for every $\Lambda = \Lambda_k := \omega_7 \cdot \theta_k$.

For $k = 0$, by volume monotonicity formula and Allard regularity, (6.5.2) holds and the upper bound could be taken to be 0.

Suppose for contradiction that there's some $k \geq 1$ such that (6.5.2) holds for $\Lambda = \Lambda_{k-1}$ but fails for Λ_k, and then there exists a family of stable minimal hypersurfaces $\Sigma_j \subset (\mathbb{B}_6, g_j)$, with $(\Sigma_j; \mathbb{B}_6, g_j) \to (\mathbf{C}; \mathbb{B}_6, g_{\mathrm{euc}})$ in $\mathscr{M}$ for some stable minimal hypercone $\mathbf{C}$ with density $\theta_{\mathbf{C}}(\mathbf{0}) = \theta_k$; But $\sharp(\mathrm{Sing}(\Sigma_j) \cap \mathbb{B}_4) \to \infty$.

Let $x_j \in \mathrm{Sing}(\Sigma_j) \subset \mathbb{B}^8_1$, $\delta_3 = \delta_3(\epsilon, 2\Lambda_k, 7) > 0$ be given by Lemma 3.2.3, where ϵ is a dimensional constant chosen sufficiently small such that any stable minimal hypersurface Σ satisfying

$$\mathbf{F}_{\mathbb{B}_2}(|\Sigma|, \mathscr{C}^7_{2\Lambda_k}) \leq \epsilon\,,$$

must be regular in $A(\mathbf{0}; 1/2, 3/2)$. Such ϵ exists by Schoen-Simon's Regularity Theorem [SS81]. We can then define

$$r_j := \inf\{r > 0 : \theta_{|\Sigma_j|}(x_j, 4) - \theta_{|\Sigma_j|}(x_j, r) \leq \delta_3/2\}\,.$$

By Schoen-Simon Compactness, $x_j \to \mathbf{0} \in \mathbb{R}^8$ and $r_j \to 0_+$. And since by our assumption, Σ_j has at least two singular points, we then have $r_j > 0$ and

$$\theta_{|\Sigma_j|}(x_j, r_j) = \theta_{|\Sigma_j|}(x_j, 4) - \delta_3/2\,. \tag{6.5.3}$$

By Proposition 3.2.1 and Schoen-Simon's Compactness, $\hat{\Sigma}_j := r_j^{-1}(\Sigma_j - x_j)$ **F**-subconverges to some multiplicity 1 stable minimal hypersurface $\Sigma_\infty \subset (\mathbb{R}^8, g_{\mathrm{euc}})$. Moreover, $\mathrm{Sing}(\Sigma_\infty) \subset \mathbb{B}_3$ is a finite set containing $\mathbf{0}$, and by volume monotonicity formula and (6.5.3) we have,

$$\theta_{|\Sigma_\infty|}(\mathbf{0}, \infty) \leq \theta_{\mathbf{C}}(\mathbf{0}) = \theta_k\,, \qquad\qquad \theta_{|\Sigma_\infty|}(\mathbf{0}, 1) \leq \theta_{\mathbf{C}}(\mathbf{0}) - \delta_3/2\,.$$

By the rigidity case in the Monotonicity formula, this implies that every singular point of Σ_∞ has density strictly less than θ_k, hence less or equal to θ_{k-1}.

Since $\sharp(\mathrm{Sing}(\hat{\Sigma}_j) \cap \mathbb{B}_4) \to +\infty$ but Σ_∞ only has finitely many singular points, there exists $\hat{x} \in \mathrm{Sing}(\Sigma_\infty) \cap \mathbb{B}_4$ and $\rho_j \to 0_+$ such that $\sharp\big(\mathbb{B}_{\rho_j}(\hat{x}) \cap \mathrm{Sing}(\hat{\Sigma}_j)\big) \to \infty$. As the tangent cone $\mathbf{C}_{\hat{x}}\Sigma_\infty$ has density $\leq \theta_{k-1}$, the blow-up picture contradicts to the inductive assumption.

By induction, we conclude the existence of such finite $N(\Lambda)$. □

Proof of Theorem 6.2.5. For all cones $\mathbf{C} \in \mathscr{C}^7$ which are not hyperplanes, we shall define $\mathbf{SCap}^*$ as follows,

$$\mathbf{SCap}^*(\mathbf{C}) := 1 + \sup\{\limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j \cap \mathbb{B}_1, \mathbb{B}_1, g_j)\}, \tag{6.5.4}$$

where the supremum is taken over all sequence of pairs (Σ_j, g_j) satisfying,

(i) g_j is a Riemannian metric on $\mathbb{B}_2$, $\Sigma_j \subset (\mathbb{B}_2, g_j)$ is a stable minimal hypersurface;

(ii) When $j \to \infty$, $g_j \to g_{\mathrm{euc}}$ in C^4_{loc}, and $|\Sigma_j|$ **F**-converges to $|\mathbf{C}|$;

(iii) For every $p_j \in \mathrm{Sing}(\Sigma_j) \cap \mathbb{B}_1$, $\theta_{|\Sigma_j|}(p_j) < \theta_{\mathbf{C}}(\mathbf{0})$.

And if there is no such a sequence of pairs for $\mathbf{C}$, then we simply define $\mathbf{SCap}^*(\mathbf{C}) = 1$.

Recall that by (5.2.2), the densities of stable minimal hypercones form a discrete subset of $\mathbb{R}$, denoted by $\{1 = \theta_0 < \theta_1 < \theta_2 < \dots\}$. We denote

$$\mathscr{C}(l) := \{\mathbf{C} \in \mathscr{C}^7 : \theta_{\mathbf{C}}(\mathbf{0}) = \theta_l\}.$$

By Lemma 3.2.2 and Schoen-Simon's Compactness Theorem [SS81], $\mathscr{C}(l)$ is $\mathbf{F}$-compact for every $l \geq 1$. We also denote $\mathscr{C}(\leq l) := \bigcup_{1 \leq j \leq l} \mathscr{C}(l)$.

Now, (6.5.4) should be viewed as an inductive definition on θ_l. For $\mathbf{C} \in \mathscr{C}(1)$, we can define $\mathbf{SCap}^*(\mathbf{C}) = 1$. Then for every $l \geq 1$, once $\mathbf{SCap}^*$ is defined for cones $\mathbf{C} \in \mathscr{C}(\leq l)$, we can define $\mathbf{SCap}^*(\Sigma, N, g)$ following (6.2.1) for every triple $(\Sigma, N, g) \in \mathscr{M}$ such that $\forall p \in \mathrm{Sing}(\Sigma)$, $\mathbf{C}_p\Sigma \in \mathscr{C}(\leq l)$. And then we can define $\mathbf{SCap}^*$ for cones $\mathbf{C} \in \mathscr{C}(l+1)$ following (6.5.4) above.

Under this inductive definition, we first verify inductively that

$$\mathbf{SCap}^*(\mathbf{C}) < +\infty$$

for every $\mathbf{C} \in \mathcal{C}$. In fact, for every $l \geq 1$, suppose this has been established for cones in $\mathscr{C}(\leq l)$. And by a diagonal argument, we can see that $\mathbf{SCap}^*$ is upper semi-continuous on $\mathscr{C}(l)$. Then it follows from the $\mathbf{F}$-compactness of $\mathscr{C}(l)$ that

$$\sup_{\mathbf{C} \in \mathscr{C}(\leq l)} \mathbf{SCap}^*(\mathbf{C}) < +\infty\,.$$

Now consider a cone $\mathbf{C} \in \mathscr{C}(l+1)$, and let $(\Sigma_j, \mathbb{B}_2, g_j)$ be a maximizing sequence of

(6.5.4) satisfying (i)-(iii) above. Then

$$\begin{aligned}\mathbf{SCap}^*(\mathbf{C}) &\leq 1 + (\limsup_{j\to\infty} \#\,\mathrm{Sing}(\Sigma_j)\cap\mathbb{B}_1)\cdot \sup_{\mathbf{C}'\in\mathscr{C}(\leq l)} \mathbf{SCap}^*(\mathbf{C}')\\ &\leq 1 + N(\omega_7\theta_{l+1})\cdot \sup_{\mathbf{C}'\in\mathscr{C}(\leq l)} \mathbf{SCap}^*(\mathbf{C}')\\ &< +\infty,\end{aligned}$$

where the second and the last inequality follow from Lemma 6.5.6.

Proof of (iii) in Theorem 6.2.5. For any $p\in\Sigma_\infty$, there exists a radius $r_p>0$, such that $\Sigma_\infty\setminus B^{g_\infty}(p,r_p)$ has index I. Since $\mathrm{index}(\Sigma_j)=I$, by Sharp's compactness, for j large enough, $\Sigma_j\setminus B^{g_\infty}(p,r_p)$ also has index I and thus $\Sigma_j\cap B^{g_\infty}(p,r_p)$ is stable.

Therefore, by rescaling, it suffices to prove the inequality when Σ_j's are stable and $(\Sigma_\infty,N,g_\infty)=(\mathbf{C},\mathbb{B}_2,g_{\mathrm{euc}})$ for some cone $\mathbf{C}\in\mathcal{C}$. By the definition (6.5.4), one only needs to verify the case when there exists some $p_j\in\mathrm{Sing}(\Sigma_j)\cap\mathbb{B}_1$ such that $\theta_{|\Sigma_j|}(p_j)=\theta_{\mathbf{C}}(\mathbf{0})$. By the volume monotonicity,

$$\limsup_{j\to\infty}\theta_{|\Sigma_j|}(p_j,1)-\theta_{|\Sigma_j|}(p_j)=\theta_{\mathbf{C}}(\mathbf{0},1)-\theta_{\mathbf{C}}(\mathbf{0})=0.$$

Hence by Lemma D.0.1 and Corollary D.0.3, $\mathrm{Sing}(\Sigma_j)\cap\mathbb{B}_1=\{p_j\}$, and $\mathbf{C}_{p_j}(\Sigma_j)$ $\mathbf{F}$-converges to $\mathbf{C}$. It follows from the upper semi-continuity of $\mathbf{SCap}^*$ on $\mathscr{C}(l)$ that

$$\limsup_{j\to\infty}\mathbf{SCap}^*(\Sigma_j,\mathbb{B}_1,g_j)=\limsup_{j\to\infty}\mathbf{SCap}^*(\mathbf{C}_{p_j}(\Sigma_j))\leq\mathbf{SCap}^*(\mathbf{C})\,.$$

Proof of (iv) in Theorem 6.2.5. By the connectedness of Σ_j and Corollary 6.5.3, we know that $m=2$ and Σ_∞ is one-sided while Σ_j are all two-sided for $j>>1$. Let $\pi:\hat{N}\to N$ be the double cover such that $\hat{\Sigma}_\infty:=\pi^{-1}(\Sigma_\infty)$ is connected and two-sided. Then for $j>>1$, $\pi^{-1}(\Sigma_j)$ has two connected components $\Sigma_j^{\pm}$, each isometric to Σ_j. Moreover, $\hat{\Sigma}_\infty$ and $\Sigma_j^{\pm}$ are two-sided stable minimal hypersurfaces in $(\hat{N},\hat{g}_j:=\pi^*g_j)$,

and $|\Sigma_j^{\pm}|$ **F**-converges to $|\hat{\Sigma}_\infty|$ when $j \to \infty$. Let $p \in \mathrm{Sing}(\hat{\Sigma}_\infty)$, and W_p be a small neighborhood of p such that $\mathrm{Sing}(\hat{\Sigma}_\infty) \cap \mathrm{Clos}(W_p) = \{p\}$. By definition (6.5.4) and Lemma 6.5.5, we have

$$\limsup_{j\to\infty} \min\{\mathbf{SCap}^*(\Sigma_j^{\pm}, W_p, \hat{g}_j)\} \leq \mathbf{SCap}^*(\hat{\Sigma}_\infty, W_p, \hat{g}_\infty) - 1\,. \tag{6.5.5}$$

WLOG, let's assume that Σ_j^+ corresponds to the one with smaller $\mathbf{SCap}^*$. Therefore, by Theorem 6.2.5 (iii) and (6.5.5), we obtain

$$\begin{aligned}
\mathbf{SCap}^*(\Sigma_\infty, N, g_\infty) &= \mathbf{SCap}^*(\hat{\Sigma}_\infty, \hat{N}, \hat{g}_\infty) \\
&\geq \limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j^+, \hat{N} \setminus \mathrm{Clos}(W_p), \hat{g}_j) + \mathbf{SCap}^*(\Sigma_j^+, W_p, \hat{g}_j) + 1 \\
&= \limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, N, g_j) + 1.
\end{aligned}$$

□

6.6 Almost Cone Decomposition

The goal for this section is to prove Countable Decomposition Theorem 6.2.9. This proof is inspired by a recent work of N. Edelen [Ede21], which we shall review in Section 6.6.1.

Then the proof is carried out in Section 6.6.2 and 6.6.3 by a two-step decomposition of $\mathcal{M}^{k,\alpha}(M)$, where the first step is a refinement of Edelen's cone decomposition while the latter is obtained by a compactness argument.

6.6.1 Cone decompositions à la Edelen

In the recent work of N. Edelen [Ede21], he proved a finite parametrization result of all the minimal hypersurfaces near a given minimal hypersurface in a closed 8-

dimensional Riemannian manifold. The main difficulty lies in the parametrization near a singular point. Therein, Edelen introduces a notion of cone decomposition to overcome this issue.

Definition 6.6.1 (Strong Cone Region, [Ede21, Definition 6.0.3])**.** Let g be a C^2 metric on $\mathbb{B}(a,R) \subset \mathbb{R}^8$, and V an integral varifold on $(\mathbb{B}(a,R),g)$. Given $\mathbf{C} \in \mathcal{C}, m \in \mathbb{N}, \beta, \tau, \sigma \in [0,1/4], \rho \in [0,R]$, we say that $V \llcorner (\mathbb{A}(a,\rho,R),g)$ is a $(\mathbf{C},m,\beta)$**-strong-cone region** if there are C^2 functions

$$\left\{u_j : (a+\mathbf{C}) \cap \mathbb{A}(a,\rho/8,R) \to \mathbf{C}^{\perp}\right\}_{j=1}^{m}$$

so that for any $r \in [\rho,R] \cap (0,\infty)$,

1. Small C^2 norms: $r^{-1}|u_j| + |\nabla u_j| + r|\nabla^2 u_j| \leq \beta$, for any $j = 1, \cdots, m$;
2. Almost constant density ratios: $m\theta_{\mathbf{C}}(0) - \beta \leq \theta_V(a,r) \leq m\theta_{\mathbf{C}}(0) + \beta$;
3. Multigraph: $V \llcorner \mathbb{A}(a,\rho/8,R) = \sum_{j=1}^{m} |\operatorname{graph}_{a+\mathbf{C}}(u_j) \cap \mathbb{A}(a,\rho/8,R)|$.

In this case, for simplicity, we will also call $\mathbb{A}(a,\rho,R)$ a strong-cone region for V.

Definition 6.6.2 (Smooth model, [Ede21, Definition 7.0.1])**.** Given $\Lambda, \gamma \geq 0, \sigma \in (0,1/3)$, a tuple $(S, \mathbf{C}, \{(\mathbf{C}_\alpha, \mathbb{B}(y_\alpha, r_\alpha))\}_\alpha)$ is called a (Λ,σ,γ)**-smooth model** if

- S is a stationary integral 7-varifold in $(\mathbb{R}^8, g_{\text{eucl}})$ with

$$\theta_S(\mathbf{0},\infty) \leq \Lambda\,;$$

- $\mathbf{C}, \{\mathbf{C}_\alpha\}_\alpha \subset \mathcal{C}_\Lambda$;
- $\{\mathbb{B}(y_\alpha, 2r_\alpha)\}$ is a finite collection of disjoint balls in $\mathbb{B}_{1-3\sigma}$;

such that the following are satisfied:

1. S can be represented by a union of disjoint closed smooth embedded minimal hypersurfaces in $\mathbb{R}^8 \setminus \{y_\alpha\}_\alpha$ with multiplicities, i.e.,

$$S = m_1|S_1| + m_2|S_2| + \cdots + m_k|S_k|\,;$$

2. $S \llcorner (\mathbb{A}(\mathbf{0}, 1, \infty), g_{\mathrm{euc}})$ is a $(\mathbf{C}, m, \gamma)$-strong-cone region;

3. For each α, there is a j so that $\operatorname{spt} S \cap \mathbb{A}(y_\alpha, 0, 2r_\alpha) = S_j \cap \mathbb{A}(y_\alpha, 0, 2r_\alpha)$, and it is a $(\mathbf{C}_\alpha, 1, \gamma)$-strong-cone region.

If there is no ambiguity, we will refer to the smooth model as S for simplicity.

Definition 6.6.3 (Smooth model scale constant, [Ede21, Definition 7.0.2])**.** Given a $(\Lambda, \sigma, \gamma)$-smooth model S, we let ϵ_S be the largest number $\leq \min(1, \min_\alpha \{r_\alpha\})$ for which the graph map

$$\operatorname{graph}_S : T^\perp(\bigcup_j S_j) \to \mathbb{R}^8, \quad \operatorname{graph}_S(x, v) := x + v\,,$$

is diffeomorphism from $\left\{(x, v) \in T^\perp(\bigcup_j S_j) : x \in \mathbb{B}_2 \setminus \bigcup_\alpha \mathbb{B}(y_\alpha, r_\alpha/8), |v| < 2\epsilon_S\right\}$ onto its image, and satisfies

$$\left|D \operatorname{graph}_S|_{(x,v)} - \mathrm{Id}\right| \leq |\epsilon_S|^{-1}|v|\,.$$

Definition 6.6.4 (Smooth region, [Ede21, Definition 7.0.3])**.** Given a smooth model S, a C^2 metric g on $\mathbb{B}(a, R) \subset \mathbb{R}^8$, and $\beta \in (0, 1)$, we say that an integral varifold $V \llcorner (\mathbb{B}(a, R), g)$ is an (S, β)**-smooth region** if for each $i = 1, \cdots, k$, there are C^2 functions $\left\{u_{ij} : S_i \to S_i^\perp\right\}_{j=1}^{m_i}$ so that

$$((\eta_{a,R})_\sharp V) \llcorner (\mathbb{B}_1 \setminus \bigcup_\alpha \mathbb{B}(y_\alpha, r_\alpha/4)) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \left[\operatorname{graph}_{S_i}(u_{ij}) \cap \mathbb{B}_1 \setminus \bigcup_\alpha \mathbb{B}(y_\alpha, r_\alpha/4)\right]_{R^{-2}(\eta_{a,R}^{-1})^* g},$$

and

$$|u_{ij}|_{C^2(S_i)} \leq \beta\epsilon_S, \quad \forall i, j\,,$$

where ϵ_S is a scale constant in the previous definition.

In this case, for simplicity, we will also call $\mathbb{B}(a, R)$ a smooth region for V.

Definition 6.6.5 (Cone decomposition, [Ede21, Definition 7.0.4])**.** Given $\theta, \gamma, \beta \in \mathbb{R}$, $\sigma \in (0, 1/3)$, and $N \in \mathbb{N}$, we let

- g be a $C^{k,\alpha}$ metric on $\mathbb{B}(x, R) \subset \mathbb{R}^8$;
- V be an integral varifold in $(\mathbb{B}(x, R), g)$;
- $\mathcal{S} = \{S_s\}_s$ be a finite collection of (θ, σ, γ)-smooth models.

A $(\theta, \beta, \mathcal{S}, N)$**-cone decomposition** of $V \llcorner (\mathbb{B}(x, R), g)$ consists of the following parameters:

- Integers N_C, N_S satisfying $N_C + N_S \leq N$, where N_C is the number of strong-cone regions and N_S the number of smooth regions and ;
- Points $\{x_a\}_a, \{x_b\}_b \subset \mathbb{B}(x, R)$, where $\{x_a\}$ are centers of strong-cone regions and $\{x_b\}$ centers of smooth regions;
- Radii $\{R_a, \rho_a | R_a \geq 2\rho_a\}_a, \{R_b\}_b$, corresponding to radii of annuli in the definition of strong-cone regions and of balls in the definition of smooth regions, respectively;
- Cones $\{\mathbf{C}_a\}_a \subset \mathcal{C}$;
- Integers $\{1 \leq m_a \leq [\theta]\}_a$;
- Indices $\{s_b\}_b$, corresponding to the smooth model S_{s_b};

where $a = 1, \cdots, N_C$ and $b = 1, \cdots, N_S$. Moreover, these parameters determine a covering of balls and annuli satisfying:

1. Every $V \llcorner (\mathbb{A}(x_a, \rho_a, R_a), g)$ is a $(\mathbf{C}_a, m_a, \beta)$-strong cone region and every $V \llcorner (\mathbb{B}(x_b, R_b), g)$ is a (S_{s_b}, β)-smooth region;

2. In (1), there is either a strong-cone region $\mathbb{A}(x_a, \rho_a, R_a)$ for V with $R_a = R$ and $x_a = x$, or a smooth region $B_{R_b}(x_b)$ for V with $R_b = R$ and $x_b = x$;

3. If $V \llcorner \mathbb{A}(x_a, \rho_a, R_a)$ is a $(\mathbf{C}_a, m_a, \beta)$-strong-cone region and $\rho_a > 0$, then there exists either a smooth region $\mathbb{B}(x_b, R_b)$ for V with $R_b = \rho_a$, or another cone region $\mathbb{A}(x_{a'}, \rho_{a'}, R_{a'})$ for V with $R_{a'} = \rho_a, x_{a'} = x_a$. If $\rho_a = 0$, then $\theta_{\mathbf{C}_a}(\mathbf{0}) > 1$;

4. If $V \llcorner (\mathbb{B}(x_b, R_b), g)$ is a smooth region with $(S, \mathbf{C}, \{\mathbf{C}_{\hat{\alpha}}, B(y_{\hat{\alpha}}, r_{\hat{\alpha}})\}_{\hat{\alpha}}) \in \mathcal{S}$, then for any $\hat{\alpha}$, there exists a point $x_{b,\hat{\alpha}}$ and a radius $R_{b,\hat{\alpha}}$ satisfying

$$|x_{b,\hat{\alpha}} - (x_b + R_b \cdot y_{\hat{\alpha}})| \leq \beta R_b r_{\hat{\alpha}}, \quad \frac{1}{2} \leq \frac{R_{b,\hat{\alpha}}}{R_b r_{\hat{\alpha}}} \leq 1 + \beta \,,$$

   and either a strong-cone region $\mathbb{A}(x_{a'}, \rho_{a'}, R_{a'})$ for V with $R_a = R_{b,\hat{\alpha}}, x_a = x_{b,\hat{\alpha}}$, or another smooth region $\mathbb{B}(x_{b'}, R_{b'})$ with $R_{b'} = R_{b,\hat{\alpha}}$ and $x_{b'} = x_{b,\hat{\alpha}}$.

By (5.2.2), we can enumerate the set of densities with multiplicities, i.e.,

$$\left\{\tilde{\theta}_i\right\}_i := \{m\theta_i : m, i \in \mathbb{N}, m > 0\} \,, \tag{6.6.1}$$

such that $1 = \tilde{\theta}_0 < \tilde{\theta}_1 < \cdots$.

Theorem 6.6.6 (Existence of Cone Decomposition, [Ede21, Theorem 7.1])**.** *Given $l, I \in \mathbb{N}$, $0 < \gamma \leq 1$, $\sigma \in (0, \frac{1}{100(I+1)}]$, there exist constants $\delta_{l,I}$, N and a finite collection of $(\tilde{\theta}_l, \sigma, \gamma)$-smooth models $\{S_s\}_s = \mathcal{S}$, all depending only on (l, I, γ, σ) with the following property.*

For any C^3 metric g on $\mathbb{B}_1$ satisfying $|g - g_{\text{euc}}|_{C^3(\mathbb{B}_1)} \leq \delta_{l,I}$, any stationary integral varifold V in $(\mathbb{B}_1, g)$ with $\operatorname{index}(\operatorname{spt} V) \leq I$, $\mathbf{C} \in \mathcal{C}$, *and $m \in \mathbb{N}$, if*

- $m\theta_{\mathbf{C}}(0) \leq \tilde{\theta}_l$;

- $dist_H(\mathrm{spt}\, V \cap B_1, \mathbf{C} \cap B_1) \leq \delta_{l,I}$;

- $(m - 1/2)\theta_{\mathbf{C}}(0) \leq \theta_V(0, 1/2)$ *and* $\theta_V(0, 1) \leq (m + 1/2)\theta_{\mathbf{C}}(0)$;

Then there exists a radius $r \in (1-20(I+1)\sigma, 1)$ *so that* $V\llcorner(B_r, g)$ *admits a* $(\tilde{\theta}_l, \gamma, \mathcal{S}, N)$*-cone decomposition.*

6.6.2 Tree representation and first decomposition

Now we go back to the proof of Theorem 6.2.9. First, we introduce a tree structure representing the cone decomposition of a minimal hypersurface near a cone.

Definition 6.6.7 (Tree representation of cone decomposition)**.** Given a $(\theta, \beta, \mathcal{S}, N)$-cone decomposition of $V\llcorner(\mathbb{B}(x, R), g)$ as in Definition 6.6.5 with parameters:

- Integers N_S, N_C satisfying $N_S + N_C \leq N$;

- Points $\{x_a\}_a, \{x_b\}_b \subset \mathbb{B}(x, R)$;

- Radii $\{R_a, \rho_a | R_a \geq 2\rho_a\}_a, \{R_b\}_b$;

- Indices $\{s_b\}_b$;

- Integers $\{1 \leq m_a \leq [\theta]\}_a$;

- Cones $\{\mathbf{C}_a\}_a \subset \mathcal{C}$,

where $a = 1, \cdots, N_C$ and $b = 1, \cdots, N_S$. The corresponding **tree representation** of the cone decomposition is a rooted tree ([CLRS22, Section B.5]) uniquely defined by:

1. There are two types of nodes: every node of *type I* is labeled with $(\mathbf{C}_a, m_a, x_a, R_a, \rho_a)$, while every node of *type II* with (S_{s_b}, x_b, R_b);

2. The root is labeled with either $(\mathbf{C}_a, m_a, x_a = x, R_a = R, \rho_a)$ or $(S_{s_b}, x_b = x, R_b = R)$;

3. For any type-I node $(\mathbf{C}_a, m_a, x_a, R_a, \rho_a)$, either $\rho_a = 0$, $\theta_{\mathbf{C}_a}(0) > 1$ and it is a leaf; Or $\rho_\alpha > 0$ and it has a unique child of either

   - type I $(\mathbf{C}_{a'}, m_{a'} = m_a, x_{a'} = x_a, R_{a'} = \rho_a, \rho_{a'})$, or
   - type II $(S_{s_{b'}}, x_{b'} = x_a, R_{b'} = \rho_a)$;

4. For any type-II node (S_{s_b}, x_b, R_b) where $S_{s_b} = (S, \mathbf{C}, \{\mathbf{C}_{\hat{\alpha}}, m_{\hat{\alpha}}, \mathbb{B}(y_{\hat{\alpha}}, r_{\hat{\alpha}})\}_{\hat{\alpha}\in I_b})$, it has $\sharp I_b$ child nodes such that for each $\hat{\alpha}$, there exists $R_{b,\hat{\alpha}}$ and $x_{b,\hat{\alpha}}$ such that

$$|x_{b,\hat{\alpha}} - (x_b + R_b \cdot y_{\hat{\alpha}})| \leq \beta R_b r_{\hat{\alpha}}, \quad \frac{1}{2} \leq \frac{R_{b,\hat{\alpha}}}{R_b r_{\hat{\alpha}}} \leq 1 + \beta\,,$$

   so that the corresponding child node is either

   - of type I $(\mathbf{C}_{a'} = \mathbf{C}_{\hat{\alpha}}, m_{a'} = m_{\hat{\alpha}}, x_{a'} = x_{b,\hat{\alpha}}, R_{a'} = R_{b,\hat{\alpha}}, \rho_{a'})$, or
   - of type II $(S_{s_{b'}}, x_{b'} = x_{b,\hat{\alpha}}, R_{b'} = R_{b,\hat{\alpha}})$;

The **coarse tree representation** of the cone decomposition is obtained by relabeling the rooted tree above, i.e., replacing the type-I node $(\mathbf{C}_a, m_a, x_a, R_a, \rho_a)$ by $(\theta_{\mathbf{C}_a}(0), m_a)$, and the type-II node (S_{s_b}, x_b, R_b) by S_{s_b}.

Definition 6.6.8. For $\gamma \in (0, 1/100)$, two $(\theta, \beta, \mathcal{S}, N)$-tree representaions with parameters

- $(N_S, N_C, \{x_a\}, \{x_b\}, \{R_a\}, \{\rho_a\}, \{R_b\}, \{m_a\}, \{\mathbf{C}_a\}\{s_b\})$,
- $(N'_S, N'_C, \{x'_a\}, \{x'_b\}, \{R'_a\}, \{\rho'_a\}, \{R'_b\}, \{m'_a\}, \{\mathbf{C}'_a\}, \{s'_b\})$,

are said to be γ**-close** if $N'_S = N_S$, $N'_C = N_C$ and they have the same coarse tree representations, s.t.

1. If the corresponding two nodes are both of type I, then

   - $dist_H(\mathbf{C}_a \cap \partial B_1, \mathbf{C}_{a'} \cap \partial B_1) \leq \gamma$,

- If $\rho_a > 0$, then
  - $|\rho_a - \rho_{a'}| \leq \gamma \min(\rho_a, \rho_{a'})$;
  - $|x_a - x_{a'}| \leq \gamma \min(\rho_a, \rho_{a'})$;
  - $|R_a - R_{a'}| \leq \gamma \min(\rho_a, \rho_{a'})$;

  Otherwise $\rho_a = 0$, then
  - $\rho_{a'} = 0$;
  - $|x_a - x_{a'}| \leq \gamma \min(R_a, R_{a'})$;
  - $|R_a - R_{a'}| \leq \gamma \min(R_a, R_{a'})$.

2. If the corresponding two nodes are both of type II, then
   - $|x_b - x_{b'}| \leq \gamma \min(R_b, R_b') \min_{\hat{\alpha} \in I_b}(r_{\hat{\alpha}})$,
   - $|R_b - R_{b'}| \leq \gamma \min(R_b, R_b') \min_{\hat{\alpha} \in I_b}(r_{\hat{\alpha}})$.

In order to deal with minimal hypersurfaces in a closed Riemannian manifold directly, we also introduce a large-scale cone decomposition for a minimal hypersurface in a closed Riemannian manifold.

