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Mechanisms of Lagrangian Analyticity in Fluids

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Mechanisms of Lagrangian Analyticity in Fluids

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

Certain systems of inviscid fluid dynamics have the property that for solutions with just a modest amount of regularity in Eulerian variables, the corresponding Lagrangian trajectories are analytic in time. We elucidate the mechanisms in fluid dynamics systems that give rise to this automatic Lagrangian analyticity, as well as mechanisms in some particular fluids systems which prevent it from occurring.

We give a conceptual argument for a general fluids model which shows that the fulfillment of a basic set of criteria results in the analyticity of the trajectory maps in time. We then apply this to the incompressible Euler equations, obtaining analyticity for vortex patch solutions in particular. We also use the method to prove the Lagrangian trajectories are analytic for solutions to the pressureless Euler-Poisson equations, for initial data with moderate regularity.

We then examine the compressible Euler equations, and find that the finite speed of propagation in the system is incompatible with the Lagrangian analyticity property. By taking advantage of this finite speed we are able to construct smooth initial data with the property that some corresponding Lagrangian trajectory is not analytic in time. We also study the Vlasov-Poisson system on $\mathbb{R}^2 \times \mathbb{R}^2$, uncovering another mechanism that damages the potential analytic properties of the trajectory map. In this instance, we find that a key nonlocal operator does not preserve analytic dependence in time. As a result, we can construct highly regular initial data with corresponding Lagrangian phase space flows that are not analytic as functions of time into the space of $C^1(\mathbb{R}^4, \mathbb{R}^4)$ diffeomorphisms.

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To my family.

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Chapter 1

Introduction

In previous studies of several systems of PDE in fluid mechanics, it has been observed that when one imposes only a small amount of regularity in the initial data, the trajectories of fluid particles for the corresponding system are in fact analytic in the time variable while the solution exists. The goal of this thesis is to illuminate the underlying mechanisms in physical systems that lead to this automatic analyticity of Lagrangian trajectories, as well as those that prevent it from occurring.

In general, one can examine this property for certain PDE involving a velocity field, a time-dependent vector function $u(x,t)$ on $\mathbb{R}^d \times \mathbb{R}$, as an unknown in the system. The corresponding trajectory map $X(\alpha,t)$ then satisfies the ODE system

$$\begin{aligned} \frac{dX}{dt}(\alpha,t) &= u(X(\alpha,t),t), \\ X(\alpha,0) &= \alpha, \end{aligned} \tag{1.1}$$

where $u(x,t)$ satisfies the original PDE one started with.

Writing the original PDE in terms of $u(X(\alpha,t),t)$, rather than $u(x,t)$, (and making the analogous replacements for other potential unknowns in the original system) yields what is often referred to as the PDE written in Lagrangian variables. The intuitive idea is that instead of thinking of the unknown quantities in the system as functions of fixed points in space $x \in \mathbb{R}^d$ (and time), one can think of them as functions with various particles (and time) as their inputs, in a sense, where the particles are labelled by their starting positions $\alpha \in \mathbb{R}^d$. More concretely, the composition of a function f with $X(\alpha,t)$ transforms the function f into one which takes an initial position $\alpha \in \mathbb{R}^d$ as the input, and returns the value of f obtained as one follows the associated particle and samples f at the particle's location.

For certain types of equations, by using the original PDE, the evolution of the tangent of a trajectory, given by $u(X(\alpha,t),t)$, can be described in terms of a nonlocal operator acting on the initial velocity field $u_0 = u|_{t=0}$ and the trajectory map X, and the equation (1.1) for X can be expressed as

$$\frac{dX}{dt}(\alpha,t) = F[X,u_0](\alpha,t), \tag{1.2}$$

where the operator F may also depend on the initial data for other potential unknowns in the original PDE. The fact that there is no dependence on $u(x,t)$ at times $t > 0$ in the right hand side of (1.2) may allow one to solve for the trajectory map X without knowledge of the solution u to the original PDE at later times. In many such PDE one observes that u and the other unknowns in the original equation are determined by the solution X to the equation (1.2) with initial data $X|_{t=0} = Id$.

Here we examine several systems of PDE that have such a Lagrangian formulation. The first result we give is a conceptual method that proves that the Lagrangian trajectories for such a system are analytic in time given that two basic criteria hold for the ODE (1.2). The first criterion is that for the particular choice of initial data u_0, the operator $F[\cdot,u_0]$ is locally bounded, in some sense, when acting on functions in a suitable function space, and the second is that the operator preserves analytic dependence in time when one inputs an analytic function of time $X(\alpha,t)$ in the corresponding argument.

As a basic application, we use these results to obtain a relatively short proof which recovers the known result of Lagrangian analyticity for solutions to the 2D incompressible Euler equations with Hölder continuous initial vorticity. With this application, we demonstrate how the tools in the previous section are used to prove the Lagrangian analyticity property for a given system in a simple setting. Specifically, for this example we consider the case of the system on the flat torus.

We then use the method to prove Lagrangian analyticity for vortex patch solutions to the 2D incompressible Euler equations, a case in which the gradient of the velocity field has discontinuities. This yields a result for lower regularity solutions than established in past studies. In the previous results the initial data is assumed to be slightly better than differentiable everywhere.

In the section that follows, we use the method to prove the analyticity of trajectories for the pressureless Euler-Poisson system for initial data with moderate spatial regularity. This system models stellar dynamics, in which particles experience a long-range attractive force, as well as the dynamics of charged fluids, in which the particles experience repulsion, when the effects of pressure are negligible. One new obstacle over the system handled in the previous section is that the flows for

the Euler-Poisson system are not necessarily incompressible. For this result, we provide formulas for taking spatial derivatives of the quantity $F[X, u_0]$ to arbitrary order and estimates bounding the L^2 norms of the results in a uniform way. While the proofs in Section 3 used techniques more specific to the setting there, and are likely sharper in terms of the regularity of the initial data, the style of proof in Section 4 used for the Euler-Poisson system, in terms of Sobolev spaces, seems more robust and likely applies formulaically with almost no changes to most fluid equations with Lagrangian form (1.2), where F is assumed to satisfy the basic criteria discussed in Section 2.

In the next section, we consider the compressible Euler equations, which turn out to be quite different from the incompressible Euler equations with regard to the analyticity of trajectories. We show that in fact one can construct C^∞ initial data which has trajectories that are not real analytic in time at $t = 0$. This appears to be the first result that gives a counterexample to Lagrangian analyticity for a fluid mechanics system, in the sense that it proves for the system that no matter how smooth one requires the initial data to be, the corresponding Lagrangian trajectories need not be analytic. We find that the finite speed of propagation for the compressible Euler equations is a key mechanism which is incompatible with the analyticity of the trajectories. This idea results in a relatively transparent counterexample of a fluid equation in which smooth data can easily lead to non-analytic trajectories.

Finally, we consider the Lagrangian formulation for the Vlasov-Poisson equation, in which the unknown $f(x, v, t) : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a probability distribution governing the velocities and spatial positions of a large number of particles under the effects of either a force of mutual attraction or a repulsive force. The natural Lagrangian formulation associated with this problem is in terms of a phase space trajectory $Z(t) = (X(t), V(t))$ in $\mathbb{R}^2 \times \mathbb{R}^2$, say with $Z(0) = (x_0, v_0)$, where $Z(t)$ satisfies an ODE of the form (1.2). There are several interesting features the Vlasov-Poisson system shares with the incompressible Euler equations. For one, f is conserved along trajectories, just as vorticity is in the 2D incompressible Euler equations. Secondly the phase space flow for Vlasov-Poisson is also volume-preserving. Third, both systems have an infinite speed of propagation. In this case we found that, despite these similarities, the situation is very different with regard to the analyticity of trajectories. For the Vlasov-Poisson system, we are able to construct examples of C^n initial data for any $n \geq 10$ with the property that the corresponding trajectory map is not uniformly analytic in time, in a sense. While this result does not guarantee the existence of a non-analytic trajectory, it does provide examples in which the flow cannot be analytic as a function of time into the space of C^1 maps on $\mathbb{R}^2 \times \mathbb{R}^2$. On the other hand, for the incompressible Euler equations and the Euler-Poisson equations, the corresponding flows are in fact analytic as functions of time into the corresponding

spaces of C^1 maps. The essential difference for the Vlasov-Poisson system is that the corresponding operator to $F[\cdot, u_0]$ in (1.2) does not preserve the analyticity of time dependence in this case.

Results showing analyticity of trajectories for solutions to fluids equations amidst low spatial regularity have been discussed in [5, 9, 11, 17, 18]. These studies treat cases in which the initial data is slightly better than C^1, showing that this typically leads to analytic trajectories in time for the solutions. In [17], the author proposes an argument for this for the incompressible Euler equations: to show that the trajectory map solves an ODE in an abstract function space which must have analytic solutions in time. In [11], a recursive formula for the time derivatives of the trajectory map is used to show analyticity for the trajectories of solutions to the 3D incompressible Euler equations on the torus. In [9], the 2D SQG equations are considered; here the authors also use a recursive formula and then apply combinatorial identities to calculate and bound the nth order time derivatives of the trajectory map. In [18], a more abstract setting is used to prove the result for the incompressible Euler equations in arbitrary dimensions, where the flows of the solutions are handled as geodesics in the space of volume-preserving diffeomorphisms. Our proofs of analyticity revolve around a conceptual strategy of showing that the trajectories for a fluid system solve a particular kind of ODE, which then implies they must be analytic in time. The basic idea is in some ways similar to that in the approach of [17], though we completely rigorously develop such a method into a systematic technique for proving Lagrangian analyticity-type results. Indeed, in Section 2 we demonstrate how the method directly generalizes to various fluid mechanics systems, given that they satisfy certain key properties. To the best of the author's knowledge, the counterexample-type results in this thesis to the Lagrangian analyticity property are the first in the literature. Overall, we hope that the work here lucidly explains the essential reasons some fluid mechanics systems do have the Lagrangian analyticity property, and certain fluids mechanics systems do not.

Chapter 2

A general ODE method for proving Lagrangian analyticity

In this section, we provide a strategy for proving Lagrangian analyticity results for fluid mechanics systems. One can often exploit the structure of such fluids equations, such as the incompressible Euler equations, to represent their corresponding trajectory maps as solutions of a relatively tractable ODE in an abstract function space. The main result in this section is a proof that if one has an ODE in an abstract function space, say for the trajectory maps, which is of a particular form, then the solutions are analytic in time. This results in a conceptual method for proving Lagrangian analyticity. Indeed, criteria leading to existence and uniqueness, as well as analyticity, of solutions to ODE are not overly technical, so the core parts of our strategy are clear-cut and easily summarized.

The first step is to reformulate the PDE in Lagrangian variables in the manner described in the introduction. Let us take the $\mathbb{R}^d$-valued function $u(x,t)$ on $\mathbb{R}^d \times \mathbb{R}$ to be the velocity field associated to the original PDE. Then the trajectory map $X(\alpha,t)$ is the solution to the equation

$$\begin{aligned} \frac{dX}{dt}(\alpha,t) &= u(X(\alpha,t),t), \\ X(\alpha,0) &= \alpha. \end{aligned} \tag{2.1}$$

For this section, we assume that one is able to rewrite this system in the form

$$\begin{aligned} \frac{dX}{dt} &= \tilde{F}(X), \\ X|_{t=0} &= Id, \end{aligned} \tag{2.2}$$

for an operator $\tilde{F}$ defined on a set of maps from $\mathbb{R}^d$ to $\mathbb{C}^d$, depending on $u_0 = u|_{t=0}$ and the other initial data prescribed for the original PDE. We give explicit derivations for the corresponding operators $\tilde{F}$ in the case of the incompressible Euler equations in Section 3 and in the case of the Euler-Poisson equation in Section 4.

We have yet to discuss some simple matters such as the domain and range of the operator $\tilde{F}$ in (2.2). We now address this, as well as give a complete discussion on the structure required for our strategy.

2.1 The basic structure

Let us consider an ODE of the form

$$\frac{dX}{dt} = \tilde{F}(X), \quad X = Id \quad \text{at } t = 0, \tag{2.3}$$

for unknown $X(t)$, with $(X(t) - Id) \in Y$ for each t, where Y is a Banach space of maps from a fixed open subset $\Omega \subset \mathbb{R}^d$ to $\mathbb{C}^d$, for some positive integer d. Moreover, we require that $\|X(t) - Id\|_Y \leq \delta$, for some $\delta > 0$, and that the operator $\tilde{F}$ is defined on the set $\{X : \Omega \to \mathbb{C}^d : \|X - Id\|_Y \leq \delta\}$, taking values in Y. We work under the assumption that Y is a Banach space of continuous maps with a norm that dominates the $C^{0,1}(\Omega)$ norm, i.e. with the property that

$$\|f\|_{C^{0,1}(\Omega)} = \sup_{x \in \Omega} |f(x)| + \sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|} \leq C\|f\|_Y. \tag{2.4}$$

In this section we give criteria for the operator $\tilde{F}$ which are relatively simple and which guarantee that solutions $X(t)$ are analytic in time in a small disc in $\mathbb{C}$ as maps into Y, with Theorem 2.2.6. We now give an overview of these criteria.

A summary of the key properties of $\tilde{F}$ that lead to analytic trajectories

- **Local boundedness**

The first criterion is that $\tilde{F}$ satisfies a local boundedness property near the identity map. To say this is to assert that for a fixed $\delta > 0$, $\tilde{F}$ sends maps X in the closed δ-ball about the identity in Y into Y, satisfying a bound of the form

$$\sup\{\|\tilde{F}(X)\|_Y : \|X - Id\|_Y \leq \delta\} \leq C, \tag{2.5}$$

for some C, possibly dependent on the initial data under consideration.

We present the following, Proposition 3.2.4, proven in the next section, which gives us this property for the operator for the incompressible Euler equations on the 2D torus in the case of Hölder continuous initial vorticity.

Proposition. *Fix $\mu \in (0,1)$. Assume $\omega := \operatorname{curl} u_0 \in C^{0,\mu}(\mathbb{T}^2)$. Let $F_T(X)$ be as defined in* (3.17)*. Then there exists a $\delta > 0$ such that*

$$\sup\{\|F_T(X)\|_{C^{1,\mu}(\mathbb{T}^2)} : \|X - Id\|_{C^{1,\mu}(\mathbb{T}^2)} \leq \delta\} \leq C, \tag{2.6}$$

where C depends only on ω.

To consider the case of vortex patch initial data for the incompressible Euler equations in Section 3, we take Y to be the set of $C^{1,\mu}$ maps on Ω, where Ω is the initial patch. We prove the corresponding local boundedness result for vortex patches with Proposition 3.3.6. In addition, in Section 4, for the pressureless Euler-Poisson system we give a proof of the corresponding bound for the analogous operator for H^s maps from $\mathbb{R}^3$ to $\mathbb{C}^3$, $s \geq 6$, with Proposition 4.3.7.

The analyticity of trajectories also relies critically on the analytic properties of the operator $\tilde{F}$, which are related to the remaining major criterion for analytic trajectories.

- **Preservation of analytic time dependence**

Now we consider the composition of $\tilde{F}$ with maps dependent on an additional, complex parameter z, which plays the role of the time variable. Precisely, we compose it with a function of z in a complex disc, $X_z : \Omega \to \mathbb{C}^d$ with $(X_z - Id) \in Y$ for each z. Let us denote $d_r := \{z \in \mathbb{C} : |z| < r\}$ for $r > 0$. The desired property of $\tilde{F}$ is that the following implication holds for any such X_z mapping into the domain of $\tilde{F}$ and any $r > 0$:

$$\begin{aligned} &\text{If for each } \alpha \in \Omega \text{ we have } X_z(\alpha) : d_r \to \mathbb{C}^d \text{ is analytic in } z, \\ &\text{then for each } \alpha \in \Omega \text{ we have } F(X_z)(\alpha) : d_r \to \mathbb{C}^d \text{ is analytic in } z. \end{aligned} \tag{2.7}$$

In this case we say the operator $\tilde{F}$ preserves analyticity ((ii) of Definition 2.2.3). Such a property is quite natural if we expect to have analytic trajectories. Note that if $X(\alpha, t)$ solves (2.3) and is analytic in t, t playing the role of z, and we take the composition $\tilde{F}(X(\cdot, t))(\alpha)$, this returns the tangent vector $\frac{dX}{dt}(\alpha, t)$, which must be analytic in t. It is verified with Lemma 3.1.4 that for vortex

patch solutions to 2D Euler, the corresponding operator F preserves analyticity in this sense. In Section 4, we show the analogous result holds for the Euler-Poisson system.

With Theorem 2.2.6 we prove that if $\tilde{F}$ is locally bounded near the identity map in Y and preserves analyticity, then there is a unique analytic Y-valued function on a disc in $\mathbb{C}$ solving the equation (2.3).

We verify that both of the above criteria are satisfied for the 2D incompressible Euler equations with Hölder continuous initial vorticity, and vortex patch solutions, in Section 3. Thus with Theorem 2.2.6, we get analyticity of trajectories for both cases. We also prove the criteria are satisfied for the 3D Euler-Poisson system in Section 4, which yields a proof of analytic trajectories for this system as well.

Additionally, we examine some examples of fluid mechanics systems for which the above criteria do not hold. We do indeed find that in these cases the systems do not have the same Lagrangian analytic properties as the incompressible Euler equations and the Euler-Poisson system, as we explain with the following remark.

Remark 2.1.1. *(i) For the compressible Euler equations, it is not clear how one might express the system as an ODE of the form* (2.3)*. In this case, there exists C^∞ initial data for which the solution has some trajectory which is not analytic in time. This is proved in Section 5.*

(ii) The Vlasov-Poisson system, examined in Section 6, has a natural Lagrangian formulation of the form (2.3)*, but the corresponding operator does not satisfy the preservation of analyticity property,* (2.7)*. For this system, we construct C^n initial data for any $n \geq 10$ such that the following holds: there is a specified compact set such that if all the trajectories starting in the set are analytic in time at $t = 0$, then there is no uniform, positive lower bound among these trajectories on the radii of analyticity. In particular, this implies that the corresponding trajectory map cannot be analytic as a function of time into the space of C^1 maps. On the other hand, for the incompressible Euler and the Euler-Poisson systems handled in Sections 3 and 4, we find the trajectory maps* are *analytic as functions of time into the corresponding spaces of C^1 maps.*

2.2 Analytic solutions in time for the ODE in an abstract function space

Now let us put the criteria discussed above on a firm footing. We write X_z to indicate a function dependent on a parameter $z \in \mathbb{C}$, with $(X_z - Id)$ taking values in a Banach space Y of continuous

maps from an open subset Ω of $\mathbb{R}^d$ to $\mathbb{C}^d$. As mentioned above, we make the assumption that the $\|\cdot\|_Y$ norm dominates the $C^{0,1}$ norm.

Assumption 2.2.1. *$Y \subset C(\Omega)$ is a Banach space of maps from $\Omega \subset \mathbb{R}^d$ to $\mathbb{C}^d$ with the property that*

$$\|f\|_{C^{0,1}(\Omega)} \leq C\|f\|_Y \text{ for all } f \in Y. \tag{2.8}$$

Moreover, we specifically consider maps within a distance δ of the identity map in the Y norm, for some $\delta > 0$.

Definition 2.2.2. *Given a Banach space Y satisfying Assumption 2.2.1 we define (i)*

$$B_\delta := \{X : \|X - Id\|_Y \leq \delta\}, \tag{2.9}$$

(ii)

$$\mathcal{Y}_r := \{Y\text{-valued maps } X_z \text{ analytic in } z \text{ on } d_r\}, \tag{2.10}$$

where $d_r = \{|z| < r\}$, and (iii) we consider the set of maps which differ from the identity by an element of $\mathcal{Y}_r$, and which map into the ball B_δ as functions of z:

$$\mathcal{B}_{\delta,r} := \{\text{maps } X_{(\cdot)} : (X_{(\cdot)} - Id) \in \mathcal{Y}_r,\ X_z \in B_\delta \text{ for all } z \in d_r\}. \tag{2.11}$$

With this, we now have a setting in which we can verify whether or not for each fixed α, $\tilde{F}(X_z)(\alpha)$ inherits analyticity in z from the functions X_z analytic in z, that is, whether the property of preservation of analyticity holds.

Definition 2.2.3. *(i) For any function $\tilde{F} : B_\delta \to \{\mathbb{C}^2\text{-valued maps on } \Omega\}$, we define $\tilde{F}$ with domain $\mathcal{B}_{\delta,r}$ in the compatible way, mapping $X_{(\cdot)} \in \mathcal{B}_{\delta,r}$ to the function $\tilde{F}(X_{(\cdot)}) : d_r \to \{\mathbb{C}^2\text{-valued maps on } \Omega\}$ defined by*[1]

$$\tilde{F}(X_{(\cdot)})(z) := \tilde{F}(X_z) \quad \text{for } z \in d_r. \tag{2.12}$$

(ii) We say that $\tilde{F}$ preserves analyticity if (for any $r > 0$ for the disc of analyticity d_r) for any $X_{(\cdot)} \in \mathcal{B}_{\delta,r}$, at each fixed $\alpha \in \Omega$, $\tilde{F}(X_z)(\alpha)$ is analytic in d_r.

Note that so far we have only discussed how analyticity of $\tilde{F}(X_z)(\alpha)$ in z in a disc d_r for each fixed $\alpha \in \Omega$ is inherited from analyticity of $X_z(\alpha)$ in z in d_r for each $\alpha \in \Omega$. Now we discuss how

[1] Occasionally we use $\tilde{F}(X_z)$ to refer to the map $\tilde{F}(X_{(\cdot)})$.

to improve this to get the seemingly stronger conclusion that $\tilde{F}(X_z)$ is analytic in z in d_r as a map into the Banach space Y.

Lemma 2.2.4. *Let B_δ, $\mathcal{B}_{\delta,r}$, Y, and $\mathcal{Y}_r$ be as in Definition 2.2.2. Consider a function $\tilde{F} : B_\delta \to Y$. If we have a bound of the form*

$$\sup_{X \in B_\delta} \|\tilde{F}(X)\|_Y \leq C < \infty, \tag{2.13}$$

and if $\tilde{F}$ preserves analyticity, then in fact for any $r > 0$ as the radius of analyticity in the definition of $\mathcal{B}_{\delta,r}$, for $X_z \in \mathcal{B}_{\delta,r}$ we have that $\tilde{F}(X_z)$ is an analytic function from d_r into Y. That is,

$$\tilde{F} : \mathcal{B}_{\delta,r} \to \mathcal{Y}_r. \tag{2.14}$$

Proof. By the Cauchy integral formula, for each $\alpha \in \Omega$ and $z \in d_r$ we have

$$\frac{d}{dz}\left(\tilde{F}(X_z)(\alpha)\right) = \frac{1}{2\pi i}\int_{|\zeta - z| = r'} \frac{\tilde{F}(X_\zeta)(\alpha)}{(\zeta - z)^2} d\zeta, \tag{2.15}$$

for $r' < \text{dist}(z, \partial d_r)$. Thus by applying $\|\cdot\|_Y$ to both sides we obtain

$$\left\| \frac{d}{dz}\tilde{F}(X_z) \right\|_Y \leq \frac{C'}{\text{dist}(z, \partial d_r)} \sup_{|\zeta| < r} \|\tilde{F}(X_\zeta)\|_Y \leq \frac{C''}{\text{dist}(z, \partial d_r)}, \tag{2.16}$$

by (2.13), where in the left hand side we are considering the norm of the function mapping a given α to $\frac{d}{dz}\left(\tilde{F}(X_z)(\alpha)\right)$. Now it just remains to prove that as h tends to zero, the difference quotient $(\tilde{F}(X_{z+h}) - \tilde{F}(X_z))/h$ converges in the Y norm to the function $\alpha \mapsto \frac{d}{dz}\left(\tilde{F}(X_z)(\alpha)\right)$. Consider h of size much smaller than the distance from z to ∂d_r, and fix $r' > |h|$, with r' smaller than the distances of both z and $z + h$ to the boundary of d_r. Then

$$\begin{aligned} \frac{1}{h}(\tilde{F}(X_{z+h})(\alpha) - \tilde{F}(X_z)(\alpha)) &= \frac{1}{2\pi i h}\left(\int_{|\zeta - z| = r'} \frac{\tilde{F}(X_\zeta)(\alpha)}{\zeta - (z+h)} d\zeta - \int_{|\zeta - z| = r'} \frac{\tilde{F}(X_\zeta)(\alpha)}{\zeta - z} d\zeta \right) \\ &= \frac{1}{2\pi i}\int_{|\zeta - z| = r'} \frac{\tilde{F}(X_\zeta)(\alpha)}{(\zeta - (z+h))(\zeta - z)} d\zeta, \end{aligned} \tag{2.17}$$

and so we have

$$\begin{aligned}\frac{1}{h}(\tilde{F}(X_{z+h})(\alpha)-\tilde{F}(X_z)(\alpha))-\frac{d}{dz}(\tilde{F}(X_z)(\alpha)) &= \frac{1}{2\pi i}\Big(\int_{|\zeta-z|=r'}\frac{\tilde{F}(X_\zeta)(\alpha)}{(\zeta-(z+h))(\zeta-z)}d\zeta \\ &\qquad -\int_{|\zeta-z|=r'}\frac{\tilde{F}(X_\zeta)(\alpha)}{(\zeta-z)^2}d\zeta\Big), \\ &= \frac{h}{2\pi i}\int_{|\zeta-z|=r'}\frac{\tilde{F}(X_\zeta)(\alpha)}{(\zeta-(z+h))(\zeta-z)^2}d\zeta. \end{aligned} \tag{2.18}$$

We apply the Y norm to both sides to get the bound

$$\begin{aligned}\left\|\frac{1}{h}(\tilde{F}(X_{z+h})(\alpha)-\tilde{F}(X_z)(\alpha))-\frac{d}{dz}(\tilde{F}(X_z)(\alpha))\right\|_Y &\leq \frac{C'h}{(\mathrm{dist}(z,\partial d_r))^2}\sup_{|\zeta|<r}\|\tilde{F}(X_\zeta)\|_Y \\ &\leq \frac{C''h}{(\mathrm{dist}(z,\partial d_r))^2}. \end{aligned} \tag{2.19}$$

Since the right hand side is $O(|h|)$, the conclusion of the lemma follows. □

In the coming ODE argument, we will also need that the operator of concern is Lipschitz on some ball about the identity in Y. With the following lemma, we show this property also follows from the local boundedness property together with preservation of analyticity.

Lemma 2.2.5. *Consider a function $\tilde{F}: B_\delta \to Y$ and assume that we have a bound of the form*

$$\sup_{X\in B_\delta}\|\tilde{F}(X)\|_Y \leq C, \tag{2.20}$$

and that $\tilde{F}$ preserves analyticity. For any $\epsilon \leq \delta/10$ we have that the function $\tilde{F}$ with domain B_ϵ is Lipschitz with constant C_0 depending only on C.

Proof. Let us define $r_0 := \delta/4$. For distinct $X, Z \in B_\epsilon$, we may write

$$\tilde{F}(X)-\tilde{F}(Z) = \int_0^{\|X-Z\|_Y}\frac{d}{d\tau}\tilde{F}(X_\tau)d\tau, \tag{2.21}$$

where we define $X_\tau := Z + \tau(X-Z)/\|X-Z\|_Y$ for any $\tau \in \{|z|<r_0\}$. Note this contains $[0, 2\epsilon]$, and thus the interval $[0, \|X-Z\|_Y]$ integrated over in (2.21). Now we check that we have a suitable bound on $\frac{d}{d\tau}(\tilde{F}(X_\tau))$, uniform in $\tau \in [0, \|X-Z\|_Y]$. Since ϵ and r_0 are each less than $\delta/2$, we know that for all $|\tau| < r_0$ we have $X_\tau \in B_\delta$. Since then $X_{(\cdot)}$, defined on the disc $\{|z|<r_0\}$, is in $\mathcal{B}_{\delta,r_0}$, we know that $\tilde{F}(X_\tau)$ is analytic in τ in $\{|z|<r_0\}$ by Lemma 2.2.4, and moreover the bound (2.16)

with r_0 in place of r shows $\|\frac{d}{d\tau}\tilde{F}(X_\tau)\|_Y$ is bounded by some constant as τ varies in the smaller disc $\{|z| \leq 2\epsilon\}$. For this, we have used the fact that $r_0 - 2\epsilon \geq \delta/20 > 0$. We observe that the resulting constant is thus determined by δ and the bounding constant C from (2.20). The argument is concluded by applying $\|\cdot\|_Y$ to both sides of (2.21) and bounding this above by moving the norm into the integrand. □

Now suppose $\tilde{F}$ satisfies the hypotheses of Lemma 2.2.5, and let us pick a constant ϵ which satisfies the hypotheses of Lemma 2.2.5:

$$\epsilon := \frac{\delta}{10}. \tag{2.22}$$

By restricting the domain of such an $\tilde{F}$ to B_ϵ, we ensure that it is Lipschitz with constant C_0, from Lemma 2.2.5, which readies us for an application of a fixed point theorem. Note that for any $r > 0$ we have a natural choice of norm for $\mathcal{Y}_r$, with

$$\|X_z\|_{\mathcal{Y}_r} := \sup_{z \in d_r} \|X_z\|_Y. \tag{2.23}$$

Now we have established all the properties of $\tilde{F}$ that are required for solving the equation

$$\begin{aligned} \tfrac{d}{dt}X_t &= \tilde{F}(X_t), \\ X_t|_{t=0} &= Id, \end{aligned} \tag{2.24}$$

in the desired setting. We prove that the solution to the equation exists as a fixed point of a contraction mapping on the space of functions analytic in time as maps into Y. For a given $\tilde{F}$ and $r > 0$, we define the following mapping on functions $X_{(\cdot)} \in \mathcal{B}_{\epsilon,r}$:

$$\Phi_{\tilde{F}}\left(X_{(\cdot)}\right)(t) := Id + \int_0^t \tilde{F}(X_\zeta)d\zeta \qquad \text{for } t \in d_r, \tag{2.25}$$

where the integration is taken along the straight path from 0 to t. This is the map we will show has a fixed point that solves the ODE (2.24).

Theorem 2.2.6. *Suppose that $\tilde{F} : B_\delta \to Y$ satisfies a bound of the form* (2.13) *and that $\tilde{F}$ preserves analyticity. Then for $\epsilon = \delta/10$ and sufficiently small $r_1 > 0$, there exists a unique fixed point $X_{(\cdot)} \in \mathcal{B}_{\epsilon,r_1}$ of $\Phi_{\tilde{F}} : \mathcal{B}_{\epsilon,r_1} \to \mathcal{B}_{\epsilon,r_1}$, and the function $X(\alpha,t) = X_t(\alpha)$ is analytic in t on d_{r_1} and the unique solution to the equation* (2.24).

Proof. It is easy to check that $\Phi_{\tilde{F}}$ maps $\mathcal{B}_{\epsilon,r_1}$ to itself, given that t is restricted to d_{r_1} and r_1 is

chosen small enough. The bound (2.13) allows us to ensure that $\Phi_{\tilde{F}}(X_{(\cdot)})$ is close to Id for small r_1. Using Lemma 2.2.5, we have the Lipschitz property of $\tilde{F}$ on B_ϵ, and so we get that $\Phi_{\tilde{F}}$ is a contraction mapping on $\mathcal{B}_{\epsilon,r_1}$ as long as $r_1 < 1/(2C_0)$. The conclusion of the theorem now follows from the contraction mapping principle. □

Having established the more general material of this section, for our proofs that the Lagrangian trajectories for solutions to incompressible Euler, and to the Euler-Poisson systems are analytic, it remains for us to verify that the local boundedness property and the preservation of analyticity property both hold for the corresponding $\tilde{F}$ operators. In the next section we do so for the incompressible Euler equations, specifically for solutions with Hölder continuous initial vorticity, and for vortex patch solutions.

Chapter 3

Lagrangian analyticity for the incompressible Euler equations

The incompressible Euler equations on $\mathbb{R}^d$ are given by

$$\begin{aligned} \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= 0, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= u_0, \end{aligned} \tag{3.1}$$

where $u(x,t)$ is a vector field defined on $\mathbb{R}^d \times \mathbb{R}$, taking values in $\mathbb{R}^d$, and the pressure $p(x,t)$ is an unknown scalar, with initial data u_0 to be prescribed, for example, as a function in $C^{1,\mu}$ for some $\mu \in (0,1)$, or as an initial velocity field corresponding to a vortex patch. For the results of this section, we treat the case $d = 2$, but we comment that the methods in this thesis can be adapted to the 3D incompressible Euler equations as well. Let us proceed by showing how one frames the 2D incompressible Euler equations written in Lagrangian variables in the form of an ODE in an abstract function space, such as (2.3).

3.1 The Lagrangian setting for the incompressible Euler equations

The first thing to observe is that by taking the scalar curl of the first equation in (3.1) in the 2D case, we arrive at the transport equation for the vorticity, $\omega = \partial_1 u_2 - \partial_2 u_1$.

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = 0. \tag{3.2}$$

Thus the vorticity is transported by the flow, which is equivalently expressed by $\frac{d}{dt}\omega(X(\alpha,t),t) = 0$, or

$$\omega(X(\alpha,t),t) = \omega_0(\alpha), \tag{3.3}$$

where the initial vorticity is denoted by $\omega_0(\alpha) = \omega(\alpha,0)$. Here we remark that the vorticity *is not* transported by the flow in the 3D case, although its evaluation along a Lagrangian trajectory does obey a simple evolution, through the Cauchy vorticity formula, allowing a similar derivation to the one presented here.

Our main tool is the Biot-Savart law, which allows us to recover the velocity vector $u(x,t)$ from the vorticity ω, with

$$u(x,t) = \frac{1}{2\pi}\int \frac{(x-y)^\perp}{|x-y|^2}\omega(y,t)dy, \tag{3.4}$$

for the 2D case. In this expression, we use the notation $y^\perp = (-y_2, y_1)$. Considering the Lagrangian point of view, we may use this to find the velocity vector tangent to a specified point along a path given by $X(\alpha,t)$. Inputting this function into the above expression and making a change of variables yields

$$\frac{dX}{dt}(\alpha,t) = \frac{1}{2\pi}\int \frac{(X(\alpha,t)-X(\beta,t))^\perp}{|X(\alpha,t)-X(\beta,t)|^2}\omega_0(\beta)d\beta, \tag{3.5}$$

where we have taken into account (3.3) along with the area preserving condition satisfied by incompressible flows, i.e. that $\det \nabla X(\alpha,t) = 1$. This formula exhibits the main necessary qualities we will need in order to prove that the solution to this equation, $X(\alpha,t)$, must be analytic in time, assuming some mild criteria hold for ω_0. We denote the kernel appearing here by K, with

$$K(y) = \frac{1}{2\pi}\frac{(-y_2,y_1)}{y_1^2+y_2^2}. \tag{3.6}$$

Note that $K(y)$ analytically extends to complex pairs (y_1,y_2) outside of the set $\{y_1 \pm iy_2 = 0\}$. Considering the right hand side of (3.5), inside of the integral, we have an expression with very nice

dependence on the quantity $X(\alpha, t)$. The idea will be to regard the right hand side as a function F of the map $X_t = X(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^2$, and extend the definition of F to a well-behaved function defined on maps more generally from $\mathbb{R}^2$ to $\mathbb{C}^2$. Here the complex output is necessitated by our complexification of the time variable. In that formulation, the equation (3.5) becomes

$$\frac{d}{dt} X_t = F(X_t), \tag{3.7}$$

together with the initial data $X_t|_{t=0} = Id$.

Now we begin with a rigorous definition of the operator F for $d = 2$. In the following, the set $\Omega \subset \mathbb{R}^2$ is just intended to be some set containing the support of some initial vorticity prescribed for (3.1), and we now and afterwards denote the initial vorticity by $\omega(\alpha) := \operatorname{curl} u_0(\alpha)$ (rather than $\omega_0(\alpha)$).

Definition 3.1.1. *Fix a scalar, real-valued function $\omega \in L^\infty \cap L^1(\Omega)$ and an open set $\Omega \subset \mathbb{R}^2$. We consider the space $C^{0,1}(\Omega)$ of $\mathbb{C}^2$-valued, Lipschitz continuous functions on Ω. For a map $X \in C^{0,1}(\Omega)$ with $\|X - Id\|_{C^{0,1}(\Omega)} \leq \delta$, for a small fixed $\delta > 0$ to be specified later, we define*

$$F(X)(\alpha) = \int_\Omega K(X(\alpha) - X(\beta))\omega(\beta) d\beta. \tag{3.8}$$

In this definition K is regarded as the complex vector valued function

$$K(z) = \frac{1}{2\pi} \frac{(-z_2, z_1)}{z_1^2 + z_2^2}, \quad z_1 \pm i z_2 \neq 0. \tag{3.9}$$

Let us note that for X sufficiently close to Id, say as in Definition 3.1.1, the denominator of $K(X(\alpha) - X(\beta))$ cannot vanish unless $\alpha = \beta$. This follows from the fact that

$$(X(\alpha) - X(\beta)) - (\alpha - \beta) = (X - Id)(\alpha) - (X - Id)(\beta) = O(|\alpha - \beta|)O(\delta). \tag{3.10}$$

Here we comment that we use the convention that vectors indicating a spatial position or displacement, such as $X(\alpha)$ or $(\alpha - \beta)$, are row vectors. Thus, as we have written it here, the $O(|\alpha - \beta|)$ factor in the rightmost expression is also a row vector, and the $O(\delta)$ vector is a 2×2 matrix acting on it. From (3.10) we get

$$\begin{aligned} |(X_1(\alpha) - X_1(\beta))^2 + (X_2(\alpha) - X_2(\beta))^2| &\geq \left|(\alpha - \beta)_1^2 + (\alpha - \beta)_2^2\right| - C\delta|\alpha - \beta|^2, \quad (3.11) \\ &\geq |\alpha - \beta|^2/2, \end{aligned}$$

for sufficiently small δ. This computation leads us to

Remark 3.1.2. *In the integral defining $F(X)(\alpha)$, we get something no more singular than $|\alpha-\beta|^{-1}$, and so $F(X)$ gives us a finite $\mathbb{C}^2$-valued function on Ω.*

Now that the operator F has been defined for the incompressible Euler equations, we consider its behavior with regard to the criterion of preservation of analytic dependence in time, discussed in Section 2.

Preservation of analyticity for the operator F

With the goal of showing that the operator F defined above preserves analyticity, given an initial vorticity ω that is not too wild, we first present a bound on the derivatives of the kernel K which is useful in the verification of this fact.

Lemma 3.1.3. *There is a small positive constant δ' such that for a matrix $M \in \mathbb{C}^{2\times 2}$ with $|M| \leq \delta'$, the function $K(z) : \mathbb{C}^2 \setminus \{z_1 \pm iz_2 = 0\} \to \mathbb{C}^2$ defined by (3.9), satisfies*

$$|(\partial_z^\gamma K)(\beta(I+M))| \leq \quad A_\gamma |\beta|^{-|\gamma|-1} \quad \text{for all nonzero } \beta \in \mathbb{R}^2 \text{ and } |\gamma| \geq 0. \tag{3.12}$$

Proof. An easy calculation shows that the derivatives of $(\partial_z^\gamma K)(z)$ satisfy the bound

$$|(\partial_z^\gamma K)(z)| \leq \frac{C_\gamma |z|^{|\gamma|+1}}{|z_1^2 + z_2^2|^{|\gamma|+1}}. \tag{3.13}$$

Using this and the estimate $|[\beta(I+M)]_1^2 + [\beta(I+M)]_2^2| \geq |\beta|^2/2$ for small enough δ' (checked in a similar manner to the bound (3.11)) one may verify (3.12). □

Now we claim the following.

Lemma 3.1.4. *Fix $\omega \in L^\infty \cap L^1(\Omega)$, $\mu \in (0,1)$, and define $Y = C^{1,\mu}(\Omega)$. Then the operator F defined in Definition 3.1.1 preserves analyticity. That is, for any $r > 0$ and $X_{(\cdot)} \in \mathcal{B}_{\delta,r}$, for fixed $\alpha \in \Omega$, $F(X_z)(\alpha)$ is analytic in d_r, where $\mathcal{B}_{\delta,r}$ is as in Definition 2.2.2.*

Proof. Fix an $r > 0$, and an $X_{(\cdot)} \in \mathcal{B}_{\delta,r}$. What we want to prove is that the limit definition of $\frac{d}{dz}\left(F(X_z)(\alpha)\right)$ converges for each $z \in d_r$, $\alpha \in \Omega$. Fixing $\alpha \in \Omega$ and $z \in d_r$ we find

$$\begin{aligned} &\frac{1}{h}\int_\Omega (K(X_{z+h}(\alpha) - X_{z+h}(\beta)) - K(X_z(\alpha) - X_z(\beta)))\omega(\beta)d\beta \\ &= \int_\Omega \frac{1}{h}\left(\int_z^{z+h} \left(\frac{d}{d\zeta}X_\zeta(\alpha) - \frac{d}{d\zeta}X_\zeta(\beta)\right)\nabla K(X_\zeta(\alpha) - X_\zeta(\beta))d\zeta\right)\omega(\beta)d\beta, \end{aligned} \tag{3.14}$$

where we take the straight path from z to $z+h$ in the integral. Since $X_{(\cdot)} \in \mathcal{B}_{\delta,r}$ implies $X_\zeta(\cdot)$ is Lipschitz with a fixed constant as ζ varies in d_r, by using the Cauchy integral formula one finds that we get a uniform Lipschitz bound on $\frac{d}{d\zeta}X_\zeta(\cdot)$, i.e.

$$\sup_{\gamma,\beta\in\Omega}\left(\frac{\left|\frac{d}{d\zeta}X_\zeta(\gamma)-\frac{d}{d\zeta}X_\zeta(\beta)\right|}{|\gamma-\beta|}\right)\leq C, \tag{3.15}$$

as ζ varies in the interval from z to $z+h$, which must be contained in d_r for small h. Moreover, since we have $(X_\zeta(\alpha)-X_\zeta(\beta))=(\alpha-\beta)(I+O(\delta))$, we can apply Lemma 3.1.3 to get a bound on the kernel in the right hand side of (3.14). Together with the Lipschitz bound on $\frac{d}{d\zeta}X_\zeta(\cdot)$, we can thus bound the quantity inside the integral over β uniformly by $C|\alpha-\beta|^{-1}|\omega(\beta)|$, then observe that the quantity within the integral taken in the ζ variable is a continuous function of ζ as long as $\beta\neq\alpha$, and then apply the dominated convergence theorem to then get

$$\frac{d}{dz}\left(F(X_z)(\alpha)\right)=\int\left(\frac{d}{dz}X_z(\alpha)-\frac{d}{dz}X_z(\beta)\right)\nabla K(X_z(\alpha)-X_z(\beta))\omega(\beta)d\beta. \tag{3.16}$$

□

In the remaining parts of this section, we finish the proofs of Lagrangian analyticity for the incompressible Euler equations for Hölder continuous initial vorticity on the 2D torus, and for vortex patch solutions on $\mathbb{R}^2$. We comment that some modifications to the above lemma ought to be made for the 2D torus, since the corresponding kernel in place of K is then slightly different. Nevertheless one finds the analogous result for the torus is verified without much additional difficulty.

3.2 Lagrangian analyticity for Hölder continuous vorticity

3.2.1 Setup

Here we provide a proof of Lagrangian analyticity for the case of the incompressible Euler equations on the 2D torus. Let ω be the initial vorticity of a given solution to the system (3.1), for which we assume $\omega\in C^{0,\mu}(\mathbb{T}^2)$ for some $\mu\in(0,1)$. For the entirety of this section, we take the Banach space Y to be $C^{1,\mu}(\mathbb{T}^2)$, and we define the operator F_T (similar to the operator F of Definition 3.1.1) by

$$F_T(X)(\alpha):=\int_{\mathbb{T}^2}K_T(X(\alpha)-X(\beta))\omega(\beta)\det\nabla X(\beta)d\beta\quad\text{for } X\in B_\delta. \tag{3.17}$$

Here, the kernel K_T is defined on $(\mathbb{C}^2/\Lambda) \setminus \big(\{(z_1, z_2) \in \mathbb{C}^2 : z_1 \pm iz_2 = 0\}/\Lambda\big)$, where we have the integer lattice denoted by $\Lambda = \mathbb{Z}^2$, and in this case

$$K_T(z_1, z_2) = \frac{1}{2\pi}\frac{(-z_2, z_1)}{z_1^2 + z_2^2} + \frac{1}{2\pi}\sum_{k\in\mathbb{Z}^2\setminus 0}\left(\frac{(-z_2 + k_2, z_1 - k_1)}{(z_1 - k_1)^2 + (z_2 - k_2)^2} + \frac{(-k_2, k_1)}{|k|^2}\right) - (-z_2, z_1). \quad (3.18)$$

The kernel K_T is just the appropriate version of the kernel for the Biot-Savart law adapted to the torus, analogous to (3.4), so that the solution to the incompressible Euler equations (3.1) on $\mathbb{T}^2$ is given by

$$u(x,t) = \int K_T(x-y)\omega(y,t)dy \quad \text{for } x \in \mathbb{T}^2, \quad (3.19)$$

where $\omega(x,t)$ satisfies

$$\begin{aligned} \partial_t\omega(x,t) + u(x,t)\cdot\nabla\omega(x,t) &= 0, \quad (3.20)\\ \omega(x,0) &= \omega(x) \end{aligned}$$

and for initial data u_0 we have $\mathrm{curl} u_0(x) = \omega(x)$. Notice that for incompressible flows we will have $\det \nabla X(\beta, t) = 1$, so (3.17) is compatible with the solution to the Euler equations.

For the above kernel we have the same type of bound as for K in Lemma 3.1.3.

Lemma 3.2.1. *There is a small positive constant δ such that for $z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2/\Lambda$ with $|y| < c\delta$, $|x_j| < 1/2$, $j = 1, 2$, and $z_1 \pm iz_2 \neq 0$, $K_T(z)$ and its derivatives at the point z satisfy the bound*

$$|(\partial_z^\gamma K_T)(z)| \le \frac{C_\gamma |z|^{|\gamma|+1}}{|z_1^2 + z_2^2|^{|\gamma|+1}}. \quad (3.21)$$

In addition, for a matrix $M \in \mathbb{C}^{2\times 2}$ with $|M| \le \delta$,

$$|(\partial_z^\gamma K_T)(\beta(I+M))| \le \quad A_\gamma |\beta|^{-|\gamma|-1} \quad \text{for all nonzero } \beta \in \mathbb{T}^2 \text{ and } |\gamma| \ge 0, \quad (3.22)$$

and

Proof. The bound (3.21) follows from the bound (3.13) in the proof of Lemma 3.1.3, since $K_T(z_1, z_2)$ is the sum of $K(z_1, z_2)$ and a harmless function on the set $\{(x_1 + iy_1, x_2 + iy_2) : |y| < c\delta,\ |x_j| < 1/2,\ j = 1, 2\}$. Once we have this, (3.22) follows. □

In this situation we also have preservation of analytic dependence. The proof is similar to that of Lemma 3.1.4.

Lemma 3.2.2. *The operator F_T preserves analyticity, where we take $Y = C^{1,\mu}(\mathbb{T}^2)$ in Definition 4.4.1. That is, for each $r > 0$ and $X_z \in \mathcal{B}_{\delta,r}$, we have $F_T(X_z)(\alpha)$ is analytic in z in d_r for each fixed $\alpha \in \mathbb{T}^2$.*

We also have the following identity, which will be useful in proving the local boundedness of the operator F_T.

Lemma 3.2.3. *There is a $\delta_0 > 0$ such that for all $X \in B_{\delta_0}$,*

$$\int K_T(X(\alpha) - X(\beta)) \det \nabla X(\beta) d\beta = 0. \tag{3.23}$$

Proof. Fix $X \in B_{\delta_0}$, where δ_0 is to be determined, and fix $\alpha \in \mathbb{T}^2$. For $\lambda \in \mathbb{C}$ with $|\lambda| < 2$, we define $X_\lambda := \mathrm{Re}(X) + \lambda \mathrm{Im}(X)$. First we will argue that for $\lambda \in [-1, 1]$,

$$\int K_T(X_\lambda(\alpha) - X_\lambda(\beta)) \det \nabla X_\lambda(\beta) d\beta = 0. \tag{3.24}$$

Suppose δ_0 is small enough that $\|X - Id\|_{C^{1,\gamma}} < \delta_0$ implies for all $|\lambda| < 2$ that $\|X_\lambda - Id\|_{C^{1,\gamma}} < \delta'$, where δ' is as in Lemma 3.2.1, so that this fact implies the left hand side of (3.24) is a convergent integral. With an argument similar to that in the proof of Lemma 3.1.4 we find this expression is analytic as a function of λ in $\{|\lambda| < 2\}$. Then, for $\lambda \in [-1, 1]$, since X_λ is real, we are permitted to make a change of variables inverting $X_\lambda(\beta)$ and then we find that (3.24) holds since K_T has average zero on the real torus.

Since the left hand side is analytic in λ in all of the disc $\{|\lambda| < 2\}$, and the equality holds for $\lambda \in [-1, 1]$, (3.24) must hold in all of the disc $\{|\lambda| < 2\}$. Taking $\lambda = i$, we have $X_\lambda = \mathrm{Re}(X) + i\mathrm{Im}(X)$, and so we have shown that (3.23) holds. □

3.2.2 Local boundedness of the operator F_T

Now we show the local boundedness property for the operator F_T holds in this setting.

Proposition 3.2.4. *Assume $\omega \in C^{0,\mu}(\mathbb{T}^2)$. Let $F_T(X)$ be as defined in (3.17). There is a $\delta_0' > 0$ such that we have the bound*

$$\sup_{X \in B_{\delta_0'}} \left\| F_T(X) \right\|_{C^{1,\mu}(\mathbb{T}^2)} \leq C, \tag{3.25}$$

for a constant C depending only on ω.

Proof. Fix $X \in B_\delta$, for some δ to be specified. We check that $F_T(X)$ is uniformly bounded in L^∞

with the use of Lemma 3.2.1, which gives

$$|F_T(X)(\alpha)| \leq \left| \int K_T(X(\alpha) - X(\beta))\omega(\beta) \det \nabla X(\beta) d\beta \right| \leq \int C|\alpha - \beta|^{-1} d\beta \leq C, \tag{3.26}$$

for a constant C depending on ω.

The next thing to check is that $F_T(X)(\alpha)$ is differentiable, with uniformly bounded derivative. We use Lemma 3.2.3 to write

$$F_T(X)(\alpha) = \int K_T(X(\alpha) - X(\beta))(\omega(\beta) - \omega(\alpha)) \det \nabla X(\beta) d\beta. \tag{3.27}$$

With the use of Lemma 3.2.3, one can calculate that for $i = 1, 2$, the partial derivatives are then given by

$$\partial_i(F_T(X)(\alpha)) = \partial_i X(\alpha) \int \nabla K_T(X(\alpha) - X(\beta))(\omega(\beta) - \omega(\alpha)) \det \nabla X(\beta) d\beta. \tag{3.28}$$

In particular, note that this converges by the bounds on the kernel given in Lemma 3.2.1. Furthermore, one can verify that the quantity in the right hand side of (3.28) is continuous in α by making the change of variables giving

$$\partial_i(F_T(X)(\alpha)) = \partial_i X(\alpha) \int \nabla K_T(X(\alpha) - X(\alpha - \beta))(\omega(\alpha - \beta) - \omega(\alpha)) \det \nabla X(\alpha - \beta) d\beta. \tag{3.29}$$

This is seen to be continuous by using the uniform bound by $C|\beta|^{\mu-2}$ on the integrand here, independent of α, and applying the dominated convergence theorem. Since for each of $i = 1, 2$ we have that $\partial_i(F_T(X)(\alpha))$ exists and is continuous, we get that $F_T(X)$ is differentiable, with gradient $\nabla_\alpha(F_T(X)(\alpha))$ given by

$$\nabla(F_T(X)(\alpha)) = \nabla X(\alpha) \int \nabla K_T(X(\alpha) - X(\beta))(\omega(\beta) - \omega(\alpha)) \det \nabla X(\beta) d\beta. \tag{3.30}$$

We thus find

$$\left\| \nabla(F_T(X)(\alpha)) \right\|_{L^\infty} \leq C, \tag{3.31}$$

for a constant C depending only on ω, by using the bounds on the kernel and the Hölder bound on ω.

Now we show a uniform Hölder bound for $\nabla_\alpha(F_T(X)(\alpha))$ holds. Note that by the formula (3.28), since $\nabla X(\alpha)$ is uniformly bounded in $C^{0,\mu}$, this reduces to proving a Hölder bound for the following

for any fixed $\alpha, \gamma \in \mathbb{T}^2$:

$$d(\alpha,\gamma) := \int \nabla K_T(X(\alpha)-X(\beta))(\omega(\beta)-\omega(\alpha))\det\nabla X(\beta)d\beta \\ - \int \nabla K_T(X(\gamma)-X(\beta))(\omega(\beta)-\omega(\gamma))\det\nabla X(\beta)d\beta. \tag{3.32}$$

We split the region of integration into the two sets

$$\begin{aligned} L &:= \{\beta : |\alpha-\beta| < 10|\alpha-\gamma|\}, \\ N &:= \{\beta : |\alpha-\beta| \geq 10|\alpha-\gamma|\}, \end{aligned} \tag{3.33}$$

which gives

$$\begin{aligned} d(\alpha,\gamma) &= \int_L \nabla K_T(X(\alpha)-X(\beta))(\omega(\beta)-\omega(\alpha))\det\nabla X(\beta)d\beta \\ &\quad - \int_L \nabla K_T(X(\gamma)-X(\beta))(\omega(\beta)-\omega(\gamma))\det\nabla X(\beta)d\beta \\ &\quad + (\omega(\gamma)-\omega(\alpha))\int_N \nabla K_T(X(\alpha)-X(\beta))\det\nabla X(\beta)d\beta \\ &\quad + \int_N (\nabla K_T(X(\alpha)-X(\beta)) - \nabla K_T(X(\gamma)-X(\beta)))(\omega(\beta)-\omega(\gamma))\det\nabla X(\beta)d\beta \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.34}$$

For I_1, we use the fact that $|\nabla K_T(X(\alpha)-X(\beta))| \leq C|\alpha-\beta|^{-2}$ to get

$$I_1 \leq C\int_L |\alpha-\beta|^{2-\mu}d\beta \leq C'|\alpha-\gamma|^{\mu}. \tag{3.35}$$

Similarly, one finds

$$I_2 \leq C|\alpha-\gamma|^{\mu}. \tag{3.36}$$

Now we address I_3. Since ω is assumed to be Hölder, all we need to show is that the integral

$$\int_N \nabla K_T(X(\alpha)-X(\beta))\det\nabla X(\beta)d\beta \tag{3.37}$$

is bounded. First, for $r > 0$, we consider

$$E(r,\alpha) := \int_{|\beta-\alpha|>r} \nabla K_T((\alpha-\beta)\nabla X(\alpha))d\beta. \tag{3.38}$$

We claim $E(r,\alpha)$ is uniformly bounded, independently of r, α, and X, for $X \in B_{\delta_0'}$. To see this we note

$$\begin{aligned}\int_{|\beta-\alpha|>r} \nabla K_T((\alpha-\beta)\nabla X(\alpha))d\beta &= -(\nabla X(\alpha))^{-1}\int_{|\beta-\alpha|>r} \nabla_\beta(K_T((\alpha-\beta)\nabla X(\alpha)))d\beta, &(3.39)\\ &= -(\nabla X(\alpha))^{-1}\int_{|\beta-\alpha|=r} \nu(\beta)K_T((\alpha-\beta)\nabla X(\alpha))d\sigma(\beta), &(3.40)\end{aligned}$$

where $\nu(\beta)$ is the outward pointing normal. By Lemma 3.2.1, the integrand in line (3.40) is bounded by $C|\alpha-\beta|^{-1} = Cr^{-1}$, while we have that the circle $\{\beta : |\beta-\alpha| < r\}$ has circumference $2\pi r$, so that we obtain

$$|E(r,\alpha)| \le C\int_{|\beta-\alpha|=r} |K_T((\alpha-\beta)\nabla X(\alpha))|d\sigma(\beta) \le C'. \tag{3.41}$$

Note for $X \in B_\delta$ that

$$(X(\alpha)-X(\beta)) - (\alpha-\beta)\nabla X(\alpha) = (\alpha-\beta)\int_0^1 (\nabla X(\tau\alpha+(1-\tau)\beta) - \nabla X(\alpha))d\tau = O(|\alpha-\beta|^{\mu+1}). \tag{3.42}$$

With this we have the bound

$$\begin{aligned}&|\nabla K_T(X(\alpha)-X(\beta)) - \nabla K_T((\alpha-\beta)\nabla X(\alpha))| &(3.43)\\ &\quad= \left|\left(\sum_{i=1}^2 [X(\alpha)-X(\beta)-(\alpha-\beta)\nabla X(\alpha)]_i\right)\right.\\ &\qquad\left.\times\int_0^1 (\partial_i\nabla K_T)(\tau(X(\alpha)-X(\beta)) + (1-\tau)(\alpha-\beta)\nabla X(\alpha))d\tau\right|,\\ &\quad\le C|\alpha-\beta|^{\mu+1}\left|\int_0^1 \nabla^2 K_T(\tau(X(\alpha)-X(\beta)) + (1-\tau)(\alpha-\beta)\nabla X(\alpha))d\tau\right|.\end{aligned}$$

Also, note for all $\tau \in [0,1]$ we have for $X \in B_\delta$ that

$$\tau(X(\alpha)-X(\beta)) + (1-\tau)(\alpha-\beta)\nabla X(\alpha) = (\alpha-\beta)(I+O(\delta)). \tag{3.44}$$

So together, (3.43), (3.44), and Lemma 3.2.1 give

$$|\nabla K_T(X(\alpha)-X(\beta)) - \nabla K_T((\alpha-\beta)\nabla X(\alpha))| \le C|\alpha-\beta|^{\mu+1-3} = C|\alpha-\beta|^{\mu-2}. \tag{3.45}$$

Using (3.45), (3.41), and Lemma 3.2.1, we then get for any $r > 0$

$$\begin{aligned}\left|\int_{|\beta-\alpha|>r} \nabla K_T(X(\alpha)-X(\beta)) \det \nabla X(\beta) d\beta\right| &\leq \Big|\int_{|\beta-\alpha|>r} \big[\nabla K_T(X(\alpha)-X(\beta)) \\ &\qquad -\nabla K_T((\alpha-\beta)\nabla X(\alpha))\big] \det \nabla X(\beta) d\beta \\ &\qquad + \int_{|\beta-\alpha|>r} \nabla K_T((\alpha-\beta)\nabla X(\alpha)) \\ &\qquad \times \big[\det \nabla X(\beta) - \det \nabla X(\alpha)\big] d\beta\Big| \\ &\qquad + |E(r,\alpha) \det \nabla X(\alpha)|, \\ &\leq C \int_{|\beta-\alpha|>r} |\alpha-\beta|^{\mu-2} d\beta + C', \\ &\leq C'', \end{aligned} \tag{3.46}$$

noting that the bounding constant C'' is independent of r. This gives us the bound we wanted for I_3. For I_4, we first make several observations that will be used to bound the differences of the kernels in the integrand. For any $\tau \in [0,1]$, we have

$$\begin{aligned}\sum_{i=1}^{2} (\tau X_i(\alpha) + (1-\tau)X_i(\gamma) - X_i(\beta))^2 &= \sum_{i=1}^{2} (X_i(\gamma) - X_i(\beta))^2 \\ &\quad + 2\tau \sum_{i=1}^{2} (X_i(\gamma) - X_i(\beta))(X_i(\alpha) - X_i(\gamma)) \\ &\quad + \tau^2 \sum_{i=1}^{2} (X_i(\alpha) - X_i(\gamma))^2. \end{aligned} \tag{3.47}$$