Definition 6.6.9 (Large-scale Cone Decomposition)**.** Given $\theta, \gamma, \beta \in \mathbb{R}_+$, $\sigma \in (0, 1/3)$, and $N \in \mathbb{N}$, let

- g_0, g be two $C^{k,\alpha}$ metrics on N;
- Σ_0, Σ be two minimal hypersurfaces in (N, g_0) and (N, g) respectively;
- $\mathcal{S} = \{S_s\}_s$ be a finite collection of (θ, σ, γ)-smooth models;

A **large-scale** $(\theta, \beta, g_0, \Sigma_0, \mathcal{S}, N)$**-cone decomposition** of Σ consists of:

- a collection of radii $\{r_{\hat{\alpha}}\}_{\hat{\alpha}}$ corresponding to the singular sets $\mathrm{Sing}(\Sigma_0) = \{p_{\hat{\alpha}}\}_{\hat{\alpha}}$, such that $\{B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})\}_{\hat{\alpha}}$ are pairwise disjoint;

- a $(\theta, \beta, \mathcal{S}, N)$-cone decomposition for each $|\Sigma| \llcorner B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$;

- a C^2 function $u : \Sigma_0 \backslash \bigcup_{p_{\hat{\alpha}} \in \mathrm{Sing}(\Sigma_0)} B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}}/2) \to \Sigma_0^{\perp}$ so that for $r_0 = \min_{\hat{\alpha}} \{r_{\hat{\alpha}}\} > 0$,

$$r_0^{-1}|u| + |\nabla u| + r_0|\nabla^2 u| \leq \beta\,,$$

  and $\Sigma \setminus B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$ coincides with $\mathrm{graph}_{\Sigma_0}(u) \setminus B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$;

Similarly, we can define the corresponding tree representation and the γ-closeness.

Definition 6.6.10 (Tree representation of Large-scale cone decomposition)**.** Given a large-scale $(\theta, \beta, g_0, \Sigma_0, \mathcal{S}, N)$-cone decomposition of $\Sigma \subset (N, g)$ with parameters:

- $\mathrm{Sing}(\Sigma_0) = \{p_{\hat{\alpha}}\}_{\hat{\alpha}}$;

- radii $\{r_{\hat{\alpha}}\}_{\hat{\alpha}}$;

- $(\theta, \beta, \mathcal{S}, N)$-cone decompositions for each $\Sigma \cap B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$;

The corresponding **tree representation** of the large-scale cone decomposition is a rooted tree uniquely defined by:

1. The root node is a labeled by a tuple $(\Sigma_0, g_0, \{p_{\hat{\alpha}}\}, \{r_{\hat{\alpha}}\})$;

2. The root node has $\sharp\,\mathrm{Sing}(\Sigma_0)$ children, indexed by $\hat{\alpha}$. The corresponding subtree rooted at the $\hat{\alpha}$-child is the tree representation of the $(\theta, \beta, \mathcal{S}, N)$-cone decomposition for each $|\Sigma| \llcorner B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$.

Similarly, the **coarse tree representation** will be the same directed rooted tree with the subtrees above replaced by their corresponding coarse trees.

Definition 6.6.11. For $\gamma \in (0, 1/100)$, two $(\theta, \beta, g_0, \Sigma_0, \mathcal{S}, N)$-tree representaions $(\beta \leq \gamma)$ are said to be γ**-close** if

- Their root nodes have the same label;

- Their subtrees corresponding to the $\hat{\alpha}$-child are γ-close for each $\hat{\alpha}$.

Theorem 6.6.12. *For any given $(g,\Sigma) \in \mathcal{M}^{k,\alpha}(M)$, $\beta \in (0, 1/100)$, and $I, \Lambda \in \mathbb{N}$ with*

$$\mathscr{H}^7(\Sigma, g) \leq \Lambda,\ \mathrm{index}(\Sigma, g) \leq I\,,$$

there exist $\delta(g,\Sigma,\beta,\Lambda,I) > 0$ and $l(g,\Sigma) > 0$ satisfying the following property.

For a δ-"neighborhood" of (g,Σ),

$$\mathcal{M}^{k,\alpha}(g,\Sigma;\Lambda,I,\beta) := \Big\{(g',\Sigma') \in \mathcal{M}^{k,\alpha}(M) : \|g'\|_{C^{k,\alpha}} \leq \Lambda; \|g-g'\|_{C^{k-1,\alpha}} + \mathbf{F}(\Sigma,\Sigma') \leq \delta\,, \\ \mathscr{H}^7(\Sigma',g') \leq \Lambda,\ \mathrm{index}(\Sigma',g') \leq I\Big\}\,.$$

we have

1. *There exist a finite collection of (θ,σ,β)-smooth models $\mathcal{S}$ and an integer N, so that any $(g',\Sigma') \in \mathcal{M}^{k,\alpha}(g,\Sigma;\Lambda,I,\beta)$ admits a large-scale $(\tilde{\theta}_l,\beta,g,\Sigma,\mathcal{S},N)$-cone decomposition, where l depends on (g,Σ) only;*

2. *There exists a countable collection $\{(g_v,\Sigma_v)\}_{v\in\mathbb{N}} \subset \mathcal{M}^{k,\alpha}(g,\Sigma;\Lambda,I,\beta)$ with fixed large-scale $(\tilde{\theta}_l,\beta,g,\Sigma,\mathcal{S},N)$-cone decompositions with the following property. Every $(g',\Sigma') \in \mathcal{M}^{k,\alpha}(g,\Sigma;\Lambda,I)$ admits a large-scale $(\tilde{\theta}_l,\beta,g,\Sigma,\mathcal{S},N)$-cone decomposition whose tree representation is β-close to that of some (g_v,Σ_v).*

Remark 6.6.13. By the second bullet of the theorem, for each v, we can define an "intermediate canonical neighborhood" $\mathcal{L}_0^{k,\alpha}(g_v,\Sigma_v;\Lambda,I,\beta)$ (we omit g and Σ here for simplicity) consisting of those pairs (g',Σ')'s satisfying:

- $\|g'\|_{C^{k,\alpha}} \leq \Lambda$;
- $\|g-g'\|_{C^{k-1,\alpha}} + \mathbf{F}(\Sigma,\Sigma') \leq \delta$;
- $\mathscr{H}^7(\Sigma',g') \leq \Lambda,\ \mathrm{index}(\Sigma',g') \leq I$;

- (g', Σ') admits a large-scale $(\Lambda, \beta, g, \Sigma, \mathcal{S}, N)$-cone decomposition whose tree representation is β-close to that of (g_v, Σ_v).

Then, the theorem implies that

$$\mathcal{M}^{k,\alpha}(g, \Sigma; \Lambda, I, \beta) = \bigcup_v \mathcal{L}_0^{k,\alpha}(g_v, \Sigma_v; \Lambda, I, \beta)\,.$$

Proof. (1) The first part follows from the existence theorem of cone decompositions, i.e., Theorem 6.6.6.

Indeed, given (g, Σ), we can take $l \in \mathbb{N}$ so that $\tilde{\theta}_l = \sup_{x \in \mathrm{Sing}\,\Sigma} \{\theta_\Sigma(x, 0)\}$. Then for each $p_{\hat{\alpha}} \in \mathrm{Sing}(\Sigma)$, there exists a radius $r_{\hat{\alpha}} \in (0, 1)$ such that

- $\{B^g(p_{\hat{\alpha}}, 2r_{\hat{\alpha}})\}$ are pairwise disjoint;
- $|(2r_{\hat{\alpha}})^{-2}(\eta^{-1}_{p_{\hat{\alpha}}, 2r_{\hat{\alpha}}})^* g - g_{\mathrm{eucl}}|_{C^3(\mathbb{B}_1)} \leq \delta_{l,I}/2$;
- There exists a cone $\mathbf{C}_{p_{\hat{\alpha}}} \in \mathcal{C}$ such that

$$dist_H(\eta_{p_{\hat{\alpha}}, 2r_{\hat{\alpha}}}(\Sigma) \cap \mathbb{B}_1, \mathbf{C}_{p_{\hat{\alpha}}} \cap \mathbb{B}_1) \leq \delta_{l,I}/2\,,$$

$$\theta_{\mathbf{C}_{p_{\hat{\alpha}}}}(0) \leq \tilde{\theta}_l\,,$$

and

$$\frac{3}{4}\theta_{\mathbf{C}_{p_{\hat{\alpha}}}}(0) \leq \theta_{\eta_{p_{\hat{\alpha}}, 2r_{\hat{\alpha}}}(\Sigma)}(0, 1/2), \quad \theta_{\eta_{p_{\hat{\alpha}}, 2r_{\hat{\alpha}}}(\Sigma)}(0, 1) \leq \frac{5}{4}\theta_{\mathbf{C}_{p_{\hat{\alpha}}}}(0)\,.$$

By taking δ small enough, we may see that for any $(g', \Sigma') \in \mathcal{M}^{k,\alpha}(g, \Sigma; \Lambda, I, \beta)$, we have

- By Allard's Regularity Theorem [All72], there exists a C^2 function $u : \Sigma \setminus \bigcup_{p_{\hat{\alpha}} \in \mathrm{Sing}(\Sigma)} B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}}/2) \to \Sigma^\perp$ so that for $r_0 = \min\{r_{\hat{\alpha}}\}$,

$$r_0^{-1}|u| + |\nabla u| + r_0|\nabla^2 u| \leq \beta\,,$$

and $\Sigma' \setminus \bigcup B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$ coincides with $\mathrm{graph}_\Sigma(u) \setminus \bigcup B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$;

- $|(r_{\hat{\alpha}})^{-2}(\eta^{-1}_{p_{\hat{\alpha}}, r_{\hat{\alpha}}})^* g' - g_{\mathrm{euc}}|_{C^3(\mathbb{B}_1)} \leq \delta_{l,I}$ for each $p_{\hat{\alpha}}$;

- $$dist_H(\eta_{p_{\hat{\alpha}}, r_{\hat{\alpha}}}(\Sigma') \cap \mathbb{B}_1, \mathbf{C}_{p_{\hat{\alpha}}} \cap \mathbb{B}_1) \leq \delta_{l,I}\,,$$

  and
  $$\frac{1}{2}\theta_{\mathbf{C}_{p_{\hat{\alpha}}}}(0) \leq \theta_{\eta_{p_{\hat{\alpha}}, r_{\hat{\alpha}}}(\Sigma')}(0, 1/2), \quad \theta_{\eta_{p_{\hat{\alpha}}, r_{\hat{\alpha}}}(\Sigma')}(0,1) \leq \frac{3}{2}\theta_{\mathbf{C}_{p_{\hat{\alpha}}}}(0)\,;$$

By Theorem 6.6.6, the last two bullets imply the existence of a $(\tilde{\theta}_l, \beta, \mathcal{S}_\alpha, N_\alpha)$-cone decomposition for each $\Sigma' \cap B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$. Together with the first bullet, we obtain a large-scale $(\tilde{\theta}_l, \beta, g, \Sigma, \mathcal{S}, N)$-cone decomposition of (Σ', g'), where $N = \sum_{\hat{\alpha}} N_{\hat{\alpha}}$ and $\mathcal{S} = \bigcup_{\hat{\alpha}} \mathcal{S}_{\hat{\alpha}}$.

(2) By (1), for each (g', Σ'), we can fix a large-scale $(\tilde{\theta}_l, \beta, g, \Sigma, \mathcal{S}, N)$-cone decomposition, and its corresponding tree representation. It follows from the finiteness of N, $\mathcal{S}$, $m_{\hat{\alpha}}$'s and the discreteness of $\tilde{\theta}_i$'s that there are only finitely many distinct coarse tree representations.

Let's fix one such a coarse tree representation $\tilde{T} = (\tilde{V}, \tilde{E})$, and consider the set $\mathcal{L}'(\tilde{T})$ of all the pairs (g', Σ') associated to this coarse tree representation. The goal will be to find a countable covering of $\mathcal{L}'(\tilde{T})$.

To begin with, we construct countable coverings for each node of the coarse tree.

(a) By the definition of coarse tree representation, all tree representations of pairs in $\mathcal{L}'$ have the same root node;

(b) Each type-I node $\tilde{u}$ associated with $S_{s_b} = (S, \mathbf{C}, \{\mathbf{C}_{\hat{\alpha}}, m_{\hat{\alpha}}, \mathbb{B}(y_{\hat{\alpha}}, r_{\hat{\alpha}})\}_{\hat{\alpha} \in I_b}) \in \mathcal{S}$ in $\tilde{V}$ corresponds to the set of type-I nodes in $\mathcal{L}'_{\tilde{u}}$

$$\mathcal{L}'_{\tilde{u}} = \left\{(S_{s_b}, x_{\hat{\beta}}, R_{\hat{\beta}})\right\}_{\hat{\beta}}\,,$$

where $x_{\hat{\beta}} \in M$ and $R_{\hat{\beta}} \in (0, \infty)$.

Let $r_0 = \min(\min_{\hat{\alpha} \in I_b}(r_{\hat{\alpha}}), 1/2)$ and define a countable collection of intervals

$$\mathrm{IC}_{\tilde{u}} := \left\{ \mathrm{Int}_k := [(1 + \frac{\gamma r_0}{2})^k, (1 + \frac{\gamma r_0}{2})^{k+1}] \right\}_{k \in \mathbb{Z}}$$

which covers $(0, \infty)$ ("I" stands for interval and "C" for covering). For each k, we associate it with a finite covering $\mathrm{BC}_{\tilde{u},k}$ of (M, g) by geodesic balls of radius no greater than

$$\min(\mathrm{injrad}(M, g), (1 + \frac{\gamma r_0}{2})^k \gamma r_0 / 10).$$

Therefore, we obtain a countable covering of the product space $M \times (0, \infty)$,

$$\mathrm{PC}_{\tilde{u}} := \{(B, \mathrm{Int}_k) : k \in \mathbb{Z}, B \in \mathrm{BC}_{\tilde{u},k}\} \ .$$

Note that if $(x_{\hat{\beta}}, R_{\hat{\beta}})$ and $(x_{\hat{\beta}'}, R_{\hat{\beta}'})$ are in the same $(B, \mathrm{Int}_k) \in \mathrm{PC}_{\tilde{u}}$, then $(S_{s_b}, x_{\hat{\beta}}, R_{\hat{\beta}})$ and $(S_{s_b}, x_{\hat{\beta}'}, R_{\hat{\beta}'})$ satisfy the inequalities in the first part of the definition of γ-closeness.

(c) Each type-II node $\tilde{w}$ associated with $(\theta_{\mathbf{C}}(\mathbf{0}), m)$ in $\tilde{V}$ corresponds to the set of type-I nodes in $\mathcal{L}'_{\tilde{w}}$

$$\mathcal{L}'_{\tilde{w}} = \left\{ (\mathbf{C}_{\hat{\beta}}, m, x_{\hat{\beta}}, R_{\hat{\beta}}, \rho_{\hat{\beta}}) \right\}$$

where $\theta_{\mathbf{C}_{\hat{\beta}}}(\mathbf{0}) = \theta_{\mathbf{C}}(\mathbf{0})$, $x_{\hat{\beta}} \in M$, $R_{\hat{\beta}} \in (0, \infty)$ and $\rho_{\hat{\beta}} \in [0, \infty)$.

By the compactness of stable cones with the same density $\theta_{\mathbf{C}}(\mathbf{0})$, we can find a finite set of cones

$$\{\mathbf{C}_i\}_{i=1}^N \ ,$$

where N depends on γ such that any other stable cone $\mathbf{C}'$ with $\theta_{\mathbf{C}'}(\mathbf{0}) = \theta_{\mathbf{C}}(\mathbf{0})$ has

$$dist_H(\mathbf{C}_{i_0} \cap \partial B_1, \mathbf{C}' \cap \partial B_1) \leq \gamma$$

for some $i_0 \in \{1, \cdots, N\}$. Thus, we can define

$$\mathrm{CC}_{\tilde{w}} := \{\{\mathbf{C}' : \theta_{\mathbf{C}'}(\mathbf{0}) = \theta_{\mathbf{C}}(\mathbf{0}),\ dist_H(\mathbf{C}_i \cap \partial B_1, \mathbf{C}' \cap \partial B_1) \leq \gamma\} : i = 1, \cdots, N\}\ .$$

To cover $[0, \infty)$ related to ρ, we first define a countable covering,

$$\mathrm{IC}_{\tilde{w}} := \{0\} \cup \left\{\mathrm{Int}_k := [(1 + \frac{\gamma}{2})^k, (1 + \frac{\gamma}{2})^{k+1}]\right\}_{k \in \mathbb{Z}}\ .$$

For the singleton $\{0\}$, we define a countable covering of $(0, \infty)$ related to R as

$$\mathrm{IC}_{\tilde{w},\{0\}} := \left\{\mathrm{Int}_{\{0\},k'} := [(1 + \frac{\gamma}{2})^{k'}, (1 + \frac{\gamma}{2})^{k'+1}]\right\}_{k' \in \mathbb{Z}}$$

and for each fixed k', we consider a finite covering $\mathrm{BC}_{\tilde{u},\{0\},k'}$ of (M, g) by geodesic balls of radius no greater than

$$\min(\mathrm{injrad}(M, g), (1 + \frac{\gamma}{2})^{k'}\gamma/10).$$

For each k, we define

$$\mathrm{IC}_{\tilde{w},k} := \left\{\mathrm{Int}_{k,k'} := [(1 + \frac{\gamma}{2}\min(1, (1 + \frac{\gamma}{2})^k))^{k'}, (1 + \frac{\gamma}{2}\min(1, (1 + \frac{\gamma}{2})^k))^{k'+1}]\right\}_{k' \in \mathbb{Z}}$$

and in this case, for each fixed k', similarly, we consider a finite covering $\mathrm{BC}_{\tilde{u},k,k'}$ of (M, g) by geodesic balls of radius no greater than

$$\min\left(\mathrm{injrad}(M, g), (1 + \frac{\gamma}{2}\min(1, (1 + \frac{\gamma}{2})^k))^{k'}\gamma\min(1, (1 + \frac{\gamma}{2})^k)/10\right)\ .$$

Therefore, for the product space $\{\mathbf{C}' : \theta_{\mathbf{C}'}(\mathbf{0}) = \theta_{\mathbf{C}}(\mathbf{0})\} \times M \times (0, \infty) \times [0, \infty)$, we

obtain a countable covering

$$\mathrm{PC}_{\tilde{w}} := \mathrm{CC}_{\tilde{w}} \otimes \Big\{ \{(B, \mathrm{Int}_{k,k'}, \mathrm{Int}_k) : k, k' \in \mathbb{Z}, B \in \mathrm{BC}_{\tilde{w},k,k'}\} \cup \big\{(B, \mathrm{Int}_{\{0\},k'}, \{0\}) : k' \in \mathbb{Z}, B \in \mathrm{BC}_{\tilde{w},\{0\},k'}\big\} \Big\}.$$

As above, note that if $(\mathbf{C}_{\hat{\beta}}, x_{\hat{\beta}}, R_{\hat{\beta}}, \rho_{\hat{\beta}})$ and $(\mathbf{C}_{\hat{\beta}'}, x_{\hat{\beta}'}, R_{\hat{\beta}'}, \rho_{\hat{\beta}'})$ are in the same element of $\mathrm{PC}_{\tilde{w}}$, then $(\mathbf{C}_{\hat{\beta}}, m, x_{\hat{\beta}}, R_{\hat{\beta}}, \rho_{\hat{\beta}})$ and $(\mathbf{C}_{\hat{\beta}'}, x_{\hat{\beta}'}, R_{\hat{\beta}'}, \rho_{\hat{\beta}})$ satisfy the inequalities in the first part of the definition of γ-closeness.

With the construction above, we see that the tree representations of all pairs in $\mathcal{L}'(\tilde{T})$ are contained in the countable union

$$\mathrm{PC}(\tilde{T}) := \bigotimes_{\tilde{u}} \mathrm{PC}_{\tilde{u}} \otimes \bigotimes_{\tilde{w}} \mathrm{PC}_{\tilde{w}}\,. \tag{6.6.2}$$

Hence, we can find a countable subsets $\mathcal{L}''(\tilde{T})$ of $\mathcal{L}'(\tilde{T})$ satisfying the following property. For each element of $PC(\tilde{T})$ containing at least one tree representation of pairs in $\mathcal{L}'(\tilde{T})$, it contains exactly one tree representation of a pair in $\mathcal{L}''(\tilde{T})$.

In summary, it follows from the finiteness of coarse tree representations and the construction above that the set of pairs

$$\bigcup_{\tilde{T}\text{: coarse tree representation}} \mathcal{L}''(\tilde{T}) \tag{6.6.3}$$

is countable and satisfies the desired property. □

For any $\Lambda > 0$ and any integer $I > 0$, let

$$\mathcal{M}^{k,\alpha}(M; \Lambda, I) := \Big\{(g, \Sigma) \in \mathcal{M}^{k,\alpha}(M) : \|g\|_{C^{k,\alpha}} \leq \Lambda; \mathscr{H}^7(\Sigma', g') \leq \Lambda,\ \mathrm{index}(\Sigma', g') \leq I\Big\},$$

and

$$\mathcal{M}^{k,\alpha}_{>1}(M;\Lambda,I,\delta) := \Big\{(g',\Sigma') \in \mathcal{M}^{k,\alpha}(M) : \|g'\|_{C^{k,\alpha}} \leq \Lambda; \mathscr{H}^7(\Sigma',g') \leq \Lambda,\ \mathrm{index}(\Sigma',g') \leq I\,,$$
$$\|g-g'\|_{C^{k-1,\alpha}} + \mathbf{F}(m|\Sigma|,|\Sigma'|) < \delta \text{ for some } (g,\Sigma) \in \mathcal{M}^{k,\alpha}(M;\Lambda,I), m \in \mathbb{N}^{>1}\,,\Big\}.$$

It follows from Sharp's compactness theorem [Sha17] that for any $\delta > 0$, $\mathcal{M}^{k,\alpha}(M;\Lambda,I)\backslash \mathcal{M}^{k,\alpha}_{>1}(M;\Lambda,I,\delta)$ is a compact set, and moreover,

$$\mathcal{M}^{k,\alpha}(M;\Lambda,I) = \bigcup_{j=1}^{\infty}\left(\mathcal{M}^{k,\alpha}(M;\Lambda,I) \setminus \mathcal{M}^{k,\alpha}_{>1}(M;\Lambda,I,\frac{1}{j})\right).$$

On the one hand, for any fixed Λ, I and j, by the compactness property, there exist a finite collection of metrics on M, $\{g_i\}$ and a finite collection of g_i-minimal hypersurfaces $\{\Sigma_i\}$, such that

$$\mathcal{M}^{k,\alpha}(M;\Lambda,I) \setminus \mathcal{M}^{k,\alpha}_{>1}(M;\Lambda,I,\frac{1}{j}) \subset \bigcup_i \mathcal{M}^{k,\alpha}(g_i,\Sigma_i;\Lambda,I,\beta)\,. \tag{6.6.4}$$

In other words, we have

$$\begin{aligned}\mathcal{M}^{k,\alpha}(M) &= \bigcup_{I=0}^{\infty}\bigcup_{\Lambda=1}^{\infty}\mathcal{M}^{k,\alpha}(M;\Lambda,I)\\ &= \bigcup_{I=0}^{\infty}\bigcup_{\Lambda=1}^{\infty}\bigcup_{j=1}^{\infty}\mathcal{M}^{k,\alpha}(M;\Lambda,I)\setminus\mathcal{M}^{k,\alpha}_{>1}(M;\Lambda,I,\frac{1}{j})\\ &= \bigcup_{I=0}^{\infty}\bigcup_{\Lambda=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcup_{i}\mathcal{M}^{k,\alpha}(g_i,\Sigma_i;\Lambda,I,\beta)\,.\end{aligned}$$

By relabeling all the indices, we obtain a countable covering

$$\mathcal{M}^{k,\alpha}(M) = \bigcup_{i=1}^{\infty}\mathcal{M}^{k,\alpha}(g_i,\Sigma_i;\Lambda_i,I_i,\beta)\,. \tag{6.6.5}$$

On the other hand, by Remark 6.6.13,

$$\mathcal{M}^{k,\alpha}(g_i, \Sigma_i; \Lambda_i, I_i, \beta) = \bigcup_{v=1}^{\infty} \mathcal{L}_0^{k,\alpha}(g_{i,v}, \Sigma_{i,v}; \Lambda_i, I_i, \beta)\,,$$

By relabeling the indices $\{i, v\}$, we have

$$\mathcal{M}^{k,\alpha}(M) = \bigcup_{i=1}^{\infty} \mathcal{L}_0^{k,\alpha}(g_i, \Sigma_i; \Lambda_i, I_i, \beta)\,. \tag{6.6.6}$$

6.6.3 Compactness and second decomposition

Proposition 6.6.14 (Sequential compactness of $\mathcal{L}_0^{k,\alpha}$)**.** *The intermediate "canonical neighborhood" $\mathcal{L}_0^{k,\alpha}(g_v, \Sigma_v; \Lambda, I, \beta)$ is sequentially compact. In other words, given a sequence $\{(g_j, \Sigma_j)\} \subset \mathcal{L}_0^{k,\alpha}(g_v, \Sigma_v; \Lambda, I, \beta)$, there exists a pair $(g_\infty, \Sigma_\infty) \in \mathcal{L}_0^{k,\alpha}(g_v, \Sigma_v; \Lambda, I, \beta)$, such that up to a subsequence, $g_j \to g_\infty$ in $C^{k-1,\alpha}$, and $\Sigma_j \to \Sigma_\infty$ in the varifold sense.*

Proof. By definition, we have

- $\|g_j\|_{C^{k,\alpha}} \leq \Lambda$;
- $\|g_j - g_v\|_{C^{k-1,\alpha}} + \mathbf{F}(\Sigma_j, \Sigma_v) \leq \delta$;
- $\mathscr{H}^7(\Sigma_j, g_j) \leq \Lambda,\ \mathrm{index}(\Sigma_j, g_j) \leq I$.

The compactness of $C^{k,\alpha} \hookrightarrow C^{k-1,\alpha}$ and Sharp's compactness [Sha17] together imply that there exists a pair $(g_\infty, \Sigma_\infty)$ such that

- $\|g_\infty\|_{C^{k,\alpha}} \leq \Lambda$ and Σ_∞ is a LSMH;
- Up to a subsequence, $g_j \to g_\infty$ in $C^{k-1,\alpha}$, and $\Sigma_j \to \Sigma_\infty$ in the varifold sense;
- $\|g_\infty - g_v\|_{C^{k-1,\alpha}} + \mathbf{F}(\Sigma_\infty, \Sigma_v) \leq \delta$;
- $\mathscr{H}^7(\Sigma_\infty, g_\infty) \leq \Lambda,\ \mathrm{index}(\Sigma_\infty, g_\infty) \leq I$.

Now, it suffices to show that $(g_\infty, \Sigma_\infty)$ has a tree representation β-close to the one of (g_v, Σ_v).

Indeed, by large-scale cone decomposition and the constancy theorem, we know that Σ_∞ should be a multiplicity one minimal hypersurface. Moreover, the β-graphic property indicates that the convergence are all smooth and thus, the coarse tree representation will be the same. Since all the closeness property will be preserved along the limiting process, $(g_\infty, \Sigma_\infty)$ has a tree representation β-close to the one of (g_v, Σ_v). □

Compactness is equivalent to the existence of a finite subcover of its open covers. Hence, a finite decomposition of intermediate canonical neighborhoods as follows.

Proposition 6.6.15 (Finite covering of $\mathcal{L}_0^{k,\alpha}$)**.** *For any $g, \Sigma, \Lambda, I, \beta$ as in (6.6.6) (with subscripts omitted), there exists a countable set of pairs $\{(g_v, \Sigma_v)\}_v \subset \mathcal{L}_0^{k,\alpha}(g, \Sigma; \Lambda, I, \beta)$ with corresponding $\{\epsilon_v\}_v$, such that*

$$\mathcal{L}_0^{k,\alpha}(g, \Sigma; \Lambda, I, \beta) \subset \bigcup_{v=1}^{V} \mathcal{L}^{k,\alpha}(g_v, \Sigma_v; \Lambda', \kappa_v)\,.$$

where $\Lambda' = \Lambda + I + 1$, $V \in \mathbb{N}$ and $\kappa_v = \kappa(g_v, \Sigma_v; \Lambda')$ defined in Theorem 6.2.9.

Proof. By Proposition 6.6.14, we know that $\mathcal{L}_0^{k,\alpha}(g_i, \Sigma_i; \Lambda_i, I, \beta)$ is compact, so it suffices to show that for any $(g', \Sigma') \in \mathcal{L}_0^{k,\alpha}(g, \Sigma; \Lambda, I, \beta)$ and $\delta' > 0$, the corresponding canonical neighborhood $\mathcal{L}^{k,\alpha}(g', \Sigma'; \Lambda', \delta')$ contains an open neighborhood of (g', Σ') in $\mathcal{L}_0^{k,\alpha}(g, \Sigma; \Lambda, I, \beta)$.

Suppose otherwise that there exists some (g', Σ') and a sequence of $\{(g_i, \Sigma_i)\} \subset \mathcal{L}_0^{k,\alpha}(g, \Sigma; \Lambda, I, \beta) \setminus \mathcal{L}^{k,\alpha}(g', \Sigma'; \Lambda', \delta')$ so that

$$(g_i, \Sigma_i) \to (g', \Sigma')\,.$$

For large enough i, we can verify that

- g_i is a $C^{k,\alpha}$ metric on M with $\|g_i\|_{C^{k,\alpha}} \leq \Lambda'$ and $\|g' - g_i\|_{C^{k-1,\alpha}} \leq \delta'$;
- Σ_i is a LSMH in (M, g_i) satisfying

$$\mathbf{F}(|\Sigma_i|_{g_i}, |\Sigma'|_{g'}) \leq \delta', \quad \text{index}(\Sigma_i) \leq \Lambda' ;$$

Furthermore, the last condition is implied by the β-closeness of tree representations where $\beta < 1/100$.

In other words, for i large enough, $(g_i, \Sigma_i) \in \mathcal{L}^{k,\alpha}(g', \Sigma'; \Lambda, I, \epsilon')$ which gives a contradiction. □

Proof of Theorem 6.2.9. By (6.6.6) and the previous covering proposition, we have

$$\begin{aligned} \mathcal{M}^{k,\alpha}(M) &= \bigcup_{i=1}^{\infty} \mathcal{L}_0^{k,\alpha}(g_i, \Sigma_i; \Lambda_i, I_i, \beta) \\ &= \bigcup_{i=1}^{\infty} \bigcup_{v=1}^{V_i} \mathcal{L}^{k,\alpha}(g_{i,v}, \Sigma_{i,v}; \Lambda_i', \kappa_{i,v}) , \end{aligned}$$

where $\Lambda_i' = \Lambda_i + I_i + 1$, $V_i \in \mathbb{N}$ and $\kappa_{i,v} = \kappa(g_{i,v}, \Sigma_{i,v}; \Lambda_i')$.

Hence, Theorem 6.2.9 follows from relabeling the indices $\{i, v\}$. □

6.7 Induced Jacobi Fields

Let $(\Sigma, \nu) \subset (M, g)$ be a two-sided minimal hypersurface with a unit normal field ν. Let $\{g_j\}$ be a family of metrics smoothly converging to g_0, V_j be a family of stationary integral varifolds in (M, g_j) **different from** $|\Sigma|$ and converging to $|\Sigma|$. Then by Allard Regularity Theorem [All72, Sim83b], the convergence is in $C^{\infty}_{loc}(\Sigma)$.

Definition 6.7.1. We call a non-zero function $\phi \in C^{\infty}(\Sigma)$ a **generalized Jacobi field induced by** $\{(V_j, g_j)\}_{j \geq 1}$, if there exist sequences $t_j \to 0_+$ and $v_j \in C^{\infty}(\Sigma)$

such that v_j/t_j converges to ϕ in $C^\infty_{loc}(\Sigma)$ and that for every $U \subset\subset M \setminus \mathrm{Sing}(\Sigma)$,

$$|\operatorname{graph}_\Sigma(v_j)|\llcorner U = V_j \llcorner U, \qquad \text{for sufficiently large } j\,.$$

Clearly by definition, if ϕ is an induced Jacobi field from $\{(V_j, g_j)\}_j$, then so is $c\phi$ for every constant $c > 0$. Also, if $g_j \equiv g$ and ϕ is induced by some $\{(V_j, g)\}_j$, then ϕ satisfies the Jacobi field equation $L_\Sigma \phi = 0$.