For any $\xi, \eta \in \mathbb{T}^2$, and $i = 1, 2$ we have

$$|X_i(\xi) - X_i(\eta)| \leq (1+\delta)|\xi - \eta|, \tag{3.48}$$

and

$$\left|\sum_{i=1}^{2} (X_i(\xi) - X_i(\eta))^2\right| \geq (1 - c\delta)|\xi - \eta|^2. \tag{3.49}$$

In addition, note that for β in the region N, i.e. such that $|\alpha - \beta| \geq 10|\alpha - \gamma|$, we have

$$\frac{|\alpha - \gamma|}{|\beta - \gamma|} \leq \frac{1}{9}. \tag{3.50}$$

Now we will show the following bound holds in the region $N = \{|\alpha - \beta| \geq 10|\alpha - \gamma|\}$ with the use

of (3.47)-(3.50) and Lemma 3.2.1.

$$\begin{aligned} |\nabla K_T(X(\alpha)-X(\beta))-\nabla K_T(X(\gamma)-X(\beta))| &\leq |X(\alpha)-X(\gamma)| \\ &\quad \times \left| \int_0^1 \nabla^2 K_T(\tau X(\alpha)+(1-\tau)X(\gamma)-X(\beta))d\tau \right|, \\ &\leq C|\alpha-\gamma||\beta-\gamma|^{-3}. \end{aligned} \tag{3.51}$$

Note this will follow if we show

$$\left|\nabla^2 K_T(\tau X(\alpha)+(1-\tau)X(\gamma)-X(\beta))d\tau\right| \leq C|\beta-\gamma|^{-3}. \tag{3.52}$$

First we check that we have a bound of the form

$$\frac{\left|\sum_{i=1}^2 (\tau X_i(\alpha)+(1-\tau)X_i(\gamma)-X_i(\beta))^2\right|}{|\gamma-\beta|^2} \geq c. \tag{3.53}$$

To see this, using (3.47), we bound the left hand side of (3.53) below by

$$\begin{aligned} \frac{\left|\sum_{i=1}^2 (X_i(\gamma)-X_i(\beta))^2\right|}{|\gamma-\beta|^2} - 2\frac{\left|\sum_{i=1}^2 (X_i(\gamma)-X_i(\beta))(X_i(\alpha)-X_i(\beta))\right|}{|\gamma-\beta|^2} - \frac{\left|\sum_{i=1}^2 (X_i(\alpha)-X_i(\gamma))^2\right|}{|\gamma-\beta|^2} \\ = A - B_1 - B_2 \end{aligned} \tag{3.54}$$

By (3.49),

$$A \geq (1-c\delta), \tag{3.55}$$

and by (3.48) and (3.50),

$$B_1 \leq \quad 2(1+c\delta)\frac{|X(\alpha)-X(\gamma)|}{|\gamma-\beta|} = 2(1+c\delta)\frac{|\alpha-\gamma|}{|\beta-\gamma|}\frac{|X(\alpha)-X(\gamma)|}{|\alpha-\gamma|} \leq 2(1+c\delta)^2\frac{1}{9}. \tag{3.56}$$

For B_2, by (3.48) and (3.50) we have the bound

$$B_2 = \frac{\left|\sum_{i=1}^2 (X_i(\alpha)-X_i(\gamma))^2\right|}{|\alpha-\gamma|^2}\frac{|\alpha-\gamma|^2}{|\beta-\gamma|^2} \leq (1+c\delta)^2\frac{1}{9^2}. \tag{3.57}$$

Together, (3.55)-(3.57) give

$$A - B_1 - B_2 > c' \tag{3.58}$$

for some positive $c' > 0$, given that δ is chosen small enough. This implies (3.53). Now, we use the

bound on the kernel (3.21) from Lemma 3.2.1, (3.53), (3.48), and (3.50) to find for any $\tau \in [0,1]$

$$\begin{aligned} |\nabla^2 K_T(\tau X(\alpha) + (1-\tau)X(\gamma) - X(\beta))| &\leq C \frac{\left(|X(\gamma) - X(\beta)| + \tau |X(\alpha) - X(\gamma)|\right)^3}{\left|\sum_{i=1}^2 (\tau X_i(\alpha) + (1-\tau)X_i(\gamma) - X_i(\beta))^2\right|^3} \\ &\leq C' \frac{(|\gamma - \beta|^3 + |\alpha - \gamma|^3)}{|\gamma - \beta|^6}, \\ &\leq C'' |\gamma - \beta|^{-3}, \end{aligned} \tag{3.59}$$

from which (3.51) follows.

In view of the definition of I_4 in (3.34), combining (3.51) with the bound $|\omega(\gamma) - \omega(\beta)| \leq C|\gamma - \beta|^\mu$ and $|\gamma - \beta|^{\mu - 3} \leq C|\alpha - \beta|^{\mu - 3}$ in the region N, we find the following bound for I_4:

$$|I_4| \leq C|\alpha - \gamma| \int_N |\alpha - \beta|^{\mu - 3} d\beta \leq C'|\alpha - \gamma|^{1 + \mu - 1} = C'|\alpha - \gamma|^\mu. \tag{3.60}$$

With the Hölder bounds on I_1, I_2, I_3, and I_4, we conclude that the quantity $d(\alpha, \gamma)$ also satisfies a bound of the form

$$|d(\alpha, \gamma)| \leq C|\alpha - \gamma|^\mu, \tag{3.61}$$

for a constant C depending only on ω. Combining this with the form for the gradient given in (3.28), this concludes the Hölder bound for $\nabla_\alpha (F_T(X)(\alpha))$. □

Corollary 3.2.5. *The Lagrangian trajectories are real analytic in time for solutions to the 2D incompressible Euler equations with initial data with Hölder continuous vorticity $\omega \in \mathbb{C}^{0,\mu}(\mathbb{T}^2)$, for some $\mu \in (0,1)$.*

Proof. From Theorem 2.2.6, together with Proposition 3.2.4 and Lemma 3.2.2, we get an analytic solution $X(\alpha, t)$ to the equation

$$\frac{dX}{dt}(\alpha, t) = F_T(X)(\alpha, t), \tag{3.62}$$

with $X(\alpha, 0) = \alpha$. Recalling the formula (3.17), once we verify $\det \nabla X(\alpha, t) = 1$, we will have guaranteed $X(\alpha, t)$ gives the trajectory of the solution to the 2D incompressible Euler equations with initial vorticity ω. Now, we claim that we know $X(\alpha, t)$ is real for t real. This holds since for Φ_{F_T} as defined by (2.25), we find that Φ_{F_T} maps the closed subspace of $\mathcal{B}_{\delta,r}$ with this property, $\{X_z \in \mathcal{B}_{\delta,r} : \overline{X_t} = X_t \text{ for real } t\}$, to itself. As in the proof of Theorem 2.2.6, we then find the solution to the corresponding ODE (3.62) also has this property, as the fixed point of this map.

Next, we define for t real and $x \in \mathbb{T}^2$,

$$u(x,t) := \frac{dX}{dt}\left(X^{-1}(x,t)\right) = \int K_T(x - X(\beta,t))\omega(\beta)\det \nabla X(\beta,t)d\beta. \tag{3.63}$$

It then follows that div$u = 0$. Since $\frac{dX}{dt} = u \circ X$, the map $X(\alpha,t)$ must preserve area and so $\det \nabla X(\alpha,t) = 1$. □

3.3 Lagrangian analyticity for vortex patch solutions

In this part, we show that one has a Lagrangian analyticity-type result in the case of vortex patch solutions to the 2D incompressible Euler equations, which are characterized by having vorticity $\omega(x,t) = \operatorname{curl} u(x,t)$ of the form

$$\omega(x,t) = \begin{cases} 0 & \text{if } x \in \Omega(t), \\ 1 & \text{otherwise,} \end{cases} \tag{3.64}$$

where $\Omega(t)$ is a bounded region that is transported by the flow. If the vorticity is initially given by an identifier function on a region Ω with smooth boundary, the solution exists globally, and ω has this form at later times, due to the fact that vorticity is transported by the flow for the 2D incompressible Euler equations. One has good existence and uniqueness properties and that the smoothness of the boundary $\partial\Omega(t)$ is maintained in time (see, for example [4, 7, 23]) despite the jump discontinuity in the gradient of the velocity across the boundary. Considering the question of the regularity in time of the particle trajectories, and thus the evolution of the set $\Omega(t)$, is a natural progression from the corresponding question for Hölder continuous vorticity, studied in the previous part of this section and in past studies of Lagrangian analyticity for the incompressible Euler equations.

The main task of this section is to give a proof of the local boundedness result, Proposition 3.3.6, for the operator F for this scenario, which is given in Definition 3.1.1. In accordance with the remarks above, in this part we treat the case where $\omega(\alpha) := \operatorname{curl} u_0(\alpha)$ is given by

$$\omega(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \Omega, \\ 0 & \text{otherwise,} \end{cases} \tag{3.65}$$

for an open set $\Omega \subset \mathbb{R}^2$.

In all that follows let us consider an arbitrary fixed $\mu \in (0,1)$, and let us use the notation F_X to

refer to $F(X)$. Proposition 3.3.6 asserts the following: under the hypothesis that the initial patch Ω is bounded, convex, and has C^2 boundary, there exists a $\delta > 0$ and a bound of the form

$$\sup_{X \in B_\delta} \|F_X\|_{C^{1,\mu}(\Omega)} \leq C, \tag{3.66}$$

for a constant $C = C(\Omega)$, where $B_\delta = \{X \in C^{1,\mu}(\Omega) : \|X - Id\|_{C^{1,\mu}(\Omega)} \leq \delta\}$. The proof of this estimate is the main difficulty in proving the Lagrangian analyticity result.

We apply the methods of Section 2, taking the Banach space of maps Y to be $C^{1,\mu}(\Omega)$. When combining the result of Proposition 3.3.6 with Lemma 3.1.4 and Theorem 2.2.6, this proves that for $\alpha \in \Omega$, the trajectories $X(\alpha, t)$ of the solutions u solving (3.1) are analytic in time, as we conclude in Corollary 3.3.8.

Observe that F_X is not difficult to bound pointwise with the use of Lemma 3.1.3:

$$|F_X(\alpha)| \leq \left| \int_\Omega K(X(\alpha) - X(\beta)) d\beta \right| \leq \int_\Omega C|\alpha - \beta|^{-1} d\beta < \infty. \tag{3.67}$$

The main estimates involve bounds on the gradient of $F_X(\alpha)$. The first task is to establish pointwise bounds on $\nabla F_X(\alpha)$ and related quantities, via Lemmas 3.3.1 and 3.3.2. With that, we are also able to justify a useful formula for the derivatives of F_X in the directions of vector fields T tangent to the boundary of the patch and N normal to the boundary, which is given in Lemma 3.3.3. With these formulas, the proof of the Hölder bound on $\nabla F_X(\alpha)$ is essentially reduced to proving the Hölder continuity of $T(F_X)(\alpha)$, which is verified with Lemma 3.3.5. Once these lemmas have been established, the proof of Proposition 3.3.6 follows without much additional difficulty.

3.3.1 Pointwise gradient bounds

We begin with Lemma 3.3.1, which guarantees in particular the differentiability of F_X.

Lemma 3.3.1. *There is a $\delta_1 > 0$ such that for all $X \in B_{\delta_1}$, $F_X(\alpha)$ is differentiable in Ω with gradient satisfying the pointwise bound*

$$|\nabla F_X(\alpha)| \leq C \quad \text{for all } \alpha \in \Omega, \tag{3.68}$$

for a constant C depending on Ω.

For the most part the proof of Lemma 3.3.1 is routine once Lemma 3.3.2 is proven. Note that if we were to simply place the gradient inside the integral in the expression for F_X and use a principal

value integral, we would obtain the expression

$$p.v. \int_\Omega \nabla X(\alpha) \nabla K(X(\alpha) - X(\beta)) d\beta. \tag{3.69}$$

Here we use the convention that $(\nabla X)_{ij} = \partial X_j / \partial \alpha_i$. The next lemma allows us to bound expressions like this, and provides the main estimate used to prove Lemma 3.3.1. It is also used crucially in Lemma 3.3.5, which in turn provides the key estimate for establishing the Hölder continuity of ∇F_X.

Lemma 3.3.2. *For sufficiently small δ_2 there exists a constant C such that that for all $X \in B_{\delta_2}$, all $\alpha \in \Omega$, and $r > 0$,*

$$\left| \int_{\Omega \setminus B_r(\alpha)} \nabla K(X(\alpha) - X(\beta)) d\beta \right| \leq C. \tag{3.70}$$

Proof. First, we claim that the quantity in the left hand side of (3.70) that we want to bound is within a bounded constant of

$$I := \int_{\Omega \setminus B_r(\alpha)} \nabla K((\alpha - \beta) \nabla X(\alpha)) d\beta. \tag{3.71}$$

To see this, observe

$$\begin{aligned} &\nabla K(X(\alpha) - X(\beta)) - \nabla K((\alpha - \beta)\nabla X(\alpha)) = \\ &\sum_{i=1}^{2} [(X(\alpha) - X(\beta)) - (\alpha - \beta)\nabla X(\alpha)]_i \int_0^1 (\partial_i \nabla K)(\tau(X(\alpha) - X(\beta)) + (1-\tau)(\alpha - \beta)\nabla X(\alpha)) d\tau, \end{aligned} \tag{3.72}$$

and

$$\begin{aligned} |(X(\alpha) - X(\beta)) - (\alpha - \beta)\nabla X(\alpha)| &= |(\alpha - \beta) \int_0^1 (\nabla X(\tau\alpha + (1-\tau)\beta) - \nabla X(\alpha)) d\tau|, \\ &\leq |\alpha - \beta| \int_0^1 |\nabla X(\tau\alpha + (1-\tau)\beta) - \nabla X(\alpha)| d\tau, \\ &\leq C|\alpha - \beta|^{1+\mu}, \end{aligned} \tag{3.73}$$

where we have used here that for each $\tau \in (0,1)$, $\tau\alpha + (1-\tau)\beta$ remains inside Ω, since Ω is convex. For the integral over τ in the right hand side of (3.72), we note $X(\alpha) - X(\beta) = (\alpha - \beta)(I + O(\delta_2))$ and $\nabla X(\alpha) = I + O(\delta_2)$, so that for any $\tau \in (0,1)$,

$$\tau(X(\alpha) - X(\beta)) + (1-\tau)(\alpha - \beta)\nabla X(\alpha) = (\alpha - \beta)(I + O(\delta_2)). \tag{3.74}$$

Thus, by using Lemma 3.1.3, for small enough δ_2 we get

$$|\nabla^2 K(\tau(X(\alpha) - X(\beta)) + (1-\tau)(\alpha-\beta)\nabla X(\alpha))| \leq c'|\alpha-\beta|^{-3}. \tag{3.75}$$

Using (3.73) and (3.75) in (3.72) we now find

$$|\nabla K(X(\alpha) - X(\beta)) - \nabla K((\alpha-\beta)\nabla X(\alpha))| \leq c''|\alpha-\beta|^{\mu-2}. \tag{3.76}$$

Now we can bound the difference:

$$\left| \int_{\Omega \setminus B_r(\alpha)} (\nabla K(X(\alpha) - X(\beta)) - \nabla K((\alpha-\beta)\nabla X(\alpha)) d\beta \right| \leq c'' \int_{\Omega} |\alpha-\beta|^{\mu-2} d\beta \leq \tilde{c}. \tag{3.77}$$

Now we have reduced the proof of the lemma to bounding I. We fix a positive constant $D > 0$ to be specified later, determined by Ω, and uniformly bound the quantity

$$I' := \int_{\Omega \cap \{|\alpha-\beta| > D\}} |\nabla K((\alpha-\beta)\nabla X(\alpha))| d\beta. \tag{3.78}$$

Indeed, we have

$$I' \leq C \int_{\Omega \cap \{|\alpha-\beta|>D\}} |\alpha-\beta|^{-2} d\beta \leq C \int_{A_D(\alpha)} |\alpha-\beta|^{-2} d\beta \leq \underline{C}, \tag{3.79}$$

where $A_D(\alpha)$ is taken to be an annulus centered at α with inner radius D and with the same area as Ω.

Now note that if $r \geq D$, we have

$$|I| \leq I' \leq \underline{C}, \tag{3.80}$$

and so in that case we are done.

On the other hand, suppose that we have $r < D$. In that case we split the integral over $\Omega \setminus B_r$ in the expression for I, getting

$$\begin{aligned} |I| &= \left| \left(\int_{\Omega \cap \{|\alpha-\beta| \geq D\}} + \int_{\Omega \cap \{D > |\alpha-\beta| > r\}} \right) \nabla K((\alpha-\beta)\nabla X(\alpha)) d\beta \right|, & (3.81) \\ &\leq I' + \left| \int_{\Omega \cap \{D > |\alpha-\beta| > r\}} \nabla K((\alpha-\beta)\nabla X(\alpha)) d\beta \right|, & (3.82) \end{aligned}$$

so that it remains to bound the integral on the right in line (3.82). Let us denote $d_\alpha := \mathrm{dist}(\alpha, \partial\Omega)$.

Consider now the case that $d_\alpha > r$. First, suppose d_α is so large that $d_\alpha \geq D$. In that case the remaining integral is over the entire annulus, which is then contained in Ω, namely

$$\int_{D>|\alpha-\beta|>r} \nabla K((\alpha-\beta)\nabla X(\alpha))d\beta. \tag{3.83}$$

However, we have that for any annulus $A(\alpha)$ centered at α,

$$\int_{A(\alpha)} \nabla K((\alpha-\beta)\nabla X(\alpha))d\beta = 0. \tag{3.84}$$

To see this, consider the following calculation, for which we take R and R' to be the inner and outer radii, respectively, of $A(\alpha)$.

(3.85)

$$\begin{aligned}
\int_{A(\alpha)} \nabla K((\alpha-\beta)\nabla X(\alpha))d\beta &= -(\nabla X(\alpha))^{-1}\int_{A(\alpha)} \nabla_\beta (K((\alpha-\beta)\nabla X(\alpha)))d\beta, \\
&= -(\nabla X(\alpha))^{-1}\int_{\partial A(\alpha)} \nu_{out}(\beta)K((\alpha-\beta)\nabla X(\alpha))d\sigma(\beta), \\
&= -(\nabla X(\alpha))^{-1}\left(\int_{|\alpha-\beta|=R'} - \int_{|\alpha-\beta|=R}\right) \qquad (3.86)\\
&\qquad\qquad n^+(\alpha-\beta)K((\alpha-\beta)\nabla X(\alpha))d\sigma(\beta), \\
&= 0,
\end{aligned}$$

where $\nu_{out} = (\nu^1_{out}, \nu^2_{out})^t$ indicates the outward pointing normal for $\partial A(\alpha)$, and $n^+(\alpha-\beta) = \frac{(\alpha-\beta)^t}{|\alpha-\beta|}$. One finds the expression on the right in (3.86) is zero by observing that the expression

$$\int_{|\alpha-\beta|=\rho} n^+(\alpha-\beta)K((\alpha-\beta)\nabla X(\alpha))d\sigma(\beta) = \int_{|\beta|=\rho} n^+(\beta)K(\beta\nabla X(\alpha))d\sigma(\beta) \tag{3.87}$$

is independent of ρ, as the expression in the integral on the right is homogenous of degree 0 in β. Thus the case $d_\alpha \geq D$ is handled. Now we consider the case that $d_\alpha < D$. Then if $d_\alpha > r$ we can split the integral on the right in (3.82) into the following two:

$$\begin{aligned}
\int_{\Omega\cap\{D>|\alpha-\beta|>r\}} \nabla K((\alpha-\beta)\nabla X(\alpha))d\beta &= \int_{\Omega\cap\{D>|\alpha-\beta|>d_\alpha\}} \nabla K((\alpha-\beta)\nabla X(\alpha))d\beta \qquad (3.88)\\
&\quad + \int_{\{d_\alpha\geq|\alpha-\beta|>r\}} \nabla K((\alpha-\beta)\nabla X(\alpha))d\beta.
\end{aligned}$$

The second integral is over an annulus centered at α, so it is zero, and we are left with just the

first integral in the right hand side in the case that $d_\alpha > r$. Thus, we will have handled both cases, $d_\alpha > r$ and $d_\alpha \leq r$, if we can bound the quantity

$$\tilde{I} := \int_{\Omega \cap \{D > |\alpha - \beta| > \rho_0\}} \nabla K((\alpha - \beta)\nabla X(\alpha)) d\beta, \tag{3.89}$$

where $\rho_0 = \max(d_\alpha, r)$. First, we choose $\hat{\alpha} \in \partial\Omega$ such that $|\alpha - \hat{\alpha}| = d_\alpha$, and we define

$$\begin{aligned} \mathcal{S}(\alpha) &:= \{\hat{u} \in S^1 : \hat{u} \cdot \nu_{in}(\hat{\alpha}) > 0\}, \\ \mathcal{U}(\alpha) &:= \{\alpha + \rho\hat{u} : \rho \in (\rho_0, D), \hat{u} \in \mathcal{S}(\alpha)\}. \end{aligned} \tag{3.90}$$

The goal will be to show that, by the choice of D we make, the regions of integration for the integral in $\tilde{I}$ and the corresponding integral over the semi-annulus $\mathcal{U}(\alpha)$ will mostly overlap, since the boundary of Ω is somewhat smooth. This will be helpful since we will be able to show that the corresponding integral over $\mathcal{U}(\alpha)$ is zero.

In the proof of Lemma 3.3.4, a C^2 function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is produced with the properties that the interior of the patch is given by $\Omega = \{\beta : \varphi(\beta) > 0\}$, and $T_\alpha = \nabla^\perp \varphi(\alpha)$. Following the strategy of [4], we use this to define D, taking

$$D := \frac{\inf_{\partial\Omega} |\nabla\varphi|}{\sup_\Omega |\nabla^2 \varphi|}. \tag{3.91}$$

Clearly the denominator is nonzero. Otherwise T_α would be constant, and Ω unbounded. We denote $A_{\rho_0, D} := \{D > |\alpha - \beta| > \rho_0\}$ and check

$$\begin{aligned} \tilde{I} &= \left(\int_{\Omega \cap A_{\rho_0, D}} - \int_{\mathcal{U}(\alpha)} + \int_{\mathcal{U}(\alpha)} \right) \nabla K((\alpha - \beta)\nabla X(\alpha)) d\beta, & (3.92) \\ &= \left(\int_{(\Omega \cap A_{\rho_0, D}) \setminus \mathcal{U}(\alpha)} - \int_{\mathcal{U}(\alpha) \setminus (\Omega \cap A_{\rho_0, D})} + \int_{\mathcal{U}(\alpha)} \right) \nabla K((\alpha - \beta)\nabla X(\alpha)) d\beta. & (3.93) \end{aligned}$$

We argue that

$$\int_{\mathcal{U}(\alpha)} \nabla K((\alpha - \beta)\nabla X(\alpha)) d\beta = 0. \tag{3.94}$$

Because $K(z)$ is odd, $\nabla K(z)$ is even, and so (3.94) follows from the corresponding fact for the full annulus. Using (3.94) in (3.93), it follows that

$$|\tilde{I}| \leq \int_{(\Omega \cap A_{\rho_0, D}) \Delta \mathcal{U}(\alpha)} |\nabla K((\alpha - \beta)\nabla X(\alpha))| d\beta. \tag{3.95}$$

We define for $\rho \in (d_\alpha, D)$

$$\begin{aligned} \mathcal{A}_\rho(\alpha) &:= \{\hat{u} \in S^1 : \alpha + \rho\hat{u} \in \Omega\}, \\ R_\rho(\alpha) &:= \mathcal{A}_\rho(\alpha)\Delta\mathcal{S}(\alpha), \end{aligned} \tag{3.96}$$

and note that we have $(\Omega \cap A_{\rho_0,D})\Delta\mathcal{U}(\alpha) = \{\alpha + \rho\hat{u} : \rho \in (\rho_0, D), \hat{u} \in R_\rho(\alpha)\}$. By using the Geometric Lemma from [4], one finds that the following bound on the measure of $R_\rho(\alpha)$ holds[1]:

$$|R_\rho(\alpha)| \leq 2\pi \left(3\frac{d_\alpha}{\rho} + 2\frac{\rho}{D}\right), \tag{3.97}$$

as long as $\rho \in (d_\alpha, D)$ and $d_\alpha < D$. Thus we get the bound

$$\begin{aligned} |\tilde{I}| &\leq C \int_{(\Omega\cap A_{\rho_0,D})\Delta\mathcal{U}(\alpha)} |\alpha - \beta|^{-2} d\beta, \\ &= C \int_{\rho_0}^{D} \int_{R_\rho(\alpha)} d\theta \rho^{-1} d\rho, \\ &= C \int_{\rho_0}^{D} |R_\rho(\alpha)| \rho^{-1} d\rho, \\ &\leq C' \left(\int_{\rho_0}^{D} d_\alpha \rho^{-2} d\rho + \int_{\rho_0}^{D} D^{-1} d\rho \right), \\ &\leq C''((d_\alpha/\rho_0 - d_\alpha/D) + (1 - \rho_0/D)). \end{aligned} \tag{3.98}$$

Now using (3.79) and (3.81)-(3.82), our bound on $\tilde{I}$, and (3.88) (needed for the case $d_\alpha > r$ only), we have bounded I. Using (3.77), this bounds the integral in the left hand side of (3.70). □

Now, we provide the proof of Lemma 3.3.1.

Proof. (Proof of Lemma 3.3.1) Consider α varying in a small open neighborhood $N \subset \Omega$ and a fixed $r > 0$ such that $B_r(\alpha) \subset \Omega$ for all α in that neighborhood. We write

$$F_X(\alpha) = \int_{\Omega \setminus B_r(\alpha)} K(X(\alpha) - X(\beta)) d\beta + \int_{B_r(\alpha)} K(X(\alpha) - X(\beta)) d\beta. \tag{3.99}$$

We denote by $D^h_{\hat{u}}$ the difference quotient operator with increment h approximating the derivative in

[1] In [4] the proof is given for patches with $C^{1,\nu}$ boundary for $\nu \in (0, 1)$ and an analogous constant replacing D in the definition of the set corresponding to $R_\rho(\alpha)$. However, the Geometric Lemma also holds in the case $\nu = 1$, which is what we use.

the direction of a given unit vector $\hat{u}$. For a fixed $\hat{u}$ we have

$$D^h_{\hat{u}}(F_X)(\alpha) = D^h_{\hat{u}}\left(\int_{\Omega\setminus B_r(\alpha)} K(X(\alpha)-X(\beta))d\beta\right) + D^h_{\hat{u}}\left(\int_{B_r(\alpha)} K(X(\alpha)-X(\beta))d\beta\right), \quad (3.100)$$
$$= I^\flat_h + I^\sharp_h.$$

To handle $I^\flat_h$, we note

$$D_{\hat{u}}\left(\int_{\Omega\setminus B_r(\alpha)} K(X(\alpha)-X(\beta))d\beta\right) = \hat{u}\cdot\nabla X(\alpha)\int_{\Omega\setminus B_r(\alpha)} \nabla K(X(\alpha)-X(\beta))d\beta \quad (3.101)$$
$$+\int_{|\beta-\alpha|=r} \hat{u}\cdot\nu_{in}(\beta)K(X(\alpha)-X(\beta))d\sigma(\beta),$$

where the second integral in the right hand side of (3.101) is easily seen to be bounded uniformly in r, since $K(X(\alpha)-X(\beta)) = O(|\alpha-\beta|^{-1}) = O(r^{-1})$ and the circumference of the ball is $2\pi r$. Furthermore, the first integral on the right hand side of (3.101) is bounded independently of α, N, and r by Lemma 3.3.2. Thus, the left hand side of (3.101) is bounded uniformly, and so we can conclude that $\lim_{h\to 0} I^\flat_h$ exists for any $\alpha \in N$ and is uniformly bounded. In addition, note that by using the dominated convergence theorem in the right hand side of (3.101), we find that the left hand side defines a continuous function of α in N. Now we handle $I^\sharp_h$. We use the notation $D^{h,\alpha}_{\hat{u}}$ to emphasize that we mean the difference quotient operator $D^h_{\hat{u}}$ specifically with respect to the α variable, to avoid ambiguity (and later we use the corresponding notation for the β variable). Observe

$$I^\sharp_h = \int_{B_r(\alpha+h\hat{u})} D^{h,\alpha}_{\hat{u}}(K(X(\alpha)-X(\beta)))d\beta + \int K(X(\alpha)-X(\beta))D^{h,\alpha}_{\hat{u}}(1_{B_r(\alpha)}(\beta))d\beta, \quad (3.102)$$

where

$$\lim_{h\to 0}\int K(X(\alpha)-X(\beta))D^{h,\alpha}_{\hat{u}}(1_{B_r(\alpha)}(\beta))d\beta = \int_{|\beta-\alpha|=r} \hat{u}\cdot\nu_{out}(\beta)K(X(\alpha)-X(\beta))d\sigma(\beta). \quad (3.103)$$

We handle this in the same way that we handled the boundary integral in the right hand side of (3.101), finding the result is bounded independently of r, α, and N, and is continuous with respect

to α in N. For the integral on the left in (3.102), we have

$$\begin{aligned}\int_{B_r(\alpha+h\hat{u})} D^{h,\alpha}_{\hat{u}}(K(X(\alpha)-X(\beta)))d\beta &= \int_{B_r(\alpha+h\hat{u})}(D^{h,\alpha}_{\hat{u}}+D^{h,\beta}_{\hat{u}})(K(X(\alpha)-X(\beta)))d\beta \\ &\quad -\int_{B_r(\alpha+h\hat{u})} D^{h,\beta}_{\hat{u}}(K(X(\alpha)-X(\beta)))d\beta, \\ &= J^1_h + J^2_h.\end{aligned} \tag{3.104}$$

Regarding J^1_h, we note that (using $D^{\alpha}_{\hat{u}}$ to denote the directional derivative in the α variable, and analogously using the notation $D^{\beta}_{\hat{u}}$)

$$\int_{B_r(\alpha)}(D^{\alpha}_{\hat{u}}+D^{\beta}_{\hat{u}})(K(X(\alpha)-X(\beta)))d\beta = \int_{B_r(\alpha)}\hat{u}\cdot(\nabla X(\alpha)-\nabla X(\beta))\nabla K(X(\alpha)-X(\beta))d\beta \tag{3.105}$$

is uniformly bounded in r, since $|\nabla X(\alpha)-\nabla X(\beta)|\leq C|\alpha-\beta|^{\mu}$ and $|\nabla K(X(\alpha)-X(\beta))|\leq C'|\alpha-\beta|^{-2}$. From this we can conclude that $\lim_{h\to 0} J^1_h$ exists and is uniformly bounded. In addition, with a change of variables we find the right hand side of (3.105) is

$$\int_{B_r(0)}\hat{u}\cdot(\nabla X(\alpha)-\nabla X(\alpha-\beta))\nabla K(X(\alpha)-X(\alpha-\beta))d\beta, \tag{3.106}$$

which we see is continuous in α by using the dominated convergence theorem. Now we consider J^2_h. Note

$$\begin{aligned}\int_{B_r(\alpha+h\hat{u})} D^{h,\beta}_{\hat{u}}(K(X(\alpha)-X(\beta)))d\beta &= \frac{1}{h}\Big(\int_{B_r(\alpha+h\hat{u})} K(X(\alpha)-X(\beta+h\hat{u}))d\beta \\ &\qquad -\int_{B_r(\alpha+h\hat{u})} K(X(\alpha)-X(\beta))d\beta\Big), \\ &= \frac{1}{h}\Big(\int_{B_r(\alpha+2h\hat{u})} K(X(\alpha)-X(\beta))d\beta \\ &\qquad -\int_{B_r(\alpha+h\hat{u})} K(X(\alpha)-X(\beta))d\beta\Big),\end{aligned} \tag{3.107}$$

and

$$\lim_{h\to 0}\frac{1}{h}\left(\int_{B_r(\alpha+h\hat{u})}-\int_{B_r(\alpha)}\right)K(X(\alpha)-X(\beta))d\beta = \int_{|\alpha-\beta|=r}\hat{u}\cdot\nu_{out}(\beta)K(X(\alpha)-X(\beta))d\sigma(\beta), \tag{3.108}$$

which as we saw before is uniformly bounded. Also, the existence of the limit (3.108) implies that the limit as h tends to zero of the expression in (3.107) exists and is the same. Note, too, that the right hand side of (3.108) is a continuous function of α. With that, we get that $\lim_{h\to 0} J^2_h$ exists and

is uniformly bounded, and we can conclude that $\lim_{h\to 0} I_h^\sharp$ in addition to $\lim_{h\to 0} I_h^\flat$ both exists and is uniformly bounded, giving us the existence of a limit $D_{\hat{u}}(F_X)(\alpha)$ in any unit direction, $\hat{u}$, which we find is a continuous function of α. Since we get the existence and continuity of the partial derivatives $\partial_i(F_X(\alpha))$ for each $i = 1, 2$ in the neighborhood N, we have that F_X is differentiable there. From the above discussion we also see that this gives a bound on $|\nabla F_X(\alpha)|$ that is independent of the chosen r, α, and N. □

3.3.2 Tangential vector fields and Hölder bounds

Now that we have justified that one can make sense of the gradient of F_X and we have some related bounds, we can turn our attention to finding a convenient way to express derivatives along certain vector fields. With the following lemma, we get a formula for differentiating along a vector field tangent to the boundary of Ω and along a vector field normal to that one. The remainder of the key steps for proving Proposition 3.3.6 are framed in terms of the resulting expressions.