When $g \equiv g_j$ and Σ is regular, after passing to subsequences, the existence of induced Jacobi field directly follows from Allard Regularity Theorem and elliptic estimates, see for example [Sha17]. However, if $\mathrm{Sing}(\Sigma) \neq \emptyset$, even when $g_j = g$, it's usually unclear whether for any such family of stationary integral varifolds $\{V_j\}$, there exists an induced Jacobi field. The key difficulty is the lack of control on growth rate toward singularities of the graphical functions. Here's a toy example in the linear case.

Example 6.7.2. *Consider a sequence of harmonic functions on the two-dimensional punctured disc $\bar{\mathbb{D}}^2 \setminus \{\mathbf{0}\}$ given by*

$$u_k(re^{i\theta}) = (kr)^{-k} \cos k\theta\,.$$

Then as $k \to \infty$, these u_k converge to 0 in $C^\infty_{loc}(\mathrm{Clos}(\mathbb{D}) \setminus \{\mathbf{0}\})$ and satisfy

$$\sup_{\partial \mathbb{D}_{1/3}} |u_k| \geq 2^k \cdot \sup_{\partial \mathbb{D}_{2/3}} |u_k|\,.$$

Thus, there don't exist normalizing constants $\{c_k > 0\}_{k\geq 1}$ such that u_k/c_k converge smoothly on $\mathbb{D}_1 \setminus \mathbb{D}_{1/4}$ to a non-zero function.

This example suggests that if we hope to find an induced Jacobi field from a converging sequence of minimal hypersurface near singularity, then we need to first bound the growth rate of graphical functions towards the singular set; And to achieve this,

the infinitesimal behavior of these minimal hypersurfaces has be carefully addressed.

This section is devoted to the existence of induced Jacobi field under different assumptions. In Section 6.7.1, we first prove several local growth estimates near strongly isolated singularity, based on the growth dichotomy Lemma 5.2.17. The key feature is that it provides link between the asymptotic rate of induced Jacobi field and appropriate blow up limit of the converging minimal hypersurfaces. In particular, based on our analysis on asymptotic conic minimal hypersurfaces in Section 5.2.2, we derive the existence of induced Jacobi fields in Section 6.7.2, as well as how their asymptotic rate reflects the information of the converging sequence, which enable us to prove the Regular Deformation Theorem 6.2.4. We also study the induced Jacobi field for converging pairs of minimal hypersurfaces in Section 6.7.3, which plays an important role in deriving the Local Sard Theorem 6.2.8 in Section 6.8.

To save the notation, we introduce the following *graphical function* between two singular hypersurfaces. Let (N^8, g) be a Riemannian manifold, not necessarily complete. For $i = 0, 1$, suppose Σ^i are hypersurfaces in N such that $\mathrm{Clos}(\Sigma^i) = \partial U^i$, where $U^i \subset N$ is an open subset. Let ν^i be the unit normal field of Σ^i under metric g, which points away from U^i.

Define the **graphical function** over Σ^0 of Σ^1 under metric g as follows. For every $x \in \Sigma^0$,

$$G^{\Sigma^1}_{\Sigma^0,g}(x) := \begin{cases} \sup\{t \geq 0 : \exp^g_x(s\nu^0(x)) \in U^1 \text{ for every } s \in [0,t]\}, & \text{if } x \in U^1; \\ \inf\{t \leq 0 : \exp^g_x(s\nu^0(x)) \in N \setminus U^1 \text{ for every } s \in [t,0]\}, & \text{if } x \notin U^1. \end{cases} \tag{6.7.1}$$

Remark 6.7.3. We abuse the notation here since the definition of $G^{\Sigma^1}_{\Sigma^0,g}$ relies also on the choice of U^i and N. In applications, these are naturally chosen and there will be no ambiguity.

Heuristically, Σ^1 is the graph of $G^{\Sigma^1}_{\Sigma^0,g}$ over Σ^0. In fact, this is the case when Σ^i are smooth hypersurfaces and are C^2 close. In general however, $G^{\Sigma^1}_{\Sigma^0,g}$ need not be even C^0 but only L^∞ along Σ^0.

6.7.1 Local growth estimate of graphical functions

In this subsection, the ambient manifold N to define graphical function is fixed to be $\mathbb{B}^{n+1=8}_2$. When two connected LSMH Σ^0, Σ^1 in $\mathbb{B}_2$ are sufficiently close, and $\Sigma^0 = \partial U^0$ for some $U^0 \subset N$, there's a unique choice of domain U^1 bounded by Σ^1 in N such that $U^0 \Delta U^1$ is also small. We shall fix this choice of U^i in defining the graphical functions.

We first recall the general dichotomy of growth rate for graphical function in Lemma 5.2.17

Lemma 6.7.4. *Let $\Lambda, \Lambda' > 1$, $\sigma > 0$, $\kappa > 0$; $K(\sigma, \Lambda, \Lambda') > 2$ be given by Corollary 5.2.3. Then there exists $\delta_0(\Lambda, \Lambda', \sigma, \kappa) > 0$ with the following significance.*

Let $\mathbf{C} \in \mathscr{C}^n_{\Lambda,\Lambda'}$ be a regular stable minimal hypercone; $\gamma \in (-\Lambda, 1)$ be such that

$$dist_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}) \cup \{-(n-2)/2\}) \geq \sigma.$$

Let $\{(g^i_j, V^i_j)\}_{1 \leq j \leq \infty, i=0,1}$ be a family of pairs such that

(a) g^i_j are C^4 Riemannian metrics on $\mathbb{B}^{n+1}_2$ such that

$$\|g^1_j - g^0_j\|_{C^4} \to 0 \quad \text{as } j \to \infty, \qquad \|g^0_j - g_{\mathrm{euc}}\|_{C^4} \leq \delta_0 \text{ for every } j \geq 1\,;$$

(b) $V^i_j \in \mathcal{IV}_n(\mathbb{B}^{n+1}_2)$ are g^i_j-stationary such that,

$$\mathbf{F}_{\mathbb{B}_2}(V^1_j, V^0_j) \to 0 \quad \text{as } j \to \infty; \qquad V^0_j = |\operatorname{graph}_{\mathbf{C}, g_{\mathrm{euc}}}(v^0_j)| \llcorner \mathbb{B}^{n+1}_2\,,$$

for some $v_j^0 \in C^4(\mathbf{C})$ *with* $\|v_j^0\|_{C^2_*,\mathbf{B}_2} \leq \delta_0$.

We further assume that $\Sigma_j := \mathrm{spt}(V_j^0)$ *is a* g_j^0*-minimal hypersurface in* $\mathbb{B}_2$ *with isolated singularity at* $\mathbf{0}$*; And let*

$$\tau_j := \inf\{t \in (0,1) : V_j^1 \llcorner \mathbb{A}(\mathbf{0};t,1) = |\operatorname{graph}_{\Sigma_j,g_j^0}(u_j)| \llcorner \mathbb{A}(\mathbf{0};t,1), \ \textit{with} \ \|u_j\|_{C^2_*} \leq \delta_0\}. \tag{6.7.2}$$

Then after passing to a subsequence, either of the following hold for $j \geq 1$,

(i) $\tau_j > 0$ *and there exists a linear map* $T \in \mathrm{GL}_{n+1}(\mathbb{R})$ δ_0*-close to* $id_{\mathbb{R}^{n+1}}$ *such that* $\tau_j^{-1} \cdot T_\sharp V_j^1$ $\mathbf{F}$*-converges to some* g_{euc}*-stationary* $\hat{V}_\infty \in \mathcal{IV}_n(\mathbb{R}^{n+1})$ *which doesn't equal to* $|\mathbf{C}|$ *and asymptotic to* $\mathbf{C}$ *near infinity, with asymptotic rate* $\mathcal{AR}_\infty(\hat{V}_\infty, \mathbf{C}) < \gamma$*. We call such* $\hat{V}_\infty$ *a* ***minimal blow-up limit*** *of* $\{V_j^1\}_{j\geq 1}$*;*

(ii) For every $t \in (2K^3\tau_j, 1)$*, we have*

$$max\{J_K^\gamma(u_j; K^{-1}t), \kappa\|g_j^1 - g_j^0\|_{C^4}\} \leq max\{J_K^\gamma(u_j;t), \kappa\|g_j^1 - g_j^0\|_{C^4}\}\,,$$

where u_j *is the graphical function in (6.7.2),* $J_K^\gamma = J^\gamma_{K;\Sigma_j^0,g_j^0}$ *is defined by*

$$J^\gamma_{K;\Sigma,g}(u;r)^2 := \int_{A^g(\mathbf{0},K^{-1}r,r)} u^2(x) \cdot |dist_g(\mathbf{0},x)|^{-2\gamma-n} \, d\|\Sigma\|_g(x)\,. \tag{6.7.3}$$

Recall that as in 5.2.29 we have,

$$C(K,\Lambda)^{-1} \leq \frac{J^\gamma_{K;\Sigma,g}(u;r) r^{\gamma+n/2}}{\|u\|_{L^2(A^g(\mathbf{0},K^{-1}r,r);\|\Sigma\|_g)}} \leq C(K,\Lambda). \tag{6.7.4}$$

The following L^2 growth estimate is an immediate consequence of Lemma 6.7.4.

Corollary 6.7.5 (L^2 growth estimate)**.** *Let* $\Lambda > 0$, $0 < \sigma < \gamma_{\mathrm{gap},\Lambda}/2$*, where* $\gamma_{\mathrm{gap},\Lambda}$ *is given by Lemma 5.2.1. Let* $K = K(\sigma,\Lambda) > 2$ *be given by Corollary 5.2.3. Then there exists* $\delta_1 = \delta_1(\sigma,\Lambda) \in (0,1)$ *with the following property,*

Let $\Sigma^i = \partial U^i$ $(i = 0, 1)$ be LSMH in $(\mathbb{B}^8_2, g)$ with area $\leq \Lambda$ and index $\leq \Lambda$. Assume the following holds for $\delta = \delta_1$,

(a) $\|g - g_{\mathrm{euc}}\|_{C^4} \leq \delta$; $\|u\|_{L^2(A^g(\mathbf{0},1/2,1),\|\Sigma^0\|_g)} \leq \delta$, where $u := G^{\Sigma^1}_{\Sigma^0,g} \in L^\infty(\Sigma^0)$;

(b) For $i = 0, 1$, there exists $x^i \in \mathrm{Sing}(\Sigma^i) \cap \mathbb{B}_{1/2}$ such that $x^0 = \mathbf{0}$ and

$$\theta_{|\Sigma^i|}(x^i, 1) - \theta_{|\Sigma^i|}(x^i) \leq \delta, \qquad \theta_{|\Sigma^1|}(x^1) = \theta_{|\Sigma^0|}(x^0).$$

Then,

(i) $\Sigma^1 \cap A^g(\mathbf{0}, K^{-3}, 3/2) = \mathrm{graph}_{\Sigma^0,g}(u) \cap A^g(\mathbf{0}, K^{-3}, 3/2)$; $x^1 \in B^g_{K^{-3}}(\mathbf{0})$ and

$$\|u\|_{C^2_*, A^g(\mathbf{0}, K^{-2}, 1)} \leq C(\sigma, \Lambda) \cdot \|u\|_{L^2(\|\Sigma^0\|_g)}.$$

(ii) We have L^2-growth estimate for u,

$$J_K^{\gamma_2^+ - \sigma}(u; K^{-1}) \leq J_K^{\gamma_2^+ - \sigma}(u; 1); \tag{6.7.5}$$

where $\gamma_2^+ := \gamma_2^+(\mathbf{C_0}\Sigma^0)$; $J^\gamma_K(u; r)$ is defined in (6.7.3) with $\Sigma = \Sigma^0$.

Proof. Suppose for contradiction, there exists a family of LSMH $\Sigma^i_j \subset (\mathbb{B}_2, g_j)$ $(i = 0, 1,\ j \geq 1)$ satisfying (a) and (b) with area $\leq \Lambda$, index $\leq \Lambda$, $\delta = 1/j$, $g_j \to g_{\mathrm{euc}}$ in C^4 but either (i) or (ii) fails. Let $u_j := G^{\Sigma^1_j}_{\Sigma^0_j, g_j}$. By unique continuation of LSMH, WLOG u_j is not identically 0 on $A^{g_j}(\mathbf{0}, 1/2, 1)$.

When $j \to \infty$, by [Sha17] and assumption (a), $|\Sigma^i_j|$ subconverges to an LSMH with multiplicity V_∞, independent of $i = 0, 1$. Moreover, by assumption (b) and Corollary D.0.3, $V_\infty = |\mathbf{C}| \llcorner \mathbb{B}_2$ for some stable minimal hypercone $\mathbf{C} \in \mathscr{C}_\Lambda$. Hence for sufficiently large j, (i) holds by Allard's Regularity Theorem [All72] and classical elliptic estimates. Thus (ii) fails for the sequence. Also by Corollary D.0.3, assumption (a) and (b) in Lemma 6.7.4 holds with $\kappa = 1$, $\gamma = \gamma_2^+(\mathbf{C}) - \sigma$, $g^i_j = g_j$ and $V^i_j = |\Sigma^i_j|_{g_j}$

when $j >> 1$. Therefore, Lemma 6.7.4 applies to deduce that appropriate blow up around $\mathbf{0}$ produces a stationary integral varifold $\hat{V}_\infty$ with $\mathcal{AR}_\infty(\hat{V}_\infty; \mathbf{C}) < \gamma_2^+(\mathbf{C}) - \sigma$. But by the density-almost-same assumption in Corollary 6.7.5 (b) and the rigidity case of Monotonicity formula, $\hat{\mathbf{V}}_\infty$ must be a non-zero translation of $\mathbf{C}$, which has asymptotic rate $= 0 > \gamma_2^+ - \sigma$. This provides a contradiction. □

By applying Corollary 6.7.5 to all suitable scales with a bootstrap argument, we get,

Corollary 6.7.6. *Let $0 < \sigma < \gamma_{\mathrm{gap},\Lambda}$, $\sigma' := \gamma_{\mathrm{gap},\Lambda} - \sigma$. Let $K = K(\sigma, \Lambda) > 2$, $\delta_1 = \delta_1(\sigma, \Lambda) > 0$ be in Corollary 6.7.5. There exists $\delta_2(\sigma, \Lambda) \in (0, \delta_1)$ with the following property.*

Suppose Σ^i $(i = 0, 1), g, \gamma_2^+$ be in Corollary 6.7.5 with (a) (b) holds for $\delta \leq \delta_2(\sigma, \Lambda)$ and $u := G^{\Sigma^1}_{\Sigma^0, g} \in L^2(\Sigma^0)$. Let

$$r_u := \|u\|^{2/(n-\sigma')}_{L^2(A^g(\mathbf{0}, K^{-1}, 1); \|\Sigma^0\|_g)}.$$

Then,

(i) $u \in C^2(\Sigma^0 \cap A^g(\mathbf{0}, r_u/2, 3/2))$ and

$$\mathrm{graph}_{\Sigma^0, g}(u) \cap A^g(\mathbf{0}, r_u/2, 3/2) = \Sigma^1 \cap A^g(\mathbf{0}, r_u/2, 3/2)\,;$$

(ii) For every $r \in [r_u, 1]$, we have

$$\|u\|_{C^2_*, A^g(\mathbf{0}, r, 1)} \leq C(\sigma, \Lambda) \|u\|_{L^2(A^g(\mathbf{0}, K^{-1}, 1); \|\Sigma^0\|_g)} \cdot r^{\gamma_2^+ - \sigma - 1} \leq C(\sigma, \Lambda) r_u^{\sigma'/2}\,.$$

Proof. Assume that u is not identically 0. Then by unique continuation of LSMH, $r_u > 0$.

For each $\tau \in (0,1]$, let $g_\tau := \tau^{-2}\eta^*_{\mathbf{0},\tau} g$, $\Sigma^i_\tau := \eta^{-1}_{\mathbf{0},\tau}(\Sigma^i)$, $x^i_\tau := \eta^{-1}_{\mathbf{0},\tau}(x^i)$, $i = 0, 1$. By the volume monotoncity formula for LSMH, we can choose $\delta_1'(\sigma, \Lambda) \in (0, \delta_1)$ small such that if (a), (b) hold for $\delta \le \delta_1'$, then for every $\tau \in (0,1]$, we have

$$\|g_\tau - g_{\mathrm{euc}}\|_{C^3(\mathbb{B}_2)} \le \delta_1, \qquad\qquad \theta_{|\Sigma^i_\tau|}(x^i_\tau, 1) - \theta_{|\Sigma^i_\tau|}(x^i_\tau) \le \delta_1.$$

Let

$$l_0 := \sup\big\{ l \in \mathbb{N}^+ : \|u\|_{L^2(A^g(\mathbf{0},\tau/K,\tau);\|\Sigma^0\|_g)} \le \delta_1' \cdot \tau^{1+n/2}, \\ \forall \tau = K^{-m} \in [K^{-l+1}, 1],\ m \text{ integer} \big\}.$$

Claim. By taking $\delta_2(\sigma, \Lambda) \in (0, \delta_1')$ small enough, we have $r_u \ge K^{-l_0}$ provided (a) and (b) hold for $\delta < \delta_2$.

Proof of the Claim. WLOG assume that $l_0 < +\infty$. Choose $\delta_2(\sigma, \Lambda) \in (0, \delta_1')$ small enough such that

$$\|u\|_{L^2(A^g(\mathbf{0},K^{-1},1);\|\Sigma^0\|_g)} < \varepsilon,$$

where $\varepsilon = \varepsilon(\sigma, \Lambda) < 1$ small enough, which will be determined later.

By definition of l_0 and Corollary 6.7.5, inductively on l we have, $\forall 1 \le l \le l_0$,

$$J_K^{\gamma_2^+ - \sigma}(u, K^{-l}) \le J_K^{\gamma_2^+ - \sigma}(u, 1) \le C(\sigma, \Lambda)\|u\|_{L^2(A^g(\mathbf{0},K^{-1},1);\|\Sigma^0\|_g)}.$$

Hence combined with (6.7.4) we have for every $1 \le l \le l_0$,

$$\|u\|_{L^2(A^g(\mathbf{0},K^{-l-1},K^{-l}))} \le C(\sigma, \Lambda)\|u\|_{L^2(A^g(\mathbf{0},K^{-1},1);\|\Sigma^0\|_g)} \cdot K^{-l(\gamma_2^+ - \sigma + n/2)}. \tag{6.7.6}$$

Therefore if $K^{-l_0} > r_u$, then we must have,

$$\begin{aligned}\|u\|_{L^2(A^g(\mathbf{0},K^{-l_0-1},K^{-l_0}))} &\leq C(\sigma,\Lambda) r_u^{\sigma'/2} \cdot r_u^{n/2-\sigma'} \cdot K^{-l_0(\gamma_2^+-\sigma+n/2)} \\ &\leq C(\sigma,\Lambda) r_u^{\sigma'/2} \cdot K^{-l_0(n+\gamma_2^+-\sigma-\sigma')} \\ &\leq \bar{C}(\sigma,\Lambda) r_u^{\sigma'/2} \cdot K^{-l_0(1+n/2)},\end{aligned} \tag{6.7.7}$$

where the first inequality follows from (6.7.6) and the definition of r_u, the second inequality follows by $K^{-l_0} > r_u$ and $\sigma' < \gamma_{\mathrm{gap},\Lambda} < (n-2)/2$, the third inequality follows by the fact that $\sigma + \sigma' = \gamma_{\mathrm{gap},\Lambda}$ and that

$$\frac{n-2}{2} + \gamma_2^+ - \gamma_{\mathrm{gap},\Lambda} \geq -\gamma_1^+ + \gamma_2^+ - \gamma_{\mathrm{gap},\Lambda} \geq 0. \tag{6.7.8}$$

Choose $\varepsilon(\sigma,\Lambda)$ small enough such that $\bar{C}(\sigma,\Lambda)\varepsilon^{\sigma'/(n-\sigma')} \leq \delta_1'/2$. (6.7.7) contradicts the choice of l_0. This concludes the proof of the Claim. ■

Now by the Claim and Corollary 6.7.5, (i) holds; (ii) also holds by combining the estimate (6.7.6) and (6.7.8). □

For later applications, we also need to compare the graphical functions of different hypersurfaces. This is also derived from Corollary 6.7.5.

Corollary 6.7.7. *Let $\Sigma^i = \partial U^i$ $(i = 0,1,2)$ be LSMH in $(\mathbb{B}_2^8, g)$ with area $\leq \Lambda$ and index $\leq \Lambda$. Assume*

(a) $\|g - g_{\mathrm{euc}}\|_{C^4} \leq \delta$, $\|u^{(i)}\|_{L^2(A^g(\mathbf{0},1/2,1);\|\Sigma^0\|_g)} \leq \delta$, where $u^{(i)} := G^{\Sigma^i}_{\Sigma^0,g} \in L^\infty(\Sigma^0)$, $i = 1,2$;

(b) For $i = 0,1,2$, there exists $x^i \in \mathrm{Sing}(\Sigma^i) \cap \mathbb{B}_{1/2}$ such that $x^0 = \mathbf{0}$ and

$$\begin{aligned}&\theta_g(x^i, 1; \|\Sigma^i\|_g) - \theta(x^i; \|\Sigma^i\|_g) \leq \delta; \\ &\theta(x^0; \|\Sigma^0\|_g) = \theta(x^1; \|\Sigma^1\|_g) = \theta(x^2; \|\Sigma^2\|_g).\end{aligned}$$

Let $0 < \sigma < \gamma_{\mathrm{gap},\Lambda}/2$, $\sigma' := \gamma_{\mathrm{gap},\Lambda} - \sigma$. *Then for* $\delta \leq \delta_3(\sigma, \Lambda)$ *and for* $r \in (40 \sup\{r_{u^{(1)}}, r_{u^{(2)}}\}, 1/4)$, *we have*

$$\|u^{(1)} - u^{(2)}\|_{L^2(A^g(\mathbf{0},r/2,r);\|\Sigma^0\|_g)} \leq C(\sigma,\Lambda)\|G^{\Sigma^2}_{\Sigma^1,g}\|_{L^2(\|\Sigma^1\|_g)} \cdot r^{n/2+\gamma_2^+(\mathbf{C}_{x^1}\Sigma^1)-\sigma}.$$

Proof. Denote for simplicity, $v := G^{\Sigma^2}_{\Sigma^1,g} \in L^\infty(\Sigma^1)$; We have,

$$\begin{aligned}
\|u^{(2)} - u^{(1)}\|_{L^2(A^g(\mathbf{0},r/2,r))} &\leq C(\Lambda) r^{n/2} \cdot \sup_{A^g(\mathbf{0},r/2,r)\cap\Sigma^0} |u^{(2)} - u^{(1)}| \\
&\leq C(\Lambda) r^{n/2} \cdot \sup_{A^g(\mathbf{0},r/4,2r)\cap\Sigma^1} |v| \\
&\leq C(\Lambda) r^{n/2+1} \cdot \|v\|_{C^2_*, A^g(x^1,r/10,1)\cap\Sigma^1} \\
&\leq C(\Lambda,\sigma) r^{n/2+1} \cdot r^{\gamma_2^+(\mathbf{C}_{x^1}\Sigma^1)-\sigma-1}\|v\|_{L^2(\|\Sigma^1\|_g)}.
\end{aligned}$$

Here, the first inequality follows by volume upper bound for minimal hypersurface Σ^0 in $A^g(\mathbf{0}, r/2, r)$; The second inequality, after taking δ sufficiently small, is given by the following geometric Lemma 6.7.8 comparing $G^{\Sigma^2}_{\Sigma^1,g}$ with $|u^2 - u^1|$; The third inequality follows directly from definition of C^2_* norm and that $A^g(\mathbf{0}, r/4, 2r) \subset A^g(x^1; r/10, 1)$; The last inequality is given by Corollary 6.7.6 (ii). □

Lemma 6.7.8. *There is a universal constant* ε_0 *with the following property. Let* $\Sigma^i = \partial U^i$ *(*$i = 0, 1, 2$*) be connected* C^2 *hypersurfaces in* $\mathbb{B}_1$. *Suppose* g *is a* C^2 *metric on* $\mathbb{B}_1$ *such that*

(a) $|A_{\Sigma^0}|_g \leq \varepsilon_0$ *and* $\mathbf{0} \in \Sigma^0$; $\|g - g_{\mathrm{euc}}\|_{C^2} \leq \varepsilon_0$;

(b) $\Sigma^i = \mathrm{graph}_{\Sigma^0}(u^{(i)})$ *with* $\|u^{(i)}\|_{C^2} \leq \varepsilon_0$, $i = 1, 2$.

Then we have

$$|u^{(2)}(\mathbf{0}) - u^{(1)}(\mathbf{0})| \leq 2 \sup_{\mathbb{B}_{1/4}} |G^{\Sigma^2}_{\Sigma^1,g}| \leq 4 \sup_{\mathbb{B}_{1/2}} |u^{(2)} - u^{(1)}|.$$

Proof. This follows directly from the triangle inequality. □

6.7.2 Induced Jacobi field in general

The first goal of this section is to prove the following,

Theorem 6.7.9. *Let (M^{n+1}, g) be a closed Riemannian manifold, (Σ, ν) be a two-sided LSMH in (M^{n+1}, g) with only strongly isolated singularities, β be a symmetric 2-tensor on M such that*

$$q(\beta) := div_{\Sigma,g}(\beta(\nu, \cdot)) - \frac{1}{2} tr_\Sigma(\nabla^M_\nu \beta) \quad \text{is not identically } 0 \text{ on } \Sigma\,.$$

Let $\{g_j\}$ be a family of Riemannian metrics such that either $g_j \equiv g$ for all j, or there exist constants $c_j \searrow 0$ and symmetric 2-tensors $\beta_j \to \beta$ in C^4 as $j \to \infty$ such that $g_j = g + c_j\beta_j$.

*Let $\{V_j \neq |\Sigma|\}_{j\geq 1}$ be a family of stationary integral varifolds in (M, g_j), V_j **F**-converges to $|\Sigma|$. Suppose for every $p \in \mathrm{Sing}(\Sigma)$, one of the following holds,*

(A) the tangent cone $\mathbf{C}_p$ of Σ at p is area-minimizing at least in one side.

(B) there exists some neighborhood $U_p \subset M$ of p such that $V_j \llcorner U_p$ are stable minimal hypersurfaces (possibly with multiplicity) in (U_p, g_j), $j >> 1$.

Then after passing to a subsequence, $\{(V_j, g_j)\}_{j\geq 1}$ induces a generalized Jacobi field ϕ satisfying $\mathcal{AR}_p(\phi) \geq \gamma_1^-(\mathbf{C}_p)$ for every $p \in \mathrm{Sing}(\Sigma)$, and

$$\begin{cases} L_\Sigma \phi = 0, & \text{if } g_j \equiv g\,; \\ L_\Sigma \phi = cq(\beta) \ \text{ for some } c \geq 0, & \text{if } g_j = g + c_j\beta_j\,. \end{cases} \tag{6.7.9}$$

Moreover, if $p \in \mathrm{Sing}(\Sigma)$ be such that $\mathcal{AR}_p(\phi) < \gamma_2^+(\mathbf{C}_p)$, then there exists some neighborhood $W_p \subset M$ of p such that $\mathrm{Sing}(V_j) \cap W_p = \emptyset$ for infinitely many j.

Proof. We first consider the simpler setting where $g_j \equiv g$. For each $p \in \mathrm{Sing}(\Sigma)$, let $\gamma_p \in (\gamma_2^-(\mathbf{C}_p), \gamma_1^-(\mathbf{C}_p)) \setminus \Gamma(\mathbf{C}_p)$ be fixed, $\sigma := \inf_{p\in\mathrm{Sing}(\Sigma)} dist_{\mathbb{R}}(\gamma_p, \Gamma(\mathbf{C}_p)) > 0$; Take $g_j^0 = g_j^1 = g$, $V_j^0 = |\Sigma|_g$, $V_j^1 = |\Sigma_j|_g$, $\kappa = 1$, Λ, Λ' be sufficiently large such that $\mathbf{C}_p \in \mathscr{C}^n_{\Lambda,\Lambda'}$ for every $p \in \mathrm{Sing}(\Sigma)$. Let $K(\sigma, \Lambda, \Lambda') > 2$, $\bar{\delta}_0(\sigma, \Lambda, \Lambda', \kappa) \in (0, 1)$ be given by Lemma 6.7.4. Working in T_pM, by Lemma 2.3.1, there exists $\tau_p(\Sigma, M, g) \in (0, \mathrm{injrad}(M, g)/4)$ such that $(\tau_p^{-1} \cdot V_j^i, \tau_p^{-2}\eta_{\tau_p}^* g_j^i)$ satisfies the asumption (a) (b) in Lemma 6.7.4 with $\delta_0 = \bar{\delta}_0$. And by a rescaling of g, WLOG $\tau_p = 1$ for every $p \in \mathrm{Sing}(\Sigma)$.

By Theorem 5.2.12 and assumption (A) and (B) in Theorem 6.7.9, any blow up limit from Lemma 6.7.4 (i) must have $\mathcal{AR}_\infty(\hat{V}_\infty, \mathbf{C}_p) \geq \gamma_1^-(\mathbf{C}_p)$, only (ii) in Lemma 6.7.4 could happen for every singularity p with $\gamma = \gamma_p$. In particular, if let $s_j \searrow 0$ and $v_j \in C^2_{loc}(\Sigma)$ be such that

$$\|v_j\|_{C^2_*, \Sigma\setminus\mathcal{B}_{s_j}(\mathrm{Sing}(\Sigma))} \leq \bar{\delta}_0$$

and that

$$V_j \llcorner M \setminus B^g_{s_j}(\mathrm{Sing}(\Sigma)) = |\,\mathrm{graph}_\Sigma(v_j)|\llcorner M \setminus B^g_{s_j}(\mathrm{Sing}(\Sigma))$$

Then for each $p \in \mathrm{Sing}(\Sigma)$ and every $r \in (2K^2 s_j, 1)$,

$$\int_{A^g(p;K^{-1}r,r)} v_j^2 \rho^{-n-2\gamma_p}\, d\|\Sigma\|(x) \leq C(\Sigma, M, g) \int_{A^g(p;K^{-1},1)} v_j^2 \rho^{-n-2\gamma_p}\, d\|\Sigma\|(x)\,, \tag{6.7.10}$$

where $\rho := dist_\Sigma(\mathrm{Sing}(\Sigma), \cdot)$. In particular, $\forall j >> 1$ and $\forall \Omega \subset \Sigma \setminus B^g_{s_j}(\mathrm{Sing}(\Sigma))$, we have

$$\int_\Omega v_j^2\, dx \leq C(\Sigma, M, g, \sigma, \Omega) \cdot \int_{\Sigma\setminus B^g_{\bar{s}_0}(\mathrm{Sing}(\Sigma))} v_j^2\, dx\,. \tag{6.7.11}$$

The key point of this estimate is that the constant on the right hand side does **NOT**

depend on j.

We take $a_j := \|v_j\|_{L^2(\Sigma\setminus B^g_{K^{-1}}(\mathrm{Sing}(\Sigma)))} > 0$ for $j >> 1$. Since $V_j \neq |\Sigma|$, by Allard Regularity Theorem [All72], unique continuation property of minimal surface equation and assumption (A) (B) on V_j, we must have $a_j > 0$ for $j >> 1$.

Let $\hat{v}_j := v_j/a_j$. By (6.7.11) and standard elliptic estimate, $\hat{v}_j \to v_\infty$ in $C^3_{loc}(\Sigma)$ and by the choice of a_j, v_∞ is not identically 0. Also by (6.7.10), the choice of γ_p and the growth rate quantization Corollary 5.2.6, we know that $\mathcal{AR}_p(v_\infty) \geq \gamma_1^-(\mathbf{C}_p)$ for all $p \in \mathrm{Sing}(\Sigma)$.

Moreover, if at some p, $\mathcal{AR}_p(v_\infty) < \gamma_2^+(\mathbf{C}_p)$, then Corollary 5.2.6 yields $\mathcal{AR}_p(v_\infty) \in \{\gamma_1^\pm(\mathbf{C}_p)\}$. Again by Lemma 6.7.4, the minimal blow-up limit $\hat{V}_\infty$ should satisfies $\mathcal{AR}_\infty(\hat{V}_\infty, \mathbf{C}_p) \leq \gamma_1^+(\mathbf{C}_p)$. Then by Theorem 5.2.12 and assumption (A), (B) above, $\hat{V}_\infty$ has no singularity, and so does the converging sequence near p by Allard Regularity Theorem.

The proof for the case where $g_j = g + c_j\beta_j$ as in the statement of Theorem is similar but more delicate in choosing the renormalizing constant. Since $q(\beta)$ is not identically 0, let $\Omega_0 \subset\subset \Sigma$ such that $\mathrm{spt}(q(\beta)) \cap \Omega_0 \neq \emptyset$. Let $v_j \in C^2(\Omega_0)$ be the graphical function of V_j over Ω, $\|v_j\|_{C^2} \to 0$.

Claim. $\liminf_{j\to\infty} \|v_j\|_{L^2(\Omega_0)}/c_j > 0$.

Proof of the Claim. Otherwise, up to a subsequence, $v_j/c_j \to 0$ in $L^2(\Omega_0)$. Since V_j is stationary in (M, g_j), by Theorem A.0.1, for every $\phi \in C^2_c(\Omega_0)$,

$$\begin{aligned} 0 &= \int_{\Omega_0} \mathscr{M}^{g_j} v_j \cdot \phi \\ &= \int_{\Omega_0} a^{ik}_{g_j}(x, v_j, \nabla v_j)\partial_i(v_j)\partial_k\phi + b^k_{g_j}(x, v_j, \nabla v_j)\partial_k\phi + \partial_z F^{g_j}(x, v_j, \nabla v_j)\phi \\ &= \int_{\Omega_0} -v_j \cdot \partial_i\big(a^{ik}_{g_j}(x, v_j, \nabla v_j)\partial_k\phi\big) + b^k_{g_j}(x, v_j, \nabla v_j)\partial_k\phi + \partial_z F^{g_j}(x, v_j, \nabla v_j)\phi\,. \end{aligned} \tag{6.7.12}$$

By multiplying both side of (6.7.12) by $1/c_j$ and taking $j \to \infty$, by Theorem A.0.1 and the assumption that $(g_j - g)/c_j \to \beta$ in $C^4(M)$, $v_j \to 0$ in $C^2(\Omega_0)$ and $v_j/c_j \to 0$ in $L^2(\Omega_0)$, we get,

$$\int_{\Omega_0} \beta(\nu, \nabla_\Sigma \phi) + \frac{1}{2} tr_\Sigma(\nabla^M_\nu \beta) \cdot \phi = 0, \quad \forall \phi \in C^2_c(\Omega_0)\,.$$

This contradicts to that $\mathrm{spt}(q(\beta)) \cap \Omega_0 \neq \emptyset$. Thus the Claim is proved. □

With this Claim, we see that $\exists \kappa > 0$ such that $\kappa \|g_j - g\|_{C^4} \leq \|v_j\|_{L^2(\Omega_0)}$. Now repeat the previous proof by applying Lemma 6.7.4 with also taking $\kappa > 0$ specified above. We still get a local-control-global typed L^2-estimate as (6.7.11), and then the conclusion holds immediately by elliptic estimate. □

Now we shall use Theorem 6.7.9 to prove the Regular Deformation Theorem 6.2.4. The upshot is that semi-nondegeneracy of Σ and appropriate choice of metric perturbation guarantees that every solution to equation (6.7.9) has asymptotic rate $\in \{\gamma_1^{\pm}(\mathbf{C}_p)\}$ for some $p \in \mathrm{Sing}(\Sigma)$.

Given a two-sided LSMH (Σ, ν) in an 8-manifold, first recall that in Section 5.3, we have introduced the function space $\mathscr{B}(\Sigma)$ to study the spectral theory of the Jacobi opreator L_Σ in place of $W^{1,2}(\Sigma)$. By Proposition 5.3.5 and Lemma 5.3.9, we have a spectral decomposition of $L^2(\Sigma)$ and $\mathscr{B}(\Sigma)$ into finite dimensional eigensubspaces of L_Σ. And Σ is called *non-degenerate* if $L_\Sigma u = 0$ has only zero solution in $\mathscr{B}(\Sigma)$.

Lemma 6.7.10. *Suppose (Σ, ν) is a two-sided LSMH in (M, g); $G < Diff(M)$ is a finite group which acts fixed-point-freely on M, such that (Σ, M, g) is G-invariant in the sense of Definition 6.2.11. Then the following $\mathscr{F}^{k,\alpha}_G$ contains an open dense subset in $C^{k,\alpha}(M)^G$,*

$$\mathscr{F}^{k,\alpha}_G := \{f \in C^{k,\alpha}(M)^G : L_{\Sigma,g} u = \nu(f) \text{ has no slower growth } G\text{-invariant solution } u\}\,.$$

Proof. When $G = \{id_M\}$, this is proved in Lemma 5.3.13. For a general G, we shall slightly modify that proof based on the following observation.

Claim. *If* $(\operatorname{Ker} L_{\Sigma,g})^G = 0$, *then for every* $f \in L^2(\Sigma)^G$, *there's a unique solution* $u \in \mathscr{B}(\Sigma)^G$ *to* $L_\Sigma u = f$. *Here* $\operatorname{Ker} L_{\Sigma,g}$ *consists of Jacobi fields in* $\mathscr{B}(\Sigma)$.

Proof of the Claim. The uniqueness follows by definition; To see the existence, first notice that for every $\phi \in \operatorname{Ker} L_{\Sigma,g}$, since f is G-invariant, we have

$$\langle \phi, f \rangle_{L^2} = \langle \frac{1}{\#G} \sum_{\gamma \in G} \gamma \cdot \phi, f \rangle_{L^2} = 0 \,.$$

Here the last equality follows from that $\sum_\gamma \gamma \cdot \phi \in (\operatorname{Ker} L_{\Sigma,g})^G = 0$. Therefore by Proposition 5.3.5, there exists $u_0 \in \mathscr{B}(\Sigma)$ solving $L_{\Sigma,g} u_0 = f$. Then

$$u := \frac{1}{\#G} \sum_{\gamma \in G} \gamma \cdot u_0$$

will be a desired G-invariant solution. □

Now we go back to the proof of Lemma 6.7.10. Following the proof of Lemma 5.3.13, for each $p \in \operatorname{Sing}(\Sigma)$, we define $\mathscr{V}_p$ by,

- $\mathscr{V}_p := \{f \in L^\infty(\Sigma)^G : u := (L_{\Sigma,g})^{-1} f \in \mathscr{B}_0(\Sigma)^G \text{ satisfies } \limsup_{x \to p} |u|(x)/\xi_p(x) > 0\}$, if $(\operatorname{Ker} L_{\Sigma,g})^G = 0$;
- $\mathscr{V}_p := \{f \in L^\infty(\Sigma)^G : L_{\Sigma,g} u = f \text{ has no solution } u \in \mathscr{B}(\Sigma)^G\}$, if $(\operatorname{Ker} L_{\Sigma,g})^G \neq 0$.

Here ξ_p is defined in the proof of Lemma 5.3.13. We can even choose $\xi_{\gamma(p)} = \gamma \cdot \xi_p$ for every $\gamma \in G$. By the same argument as Lemma 5.3.13 with aid of the Claim, we have

$$\mathscr{F}^{k,\alpha}_G \supset \bigcup_{p \in \operatorname{Sing}(\Sigma)} \{f \in C^{k,\alpha}(M)^G : \nu_\Sigma(f) \in \mathscr{V}_p\} \,,$$

and the right hand side is open and dense in $C^{k,\alpha}(M)^G$. □

Remark 6.7.11. If Σ is semi-nondegenerate, by adapting the proof of Lemma 6.8.1 in the next section, one can also show that

$$\{f \in C^{k,\alpha}(M) : L_{\Sigma,g}u = \nu(f) \text{ has no solution } u \text{ of slower growth}\},$$

is open and dense in $C^{k,\alpha}(M)$.

Proof of Theorem 6.2.4. First assume that Σ is two-sided. Let $\mathscr{F}^{k,\alpha}$ be given by Lemma 6.7.10 with $G = \{id_M\}$. Also let $f \in \mathscr{F}^{k,\alpha}$ and Σ_j, g_j be given by Theorem 6.2.4 (a), (b). Then by Theorem 6.7.9, the induced Jacobi field $u \not\equiv 0$ satisfies $L_{\Sigma,g}u = c\nu(f)$ for some $c \geq 0$. Hence either $c \neq 0$, or $c = 0$ and thus $u \in \operatorname{Ker} L_{\Sigma,g}$. By definition of $\mathscr{F}^{k,\alpha}$ and semi-nondegeneracy of Σ, we know that in both cases, u is not a function of slower growth on Σ, in other words, there exists $p \in \operatorname{Sing}(\Sigma)$ such that $\mathcal{AR}_p(u) < \gamma_2^+(\mathbf{C}_p\Sigma)$. Hence by Lemma 6.2.1, $\mathcal{AR}_p(u) \in \{\gamma_1^{\pm}(\mathbf{C}_p\Sigma)\}$ and Theorem 6.7.9 applies to conclude Theorem 6.2.4.

When Σ is one-sided, let $\hat{\Sigma} \subset (\hat{M}, \hat{g})$ be its two-sided double cover. Then using Lemma 6.7.10 with $G = \mathbb{Z}_2 < \operatorname{Diff}(\hat{M})$ to be the deck transformation, the same argument as above proves Theorem 6.2.4. □

6.7.3 Induced Jacobi field for pairs

Let Σ be a connected two-sided LSMH in a closed manifold (M^8, g), with $\nu = \nu_{\Sigma,g}$ to be a global unit normal field. We also fix an integer $k \geq 4$ and $\alpha \in (0, 1)$.

Since near each singularity, Σ is a $C^{1,\alpha}$ diffeomorphic to a minimal hypercone restricted in unit ball, we know that there exists an open subset $\mathcal{N} \subset M$ being a sufficiently small neighborhood of $\operatorname{Clos}(\Sigma)$ such that $\mathcal{N} \setminus \operatorname{Clos}(\Sigma)$ has two connected component $\mathcal{N}_+ \sqcup \mathcal{N}_-$ and ν points into $\mathcal{N}_+$. Moreover, for every $(g', \Sigma') \in \mathcal{M}^{k,\alpha}(M)$ close enough to (g, Σ), we know that $\operatorname{Clos}(\Sigma') \subset \mathcal{N}$ and that there is a unique

connected component $\mathcal{N}'_-$ of $\mathcal{N} \setminus \operatorname{Clos}\Sigma'$ such that

$$\mathcal{N}'_- \Delta \mathcal{N}_- \subset\subset \mathcal{N}.$$

In this subsection, to define a graphical function as in (6.7.1), the ambient manifold N is chosen to be $\mathcal{N}$. Let $\delta_4 = \delta_4(g, \Sigma, \Lambda) > 0$ such that for every $(g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_4)$, $\operatorname{Clos}(\Sigma') \subset \mathcal{N}$ and bounds a unique $\mathcal{N}_-$ as above. When defining graphical functions between any two elements $(g^1, \Sigma^1), (g^2, \Sigma^2) \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_4)$ as in (6.7.1), we always choose $U^i := \mathcal{N}^i_-$.

We define a semi-metric on $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_4)$ by

$$\mathbf{D}^{\mathcal{L}}[(g_1, \Sigma_1), (g_2, \Sigma_2)] := \|g_1 - g_2\|_{L^\infty(M)} + \|G^{\Sigma^2}_{\Sigma_1, g_1}\|_{L^2(\|\Sigma_1\|_{g_1})} + \|G^{\Sigma^1}_{\Sigma_2, g_2}\|_{L^2(\|\Sigma_2\|_{g_2})}.$$

Here, norms of tensors are taken with respect to metric g which we have fixed at the beginning. By [SS81], (g_j, Σ_j) converges to $(g_\infty, \Sigma_\infty)$ in $\mathcal{L}^{k,\alpha}$ if and only if $\mathbf{D}^{\mathcal{L}}[(g_j, \Sigma_j), (g_\infty, \Sigma_\infty)] \to 0$ and $\|g_j - g_\infty\|_{C^{k-1,\alpha}(M)} \to 0$ when $j \to \infty$.

The following Lemma is an application of Corollary 6.7.7.

Lemma 6.7.12. *Let $r \in (0, \operatorname{injrad}(g, \Sigma))$. Suppose for $1 \leq j < \infty$,*

(a) $(g^0_j, \Sigma^0_j) \neq (g^1_j, \Sigma^1_j) \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_1)$ are distinct pairs, and $(\bar{g}_j, \bar{\Sigma}_j) \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_1)$ such that when $j \to \infty$, $(g^i_j, \Sigma^i_j) \to (g, \Sigma)$ in $\mathcal{L}^{k,\alpha}$, $i = 0, 1$ and $(\bar{g}_j, \bar{\Sigma}_j) \to (g, \Sigma)$ also in $\mathcal{L}^{k,\alpha}$;

(b) For $i = 0, 1$, $g^i_j = (1 + t_j f^i_j)\bar{g}_j$, where $t_j \searrow 0$, $f^i_j \in C^{k,\alpha}(M)$ with uniformly bounded $C^{k,\alpha}$ norms, such that $\operatorname{spt}(f^i_j) \cap B^g(\operatorname{Sing}(\Sigma), r) = \emptyset$, and that either $f^0_j \equiv f^1_j$ (in which case we set $f_\infty \equiv 0$), or $f^1_j - f^0_j \to f_\infty$ in $C^{k,\alpha}(M)$ satisfying that $\nu_{\Sigma,g}(f_\infty)$ is not identically 0 along Σ.

Let $d_j := \mathbf{D}^{\mathcal{L}}[(g^0_j, \Sigma^0_j), (g^1_j, \Sigma^1_j)] > 0$ and $u^{(i)}_j := G^{\Sigma^i_j}_{\bar{\Sigma}_j, \bar{g}_j} \in L^2(\bar{\Sigma}_j)$. Then we have, after

passing to a subsequence of $j \to \infty$,

$$(u_j^{(1)} - u_j^{(0)})/d_j \to \hat{u}_\infty \ \text{ in } C^2_{loc}(\Sigma); \qquad t_j(f_j^1 - f_j^0)/d_j \to \hat{f}_\infty \ \text{ in } C^2(M),$$

where $\hat{f}_\infty \in \mathbb{R}_{\geq 0}\langle f_\infty \rangle$, $\hat{u}_\infty$ *is a nonzero solution of*

$$L_{\Sigma,g}\hat{u}_\infty = \frac{n}{2}\nu_{\Sigma,g}(\hat{f}_\infty);$$

Moreover, $\hat{u}_\infty$ *satisfies* $\mathcal{AR}_p(\hat{u}_\infty) \geq \gamma_2^+(\mathbf{C}_p\Sigma)$ *for every* $p \in \mathrm{Sing}(\Sigma)$. *In other words,* $\hat{u}_\infty$ *is a function of slower growth.*

Proof. Fix $\sigma \in (0, \gamma_{\mathrm{gap},\Lambda}/2)$. By Corollary 6.7.6 and 6.7.7, there exists $r_0(\sigma, g, \Sigma, \Lambda) \in (0, r/4)$ and $r_j \searrow 0$ such that

(i) For $i = 0, 1$, $\Sigma_j^i \setminus B^{\bar{g}_j}(\mathrm{Sing}(\bar{\Sigma}_j), r_j) = \mathrm{graph}_{\bar{\Sigma}_j,\bar{g}_j}(u_j^{(i)}) \setminus B^{\bar{g}_j}(\mathrm{Sing}(\bar{\Sigma}_j), r_j)$, and when $j \to \infty$,

$$\|u_j^{(i)}\|_{C^2_*,\bar{\Sigma}_j \setminus B^{\bar{g}_j}(\mathrm{Sing}(\bar{\Sigma}_j), r_j)} \to 0.$$

(ii) For every $p \in \mathrm{Sing}(\bar{\Sigma}_j)$ and every $\tau \in [2r_j, 2r_0]$, let $p^0 \in \mathrm{Sing}(\Sigma_j^0) \cap B^{g_j^0}(p, r_j)$. Then $w_j := u_j^{(1)} - u_j^{(0)}$ satisfies

$$\|w_j\|_{L^2(A^{\bar{g}_j}(p,\tau/2,\tau);\|\bar{\Sigma}_j\|_{\bar{g}_j})} \leq C(g, \Sigma, \Lambda, \sigma, r_0) d_j \cdot \tau^{n/2+\gamma_2^+(\mathbf{C}_{p^0}\Sigma_j^0)-\sigma}. \tag{6.7.13}$$

Also by Theorem A.0.1, for $j >> 1$, w_j satisfies the following equation on $\bar{\Sigma}_j \setminus B^{\bar{g}_j}(\mathrm{Sing}(\bar{\Sigma}_j), r_j)$

$$-L_{\bar{\Sigma}_j,\bar{g}_j} w_j + \frac{n}{2} t_j \nu_{\bar{\Sigma}_j,\bar{g}_j}(f_j^1 - f_j^0) = \mathrm{div}_{\bar{\Sigma}_j,\bar{g}_j}(\mathcal{E}_j^1) + \frac{1}{\rho_j}\mathcal{E}_j^2, . \tag{6.7.14}$$

where ρ_j be the distant function under $\bar{g}_j$ to $\mathrm{Sing}(\bar{\Sigma})$, $\mathcal{E}_j^1$ and $\mathcal{E}_j^2$ are error terms with

pointwise estimate,

$$|\mathcal{E}_j^1|, |\mathcal{E}_j^2| \leq C(\Sigma, g) \cdot \left[\sum_{i=0}^{1} \left(\frac{|u_j^{(i)}|}{\rho_j} + |\nabla u_j^{(i)}| + t_j \sum_{l=0}^{2} |\nabla_M^l f_j^i| \right) \right] \cdot \left(\frac{|w_j|}{\rho_j} + |\nabla w_j| + t_j \sum_{l=0}^{2} |\nabla_M^l (f_j^1 - f_j^0)| \right). \tag{6.7.15}$$

Note that if $f_j^0 \neq f_j^1$, then by definition, for $j >> 1$,

$$d_j \geq \|g_j^1 - g_j^0\|_{L^\infty} \geq C(g) t_j \cdot \|f_\infty\|_{L^\infty}. \tag{6.7.16}$$

Thus by assumption (b), after passing to a subsequence, we have,

$$\frac{1}{d_j} \cdot t_j (f_j^1 - f_j^0) \to \hat{f}_\infty \in \mathbb{R}_{\geq 0} \langle f_\infty \rangle \quad \text{in } C^k(M).$$

While if $f_j^0 \equiv f_j^1$, then (6.7.16) and the convergence above hold automatically.

Let $d_j' := \|w_j\|_{L^2(\bar{\Sigma}_j \setminus B^{\bar{g}_j}(\mathrm{Sing}(\bar{\Sigma}_j), r_0); \|\bar{\Sigma}_j\|_{\bar{g}_j})}$.

Claim. $\liminf_{j \to \infty} d_j / d_j' \in (0, +\infty)$.

Once the claim is established, recall by (6.7.13), (6.7.16) and elliptic estimate for (6.7.14), up to a subsequence, we have for $j >> 1$,

$$\|w_j\|_{C^2, \bar{\Sigma}_j \setminus B^{\bar{g}_j}(\mathrm{Sing}(\bar{\Sigma}_j), 2r_0)} \leq C(g, \Sigma, r_0)(d_j' + d_j + t_j \|f_j^1 - f_j^0\|_{C^2(M)}).$$

Thus combined with (6.7.13), (6.7.16) and the Claim, we have, after passing to a subsequence of $j \to \infty$, $w_j / d_j \to \hat{u}_\infty$ in $C^2_{loc}(\Sigma)$. By (6.7.14), (6.7.15) and the Claim, we know that $\hat{u}_\infty \neq 0$ and satisfies

$$-L_{\Sigma, g} \hat{u}_\infty + \frac{n}{2} \nu_{\Sigma, g}(\hat{f}_\infty) = 0.$$

Also by (6.7.13) and Corollary D.0.3, for every $p \in \mathrm{Sing}(\Sigma)$ and every $\tau \in (0, r_0)$,

$$\|\hat{u}_\infty\|_{L^2(A^g(p;\tau/2,\tau);\|\Sigma\|_g)} \leq C\tau^{n/2+\gamma_2^+(\mathbf{C}_p\Sigma)-\sigma}.$$

By definition of $\mathcal{AR}_p$, this means $\mathcal{AR}_p(\hat{u}_\infty) \geq \gamma_2^+(\mathbf{C}_p\Sigma) - \sigma$. Therefore, by Lemma 6.2.1 we have,

$$\mathcal{AR}_p(\hat{u}_\infty) \geq \gamma_2^+(\mathbf{C}_p\Sigma).$$

Proof of Claim. Suppose otherwise, $c := \liminf_{j\to\infty} d_j/d'_j$ is either 0 or $+\infty$.

<u>*Case I.*</u> If $c = 0$, then using (6.7.13), (6.7.14), (6.7.15) and (6.7.16), we can apply the same argument as above to obtain, after passing to a subsequence of $j \to \infty$,

$$\frac{w_j}{d'_j} \to \tilde{w}_\infty \quad \text{in } C^2_{loc}(\Sigma), \quad \frac{t_j(f_j^1 - f_j^0)}{d'_j} \to 0 \quad \text{in } C^k(M)\,.$$

And $-L_{\Sigma,g}\tilde{w}_\infty = 0$, $\|\tilde{w}_\infty\|_{L^2(\Sigma\setminus B^g(\mathrm{Sing}(\Sigma)),r_0)} = 1$. But by (6.7.13), $\tilde{w}_\infty \equiv 0$ on $\Sigma \cap B^g(\mathrm{Sing}(\Sigma), 2r_0)$. This violates the unique continuation property for solutions to elliptic equations.

<u>*Case II.*</u> If $c = +\infty$, then again using (6.7.13), (6.7.14), (6.7.15) and (6.7.16), after passing to a subsequence of $j \to \infty$, we have

$$\frac{w_j}{d_j} \to \tilde{w}_\infty \quad \text{in } C^2_{loc}(\Sigma), \quad \frac{t_j(f_j^1 - f_j^0)}{d_j} \to \hat{f}_\infty\,.$$

And $-L_{\Sigma,g}\tilde{w}_\infty + \frac{n}{2}\nu_{\Sigma,g}(\hat{f}_\infty) = 0$. But $\tilde{w}_\infty \equiv 0$ on $\Sigma \setminus B^g(\mathrm{Sing}(\Sigma), r_0)$, and by the choice of r_0, $\mathrm{spt}(\hat{f}_\infty) \cap B^g(\mathrm{Sing}(\Sigma), 2r_0) = \emptyset$. Hence by unique continuation property, $\tilde{w}_\infty \equiv 0$, and then since $\nu_{\Sigma,g}(f_\infty)$ is not identically 0, we must have $\hat{f}_\infty \equiv 0$, in other words,

$$\|g_j^1 - g_j^0\|_{L^\infty}/d_j \to 0. \tag{6.7.17}$$

We now show that this contradicts to the definition of d_j.

Denote for simplicity

$$u_j^{(0,1)} := G^{\Sigma_j^1}_{\Sigma_j^0, g_j^0} \in L^\infty(\Sigma_j^0), \quad u_j^{(1,0)} := G^{\Sigma_j^0}_{\Sigma_j^1, g_j^1} \in L^\infty(\Sigma_j^1),$$
$$\tilde{d}_j := \|u_j^{(0,1)}\|_{L^2(\Sigma_j^0 \setminus B^{g_j^0}(\mathrm{Sing}(\Sigma_j^0)), r_0/K)} + \|u_j^{(1,0)}\|_{L^2(\Sigma_j^1 \setminus B^{g_j^1}(\mathrm{Sing}(\Sigma_j^1)), r_0/K)},$$

where $K = K(\sigma, \Lambda) > 2$ is given by Lemma C.0.6. First of all, by Lemma 6.7.8 and the fact that w_j/d_j subconverges to 0 in $C^2_{loc}(\Sigma)$, we know that $u_j^{(0,1)}/d_j$ and $u_j^{(1,0)}/d_j$ both subconverges to 0 in $C^2_{loc}(\Sigma)$. In particular,

$$\liminf_{j\to\infty} \tilde{d}_j/d_j = 0. \tag{6.7.18}$$

Also when $j >> 1$, by Corollary 6.7.6 (ii), on $\Sigma_j^0 \setminus B^{g_j^0}(\mathrm{Sing}(\Sigma_j^0), s_j)$, $u_j^{(0,1)}$ is C^2 and pointwisely,

$$|u_j^{(0,1)}| \leq C(\Sigma, g, \Lambda, \sigma, r_0)\tilde{d}_j \cdot \rho_{j,0}^{-n/2+1},$$

where $s_j \leq C(\Sigma, g, r_0)\tilde{d}_j^{\,2/(n-\sigma')}$, and $\rho_{j,0}$ denotes the distance function to $\mathrm{Sing}(\Sigma_j^0)$ with respect to g_j^0. Here we are using the fact that $\gamma_2^+ - \sigma \geq \gamma_1^+ \geq -n/2 + 1$. In particular,

$$\begin{aligned}
\|u_j^{(0,1)}\|_{L^2(\Sigma_j^0)} &\leq \tilde{d}_j + \|u_j^{(0,1)}\|_{L^2(A^{g_j^0}(\mathrm{Sing}(\Sigma_j^0)), s_j, r_0)} + \|u_j^{(0,1)}\|_{L^2(B^{g_j^0}(\mathrm{Sing}(\Sigma_j^0)), s_j)} \\
&\leq \tilde{d}_j + C(\Sigma, g, \Lambda, \sigma, r_0)\tilde{d}_j \cdot \|\rho_{j,0}^{-n/2+1}\|_{L^2(A^{g_j^0}(\mathrm{Sing}(\Sigma_j^0)), s_j, r_0)} \\
&\quad + C(g)\|\Sigma_j^0\|(B^{g_j^0}(\mathrm{Sing}(\Sigma_j^0)), s_j)^{1/2} \\
&\leq \tilde{d}_j\Big(1 + C(\Sigma, g, \Lambda, \sigma, r_0) + C(g, \Sigma, \Lambda)\tilde{d}_j^{\,\sigma'/(n-\sigma')}\Big) = O(\tilde{d}_j).
\end{aligned} \tag{6.7.19}$$

Similarly,

$$\|u_j^{(1,0)}\|_{L^2(\Sigma_j^1)} \leq O(\tilde{d}_j). \tag{6.7.20}$$

Together with (6.7.18) and (6.7.17), (6.7.19) and (6.7.20) contract the definition of d_j. This finishes the proof of Claim. ■

□

6.8 Generic Semi-nondegeneracy

The goal of this section is to prove Theorem 6.2.8. Throughout this section, let $\Sigma \subset (M, g)$ be a connected G-invariant two-sided LSMH, and $\delta_0 = \delta_0(g, \Sigma, \Lambda) > 0$ be as in Lemma 6.2.7.

Lemma 6.8.1 (Compactness of Jacobi fields of slower growth)**.** *Suppose that there exists a sequence* $\{(g_j, \Sigma_j)\}_{j=1}^{\infty} \subset \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_0)^G$ *satisfying*

(a) $(g_j, \Sigma_j) \to (g_\infty, \Sigma_\infty) \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_0)^G$ *in* $\mathcal{L}^{k,\alpha}$*;*

(b) There exists $0 \neq u_j \in (\mathrm{Ker}^+ L_{\Sigma_j, g_j})^G$ *a* G*-invariant Jacobi field of slower growth on* Σ_j *with* $\|u_j\|_{L^2(\Sigma_j)} = 1$.

Then u_j *subconverges to some* G*-invariant Jacobi field of slower growth* u_∞ *in* $C^2_{loc}(\Sigma_\infty)$*, with* $\|u_\infty\|_{L^2(\Sigma_\infty)} = 1$.

Proof. Fix $\sigma \in (0, \gamma_{\mathrm{gap},\Lambda}/2)$, where $\gamma_{\mathrm{gap},\Lambda}$ is defined in Lemma 5.2.1. Let $K = K(\sigma, \Lambda) > 2$ be given by Lemma C.0.6.

By standard elliptic estimates, for every $r > 0$, there exist an integer $j(r) > 0$ and a constant $C(\Sigma, g, r) > 0$ such that when $j \geq j(r)$, we have

$$\|u_j\|_{C^3(\Sigma_j \setminus B^g(\mathrm{Sing}(\Sigma), r))} \leq C(\Sigma, g, r)\,.$$

Hence u_j subconverges to some Jacobi field u_∞ in $C^2_{loc}(\Sigma)$. Since all u_j's are G-invariant, so is u_∞.

The major effort here is to show $\|u_\infty\|_{L^2(\|\Sigma\|_g)} = 1$ and that u_∞ is of slower growth.

First notice that for every $p \in \mathrm{Sing}(\Sigma)$, let $p_j \in \mathrm{Sing}(\Sigma_j)$ such that $p_j \to p$ as $j \to \infty$, then by the definition of $\mathcal{L}^{k,\alpha}$ convergence and the monotonicity formula of volume ratio, we have

$$\limsup_{r\to 0} \limsup_{j\to\infty} \theta_{|\Sigma_j|}(p_j, r) - \theta_{|\Sigma_j|}(p_j) = 0\,. \tag{6.8.1}$$

Hence by Lemma D.0.1, we can choose $r_0 = r_0(\Sigma_\infty, g_\infty) \in (0, \mathrm{injrad}(g, \Sigma))$ such that for $j >> 1$ and every $p_j \in \mathrm{Sing}(\Sigma_j)$, we can parametrize $\Sigma_j \cap \mathbb{B}_{r_0}$ by $\mathbf{C}_{p_j}\Sigma_j$ using

$$F_j(x) := x + \phi_j(x)\nu_j(x),$$

where ν_j be the unit normal field of $\mathbf{C}_{p_j}\Sigma_j$, i.e.

$$\Sigma_j \cap \mathbb{B}_{r_0} = \mathrm{graph}_{\mathbf{C}_{p_j}\Sigma_j}(\phi_j)\,.$$

Furthermore, $\phi_j \in C^2_{loc}(\mathbf{C}_{p_j}\Sigma_j \cap \mathbb{B}_{r_0})$ is the graphical function with estimates

$$\|\phi_j\|_{C^2_*, \mathbb{B}_s} \leq \Psi(s|g_\infty, \Sigma_\infty), \quad \forall s \in (0, r_0),\ \forall j \geq j_s = j(s, g_\infty, \Sigma_\infty)\,. \tag{6.8.2}$$

Here $\Psi(s|g_\infty, \Sigma_\infty)$ is some positive constant depending only on $s, g_\infty, \Sigma_\infty$, increasing in s and tending to 0 when $s \searrow 0$. We emphasis that the right hand side does not depend on j provided j is sufficiently large. In addition, for every function u_j on Σ_j, we let $\tilde{u}_j := u_j \circ F_j$.