Lemma 3.3.3. *Let $T = (T^1, T^2) : \overline{\Omega} \to \mathbb{R}^2$ be a continuously differentiable, divergence-free vector field tangent to $\partial\Omega$. Then for $X \in B_\delta$ we have (i)*

$$T(F_X)(\alpha) = \int_\Omega (T_\alpha \nabla X(\alpha) - T_\beta \nabla X(\beta)) \nabla K(X(\alpha) - X(\beta)) d\beta \quad \text{for all } \alpha \in \Omega. \tag{3.109}$$

Here, we have denoted the value of the vector field at the point α by $T_\alpha = (T^1(\alpha), T^2(\alpha))$, and when the matrix-valued function $\nabla X(\cdot)$ appears to the right of $T_{(\cdot)}$, we mean that it acts on $T_{(\cdot)}$ via multiplication.

(ii) Taking $N = (N^1, N^2) : \overline{\Omega} \to \mathbb{R}^2$ to be the orthogonal vector field given by $N_\alpha = T_\alpha^\perp$, we have

$$N(F_X)(\alpha) = (T(F_X)(\alpha) P_X(\alpha) - \det \mathcal{O}(\alpha) e_1) (Q_X(\alpha))^{-1}, \tag{3.110}$$

for all $\alpha \in \Omega$ such that $T_\alpha \neq 0$, where $e_1 = (1, 0)$, $\mathcal{O}(\alpha) := \begin{pmatrix} T_\alpha \\ N_\alpha \end{pmatrix}$, and

$$P_X = \begin{pmatrix} N(X_1) & -N(X_2) \\ N(X_2) & N(X_1) \end{pmatrix}, \qquad Q_X = \begin{pmatrix} T(X_1) & -T(X_2) \\ T(X_2) & T(X_1) \end{pmatrix}. \tag{3.111}$$

Proof. One verifies the statement of (i) with a computation using the fact that T is divergence free and tangent to $\partial\Omega$ in particular. Now we show that (ii) holds. At first, let us prove the formula

(3.110) holds for $X \in B_{\delta'}$ for small δ' with X taking values in $\mathbb{R}^2$, rather than $\mathbb{C}^2$. We then define

$$G_X(x) := \int_\Omega K(x - X(\beta))d\beta. \tag{3.112}$$

Then we claim that

$$\begin{aligned} (\mathrm{div} G_X)(X(\alpha)) &= 0, & (3.113)\\ (\mathrm{curl} G_X)(X(\alpha)) &= \det(\nabla X(\alpha))^{-1}, \quad \text{for } \alpha \in \Omega. & (3.114) \end{aligned}$$

This follows if we observe that

$$G_X(x) = \int_{X(\Omega)} K(x-y)|\det(\nabla X(X^{-1}(y)))^{-1}|dy, \tag{3.115}$$

so $\mathrm{div} G_X = 0$ and

$$(\mathrm{curl} G_X)(X(\alpha)) = \int_{X(\Omega)} \delta(X(\alpha)-y)|\det\left(\nabla X(X^{-1}(y))\right)^{-1}|dy = \det\left(\nabla X(\alpha)\right)^{-1}, \tag{3.116}$$

noting that we can ensure the determinant is positive for real $X \in B_{\delta'}$ if δ' is small enough. From (3.113) and (3.114) we find

$$(\partial_{x_2} G_X)(X(\alpha)) = (\partial_{x_1} G)(X(\alpha)) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \det(\nabla X(\alpha))^{-1} e_1. \tag{3.117}$$

Meanwhile,

$$(\mathcal{O}(\alpha)\nabla X(\alpha))^{-1} \begin{pmatrix} T(F_X)(\alpha) \\ N(F_X)(\alpha) \end{pmatrix} = (\nabla G_X)(X(\alpha)). \tag{3.118}$$

This gives us two ways of calculating $\partial_{x_2} G_X(X(\alpha))$. For the first, we simply look at the second row in the above expression. For the second, we compute the first row, $\partial_{x_1} G_X(X(\alpha))$, and then use (3.117) to get another expression for $\partial_{x_2} G_X(X(\alpha))$ in terms of this. Equating these gives us

$$\begin{aligned} & e_2(\mathcal{O}(\alpha)\nabla X(\alpha))^{-1} \begin{pmatrix} T(F_X)(\alpha) \\ N(F_X)(\alpha) \end{pmatrix} & (3.119)\\ & = e_1 \left((\mathcal{O}(\alpha)\nabla X(\alpha))^{-1} \begin{pmatrix} T(F_X)(\alpha) \\ N(F_X)(\alpha) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \det(\nabla X(\alpha))^{-1} I \right). \end{aligned}$$

After a little computation and rearrangement this gives

$$N(F_X)(\alpha)Q_X(\alpha) = T(F_X)(\alpha)P_X(\alpha) - \det \mathcal{O}(\alpha)e_1, \tag{3.120}$$

and then multiplying on the right by $(Q_X(\alpha))^{-1}$ gives the desired formula. Thus, (3.110) holds for $X \in B_{\delta'}$ taking values in $\mathbb{R}^2$. To get the same result for δ sufficiently small and any $X \in B_\delta$, not necessarily real, we fix such an X and define the map $X_\lambda := \mathrm{Re}(X) + \lambda \mathrm{Im}(X)$ for λ in a disc in $\mathbb{C}$ of radius 2 about 0. Then observe that $X \in B_\delta$ implies X_λ satisfies $\|X_\lambda - Id\|_Y \leq C\delta$. Thus, as long as $\delta < \frac{\delta'}{C}$, $X_\lambda \in B_{C\delta}$ implies that the formula holds for X_λ for all λ in the interval $[-1, 1]$. Now we claim that if we insert X_λ in the right hand side of (3.110), the result is an analytic function of λ in $\{|\lambda| < 2\}$, given $X \in B_\delta$ for small enough δ. To verify this, we first fix an arbitrary in $\alpha \in \Omega$. Now we consider the spatial difference quotients approximating $T(F_{X_\lambda})(\alpha)$. For sufficiently small $h \neq 0$ and a differentiable function $g : \Omega \to \mathbb{R}^2$, we define

$$T^h(g)(\alpha) := \frac{1}{h}(g(\alpha + hT_\alpha) - g(\alpha)). \tag{3.121}$$

Note that as long as δ is small enough, $X_\lambda \in B_{C\delta}$ satisfies the hypothesis of Lemma 3.3.1. We find that as h varies in some small interval, say $h \neq 0$ with $|h| < h_{max}(\alpha)$ for some $h_{max} > 0$, the $T^h(F_{X_\lambda})(\alpha)$ are all analytic functions of λ, and moreover they are convergent and uniformly bounded as h tends to zero, by Lemma 3.3.1. This implies that $T(F_{X_\lambda})(\alpha)$ is analytic in $\{|\lambda| < 2\}$. In fact, if we consider (3.110) with X_λ in place of X, which we shall refer to as equation $(3.110)_\lambda$, we find that all the terms in the right hand side are analytic in λ as long as $T_\alpha \neq 0$. Similarly, the spatial difference quotients approximating $N(F_{X_\lambda})(\alpha)$ are all analytic functions of λ in $\{|\lambda| < 2\}$, also uniformly bounded, so that $N(F_{X_\lambda})(\alpha)$ is also analytic in λ. Recall that we already have the equality $(3.110)_\lambda$ for λ in $[-1, 1]$, and so we must have that $(3.110)_\lambda$ holds for all $|\lambda| < 2$. In particular it holds for $\lambda = i$, which proves the formula for $X = \mathrm{Re}(X) + i\mathrm{Im}(X)$. □

Note that we can try to recover ∇F_X from the expressions for $T(F_X)$ and $N(F_X)$ via the relation

$$\mathcal{O}(\alpha)\nabla F_X(\alpha) = \begin{pmatrix} T(F_X)(\alpha) \\ N(F_X)(\alpha) \end{pmatrix}. \tag{3.122}$$

Of course, this does not work if $\mathcal{O}(\alpha)$ is not invertible, which happens exactly at points where T_α is zero. The next lemma provides a pair of vector fields T and T' which are not zero at the same

points, so that when it is used in combination with Lemma 3.3.3 we get a complete description of the gradient of $F_X(\alpha)$ at every point $\alpha \in \Omega$.

Lemma 3.3.4. *There exists a pair of C^1, divergence-free vector fields $T, T' : \overline{\Omega} \to \mathbb{R}^2$ tangent to $\partial\Omega$ and such that each of T and T' has a unique zero belonging to the interior, p and p' respectively, with $p \neq p'$.*

Proof. Since Ω is convex and $\partial\Omega$ is C^2, it is not hard to construct a C^2 bijection with C^2 inverse, say $\psi : \mathbb{R}^2 \to \mathbb{R}^2$, mapping $\partial\Omega$ to S^1, and Ω to the interior. Then we may take a function with the desired properties on the unit ball and pull it back by ψ to get a new function φ, which is then C^2, zero on $\partial\Omega$, and has zero gradient at exactly one point inside Ω. Similarly, one can construct another function φ' which satisfies the same properties, but has zero gradient at a different point in Ω. Taking $T = \nabla^\perp\varphi$ and $T' = \nabla^\perp\varphi'$ then gives vector fields with the desired properties. □

The main remaining task is to prove the Hölder seminorm bound for $T(F_X)$ and $T'(F_X)$. Once we have this, in combination with using (3.122) to express the gradient ∇F_X in terms of the derivatives along T and T', it will not be hard to prove the Hölder continuity of ∇F_X.

Lemma 3.3.5. *Let T and T' be the vector fields asserted by Lemma 3.3.4. There exists a constant $\tilde{C}$ depending only on Ω, T, and T' such that for all $X \in B_\delta$*

$$|T(F_X)(\alpha) - T(F_X)(\gamma)| \leq \tilde{C}|\alpha - \gamma|^\mu \quad \text{for } \alpha, \gamma \in \Omega. \tag{3.123}$$

The same bound holds with T' in place of T.

Proof. Here we prove the bound for the vector field T, though T is straightforwardly replaced by T' for the other bound.

Fix α, $\gamma \in \Omega$. First we split Ω into the regions

$$\Omega_{near} := \Omega \cap \{\beta : |\alpha - \beta| < 10|\alpha - \gamma|\}, \qquad \Omega_{far} := \Omega \cap \{\beta : |\alpha - \beta| > 10|\alpha - \gamma|\}. \tag{3.124}$$

Thus, using Lemma 3.3.3, we can write

$$\begin{aligned} T(F_X)(\alpha) - T(F_X)(\gamma) &= \int_{\Omega_{near}\cup\Omega_{far}} (T_\alpha \nabla X(\alpha) - T_\beta \nabla X(\beta)) \nabla K(X(\alpha) - X(\beta)) d\beta \\ &\quad - \int_{\Omega_{near}\cup\Omega_{far}} (T_\gamma \nabla X(\gamma) - T_\beta \nabla X(\beta)) \nabla K(X(\gamma) - X(\beta)) d\beta, \\ &= I_{near}(\alpha) - I_{near}(\gamma) + I_{far}(\alpha) - I_{far}(\gamma), \end{aligned} \tag{3.125}$$

where

$$I_{near}(\zeta) := \int_{\Omega_{near}} (T_\zeta \nabla X(\zeta) - T_\beta \nabla X(\beta)) \nabla K(X(\zeta) - X(\beta)) d\beta \quad \text{for } \zeta \in \Omega, \tag{3.126}$$

and I_{far} is defined analogously. First we will show that $|I_{near}(\alpha)|$ and $|I_{near}(\gamma)|$ are both $O(|\alpha - \gamma|^\mu)$. Observe that

$$\begin{aligned} |I_{near}(\alpha)| &\leq \int_{\Omega_{near}} |T_\alpha \nabla X(\alpha) - T_\beta \nabla X(\beta)| |\nabla K(X(\alpha) - X(\beta))| d\beta, \\ &\leq C' \int_{|\alpha-\beta|<10|\alpha-\gamma|} |\alpha - \beta|^{\mu-2} d\beta, \\ &\leq C'' |\alpha - \gamma|^\mu. \end{aligned} \tag{3.127}$$

Similarly,

$$\begin{aligned} |I_{near}(\gamma)| &\leq \int_{\Omega_{near}} |T_\gamma \nabla X(\gamma) - T_\beta \nabla X(\beta)| |\nabla K(X(\gamma) - X(\beta))| d\beta, \\ &\leq C' \int_{|\alpha-\beta|<10|\alpha-\gamma|} |\gamma - \beta|^{\mu-2} d\beta, \\ &\leq C' \int_{|\gamma-\beta|<10|\alpha-\gamma|} |\gamma - \beta|^{\mu-2} d\beta, \\ &\leq C'' |\alpha - \gamma|^\mu. \end{aligned} \tag{3.128}$$

Now that these quantities have been bounded, we are left with handling

$$\begin{aligned} I_{far}(\alpha) - I_{far}(\gamma) &= \int_{\Omega_{far}} (T_\alpha \nabla X(\alpha) - T_\gamma \nabla X(\gamma)) \nabla K(X(\alpha) - X(\beta)) d\beta \\ &\quad + \int_{\Omega_{far}} (T_\gamma \nabla X(\gamma) - T_\beta \nabla X(\beta)) \\ &\qquad \times (\nabla K(X(\alpha) - X(\beta)) - \nabla K(X(\gamma) - X(\beta))) d\beta, \\ &= I^i_{far} + I^{ii}_{far}. \end{aligned} \tag{3.129}$$

Note that

$$\begin{aligned} |I^i_{far}| &\leq |T_\alpha \nabla X(\alpha) - T_\gamma \nabla X(\gamma)| \left| \int_{\Omega_{far}} \nabla K(X(\alpha) - X(\beta)) d\beta \right|, \\ &\leq C |\alpha - \gamma|^\mu \left| \int_{\Omega_{far}} \nabla K(X(\alpha) - X(\beta)) d\beta \right|, \end{aligned} \tag{3.130}$$

where $\Omega_{far} = \Omega \cap \{|\alpha - \beta| > 10|\alpha - \gamma|\}$, so an application of Lemma 3.3.2 gives the uniform bound

$$|I^i_{far}| \leq \tilde{C}|\alpha - \gamma|^\mu \quad \text{for all } \alpha, \gamma \in \Omega. \tag{3.131}$$

Now we proceed with the bound for I^{ii}_{far}. We have

$$I^{ii}_{far} = \int_{\Omega_{far}} (T_\gamma \nabla X(\gamma) - T_\beta \nabla X(\beta))(\nabla K(X(\alpha) - X(\beta)) - \nabla K(X(\gamma) - X(\beta)))d\beta. \tag{3.132}$$

We rewrite

$$\begin{aligned} &\nabla K(X(\alpha) - X(\beta)) - \nabla K(X(\gamma) - X(\beta)) \\ &\qquad = \sum_{i=1}^{2} (X_i(\alpha) - X_i(\gamma)) \int_0^1 (\partial_i \nabla K)(\tau X(\alpha) + (1-\tau)X(\gamma) - X(\beta))d\tau. \end{aligned} \tag{3.133}$$

Now we claim that for small enough δ there is a universal constant $c > 0$ such that for any $\tau \in [0, 1]$, and $\alpha, \beta, \gamma \in \Omega$,

$$\left| \sum_{i=1}^{2} (\tau X_i(\alpha) + (1-\tau)X_i(\gamma) - X_i(\beta))^2 \right| \geq c|\gamma - \beta|^2 \quad \text{if} \quad |\alpha - \beta| > 10|\alpha - \gamma|. \tag{3.134}$$

Regarding the verification of this, note that we have

$$\begin{aligned} &|X_i(\xi) - X_i(\eta)| \leq (1 + \delta)|\xi - \eta| \\ \text{and} \quad &\left| \sum_{i=1}^{2} (X_i(\xi) - X_i(\eta))^2 \right| \geq (1 - c\delta)|\xi - \eta|^2 \quad \text{for any } \xi, \eta \in \Omega, \end{aligned} \tag{3.135}$$

and that

$$\frac{|\alpha - \gamma|}{|\beta - \gamma|} < \frac{1}{9} \quad \text{if} \quad |\alpha - \beta| > 10|\alpha - \gamma|. \tag{3.136}$$

One uses (3.135) and (3.136), and performs the same computation used to establish the bound (3.53), via (3.54)-(3.58) to get (3.134).

Using (3.134) along with (3.133) in (3.132) yields

$$\begin{aligned}
|I^{ii}_{far}| &\leq \int_{\Omega_{far}} |T_\gamma \nabla X(\gamma) - T_\beta \nabla X(\beta)||\nabla K(X(\alpha) - X(\beta)) - \nabla K(X(\gamma) - X(\beta))| d\beta \\
&\leq C' \int_{\Omega_{far}} |\gamma - \beta|^\mu |\gamma - \beta|^{-3} |X(\alpha) - X(\gamma)| d\beta, \qquad (3.137) \\
&\leq C'' |\alpha - \gamma| \int_{\Omega_{far}} |\gamma - \beta|^{\mu - 3} d\beta, \\
&\leq C'' |\alpha - \gamma| \int_{|\alpha - \beta| > 10|\alpha - \gamma|} |\gamma - \beta|^{\mu - 3} d\beta, \\
&\leq C'' |\alpha - \gamma| \int_{|\gamma - \beta| > |\alpha - \gamma|} |\gamma - \beta|^{\mu - 3} d\beta, \\
&= C''' |\alpha - \gamma|^\mu.
\end{aligned}$$

Using our bound on I^{ii}_{far} along with (3.131), (3.129), (3.128), and (3.127), in (3.125), we conclude with (3.123). □

3.3.3 Proofs of the main bound and of the analyticity of trajectories

Having established the lemmas of the previous sections, we are ready to give the proof of the main proposition.

Proposition 3.3.6. *Fix $\mu \in (0, 1)$. Assume Ω is bounded, convex, and has C^2 boundary, and let F be as in Definition 3.1.1, with*

$$\omega(\alpha) := \begin{cases} 1 & \text{if } \alpha \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \qquad (3.138)$$

Then there exists a $\delta > 0$ such that

$$\sup_{X \in B_\delta} \|F(X)\|_{C^{1,\mu}(\Omega)} \leq C, \qquad (3.139)$$

where C depends only on Ω.

Proof. Let $X \in B_\delta$. To bound $\|F_X\|_{L^\infty(\Omega)}$, we simply observe that (3.67) gives a uniform bound. For the $C^1(\Omega)$ norm, we use the uniform bound on $|\nabla F_X(\alpha)|$ following from Lemma 3.3.1. Thus it only remains to give the $[\nabla F_X]_\mu$ estimate. First we establish the pair of estimates

$$|T(F_X)(\alpha) - T(F_X)(\gamma)| \leq \tilde{C} |\alpha - \gamma|^\mu, \qquad (3.140)$$

$$|N(F_X)(\alpha) - N(F_X)(\gamma)| \leq \tilde{C} |\alpha - \gamma|^\mu, \qquad (3.141)$$

for $\alpha, \gamma \in \Omega \setminus \mathcal{N}$, where $\mathcal{N}$ is a neighborhood of p, for T and p as in Lemma 3.3.4, and $N = T^{\perp}$.

Lemma 3.3.5 establishes the Hölder bound for $T(F_X)$ with (4.89). Recall the formula for $N(F_X)$ given by Lemma 3.3.3. From that, along with the observation that since T is C^1 and ∇X is Hölder continuous, we get the desired Hölder continuity of $\mathcal{O}$, P_X, and Q_X^{-1} away from p, we find that (3.140) also implies the estimate (3.141) for $\alpha, \gamma \in \Omega \setminus \mathcal{N}$.

Now note $\mathcal{O}(\alpha)$ is invertible at any $\alpha \in \Omega \setminus \{p\}$, and

$$\nabla F_X(\alpha) = (\mathcal{O}(\alpha))^{-1} \begin{pmatrix} T(F_X)(\alpha) \\ N(F_X)(\alpha) \end{pmatrix}. \tag{3.142}$$

With (3.142), and the Hölder continuity of $\mathcal{O}^{-1}$ away from p, we then find that this implies

$$|\nabla F_X(\alpha) - \nabla F_X(\gamma)| \leq \tilde{C}' |\alpha - \gamma|^{\mu} \tag{3.143}$$

for all $\alpha, \gamma \in \Omega \setminus \mathcal{N}$. Applying Lemma 3.3.5 and Lemma 3.3.3 with T' and N' gives us the same estimate for all $\alpha, \gamma \in \Omega \setminus \mathcal{N}'$, for a neighborhood $\mathcal{N}'$ of p'. Then, since we are free to choose the neighborhoods $\mathcal{N}$ and $\mathcal{N}'$ to be disjoint, we have the Hölder estimate (3.143) in all of Ω. We conclude

$$\|F_X\|_{C^{1,\mu}(\Omega)} \leq C. \tag{3.144}$$

□

Corollary 3.3.7. *Take $Y = C^{1,\mu}(\Omega)$ and suppose $\omega = 1$ on Ω, where Ω satisfies the hypotheses of Proposition 3.3.6. Then for δ as in Proposition 3.3.6 and $X_z \in \mathcal{B}_{\delta,r}$, $F(X_z)$ is an analytic function in z on d_r as a map into $C^{1,\mu}(\Omega)$.*

Proof. This follows from Lemmas 3.1.4 and 2.2.4 and Proposition 3.3.6. □

Now applying Proposition 3.3.6 in the case $Y = C^{1,\mu}(\Omega)$, we get the Lagrangian analyticity result for vortex patches as a corollary.

Corollary 3.3.8. *Let $Y = C^{1,\mu}(\Omega)$ for Ω and ω as in Proposition 3.3.6. Then there exists an $\epsilon > 0$, an $r > 0$, and a unique function $X(\alpha, t)$ in $\mathcal{B}_{\epsilon,r}$ solving (3.7), and the solution is then analytic in the time variable on the disc d_r.*

Remark 3.3.9. *Corollary 3.3.8 implies directly that the points inside the patch $\Omega(t) = X(\Omega, t)$ have analytic trajectories. One can extend this to the analyticity of the particle trajectories at the*

boundary. To see this, consider an $\alpha \in \partial\Omega$. *Taking a sequence of* α_k *tending to* α, *the analytic functions* $X(\alpha_k, t)$ *are uniformly bounded and converge to* $X(\alpha, t)$, *which then must also be analytic.*

Chapter 4

Lagrangian analyticity for the Euler-Poisson system

The pressureless Euler-Poisson equation gives a model for the dynamics of a fluid in which the fluid particles experience long-range interaction. In particular, it models stellar dynamics, in which case the force between particles is attractive, or the dynamics of a charged fluid, in which case it is repulsive. Specifically, we consider the three dimensional case. The system is given by

$$\begin{aligned} \partial_t \rho + u \cdot \nabla \rho &= -\rho \mathrm{div} u, && (4.1) \\ \partial_t u + u \cdot \nabla u &= q \nabla \Delta^{-1} \rho, && (4.2) \\ (\rho, u) &= (\rho_0, u_0) \text{ at } t = 0. \end{aligned}$$

where $q = \pm 1$, with unknowns density $\rho : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^+$ and velocity $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$, and with initial data $\rho_0 \in C_c^s$ and $u_0 \in H^s$ for $s \geq 6$. In the case $q = 1$, the force is repulsive, and in the case $q = -1$, it is attractive. Here we explicitly handle the case of a repulsive force, i.e. $q = 1$, but the results carry over to the $q = -1$ case with essentially no changes to the proofs.

In this section, using a strategy based on that in Section 2, with minor modifications, we will demonstrate that we also have analyticity of the Lagrangian trajectories in t for this fluid mechanics model, for initial data of the regularity prescribed above.

We remark that in the study [11], a Lagrangian analyticity-type property is verified for a different Euler-Poisson system, modeling the dynamics of a self-gravitating fluid in a cosmological setting. There the authors use recursive relations for Taylor coefficients to prove the trajectories are analytic

as functions of a sort of time parameter, as long as the initial data for the system is slightly better than differentiable. For several reasons the results in that study do not directly apply to the Lagrangian analyticity problem treated here in this section. The systems studied here and there are fundamentally different. In particular, in that study, analyticity is proven not with respect to the typical time variable t, but instead with respect to a parameter τ proportional to $t^{2/3}$, and their results do not imply analyticity of the corresponding trajectories in t.

4.1 Lagrangian formulation

Consider a solution (ρ, u) to the Euler-Poisson equations. For each $\alpha \in \mathbb{R}^3$ the Lagrangian trajectory $X(\alpha, t)$ is then defined as the solution of the ODE

$$\begin{aligned} \frac{dX}{dt}(\alpha, t) &= u(X(\alpha, t), t), \\ X(\alpha, 0) &= \alpha. \end{aligned} \tag{4.3}$$

One can show that a consequence of (4.1) together with (4.3) is that

$$\rho(X(\alpha, t), t)) = (\det \nabla X(\alpha, t))^{-1} \rho_0(\alpha). \tag{4.4}$$

For the kernel

$$K(y) = \frac{1}{4\pi} \frac{y}{|y|^3}, \tag{4.5}$$

from (4.2), we find

$$\begin{aligned} \frac{d^2X}{dt^2}(\alpha, t) &= \int K(X(\alpha, t) - y)\rho(y, t)dy, \\ &= \int K(X(\alpha, t) - X(\beta, t))\rho(X(\beta, t), t) \det \nabla X(\beta, t) d\beta, \\ &= \int K(X(\alpha, t) - X(\beta, t))\rho_0(\beta)d\beta. \end{aligned} \tag{4.6}$$

Thus, we can write the Lagrangian formulation as a first order ODE in the following way:

$$\begin{aligned} \frac{dX}{dt}(\alpha, t) &= V(\alpha, t), \\ \frac{dV}{dt}(\alpha, t) &= \int K(X(\alpha, t) - X(\beta, t))\rho_0(\beta)d\beta, \end{aligned} \tag{4.7}$$

with initial data $X|_{t=0} = Id$, $V|_{t=0} = u_0$. In the next section, we will define the corresponding operator F on the appropriate Banach space so that the above system can be written as

$$\begin{aligned} \frac{dX}{dt}(\alpha, t) &= V(\alpha, t), \\ \frac{dV}{dt}(\alpha, t) &= F(X)(\alpha, t), \\ (X, V)|_{t=0} &= (Id, u_0), \end{aligned} \tag{4.8}$$

given initial data $\rho_0 \in C_c^s$ and $u_0 \in H^s$ for $s \geq 6$ for (4.1)-(4.2).