By (6.8.2) and Remark 5.2.4, there exists $s_0 = s_0(g_\infty, \Sigma_\infty) \in (0, r_0)$ such that when $j \geq j_{s_0}$, Corollary 5.2.3 applies to $\tilde{u}_j$ on each $\mathbb{A}(\mathbf{0}, K^{-3}s, s) \cap \mathbf{C}_{p_j}\Sigma_j$ and we obtain

$$J^{\gamma_j}_{j,K}(\tilde{u}_j; K^{-2}s) - 2J^{\gamma_j}_{j,K}(\tilde{u}_j; K^{-1}s) + J^{\gamma_j}_{j,K}(\tilde{u}_j; s) > 0 \tag{6.8.3}$$

for every $s \in (0, s_0)$. Here $\gamma_j := \gamma_2^+(\mathbf{C}_{p_j}\Sigma_j) - \sigma$ and

$$J^{\gamma}_{j,K}(v;\tau)^2 := \int_{\mathbb{A}(\mathbf{0},K^{-1}\tau,\tau)} v(x)^2|x|^{-n-2\gamma}\, d\|\mathbf{C}_{p_j}\Sigma_j\|.$$

But since u_j is of slower growth at p_j, and $\gamma_j < \gamma_2^+(\mathbf{C}_{p_j}\Sigma_j) \leq \mathcal{AR}_{p_j}(u_j)$, we know that

$$\lim_{s\to 0} J^{\gamma_j}_{j,K}(\tilde{u}_j; s) = 0\,.$$

Thus for any $N \in \mathbb{N}^+$, there is some $N_0 > N$ and $s = s_0 \cdot K^{-N_0}$ such that

$$J^{\gamma_j}_{j,K}(\tilde{u}_j; K^{-1}s) < J^{\gamma_j}_{j,K}(\tilde{u}_j; s).$$

And then inductively, (6.8.3) implies,

$$J^{\gamma_j}_{j,K}(\tilde{u}_j; K^{-l}s_0) < J^{\gamma_j}_{j,K}(\tilde{u}_j; s_0), \quad \forall l \geq 1,\ \forall j \geq j_{s_0}.$$

By taking sum over l and change the parametrization, this implies (recall $\|u_j\|_{L^2(\|\Sigma_j\|_{g_j})} = 1$),

$$\|u_j\|_{L^2(B^{g_j}(p_j,\tau);\|\Sigma_j\|_{g_j})} \leq C(\Sigma, g, s_0)\tau^{n/2+\gamma_2^+(\mathbf{C}_{p_j}\Sigma_j)-\sigma}, \quad \forall \tau \in (0, s_0). \tag{6.8.4}$$

The key is that the coefficients on RHS does not depend on j and τ.

Claim. *If $p_j \to p \in \mathrm{Sing}(\Sigma)$, then up to a subsequence,*

$$\gamma_2^+(\mathbf{C}_{p_j}\Sigma_j) \to \gamma_2^+(\mathbf{C}_p\Sigma)\,.$$

Proof of Claim. It suffices to show that $\mathbf{C}_{p_j}\Sigma_j$ **F**-subconverges to $\mathbf{C}_p\Sigma$.

By the fact that Σ_j **F**-converges to Σ and (6.8.1), there exists a sequence $\{r_k\}$

with $r_k \to 0$ such that

$$\eta_{p_{j_k}, r_k}(\Sigma_{j_k}) \to \mathbf{C}_p\Sigma$$

in $\mathbb{B}_3$, and furthermore,

$$\lim_k \theta_{|\eta_{p_{j_k}, r_k}(\Sigma_{j_k})|}(p_{j_k}, 2) - \theta_{|\eta_{p_{j_k}, r_k}(\Sigma_{j_k})|}(p_{j_k}) = 0$$

Therefore, by Corollary D.0.3, we know that $\mathbf{C}_{p_j}\Sigma_j$ $\mathbf{F}$-converges to $\mathbf{C}_p\Sigma$. ■

Now, since

$$\gamma_2^+(\mathbf{C}_{p_j}\Sigma_j) - \sigma > \gamma_1^+(\mathbf{C}_{p_j}\Sigma_j) \geq -\frac{n-2}{2}\,,$$

(6.8.4) then guarantees that there is no concentration of L^2 norm for $\{u_j\}_{j\geq 1}$ near $\mathrm{Sing}(\Sigma)$, i.e.

$$\limsup_{\tau\to 0} \limsup_{j\to\infty} \|u_j\|_{L^2(B^g(\mathrm{Sing}(\Sigma_j)),\tau)} = 0\,,$$

and thus $\|u_\infty\|_{L^2} = 1$. It also implies that for every $p \in \mathrm{Sing}(\Sigma)$,

$$\mathcal{AR}_p(u_\infty) \geq \gamma_2^+(\mathbf{C}_p\Sigma) - \sigma\,.$$

Then by Lemma 6.2.1, u_∞ is of slower growth. □

Corollary 6.8.2. *There exists $\kappa_1(g, \Sigma, \Lambda, G) \in (0, \delta_0)$ such that for every $(g', \Sigma') \in \mathcal{L}^k(g, \Sigma; \Lambda, \kappa_1)^G$, we have* $\dim(\mathrm{Ker}^+ L_{\Sigma', g'})^G \leq \dim(\mathrm{Ker}^+ L_{\Sigma, g})^G$.

Proof. Suppose otherwise, there exist $(g_j, \Sigma_j) \to (g, \Sigma)$ in $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_0)^G$ with

$$\dim(\mathrm{Ker}^+ L_{\Sigma_j, g_j})^G > \dim(\mathrm{Ker}^+ L_{\Sigma, g})^G =: I.$$

For each $j \geq 1$, let $u_j^{(1)}, ..., u_j^{(I+1)}$ be an orthonormal class in $(\mathrm{Ker}^+ L_{\Sigma_j, g_j})^G$, i.e. $\langle u_j^{(s)}, u_j^{(t)} \rangle_{L^2(\|\Sigma_j\|_{g_j})} = \delta^{st}$, $1 \leq s, t \leq I+1$.

By Lemma 6.8.1, for each $1 \leq s \leq I+1$, $u_j^{(s)}$ subconverges to some slower growth

Jacobi field $u^{(s)}$ in $C^2_{loc}(\Sigma)$, and

$$\limsup_{r\to 0}\limsup_{j\to\infty}\|u_j^{(s)}\|_{L^2(B^g(\mathrm{Sing}(\Sigma),r);\|\Sigma_j\|_{g_j})}=0.$$

Hence $\{u^{(s)}\}_{1\le s\le I+1}\subset(\mathrm{Ker}^+ L_{\Sigma,g})^G$ are also orthonormal, contradicting to that $\dim(\mathrm{Ker}^+ L_{\Sigma,g})^G=I$. □

Let

$$\mathcal{L}^{k,\alpha}_{top}(g,\Sigma;\Lambda,\delta)^G:=\{(g',\Sigma')\in\mathcal{L}^{k,\alpha}(g,\Sigma;\Lambda,\delta)^G:\dim(\mathrm{Ker}^+ L_{\Sigma',g'})^G=\dim(\mathrm{Ker}^+ L_{\Sigma,g})^G\}.$$

Lemma 6.8.1 and Corollary 6.8.2 implies that $\mathcal{L}^{k,\alpha}_{top}(g,\Sigma;\Lambda,\delta)^G$ is closed in $\mathcal{L}^{k,\alpha}(g,\Sigma;\Lambda,\delta)^G$ for $\delta\le\kappa_1$.

Let $I:=\dim(\mathrm{Ker}^+ L_{\Sigma,g})^G<\infty$. For $(\bar g,\bar\Sigma)\in\mathcal{L}^{k,\alpha}_{top}(g,\Sigma;\Lambda,\kappa_1)^G$, let $\pi^{L^2,G}_{\bar\Sigma,\bar g}:L^2(\bar\Sigma)\to(\mathrm{Ker}^+ L_{\bar\Sigma,\bar g})^G$ be the L^2-orthogonal projection to the finite dimensional subspace. The following Lemma guarantees that we can parametrize slices of $\mathcal{L}^{k,\alpha}_{top}$ by compact subset of $(\mathrm{Ker}^+ L)^G$.

Lemma 6.8.3. *Let κ_1 be as in Corollary 6.8.2. Then there exist constants $\kappa_2(g,\Sigma,\Lambda,G)\in(0,\kappa_1)$, $r_0(g,\Sigma,\Lambda,G)>0$ and an I-dimensional linear subspace $\mathcal{F}\subset C^{k,\alpha}_c(M\setminus B^g(\mathrm{Sing}(\Sigma),10r_0))^G$ also depending only on Σ,g,Λ,G with the following property.*

(i) For every $(\bar g,\bar\Sigma)\in\mathcal{L}^{k,\alpha}_{top}(g,\Sigma;\Lambda,\kappa_2)^G$, the map

$$\mathcal{F}\to(\mathrm{Ker}^+ L_{\bar\Sigma,\bar g})^G,\quad f\mapsto\pi^{L^2,G}_{\bar\Sigma,\bar g}(\nu_{\bar\Sigma,\bar g}(f)),\tag{6.8.5}$$

is a linear isomorphism with biLipschitz constant $\le C(g,\Sigma,\Lambda,k,\alpha,G)$.

In particular, for every nonzero $f\in\mathcal{F}$, there is no G-invariant solution of slower growth v to $L_{\bar\Sigma,\bar g}v=\nu_{\bar\Sigma,\bar g}(f)$.

(ii) Denote for simplicity $\mathcal{F}\cdot\bar{g} := \{(1+f)\bar{g} : f \in \mathcal{F}\}$. *For every* $(\bar{g},\bar{\Sigma}) \in \mathcal{L}^{k,\alpha}_{top}(g,\Sigma;\Lambda,\kappa_2)^G$, *the map*

$$P_{\bar{g},\bar{\Sigma}} : \mathcal{L}^{k,\alpha}_{top}(g,\Sigma;\Lambda,\kappa_2)^G \cap \Pi^{-1}(\mathcal{F}\cdot\bar{g}) \to (\mathrm{Ker}^+ L_{\bar{\Sigma},\bar{g}})^G,$$
$$(g',\Sigma') \mapsto \pi^{L^2,G}_{\bar{\Sigma},\bar{g}}\big(G^{\Sigma'}_{\bar{\Sigma},\bar{g}}\cdot\zeta_{\bar{\Sigma},\bar{g},r_0}\big),$$

is bi-Lipschitz onto its image with bi-Lipschitz constant $\leq C(g,\Sigma,\Lambda,G)$ *and thus, injective, where*

$$\zeta_{\bar{\Sigma},\bar{g},r_0}(x) := \zeta(dist_{\bar{g}}(x,\mathrm{Sing}(\bar{\Sigma}))/r_0) \tag{6.8.6}$$

and $\zeta \in C^\infty(\mathbb{R};[0,1])$ *is a cut-off function such that* $\zeta \equiv 0$ *on* $(-\infty,1]$ *and* $\zeta \equiv 1$ *on* $[2,+\infty)$.

Remark 6.8.4. In this lemma, we choose an $L^2(M)$-metric on $\mathcal{F}$. Since $\mathcal{F}$ is of finite dimensional, this metric is equivalent to the $C^{k,\alpha}(M)$ metric. The semi-metric we use on $\mathcal{L}^{k,\alpha}_{top}$ is $\mathbf{D}^{\mathcal{L}}$.

Proof. Fix a cut-off function $\zeta \in C^\infty(\mathbb{R};[0,1])$ such that $\zeta \equiv 0$ on $(-\infty,1]$ and $\zeta \equiv 1$ on $[2,+\infty)$.

To prove (i), we start with specifying the choice of r_0 and $\mathcal{F}$. Take $r_0(g,\Sigma,\zeta) \in (0,\mathrm{injrad}(g,\Sigma)/40)$ small enough such that

$$(\mathrm{Ker}^+ L_{\Sigma,g})^G \to (\mathrm{Ker}^+ L_{\Sigma,g})^G, \quad v \mapsto \pi^{L^2,G}_{\Sigma,g}(\zeta_{\Sigma,g,r_0}\cdot v)$$

is an isomorphism, where ζ_{Σ,g,r_0} is defined in (6.8.6). Take $u_1,...,u_I$ be an L^2-orthonormal basis of $(\mathrm{Ker}^+ L_{\Sigma,g})^G$ and choose $f_i \in C^{k,\alpha}_c(M\setminus\mathrm{Sing}(\Sigma))^G$ such that $\nu_{\Sigma,g}(f_i) = \zeta_{\Sigma,g,r_0}\cdot u_i$ along Σ. We let

$$\mathcal{F} := \mathrm{span}_{\mathbb{R}}\langle f_i : 1\leq i\leq I\rangle\,.$$

To prove (6.8.5) is an isomorphism, since $\dim(\mathrm{Ker}^+ L_{\bar\Sigma,\bar g})^G = \dim \mathcal{F}$, it suffices to verify that for small δ, for every $f \in \mathcal{F}$ and every $(\bar\Sigma, \bar g) \in \mathcal{L}^{k,\alpha}_{top}(g, \Sigma; \Lambda, \delta)^G$,

$$\|\pi^{L^2,G}_{\bar\Sigma,\bar g}(\nu_{\bar\Sigma,\bar g}(f))\|_{L^2(\bar\Sigma)} \geq (L+1)^{-1}\|f\|_{L^2(M)},$$

where L is the Lipschitz constant of the inverse of $\pi^{L^2,G}_{\Sigma,g} \circ \nu_{\Sigma,g}\big|_{\mathcal{F}}$ on $(\mathrm{Ker}^+ L_{\Sigma,g})^G$.

Suppose otherwise, there exists $(g_j, \Sigma_j) \to (g, \Sigma)$ in $\mathcal{L}^{k,\alpha}_{top}(g, \Sigma; \Lambda, \kappa_1)^G$ and $L^2(M)$-unit $f_j \in \mathcal{F}$ such that $v_j := \pi^{L^2,G}_{\Sigma_j,g_j}(\nu_{\Sigma_j,g_j}(f_j))$ has $L^2(\Sigma_j)$-norm $< (L+1)^{-1}$. Suppose WLOG $f_j \to f_\infty \in \mathcal{F}$ in $C^{k,\alpha}(M)$ and by Lemma 6.8.1, v_j subconverges to $v_\infty \in (\mathrm{Ker}^+ L_{\Sigma,g})^G$ in $C^2_{loc}(\Sigma)$, where by Fatou's Lemma $\|v_\infty\|_{L^2(\Sigma)} \leq (L+1)^{-1}$. Also by Lemma 6.8.1, orthonormal bases of $(\mathrm{Ker}^+ L_{\Sigma_j,g_j})^G$ subconverge to an orthonormal basis of $(\mathrm{Ker}^+ L_{\Sigma,g})^G$. Thus

$$\pi^{L^2,G}_{\Sigma,g}(\nu_{\Sigma,g}(f_\infty)) = \lim_{j\to\infty} v_j = v_\infty,$$

which contradicts the choice of L.

To prove the second part, it suffices to show that *if a slower growth function v solves $L_{\bar\Sigma,\bar g} v = \nu_{\bar\Sigma,\bar g}(f)$ for some $f \in \mathcal{F}$, then $f \equiv 0$.* Multiply by elements of $(\mathrm{Ker}^+ L_{\bar\Sigma,\bar g})^G$ on both sides of the equation and integrate over $\bar\Sigma$. By Lemma 6.2.1 (iii), we can use integration by parts to derive

$$\nu_{\bar\Sigma,\bar g}(f) \perp (\mathrm{Ker}^+ L_{\bar\Sigma,\bar g})^G, \text{ i.e. } \pi^{L^2,G}_{\bar\Sigma,\bar g}(\nu_{\bar\Sigma,\bar g}(f)) = 0\,,$$

which implies $f \equiv 0$ by the isomorphism (6.8.5).

To prove (ii), we also argue by contradiction. Suppose $P_{\bar g_j, \bar\Sigma_j}$ is not uniformly bi-Lipschitz for $(\bar g_j, \bar\Sigma_j) \in \mathcal{L}^{k,\alpha}_{top}(g, \Sigma; \Lambda, \kappa_1)^G$ with $(\bar g_j, \bar\Sigma_j) \to (g, \Sigma)$ in $\mathcal{L}^{k,\alpha}$. Then there exists a sequence of pairs $(g^0_j, \Sigma^0_j), (g^1_j, \Sigma^1_j) \to (g, \Sigma)$ in $\mathcal{L}^{k,\alpha}_{top}(g, \Sigma; \Lambda, \delta)^G$ such that

(a) $g^i_j = (1 + f^i_j)\bar g_j$, $i = 0, 1$, $f^i_j \in \mathcal{F}$;

(b) One of the following holds,

$$\text{either} \quad \|\pi^{L^2,G}_{\bar{\Sigma}_j,\bar{g}_j}((u_j^{(1)}-u_j^{(0)})\zeta_j)\|_{L^2(\bar{\Sigma}_j)} \leq \frac{d_j}{j}, \tag{6.8.7}$$

$$\text{or} \quad \|\pi^{L^2,G}_{\bar{\Sigma}_j,\bar{g}_j}((u_j^{(1)}-u_j^{(0)})\zeta_j)\|_{L^2(\bar{\Sigma}_j)} \geq j\cdot d_j; \tag{6.8.8}$$

where $u_j^{(i)} := G^{\Sigma_j^i}_{\bar{\Sigma}_j,\bar{g}_j}$ is G-invariant; $\zeta_j := \zeta_{\bar{\Sigma}_j,\bar{g}_j,r_0}$, $d_j := \mathbf{D}^{\mathcal{L}}((g_j^0,\Sigma_j^0),(g_j^1,\Sigma_j^1))$.

Since $\mathcal{F}$ is of finite dimension and (6.8.5) is an isomorphism, Lemma 6.7.12 applies to conclude that when $j\to\infty$, $(f_j^1-f_j^0)/d_j$ subconverges in $C^{k,\alpha}(M)$ to some $\hat{f}_\infty\in\mathcal{F}$; $(u_j^{(1)}-u_j^{(0)})/d_j$ subconverges to some non-zero G-invariant slower growth $\hat{u}_\infty$ in C^2_{loc}, solving

$$L_{\Sigma,g}\hat{u}_\infty = \frac{n}{2}\nu_{\Sigma,g}(\hat{f}_\infty). \tag{6.8.9}$$

And since ζ_j is supported away from $\mathrm{Sing}(\Sigma)$, by the convergence above and Lemma 6.8.1, we know that (6.8.8) can not hold for $j >> 1$.

Also, by (6.8.9) and (i), we derive $\hat{f}_\infty = 0$, in other words, $\mathbf{0}\neq\hat{u}_\infty\in(\mathrm{Ker}^+ L_{\Sigma,g})^G$. Then since $\zeta_j(u_j^{(1)}-u_j^{(0)})/d_j$ subconverges in $L^2(\Sigma)$ to $\hat{u}_\infty\cdot\zeta_{\Sigma,g,r_0}$, by (6.8.7), $\pi^{L^2,G}_{\Sigma,g}(\hat{u}_\infty\cdot\zeta_{\Sigma,g,r_0})=0$. This violates the choice of r_0 in (i). □

Having established this, we can prove Theorem 6.2.8 by unwrapping the proof of Sard Theorem. Let us start with the following abstract finite dimensional version of the theorem.

Let $Z\subset\mathbb{R}^n$ be a compact subset; $f: Z\to\mathbb{R}^n$ be a Lipschitz map. we call $x\in Z$ a **critical point** of f if there exists some proper linear subspace $T_x\subset\mathbb{R}^n$ such that for every sequence of distinct points $Z\ni x_j\to x$, if $(f(x_j)-f(x))/|x_j-x|\to v$ in $\mathbb{R}^n$, then $v\in T_x$. Let Crit_f be the set of critical points of f on Z.

Lemma 6.8.5 (Sard-Smale-type Theorem for Lipschitz Maps)**.** *Let $Z\subset\mathbb{R}^n$ be a*

compact subset and $f : Z \to \mathbb{R}^n$ be a Lipschitz map. Then we have

$$\mathscr{H}^n(f(\mathrm{Crit}_f)) = 0.$$

Proof. Fix some $\epsilon > 0$. For every $x \in \mathrm{Crit}_f$, let $\mathcal{B}_x$ be the collection of balls B in $\mathbb{R}^n$ centered at x with radius ≤ 1 and such that $\mathscr{H}^n(f(5B \cap Z)) \leq \epsilon \mathscr{H}^n(B)$, where $5B$ is the ball with same center and five times radius of B. The definition of Crit_f guarantees that $\mathcal{B}_x$ contains balls with arbitrarily small radius.

By the Vitali covering lemma, there exists a disjoint family of balls $\{B_j\}_{j\geq 1} \subset \bigcup_{x\in \mathrm{Crit}_f} \mathcal{B}_x$ such that $\mathrm{Crit}_f \subset \bigcup_{j\geq 1} 5B_j$. Thus,

$$\mathscr{H}^n(f(\mathrm{Crit}_f)) \leq \sum_{j\geq 1} \mathscr{H}^n(f(5B_j \cap Z)) \leq \epsilon \sum_{j\geq 1} \mathscr{H}^n(B_j) \leq \epsilon \mathscr{H}^n(\mathbb{B}_1(Z)).$$

Take $\epsilon \to 0$, we derive $\mathscr{H}^n(f(\mathrm{Crit}_f)) = 0$. □

Theorem 6.8.6. *Let $G < Diff(M)$ be a fixed finite subgroup, $\Sigma \subset (M, g)$ be a connected G-invariant two-sided LSMH, $\Lambda > 0$, and $\kappa_2 = \kappa_2(g, \Sigma, \Lambda, G) > 0$ be given by Lemma 6.8.3. Then for every $\delta \in (0, \kappa_2)$,*

$$\begin{aligned} \mathcal{G}(g, \Sigma; \Lambda, \delta)^G := \big\{ g' \in \mathcal{G}^{k,\alpha}(M)^G : & \text{ either } g' \notin \Pi(\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)^G); \\ & \text{or } \forall (g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)^G, \\ & \text{we have } (\mathrm{Ker}^+ L_{\Sigma', g'})^G = \{0\}. \big\} \end{aligned} \tag{6.8.10}$$

is both open and dense in $\mathcal{G}^{k,\alpha}(M)^G := \{g \in \mathcal{G}^{k,\alpha}(M) : \gamma^* g = g, \forall \gamma \in G\}$.

Proof. We shall prove it inductively on $I := \dim(\mathrm{Ker}^+ L_{\Sigma,g})^G \geq 0$. Note that by Corollary 6.8.2, when $I = 0$, we always have $\mathcal{G}(g, \Sigma; \Lambda, \delta)^G = \mathcal{G}^{k,\alpha}(M)^G$, which is definitely open and dense. Now consider $I \geq 1$.

Openness. If $g_j \notin \mathcal{G}(g, \Sigma; \Lambda, \delta)^G$ for $1 \leq j < +\infty$ and $g_j \to g_\infty$ in $\mathcal{G}^{k,\alpha}(M)^G$, then

by definition, there exists $(g_j, \Sigma_j) \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)^G$ with a non-trivial G-invariant Jacobi field of slower growth $u_j \in (\mathrm{Ker}^+ L_{\Sigma_j, g_j})^G$. By the compactness Lemma 6.2.7, up to a subsequence, $(g_j, \Sigma_j) \to (g_\infty, \Sigma_\infty)$ in $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)^G$, and by Lemma 6.8.1, after renormalization, u_j subconverges to some non-trivial G-invariant slower growth Jacobi field $u_\infty \in (\mathrm{Ker}^+ L_{\Sigma_\infty, g_\infty})^G$, which implies $g_\infty \notin \mathcal{G}(g, \Sigma; \Lambda, \delta)^G$. Therefore, $\mathcal{G}(g, \Sigma; \Lambda, \delta)^G$ is an open subset.

Denseness. By induction, we can assume that the denseness has been established for every pair (g, Σ) with $\dim(\mathrm{Ker}^+ L_{\Sigma, g})^G \leq I - 1$. The goal now is to prove it for $I(\geq 1)$.

Claim 1. For each $\bar{g} \in \mathcal{G}^{k,\alpha}(M)^G \setminus \mathcal{G}(g, \Sigma; \Lambda, \delta)^G$, there exists a sequence of metrics $g_j \in \mathcal{F} \cdot \bar{g} \setminus \Pi(\mathcal{L}^{k,\alpha}_{top}(g, \Sigma; \Lambda, \delta)^G)$ such that $g_j \to \bar{g}$ in $C^{k,\alpha}(M)$ when $j \to \infty$.

We first finish the proof of denseness assuming this claim. It suffices to show that each g_j in the claim can be approximated by metrics in $\mathcal{G}(g, \Sigma; \Lambda, \delta)^G$.

Let $j \geq 1$ be fixed. Suppose WLOG that $g_j \notin \mathcal{G}(g, \Sigma; \Lambda, \delta)^G$. Hence $\Pi^{-1}(g_j) \cap \mathcal{L}^k(g, \Sigma; \Lambda, \delta)^G \neq \emptyset$. By definition of $(\mathcal{L}^{k,\alpha}_{top})^G$ and Corollary 6.8.2 we have, for every $(g_j, \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)^G$, $\dim(\mathrm{Ker}^+ L_{\Sigma', g_j})^G \leq I - 1$. Hence by the induction assumption, there exists $\delta_{g_j, \Sigma'} = \delta(g_j, \Sigma', \Lambda) > 0$ such that $\mathcal{G}(g_j, \Sigma'; \Lambda, \delta_{g_j, \Sigma'})^G$ is open and dense in $\mathcal{G}^{k,\alpha}(M)^G$. Since $\mathcal{L}^{k,\alpha}(g_j, \Sigma'; \Lambda, \delta_{g_j, \Sigma'})^G$ contains an open neighborhood of (g_j, Σ') in $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)^G$, and by Lemma 6.2.7, $\Pi^{-1}(g_j) \cap \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)^G$ is compact, we thus know that there exists a finite sequence of pairs $(g_j, \Sigma^{(1)})$, $(g_j, \Sigma^{(2)})$, $\ldots$, $(g_j, \Sigma^{(q)}) \in \Pi^{-1}(g_j) \cap \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)^G$, $1 \leq q < +\infty$, such that

$$\bigcup_{l=1}^{q} \mathcal{L}^{k,\alpha}(g_j, \Sigma^{(l)}; \Lambda, \delta_{g_j, \Sigma^{(l)}})^G \supset \Pi^{-1}(g_j) \cap \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)^G. \tag{6.8.11}$$

Since $\mathcal{G}_j := \bigcap_{l=1}^{q} \mathcal{G}(g_j, \Sigma^{(l)}; \Lambda, \delta_{g_j, \Sigma^{(l)}})^G$ is still open and dense in $\mathcal{G}^(M)^G$, there exists a sequence $\{g'_m\}_{m \geq 1} \subset \mathcal{G}_j$ such that $g'_m \to g_j$ in $C^{k,\alpha}$ as $m \to \infty$.

We now verify that $g'_m \in \mathcal{G}(g,\Sigma;\Lambda,\delta)^G$ for $m >> 1$. Suppose otherwise, for infinitely many m, there exists $(g'_m,\Sigma'_m) \in \mathcal{L}^{k,\alpha}(g,\Sigma;\Lambda,\delta)^G$ with $(\mathrm{Ker}^+ L_{\Sigma'_m,g'_m})^G \neq \{0\}$. By Lemma 6.2.7, up to a subsequence of $m \to \infty$, $(g'_m,\Sigma'_m) \to (g_j,\Sigma')$ in $\mathcal{L}^{k,\alpha}(g,\Sigma;\Lambda,\delta)^G$. By (6.8.11), WLOG $(g_j,\Sigma') \in \mathcal{L}^{k,\alpha}(g_j,\Sigma^{(1)};\Lambda,\delta_{g_j,\Sigma^{(1)}})^G$, and so does (g'_m,Σ'_m) for $m >> 1$. But since $g'_m \in \mathcal{G}_j \subset \mathcal{G}(g_j,\Sigma^{(1)};\Lambda,\delta_{g_j,\Sigma^{(1)}})^G$, by its definition (6.8.10), we must have $(\mathrm{Ker}^+ L_{\Sigma'_m,g'_m})^G = \{0\}$ for those m, which contradicts the choice of (g'_m,Σ'_m). This concludes the proof of denseness.

Proof of Claim 1. Suppose WLOG that $(\bar{g},\bar{\Sigma}) \in \mathcal{L}^{k,\alpha}_{top}(g,\Sigma;\Lambda,\delta)^G$, otherwise just taking $g_j \equiv \bar{g}$ proves the Claim.

Recall by Lemma 6.8.3, $\mathcal{L}^k_{top}(g,\Sigma;\Lambda,\delta)^G \cap \Pi^{-1}(\mathcal{F}\cdot\bar{g})$ is bi-Lipschitzly embedded as a compact subset of finite dimensional Euclidean space $(\mathrm{Ker}^+ L_{\bar{\Sigma},\bar{g}})^G$. We denote for simplicity $\mathcal{Z} := P_{\bar{g},\bar{\Sigma}}(\mathcal{L}^k_{top}(g,\Sigma;\Lambda,\delta)^G \cap \Pi^{-1}(\mathcal{F}\cdot\bar{g}))$. Consider the projection map,

$$\tilde{\Pi} : \mathcal{Z} \cong \mathcal{L}^k_{top}(g,\Sigma;\Lambda,\delta)^G \cap \Pi^{-1}(\mathcal{F}\cdot\bar{g}) \overset{\Pi}{\to} \mathcal{F}\cdot\bar{g} \cong \mathcal{F}.$$

More precisely, $\tilde{\Pi}$ is a Lipschitz map between (compact subset of) vector spaces of same dimension, given by $\tilde{\Pi}(P_{\bar{g},\bar{\Sigma}}((1+f)\bar{g},\Sigma')) = f$.

We shall show that $\mathcal{Z}$ consists of critical points in the following Claim 2. Then by Lemma 6.8.5, we see that $\Pi(\mathcal{L}^{k,\alpha}_{top}(g,\Sigma;\Lambda,\delta)^G)$ is a compact subset of I-dimensional measure 0 in $\mathcal{F}\cdot\bar{g} \cong \mathcal{F}$. Therefore, we can find $g_j \in \mathcal{F}\cdot\bar{g} \setminus \Pi(\mathcal{L}^{k,\alpha}_{top}(g,\Sigma;\Lambda,\delta)^G)$ to approximate $\bar{g}$, and thus, finish the proof of Claim 1. ■

Claim 2. $\mathcal{Z} \subset \mathrm{Crit}_{\tilde{\Pi}}$.