Before continuing, we make some comparison and contrast with the situation for the incompressible Euler equations. The first is that this model is intended to handle compressible fluids, as the density ρ is free to be a nonconstant function. Thus the volume of a fixed set of fluid particles is not necessarily preserved under the flow map $X(\alpha, t)$, and we do not have the identity $\det \nabla X(\alpha, t) = 1$. On the other hand, from (4.4), we do have that the quantity

$$\rho(X(\alpha, t), t))(\det \nabla X(\alpha, t)) = \rho_0(\alpha) \tag{4.9}$$

is independent of time. This is what allowed us to write the right hand side of (4.6) without explicit dependence on $\rho(x, t)$ at later times $t > 0$. For the incompressible Euler equations we were also able to write the analogous right hand side without explicit dependence of on any unknowns at later times besides the trajectory map.

4.2 Definitions of operators

4.2.1 Defining the operator $F(X)$

In the following we have a fixed $\rho \in C_c^s(\mathbb{R}^3)$, with some fixed $s \geq 6$, where ρ plays the role of the initial density for the system (4.1).

Now we define the F for the case of the 3D Euler-Poisson system. We consider the vector-valued functions on $\mathbb{R}^3$ which take values in $\mathbb{C}^3$, in particular such maps sufficiently close to the identify map Id in the Sobolev norm $H^s(\mathbb{R}^3)$, $s \geq 6$.

$$B_\delta = \{\mathbb{C}^3\text{-valued } X \text{ defined on } \mathbb{R}^3 : \|X - Id\|_{H^s(\mathbb{R}^3)} \leq \delta\}. \tag{4.10}$$

We have the kernel

$$K(z) = \frac{1}{4\pi} \frac{(z_1, z_2, z_3)}{(z_1^2 + z_2^2 + z_3^2)^{3/2}}, \quad z_1^2 + z_2^2 + z_3^2 \neq 0, \tag{4.11}$$

which extends analytically; for $z \in \mathbb{C}^3$ we choose to take the branch cut for the square root along the negative reals. Now we record the fact

$$X(\alpha) - X(\beta) = (\alpha - \beta) \int_0^1 \nabla X(\tau\alpha + (1-\tau)\beta) d\tau = (\alpha - \beta)(I + O(\delta)). \tag{4.12}$$

We use the convention that vectors indicating a spatial position, like X, are row vectors, a spatial gradient is a column vector, and, compatible with these conventions, $(\nabla X)_{ij} = \partial X_j / \partial \alpha_i$.

We verify with the following computation that it is compatible with the chosen branch cut to plug $z = X(\alpha) - X(\beta)$ into K for any $X \in B_\delta$, and $\alpha, \beta \in \mathbb{R}^3$.

$$\begin{aligned} \sum_{i=1}^3 (X_i(\alpha) - X_i(\beta))^2 &= \sum_{i=1}^3 [(\alpha - \beta) \int_0^1 \nabla X(\tau\alpha + (1-\tau)\beta) d\tau]_i^2, \\ &= \sum_{i=1}^3 [(\alpha - \beta)(I + O(\delta))]_i^2 = |\alpha - \beta|^2 (1 + O(\delta)). \end{aligned} \tag{4.13}$$

Taking the real part of both sides of the equation gives

$$\mathrm{Re} \left(\sum_{i=1}^3 (X_i(\alpha) - X_i(\beta))^2 \right) = |\alpha - \beta|^2 \mathrm{Re}(1 + O(\delta)) \geq |\alpha - \beta|^2 / 2, \tag{4.14}$$

for small enough δ.

One verifies from (4.13) that the singularity along the diagonal in $K(X(\alpha) - X(\beta))$ is no worse than that of $|\alpha - \beta|^{-2}$, i.e.

$$|K(X(\alpha) - X(\beta))| \leq C |\alpha - \beta|^{-2} \tag{4.15}$$

for some constant C, which is an integrable singularity in three dimensions. We comment that we will encounter singularities that are not integrable in the later content of this section, which will require a careful analysis involving singular integral estimates, but for the moment, this is not necessary.

Definition 4.2.1. *Fix a scalar, real-valued function $\rho \in H^s(\mathbb{R}^3) \cap C_c(\mathbb{R}^3)$. For $X \in B_\delta$ we define*

$F(X)(\alpha)$ *for each* $\alpha \in \mathbb{R}^3$ *by*

$$F(X)(\alpha) = \int K(X(\alpha) - X(\beta))\rho(\beta)d\beta. \tag{4.16}$$

In this definition, ρ plays the role of the initial density. Note that in the integral defining $F(X)(\alpha)$, we get something no more singular than $|\alpha - \beta|^{-2}$, and so $F(X)$ gives us a finite $\mathbb{C}^3$-valued function on $\mathbb{R}^3$.

4.2.2 Relevant integral kernel operators

Taking spatial derivatives of $F(X)(\alpha)$ naturally leads us to some singular integral operators of the form

$$(T_{\mathcal{K}} f)(\alpha) = p.v. \int f(\beta)\mathcal{K}(\alpha, \beta)d\beta, \tag{4.17}$$

for a scalar $f : \mathbb{R}^3 \to \mathbb{C}$ in $C^1(\mathbb{R}^3)$, and kernel $\mathcal{K} : (\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{\alpha = \beta\}) \to \mathbb{C}$, where $|\mathcal{K}(\alpha, \beta)| \leq C|\alpha - \beta|^{-3}$. The $T_{\mathcal{K}} f$ will then be defined pointwise, given basic assumptions are satisfied for a given $\mathcal{K}(\alpha, \beta)$. We extend this definition naturally for vector valued kernels or matrix valued kernels. For example, we will have kernels of the form $\mathcal{K} : (\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{\alpha = \beta\}) \to (\mathbb{C}^3)^2$, with corresponding $T_{\mathcal{K}}$ applied to a vector valued C^1 function $f : \mathbb{R}^3 \to \mathbb{C}^3$, yielding $T_{\mathcal{K}} f \in \mathbb{R}^3 \to \mathbb{C}^3$. In particular, we will deal with the operator $T_{\tilde{K}}$, where

$$\tilde{K}(\alpha, \beta) = -\nabla_\beta (K(X(\alpha) - X(\beta))) = \nabla X(\beta) \nabla K(X(\alpha) - X(\beta)). \tag{4.18}$$

We will also handle related operators from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. It will assist us to have the following facts and definition recorded, which give bounds for these kinds of integral kernels and integral kernel operators. The proofs of the boundedness of these operators are delegated to section on integral kernel operator bounds at the end of the section.

Lemma 4.2.2. *For* $M \in (\mathbb{C}^3)^2$ *with* $|M| \leq \delta'$*, where* δ' *is a small constant, the function* $K(z) : \mathbb{C}^3 \setminus 0 \to \mathbb{C}^3$ *defined by* (4.11)*, satisfies*

$$|(\partial_z^\gamma K)(\beta(I + M))| \leq \quad A_\gamma |\beta|^{-2-|\gamma|} \quad \textit{for all } \beta \in \mathbb{R}^3 \textit{ and } |\gamma| \geq 0. \tag{4.19}$$

Proof. The proof of this is is a straightforward calculation. □

Definition 4.2.3. *(i) For each $k = 1, 2, 3$, define*

$$\tilde{K}^k(\alpha, \beta) = \sum_{i=1}^{3} \partial_k X_i(\beta)(\partial_i K)(X(\alpha) - X(\beta)). \tag{4.20}$$

(ii) We define for $f \in L^2 \cap C^1$

$$(T_{X,k} f)(\alpha) = p.v. \int \tilde{K}^k(\alpha, \beta) f(\beta) d\beta \quad \text{for } \alpha \in \mathbb{R}^3. \tag{4.21}$$

(iii) We define for $f \in L^2$ and $\epsilon > 0$

$$(T^{\epsilon}_{X,k} f)(\alpha) = \int_{|\alpha - \beta| > \epsilon} \tilde{K}^k(\alpha, \beta) f(\beta) d\beta. \tag{4.22}$$

It is proved in Lemma 4.5.4 that $T^{\epsilon}_{X,k} f$ converges in L^2 as ϵ tends to zero. We define

$$T^0_{X,k} f = \lim_{\epsilon \to 0} T^{\epsilon}_{X,k} f \quad \text{in } L^2. \tag{4.23}$$

Remark 4.2.4. *It is proved in the subsection on integral kernel operator bounds that the p.v. limit defining $T_{X,k}$ converges uniformly in α, and that $T^{\epsilon}_{X,k}$ and $T^0_{X,k}$ as defined are bounded operators from L^2 to L^2, with Lemmas 4.5.3 and 4.5.4. Also given by these lemmas are the bounds*

$$\|T^{\epsilon}_{X,k} f\|_{L^\infty} \leq C(\|f\|_{L^2} + \|f\|_{C^1}) \quad \text{for all } f \in L^2 \cap C^1, \tag{4.24}$$

and

$$\|T_{X,k} f\|_{L^\infty} \leq C(\|f\|_{L^2} + \|f\|_{C^1}) \quad \text{for all } f \in L^2 \cap C^1, \tag{4.25}$$

and the L^2 bounds

$$\|T^{\epsilon}_{X,k} f\|_{L^2} \leq C\|f\|_{L^2} \quad \text{for all } f \in L^2, \tag{4.26}$$

for $\epsilon > 0$, and

$$\|T^0_{X,k} f\|_{L^2} \leq C\|f\|_{L^2} \quad \text{for all } f \in L^2. \tag{4.27}$$

4.3 Differentiation formulae

Fix $k = 1, 2$ or 3. Observe that we may regard the quantity

$$\mathcal{K}(\alpha, \beta) := \partial_{\beta_k} \left(K(X(\alpha) - X(\alpha - \beta)) \right) \tag{4.28}$$

as a kernel, with

$$|\mathcal{K}(\alpha, \beta)| \leq C|\beta|^{-3}. \tag{4.29}$$

Objects like this occur in integrals like in (4.21) and their derivatives, which is apparent if one first makes a change of variable in the integral replacing β with $(\alpha - \beta)$, then differentiates with respect to some α component, and manipulates the resulting expression. We elaborate on this technique further in what follows; it provides a useful technique for calculating high order derivatives of the Biot-Savart type integrals composed with trajectory maps, such as (4.21), in particular, and we remark that it can be adapted to the similar Biot-Savart type trajectory map operators studied in Section 3 on the incompressible Euler equations. The goal of this part of the section is to give a formula for taking any number of derivatives of $F(X)$, as in Definition 4.2.1, resulting in a compact expression well-suited for making L^2 estimates. Using this, together with the singular integral operator estimates given at the end of the section we readily bound the Sobolev norms of $F(X)$ as a corollary, which then gives us the local boundedness estimate for the operator F. The proof of analyticity of trajectories is easily shown once that is verified.

One can take derivatives of integral convolution type expressions with kernel $\mathcal{K}(\alpha, \beta)$ as written in (4.28), potentially repeatedly, in a way that yields a sum of expressions of a similar form, with nearly the same kernel as that in (4.28), but perhaps with different indices. We give a rigorous derivation of the relevant differentiation formula with Proposition 4.3.1. In the following discussion leading up to the statement of the proposition, we sketch out the main idea of the way the manipulations work, to motivate Proposition 4.3.1 and lemmas leading up to the differentiation formula.

We begin by considering functions $f \in C_c^2$, $g \in H^1$. As our candidate expression to be differentiated, let us take the following, without being too careful about the precise meaning of the principal value integral in this instance.

$$T(\alpha) := \text{p.v.} \int (g(\alpha) - g(\alpha - \beta)) f(\alpha - \beta) \partial_{\beta_k} \left(K(X(\alpha) - X(\alpha - \beta)) \right) d\beta. \tag{4.30}$$

Then if we differentiate with respect to α_l, for some $l = 1, 2$, or 3, and, without justification at the

moment, move the differentiation under the integral sign, we find

$$\begin{aligned}\partial_l T(\alpha) &= \text{p.v.} \int \partial_{\alpha_l} \left[(g(\alpha) - g(\alpha-\beta))f(\alpha-\beta)\right] \partial_{\beta_k} \left(K(X(\alpha) - X(\alpha-\beta))\right) d\beta \\ &\quad + \text{p.v.} \int (g(\alpha) - g(\alpha-\beta))f(\alpha-\beta) \partial_{\beta_k} \left(\partial_{\alpha_l}[K(X(\alpha) - X(\alpha-\beta))]\right) d\beta.\end{aligned} \tag{4.31}$$

Now we integrate by parts in the second integral on the right. This in fact yields no boundary term; we justify this for these types of calculations in the rigorous derivations coming later in the section.

$$\begin{aligned}\partial_l T(\alpha) &= \text{p.v.} \int \partial_{\alpha_l} \left[(g(\alpha) - g(\alpha-\beta))f(\alpha-\beta)\right] \partial_{\beta_k} \left(K(X(\alpha) - X(\alpha-\beta))\right) d\beta \\ &\quad - \text{p.v.} \int \partial_{\beta_k} \left[(g(\alpha) - g(\alpha-\beta))f(\alpha-\beta)\right] \partial_{\alpha_l}[K(X(\alpha) - X(\alpha-\beta))] d\beta.\end{aligned} \tag{4.32}$$

Now, we use the next identity, following just from the chain rule

$$\begin{aligned}\nabla_\alpha (K(X(\alpha) - X(\alpha-\beta))) &= (\nabla X(\alpha) - \nabla X(\alpha-\beta)) \nabla K(X(\alpha) - X(\alpha-\beta)), \\ &= (\nabla X(\alpha) - \nabla X(\alpha-\beta))(\nabla X(\alpha-\beta))^{-1} \nabla_\beta (K(X(\alpha) - X(\alpha-\beta))).\end{aligned}$$

Using this we obtain

$$\begin{aligned}\partial_l T(\alpha) &= \text{p.v.} \int \partial_{\alpha_l} \left[(g(\alpha) - g(\alpha-\beta))f(\alpha-\beta)\right] \partial_{\beta_k} \left(K(X(\alpha) - X(\alpha-\beta))\right) d\beta \\ &\quad - \sum_{i,j=1}^{3} \text{p.v.} \int \partial_{\beta_k} \left[(g(\alpha) - g(\alpha-\beta))f(\alpha-\beta)\right] \\ &\qquad \times (\partial_l X_i(\alpha) - \partial_l X_i(\alpha-\beta))(\nabla X(\alpha-\beta))^{-1}_{ij} \partial_{\beta_j} (K(X(\alpha) - X(\alpha-\beta)) d\beta.\end{aligned} \tag{4.33}$$

Thus we have an expression for the derivative in terms of integrals with kernels just of the form (4.28). Given higher differentiability in f and g, one can repeat the process over and over to calculate higher order derivatives of $T(\alpha)$, in terms of the corresponding derivatives of f, g, and X, integrated against the one type of integral kernel. The main gain of using this strategy is just that we essentially only need to verify L^2 estimates for the one kernel that appears.

This outlines the idea behind the proposition that follows. The proposition is used to verify that computations that arise for us in our estimates of $\|F(X)\|_{H^s}$ can basically be handled in the same way, with the resulting principal value integrals converging in the L^2 sense.

Proposition 4.3.1. *Fix $X \in B_\delta$ and $k = 1, 2,$ or 3. We define the spaces*

$$\begin{aligned} \mathcal{Z}_1 &= C^1 \times C^1 \times H^1, \quad \mathcal{Z}_2 = C^1 \times H^1 \times C^2, \quad \mathcal{Z}_3 = H^1 \times C^2 \times C^2, \\ \mathcal{Z}_1' &= L^\infty \times L^\infty \times L^2, \quad \mathcal{Z}_2' = L^\infty \times L^2 \times C^1, \quad \mathcal{Z}_3' = L^2 \times C^1 \times C^1, \end{aligned} \tag{4.34}$$

where by C^k we mean the set of k-times continuously differentiable functions on $\mathbb{R}^3$ with bounded partial derivatives up to order k, and we define for $i = 1, 2,$ and 3,

$$\|(h,g,f)\|_{\mathcal{Z}_i'} := \begin{cases} \|h\|_{L^\infty} \cdot \|g\|_{L^\infty} \cdot \|f\|_{L^2} & \text{if } i = 1, \\ \|h\|_{L^\infty} \cdot \|g\|_{L^2} \cdot \|f\|_{C^1} & \text{if } i = 2, \\ \|h\|_{L^2} \cdot \|g\|_{C^1} \cdot \|f\|_{C^1} & \text{if } i = 3, \end{cases} \qquad \text{for } (h,g,f) \in \mathcal{Z}_i'. \tag{4.35}$$

Fix a triple $(h,g,f) \in \mathcal{Z}_1'$, $\mathcal{Z}_2'$, or $\mathcal{Z}_3'$ for which supp(f) is bounded, and with $h, g,$ and f functions defined almost everywhere on $\mathbb{R}^3$. We denote

$$\theta(\alpha, \beta) := X(\alpha) - X(\alpha - \beta). \tag{4.36}$$

For any $\epsilon > 0$ we write B_ϵ^c to indicate the complement in $\mathbb{R}^3$ of the ball $B_\epsilon(0)$ about the origin. (i) For $\epsilon > 0$ we have the a.e. pointwise-defined function

$$(T_{X,k}^\epsilon[h,g,f])(\alpha) := h(\alpha) \int_{B_\epsilon^c} (g(\alpha) - g(\alpha - \beta)) f(\alpha - \beta) \partial_{\beta_k}(K(\theta(\alpha, \beta))) d\beta, \tag{4.37}$$

and we have

$$\lim_{\epsilon \to 0} (T_{X,k}^\epsilon[h,g,f]) \quad \text{converges in } L^2. \tag{4.38}$$

Defining the limit

$$(T_{X,k}^0[h,g,f]) := \lim_{\epsilon \to 0} (T_{X,k}^\epsilon[h,g,f]) \quad \text{in } L^2, \tag{4.39}$$

we have that $(h,g,f) \in \mathcal{Z}_i$, for $i = 1, 2,$ or 3, implies

$$\|T_{X,k}^0[h,g,f]\|_{L^2} \leq C(\text{supp}(f)) \|(h,g,f)\|_{\mathcal{Z}_i'}, \tag{4.40}$$

where the factor in front depends only on the support of f.

(ii) If in addition the triple (h,g,f) is in $\mathcal{Z}_i$ for $i = 1, 2,$ or 3, we have that $(T_{X,k}^0[h,g,f])$ is in

H^1, *and we have the formula for* $l = 1, 2,$ *and* 3,

$$
\begin{aligned}
\partial_l((T^0_{X,k}[h,g,f])(\alpha)) \quad = \quad & \lim_{\epsilon\to 0}\Big(\int_{B^c_\epsilon} \partial_{\alpha_l}\big(h(\alpha)(g(\alpha)-g(\alpha-\beta))f(\alpha-\beta)\big)\partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta \\
& -h(\alpha)\sum_{i,j=1}^{3}\int_{B^c_\epsilon}\partial_{\beta_k}\big((g(\alpha)-g(\alpha-\beta))f(\alpha-\beta)\big)(\partial_l X_i(\alpha)-\partial_l X_i(\alpha-\beta)) \\
& \times(\nabla X(\alpha-\beta))^{-1}_{ij}\partial_{\beta_j}(K(\theta(\alpha,\beta)))d\beta\Big) \quad \textit{in } L^2.
\end{aligned} \tag{4.41}
$$

Proof. Fix a triple $(h,g,f) \in \mathcal{Z}'_1$, $\mathcal{Z}'_2$, or $\mathcal{Z}'_3$, with each of f, g, and h defined almost everywhere, and with supp(f) bounded. Fix an $l = 1, 2$, or 3. The first thing we note is that for each $\epsilon > 0$, for each α for which both $h(\alpha)$ and $g(\alpha)$ are defined (guaranteed to hold for almost all α), the integrand in the definition of $(T^\epsilon_{X,k}[h,g,f])(\alpha)$ is clearly integrable over the region of integration. Thus this gives a pointwise defined function almost everywhere.

For (4.38), notice that for each $\epsilon > 0$, a change of variables gives

$$
\begin{aligned}
(T^\epsilon_{X,k}[h,g,f])(\alpha) \quad = \quad & h(\alpha)g(\alpha)\int_{B^c_\epsilon} f(\alpha-\beta)\partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta \\
& -h(\alpha)\int_{B^c_\epsilon} g(\alpha-\beta)f(\alpha-\beta)\partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta, \\
= \quad & h(\alpha)g(\alpha)\int_{(B_\epsilon(\alpha))^c} f(\beta)\tilde{K}^k(\alpha,\beta)d\beta - h(\alpha)\int_{(B_\epsilon(\alpha))^c} g(\beta)f(\beta)\tilde{K}^k(\alpha,\beta)d\beta, \\
= \quad & h(\alpha)g(\alpha)(T^\epsilon_{X,k}f)(\alpha) - h(\alpha)(T^\epsilon_{X,k}(gf))(\alpha).
\end{aligned} \tag{4.42}
$$

Suppose $(h,g,f) \in \mathcal{Z}'_1$. Then we find $f \in L^2$ and $gf \in L^2$, so $T^\epsilon_{X,k}f$ and $T^\epsilon_{X,k}(gf)$ both converge in L^2 as ϵ tends to zero, in view of Definition 4.2.3 and Remark 4.2.4. At the same time h and g are in L^∞, so the entire right hand side of the last equality of (4.42) converges in L^2 as ϵ tends to zero. One easily verifies then

$$
\|T^0_{X,k}[h,g,f]\|_{L^2} \le C\|(h,g,f)\|_{\mathcal{Z}'_1} \tag{4.43}
$$

Suppose $(h,g,f) \in \mathcal{Z}'_2$. Then $f \in C^1_c$ and $hg \in L^2$. So by Definition 4.2.3 and Remark 4.2.4, $T^\epsilon_{X,k}f$ converges in L^∞ and thus $(hg \cdot (T^\epsilon_{X,k}f))$ converges in L^2 norm as ϵ tends to zero. Also, we have $h \in L^\infty$ and $gf \in L^2$, so $(h \cdot (T^\epsilon_{X,k}(gf))$ also converges in L^2 in the limit, and again we find the

entire right hand side converges in L^2. In this case we find

$$\begin{aligned} \|T^0_{X,k}[h,g,f]\|_{L^2} &\leq \|h\|_{L^\infty}\|g\|_{L^2}(\|f\|_{C^1}+\|f\|_{L^2})+\|h\|_{L^\infty}\|g\|_{L^2}\|f\|_{L^\infty}, \\ &\leq C(\mathrm{supp}(f))\|(h,g,f)\|_{\mathcal{Z}'_2}. \end{aligned} \tag{4.44}$$

Now suppose $(h,g,f) \in \mathcal{Z}'_3$. Then we have $hg \in L^2$ and $f \in C^1_c$, and we have $h \in L^2$ and $gf \in C^1_c$, so that we find the right hand side converges in L^2 as ϵ tends to zero similarly. Finally, we have

$$\begin{aligned} \|T^0_{X,k}[h,g,f]\|_{L^2} &\leq \|h\|_{L^2}\|g\|_{L^\infty}(\|f\|_{C^1}+\|f\|_{L^2})+\|h\|_{L^2}(\|g\|_{C^1}\|f\|_{C^1}+\|g\|_{L^\infty}\|f\|_{L^2}), \\ &\leq C(\mathrm{supp}(f))\|(h,g,f)\|_{\mathcal{Z}'_3}. \end{aligned} \tag{4.45}$$

This completes the proof of part (i). Now we begin with part (ii).

Suppose the triple (h,g,f) is in one of $\mathcal{Z}_1$, $\mathcal{Z}_2$, or $\mathcal{Z}_3$. First we consider the ∂_l derivative for a fixed $\epsilon > 0$ of $(T^\epsilon_{X,k}[h,g,f])$. We claim that for any test function $\varphi \in C^\infty_c$ the following holds, where $\langle \cdot,\cdot \rangle$ indicates the L^2 inner product.

$$\begin{aligned} -\langle \partial_l\varphi, (T^\epsilon_{X,k}[h,g,f])\rangle = & \\ &\int \varphi(\alpha)\int_{B^c_\epsilon} \partial_{\alpha_l}\big(h(\alpha)(g(\alpha)-g(\alpha-\beta))f(\alpha-\beta)\big)\partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta d\alpha \\ &-\sum_{i,j=1}^{3}\int\Bigg(\varphi(\alpha)h(\alpha)\int_{B^c_\epsilon}\partial_{\beta_k}\big((g(\alpha)-g(\alpha-\beta))f(\alpha-\beta)\big) \\ &\qquad\qquad \times(\partial_l X_i(\alpha)-\partial_l X_i(\alpha-\beta))(\nabla X(\alpha-\beta))^{-1}_{ij}\partial_{\beta_j}(K(\theta(\alpha,\beta)))d\beta\Bigg)d\alpha \\ &+\langle\varphi, \tilde{B}^\epsilon_{boundary}\rangle, \end{aligned} \tag{4.46}$$

where $\tilde{B}^\epsilon_{boundary}$ is a boundary term resulting from an integration by parts in an integral over β that arises in the calculation. Compare this with (4.41). We need to verify that (4.46) holds, where the right hand side forms an L^2 pairing of φ with a function in L^2. We also need to verify that this function that is paired with φ converges to an element in L^2 as ϵ tends to zero, and that $\tilde{B}^\epsilon_{boundary}$ tends to zero in L^2. If we can show these things, it follows that

$$-\langle \partial_l\varphi, (T^0_{X,k}[h,g,f])\rangle = -\lim_{\epsilon\to 0}\langle \partial_l\varphi, (T^\epsilon_{X,k}[h,g,f])\rangle = \langle \varphi, \text{RHS of (4.41)}\rangle. \tag{4.47}$$

Since φ was an arbitrary test function, this implies that the right hand side of (4.41) is the weak

derivative of $T^0_{X,k}[h,g,f]$, and that this derivative is in L^2. This allows us to conclude the proof of part (ii).

Now we give the calculation for (4.46). For $\epsilon > 0$, in the following we use χ_ϵ to indicate the identifier function $1_{B^c_\epsilon}$ for the set $B^c_\epsilon = \mathbb{R}^3 \setminus (B_\epsilon(0))$.

$$
\begin{aligned}
&-\langle \partial_l \varphi, T^\epsilon_{X,k}[h,g,f]) \rangle \qquad (4.48)\\
&\quad = -\int \partial_l \varphi(\alpha) h(\alpha) \int_{B^c_\epsilon} (g(\alpha) - g(\alpha-\beta)) f(\alpha-\beta) \partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta d\alpha,\\
&\quad = -\int\int \partial_l \varphi(\alpha) h(\alpha)(g(\alpha) - g(\alpha-\beta)) f(\alpha-\beta)\chi_\epsilon(\beta) \partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta d\alpha,\\
&\quad = -\int\int \partial_l \varphi(\alpha) h(\alpha)(g(\alpha) - g(\alpha-\beta)) f(\alpha-\beta)\chi_\epsilon(\beta) \partial_{\beta_k}(K(\theta(\alpha,\beta)))d\alpha d\beta,\\
&\quad = \int\int \varphi(\alpha) \partial_{\alpha_l} \Big(h(\alpha)(g(\alpha) - g(\alpha-\beta)) f(\alpha-\beta)\chi_\epsilon(\beta) \partial_{\beta_k}(K(\theta(\alpha,\beta))) \Big) d\alpha d\beta,\\
&\quad = \int\int \varphi(\alpha) \partial_{\alpha_l} \Big(h(\alpha)(g(\alpha) - g(\alpha-\beta)) f(\alpha-\beta) \Big) \chi_\epsilon(\beta) \partial_{\beta_k}(K(\theta(\alpha,\beta)))d\alpha d\beta\\
&\qquad + \int\int \varphi(\alpha)(h(\alpha)(g(\alpha) - g(\alpha-\beta)) f(\alpha-\beta))\chi_\epsilon(\beta) \partial_{\beta_k}(\partial_{\alpha_l}(K(\theta(\alpha,\beta))))d\alpha d\beta,\\
&\quad = A^\epsilon + B^\epsilon. \qquad (4.49)
\end{aligned}
$$

Regarding the third line in (4.48), since the triple (h,g,f) is in $\mathcal{Z}_1$, $\mathcal{Z}_2$, or $\mathcal{Z}_3$, and f is compactly supported, changing the order of integration is justified.