Proof of Claim 2. By the definition of Crit_f and biLipschitz property of $P_{\bar{g},\bar{\Sigma}}$, it suffices to show that *whenever* $((1+f_j)\bar{g},\Sigma'_j) \to ((1+f)\bar{g},\bar{\Sigma}')$ *in* $\mathcal{L}^{k,\alpha}_{top}(g,\Sigma;\Lambda,\delta)^G$, *then we always have when* $j \to \infty$,

$$(f_j - f)/d_j \to 0\,, \tag{6.8.12}$$

where with $d_j := \mathbf{D}^{\mathcal{L}}(((1+f_j)\bar{g}, \Sigma'_j), ((1+f)\bar{g}, \bar{\Sigma}')) > 0$.

To see this, recall by Lemma 6.7.12, after passing to a subsequence,

$$\begin{cases} u_j/d_j \to \hat{u}_\infty & \text{in } C^2_{loc}(\bar{\Sigma}'); \\ (f_j - f)/((1+f)d_j) \to \hat{f}_\infty & \text{in } C^{k,\alpha}(M). \end{cases}$$

where $u_j := G^{\Sigma'_j}_{\bar{\Sigma}',(1+f)\bar{g}}$ is the graphical function of Σ'_j over $\bar{\Sigma}'$, $\hat{u}_\infty$ is of slower growth, and $\hat{f}_\infty$ solves the equation

$$L_{\bar{\Sigma}',(1+f)\bar{g}}\hat{u}_\infty = \frac{n}{2}\nu_{\bar{\Sigma}',(1+f)\bar{g}}(\hat{f}_\infty).$$

But by Lemma 6.8.3 (i), this equation has no slower growth solution unless $\hat{f}_\infty \equiv 0$, which proves (6.8.12). ■

□

Proof of Theorem 6.2.8. The case where Σ is two-sided follows immediately from Theorem 6.8.6 with $G = \{id_M\}$.

When Σ is one-sided, by Remark 6.2.12 (ii), we can consider the two-sided double cover $\hat{\Sigma} \subset (\hat{M}, \hat{g})$, a $\mathbb{Z}_2$-invariant LSMH. Recall that by definition, one-sided $(g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ is semi-nondegenerate if and only if its double cover $\hat{\Sigma}'$ is $\mathbb{Z}_2$-semi-nondegenerate. Hence Theorem 6.8.6 also implies Theorem 6.2.8 with $G = \mathbb{Z}_2$. □

Appendix A

Minimal Surface Equation for Graph

Let (M^{n+1}, g) be a Riemannian manifold without boundary (not necessarily complete) and $\Sigma \subset M$ be an embedded two-sided C^3 hypersurface (not necessarily proper) with a unit normal field ν. Recall that for every $x \in \Sigma$, we have introduced the **regularity scale** $r_{\mathcal{S}}(x) := r_{\mathcal{S}}(x; M, g, \Sigma)$ of Σ at x in Definition 2.1.1. With aid of this, we defined the C^k_*-norms of functions over Σ, which is invariant under rescalings.

For $f \in C^k(M)$ ($0 \leq k \leq 3$ integer), $\beta \in \mathrm{Sym}(T^*M \otimes T^*M)$, and for every $x \in \Sigma$, we also define

$$[f]_{x,g,C^k_*} := \sum_{j=0}^{k} r_{\mathcal{S}}(x)^j \sup_{B^g_{r_{\mathcal{S}}(x)}} |\nabla^j_g f|; \qquad [\beta]_{x,g,C^k_*} := \sum_{j=0}^{k} r_{\mathcal{S}}(x)^j \sup_{B^g_{r_{\mathcal{S}}(x)}} |\nabla^j_g \beta|$$

This is also a norm invariant under rescaling, more precisely, for every $\lambda > 0$, we have

$$[f]_{x,\lambda^2 g,C^k_*} = [f]_{x,g,C^k_*}; \qquad [\lambda^2 \beta]_{x,\lambda^2 g,C^k_*} = [\beta]_{x,g,C^k_*}\,.$$

We shall omit the subscript g if there's no ambiguity.

Throughout this section, suppose $(M, g, \Sigma) \neq (\mathbb{R}^{n+1}, g_{\mathrm{euc}}, \mathbb{R}^n)$, and thus $r_{\mathcal{S}} < +\infty$

on Σ.

Theorem A.0.1. *There exists $\delta = \delta(n) \in (0,1)$ and $C = C(n) >> 1$ with the following properties.*

(i) If $u \in C^1(\Sigma)$ with $\|u\|_{C^1_*} \leq \delta$, then

$$\Phi^u : \Sigma \to M, \quad x \mapsto \exp^g_x(u(x) \cdot \nu(x)),$$

is a C^1 embedding;

(ii) There exists a C^1 area density function $F^f = F^f(x, z, \xi)$, where $x \in \Sigma$, $z \in \mathbb{R}$ with $r_{\mathcal{S}}^{-1}|z| < 1$, $\xi \in T^*_x\Sigma$ with $|\xi|_g < 1$, with pointwise estimate

$$|F^f(x, z, \xi) - 1| \leq C(n)(r_{\mathcal{S}}(x)^{-1}|z| + |\xi| + [f]_{x,C^2_*});$$

And such that for every $u \in C^2(\Sigma)$ with $\|u\|_{C^2_*} \leq \delta$, every $f \in C^2(M)$ with $[f]_{x,C^2_*} \leq \delta$, $\forall x \in \Sigma$, and every $\varphi \in C^0_c(M \setminus \mathrm{Sing}(\Sigma))$, we have

$$\int_M \varphi(x)\ d\|\operatorname{graph}_\Sigma(u)\|_{(1+f)g}(x) = \int_M \varphi \circ \Phi^u(x) \cdot F^f(x, u(x), du(x))\ d\|\Sigma\|_g.$$

(iii) Let F^f be in (ii); let $\mathscr{M}^f : C^2_*(\Sigma) \to C^0_{loc}(\Sigma)$ be the minimal surface operator ($\mathscr{M}^f$ is only defined in the δ-neighborhood of $\mathbf{0}$), in other words, for every $\varphi \in C^1_c(\Sigma)$,

$$\int_\Sigma \mathscr{M}^f(u) \cdot \varphi\ d\|\Sigma\|_g := \frac{d}{dt}\Big|_{t=0} \int_\Sigma F^f\ (x, u + t\varphi, d(u + t\varphi))\ \ d\|\Sigma\|_g.$$

Then for every pair $f^\pm \in C^2(M)$ with $[f^\pm]_{x,C^2_*} \leq \delta$ along $x \in \Sigma$ and every pair

$\|u^{\pm}\|_{C^2_*} \leq \delta$, *we have*

$$\begin{aligned}\mathscr{M}^{f^+}(u^+) - \mathscr{M}^{f^-}(u^-) = &- L_{\Sigma,g}(u^+ - u^-) + \frac{n}{2}\nu(f^+ - f^-) \\ &+ div_{\Sigma,g}(\mathcal{E}_1) + r_{\mathcal{S}}^{-1}\mathcal{E}_2,\end{aligned}$$

where $\mathcal{E}_1, \mathcal{E}_2$ *are functions on* Σ *satisfying the pointwise bound along* $x \in \Sigma$,

$$\begin{aligned}|\mathcal{E}_1(x)| + |\mathcal{E}_2(x)| \leq C(n) &\left(\sum_{i=\pm}[f^i]_{x,C^2_*} + r_{\mathcal{S}}(x)^{-1}|u^i|(x) + |du^i(x)|\right) \\ &\cdot \left([f^+ - f^-]_{x,C^2_*} + r_{\mathcal{S}}(x)^{-1}|u^+ - u^-|(x) + |d(u^+ - u^-)|(x)\right).\end{aligned}$$

(iv) Let $u \in C^2(\Sigma)$ *such that* $\|u\|_{C^2_*} \leq \delta$, $\tilde{g}$ *be a metric on* M *such that* $[\tilde{g} - g]_{x,C^3_*} \leq \delta$ *for every* $x \in \Sigma$. *Denote for simplicity* $\Sigma_u = \mathrm{graph}_{\Sigma,g}(u)$. *Then we have pointwise estimate,*

$$\left||A_{\Sigma_u,\tilde{g}}|^2_{\tilde{g}} \circ \Phi^u - |A_\Sigma|^2_g\right| \leq C(n)\left([\tilde{g} - g]_{x,C^3_*} + \sum_{j=0}^{2} r_{\mathcal{S}}^{j-1}|\nabla^j_{\Sigma,g}u|\right) \cdot r_{\mathcal{S}}^{-2}\,;$$

And for every $\psi \in C^2_{loc}(\Sigma_u)$, *we have*

$$(\Delta_{\Sigma_u}\psi) \circ \Phi^u - \Delta_\Sigma(\psi \circ \Phi^u) = \mathrm{div}_\Sigma(\vec{B}_0) + r_{\mathcal{S}}^{-1} \cdot B_1\,,$$

with pointwise estimate on error terms,

$$\begin{aligned}|\vec{B}_0| + |B_1| &\leq C(n)\left([\tilde{g} - g]_{x,C^3_*} + \sum_{j=0}^{2} r_{\mathcal{S}}^{j-1}|\nabla^j_{\Sigma,g}u|\right) \cdot |d(\psi \circ \Phi^u)|_g\,; \\ |div_\Sigma \vec{B}_0| &\leq C(n)\left([\tilde{g} - g]_{x,C^3_*} + \sum_{j=0}^{2} r_{\mathcal{S}}^{j-1}|\nabla^j_{\Sigma,g}u|\right) \cdot \left(|\nabla^2_\Sigma(\psi \circ \Phi^u)|_g + r_{\mathcal{S}}^{-1}|d(\psi \circ \Phi^u)|_g\right).\end{aligned}$$

Proof. First one can easily check that the Theorem is formulated invariant under rescalings. Hence (i) follows immediately by renormalizing the metric with $r_{\mathcal{S}}^{-2}$ near

each point. To prove (ii)-(iv), by taking the Fermi coordinates, it suffices to work under the following further assumptions.

(a) $M \supset M_0 := \mathbb{B}_1^n \times (-1,1)$ with global coordinates $\{x^1, x^2, \dots, x^n, x^{n+1} = z\}$ and metric $g|_{M_0} = g^z + dz^2$, where g^z is a family of metric on $\mathbb{B}_1^n$ parametrized by $z \in (0,1)$ satisfying that,

- $\|g - g_{\text{euc}}\|_{C^3,M} < 1/10$;
- $g^0_{ij} = \delta_{ij}$, $\nabla^{g^0}_{\partial_i}\partial_j = 0$ both at $\mathbf{0} \in \mathbb{B}_1^n$, $\forall 1 \le i,j \le n$.

(b) $\Sigma_0 := \Sigma \cap M_0 = \mathbb{B}_1^n \times \{0\} \subset M_0$, and thus $\Phi^u(x) = (x, u(x))$ on Σ_0;

(c) $1 \le r_{\mathcal{S}} \le C_n$ on Σ_0;

(d) When estimating geometric quantities of $\text{graph}_\Sigma(u)$, we only need to work at $x = \mathbf{0}$ and assuming $\|u\|_{C^2,\mathbb{B}_1} + \|f\|_{C^2,M_0} \le \delta_n$.

First note that, the smooth family of metric $\{g^z\}_{z\in(0,1)}$ satisfies the ODE

$$\begin{cases} \partial_z g^z_{ij} = -2A^z_{ij}; \\ \partial_z A^z_{ij} = -A^z_{ik}A^z_{jl}g_z^{kl} + R^z_{ij}, \end{cases} \tag{A.0.1}$$

where $[g_z^{ij}]$ denotes the inverse matrix of $[g^z_{ij}]$; $R^z_{ij} := R_g(\partial_i, \partial_z, \partial_z, \partial_j)|_{(x,z)}$; $A^0_{ij} = A_\Sigma(\partial_i, \partial_j)$.

To prove (ii), note that

$$\begin{aligned} \partial_i \Phi^u(x) &= (\partial_i + u_i(x)\partial_z)|_{(x,u(x))}, \quad \forall 1 \le i \le n\,; \\ \nu^{u,f}(x,u(x)) &= \frac{1}{\sqrt{(1+f)(1+g_u^{ij}u_iu_j)}} \cdot (\partial_z - g_u^{ij}u_i(x)\partial_j)\,. \end{aligned}$$

where $\nu^{u,f}$ is the upward unit normal field of $\Phi^u(\Sigma_0)$ under $(1+f)g$, $u_j := \partial_j u$. Hence the area element of $\Phi^u(\Sigma_0)$ at $(x,u(x))$ under metric $(1+f)g$ is given by

$F^f(x, u(x), du(x))$, where

$$F^f(x, z, \xi) := \sqrt{(1 + f(x, z))^n \cdot \det{}_{g^0} \left[g^z + \xi \otimes \xi\right](x)}.$$

Together with (A.0.1), this proves the estimate in (ii).

To prove (iii), notice that

$$\mathscr{M}^f(u)(x) = -\operatorname{div} \partial_\xi F^f(x, u(x), du(x)) + \partial_z F^f(x, u(x), du(x)), \tag{A.0.2}$$

and

$$\begin{aligned}
\partial_\xi F^f(x, z, \xi) &= F^f(x, z, \xi) \cdot g^0(x) G(x, z, \xi)^{-1} \xi, \\
\partial_z F^f(x, z, \xi) &= F^f(x, z, \xi) \cdot \left(\frac{n \partial_z f}{2(1+f)}\Big|_{(x,z)} + tr(G^{-1} \partial_z G)|_{(x,z,\xi)} \right),
\end{aligned}$$

where $G(x, z, \xi) := [g^z(x) + \xi \otimes \xi]$. Thus again by (A.0.1) we have,

$$\left| (\partial_\xi F^{f^+}|_{(x,z^-,\xi^-)}^{(x,z^+,\xi^+)}) - (\xi^+ - \xi^-)^\sharp \right| \lesssim_n \mathbf{O}(x, z^\pm, \xi^\pm, f^\pm); \tag{A.0.3}$$

where

$$\begin{aligned}
\mathbf{O}(x, z^\pm, \xi^\pm, f^\pm) := & \left(\sum_{i \in \{\pm\}} |z^i| + |\xi^i| + \|f^i\|_{C^2, M_0} \right) \\
& \cdot \left(|z^+ - z^-| + |\xi^+ - \xi^-| + \|f^+ - f^-\|_{C^2, M_0} \right).
\end{aligned}$$

Also, since

$$\left| tr(G^{-1} \partial_z G)|_{(x,z^-,\xi^-)}^{(x,z^+,\xi^+)} + \left(|A_\Sigma|_{g^0}^2 + \operatorname{Ric}_g(\partial_z, \partial_z) \right) \cdot (z^+ - z^-) \right| \lesssim_n \mathbf{O}(x, z^\pm, \xi^\pm),$$

We thus have,

$$\left|(\partial_z F^{f^+}|^{(x,z^+,\xi^+)}_{(x,z^-,\xi^-)}) - \left[\frac{n\partial_z(f^+ - f^-)}{2}\Big|_{(x,0)} - \left(|A_\Sigma|^2 + \mathrm{Ric}_g(\partial_z,\partial_z)\right)(z^+ - z^-)\right]\right|$$
$$\lesssim_n \mathbf{O}(x, z^\pm, \xi^\pm, f^\pm)\,; \tag{A.0.4}$$

(iii) is proved by combining (A.0.2), (A.0.3) and (A.0.4).

To prove (iv), the second fundamental form of Σ_u under $\tilde{g}$ is given by

$$A_{\Sigma_u,\tilde{g}}(\partial_i\Phi^u, \partial_j\Phi^u) = \langle \nabla^{\tilde{g}}_{\partial_i + u_i\partial_z}(\partial_j + u_j\partial_z), \tilde{\nu}\rangle\,,$$

where $\tilde{\nu} = \tilde{\nu}^j\cdot\partial_j + \tilde{\nu}^z\cdot\partial_z$ is the unit normal field of Σ_u under $\tilde{g}$, satisfying the pointwise estimate

$$\sum_{j=1}^n |\tilde{\nu}^j| + |\tilde{\nu}^z - 1| \le C_n([\tilde{g} - g]_{C^3,M_0} + |u| + |du|)\,; \tag{A.0.5}$$

Also by (A.0.1), the pull pack metric $\tilde{g}_u := (\Phi^u)^*\tilde{g}$ satisfies the pointwise estimate

$$|\tilde{g}_u - g^0|_{g_0} + |\nabla_{g^0}\tilde{g}_u|_{g^0} \le C_n\left(\|\tilde{g} - g\|_{C^3,M_0} + |u| + |du|_{g^0} + |\nabla^2 u|_{g^0}\right)\,; \tag{A.0.6}$$

And the Christoffel symbols of $\tilde{g}$ and g satisfies

$$|\tilde{\Gamma}^K_{IJ}(\mathbf{0}, z) - \Gamma^K_{IJ}(\mathbf{0}, 0)| \le C_n([\tilde{g} - g]_{C^3,M_0} + |z|), \quad 1 \le I, J, K \le n+1\,. \tag{A.0.7}$$

Combining (A.0.5), (A.0.6) and (A.0.7) one proves the estimate on difference of second fundamental forms.

To estimate Laplacian, note that $\Delta_{\tilde{g}_u}\psi - \Delta_{g^0}\psi = \mathrm{div}_{g^0}(\vec{B_0}) + B_1$, where

$$\vec{B_0}(\mathbf{0}) = (\tilde{g}^{ij}_u - g^{ij})\partial_j\psi\cdot\partial_i|_{\mathbf{0}}; \qquad B_1(\mathbf{0}) = \frac{1}{2}\tilde{g}^{ij}_u\partial_i(\log\det[\tilde{g}_u])\partial_j\psi(\mathbf{0})\,,$$

and hence by (A.0.6), they satisfies the estimate in (iv). □

Appendix B

Improved Regularity for Obstacle Problems

The goal of this chapter is to prove Theorem 4.2.11 and 4.2.12. First recall that under local coordinate chart $(\mathcal{U}, x^i)$ of ∂M, if $u \in C^1(\mathcal{U}, (-r_M/2, r_M/2))$, then $\mathscr{H}^n(\text{graph}_{\partial M}(u)) = \int_{\mathcal{U}} F(x, u, du)$, where $0 < F(x, z, p) \in C^\infty(\mathcal{U} \times (-r_M, r_M) \times \mathbb{R}^n)$ satisfies

$$\partial^2_{pp} F(x, z, p)^{ij} \xi_i \xi_j \geq \lambda(x, z, p)|\xi|^2 > 0, \qquad \forall \xi \in \mathbb{R}^n \tag{B.0.1}$$

$$\|F\|_{C^4(\mathcal{U} \times [-r_M/2, r_M/2] \times \mathbb{B}^n_R)} \leq C(M, g, R). \tag{B.0.2}$$

If $u \geq 0$ and $\text{graph}_{\partial M}(u)$ is constrained stationary in M, then

$$\begin{aligned} 0 \leq & \frac{d}{dt}\Big|_{t=0} \int_{\mathcal{U}} F(x, u + t\phi, du + td\phi)\ dx \\ \leq & \int_{\mathcal{U}} \partial_p F(x, u, du) \cdot d\phi + \partial_z F(x, u, du)\phi\ dx, \end{aligned} \tag{B.0.3}$$

for every $\phi \in C^1_c(\mathcal{U})$ such that $0 < t << 1$, $u + t\phi \geq 0$.

We shall deal with more general quasi-linear 2nd order elliptic operator of di-

vergence form. For simplicity, write $B_r(x) := \mathbb{B}^n_r(x) \subset \mathbb{R}^n$, $B_r := B_r(0)$ and $\Omega_r := B_r \times [-1,1] \times B_1$. Let $\lambda, \Lambda > 0$, $\alpha \in (0,1)$ be constant, $A = A(x,z,p) \in C^3(B_2 \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $B = B(x,z,p) \in C^2(B_2 \times \mathbb{R} \times \mathbb{R}^n)$ satisfying

$$\begin{aligned} &\partial_p A^{ij}|_{B_2 \times [-1,1] \times B_1} \cdot \xi_i \xi_j \geq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n; \\ &\|A\|_{C^3(\Omega_2)} + \|B\|_{C^2(\Omega_2)} \leq \Lambda. \end{aligned} \tag{B.0.4}$$

Let $\mathscr{M} u := -div(A(x,u,\nabla u)) + B(x,u,\nabla u)$. Consider the variational inequality for a function $u \in C^1(B_2, \mathbb{R}_+)$,

$$\begin{aligned} &\int_{B_2} A(x,u,\nabla u) \cdot \nabla \phi + B(x,u,\nabla u)\phi \, dx \geq 0, \quad \forall \phi \in C^1_c(B_2, \mathbb{R}_+); \\ &\int_{B_2} A(x,u,\nabla u) \cdot \nabla \phi + B(x,u,\nabla u)\phi \, dx = 0, \quad \forall \phi \in C^1_c(\{u > 0\}). \end{aligned} \tag{B.0.5}$$

Clearly, (B.0.3) implies (B.0.5), where $A = \partial_p F$ and $B = \partial_z F$.

Theorem B.0.1. *$\exists \delta_0 = \delta_0(\lambda, \Lambda, n, \alpha) \in (0,1)$ such that if A, B satisfies (B.0.4), $0 \leq u \in C^{1,\alpha}_{loc}(B_2 \setminus \{0\}) \cap C^1(B_2)$ satisfies (B.0.5) and that $\|u\|_{C^1(B_2)} \leq \delta_0$, then $u \in C^{1,1}_{loc}(B_2)$ and*

$$\|u\|_{C^{1,1}(B_1)} \leq C(\lambda, \Lambda, n, \alpha).$$

In view of the proof of Lemma 4.3.4, Theorem 4.2.11 follows from Theorem B.0.1. The major effort is made to prove the following lemma,

Lemma B.0.2. *There exists $\delta_0 = \delta_0(\lambda, \Lambda, n, \alpha) \in (0,1)$, $\eta_0 = \eta_0(\lambda, \Lambda, n, \alpha) \in (0, \delta_0)$ s.t. if u be in Theorem B.0.1 and further satisfying*

$$\|u\|_{C^{1,\alpha}(B_2)} \leq 2\delta_0, \tag{B.0.6}$$

and $\eta < \eta_0$ such that the elliptic coefficients satisfies

$$\|\partial_x A\|_{C^2(\Omega_2)} + \|\partial_z A\|_{C^2(\Omega_2)} + \|B\|_{C^2(\Omega_2)} \leq \eta. \tag{B.0.7}$$

Then $\|u\|_{C^{1,1}(B_1)} \leq C(\lambda, \Lambda, n, \alpha)(\|u\|_{C^{1,\alpha}(B_2)} + \eta)$.

Proof of Theorem B.0.1. Observe that if $x_0 \in B_1$, $r \in (0,1)$, let $\tilde{u}(y) := u(x_0 + ry)/r$, then

$$\mathscr{M} u(x) = \frac{1}{r}[-div\tilde{A}(y, \tilde{u}(y), \nabla\tilde{u}(y)) + \tilde{B}(y, \tilde{u}(y), \nabla\tilde{u}(y))]_{y=(x-x_0)/r},$$

where

$$\tilde{A}(y,z,p) := A(x_0 + ry, rz, p);$$
$$\tilde{B}(y,z,p) := rB(x_0 + ry, rz, p).$$

Hence, by Lemma B.0.2, for u satisfying the assumptions in Theorem B.0.1, $u \in C^{1,1}_{loc}(B_2 \backslash \{0\})$. Thus, for a.e. $x \in B_2 \backslash \{0\}$, $\mathscr{M} u(x) = B(x, 0, 0)$. Together with (B.0.5) and that $\|u\|_{C^1(B_2)} \leq 1$, by a cutting off argument, $\mathscr{M} u(x) = B(x, 0, 0) \cdot \chi_{\{u=0\}}$ in the weak sense on B_2. Hence by standard elliptic estimate [GT01, Theorem 13.1], $\exists \beta = \beta(\lambda, \Lambda, n) \in (0,1)$ such that $\|u\|_{C^{1,\beta}(B_{3/2})} \leq C(\lambda, \Lambda, n)$. Then repeat the rescaling argument above, by Lemma B.0.2, Theorem B.0.1 is proved. □

Now we start to prove Lemma B.0.2.

Lemma B.0.3. *$\exists \eta_1 = \eta_1(\lambda, \Lambda, n, \alpha) \in (0,1)$ such that if $\eta \leq \eta_1$ and A, B satisfies (B.0.4) and (B.0.7), $\|\phi\|_{C^{1,\alpha}(\partial B_1)} \leq \eta_1$. Then the equation*

$$\begin{cases} \mathscr{M} u = 0 & \text{in } B_1 \\ u = \phi & \text{on } \partial B_1 \end{cases} \tag{B.0.8}$$

has a unique solution $u \in C^{1,\alpha}(Clos(B_1))$ *satisfying*

$$\|u\|_{C^{1,\alpha}} \leq C(\lambda, \Lambda, n, \alpha)(\|\phi\|_{C^{1,\alpha}} + \eta). \tag{B.0.9}$$

The uniqueness follows directly from comparison principle, see [GT01, Chapter 10]; We assert that the existence result does not follow directly from the classical Schauder estimate, since the latter requires boundary value to be $C^{2,\alpha}$. Instead, one need to work in the space $H_a^{(b)}$ introduced [GH80], where $a = 2 + \alpha$, $b = 1 + \alpha$. By [GH80, Theorem 5.1] and a fixed point argument, there exists $u \in H_a^{(b)}(B_1)$ satisfying (B.0.8) with uniformly bounded $H_a^{(b)}$-norm; and (B.0.9) follows from [GH80, Lemma 2.1].

Proof of Lemma B.0.2. Let $\delta_0, \eta_0 \in (0, \eta_1)$ to be determined; u be in Lemma B.0.2.

Step 1 $\forall x_0 \in B_{3/2} \cap \{u = 0\}$, $\forall r \in (0, 1/2)$, let $v = v_{x_0,r} \in C^{1,\alpha}(B_r(x_0))$ be the solution of

$$\begin{cases} \mathscr{M} v = 0 & \text{in } B_r(x_0) \\ v = u & \text{on } \partial B_r(x_0) \end{cases}$$

given by Lemma B.0.3. Also, let $\bar{v}$ be the solution of

$$\begin{cases} \mathscr{M} \bar{v} = 0 & \text{in } B_r(x_0) \\ \bar{v} = 0 & \text{on } \partial B_r(x_0). \end{cases}$$

Also note that $\|u\|_{C^1(B_r)}, \|v\|_{C^1(B_r)}, \|\bar{v}\|_{C^1(B_r)} \leq C(\lambda, \Lambda, n, \alpha)(\eta_0 + \delta_0)$ and $\mathscr{M} u \geq 0$ in the weak sense. Hence, by taking $\eta_0, \delta_0 << 1$ and using the maximum principle

[GT01, Theorem 10.10],

$$0 \le u(x) - v(x) \le C(\lambda, \Lambda, n, \alpha) \cdot \sup_{B_r(x) \cap \partial\{u>0\}} (u - v) \le C(\lambda, \Lambda, n, \alpha) \|v^-\|_{C^0(B_r(x_0))}, \tag{B.0.10}$$

for every $x \in B_r(x_0)$; And $\bar{v} \le v$ on $B_r(x_0)$. Applying the classical C^0 estimate on $\tilde{v}(y) := \bar{v}(x_0 + ry)/r$, one get

$$\|\bar{v}\|_{C^0(B_r(x_0))} \le C(\lambda, \Lambda, n, \alpha) r^2 \|B(\cdot, 0, 0)\|_{C^0(B_r)}. \tag{B.0.11}$$

Since $v - \bar{v} \ge 0$ on $B_r(x_0)$, $\mathscr{M} v = \mathscr{M} \bar{v} = 0$ and $v(x_0) \le u(x_0) = 0$, by (B.0.4) and Harnack inequality,

$$\|v - \bar{v}\|_{C^0(B_{r/2}(x_0))} \le C(\lambda, \Lambda, n, \alpha)(v(x_0) - \bar{v}(x_0)) \le C(\lambda, \Lambda, n, \alpha)|\bar{v}(x_0)|. \tag{B.0.12}$$

Combining (B.0.10), (B.0.11) and (B.0.12) one derive

$$0 \le u \le C(\lambda, \Lambda, n, \alpha) \|B(\cdot, 0, 0)\|_{C^0(B_r(x_0))} r^2 \quad \text{on } B_{r/2}(x_0). \tag{B.0.13}$$

Step 2 $\forall x_0 \in \{u = 0\}$, $\nabla u(x_0) = 0$; $\forall x_0 \in \{u > 0\} \cap B_{3/2}$, if $l := dist(x_0, \{u = 0\}) < 1/8$, let $y_0 \in \{u = 0\}$ such that $|x_0 - y_0| = l$. Then by (B.0.13),

$$\|u\|_{C^0(B_l(x_0))} \le \|u\|_{C^0(B_{2l}(y_0))} \le C(\lambda, \Lambda, n, \alpha) l^2.$$

Hence by interior gradient estimate,

$$\frac{1}{l} \|\nabla u\|_{C^0(B_{l/2}(x_0))} + \|\nabla^2 u\|_{C^0(B_{l/2}(x_0))} \le C(\lambda, \Lambda, n, \alpha). \tag{B.0.14}$$

Now that, $\forall x_1, x_2 \in B_1$, $\rho := |x_1 - x_2| < 1/20$, $l_i := dist(x_i, \{u = 0\})$, $l_1 \le l_2$.

(i) If $l_2 \geq 1/8$, then directly by gradient estimate

$$|\nabla u(x_1) - \nabla u(x_2)| \leq \rho \sup_{B_\rho(x_2)} \|\nabla^2 u\| \leq C(\lambda, \Lambda, n, \alpha)|x_1 - x_2|;$$

(2) If $2\rho \leq l_2 < 1/8$, then by (B.0.14),

$$|\nabla u(x_1) - \nabla u(x_2)| \leq \rho \sup_{B_{l_2/2}(x_2)} \|\nabla^2 u\| \leq C(\lambda, \Lambda, n, \alpha)|x_1 - x_2|;$$

(3) If $l_2 \leq 2\rho$, then also by (B.0.14),

$$\begin{aligned} |\nabla u(x_1) - \nabla u(x_2)| &\leq |\nabla u(x_1)| + |\nabla u(x_2)| \\ &\leq C(\lambda, \Lambda, n, \alpha)(l_1 + l_2) \leq C(\lambda, \Lambda, n, \alpha)|x_1 - x_2|. \end{aligned}$$

This proves the $C^{1,1}$ bound of u. □

Proof of Theorem 4.2.12. The regularity results in literature focus on minimizing boundaries; For general closed integral current, the strategy is to first decompose them into sum of boundary of Caccioppoli sets. One subtlety is that a priori the graphs may have incompatible orientations. We rule out this by choosing U_y small enough.