For A^ϵ, we have

$$
\begin{aligned}
A^\epsilon \quad = \quad & \int \varphi(\alpha) \int_{B^c_\epsilon} \Big(\partial_l h(\alpha)(g(\alpha) - g(\alpha-\beta)) f(\alpha-\beta) + h(\alpha)(\partial_l g(\alpha) - \partial_l g(\alpha-\beta)) f(\alpha-\beta)\\
&\qquad + h(\alpha)(g(\alpha) - g(\alpha-\beta))\partial_l f(\alpha-\beta) \Big) \partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta d\alpha,\\
= \quad & \int \varphi(\alpha) \Big((\partial_l h(\alpha) g(\alpha) + h(\alpha)\partial_l g(\alpha)) \int_{B^c_\epsilon} f(\alpha-\beta)\partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta\\
&\qquad - \partial_l h(\alpha) \int_{B^c_\epsilon} g(\alpha-\beta) f(\alpha-\beta)\partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta\\
&\qquad - h(\alpha) \int_{B^c_\epsilon} (\partial_l g(\alpha-\beta) f(\alpha-\beta) + g(\alpha-\beta)\partial_l f(\alpha-\beta))\partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta\\
&\qquad + h(\alpha) g(\alpha) \int_{B^c_\epsilon} \partial_l f(\alpha-\beta)\partial_{\beta_k}(K(\theta(\alpha,\beta)))d\beta \Big) d\alpha,\\
= \quad & \int \varphi(\alpha) \left(A^\epsilon_1(\alpha) + A^\epsilon_2(\alpha) + A^\epsilon_3(\alpha) + A^\epsilon_4(\alpha) \right) d\alpha.
\end{aligned} \qquad (4.50)
$$

After making the change of variables replacing β with $(\alpha - \beta)$ one then finds

$$\begin{aligned} A_1^\epsilon &= (\partial_l h \cdot g + h \cdot \partial_l g) \cdot T_{X,k}^\epsilon f, \\ A_2^\epsilon &= -\partial_l h \cdot T_{X,k}^\epsilon (g \cdot f), \\ A_3^\epsilon &= -h \cdot T_{X,k}^\epsilon (\partial_l g \cdot f + g \cdot \partial_l f), \\ A_4^\epsilon &= h \cdot g \cdot T_{X,k}^\epsilon (\partial_l f). \end{aligned} \tag{4.51}$$

One uses Remark 4.2.4 to verify that since $(h, g, f) \in \mathcal{Z}_i$ for $i = 1, 2$, or 3, with $\mathrm{supp}(f)$ bounded, each of the A_j^ϵ, $j = 1, \ldots, 4$, is in L^2.

For B^ϵ in (4.49), we have

$$\begin{aligned} B^\epsilon &= \int\int_{B_\epsilon^c} \varphi(\alpha) h(\alpha)(g(\alpha) - g(\alpha - \beta)) f(\alpha - \beta) \\ &\qquad \times \partial_{\beta_k} \Big((\partial_l X(\alpha) - \partial_l X(\alpha - \beta))(\nabla K)(\theta(\alpha, \beta)) \Big) d\beta d\alpha, \\ &= -\int\int_{B_\epsilon^c} \varphi(\alpha) h(\alpha) \partial_{\beta_k} \Big((g(\alpha) - g(\alpha - \beta)) f(\alpha - \beta) \Big) \\ &\qquad \times (\partial_l X(\alpha) - \partial_l X(\alpha - \beta))(\nabla K)(\theta(\alpha, \beta)) d\beta d\alpha \\ &\quad + \int\int_{|\beta| = \epsilon} \varphi(\alpha) h(\alpha)(g(\alpha) - g(\alpha - \beta)) f(\alpha - \beta) \\ &\qquad \times (\partial_l X(\alpha) - \partial_l X(\alpha - \beta))(\nabla K)(\theta(\alpha, \beta)) \nu_{out}^k(\beta) d\sigma(\beta) d\alpha, \\ &= B_{int}^\epsilon + B_{boundary}^\epsilon. \end{aligned} \tag{4.52}$$

In the above expression, we denote by ν_{out} the outward pointing normal. We note that the integration by parts above works since we can guarantee that $(g(\alpha) - g(\alpha - \beta)) f(\alpha - \beta)$ is at least in H_β^1 or in C_β^1. Now we consider B_{int}^ϵ. Using the chain rule and the fact that $\nabla_\beta \theta(\alpha, \beta) = \nabla X(\alpha - \beta)$ is invertible for any α, β, we can write this as

$$\begin{aligned} B_{int}^\epsilon &= -\int\int_{B_\epsilon^c} \varphi(\alpha) h(\alpha) \partial_{\beta_k} \Big((g(\alpha) - g(\alpha - \beta)) f(\alpha - \beta) \Big) (\partial_l X(\alpha) - \partial_l X(\alpha - \beta)) \\ &\qquad \times (\nabla_\beta \theta(\alpha, \beta))^{-1} \nabla_\beta \big(K(\theta(\alpha, \beta)) \big) d\beta d\alpha \\ &= -\sum_{i,j=1}^{3} \int \varphi(\alpha) h(\alpha) \int_{B_\epsilon^c} \partial_{\beta_k} \Big((g(\alpha) - g(\alpha - \beta)) f(\alpha - \beta) \Big) (\partial_l X_i(\alpha) - \partial_l X_i(\alpha - \beta)) \\ &\qquad \times (\nabla X(\alpha - \beta))_{ij}^{-1} \partial_{\beta_j} (K(\theta(\alpha, \beta))) d\beta d\alpha. \end{aligned} \tag{4.53}$$

Now we compute for $i = 1, 2$, and 3,

$$\begin{aligned}
&\partial_{\beta_k}\big[(g(\alpha) - g(\alpha-\beta))f(\alpha-\beta)\big](\partial_l X_i(\alpha) - \partial_l X_i(\alpha-\beta)) \qquad (4.54)\\
&= \big[\partial_k g(\alpha-\beta)f(\alpha-\beta) - g(\alpha)\partial_k f(\alpha-\beta) + g(\alpha-\beta)\partial_k f(\alpha-\beta)\big](\partial_l X_i(\alpha) - \partial_l X_i(\alpha-\beta)),
\end{aligned}$$

and then distributing multiplication gives

$$\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.55)\\
\text{RHS of (4.54)} = \quad &\partial_l X_i(\alpha)\partial_k g(\alpha-\beta)f(\alpha-\beta) \quad - \quad \partial_k g(\alpha-\beta)f(\alpha-\beta)\partial_l X_i(\alpha-\beta)\\
&-g(\alpha)\partial_l X_i(\alpha)\partial_k f(\alpha-\beta) \quad + \quad g(\alpha)\partial_k f(\alpha-\beta)\partial_l X_i(\alpha-\beta)\\
&+\partial_l X_i(\alpha)g(\alpha-\beta)\partial_k f(\alpha-\beta) \quad - \quad g(\alpha-\beta)\partial_k f(\alpha-\beta)\partial_l X_i(\alpha-\beta).
\end{aligned}$$

We arrange this as

$$\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.56)\\
\text{RHS of (4.55)} \quad &= \quad \partial_l X_i(\alpha)\big[\partial_k g(\alpha-\beta)f(\alpha-\beta) + g(\alpha-\beta)\partial_k f(\alpha-\beta)\big]\\
&\quad -\big[\partial_k g(\alpha-\beta)f(\alpha-\beta) + g(\alpha-\beta)\partial_k f(\alpha-\beta)\big]\partial_l X_i(\alpha-\beta)\\
&\quad +g(\alpha)\partial_k f(\alpha-\beta)\partial_l X_i(\alpha-\beta)\\
&\quad -g(\alpha)\partial_l X_i(\alpha)\partial_k f(\alpha-\beta).
\end{aligned}$$

Using (4.54)-(4.56) in (4.53), we get

$$\begin{aligned}
B^{\epsilon}_{int} = -\sum_{i,j=1}^{3}\int \varphi(\alpha)\Big[&h(\alpha)\partial_l X_i(\alpha) \qquad (4.57)\\
&\times\int_{B^c_\epsilon}(\partial_k g(\alpha-\beta)f(\alpha-\beta) + g(\alpha-\beta)\partial_k f(\alpha-\beta))(\nabla X(\alpha-\beta))^{-1}_{ij}\partial_{\beta_j}(K(\theta(\alpha,\beta)))d\beta\\
&-h(\alpha)\int_{B^c_\epsilon}(\partial_k g(\alpha-\beta)f(\alpha-\beta) + g(\alpha-\beta)\partial_k f(\alpha-\beta))\partial_l X_i(\alpha-\beta)\\
&\qquad\times(\nabla X(\alpha-\beta))^{-1}_{ij}\partial_{\beta_j}(K(\theta(\alpha,\beta)))d\beta\\
&+h(\alpha)g(\alpha)\int_{B^c_\epsilon}\partial_k f(\alpha-\beta)\partial_l X_i(\alpha-\beta)(\nabla X(\alpha-\beta))^{-1}_{ij}\partial_{\beta_j}(K(\theta(\alpha,\beta)))d\beta\\
&-h(\alpha)g(\alpha)\partial_l X_i(\alpha)\int_{B^c_\epsilon}\partial_k f(\alpha-\beta)(\nabla X(\alpha-\beta))^{-1}_{ij}\partial_{\beta_j}(K(\theta(\alpha,\beta)))d\beta\Big]d\alpha,\\
= -\sum_{i,j=1}^{3}\int \varphi(\alpha)&(B^{\epsilon}_{1;ij}(\alpha) + B^{\epsilon}_{2;ij}(\alpha) + B^{\epsilon}_{3;ij}(\alpha) + B^{\epsilon}_{4;ij}(\alpha))d\alpha.
\end{aligned}$$

With the change of variables replacing β with $(\alpha-\beta)$, we see that the $B^{\epsilon}_{m;ij}$, $i,j=1,2,3$, $m=1,\ldots,4$ are thus given by

$$\begin{aligned} B^{\epsilon}_{1;ij} &= h\cdot\partial_l X_i\cdot T^{\epsilon}_{X,j}\Big((\partial_k g\cdot f+g\cdot\partial_k f)\cdot(\nabla X)^{-1}_{ij}\Big) \\ B^{\epsilon}_{2;ij} &= -h\cdot T^{\epsilon}_{X,j}\Big((\partial_k g\cdot f+g\cdot\partial_k f)\cdot\partial_l X_i\cdot(\nabla X)^{-1}_{ij}\Big) \\ B^{\epsilon}_{3;ij} &= h\cdot g\cdot T^{\epsilon}_{X,j}\Big(\partial_k f\cdot\partial_l X_i\cdot(\nabla X)^{-1}_{ij}\Big) \\ B^{\epsilon}_{4;ij} &= -h\cdot g\cdot\partial_l X_i\cdot T^{\epsilon}_{X,j}\Big(\partial_k f\cdot(\nabla X)^{-1}_{ij}\Big) \end{aligned} \tag{4.58}$$

One can consider each of the cases $(h,g,f)\in\mathcal{Z}_1$, $(h,g,f)\in\mathcal{Z}_2$, and $(h,g,f)\in\mathcal{Z}_3$, and use the facts that $\mathrm{supp}(f)$ is bounded and that ∇X and $(\nabla X)^{-1}$ are bounded in C^1, to bound each $B^{\epsilon}_{m;ij}$ in L^2.

Now we return to (4.52) to handle $B^{\epsilon}_{boundary}$. We have

$$\begin{aligned} B^{\epsilon}_{boundary} &= \int\varphi(\alpha)h(\alpha) \\ &\times\int_{|\beta|=\epsilon}(g(\alpha)-g(\alpha-\beta))f(\alpha-\beta)(\partial_l X(\alpha)-\partial_l X(\alpha-\beta))(\nabla K)(\theta(\alpha,\beta))\nu^k_{out}(\beta)d\sigma(\beta)d\alpha, \\ &= \int\varphi(\alpha)\tilde{B}^{\epsilon}_{boundary}(\alpha)d\alpha, \end{aligned} \tag{4.59}$$

where

$$\begin{aligned} &\tilde{B}^{\epsilon}_{boundary}(\alpha)=h(\alpha)C^{\epsilon}(\alpha), \\ &C^{\epsilon}(\alpha)=\int_{|\beta|=\epsilon}(g(\alpha)-g(\alpha-\beta))f(\alpha-\beta)(\partial_l X(\alpha)-\partial_l X(\alpha-\beta))(\nabla K)(\theta(\alpha,\beta))\nu^k_{out}(\beta)d\sigma(\beta), \end{aligned} \tag{4.60}$$

We comment that if we are in the case that the triple (h,g,f) is in $\mathcal{Z}_2$, we only assert $C^{\epsilon}(\alpha)$ is defined for almost all α. Note that a similar caveat must be made about $\tilde{B}^{\epsilon}_{boundary}(\alpha)$.

Now we check that $\tilde{B}^{\epsilon}_{boundary}(\alpha)$ as defined is in fact in L^2. We consider first the cases that $(h,g,f)\in\mathcal{Z}_1$ or $\mathcal{Z}_2$.

$$\begin{aligned} &\|\tilde{B}^{\epsilon}_{boundary}\|_{L^2}\le\|h\|_{L^\infty}\|C^{\epsilon}\|_{L^2}, \\ &\|C^{\epsilon}\|_{L^2}\le\int_{|\beta|=\epsilon}\|(g(\alpha)-g(\alpha-\beta))f(\alpha-\beta)\|_{L^2_\alpha} \\ &\qquad\times\|(\partial_l X(\alpha)-\partial_l X(\alpha-\beta))(\nabla K)(\theta(\alpha,\beta))\|_{L^\infty_\alpha}d\sigma(\beta), \end{aligned} \tag{4.61}$$

where for the bound on $\|C^{\epsilon}\|_{L^2}$ we have majorized the quantity by moving the L^2_α norm inside the

surface integral over β, using a generalization of the Minkowski inequality. Then, by using Lemma 4.2.2 together with the fact that $\theta(\alpha,\beta) = (X(\alpha) - X(\alpha-\beta)) = \beta(I + O(\delta))$ and $(\partial_l X(\alpha) - \partial_l X(\alpha - \beta)) = O(|\beta|)$, we get for all $\beta \neq 0$ that

$$\|(\partial_l X(\alpha) - \partial_l X(\alpha-\beta))(\nabla K)(\theta(\alpha,\beta))\|_{L^\infty_\alpha} \leq \frac{C}{|\beta|^2}. \tag{4.62}$$

Meanwhile, the surface measure of the ball of radius ϵ is proportional to ϵ^2, and so by using (4.62) in (4.61) we find

$$\|C^\epsilon\|_{L^2} \leq C' \sup_{|\beta|=\epsilon} \|(g(\alpha) - g(\alpha-\beta))f(\alpha-\beta)\|_{L^2_\alpha}. \tag{4.63}$$

Suppose we are in the case that $(h,g,f) \in \mathcal{Z}_1$. Then we can bound the right hand side of (4.63) by the finite quantity

$$c_1(\epsilon) := C' \sup_{|\beta|=\epsilon} \|g(\alpha) - g(\alpha-\beta)\|_{L^\infty_\alpha} \|f\|_{L^2}. \tag{4.64}$$

Moreover, note that $c_1(\epsilon)$ tends to zero as ϵ tends to zero, where we have used that g is Lipschitz in this case. Thus we find that $\lim_{\epsilon\to 0} C^\epsilon = 0$ in L^2.

Suppose $(h,g,f) \in \mathcal{Z}_2$. Then the right hand side of (4.63) is bounded by the quantity

$$C' \sup_{|\beta|=\epsilon} \|g(\alpha) - g(\alpha-\beta)\|_{L^2_\alpha} \|f\|_{L^\infty} \leq C''\epsilon \|\nabla g\|_{L^2} \|f\|_{L^\infty} =: c_2(\epsilon). \tag{4.65}$$

Note $c_2(\epsilon)$ tends to zero as ϵ tends to zero. This implies $\lim_{\epsilon\to 0} C^\epsilon = 0$ in L^2.

Now we return to (4.60) in the case that $(h,g,f) \in \mathcal{Z}_3$. Then we have the bound for all $\alpha \in R^3$

$$\begin{aligned} &\|\tilde{B}^\epsilon_{boundary}\|_{L^2} \leq \|h\|_{L^2} \|C^\epsilon\|_{L^\infty}, \\ &|C^\epsilon(\alpha)| \leq \int_{|\beta|=\epsilon} |(g(\alpha) - g(\alpha-\beta))f(\alpha-\beta)(\partial_l X(\alpha) - \partial_l X(\alpha-\beta))(\nabla K)(\theta(\alpha,\beta))| d\sigma(\beta), \end{aligned} \tag{4.66}$$

and since $|(\partial_l X(\alpha) - \partial_l X(\alpha-\beta))(\nabla K)(\theta(\alpha,\beta))| \leq C/|\beta|^2$ as we saw before, we then have the following bound.

$$\begin{aligned} |C^\epsilon(\alpha)| &\leq \int_{|\beta|=\epsilon} |g(\alpha) - g(\alpha-\beta)||f(\alpha-\beta)| \frac{C}{|\beta|^2} d\sigma(\beta), \\ &\leq C' \sup_{|\beta|=\epsilon} \|g(\alpha) - g(\alpha-\beta)\|_{L^\infty_\alpha} \|f\|_{L^\infty}, \\ &=: c_3(\epsilon). \end{aligned} \tag{4.67}$$

Using Lipschitz continuity of g, we find that both $c_3(\epsilon)$ and thus $|C^\epsilon(\alpha)|$ are finite, and tend to zero

as ϵ tends to zero, with the limit for $|C^\epsilon(\alpha)|$ uniform in α. So $\lim_{\epsilon\to 0} C^\epsilon = 0$ in L^∞.

Moreover, in each case among $(h,g,f) \in \mathcal{Z}_1$, $\mathcal{Z}_2$, and $\mathcal{Z}_3$, the corresponding bound (4.61) or (4.66) on $C^\epsilon(\alpha)$ implies that $\tilde{B}^\epsilon_{boundary}(\alpha)$ is in L^2 and tends to zero as ϵ tends to zero.

Now recall from (4.48) that we have

$$-\langle \partial_l \varphi, T^\epsilon_{X,k}[h,g,f]\rangle = A^\epsilon + B^\epsilon, \tag{4.68}$$

where

$$A^\epsilon = \sum_{m=1}^{4} \langle \varphi, A^\epsilon_m \rangle, \tag{4.69}$$

with the A^ϵ_m given by (4.51). In addition we have $B^\epsilon = B^\epsilon_{int} + B^\epsilon_{boundary}$, where

$$B^\epsilon_{int} = -\sum_{m=1}^{4} \sum_{i,j=1}^{3} \langle \varphi, B^\epsilon_{m;ij} \rangle, \tag{4.70}$$

for the $B^\epsilon_{m;ij}$ given by (4.58), and where

$$B^\epsilon_{boundary} = \langle \varphi, \tilde{B}^\epsilon_{boundary} \rangle, \tag{4.71}$$

so

$$-\langle \partial_l \varphi, T^\epsilon_{X,k}[h,g,f]\rangle = \sum_{m=1}^{4} \langle \varphi, A^\epsilon_m \rangle - \sum_{m=1}^{4} \sum_{i,j=1}^{3} \langle \varphi, B^\epsilon_{m;ij} \rangle + \langle \varphi, \tilde{B}^\epsilon_{boundary} \rangle. \tag{4.72}$$

Recall that we verified that each of the functions paired with φ in the above L^2 pairings is indeed in L^2. We also verified the L^2 limit $\lim_{\epsilon\to 0} \tilde{B}^\epsilon_{boundary} = 0$. Now we must verify that the remaining quantities paired with φ, i.e. the A^ϵ_m and $B^\epsilon_{m;ij}$, converge to elements in L^2 as ϵ tends to zero. This can be checked by considering each of the cases $(h,g,f) \in \mathcal{Z}_1$, $(h,g,f) \in \mathcal{Z}_2$, and $(h,g,f) \in \mathcal{Z}_3$, and applying Definition 4.2.3 and Remark 4.2.4. The last observation to make is that one can take the quantities paired with φ in (4.72), excluding the $\tilde{B}^\epsilon_{boundary}$ term, and add them in the corresponding way to produce the quantity in the right hand side of (4.41), within the limit as ϵ tends to zero that is written there, in part (ii) of the statement of the proposition. Using these observations together, one concludes that the statement of part (ii) holds, finishing the proof. □

Definition 4.3.2. *Fix $X \in B_\delta$. For $l = 1, \ldots, d$ we define $\Theta_l(\alpha,\beta) := [\nabla X(\alpha)(\nabla X(\alpha-\beta))^{-1} - I]_l$, where $[\cdot]_l$ denotes the lth row of the matrix in brackets. For $\sigma = (\sigma_1, \ldots, \sigma_n)$ for an $n = 1, \ldots, s$, with each $\sigma_i = 1, 2$, or 3, we define the $\mathbb{C}^3$-valued functions $P^\sigma_m(\alpha,\beta)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ for $m = 1, \ldots, n$*

recursively by

$$\begin{aligned} P_1^\sigma(\alpha,\beta) &= \Theta_{\sigma_1}(\alpha,\beta)\rho(\alpha-\beta), \\ P_{m+1}^\sigma(\alpha,\beta) &= \partial_{\alpha_{\sigma_{m+1}}} P_m^\sigma(\alpha,\beta) - \Theta_{\sigma_{m+1}}(\alpha,\beta)\mathrm{div}_\beta P_m^\sigma(\alpha,\beta) + \Theta_{\sigma_{m+1}}(\alpha,\beta)(\partial_{\sigma_m}\dots\partial_{\sigma_1}\rho)(\alpha-\beta), \\ &\qquad\qquad \text{for } 1 \le m \le n-1. \end{aligned}$$

Lemma 4.3.3. *Consider fixed $X \in B_\delta$, $\sigma = (\sigma_1,\dots,\sigma_n)$, some $n = 1,\dots,s$, and $P_m^\sigma(\alpha,\beta)$ as defined in Definition 4.3.2 for $1 \le m \le n$. Then we can express*

$$\begin{aligned} P_m^\sigma(\alpha,\beta) &= \sum_{|\Gamma|=m-1} \Bigg[\mathit{coeff} \left(\partial^\eta \partial_i X_j(\alpha) - \partial^\eta \partial_i X_j(\alpha-\beta) \right) \left(\prod_{l=1}^{N_1(\Gamma)} \partial^{\mu_l} (\nabla X(\alpha))_{(\iota_1,\iota_2)_l} \right) \\ &\qquad \times \left(\prod_{l=1}^{N_2(\Gamma)} \partial_\alpha^{\nu_l} (\nabla X(\alpha-\beta))^{-1}_{(\iota_3,\iota_4)_l} \right) (\partial^\gamma \rho)(\alpha-\beta) e_k \Bigg]. \end{aligned} \tag{4.73}$$

In the above expression we use the index Γ to store the information of the various other indices, in that Γ represents the array

$$\begin{aligned} &\Gamma = (\eta,\mu,\nu,\gamma;i,j,k,\vec{\iota}), \\ &\text{where} \quad \vec{\iota} = ((\iota_1,\iota_2)_1,\dots,(\iota_1,\iota_2)_{N_1(\Gamma)};(\iota_3,\iota_4)_1,\dots,(\iota_3,\iota_4)_{N_2(\Gamma)}), \end{aligned}$$

where each we have indices i, j, $k = 1, 2$, or 3, and each component of $\vec{\iota}$, is $1, 2$, or 3, where we have multiindices $\eta,\gamma \in (\mathbb{N}_0)^3$, and arrays

$$\begin{aligned} \mu &= \left(\mu_1,\dots,\mu_{N_1(\Gamma)}\right), \\ \nu &= \left(\nu_1,\dots,\nu_{N_2(\Gamma)}\right), \end{aligned} \tag{4.74}$$

where each of the μ_l, ν_l is a multiindex in $(\mathbb{N}_0)^3$, and we define

$$|\Gamma| = |\eta| + |\gamma| + \sum_{l=1}^{N_1(\Gamma)} |\mu_l| + \sum_{l=1}^{N_2(\Gamma)} |\nu_l|. \tag{4.75}$$

In the summation in (4.73), by coeff we denote a coefficient which depends on Γ; its exact value is not needed. Additionally, we have $N_1(\Gamma) = 0$ for some Γ, but we have $N_2(\Gamma) \ge 1$ for all Γ.

Corollary 4.3.4. *Fix $X \in B_\delta$ and $\sigma = (\sigma_1,\dots,\sigma_n)$, some $n = 1,\dots,s$. For $P_m^\sigma(\alpha,\beta)$, $m = 1,\dots,n$,*

as in Definition 4.3.2, we have

$$P_m^\sigma(\alpha,\beta) = \sum_{|\Gamma|=m-1} h_\Gamma(\alpha)(g_\Gamma(\alpha) - g_\Gamma(\alpha-\beta))f_\Gamma(\alpha-\beta)e_{k(\Gamma)}, \tag{4.76}$$

where for each Γ *in the sum,* $k(\Gamma) = 1, 2,$ *or* 3*, we have that supp*(f_Γ) *bounded, and*

(i) if $m \leq n-1$*, then* $(h_\Gamma, g_\Gamma, f_\Gamma) \in \mathcal{Z}_1$*,* $\mathcal{Z}_2$*, or* $\mathcal{Z}_3$*, as defined in Proposition 4.3.1, and*

(ii) if $m = n$*, we have* $(h_\Gamma, g_\Gamma, f_\Gamma) \in \mathcal{Z}_1'$*,* $\mathcal{Z}_2'$*, or* $\mathcal{Z}_3'$*, as defined in Proposition 4.3.1.*

Proof. For each $m = 1, \ldots, n$ we use the decomposition for $P_m^\sigma(\alpha,\beta)$ given in Lemma 4.3.3, defining for each Γ in the sum in (4.73)

$$\begin{aligned} h_\Gamma(\alpha) &:= \textit{coeff}(\Gamma) \prod_{l=1}^{N_1(\Gamma)} \partial^{\mu_l}(\nabla X(\alpha))_{(\iota_1,\iota_2)_l}, \\ g_\Gamma(\alpha) &:= \partial^\eta \partial_i X_j(\alpha), \\ f_\Gamma(\alpha) &:= \left(\prod_{l=1}^{N_2(\Gamma)} \partial^{\nu_l}(\nabla X(\alpha))^{-1}_{(\iota_3,\iota_4)_l} \right) \partial^\gamma \rho(\alpha), \end{aligned} \tag{4.77}$$

where *coeff*(Γ) and $k(\Gamma)$ are precisely the *coeff* and k appearing in (4.73), respectively. Note that since ρ is assumed to have compact support, we get supp(f_Γ) is bounded for each Γ. Now we verify each triple $(h_\Gamma, g_\Gamma, f_\Gamma)$ is in at least one of $\mathcal{Z}_1$, $\mathcal{Z}_2$, or $\mathcal{Z}_3$.

First, we consider the case that $m = 1, \ldots, n-1$.

We fix a Γ with $|\Gamma| = m-1$. If $m = 1$ or 2, the triple is in $\mathcal{Z}_1$. If $m \geq 3$, but each of $|\eta|$, the $|\mu_l|$, $|\nu_l|$, and $|\gamma|$, is $m-3$ or less, the triple is also in $\mathcal{Z}_1$.

Now suppose $m \geq 3$, and at least one of $|\eta|$, $|\gamma|$, the $|\mu_l|$, or the $|\nu_l|$ is $m-2$ or greater. We cover this by checking the following cases.

- If $|\eta| \geq m-2$, (note then $|\eta| \neq 0$) the triple is in $\mathcal{Z}_2$.
- If $\max_l |\nu_l| \geq m-2$, (note then $\max_l |\nu_l| \neq 0$) the triple is in $\mathcal{Z}_1$.
- If $|\gamma| \geq m-2$, the triple is in $\mathcal{Z}_1$.
- If $N_1(\Gamma) \neq 0$, with $\max_l |\mu_l| \geq m-2$ (so that then $\max_l |\mu_l| \neq 0$), then the triple is in $\mathcal{Z}_3$.

This takes care of every possibility for $m = 1, \ldots, n-1$. For the case $m = n$, one again considers each Γ with $|\Gamma| = m-1$ and makes word for word the same set of observations as above, but with $\mathcal{Z}_i'$ in place of $\mathcal{Z}_i$ for $i = 1, 2$, and 3, at each step. □

Lemma 4.3.5. *Fix a $\gamma \in (\mathbb{N}_0)^3$ with $0 \leq |\gamma| \leq s$. Then the integral, defined pointwise for each α in $\mathbb{R}^3$,*

$$\int K(\theta(\alpha,\beta))(\partial^\gamma \rho)(\alpha-\beta)d\beta \quad \text{is in } L^2, \tag{4.78}$$

with L^2 norm bounded by a constant depending only on ρ, and

$$\lim_{\epsilon\to 0}\int_{|\beta|\geq\epsilon} K(\theta(\alpha,\beta))(\partial^\gamma \rho)(\alpha-\beta)d\beta \quad \text{converges in } L^2 \text{ to (4.78).} \tag{4.79}$$

Proof. It is straightforward to verify that the integral in (4.78) is defined pointwise for each α in $\mathbb{R}^3$. Let us define for all α in $\mathbb{R}^3$

$$g(\alpha) := \int K(\theta(\alpha,\beta))(\partial^\gamma \rho)(\alpha-\beta)d\beta. \tag{4.80}$$

First we claim that we have the following bound for all α in $\mathbb{R}^3$, uniform in $\epsilon > 0$.

$$\left|\int_{|\beta|\geq\epsilon} K(\theta(\alpha,\beta))(\partial^\gamma \rho)(\alpha-\beta)d\beta\right| \leq \frac{C(\rho)}{(1+|\alpha|)^2}, \tag{4.81}$$

for a constant $C(\rho)$ depending on ρ. To show this, first we choose an $R > 0$ with $\mathrm{supp}(\rho) \subset B_R(0)$. Then we write

$$\begin{aligned}\int_{|\beta|\geq\epsilon} K(\theta(\alpha,\beta))(\partial^\gamma \rho)(\alpha-\beta)d\beta &= \int_{|\alpha-\beta|\geq\epsilon} K(X(\alpha)-X(\beta))\partial^\gamma\rho(\beta)d\beta, &(4.82)\\ &= \int_{\epsilon\leq|\alpha-\beta|\leq 100R} K(X(\alpha)-X(\beta))\partial^\gamma\rho(\beta)d\beta \\ &\quad + \int_{|\alpha-\beta|>100R} K(X(\alpha)-X(\beta))\partial^\gamma\rho(\beta)d\beta, \\ &= I_1 + I_2. &(4.83)\end{aligned}$$

For I_1 we have

$$\begin{aligned}|I_1| &\leq \int_{\epsilon\leq|\alpha-\beta|\leq 100R} |K(X(\alpha)-X(\beta))||\partial^\gamma\rho(\beta)|d\beta, &(4.84)\\ &\leq C\int_{\epsilon\leq|\alpha-\beta|\leq 100R} |\alpha-\beta|^{-2}d\beta\left(\sup_{B_{100R}(\alpha)}|\partial^\gamma\rho|\right),\end{aligned}$$

where the right hand side is bounded by a constant depending on ρ and has compact support in α. This allows us to bound $|I_1|$ by something as in the right hand side of (4.81).

For I_2, using that $\text{supp}(\rho) \subset B_R(\rho)$, we have

$$|I_2| \leq \int_{|\alpha-\beta|>100R} |K(X(\alpha)-X(\beta))||\partial^\gamma \rho(\beta)| d\beta \leq C'(\rho) \int_{S_\alpha} |\alpha-\beta|^{-2} d\beta. \tag{4.85}$$

where $C'(\rho)$ depends only on ρ, and $S_\alpha = \{\beta : |\alpha-\beta| > 100R \text{ and } |\beta| < R\}$. Suppose for a given α that S_α is nonempty. For any $\beta \in S_\alpha$, we have $|\beta| < R$ and $|\alpha-\beta| > 100R$. This requires $|\alpha| > 99R$, so moreover, $\beta \in S_\alpha$ implies $|\beta| < |\alpha|/99$, and thus $|\alpha-\beta|^{-1} \leq C|\alpha|^{-1}$. Thus from (4.85) we now find

$$|I_2| \leq \begin{cases} C''(\rho) \int_{|\beta|<R} |\alpha|^{-2} d\beta & \text{if } |\alpha| > 99R, \\ 0 & \text{if } |\alpha| \leq 99R, \end{cases} \tag{4.86}$$

which is bounded above by an expression as in the right hand side of (4.81). Combining this with the bound for $|I_1|$, we get the bound (4.81).

Now let us define

$$g_\epsilon(\alpha) := \int_{|\beta|\geq\epsilon} K(\theta(\alpha,\beta))(\partial^\gamma \rho)(\alpha-\beta) d\beta \tag{4.87}$$

Since the right hand side of (4.81) is in L^2, we can bound $|g_\epsilon(\alpha)|^2$ by an integrable function of α, independent of ϵ. It then follows from the dominated convergence theorem that $g(\alpha)$ is in L^2, with the L^2 norm bounded by a constant depending only on ρ. We also find $|g_\epsilon(\alpha) - g(\alpha)|^2$ is bounded by an integrable function of α, independent of ϵ, and so with the dominated convergence theorem we get that g_ϵ tends to g in L^2 as ϵ tends to zero. This concludes the proof of the lemma. □

Now we are ready to prove the differentiation formula, used for calculating various derivatives of $F(X)(\alpha)$ as in Definition 4.2.1 in a manner suited for L^2 estimates. Once we have this, the local boundedness result for the operator F for Sobolev spaces H^s will readily follow.

Proposition 4.3.6. *For $X \in B_\delta$, the following formula holds.*

(4.88)

For $\gamma \in (\mathbb{N}_0)^3$ with $0 \leq |\gamma| \leq s$, and $\sigma = (\sigma_1, \ldots, \sigma_{|\gamma|})$ with $\partial_{\sigma_{|\gamma|}} \ldots \partial_{\sigma_1} = \partial^\gamma$,

$$\partial^\gamma [F(X)(\alpha)] = \lim_{\epsilon\to 0} \int_{|\beta-\alpha|\geq\epsilon} P^\sigma_{|\gamma|}(\alpha, \alpha-\beta)\tilde{K}(\alpha,\beta) d\beta + \int K(X(\alpha)-X(\beta))(\partial^\gamma\rho)(\beta) d\beta \quad \text{in } L^2.$$

Here, for the case $\gamma = 0$ and σ empty, we define $P^\sigma_0(\alpha,\beta) := 0$.

Proof. Fix and $X \in B_\delta$. We prove that the formula in (4.88) holds for $\gamma \in (\mathbb{N}_0)^3$ by induction on $|\gamma|$. Observe that it holds for $|\gamma| = 0$, with σ empty, since we have the result of Lemma 4.3.5. Now we

assume that (4.88) is true for all γ with $|\gamma| \leq m$, for some $0 \leq m \leq s-1$. Consider a $\gamma \in (\mathbb{N}_0)^3$ with $|\gamma| = m+1$, and a $\sigma = (\sigma_1, \ldots, \sigma_{m+1})$ with $\partial_{\sigma_{m+1}} \ldots \partial_{\sigma_1} = \partial^\gamma$. Let us define $\sigma_\flat := (\sigma_1, \ldots, \sigma_m)$, and define $\gamma_\flat$ by $\partial^{\gamma_\flat} = \partial_{\sigma_m} \ldots \partial_{\sigma_1}$. Then using the formula (4.88) and making a change of variables shows

$$\begin{aligned} \partial^{\gamma_\flat}[F(X)(\alpha)] &= \lim_{\epsilon \to 0} \left(\int_{|\beta - \alpha| \geq \epsilon} P_m^{\sigma_\flat}(\alpha, \beta) \nabla_\beta (K(\theta(\alpha, \beta))) d\beta \right) \\ &\quad + \int K(\theta(\alpha, \beta))(\partial^{\gamma_\flat} \rho)(\alpha - \beta) d\beta \quad \text{in } L^2. \end{aligned} \tag{4.89}$$

Note we have $P_m^{\sigma_\flat}(\alpha, \beta) = P_m^{\sigma}(\alpha, \beta)$, by the definition of these quantities. Let us make the corresponding replacement in the expression within the limit in (4.89). Next, for each $\epsilon > 0$, we use part (i) of Corollary 4.3.4, taking $n := m+1$, to express the quantity within the limit as

$$\sum_{|\Gamma| = m-1} \left(\int_{|\beta| \geq \epsilon} h_\Gamma(\alpha)(g_\Gamma(\alpha) - g_\Gamma(\alpha - \beta)) f_\Gamma(\alpha - \beta) \partial_{\beta_{k(\Gamma)}} (K(\theta(\alpha, \beta)) d\beta \right), \tag{4.90}$$

where for each Γ, $\mathrm{supp}(f_\Gamma)$ is compact, and each of the triples $(h_\Gamma, g_\Gamma, f_\Gamma)$ belongs to one of $\mathcal{Z}_1$, $\mathcal{Z}_2$, or $\mathcal{Z}_3$. We use this in (4.89) to get

$$\begin{aligned} \partial^{\gamma_\flat}[F(X)(\alpha)] &= \sum_{|\Gamma| = m-1} \lim_{\epsilon \to 0} \left(\int_{|\beta - \alpha| \geq \epsilon} h_\Gamma(\alpha)(g_\Gamma(\alpha) - g_\Gamma(\alpha - \beta)) f_\Gamma(\alpha - \beta) \partial_{\beta_{k(\Gamma)}} (K(\theta(\alpha, \beta)) d\beta \right) \\ &\quad + \int K(\theta(\alpha, \beta))(\partial^{\gamma_\flat} \rho)(\alpha - \beta) d\beta \quad \text{in } L^2. \end{aligned} \tag{4.91}$$

Here we have used (i) of Proposition 4.3.1 to ensure convergence in L^2 of the summands in (4.91). Note by part (ii) of Proposition 4.3.1, the expression in the first line in the right hand side of (4.91) is in H^1. Now we claim that

$$\int K(\theta(\alpha, \beta))(\partial^{\gamma_\flat} \rho)(\alpha - \beta) d\beta. \tag{4.92}$$

is also in H^1. To check this, let us test it against an arbitrary $\varphi \in C_c^\infty$. Fix an $l = 1, 2$, or 3. Then

$$\begin{aligned} &- \int \partial_l \varphi(\alpha) \int K(\theta(\alpha, \beta))(\partial^{\gamma_\flat} \rho)(\alpha - \beta) d\beta d\alpha \\ &\quad = - \lim_{\epsilon \to 0} \int \partial_l \varphi(\alpha) \int_{|\beta| \geq \epsilon} K(\theta(\alpha, \beta))(\partial^{\gamma_\flat} \rho)(\alpha - \beta) d\beta d\alpha, \end{aligned} \tag{4.93}$$

by Lemma 4.3.5. Consider the quantity in the right hand side of (4.93) within the limit.

(4.94)

$$\begin{aligned}
&\int \partial_l\varphi(\alpha)\int_{|\beta|\geq\epsilon} K(\theta(\alpha,\beta))(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta d\alpha \\
&\qquad = \int_{|\beta|\geq\epsilon}\int \partial_l\varphi(\alpha)K(\theta(\alpha,\beta))(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\alpha d\beta, \\
&\qquad = -\sum_{i=1}^{3}\int_{|\beta|\geq\epsilon}\int \varphi(\alpha)(\partial_l X_i(\alpha)-\partial_l X_i(\alpha-\beta))(\partial_i K)(\theta(\alpha,\beta))(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\alpha d\beta \\
&\qquad\quad -\int_{|\beta|\geq\epsilon}\int \varphi(\alpha)K(\theta(\alpha,\beta))(\partial_l\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\alpha d\beta, \\
&\qquad = -\sum_{i=1}^{3}\int \varphi(\alpha)\bigg(\partial_l X_i(\alpha)\int_{|\beta|\geq\epsilon}(\partial_i K)(\theta(\alpha,\beta))(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta\bigg)d\alpha \\
&\qquad\quad +\sum_{i=1}^{3}\int \varphi(\alpha)\bigg(\int_{|\beta|\geq\epsilon}\partial_l X_i(\alpha-\beta)(\partial_i K)(\theta(\alpha,\beta))(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta\bigg)d\alpha \\
&\qquad\quad -\int \varphi(\alpha)\bigg(\int_{|\beta|\geq\epsilon}K(\theta(\alpha,\beta))(\partial_l\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta\bigg)d\alpha, \\
&\qquad = I_\epsilon^1+I_\epsilon^2+I_\epsilon^3
\end{aligned} \tag{4.95}$$

It is not difficult to check that for $i=1,2$, and 3,

$$(\partial_i K)(\theta(\alpha,\beta)) = \sum_{j=1}^{3}(\nabla X(\alpha-\beta))^{-1}_{ij}\tilde{K}^j(\alpha,\alpha-\beta). \tag{4.96}$$

So for I_ϵ^1 we have

$$I_\epsilon^1 = -\sum_{i,j=1}^{3}\int \varphi(\alpha)\partial_l X_i(\alpha)\int_{|\beta|\geq\epsilon}(\nabla X(\alpha-\beta))^{-1}_{ij}\tilde{K}^j(\alpha,\alpha-\beta)(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta. \tag{4.97}$$

Meanwhile, regarding I_ϵ^2,

$$I_\epsilon^2 = \int \varphi(\alpha)\int_{|\beta|\geq\epsilon}\tilde{K}^l(\alpha,\alpha-\beta)(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta, \tag{4.98}$$

Observe that (4.97) and (4.98) can be expressed in the form $I_\epsilon^1=\langle\varphi,v_1(\epsilon)\rangle$ and $I_\epsilon^2=\langle\varphi,v_2(\epsilon)\rangle$, with $v_1(\epsilon)$ and $v_2(\epsilon)$ both converging in L^2 as ϵ tends to zero, in view of Definition 4.2.3 and Remark 4.2.4, since $\partial^{\gamma_\flat}\rho$ is in L^2 and $(\nabla X)^{-1}$ is bounded.

Observe also that I_ϵ^3 in (4.95) has the form $I_\epsilon^3=\langle\varphi,v_3(\epsilon)\rangle$ where $v_3(\epsilon)$ converges in L^2 as ϵ tends

to zero, by Lemma 4.3.5. From (4.93) and (4.94) we then find

$$-\int \partial_l \varphi(\alpha) \int K(\theta(\alpha,\beta))\partial^{\gamma_\flat}\rho(\alpha-\beta)d\beta d\alpha = -\langle \varphi, \lim_{\epsilon\to 0}[v_1(\epsilon)+v_2(\epsilon)+v_3(\epsilon)]\rangle \tag{4.99}$$

Let us define $h := \lim_{\epsilon\to 0}[v_1(\epsilon)+v_2(\epsilon)+v_3(\epsilon)]$. Then it follows that h is in L^2 and the weak ∂_l-derivative of the expression in (4.92). This allows us to conclude that indeed, the expression in (4.92) is in H^1. In addition, we can conclude from the above discussion, by summing $v_1(\epsilon)$, $v_2(\epsilon)$, and $v_3(\epsilon)$, that the ∂_l-derivative of (4.92) for $l = 1, 2, 3$ is given by

$$\begin{aligned}\lim_{\epsilon\to 0}\Big(\int_{|\beta|\geq\epsilon} &(\partial_l X(\alpha)-\partial_l X(\alpha-\beta))(\nabla X(\alpha-\beta))^{-1}\tilde{K}(\alpha,\alpha-\beta)(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta \\ &+\int_{|\beta|\geq\epsilon} K(\theta(\alpha,\beta))(\partial_l\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta\Big) \quad \text{in } L^2.\end{aligned} \tag{4.100}$$

Now we return to (4.91), and apply $\partial_{\sigma_{m+1}}$ to both sides. For this we use our calculation of the derivative (4.100) of (4.92), taking $l = \sigma_{m+1}$, along with (ii) of Proposition 4.3.1. This gives

$$\begin{aligned}&\partial^\gamma[F(X)(\alpha)] = \\ &\lim_{\epsilon\to 0}\Bigg[\sum_{|\Gamma|=m-1}\Big(\int_{|\beta|\geq\epsilon}\partial_{\alpha_{\sigma_{m+1}}}\big[h_\Gamma(\alpha)(g_\Gamma(\alpha)-g_\Gamma(\alpha-\beta))f_\Gamma(\alpha-\beta)\big]\partial_{\beta_{k(\Gamma)}}(K(\theta(\alpha,\beta)))d\beta \\ &\quad -\sum_{i,j=1}^{3}\int_{|\beta|\geq\epsilon}\partial_{\beta_{k(\Gamma)}}\big[h_\Gamma(\alpha)(g_\Gamma(\alpha)-g_\Gamma(\alpha-\beta))f_\Gamma(\alpha-\beta)\big] \\ &\qquad\qquad \times(\partial_{\sigma_{m+1}}X_i(\alpha)-\partial_{\sigma_{m+1}}X_i(\alpha-\beta))(\nabla X(\alpha-\beta))^{-1}_{ij}\partial_{\beta_j}(K(\theta(\alpha,\beta)))d\beta\Big) \\ &\quad +\int_{|\beta|\geq\epsilon}(\partial_{\sigma_{m+1}}X(\alpha)-\partial_{\sigma_{m+1}}X(\alpha-\beta))(\nabla X(\alpha-\beta))^{-1}\tilde{K}(\alpha,\alpha-\beta)(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta \\ &\qquad +\int_{|\beta|\geq\epsilon}K(\theta(\alpha,\beta))(\partial^\gamma\rho)(\alpha-\beta)d\beta\Bigg] \quad \text{in } L^2.\end{aligned} \tag{4.101}$$

The right hand side can be rearranged to give

$$\begin{aligned}
&\partial^\gamma[F(X)(\alpha)] = \qquad (4.102)\\
&\lim_{\epsilon\to 0}\Bigg[\int_{|\beta|\geq\epsilon}\partial_{\alpha_{\sigma_{m+1}}}\left(\sum_{|\Gamma|=m-1}h_\Gamma(\alpha)(g_\Gamma(\alpha)-g_\Gamma(\alpha-\beta))f_\Gamma(\alpha-\beta)e_{k(\Gamma)}\right)\nabla_\beta(K(\theta(\alpha,\beta)))d\beta\\
&\qquad-\int_{|\beta|\geq\epsilon}\mathrm{div}_\beta\left(\sum_{|\Gamma|=m-1}h_\Gamma(\alpha)(g_\Gamma(\alpha)-g_\Gamma(\alpha-\beta))f_\Gamma(\alpha-\beta)e_{k(\Gamma)}\right)\\
&\qquad\qquad\times(\partial_{\sigma_{m+1}}X(\alpha)-\partial_{\sigma_{m+1}}X(\alpha-\beta))(\nabla X(\alpha-\beta))^{-1}\nabla_\beta(K(\theta(\alpha,\beta)))d\beta\\
&\qquad+\int_{|\beta|\geq\epsilon}(\partial_{\sigma_{m+1}}X(\alpha)-\partial_{\sigma_{m+1}}X(\alpha-\beta))(\nabla X(\alpha-\beta))^{-1}\tilde{K}(\alpha,\alpha-\beta)(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta\\
&\qquad+\int_{|\beta|\geq\epsilon}K(\theta(\alpha,\beta))(\partial^\gamma\rho)(\alpha-\beta)d\beta\Bigg]\quad\text{in } L^2.
\end{aligned}$$

Recalling the $\Theta_n(\alpha,\beta)$ notation given in Definition 4.3.2, we then find

$$\begin{aligned}
\partial^\gamma[F(X)(\alpha)] &= \lim_{\epsilon\to 0}\Bigg[\int_{|\beta|\geq\epsilon}\partial_{\alpha_{\sigma_{m+1}}}P^\sigma_m(\alpha,\beta)\nabla_\beta(K(\theta(\alpha,\beta)))d\beta \qquad (4.103)\\
&\qquad-\int_{|\beta|\geq\epsilon}\mathrm{div}_\beta P^\sigma_m(\alpha,\beta)\Theta_{\sigma_{m+1}}(\alpha,\beta)\nabla_\beta(K(\theta(\alpha,\beta)))d\beta\\
&\qquad+\int_{|\beta|\geq\epsilon}\Theta_{\sigma_{m+1}}(\alpha,\beta)\nabla_\beta(K(\theta(\alpha,\beta)))(\partial^{\gamma_\flat}\rho)(\alpha-\beta)d\beta\\
&\qquad+\int_{|\beta|\geq\epsilon}K(\theta(\alpha,\beta))(\partial^\gamma\rho)(\alpha-\beta)d\beta\Bigg],\\
&= \lim_{\epsilon\to 0}\left[\int_{|\beta|\geq\epsilon}P^\sigma_{m+1}(\alpha,\beta)\nabla_\beta(K(\theta(\alpha,\beta)))d\beta+\int_{|\beta|\geq\epsilon}K(\theta(\alpha,\beta))(\partial^\gamma\rho)(\alpha-\beta)d\beta\right],\\
&= \lim_{\epsilon\to 0}\int_{|\beta|\geq\epsilon}P^\sigma_{|\gamma|}(\alpha,\alpha-\beta)\tilde{K}(\alpha,\beta)d\beta+\int K(X(\alpha)-X(\beta))(\partial^\gamma\rho)(\beta)d\beta\quad\text{in } L^2,
\end{aligned}$$

where we have applied Lemma 4.3.5 in the last line. This concludes the inductive step. □

Proposition 4.3.7. *Let $X\in B_\delta$. Let the operator F be as in Definition 4.2.1. Then*

$$\|F(X)\|_{H^s}\leq C, \qquad (4.104)$$

where C only depends on ρ.

Proof. We consider some $\gamma\in(\mathbb{N}_0)^3$ with $0\leq|\gamma|\leq s$, define $n:=|\gamma|$, and choose $\sigma=(\sigma_1,\ldots,\sigma_n)$

with $\partial^\gamma = \partial_{\sigma_n} \dots \partial_{\sigma_1}$. By Proposition 4.3.6,

$$\partial^\gamma [F(X)(\alpha)] = \lim_{\epsilon \to 0} \int_{|\beta-\alpha| \geq \epsilon} P_n^\sigma(\alpha, \alpha-\beta) \tilde{K}(\alpha,\beta) d\beta + \int K(X(\alpha)-X(\beta))(\partial^\gamma \rho)(\beta) d\beta \quad \text{in } L^2. \tag{4.105}$$

If $n = 0$ we note the first term in the right hand side of (4.105) is zero, and apply Lemma 4.3.5 to get that the remaining term is bounded in L^2 by a constant depending on ρ. Suppose $n \geq 1$. For each $\epsilon > 0$, we consider the quantity within the limit in (4.105). We make a change of variables and apply Corollary 4.3.4, producing for $P_n^\sigma(\alpha,\beta)$ a collection of triples $(h_\Gamma, g_\Gamma, f_\Gamma)$, with each triple in $\mathcal{Z}_1'$, $\mathcal{Z}_2'$, or $\mathcal{Z}_3'$, and we find

$$\begin{aligned} \int_{|\beta-\alpha|\geq\epsilon} P_n^\sigma(\alpha,\alpha-\beta)\tilde{K}(\alpha,\beta)d\beta &= \int_{|\beta-\alpha|\geq\epsilon} P_n^\sigma(\alpha,\beta)\nabla_\beta(K(\theta(\alpha,\beta)))d\beta, \\ &= \sum_{|\Gamma|=n-1} h_\Gamma(\alpha) \int_{|\beta-\alpha|\geq\epsilon} (g_\Gamma(\alpha)-g_\Gamma(\alpha-\beta)) \\ &\qquad \times f_\Gamma(\alpha-\beta)\partial_{\beta_{k(\Gamma)}}(K(\theta(\alpha,\beta)))d\beta. \end{aligned} \tag{4.106}$$

Now we apply the result of part (i) for Proposition 4.3.1, which yields convergence in L^2 as ϵ tends to zero.

$$\lim_{\epsilon\to 0} \int_{|\beta-\alpha|\geq\epsilon} P_s^\sigma(\alpha,\alpha-\beta)\tilde{K}(\alpha,\beta)d\beta = \sum_{|\Gamma|=s-1} (T^0_{X,k(\Gamma)}[h_\Gamma,g_\Gamma,f_\Gamma])(\alpha) \quad \text{in } L^2. \tag{4.107}$$

Now we observe by examining the forms (4.77) for h_Γ, g_Γ, and f_Γ given in the proof of Corollary 4.3.4, and using the fact that $X \in B_\delta$, that we must have for each of the Γ

$$\|(h_\Gamma, g_\Gamma, f_\Gamma)\|_{\mathcal{Z}_i'} \leq C(\rho). \tag{4.108}$$

Taking the L^2 norm of both sides of (4.107), and using part (i) of Proposition 4.3.1 along with (4.108), we find

$$\left\| \lim_{\epsilon\to 0} \int_{|\beta-\alpha|\geq\epsilon} P_s^\sigma(\alpha,\alpha-\beta)\tilde{K}(\alpha,\beta)d\beta \right\|_{L^2} \leq C_1(\rho). \tag{4.109}$$

Meanwhile, from Lemma 4.3.5 we get that

$$\left\| \int K(X(\alpha)-X(\beta))(\partial^\gamma\rho)(\beta)d\beta \right\|_{L^2} \leq C_2(\rho). \tag{4.110}$$

Using (4.109) and (4.110) to bound the L^2 norm of the right hand side of (4.105), we conclude

$$\|\partial^\gamma F(X)\|_{L^2} \leq C_3(\rho). \tag{4.111}$$

Since we have such a bound for all γ with $0 \leq |\gamma| \leq s$, (4.104) follows. □

Now that we have the desired local boundedness of the operator F in this context, we must verify the other criterion of preservation of analyticity, as discussed in Section 2.

4.4 Analyticity of trajectories

Our proof of Lagrangian analyticity for the Euler-Poisson system largely follows the strategy outlined in Section 2 for proving Lagrangian analyticity for fluid mechanics systems. We also need to make some slight modifications to the techniques of that section when we prove the existence, uniqueness, and analyticity of solutions to the ODE system (4.8), to accommodate for the fact that we have initial data of the type $(X, V)|_{t=0} = (Id, u_0)$, rather than simply the identity map, but the changes are not especially substantial. We continue by providing the analogous abstract function space setting for the Euler-Poisson problem.

Definition 4.4.1. *Taking the Banach space $Y = H^s(\mathbb{R}^3)$, we recall the definition*

$$B_\delta := \{X : \|X - Id\|_Y \leq \delta\}, \tag{4.112}$$

and for $r > 0$, we define

$$\mathcal{Y}_r := \{Y\text{-valued maps } X_z \text{ analytic in } z \text{ on } d_r\}, \tag{4.113}$$

where $d_r = \{|z| < r\}$, and we denote

$$\mathcal{B}_{\delta,r} := \{\text{maps } X_{(\cdot)} : (X_{(\cdot)} - Id) \in \mathcal{Y}_r,\ X_z \in B_\delta \text{ for all } z \in d_r\}. \tag{4.114}$$

Lemma 4.4.2. *For $\rho \in C_c^s(\mathbb{R}^3)$, the operator F preserves analyticity. That is, for any $r > 0$ and $X_{(\cdot)} \in \mathcal{B}_{\delta,r}$, for fixed $\alpha \in \mathbb{R}^3$, $F(X_z)(\alpha)$ is analytic in d_r.*

Proof. Fix an $r > 0$, and an $X_{(\cdot)} \in \mathcal{B}_{\delta,r}$. What we want to prove is that the limit definition of $\frac{d}{dz}\left(F(X_z)(\alpha)\right)$ converges for each $z \in d_r$, $\alpha \in \mathbb{R}^3$. Fixing $\alpha \in \mathbb{R}^3$ and $z \in d_r$ we find for small

complex h

$$\begin{aligned}&\frac{1}{h}\int (K(X_{z+h}(\alpha)-X_{z+h}(\beta))-K(X_z(\alpha)-X_z(\beta)))\rho(\beta)d\beta \\ &=\int \frac{1}{h}\int_z^{z+h}\left(\frac{d}{d\zeta}X_\zeta(\alpha)-\frac{d}{d\zeta}X_\zeta(\beta)\right)\nabla K(X_\zeta(\alpha)-X_\zeta(\beta))d\zeta\rho(\beta)d\beta,\end{aligned} \tag{4.115}$$

where we take the straight path from z to $z+h$ in the integral. Since $X_{(\cdot)} \in \mathcal{B}_{\delta,r}$ implies $X_\zeta(\cdot)$ is Lipschitz with a fixed constant as ζ varies in d_r, by using the Cauchy integral formula one finds that we get a Lipschitz bound on $\frac{d}{d\zeta}X_\zeta(\cdot)$, i.e.

$$\sup_{\gamma,\beta\in\mathbb{R}^3}\left(\frac{\left|\frac{d}{d\zeta}X_\zeta(\gamma)-\frac{d}{d\zeta}X_\zeta(\beta)\right|}{|\gamma-\beta|}\right)\leq C, \tag{4.116}$$

uniform in ζ, as ζ varies in the interval from z to $z+h$, which must be contained in d_r for small h. Now, we recall that we have $(X_\zeta(\alpha)-X_\zeta(\beta))=(\alpha-\beta)(I+O(\delta))$ in the kernel in the right hand side of (4.115). Together with the Lipschitz bound (4.116) on $\frac{d}{d\zeta}X_\zeta(\cdot)$, we can thus bound the quantity inside the integral over β uniformly in h by $\big(C|\alpha-\beta|^{-2}|\rho(\beta)|\big)$, an expression which is integrable in β. Thus dominated convergence theorem tells us that when we calculate the limit as h tends to zero of the expression in (4.115), we can move the limit in h inside the β integral. Now observe that the quantity inside the ζ integral is a continuous function of ζ as long as $\beta\neq\alpha$. Thus, taking the limit as h tends to zero of the expression inside the β integral gives

$$\frac{d}{dz}\left(F(X_z)(\alpha)\right)=\int\left(\frac{d}{dz}X_z(\alpha)-\frac{d}{dz}X_z(\beta)\right)\nabla K(X_z(\alpha)-X_z(\beta))\rho(\beta)d\beta. \tag{4.117}$$

□

The following lemma follows immediately from an application of Lemma 2.2.4, since Y satisfies Assumption 2.2.1.

Lemma 4.4.3. *Consider B_δ, $\mathcal{B}_{\delta,r}$, Y, and $\mathcal{Y}_r$ as above. Consider a function $\tilde{F}: B_\delta \to Y$. If we have a bound of the form*

$$\sup_{X\in B_\delta}\|\tilde{F}(X)\|_Y\leq C, \tag{4.118}$$

and if $\tilde{F}$ preserves analyticity, then in fact for any $r>0$, for $X_z\in\mathcal{B}_{\delta,r}$ we have that $\tilde{F}(X_z)$ is an

analytic function from d_r into Y. That is,

$$\tilde{F} : \mathcal{B}_{\delta,r} \to \mathcal{Y}_r. \tag{4.119}$$

Now we proceed to use what has been established above to solve the following equation for $Z_t = (X_t, V_t) : \mathbb{R}^3 \times [-T, T] \to \mathbb{R}^6$.

$$\begin{aligned} \tfrac{d}{dt} Z_t &= \begin{pmatrix} V_t \\ F(X_t) \end{pmatrix}, \\ Z_t|_{t=0} &= \begin{pmatrix} Id \\ u_0 \end{pmatrix}, \end{aligned} \tag{4.120}$$

in the desired setting, where u_0 is in H^s and plays the role of our initial velocity for the Euler-Poisson system. We prove that the solution to the equation exists as a fixed point of a contraction mapping. For $\delta > 0$ and $r > 0$, we define

$$\mathcal{D}_{\delta,r} := \{Z_{(\cdot)} = (X_{(\cdot)}, V_{(\cdot)}) : \ X_{(\cdot)} \in \mathcal{B}_{\delta,r}, \ V_{(\cdot)} \in \mathcal{Y}_r, \text{ and } \|V_z - u_0\|_Y \leq \delta \text{ for all } z \in d_r\}, \tag{4.121}$$

and

$$\|(X_{(\cdot)}, V_{(\cdot)})\|_{\mathcal{D}_{\delta,r}} := \sup_{z \in d_r} \left(\|X_z - Id\|_Y + \|V_z - u_0\|_Y \right). \tag{4.122}$$

We recall Lemma 2.2.5, which directly implies the following lemma, which in turn applies to our operator F, in light of Proposition 4.3.7 and Lemma 4.4.2:

Lemma 4.4.4. *Consider $\delta > 0$ and a function $\tilde{F} : B_\delta \to Y$, where Y is a Banach space of maps from $\mathbb{R}^3$ to $\mathbb{R}^3$ with norm dominating the C^1 norm, and B_δ is the ball about the identity map of radius δ in Y. If we have a bound of the form*

$$\sup_{X \in B_\delta} \|\tilde{F}(X)\|_Y \leq C, \tag{4.123}$$

and that $\tilde{F}$ preserves analyticity, then for any $\epsilon \leq \delta/10$ we have that the function $\tilde{F}$ with domain B_ϵ is Lipschitz with constant C_0 depending only on C.