Recall that $M^{n+1} \subset \mathbb{R}^L$ is a compact submanifold with boundary, extended to a closed manifold $\tilde{M}^{n+1} \subset \mathbb{R}^L$; $\tilde{U} \subset \tilde{M}$ is relatively open and $U := \tilde{U} \cap M$. Suppose $U \cap \partial M \neq \emptyset$. $T \in \mathcal{Z}_n(U)$ minimize mass among $\{S \in \mathcal{Z}_n(U) : \mathrm{spt}(T - S) \subset\subset U\}$. Call such T **constrained minimizing** in U.

WLOG $U, \tilde{U}, V := \tilde{U} \setminus U$ are contractible and $\mathbf{M}_U(T) > 0$. By decomposition theorem [Sim83b, Chap 6, 27.6], there's a decreasing sequence of $\mathscr{H}^{n+1}$-measurable

subsets $\{U_j \subset \tilde{U}\}_{j\in\mathbb{Z}}$ such that

$$T = \sum_{j\in\mathbb{Z}} \partial[U_j]; \qquad \|T\| = \sum_{j\in\mathbb{Z}} \|\partial[U_j]\|.$$

In particular, $\mathbf{M}_W(T) = \sum_{j\in\mathbb{Z}} \mathbf{M}_W(\partial[U_j])$ for all $W \subset\subset \tilde{U}$, $\mathrm{spt}(\partial[U_j]) \subset U$ and $T_j := \partial[U_j]$ are also constrained minimizers in U (along ∂M). Thus for each j, either $\mathrm{spt}[U_j]$ or $\mathrm{spt}([\tilde{U}] - [U_j])$ contains V. WLOG, $\mathrm{spt}(\partial[U_0]) \cap \partial M \neq \emptyset$ and $\mathrm{spt}[U_0] \supset V$. Hence $\forall j \leq 0$, $\mathrm{spt}[U_j] \supset V$; and $\forall j > 0$, either $\mathrm{spt}[U_j] \supset V$ or $\mathrm{spt}[U_j] \subset \mathrm{spt}[U_0] \setminus V$.

Claim: $\forall \mathcal{K} \subset U \cap \partial M$ compact, $\exists\ r_1 = r_1(M, U, \mathcal{K}) > 0$ such that if $T_0 = \partial[U_0]$, $T_1 = \partial[U_1]$ are constrained minimizers in U and $U_0 \supset V$, $U_1 \subset U_0 \setminus V$; $y \in \mathcal{K} \cap \mathrm{spt}(T_0)$. Then $U_1 \cap B_{r_1}(y) = \emptyset$.

Clearly, with this claim, for each $j \in \mathbb{Z}$, $T_j \llcorner B_{r_1}(y) = \partial[U_j'] \llcorner B_{r_1}(y)$ is constrained minimizing in $B_{r_1}(y)$, where $U_j' := U_j \cap B_{r_1}(y)$ is either empty or containing V. Hence, by [Mir71] and [Tam82], $T_j \llcorner B_{r_1}(y)$ is $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Then by Theorem 4.2.11, T is $C^{1,1}$ multiple graphs restricting to a smaller subset U_y. This proves (1) of Theorem 4.2.12; and (2) follows from unique continuation of minimal hypersurfaces.

Proof of the claim: By [Mir71], for every $\delta \in (0,1)$, $\exists\ r_2 = r_2(M, U, \mathcal{K}, \delta) > 0$ and $u_0 \in C^1(U \cap \partial M)$ such that $\forall y \in \mathcal{K} \cap \mathrm{spt}(T_0)$,

$$T_0 \llcorner B_{r_2}(y) = [\mathrm{graph}_{\partial M}(u)] \llcorner B_{r_2}(y) \ \text{ with } \ |\nabla u| \leq \delta. \tag{B.0.15}$$

Assuming the claim fails, then $\exists\ y_j \in \mathcal{K} \cap \mathrm{spt}(T_0^{(j)})$, $T_i^{(j)} = \partial[U_i^{(j)}]$ being constrained minimizing boundary in U, where $i = 0, 1$, $j \geq 1$, $U_0^{(j)} \supset V$, $U_1^{(j)} \subset U_0^{(j)} \setminus V$ and $U_1^{(j)} \cap B_{1/j}(y_j) \neq \emptyset$. Then $\mathrm{spt}(T_1^{(j)}) \cap B_{1/j}(y_j) \neq \emptyset$.

Let $j \to \infty$, $y_j \to y_\infty \in \mathcal{K}$, and consider $(\eta_{y_j,1/j})_\sharp T_1^{(j)} \to T_1^{(\infty)}$ being constrained minimizing in $T^+_{y_\infty} M \cong \mathbb{R}^{n+1}_+$. The convergent is both in current and in varifold sense.

Therefore, by (B.0.15) and that $y_j \in \mathcal{K} \cap \operatorname{spt}(T_0^{(j)}) \subset \partial M$, $U_1^{(j)} \subset U_0^{(j)} \setminus V$, we have $T_1^{(\infty)} = \partial[U_1^{(\infty)}]$ for some $\operatorname{spt}(U_1^{(\infty)}) \subset T_{y_\infty}\partial M$, which is $n+1$ negligible. In particular, $T_1^{(\infty)} = 0$; On the other hand, by $\operatorname{spt}(T_1^{(j)}) \cap B_{1/j}(y_j) \neq \emptyset$, Lemma 4.2.3 and Corollary 4.2.4, $T_1^{(\infty)} \neq 0$ and we get a contradiction. □

Appendix C

Jacobi Fields on Minimal Hypercone

In this chapter, we disucss some facts about Jacobi fields on a regular stable minimal hypercone in $\mathbb{R}^{n+1}$. Most results here are well-known to experts, and many of them can be found in [Sim83a, CHS84].

Let $\mathbf{C} \subset \mathbb{R}^{n+1}$ be a regular stable minimal hypercone. Thus $S := \mathbf{C} \cap \mathbb{S}^n$ is a regular minimal hypersurface in $\mathbb{S}^n$. Recall that we parametrize $\mathbf{C}$ as in (5.2.3) by

$$(0, +\infty) \times S \to \mathbf{C}, \quad (r, \omega) \mapsto x = r\omega.$$

And denote for simplicity $\mathbf{B}_r := \mathbb{B}^{n+1}_r(\mathbf{0}) \cap \mathbf{C}$, and $\mathbf{A}(r, s) := \mathbb{A}_{r,s}(\mathbf{0}) \cap \mathbf{C}$. Recall by [Sim68], the Jacobi operator of $\mathbf{C}$ is decomposed under this parametrization as

$$L_{\mathbf{C}} = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}(\Delta_S + |A_S|^2)\,,$$

where A_S is the second fundamental form of S in $\mathbb{S}^n$.

To describe the space of Jacobi fields on $\mathbf{C}$, let

$$\mu_1 < \mu_2 \leq \mu_3 \leq \cdots \nearrow +\infty$$

be the eigenvalues of $-(\Delta_S + |A_S|^2)$, and $w_1, w_2, \ldots$ be the corresponding $L^2(S)$-orthonormal eigenfunctions, where $w_1 > 0$. By [Sim68], the stability of $\mathbf{C}$ is equivalent to that $\mu_1 \geq -(n-2)^2/4$. Recall that $\mathbf{C}$ is called *strictly stable* if $\mu_1 > -(n-2)^2/4$. We have introduced the asymptotic spectrum of $\mathbf{C}$ in (5.2.7), $\Gamma(\mathbf{C}) := \{\gamma_j^{\pm}\}$, which consists of all possible degrees of homogeneous Jacobi field on $\mathbf{C}$, where

$$\gamma_j^{\pm} = \gamma_j^{\pm}(\mathbf{C}) := -\frac{n-2}{2} \pm \sqrt{\mu_j + \frac{(n-2)^2}{4}}\,, \tag{C.0.1}$$

are solutions to $\gamma^2 + (n-2)\gamma - \mu_j = 0$. For later reference, we also denote for simplicity

$$b_j := \frac{\gamma_j^+ - \gamma_j^-}{2} = \sqrt{\mu_j + \frac{(n-2)^2}{4}}\,.$$

Note that $\mu_j = -(n-2)^2/4$ if and only if $j = 1$ and $\gamma_1^+ = \gamma_1^- = -(n-2)/2$. For every $\gamma \in \Gamma_C$, recall that as in (5.2.8), $W_\gamma = span_{\mathbb{R}}\{w_j : \gamma_j^{\pm} = \gamma\}$ is the corresponding eigensubspace of $(\Delta_S + |A_S|^2)$.

By [CHS84, Section 1], for every $0 \leq R_1 < R_2 \leq +\infty$ such that $1 \in [R_1, R_2]$, a general solution of $L_{\mathbf{C}} u = f$ on $\mathbf{A}_{R_1,R_2}(\mathbf{0})$ is given by

$$u(r,\omega) = \sum_{k\geq 1} \left(u_k(r) + v_k(r)\right) w_k(\omega)\,, \tag{C.0.2}$$

where for each k,

$$u_k(r,f) := \begin{cases} \dfrac{-1}{2b_k}\left(r^{\gamma_k^+}\displaystyle\int_r^1 s^{-\gamma_k^+ +1} f_k(s)\ ds - r^{\gamma_k^-}\int_r^1 s^{-\gamma_k^- +1} f_k(s)\ ds\right), & b_k \neq 0\,, \\ r^{-\frac{n-2}{2}}\left(\log r \displaystyle\int_1^r s^{\frac{n}{2}} f_k(s)\ ds - \int_1^r s^{\frac{n}{2}} \log s \cdot f_k(s)\ ds\right), & b_k = 0\,; \end{cases} \tag{C.0.3}$$

is a special solution,

$$f_k(r) := \int_S f(r,\omega) w_k(\omega)\ d\omega,$$

is the Fourier coefficients of f; and

$$v_k(r) = v_k(r; c_k^+, c_k^-) := \begin{cases} c_k^+ r^{\gamma_k^+} + c_k^- r^{\gamma_k^-} = r^{-\frac{n-2}{2}}(c_k^+ r^{b_k} + c_k^- r^{-b_k}), & b_k \neq 0; \\ r^{-\frac{n-2}{2}}(c_k^+ + c_k^- \log r), & b_k = 0. \end{cases} \tag{C.0.4}$$

are the general homogeneous solutions, $c_k^\pm \in \mathbb{R}$.

We list some corollaries of this expansion formula.

Lemma C.0.1. *For each $\phi \in C_c^\infty(\mathbf{C})$ we have,*

$$\int_{\mathbf{C}} |\nabla_{\mathbf{C}}\phi|^2 - |A_{\mathbf{C}}|^2\phi^2 \geq \left(\mu_1 + (\frac{n-2}{2})^2\right)\int_{\mathbf{C}} \frac{1}{r^2}\phi^2.$$

Proof. Write $\phi(r,\omega) = \sum_{k\geq 1}\phi_k(r)w_k(\omega)$, plug in the integration and change variables. The Lemma follows from the Hardy inequality

$$\int_0^{+\infty} 4a'(r)^2 - a(r)^2/r^2\ dr \geq 0, \qquad \forall a \in C_c^1(0,+\infty).$$

□

• **Growth estimate of Jacobi fields**

Lemma C.0.2. *If $\gamma \in \mathbb{R} \setminus \Gamma(\mathbf{C})$, u is a Jacobi field on $\mathbf{B}_R$, i.e. $L_{\mathbf{C}}u = 0$ on $\mathbf{B}_R$, (resp. $\mathbf{C} \setminus \mathbf{B}_R$). Suppose*

$$I_u^2(r) := \int_{\mathbf{A}(r,2r)} u^2(t,\omega) t^{-n}\, dx,$$

satisfies that for some $r_0 < R$ (resp. $r_0 > R$) and $C > 0$,

$$I_u^2(r) \le C r^{2\gamma}, \quad \forall\ r < r_0 \quad (\textit{resp. } \forall\ r > r_0)\,. \tag{C.0.5}$$

Then

$$u(r,\omega) = \sum_{k\ge 1} v_k(r; c_k^+, c_k^-) w_k(\omega),$$

where $c_k^\pm = 0$ if $\gamma_k^\pm < \gamma$ (resp. $\gamma_k^\pm > \gamma$); $v_k(r; c_k^+, c_k^-)$ is defined in (C.0.4).

Proof. We only prove for Jacobi fields on $\mathbf{B}_R$ since the proof for Jacobi fields on $\mathbf{C} \setminus \mathbf{B}_R$ is similar. By (C.0.2),

$$v_k(r; c_k^+, c_k^-) = \int_S u(r,\omega) w_k(\omega)\, d\omega \le (\int_S u(r,\omega)^2\, d\omega)^{1/2}.$$

Hence, for all $r \in (0, r_0)$,

$$\int_r^{2r} t^{-1} v_k(t; c_k^+, c_k^-)^2\, dt \le \int_r^{2r} t^{-1} dt \int_S u(t,\omega)^2\, d\omega \le C(n) I_u^2(r) \le C r^{2\gamma+1}.$$

And then by the expression (C.0.4) of $v_k(r; c_k^+, c_k^-)$, $c_k^\pm = 0$ if $\gamma_k^\pm < \gamma$. □

Combined with Harnack inequality and that w_k changes signs when $k \ge 2$, the following Lemma is a corollary of Lemma C.0.2, and the proof is left to readers. See also [Whi89, Lemma 4.2].

Lemma C.0.3. *If u is a positive Jacobi field on $\mathbf{B}_R$ (resp. $\mathbf{C}\setminus\mathbf{B}_R$), then*

$$u(r,\omega)=v_1(r)w_1(\omega)+\sum_{k\geq 2}c_k r^{\gamma_k^+}w_k(\omega).$$
$$(\ \textit{resp.}\ u(r,\omega)=v_1(r)w_1(\omega)+\sum_{k\geq 2}c_k r^{\gamma_k^-}w_k(\omega).\ )$$

for some $c_k\in\mathbb{R}$, where v_1 is defined in (C.0.4) for some $c_1^{\pm}$ such that $v_1>0$ on $\mathbf{B}_R$ (resp. on $\mathbf{C}\setminus\mathbf{B}_R$).

In particular, if u is a positive Jacobi field on $\mathbf{C}$, then

$$u=\begin{cases}(c^+r^{\gamma_1^+}+c^-r^{\gamma_1^-})w_1, & \text{if } \gamma_1^+>\gamma_1^-;\\ cr^{-\frac{n-2}{2}}w_1, & \text{if } \gamma_1^+=\gamma_1^-.\end{cases}$$

for some $c^{\pm},c\geq 0$.

(C.0.2) also guarantees the following existence of solutions with prescribed growth or decaying bound, which is analogous to [CHS84, Theorem 1.1]. We start with some notations.

For $\sigma\in\mathbb{R}\setminus\Gamma(\mathbf{C})$, define Π_σ^+ (resp. Π_σ^-) : $L^2(S)\to L^2(S)$ to be the orthogonal projection onto $\bigoplus_{\gamma_j^+>\sigma}W_{\gamma_j^+}$ (resp. $\bigoplus_{\gamma_j^-<\sigma}W_{\gamma_j^-}$). Note that $1-\Pi_\sigma^{\pm}$ are the projection onto their L^2-orthogonal complement; When $\sigma<\gamma_1^+$ (resp. $\sigma>\gamma_1^-$), $\Pi_\sigma^+=id$ (resp. $\Pi_\sigma^-=id$).

For $E\subset\mathbf{C}$ and $f\in L^2_{loc}(E)$, define

$$\|f\|^2_{L^2_\sigma(E)}:=\int_E|f(x)|^2\cdot|x|^{-2\sigma-n}\,dx. \tag{C.0.6}$$

Also define for $k\geq 1$, $\|f\|^2_{W^{k,2}_\sigma(E)}:=\sum_{j=0}^k\||\nabla^j f|\|^2_{L^2_{\sigma-j}}$. $L^2_\sigma(E)$ and $W^{k,2}_\sigma(E)$ will be the space of functions with finite $\|\cdot\|_{L^2_\sigma}$ and $\|\cdot\|_{W^{k,2}_\sigma}$ norm respectively.

Lemma C.0.4. *Suppose $\sigma\in\mathbb{R}\setminus\Gamma(\mathbf{C})$; $\varphi,\psi\in L^2(S)$; $f\in L^2_{\sigma-2}(\mathbf{B}_1)$ (resp. $f\in$*

$L^2_{\sigma-2}(\mathbf{C}\backslash\mathbf{B}_1)$). *Then there's a unique* $u \in L^2_\sigma \cap W^{2,2}_{loc}(\mathbf{B}_1)$ *(resp.* $u \in L^2_\sigma \cap W^{2,2}_{loc}(\mathbf{C}\backslash\mathbf{B}_1)$*) such that*

$$\begin{cases} L_{\mathbf{C}}u = f & \text{on } \mathbf{B}_1 \qquad \textit{(resp. } L_{\mathbf{C}}u = f \quad \textit{on } \mathbf{C}\setminus\mathbf{B}_1\textit{)}, \\ \Pi^+_\sigma u = \Pi^+_\sigma \varphi & \text{on } \partial\mathbf{B}_1 \qquad \textit{(resp. } \Pi^-_\sigma u = \Pi^-_\sigma \varphi \quad \textit{on } \partial\mathbf{B}_1\textit{)}, \\ (1-\Pi^-_\sigma)\partial_r u = (1-\Pi^-_\sigma)\psi & \text{on } \partial\mathbf{B}_1 \qquad \textit{(resp. } (1-\Pi^+_\sigma)\partial_r u = (1-\Pi^+_\sigma)\psi \quad \textit{on } \partial\mathbf{B}_1\textit{)}. \end{cases}$$

Moreover, u *satisfies the estimate,*

$$\begin{aligned} &\sup_{t\in(0,1)} \|u(t,\cdot)\|_{L^2(S)} \cdot t^{-\sigma} + \|u\|_{L^2_\sigma(\mathbf{B}_1)} \\ &\qquad \le C(\mathbf{C},\sigma)(\|f\|_{L^2_{\sigma-2}(\mathbf{B}_1)} + \|\varphi\|_{L^2(S)} + \|\psi\|_{L^2(S)}), \\ \big(\textit{resp.} \quad &\sup_{t\in(1,+\infty)} \|u(t,\cdot)\|_{L^2(S)} \cdot t^{-\sigma} + \|u\|_{L^2_\sigma(\mathbf{C}\backslash\mathbf{B}_1)} \\ &\qquad \le C(\mathbf{C},\sigma)(\|f\|_{L^2_{\sigma-2}(\mathbf{C}\backslash\mathbf{B}_1)} + \|\varphi\|_{L^2(S)} + \|\psi\|_{L^2(S)})\,.\big) \end{aligned}$$

The proof is almost the same as [CHS84, Theorem 1.1] and is left to readers.

The following perturbed version of Lemma C.0.4 is used in Section 5.4.

Corollary C.0.5. *Suppose* $\sigma \in \mathbb{R} \setminus \Gamma(\mathbf{C})$. *Then there exists* $\varepsilon = \varepsilon(\mathbf{C},\sigma) > 0$ *such that, if* $f \in L^2_{\sigma-2}(\mathbf{B}_1)$*;* $\mathscr{R}$ *is a second order linear differential operator satisfying*

$$\|\mathscr{R}u\|_{L^2_{\sigma-2}(\mathbf{B}_1)} \le \varepsilon \|u\|_{W^{2,2}_\sigma(\mathbf{B}_1)}, \quad \forall u \in W^{2,2}_\sigma(\mathbf{B}_1).$$

Then there exists a unique solution $u \in W^{2,2}_\sigma(\mathbf{B}_1)$ *to the equation*

$$\begin{cases} L_{\mathbf{C}}u + \mathscr{R}u = f & \text{on } \mathbf{B}_1, \\ \Pi^+_\sigma u = 0 & \text{on } \partial\mathbf{B}_1, \\ (1-\Pi^-_\sigma)\partial_r u = 0 & \text{on } \partial\mathbf{B}_1. \end{cases} \tag{C.0.7}$$

Moreover, u satisfies the estimate,

$$\sup_{t\in(0,1)} \|u(t,\cdot)\|_{L^2(S)} \cdot t^{-\sigma} + \|u\|_{L^2_\sigma(\mathbf{B}_1)} \leq C(\mathbf{C},\sigma)\|f\|_{L^2_{\sigma-2}(\mathbf{B}_1)}\,.$$

Proof. Apply Banach fixed point theorem on $T : W^{2,2}_\sigma(\mathbf{B}_1) \to W^{2,2}_\sigma(\mathbf{B}_1)$, where $v = Tu$ solves,

$$\begin{cases} L_{\mathbf{C}} v = f - \mathscr{R} u & \text{on } \mathbf{B}_1, \\ \Pi^+_\sigma v = 0 & \text{on } \partial\mathbf{B}_1, \\ (1-\Pi^-_\sigma)\partial_r v = 0 & \text{on } \partial\mathbf{B}_1. \end{cases}$$

Combining Lemma C.0.4 and the L^2 estimate for elliptic PDE, one can show that T is a contraction map provided ε is small enough. □

• **Growth rate monotonicity** The following growth rate monotonicity, as an application of the decomposition (C.0.2), which was in fact first introduced in [Sim83a] in a different form, plays a central role in studying the behavior of Jacobi fields near singularities in the thesis.

Lemma C.0.6 (Growth Rate Monotonicity)**.** *For $\sigma > 0$, $\Lambda, \Lambda' > 1$, there exists $K = K(\sigma, \Lambda, \Lambda') > 2$ with the following property.*

Let $\gamma \in \mathbb{R}$, $\mathbf{C} \in \mathscr{C}^n_{\Lambda,\Lambda'}$ be a stable minimal hypercone (see (5.2.1)) satisfying

$$dist_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}) \cup \{-(n-2)/2\}) \geq \sigma\,.$$

Suppose that $u \in C^2_{loc}(\mathbf{A}(K^{-3},1)) \cap L^2(\mathbf{A}(K^{-3},1))$ solves $L_{\mathbf{C}} u = 0$. Then

$$J^\gamma_K(u;r) := \int_{\mathbf{A}(K^{-1}r,r)} u^2 \cdot |x|^{-2\gamma-n} d\|\mathbf{C}\|\,,$$

satisfies

$$J_K^\gamma(u; K^{-2}) - 2J_K^\gamma(u; K^{-1}) + J_K^\gamma(u; 1) \geq 0\,.$$

Moreover, the equality holds if and only if $u \equiv 0$.

Proof. By (C.0.2), we can write

$$u(r, \omega) = \sum_{j \geq 1} v_j(r; c_j^+, c_j^-) \cdot w_j(\omega)\,,$$

and by (C.0.4) and definition of $J_K^\gamma(u; r)$, we have

$$\begin{aligned} J_K^\gamma(u; r) &= \int_{K^{-1}r}^r t^{n-1-n-2\gamma}\, dt \int_{\mathbf{C}\cap\mathbb{S}^n} u(t, \omega)^2\, d\omega \\ &= \sum_{j \geq 1} \int_{K^{-1}r}^r t^{-1-2\gamma} \cdot v_j(t; c_j^+, c_j^-)^2\, dt, \end{aligned}$$

where

$$\begin{aligned} &\int_{K^{-1}r}^r t^{-1-2\gamma} \cdot v_j(t; c_j^+, c_j^-)^2\, dt \\ &= \begin{cases} \int_{K^{-1}r}^r (c_j^+ t^{\gamma_j^+ - \gamma} + c_j^- t^{\gamma_j^- - \gamma})^2 t^{-1}\, dt, & \text{if } \gamma_j^+ \neq \gamma_j^-; \\ \int_{K^{-1}r}^r (c_j^+ t^{-(n-2)/2-\gamma} + c_j^- t^{-(n-2)/2-\gamma} \log t)^2 t^{-1}\, dt, & \text{if } \gamma_j^+ = \gamma_j^-. \end{cases} \end{aligned}$$

Hence Lemma C.0.6 is an immediate corollary of the following Lemma C.0.7 and C.0.8. □

Lemma C.0.7. *Suppose* $\sigma > 0$. *Then there exists a real number* $K_1 = K_1(\sigma) > 2$ *with the following property. For every* $\alpha \neq \beta \in \mathbb{R}$ *such that* $|\alpha|, |\beta|, |\alpha + \beta| \geq \sigma$ *and every* $K \geq K_1$, *the following*

$$\mathcal{I}_K(r; c, c') := \int_r^{Kr} (cs^\alpha + c's^\beta)^2 s^{-1}\, ds,$$

satisfies that for every $r > 0$ and every $(0,0) \neq (c,c') \in \mathbb{R}^2$ we have,

$$\mathcal{I}_K(K^2 r; c, c') - 2\mathcal{I}_K(Kr; c, c') + \mathcal{I}_K(r; c, c') > 0.$$

Proof.

$$\begin{aligned}\mathcal{I}_K(r; c, c') &= \int_r^{Kr} (c^2 s^{2\alpha-1} + 2cc' s^{\alpha+\beta-1} + c'^2 s^{2\beta-1})\, ds \\ &= c^2 r^{2\alpha} \cdot \frac{K^{2\alpha}-1}{2\alpha} + 2cc' r^{\alpha+\beta} \cdot \frac{K^{\alpha+\beta}-1}{\alpha+\beta} + c'^2 r^{2\beta} \cdot \frac{K^{2\beta}-1}{2\beta}.\end{aligned}$$

Thus,

$$\begin{aligned}&\mathcal{I}_K(K^2 r; c, 1) - 2\mathcal{I}_K(Kr; c, 1) + \mathcal{I}_K(r; c, 1) \\ &= c^2 r^{2\alpha} \cdot \frac{(K^{2\alpha}-1)^3}{2\alpha} + 2cc' r^{\alpha+\beta} \cdot \frac{(K^{\alpha+\beta}-1)^3}{\alpha+\beta} + c'^2 r^{2\beta} \cdot \frac{(K^{2\beta}-1)^3}{2\beta}.\end{aligned}$$

And hence to prove the Lemma, it suffices to show that when $K \geq K_1(\sigma)$, we have

$$\Delta(K; \alpha, \beta) := \Big[\frac{(K^{\alpha+\beta}-1)^3}{\alpha+\beta}\Big]^2 - \frac{(K^{2\alpha}-1)^3}{2\alpha} \cdot \frac{(K^{2\beta}-1)^3}{2\beta} < 0.$$

Since $\Delta(K; \alpha, \beta) = \Delta(K; \beta, \alpha) = K^{6\alpha+6\beta} \cdot \Delta(K; -\alpha, -\beta)$, we can assume WLOG that $\alpha > \beta$ and $\alpha + \beta > \sigma > 0$.

<u>*Case I.*</u> $0 < \beta < \alpha$. By a simple calculation, we have

$$\begin{aligned}\Delta(K; \alpha, \beta) = \frac{-(\alpha-\beta)^2}{4\alpha\beta(\alpha+\beta)^2}\Big[&K^{6\alpha+6\beta} - K^{5\alpha+5\beta}\big(6 + 3(\alpha+\beta)^2\Lambda_1\big) \\ &+ K^{4\alpha+4\beta}\big(15 + 3(\alpha+\beta)^2\Lambda_2\big) \\ &- K^{3\alpha+3\beta}\big(20 + (\alpha+\beta)^2(\Lambda_3 + 9\Lambda_1)\big) \\ &+ K^{2\alpha+2\beta}\big(15 + 3(\alpha+\beta)^2\Lambda_2\big) \\ &- K^{\alpha+\beta}\big(6 + 3(\alpha+\beta)^2\Lambda_1\big) + 1\Big],\end{aligned}$$

where for $i = 1, 2, 3$,

$$0 \leq \Lambda_i := \frac{K^{i\alpha - i\beta} - 2 + K^{i\beta - i\alpha}}{(\alpha - \beta)^2} \leq K^{i\alpha - i\beta} \cdot (i \log K)^2.$$

Hence by taking $K \geq K_1(\sigma) >> 1$, $K^{6\alpha+6\beta}$ dominates other terms in $\Delta(K; \alpha, \beta)$. In particular, we have $\Delta(K; \alpha, \beta) < 0$.

Case II. $\beta < 0 < \alpha$. Then when $K^{2\sigma} > 2$, we have $K^{2\alpha} - 1 > K^{2\alpha}/2$, $1 - K^{2\beta} > 1/2$, and hence,

$$\Delta(K; \alpha, \beta) \leq \frac{K^{6\alpha+6\beta}}{(\alpha+\beta)^2} - \frac{K^{6\alpha}}{256|\alpha\beta|}. \tag{C.0.8}$$

And since

$$\frac{(\alpha+\beta)^2}{|\alpha\beta|} \geq \begin{cases} \frac{\sigma}{2|\beta|}, & \text{if } \alpha + \beta > \alpha/2; \\ \frac{\sigma^2}{2|\beta|^2}, & \text{if } \alpha + \beta \leq \alpha/2. \end{cases}$$

(C.0.8) then implies $\Delta(K; \alpha, \beta) < 0$ by taking $K \geq K_1(\sigma) >> 1$. □

Lemma C.0.8. *Suppose $\sigma > 0$. Then there exists a real number $K_2 = K_2(\sigma) > 2$ with the following property. For every $\alpha' \in \mathbb{R}$ such that $|\alpha'| \geq \sigma$ and every $K \geq K_2$, the following*

$$\tilde{\mathcal{I}}_K(r; c, c') := \int_r^{Kr} (cs^{\alpha'} + c's^{\alpha'} \log s)^2 s^{-1} \, ds,$$

satisfies that for every $r > 0$ and every $(0, 0) \neq (c, c') \in \mathbb{R}^2$ we have,

$$\tilde{\mathcal{I}}_K(K^2 r; c, c') - 2\tilde{\mathcal{I}}_K(Kr; c, c') + \tilde{\mathcal{I}}_K(r; c, c') > 0.$$

Proof. First note that the Lemma holds trivially if $c' = 0$; Also by arbitrariness of $r > 0$, we can assume WLOG that $(c, c') = (0, 1)$. Also for simplicity of notations, let

$\alpha := 2\alpha'$. Then,

$$\begin{aligned}\tilde{\mathcal{I}}_K(r;0,1) &= \int_r^{Kr} s^{\alpha-1}(\log s)^2\, ds \\ &= r^\alpha(\log r)^2 \cdot \frac{K^\alpha - 1}{\alpha} + r^\alpha \log r \cdot \Big(\frac{2K^\alpha \log K}{\alpha} - \frac{2(K^\alpha-1)}{\alpha^2}\Big) \\ &\quad + r^\alpha \cdot \Big(\frac{K^\alpha(\log K)^2}{\alpha} - \frac{2K^\alpha \log K}{\alpha^2} + \frac{2(K^\alpha-1)}{\alpha^3}\Big).\end{aligned}$$

A direct calculation provides,

$$\begin{aligned}&\tilde{\mathcal{I}}_K(K^2r;0,1) - 2\tilde{\mathcal{I}}_K(Kr;0,1) + \tilde{\mathcal{I}}_K(r;0,1) \\ =\ & r^\alpha(\log r)^2 \cdot \frac{(K^\alpha-1)^3}{\alpha} \\ +\ & 2r^\alpha(\log r) \cdot \Big[3K^\alpha \log K \cdot \frac{(K^\alpha-1)^2}{\alpha} - \frac{(K^\alpha-1)^3}{\alpha^2}\Big] \\ +\ & r^\alpha \cdot \Big[K^\alpha(\log K)^2 \cdot \frac{(K^\alpha-1)(9K^\alpha-3)}{\alpha} - K^\alpha \log K \cdot \frac{6(K^\alpha-1)^2}{\alpha^2} + \frac{2(K^\alpha-1)^3}{\alpha^3}\Big].\end{aligned}$$

Hence to prove the Lemma, it suffices to show that

$$\begin{aligned}\Delta(K;\alpha) := & \Big[3K^\alpha \log K \cdot \frac{(K^\alpha-1)^2}{\alpha} - \frac{(K^\alpha-1)^3}{\alpha^2}\Big]^2 - \frac{(K^\alpha-1)^3}{\alpha} \cdot \\ & \Big[K^\alpha(\log K)^2 \cdot \frac{(K^\alpha-1)(9K^\alpha-3)}{\alpha} - K^\alpha \log K \cdot \frac{6(K^\alpha-1)^2}{\alpha^2} + \frac{2(K^\alpha-1)^3}{\alpha^3}\Big] \\ = & \frac{(K^\alpha-1)^4}{\alpha^2} \cdot \Big[3K^\alpha(\log K)^2 - \frac{(K^\alpha-1)^2}{\alpha^2}\Big].\end{aligned}$$

is less than 0 provided $K \geq K_2(\sigma) >> 1$. This is clear since when $\alpha > 0$ (and then by assumption, $\alpha \geq 2\sigma$), $-K^{2\alpha}/\alpha^2$ dominates the other terms, and when $\alpha < 0$ (and again by assumption, $\alpha \leq -2\sigma$), $-1/\alpha^2$ dominates the other terms. $\square$

Appendix D

Quantitative Uniqueness of Tangent Cone

In [Ede21, Theorem 6.3], Edelen proved a quantitative version of the uniqueness of tangent cones for minimal hypersurfaces. We summarize this result below with a different formulation.

Lemma D.0.1. *Given $\epsilon > 0$, integer $I \geq 0$ and $\mathbf{C}$ a stable minimal hypercone in $\mathbb{R}^{n+1=8}$, there exists $\delta = \delta(\epsilon, \mathbf{C}, I) > 0$ with the following property.*

Let g be a C^4 metric on $\mathbb{B}_4^{n+1}$, $V \in \mathcal{R}_n(\mathbb{B}_4)$ be an stationary integral varifold supported on LSMH under g with $\operatorname{index}(\operatorname{spt}(V), g) \leq I$, *and let $r \in (0, 1/16)$. Suppose:*

(a) $\|g - g_{\text{euc}}\|_{C^4} \leq \delta$;

(b) $\theta_V(\mathbf{0}, 2) \leq \theta_V(\mathbf{0}, r) + \delta$;

(c) There exists $s \in (2r, 1/2)$ and $u \in C^2(\mathbf{C})$ with $\|u\|_{C^2_*} \leq \delta$ such that

$$V \llcorner A^g(\mathbf{0}; s, 2s) = |\operatorname{graph}_{\mathbf{C}}(u)| \llcorner A^g(\mathbf{0}; s, 2s).$$

Then, there exists $\tilde{u} \in C^2(\mathbf{C})$ with $\|\tilde{u}\|_{C^2_*} \leq \epsilon$ such that

$$V \llcorner A^g(\mathbf{0}; 2r, 1) = |\operatorname{graph}_{\mathbf{C}}(\tilde{u})| \llcorner A^g(\mathbf{0}; 2r, 1).$$

Proof. It suffices to check the condition in [Ede21, Theorem 6.3] holds, i.e. $V \llcorner A^g(\mathbf{0}; 8r, 1)$ is a $(\mathbf{C}, 1, \Psi(\delta|\mathbf{C}, I))$-cone region. Denote for simplicity $V_t := \eta_{\mathbf{0},t\sharp} V$.

First of all, by (a) (b) and volume monotonicity, for every $t \in (8r, 1)$, there exists $m_t \in \mathbb{N}$ and $\mathbf{C}_t \in \mathscr{C}$ such that

$$\mathbf{F}_{\mathbb{B}_2}(V_t, m_t|\mathbf{C}_t|) \leq \Psi(\delta|\mathbf{C}, I)\,, \tag{D.0.1}$$

And by (c), one can choose $(\mathbf{C}_s, m_s) = (\mathbf{C}, 1)$. Since for every $r \in (0, 1)$ and every varifolds V_1, V_2,

$$\mathbf{F}_{\mathbb{B}_2}(\eta_{\mathbf{0},r\sharp} V_1, \eta_{\mathbf{0},r\sharp} V_2) \leq \mathbf{F}_{\mathbb{B}_2}(V_1, V_2) \cdot r^{-1}\,,\,,$$

we then have for every $t \in (8r, 1/2)$,

$$\begin{aligned} \mathbf{F}_{\mathbb{B}_2}(m_{2t}|\mathbf{C}_{2t}|, m_t|\mathbf{C}_t|) &\leq \mathbf{F}_{\mathbb{B}_2}(m_{2t}|\mathbf{C}_{2t}|, V_t) + \mathbf{F}_{\mathbb{B}_2}(m_t|\mathbf{C}_t|, V_t) \\ &\leq 2\mathbf{F}_{\mathbb{B}_2}(m_{2t}|\mathbf{C}_{2t}|, V_{2t}) + \mathbf{F}_{\mathbb{B}_2}(m_t|\mathbf{C}_t|, V_t) \\ &\leq 3\Psi(\delta|\mathbf{C}, I). \end{aligned} \tag{D.0.2}$$

By Lemma 3.2.2, for every $\mathbf{C}_1, \mathbf{C}_2 \in \mathscr{C}_\Lambda$ and every integer $m \geq 2$, we have

$$\mathbf{F}_{\mathbb{B}_2}(|\mathbf{C}_1|, m|\mathbf{C}_2|) \geq \epsilon(\Lambda)\,,$$

Hence by taking $\delta << 1$, $m_s = 1$ and (D.0.2) guarantees that $m_t = 1$ for every $t \in (8r, 1)$. Then by Allard Regularity [All72], (D.0.1) implies that $V \llcorner A^g(\mathbf{0}; t/8, t)$ is a one-sheet $\Psi(\delta|\mathbf{C}, I)$-graph over $\mathbf{C}_t$. Edelen's Theorem thus applies by taking $\delta \leq \delta(\epsilon, \mathbf{C}, I) << 1$. □

Remark D.0.2. Using a similar argument as [Ede21], one can actually drop the bounded-index assumption in Lemma D.0.1, but this is out of the scope of this paper.

This in particular implies the following result of convergence in all scales.

Corollary D.0.3. *Let $\{\Sigma_j\}_{j\geq 1}$ be a sequence of two-sided LSMH in $(\mathbb{B}^8_4, g_j)$ containing $\mathbf{0}$, with $\mathrm{index}(\Sigma_j, g_j) \leq \Lambda$, $\|\Sigma_j\|(\mathbb{B}_4) \leq \Lambda$. Suppose when $j \to \infty$,*

(a) $\|g_j - g_{\mathrm{euc}}\|_{C^4} \to 0$;

(b) $\theta_{|\Sigma_j|}(\mathbf{0}, 2) - \theta_{|\Sigma_j|}(\mathbf{0}) \to 0$.

Then there exists some stable minimal hypercone $\mathbf{C} \in \mathscr{C}_{4^n\Lambda}$ such that after passing to a subsequence,

$$\mathbf{F}_{\mathbb{B}_3}(|\Sigma_j|, |\mathbf{C}|) \to 0, \qquad \mathbf{F}_{\mathbb{B}_3}(|\mathbf{C_0}\Sigma_j|, |\mathbf{C}|) \to 0,$$

where $\mathbf{C_0}\Sigma_j$ is the tangent cone of Σ_j at $\mathbf{0}$.

Proof. First recall that by [Ilm96, Theorem B], tangent cones at a singular point of a LSMH are all of multiplicity 1. Denote for simplicity by $\mathbf{C}_j \in \mathscr{C}_{4^n\Lambda}$ the tangent cone of Σ_j at $\mathbf{0}$, and suppose when $j \to \infty$, $\mathbf{C}_j$ $\mathbf{F}$-converges to some $\mathbf{C} \in \mathscr{C}_{4^n\Lambda}$. By Allard's Regularity Theorem [All72], the convergence is in C^2 sense away from $\mathbf{0}$. Hence by Lemma D.0.1, $|\Sigma_j|$ also $\mathbf{F}$-subconverges to $|\mathbf{C}|$. □

Bibliography

[AAJ81] William K Allard and Frederick J Almgren Jr. On the radial behavior of minimal surfaces and the uniqueness of their tangent cones. *Annals of Mathematics*, pages 215–265, 1981.

[ACS17] Lucas Ambrozio, Alessandro Carlotto, and Ben Sharp. Compactness analysis for free boundary minimal hypersurfaces. *Calculus of Variations and Partial Differential Equations*, 57(1):22, December 2017.

[AIC95] Sigurd Angenent, Tom Ilmanen, and David L Chopp. A computed example of nonuniqueness of mean curvature flow in r3. *Communications in Partial Differential Equations*, 20(11-12):1937–1958, 1995.

[All72] William K. Allard. On the first variation of a varifold. *Annals of Mathematics*, 95(3):417–491, 1972.

[Alm62] Frederick J. Almgren. The homotopy groups of the integral cycle groups. *Topology*, 1(4):257 – 299, 1962.

[Alm65] Frederick J Almgren. *The Theory of Varifolds: A Variational Calculus in the Large for the k-Dimensional Area Integrand*. Princeton; Institute for Advanced Study, 1965.

[Alm75] F. J. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Bull. Amer. Math. Soc.*, 81(1):151–154, 01 1975.

[BDGG69] E. Bombieri, E. De Giorgi, and E. Giusti. Minimal cones and the Bernstein problem. *Inventiones mathematicae*, 7(3):243–268, September 1969.

[BG72] Enrico Bombieri and E Giusti. Harnack's inequality for elliptic differential equations on minimal surfaces. *Inventiones mathematicae*, 15(1):24–46, 1972.

[BKL74] Haïm Brézis, David Kinderlehrer, and Hans Lewy. The smoothness of solutions to nonlinear variational inequalities. *Indiana university mathematics journal*, 23(9):831–844, 1974.

[BM82] Elisabetta Barozzi and U. Massari. Regularity of minimal boundaries with obstacles. *Rendiconti del Seminario Matematico della Università di Padova*, 66:129–135, 1982.

[Bom82] Enrico Bombieri. Regularity theory for almost minimal currents. *Archive for Rational Mechanics and Analysis*, 78(2):99–130, 1982.

[Bra01] Hubert L Bray. Proof of the riemannian penrose inequality using the positive mass theorem. *Journal of Differential Geometry*, 59(2):177–267, 2001.

[Bro86] JE Brothers. Some open problems in geometric measure theory and its applications suggested by participants. *Geometric Measure Theory and the Calculus of Variations*, 44:441, 1986.

[BW18] Jacob Bernstein and Lu Wang. An integer degree for asymptotically conical self-expanders. *arXiv preprint arXiv:1807.06494*, 2018.

[BW19a] Jacob Bernstein and Lu Wang. Relative expander entropy in the presence of a two-sided obstacle and applications. *arXiv preprint arXiv:1906.07863*, 2019.

[BW19b] Jacob Bernstein and Lu Wang. Topological uniqueness for self-expanders of small entropy. *arXiv preprint arXiv:1902.02642*, 2019.

[BW20a] Costante Bellettini and Neshan Wickramasekera. Stable prescribed-mean-curvature integral varifolds of codimension 1: Regularity and compactness. *arXiv:1902.09669 [math]*, October 2020.

[BW20b] Jacob Bernstein and Lu Wang. A mountain-pass theorem for asymptotically conical self-expanders. *arXiv preprint arXiv:2003.13857*, 2020.

[BW21a] Jacob Bernstein and Lu Wang. Smooth compactness for spaces of asymptotically conical self-expanders of mean curvature flow. *International Mathematics Research Notices*, 2021(12):9016–9044, 2021.

[BW21b] Jacob Bernstein and Lu Wang. The space of asymptotically conical self-expanders of mean curvature flow. *Mathematische Annalen*, 380(1):175–230, 2021.

[CES17] Maria Colombo, Nick Edelen, and Luca Spolaor. The singular set of minimal surfaces near polyhedral cones. *arXiv preprint arXiv:1709.09957*, 2017.

[CGG91] Yun Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *Journal of differential geometry*, 33(3):749–786, 1991.

[Cha97] Claire C. Chan. Complete minimal hypersurfaces with prescribed asymptotics at infinity. *J. Reine. Angew. Math*, 1997.

[CHS84] Luis Caffarelli, Robert Hardt, and Leon Simon. Minimal surfaces with isolated singularities. *manuscripta mathematica*, 48(1):1–18, 1984.

[CK80] L. A. Caffareli and David Kinderlehrer. Potential methods in variational inequalities. *Journal d'Analyse Mathématique*, 37(1):285–295, 1980.

[CLRS22] Thomas H Cormen, Charles E Leiserson, Ronald L Rivest, and Clifford Stein. *Introduction to algorithms*. MIT press, 2022.

[CLS20] Otis Chodosh, Yevgeny Liokumovich, and Luca Spolaor. Singular behavior and generic regularity of min-max minimal hypersurfaces. *arXiv preprint arXiv:2007.11560*, 2020.

[CM19] Otis Chodosh and Christos Mantoulidis. Minimal Hypersurfaces with Arbitrarily Large Area. *International Mathematics Research Notices*, July 2019.

[CMS23] Otis Chodosh, Christos Mantoulidis, and Felix Schulze. Generic regularity for minimizing hypersurfaces in dimensions 9 and 10. *arXiv preprint arXiv:2302.02253*, 2023.

[CN13] Jeff Cheeger and Aaron Naber. Quantitative Stratification and the Regularity of Harmonic Maps and Minimal Currents. *Communications on Pure and Applied Mathematics*, 66(6):965–990, June 2013.

[CSZ97] Huai-Dong Cao, Ying Shen, and Shunhui Zhu. The structure of stable minimal hypersurfaces in $\mathbb{R}^n$. *Mathematical Research Letters*, 4(5):637–644, 1997.

[Dey19] Akashdeep Dey. Compactness of certain class of singular minimal hypersurfaces. *arXiv preprint arXiv:1901.05840*, 2019.

[DG61] Ennio De Giorgi. *Frontiere Orientate di Misura Minima*. Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61. Editrice Tecnico Scientifica, Pisa, 1961.

[Din20] Qi Ding. Minimal cones and self-expanding solutions for mean curvature flows. *Mathematische Annalen*, 376(1):359–405, 2020.

[DLR18] Camillo De Lellis and Jusuf Ramic. Min-max theory for minimal hypersurfaces with boundary. In *Annales de l'Institut Fourier*, volume 68, pages 1909–1986, 2018.

[Dou31] Jesse Douglas. Solution of the Problem of Plateau. *Transactions of the American Mathematical Society*, 33(1):263–321, 1931.

[Ede21] Nick Edelen. Degeneration of 7-dimensional minimal hypersurfaces which are stable or have bounded index. *arXiv:2103.13563 [math]*, March 2021.

[EH89] Klaus Ecker and Gerhard Huisken. Mean curvature evolution of entire graphs. *Annals of Mathematics*, 130(3):453–471, 1989.

[EH91] Klaus Ecker and Gerhard Huisken. Interior estimates for hypersurfaces moving by mean curvature. *Inventiones mathematicae*, 105(1):547–569, 1991.

[ES91] Lawrence C Evans and Joel Spruck. Motion of level sets by mean curvature. i. *Journal of Differential Geometry*, 33(3):635–681, 1991.

[ES19] Nick Edelen and Luca Spolaor. Regularity of minimal surfaces near quadratic cones. *arXiv preprint arXiv:1910.00441*, 2019.

[ES23] Nick Edelen and Gábor Székelyhidi. A liouville-type theorem for cylindrical cones. *arXiv preprint arXiv:2301.05967*, 2023.

[Fed69] Herbert Federer. *Geometric Measure Theory*, volume 153 of *Grundlehren der Math. Wiss.* Springer-Verlag, New York, 1969.

[Fed70] Herbert Federer. The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. *Bulletin of the American Mathematical Society*, 76(4):767–771, July 1970.

[FF60] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.

[FK85] Dirk Ferus and Hermann Karcher. Non-rotational minimal spheres and minimizing cones. *Commentarii Mathematici Helvetici*, 60(1):247–269, 1985.

[FKM81] Dirk Ferus, Hermann Karcher, and Hans-Friedrich Münzner. Cliffordalgebren und neue isoparametrische hyperflächen. *Mathematische Zeitschrift*, 177(4):479–502, 1981.

[GcLZ20] Qiang Guang, Martin Man chun Li, and Xin Zhou. Curvature estimates for stable free boundary minimal hypersurfaces. *Journal für die reine und angewandte Mathematik*, 2020(759):245 – 264, 2020.

[GH80] David Gilbarg and Lars Hörmander. Intermediate schauder estimates. *Archive for Rational Mechanics and Analysis*, 74(4):297–318, 1980.

[Giu11] E. Giusti. *Minimal Surfaces with Obstacles*, pages 119–153. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.

[GP71] Mariano Giaquinta and L. Pepe. Esistenza e regolarità per il problema dell'area minima con ostacoli in n variabili. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 3, 25(3):481–507, 1971.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, New York, 2001. reprint of the 1998 edition.

[HI01] Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the riemannian penrose inequality. *Journal of Differential Geometry*, 59(3):353–437, 2001.

[HLJ71] Wu-yi Hsiang and H Blaine Lawson Jr. Minimal submanifolds of low cohomogeneity. *Journal of Differential Geometry*, 5(1-2):1–38, 1971.

[HS79] Robert Hardt and Leon Simon. Boundary regularity and embedded solutions for the oriented plateau problem. *Annals of Mathematics*, 110(3):439–486, 1979.

[HS85] Robert Hardt and Leon Simon. Area minimizing hypersurfaces with isolated singularities. *Journal für die reine und angewandte Mathematik*, 362:102–129, 1985.

[Ilm92] Tom Ilmanen. Generalized flow of sets by mean curvature on a manifold. *Indiana University mathematics journal*, pages 671–705, 1992.

[Ilm93] Tom Ilmanen. The level-set flow on a manifold. *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, 54:193–204, 1993.

[Ilm94] Tom Ilmanen. *Elliptic regularization and partial regularity for motion by mean curvature*, volume 520. American Mathematical Soc., 1994.

[Ilm96] T. Ilmanen. A strong maximum principle for singular minimal hypersurfaces. *Calculus of Variations and Partial Differential Equations*, 4(5):443–467, August 1996.

[Ilm98] Tom Ilmanen. Lectures on mean curvature flow and related equations. 1998.

[IMN18] Kei Irie, Fernando C. Marques, and André Neves. Density of minimal hypersurfaces for generic metrics. *Ann. of Math. (2)*, 187(3):963–972, 2018.

[ISZ98] Tom Ilmanen, Peter Sternberg, and William P Ziemer. Equilibrium solutions to generalized motion by mean curvature. *The Journal of Geometric Analysis*, 8(5):845–858, 1998.

[Law69] H Blaine Lawson. Local rigidity theorems for minimal hypersurfaces. *Annals of Mathematics*, pages 187–197, 1969.

[Law72] H Blaine Lawson. The equivariant plateau problem and interior regularity. *Transactions of the American Mathematical Society*, 173:231–249, 1972.

[Law91] Gary Reid Lawlor. *A sufficient criterion for a cone to be area-minimizing*, volume 446. American Mathematical Soc., 1991.

[Li19] Yangyang Li. Existence of Infinitely Many Minimal Hypersurfaces in Higher-dimensional Closed Manifolds with Generic Metrics. *arXiv:1901.08440 [math], To appear in Journal of Differential Geometry*, January 2019.

[Li20a] Yangyang Li. An Improved Morse Index Bound of Min-Max Minimal Hypersurfaces. *arXiv:2007.14506 [math]*, August 2020.

[Li20b] Yangyang Li. On the Existence of Minimal Hypersurfaces with Arbitrarily Large Area and Morse Index. *arXiv:2001.07689 [math], to appear in Geometry & Topology*, January 2020.

[Lin85] Fang Hua Lin. *Regularity for a Class of Parametric Obstacle Problems (Integrand, Integral Current, Prescribed Mean Curvature, Minimal Surface System)*. PhD thesis, 1985.

[Lin87a] Fang Hua Lin. Approximation by smooth embedded hypersurfaces with positive mean curvature. *Bulletin of the Australian Mathematical Society*, 36(2):197–208, 1987.

[Lin87b] Fang Hua Lin. Minimality and stability of minimal hypersurfaces in $\mathbb{R}^n$. *Bulletin of the Australian Mathematical Society*, 36(2):209–214, 1987.

[Liu21] Zhenhua Liu. On a conjecture of almgren: area-minimizing surfaces with fractal singularities. *arXiv preprint arXiv:2110.13137*, 2021.

[Liu22] Zhenhua Liu. Homologically area minimizing surfaces with non-smoothable singularities. *arXiv preprint arXiv:2206.08315*, 2022.

[LMN18] Yevgeny Liokumovich, Fernando C. Marques, and André Neves. Weyl law for the volume spectrum. *Ann. of Math. (2)*, 187(3):933–961, 2018.

[Loh18] Joachim Lohkamp. Minimal smoothings of area minimizing cones. *arXiv preprint arXiv:1810.03157*, 2018.

[Loh19] Joachim Lohkamp. Potential theory on minimal hypersurfaces i: Singularities as martin boundaries. *Potential Analysis*, pages 1–36, 2019.

[LS71] H. Lewy and G. Stampacchia. On existence and smoothness of solutions of some non-coercive variational inequalities. *Arch. Rational Mech. Anal.*, 1971.

[LW20] Yangyang Li and Zhihan Wang. Generic regularity of minimal hypersurfaces in dimension 8. *arXiv preprint arXiv:2012.05401*, 2020.

[LW22] Yangyang Li and Zhihan Wang. Minimal hypersurfaces for generic metrics in dimension 8. *arXiv preprint arXiv:2205.01047*, 2022.

[LZ21] Martin Man-Chun Li and Xin Zhou. Min-max theory for free boundary minimal hypersurfaces, i: Regularity theory. *Journal of Differential Geometry*, 118(3):487–553, 2021.

[Maz17] Laurent Mazet. Minimal hypersurfaces asymptotic to simons cones. *Journal of the Institute of Mathematics of Jussieu*, 16(1):39–58, 2017.

[Mir71] Mario Miranda. Frontiere minimali con ostacoli. *Annali dell'Universita di Ferrara*, 16(1):29–37, 1971.

[MMN20] Fernando C Marques, Rafael Montezuma, and André Neves. Morse inequalities for the area functional. *arXiv preprint arXiv:2003.01301*, 2020.

[MN14] Fernando C. Marques and André Neves. Min-max theory and the Willmore conjecture. *Ann. of Math. (2)*, 179(2):683–782, 2014.

[MN16] Fernando C. Marques and André Neves. Morse index and multiplicity of min-max minimal hypersurfaces. *Camb. J. Math.*, 4(4):463–511, 2016.

[MN17] Fernando C. Marques and André Neves. Existence of infinitely many minimal hypersurfaces in positive Ricci curvature. *Invent. Math.*, 209(2):577–616, 2017.

[MN21] Fernando C. Marques and André Neves. Morse index of multiplicity one min-max minimal hypersurfaces. *Adv. Math.*, 378:Paper No. 107527, 58, 2021.

[MNS19] Fernando C. Marques, André Neves, and Antoine Song. Equidistribution of minimal hypersurfaces for generic metrics. *Invent. Math.*, 216(2):421–443, 2019.

[Mon20] Rafael Montezuma. A mountain pass theorem for minimal hypersurfaces with fixed boundary. *Calculus of Variations and Partial Differential Equations*, 59(6):1–30, 2020.

[MS94] Rafe Mazzeo and Nathan Smale. Perturbing away higher dimensional singularities from area minimizing hypersurfaces. *Communications in Analysis and Geometry*, 2(2):313–336, 1994.

[NV20] Aaron Naber and Daniele Valtorta. The singular structure and regularity of stationary varifolds. *Journal of the European Mathematical Society*, 22(10):3305–3382, July 2020.

[Per02] Oscar Perdomo. First stability eigenvalue characterization of clifford hypersurfaces. *Proceedings of the American Mathematical Society*, 130(11):3379–3384, 2002.

[Pet06] P. Petersen. *Riemannian Geometry*. NY: Springer Science + Business Media, LLC, 2006.

[Pit81] Jon T. Pitts. *Existence and Regularity of Minimal Surfaces on Riemannian Manifolds*, volume 27 of *Math. Notes*. Princeton Univ. Press, Princeton, N.J., 1981.

[Rad30] Tibor Rado. On Plateau's Problem. *Annals of Mathematics*, 31(3):457–469, 1930.

[Rei60] Ernst Robert Reifenberg. Solution of the plateau problem for m-dimensional surfaces of varying topological type. *Acta Mathematica*, 104(1-2):1–92, 1960.

[Sha17] Ben Sharp. Compactness of minimal hypersurfaces with bounded index. *J. Differential Geom.*, 106(2):317–339, 2017.

[Sim68] James Simons. Minimal Varieties in Riemannian Manifolds. *Annals of Mathematics*, 88(1):62–105, 1968.

[Sim74] Plinio Simoes. On a class of minimal cones in $\mathbb{R}^n$. *Bull. Am. murh. Sot. 80, 488*, 489, 1974.

[Sim83a] Leon Simon. Asymptotics for a Class of Non-Linear Evolution Equations, with Applications to Geometric Problems. *Annals of Mathematics*, 118(3):525–571, 1983.

[Sim83b] Leon Simon. *Lectures on Geometric Measure Theory*, volume 3 of *Proc. Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[Sim87] Leon Simon. A strict maximum principle for area minimizing hypersurfaces. *Journal of Differential Geometry*, 26(2):327–335, 1987.

[Sim93] Leon Simon. Rectifiability of the singular sets of multiplicity 1 minimal surfaces and energy minimizing maps. *Surveys in differential geometry*, 2(1):246–305, 1993.

[Sim08] Leon Simon. A general asymptotic decay lemma for elliptic problems. *arXiv preprint arXiv:0806.0462*, 2008.

[Sim21a] Leon Simon. A liouville-type theorem for stable minimal hypersurfaces. *arXiv preprint arXiv:2101.06404*, 2021.

[Sim21b] Leon Simon. Stable minimal hypersurfaces in $\mathbb{R}^{N+1+\ell}$ with singular set an arbitrary closed k in $\{0\} \times \mathbb{R}^{\ell}$. *arXiv preprint arXiv:2101.06401*, 2021.

[Sma93] Nathan Smale. Generic regularity of homologically area minimizing hypersurfaces in eight dimensional manifolds. *Communications in Analysis and Geometry*, 1(2):217–228, 1993.

[Sma99] Nathan Smale. Singular homologically area minimizing surfaces of codimension one in riemannian manifolds. *Inventiones mathematicae*, 135(1):145–183, 1999.

[Sma00] Nathan Smale. A construction of homologically area minimizing hypersurfaces with higher dimensional singular sets. *Transactions of the American Mathematical Society*, 352(5):2319–2330, 2000.

[Son18] Antoine Song. Existence of infinitely many minimal hypersurfaces in closed manifolds. *arXiv preprint arXiv:1806.08816*, 2018.

[SS81] Richard Schoen and Leon Simon. Regularity of stable minimal hypersurfaces. *Comm. Pure Appl. Math.*, 34(6):741–797, 1981.

[SS82] Richard Schoen and Leon Simon. A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals. *Indiana University Mathematics Journal*, 31(3):415–434, 1982.

[SS86] Leon Simon and Bruce Solomon. Minimal hypersurfaces asymptotic to quadratic cones in $\mathbb{R}^{n+1}$. *Inventiones mathematicae*, 86(3):535–551, 1986.

[SW89] Bruce Solomon and Brian White. A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals. *Indiana University Mathematics Journal*, 38(3):683–691, 1989.

[SY79] Richard Schoen and Shing-Tung Yau. On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*, 65(1):45–76, 1979.

[SY81] Richard Schoen and Shing-Tung Yau. Proof of the positive mass theorem. ii. *Communications in Mathematical Physics*, 79(2):231–260, 1981.

[SY17] Richard Schoen and Shing-Tung Yau. Positive scalar curvature and minimal hypersurface singularities. *arXiv preprint arXiv:1704.05490*, 2017.

[Szé21] Gábor Székelyhidi. Minimal hypersurfaces with cylindrical tangent cones. *arXiv preprint arXiv:2107.14786*, 2021.

[Tam82] Italo Tamanini. Boundaries of caccioppoli sets with hölder-continuois normal vector. *Journal für die reine und angewandte Mathematik*, 334:27–39, 1982.

[TZ20] Zizhou Tang and Yongsheng Zhang. Minimizing cones associated with isoparametric foliations. *Journal of Differential Geometry*, 115(2):367–393, 2020.

[Wan20] Zhihan Wang. Deformations of singular minimal hypersurfaces i, isolated singularities. *arXiv preprint arXiv:2011.00548*, 2020.

[Wan22a] Zhihan Wang. Mean convex smoothing of mean convex cones. *arXiv preprint arXiv:2202.07851*, 2022.

[Wan22b] Zhihan Wang. Min–max minimal hypersurfaces with obstacle. *Calculus of Variations and Partial Differential Equations*, 61(5):1–26, 2022.

[Whi89] Brian White. New applications of mapping degrees to minimal surface theory. *Journal of Differential Geometry*, 29(1):143–162, 1989.

[Whi91] Brian White. The space of minimal submanifolds for varying Riemannian metrics. *Indiana Univ. Math. J.*, 40(1):161–200, 1991.

[Whi94] Brian White. A strong minimax property of nondegenerate minimal submanifolds. *Journal für die reine und angewandte Mathematik*, 457:203–218, 1994.

[Whi95] Brian White. The topology of hypersurfaces moving by mean curvature. *Communications in analysis and geometry*, 3(2):317–333, 1995.

[Whi10] Brian White. The maximum principle for minimal varieties of arbitrary codimension. *Communications in Analysis and Geometry*, 18(3):421–432, 2010.

[Whi17] Brian White. On the bumpy metrics theorem for minimal submanifolds. *Amer. J. Math.*, 139(4):1149–1155, 2017.

[Wic14] Neshan Wickramasekera. A general regularity theory for stable codimension 1 integral varifolds. *Annals of Mathematics*, 179(3):843–1007, May 2014.

[WS94] Qi-Ming Wang and I Sterling. On a class of minimal hypersurfaces inr n. *Mathematische Annalen*, 298(1):207–251, 1994.

[Yau82] Shing Tung Yau. Problem section. In *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 669–706. Princeton Univ. Press, Princeton, N.J., 1982.

[Zha18] Yongsheng Zhang. On realization of tangent cones of homologically area-minimizing compact singular submanifolds. *Journal of Differential Geometry*, 109(1):177–188, 2018.

[Zho20] Xin Zhou. On the multiplicity one conjecture in min-max theory. *Ann. of Math. (2)*, 192(3):767–820, 2020.

[Zhu18] Jonathan J Zhu. First stability eigenvalue of singular minimal hypersurfaces in spheres. *Calculus of Variations and Partial Differential Equations*, 57(5):1–13, 2018.

[ZZ20] Xin Zhou and Jonathan Zhu. Existence of hypersurfaces with prescribed mean curvature I—generic min-max. *Camb. J. Math.*, 8(2):311–362, 2020.