Now we choose a δ sufficiently small to satisfy the hypotheses required for all the above results,

and fix $\epsilon = \delta/10$. With that, for any $r > 0$ we define the following mapping on elements $Z_{(\cdot)} \in \mathcal{D}_{\epsilon,r}$:

$$\Phi\left(Z_{(\cdot)}\right)(t) := \begin{pmatrix} Id \\ u_0 \end{pmatrix} + \int_0^t \begin{pmatrix} V_\zeta \\ F(X_\zeta) \end{pmatrix} d\zeta \qquad \text{for } t \in d_r, \tag{4.124}$$

where the integration is taken along the straight path from 0 to t. This is the map we will show has a fixed point, which solves the ODE (4.120).

Theorem 4.4.5. *For $\epsilon = \delta/10$ and sufficiently small $r_1 > 0$, there exists a unique fixed point $Z_{(\cdot)} \in \mathcal{D}_{\epsilon,r_1}$ of $\Phi : \mathcal{D}_{\epsilon,r_1} \to \mathcal{D}_{\epsilon,r_1}$, and the function $Z(\alpha,t) = Z_t(\alpha)$ is analytic in t on d_{r_1} and the unique solution to the equation* (4.120).

Proof. First we check that Φ maps $\mathcal{D}_{\epsilon,r_1}$ to itself. Note that t is restricted to d_{r_1} and r_1 can be chosen as small as we like, while the bound

$$\sup_{X \in B_\delta} \|F(X)\|_Y \leq C, \tag{4.125}$$

in particular allows us to ensure that $\Phi(Z_{(\cdot)})$ is close to $(Id, u_0)^t$ for small r_1. The verification that Φ maps $\mathcal{D}_{\epsilon,r_1}$ to itself is completed with the aid of Lemmas 4.4.2 and 4.4.3. Using Lemma 4.4.4, we have the Lipschitz property of F on B_ϵ; using this we can show that Φ is a contraction mapping on $\mathcal{D}_{\epsilon,r_1}$ as long as r_1 is small enough. The conclusion of the theorem now follows from the contraction mapping principle. □

Consider a solution (ρ, u) to the Euler-Poisson equations (4.1)-(4.2) on the time interval $[-T, T]$ and define $X(\alpha, t)$ as the solution of the ODE

$$\frac{dX}{dt}(\alpha, t) = u(X(\alpha, t), t), \tag{4.126}$$

on the time interval $[-T, T]$. Recall again that using (4.1) and (4.126), one finds that $\rho(X(\alpha,t),t)) = (\det \nabla X(\alpha,t))^{-1}\rho_0(\alpha)$. By (4.2),

$$\begin{aligned} \partial_t(u(X(\alpha,t),t)) &= \int K(X(\alpha,t) - y)\rho(y,t)dy, \\ &= \int K(X(\alpha,t) - X(\beta,t))\rho(X(\beta,t),t)\det \nabla X(\beta,t)d\beta, \\ &= \int K(X(\alpha,t) - X(\beta,t))\rho_0(\beta)d\beta, \end{aligned} \tag{4.127}$$

and so, for $V(\alpha,t) := \frac{dX}{dt}(\alpha,t)$, we have a solution to the ODE (4.120) $t \in [-T,T]$.

Now suppose we have a solution (X_t, V_t) to (4.120) for $t \in [-T, T]$, where for the $\rho(\beta)$ in the definition of F we take $\rho_0(\beta) \in C_c^s$, and we have $u_0 \in H^s$, $s \geq 6$. Then we define for all $x \in \mathbb{R}^3$ and $t \in [-T, T]$

$$\begin{aligned} u(x,t) &:= V(X^{-1}(x,t),t), \\ \rho(x,t) &:= (\det \nabla X(X^{-1}(x,t)))^{-1}. \end{aligned} \tag{4.128}$$

One finds that this then yields a solution for the system (4.1)-(4.2) on $[-T, T]$ with initial data (ρ_0, u_0).

This demonstrates the equivalence of local existence results for the Euler-Poisson system in Eulerian variables to our Lagrangian formulation of the problem.

Thus, we conclude in particular from Theorem 4.4.5 the local existence and uniqueness of solutions to the system (4.1)-(4.2), and that the corresponding Lagrangian trajectories $X(\alpha, t)$ are analytic in the time variable, as long as the initial data (ρ_0, u_0) is in $C_c^s \times H^s$ for $s \geq 6$.

4.5 Integral kernel operator bounds

In this part of the section, we direct our efforts to delineating conditions under which an integral kernel of the form $\mathcal{K}(\alpha, \beta)$, $\mathcal{K} : (\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{\alpha = \beta\}) \to \mathbb{C}$, with

$$|\mathcal{K}(\alpha, \beta)| \leq A|\alpha - \beta|^{-3}, \tag{4.129}$$

has a corresponding operator that is bounded on L^2 in particular.

Lemma 4.5.1. *Suppose* $\mathcal{K} : (\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{\alpha = \beta\}) \to \mathbb{C}$ *satisfies* (4.129) *and*

$$\begin{aligned} |\mathcal{K}(\alpha,\beta) - \mathcal{K}(\alpha',\beta)| &\leq A\frac{|\alpha - \alpha'|}{|\alpha - \beta|^4}, \quad \text{if } |\alpha - \alpha'| \leq |\alpha - \beta|/2, \\ |\mathcal{K}(\alpha,\beta) - \mathcal{K}(\alpha,\beta')| &\leq A\frac{|\beta - \beta'|}{|\alpha - \beta|^4}, \quad \text{if } |\beta - \beta'| \leq |\alpha - \beta|/2, \end{aligned} \tag{4.130}$$

and, defining the quantities

$$I_{\epsilon,R}(\alpha) = \int_{\epsilon<|\alpha-\beta|<R} \mathcal{K}(\alpha,\beta)d\beta, \qquad I^*_{\epsilon,R}(\alpha) = \int_{\epsilon<|\alpha-\beta|<R} \overline{\mathcal{K}}(\beta,\alpha)d\beta,$$

suppose we have in addition

$$|I_{\epsilon,R}(\alpha)| \leq A \quad \textit{for all } \epsilon,\ R > 0 \textit{ and all } \alpha \in \mathbb{R}^3, \tag{4.131}$$

and an analogous uniform bound on $I^*_{\epsilon,R}(\alpha)$ holds. Then the operators $T^\epsilon_{\mathcal{K}}$ defined by

$$(T^\epsilon_{\mathcal{K}} f)(\alpha) = \int_{|\beta-\alpha|\geq\epsilon} \mathcal{K}(\alpha,\beta) f(\beta) d\beta, \tag{4.132}$$

are bounded from L^2 to L^2 and satisfy

$$\|T^\epsilon_{\mathcal{K}} f\|_{L^2} \leq C\|f\|_{L^2}, \tag{4.133}$$

independent of ϵ.

Proof. The above result follows from an application of Theorem 4 in Section 7.3 of [22] together with Proposition 1 in Section 1.7 of [22]. □

The next lemma gives some basic conditions for $\mathcal{K}$ that lead to pointwise convergence of the operators $T^\epsilon_{\mathcal{K}} f$ as ϵ tends to zero when f has a little smoothness, along with related operator bounds.

Lemma 4.5.2. *Fix $X \in B_\delta$. Fix $i = 1, 2,$ or 3. We define the kernel $\mathcal{K} : (\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{\alpha = \beta\}) \to \mathbb{C}$ by*

$$\mathcal{K}(\alpha,\beta) = \partial_{\beta_i} K^\flat(\alpha,\beta), \tag{4.134}$$

where for a fixed $j = 1, 2,$ or 3,

$$K^\flat(\alpha,\beta) = K^j(X(\alpha) - X(\beta)). \tag{4.135}$$

Then we have that

$$|K^\flat(\alpha,\beta)| \leq A|\alpha-\beta|^{-2} \quad \textit{for all } \alpha, \tag{4.136}$$

and $\mathcal{K}(\alpha,\beta)$ satisfies (4.129)*.*

Additionally, for $f \in L^2 \cap C^1$, the following limit converges uniformly in α.

$$(T_{\mathcal{K}} f)(\alpha) := p.v. \int \mathcal{K}(\alpha,\beta) f(\beta) d\beta, \tag{4.137}$$

and for $\epsilon, R > 0$, the operators defined by

$$(T_{\mathcal{K}}^{\epsilon,R} f)(\alpha) := \int_{\epsilon \le |\beta-\alpha| \le R} \mathcal{K}(\alpha,\beta) f(\beta) d\beta, \tag{4.138}$$

and

$$(T_{\mathcal{K}}^{\epsilon} f)(\alpha) := \int_{|\beta-\alpha| \ge \epsilon} \mathcal{K}(\alpha,\beta) f(\beta) d\beta, \tag{4.139}$$

we have the bounds for all $f \in L^2 \cap C^1$

$$\begin{aligned} \|T_{\mathcal{K}}^{\epsilon,R} f\|_{L^\infty} &\le A'(\|f\|_{L^2} + \|f\|_{C^1}), & (4.140)\\ \|T_{\mathcal{K}}^{\epsilon} f\|_{L^\infty} &\le A'(\|f\|_{L^2} + \|f\|_{C^1}), & (4.141)\\ \|T_{\mathcal{K}} f\|_{L^\infty} &\le A'(\|f\|_{L^2} + \|f\|_{C^1}), & (4.142) \end{aligned}$$

for a universal constant A'.

Proof. Fix $\epsilon, R > 0$, and an $f \in L^2 \cap C^1$. Note for $R \le \epsilon$, (4.140) is trivial, so let us assume $R > \epsilon$. Let $r = \min(1, R)$. We split the integral for $(T_{\mathcal{K}}^{\epsilon,R} f)(\alpha)$ into two, with

$$\int_{\epsilon \le |\beta-\alpha| \le R} \mathcal{K}(\alpha,\beta) f(\beta) d\beta = \int_{\epsilon \le |\beta-\alpha| < r} \mathcal{K}(\alpha,\beta) f(\beta) d\beta + \int_{1 \le |\beta-\alpha| \le R} \mathcal{K}(\alpha,\beta) f(\beta) d\beta. \tag{4.143}$$

The integral on the right is bounded by a constant multiple of $\|f\|_{L^2}$ since $\|1_{1 \le |\beta-\alpha| \le R} \mathcal{K}(\alpha,\beta)\|_{L^2_\beta}$ is bounded by a constant, by (4.129). For the integral on the left, we have

$$\int_{\epsilon \le |\beta-\alpha| < r} \mathcal{K}(\alpha,\beta) f(\beta) d\beta = \int_{\epsilon \le |\beta-\alpha| < r} \mathcal{K}(\alpha,\beta)(f(\beta) - f(\alpha)) d\beta + f(\alpha) \int_{\epsilon \le |\beta-\alpha| < r} \mathcal{K}(\alpha,\beta) d\beta, \tag{4.144}$$

where, for the first piece we have

$$\left| \int_{\epsilon \le |\beta-\alpha| < r} \mathcal{K}(\alpha,\beta)(f(\beta) - f(\alpha)) d\beta \right| \le \int_{|\beta-\alpha|<1} A|\alpha-\beta|^{-3} \|f\|_{C^1} |\alpha-\beta| d\beta \le C\|f\|_{C^1}, \tag{4.145}$$

and the magnitude of the second piece is equal to

$$\begin{aligned}\left|f(\alpha)\int_{\epsilon\leq|\beta-\alpha|<r}\partial_{\beta_i}K^\flat(\alpha,\beta)d\beta\right| &= \Big|f(\alpha)\Big(-\int_{|\alpha-\beta|=\epsilon}K^\flat(\alpha,\beta)\nu_i(\beta)d\sigma(\beta)\\ &\qquad +\int_{|\alpha-\beta|=r}K^\flat(\alpha,\beta)\nu_i(\beta)d\sigma(\beta)\Big)\Big|,\\ &\leq |f(\alpha)|\Big(\int_{|\alpha-\beta|=\epsilon}A|\alpha-\beta|^{-2}d\sigma(\beta)\\ &\qquad +\int_{|\alpha-\beta|=r}A|\alpha-\beta|^{-2}d\sigma(\beta)\Big),\\ &\leq C\|f\|_{L^\infty},\end{aligned} \tag{4.146}$$

where ν denotes the unit normal. This concludes the proof of the estimate for $T_{\mathcal{K}}^{\epsilon,R}$. By taking the limit as R tends to infinity, we also get the estimate for $T_{\mathcal{K}}^{\epsilon}$. Now, regarding the definition of $T_{\mathcal{K}}f$ for $f\in C^1\cap L^2$, we use the above decomposition to verify that $T_{\mathcal{K}}^{\epsilon}f(\alpha)$ converges for each $\alpha\in\mathbb{R}^3$ and in fact in the sup norm, as ϵ tends to zero. It suffices to prove uniform convergence for

$$\int_{\epsilon\leq|\beta-\alpha|\leq 1}\mathcal{K}(\alpha,\beta)f(\beta)d\beta \tag{4.147}$$

which we split as in (4.144). For the piece

$$\int_{\epsilon\leq|\beta-\alpha|<1}\mathcal{K}(\alpha,\beta)(f(\beta)-f(\alpha))d\beta \tag{4.148}$$

we note from the previous observations that the singularity in β at $\beta=\alpha$ in the integrand is actually integrable, and that

$$\int_{|\beta|<\epsilon}\mathcal{K}(\alpha,\alpha-\beta)(f(\alpha-\beta)-f(\alpha))d\beta \tag{4.149}$$

tends to zero uniformly in α as ϵ tends to zero.

For the piece

$$f(\alpha)\int_{\epsilon\leq|\beta-\alpha|<1}\mathcal{K}(\alpha,\beta)d\beta \tag{4.150}$$

we observe that

$$\int_{\epsilon\leq|\beta-\alpha|<1}\mathcal{K}(\alpha,\beta)d\beta=-\int_{|\alpha-\beta|=\epsilon}K^\flat(\alpha,\beta)\nu_i(\beta)d\sigma(\beta)+\int_{|\alpha-\beta|=1}K^\flat(\alpha,\beta)\nu_i(\beta)d\sigma(\beta), \tag{4.151}$$

and so for convergence as ϵ tends to zero we just have to deal with the quantity

$$\begin{aligned}\int_{|\alpha-\beta|=\epsilon} K^\flat(\alpha,\beta)\nu_i(\beta)d\sigma(\beta) &= \int_{|\beta|=\epsilon} K^\flat(\alpha,\alpha-\beta)\nu_i(\beta)d\sigma(\beta), \\ &= \int_{|\beta|=\epsilon} K^j(X(\alpha)-X(\alpha-\beta))\nu_i(\beta)d\sigma(\beta).\end{aligned} \tag{4.152}$$

Now note that for any β with $|\beta| = \epsilon$ we have $X(\alpha) - X(\alpha-\beta) = \beta(I + O(\delta))$ and $\beta\nabla X(\alpha) = \beta(I+O(\delta))$. Thus as long as δ is small enough, the straight line segment connecting $(X(\alpha)-X(\alpha-\beta))$ and $\beta\nabla X(\alpha)$ does not intersect zero for any such β, for any $\epsilon > 0$. We may use this to apply the fundamental theorem of calculus in the following observation.

$$\begin{aligned}&\int_{|\beta|=\epsilon} K^j(X(\alpha)-X(\alpha-\beta))\nu_i(\beta)d\sigma(\beta) - \int_{|\beta|=\epsilon} K^j(\beta\nabla X(\alpha))\nu_i(\beta)d\sigma(\beta) \\ &= \int_{|\beta|=\epsilon} \big((X(\alpha)-X(\alpha-\beta)) - \beta\nabla X(\alpha)\big) \\ &\qquad \times \int_0^1 (\nabla K^j)(\tau(X(\alpha)-X(\alpha-\beta)) + (1-\tau)\beta\nabla X(\alpha))d\tau d\sigma(\beta) \\ &= \int_{|\beta|=\epsilon} O(|\beta|^2)\int_0^1 O(|\beta|^{-3})d\tau d\sigma(\beta), \\ &= O(\epsilon).\end{aligned} \tag{4.153}$$

and so the difference between the two quantities we started with tends to zero as ϵ tends to zero uniformly in α, but note that by the order of homogeneity of the integrand in

$$\int_{|\beta|=\epsilon} K^j(\beta\nabla X(\alpha))\nu_i(\beta)d\sigma(\beta), \tag{4.154}$$

we get that the above is independent of ϵ, and so the limit of the left hand side of (4.152) as ϵ tends to zero is

$$\int_{|\beta|=1} K^j(\beta\nabla X(\alpha))\nu_i(\beta)d\sigma(\beta). \tag{4.155}$$

Moreover we have convergence uniform in α.

The estimate for $T_{\mathcal{K}}f$ is then retained in the limit as ϵ tends to zero from the estimate for $T^\epsilon_{\mathcal{K}}f$, uniform in ϵ. □

Lemma 4.5.3. *For $X \in B_\delta$, for each $k = 1,\dots,d$, the function $\tilde{K}^k : (\mathbb{R}^3\times\mathbb{R}^3 \setminus \{\alpha=\beta\}) \to \mathbb{C}^3$ defined by*

$$\tilde{K}^k(\alpha,\beta) = \sum_{i=1}^3 \partial_k X_i(\beta)(\partial_i K)(X(\alpha)-X(\beta)), \tag{4.156}$$

has corresponding operators $T^{\epsilon}_{X,k} : C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ *for* $\epsilon > 0$ *and* $T_{X,k} : C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$*, defined by*

$$T^{\epsilon}_{X,k} f(\alpha) = \int_{|\beta-\alpha|\geq\epsilon} \tilde{K}^k(\alpha,\beta) f(\beta) d\beta, \tag{4.157}$$

and (with the p.v. limit converging uniformly in α*)*

$$T_{X,k} f(\alpha) = p.v. \int \tilde{K}^k(\alpha,\beta) f(\beta) d\beta, \tag{4.158}$$

for $\mathbb{C}$*-valued* $f \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$*. We have the following bounds for all* $f \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$*:*

$$\|T^{\epsilon}_{X,k} f\|_{L^\infty} \leq C(\|f\|_{L^2} + \|f\|_{C^1}), \tag{4.159}$$

$$\|T_{X,k} f\|_{L^\infty} \leq C(\|f\|_{L^2} + \|f\|_{C^1}). \tag{4.160}$$

Proof. First, we fix $k = 1, 2,$ or 3, and note

$$\tilde{K}^k(\alpha,\beta) = -\partial_{\beta_k}(K(X(\alpha) - X(\beta))). \tag{4.161}$$

In an application of Lemma 4.5.2, a choice of i and j may be prescribed, for i, j as in the statement of the lemma. We will take $i = k$, and a $j = 1, 2,$ or 3. In that case, we find for the corresponding $\mathcal{K}(\alpha,\beta)$ of Lemma 4.5.2

$$(\tilde{K}^k(\alpha,\beta))_j = -\mathcal{K}(\alpha,\beta) \tag{4.162}$$

and for $f \in C^1 \cap L^2$, and $\epsilon > 0$,

$$\left(T^{\epsilon}_{X,k} f(\alpha)\right)_j = -(T^{\epsilon}_{\mathcal{K}} f)(\alpha). \tag{4.163}$$

Thus, by Lemma 4.5.2, we have uniform convergence of the above as ϵ tends to zero, as well as the bound (4.159), and similarly the bound (4.160) follows. □

Lemma 4.5.4. *Consider any* $X \in B_\delta$*. For each* $\epsilon > 0$*, and* $k = 1, 2,$ *and* 3*, (i) we have a bounded operator* $T^{\epsilon}_{X,k} : L^2 \to L^2$*, mapping a* $\mathbb{C}$*-valued function* $f \in L^2(\mathbb{R}^3)$ *to the* $\mathbb{C}^3$*-valued function defined by*

$$T^{\epsilon}_{X,k} f(\alpha) = \int_{|\beta-\alpha|\geq\epsilon} \tilde{K}^k(\alpha,\beta) f(\beta) d\beta \quad \text{for all } f \in L^2, \tag{4.164}$$

where we have the $\mathbb{C}^3$-valued kernel $\tilde{K}^k(\alpha,\beta)$ given by

$$\tilde{K}^k(\alpha,\beta)=\sum_{i=1}^{3}\partial_k X_i(\beta)(\partial_i K)(X(\alpha)-X(\beta)). \tag{4.165}$$

We have the bound

$$\|T^{\epsilon}_{X,k}f\|_{L^2}\leq C\|f\|_{L^2}, \tag{4.166}$$

for a universal constant C.

(ii) For $f\in L^2$, $T^{\epsilon}_{X,k}f$ converges in L^2 as ϵ tends to zero, and we define the operator $T^0_{X,k}$ on L^2 by

$$T^0_{X,k}f=\lim_{\epsilon\to 0}T^{\epsilon}_{X,k}f \quad \text{in } L^2. \tag{4.167}$$

This gives a bounded operator from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ satisfying the bound

$$\|T^0_{X,k}f\|_{L^2}\leq C\|f\|_{L^2}, \tag{4.168}$$

where the constant C is the same as the one in (4.166).

Proof. Let us define $K':\mathbb{R}^3\times\mathbb{R}^3\setminus\{\alpha=\beta\}\to\left(\mathbb{C}^3\right)^2$ by

$$K'(\alpha,\beta):=(\nabla K)(X(\alpha)-X(\beta)). \tag{4.169}$$

Now we check the conditions (4.130) in Lemma 4.5.1 hold for each of the components $(K')_{ij}$ of K', $i,j=1,2,3$. Observe for $l=1,2$, and 3

$$\begin{aligned}\partial_{\alpha_l}(K'(\alpha,\beta)) &= \sum_{k=1}^{3}\partial_{\alpha_l}X_k(\alpha)(\partial_k\nabla K)(X(\alpha)-X(\beta)),\\ \partial_{\beta_l}(K'(\alpha,\beta)) &= -\sum_{k=1}^{3}\partial_{\beta_l}X_k(\beta)(\partial_k\nabla K)(X(\alpha)-X(\beta)).\end{aligned} \tag{4.170}$$

With the use of Lemma 4.2.2, these imply

$$\begin{aligned}|\partial_{\alpha_l}(K'(\alpha,\beta))| &\leq \frac{C}{|\alpha-\beta|^4},\\ |\partial_{\beta_l}(K'(\alpha,\beta))| &\leq \frac{C}{|\alpha-\beta|^4},\end{aligned} \tag{4.171}$$

which is sufficient for the conditions (4.130) to hold for the components of K'.

Now we check a bound such as (4.131) holds for $(I_{\epsilon,R})_{ij}$ and $(I^*_{\epsilon,R})_{ij}$ for fixed $i,j=1,2,3$, the

quantities corresponding to the $I_{\epsilon,R}$ and $I^*_{\epsilon,R}$ in the statement of Lemma 4.5.1 for the entry $(K')_{ij}$.

Note

$$(K'(\alpha,\beta))_{ij} = -\sum_{l=1}^{3}(\nabla X(\beta))^{-1}_{il}\partial_{\beta_l}(K^j(X(\alpha)-X(\beta))). \tag{4.172}$$

We may write $(I_{\epsilon,R})_{ij} = -\sum_{l=1}^{3} J^{ijl}_{\epsilon,R}$, where for $l = 1, 2$, and 3

$$\begin{aligned} J^{ijl}_{\epsilon,R} &= \int_{\epsilon<|\alpha-\beta|<R}(\nabla X(\beta))^{-1}_{il}\partial_{\beta_l}(K^j(X(\alpha)-X(\beta)))d\beta \\ &= \int_{\epsilon<|\alpha-\beta|<R}((\nabla X(\beta))^{-1}-I)_{il}\partial_{\beta_l}(K^j(X(\alpha)-X(\beta)))d\beta \\ &\quad + \int_{\epsilon<|\alpha-\beta|<R}\delta_{il}\partial_{\beta_l}(K^j(X(\alpha)-X(\beta)))d\beta \\ &= I^l_1 + I^l_2, \end{aligned} \tag{4.173}$$

where the first integral, I^l_1, is bounded with an application of Lemma 4.5.2, with the estimate (4.140), since $((\nabla X(\alpha))^{-1} - I)$ is bounded in both L^2 and C^1. For I^l_2 we note

$$\begin{aligned} \int_{\epsilon<|\alpha-\beta|<R}\partial_{\beta_l}(K^j(X(\alpha)-X(\beta))d\beta &= \left(\int_{|\alpha-\beta|=\epsilon}+\int_{|\alpha-\beta|=R}\right)K^j(X(\alpha)-X(\beta))\nu_l(\beta)d\sigma(\beta), \\ &= \int_{|\alpha-\beta|=\epsilon}O(\epsilon^{-2})d\sigma(\beta)+\int_{|\alpha-\beta|=R}O(R^{-2})d\sigma(\beta), \\ &= O(1). \end{aligned} \tag{4.174}$$

By summing over l, we get a uniform bound on $(I_{\epsilon,R})_{ij}$. Now, for $(I^*_{\epsilon,R})_{ij}$, we must consider $(\overline{(K')(\beta,\alpha)})_{ij}$. We note that since $(\nabla K)(z)$ is even, $K'(\alpha,\beta) = K'(\beta,\alpha)$, and so from the previous estimate on $(I_{\epsilon,R})_{ij}$ and the fact that complex conjugation does not affect the bound, we also have that $(I^*_{\epsilon,R})_{ij}$ is uniformly bounded.

Since the $(K'(\alpha,\beta))_{ij}$ satisfy the hypotheses of Lemma 4.5.1, for each $i, j = 1, 2, 3$, and all $\epsilon > 0$, we have that the operators $T^\epsilon_{(K')_{ij}}$ are bounded from L^2 to L^2, and satisfy a bound of the form

$$\|T^\epsilon_{(K')_{ij}}f\|_{L^2} \le C\|f\|_{L^2} \quad \text{for all } f \in L^2. \tag{4.175}$$

With the aim of using this to get a bound for the $T^\epsilon_{X,k}$ for each $k = 1, 2, 3$ we note that for each

$f \in L^2(\mathbb{R}^3)$ and $j = 1, 2, 3$,

$$\begin{aligned}((T^\epsilon_{X,k} f)(\alpha))_j &= \sum_{i=1}^{3} \int_{|\beta-\alpha|\geq\epsilon} \partial_k X_i(\beta)(\partial_i K^j)(X(\alpha) - X(\beta)) f(\beta) d\beta, \\ &= \sum_{i=1}^{3} \int_{|\beta-\alpha|\geq\epsilon} (K'(\alpha,\beta))_{ij} \partial_k X_i(\beta) f(\beta) d\beta, \\ &= \sum_{i=1}^{3} (T^\epsilon_{(K')_{ij}}((\partial_k X_i) f))(\alpha).\end{aligned} \tag{4.176}$$

With this we find by using the uniform L^∞ bound on $\nabla X(\alpha)$

$$\|T^\epsilon_{X,k} f\|_{L^2} \leq \sum_{i,j=1}^{3} \|T^\epsilon_{(K')_{ij}}((\partial_k X_i) f)\|_{L^2} \leq \sum_{i=1}^{3} C \|(\partial_k X_i) f\|_{L^2} \leq C' \|f\|_{L^2}. \tag{4.177}$$

This proves part (i). For part (ii), first we show that for any f in the dense subspace C_c^∞ of L^2, we have convergence of the limit $\lim_{\epsilon\to 0} T^\epsilon_{X,k} f$ in L^2. The first thing to note is that for fixed $f \in C_c^\infty(\mathbb{R}^3)$ we have pointwise convergence, by Lemma 4.5.3:

$$\lim_{\epsilon\to 0} T^\epsilon_{X,k} f(\alpha) = g(\alpha) \quad \text{for all } \alpha \in \mathbb{R}^3, \tag{4.178}$$

for some function $g \in L^\infty$. Meanwhile, since the kernel $\tilde{K}^k(\alpha,\beta)$ in the definition of $T^\epsilon_{X,k}$ satisfies the bound

$$|\tilde{K}^k(\alpha,\beta)| \leq \frac{C}{|\alpha-\beta|^3}, \tag{4.179}$$

for the fixed $f \in C_c^\infty$ we have a bound of the form

$$|T^\epsilon_{X,k} f(\alpha)| \leq \frac{C_f}{(1+|\alpha|)^3} \quad \text{for all } \alpha \in \mathbb{R}^3. \tag{4.180}$$

To see this, fix such an f and take $R > 0$ large enough that $\text{supp} f \subset B_R(0)$. Then

$$\begin{aligned}T^\epsilon_{X,k} f(\alpha) &= \int_{|\beta-\alpha|\geq\epsilon} \tilde{K}^k(\alpha,\beta) f(\beta) d\beta, \\ &= \left(\int_{\epsilon\leq|\beta-\alpha|\leq 100R} + \int_{|\beta-\alpha|>100R} \right) \tilde{K}^k(\alpha,\beta) f(\beta) d\beta, \\ &= I_1 + I_2,\end{aligned} \tag{4.181}$$

for which we first note that we have

$$I_1 = \int_{\epsilon \leq |\beta-\alpha| \leq 100R} \tilde{K}^k(\alpha,\beta)(f(\beta)-f(\alpha))d\beta + f(\alpha)\int_{\epsilon \leq |\beta-\alpha| \leq 100R} \tilde{K}^k(\alpha,\beta)d\beta. \tag{4.182}$$

Regarding the first term in the right hand side of (4.182), note that

$$\begin{aligned} \left| \int_{\epsilon \leq |\beta-\alpha| \leq 100R} \tilde{K}^k(\alpha,\beta)(f(\beta)-f(\alpha))d\beta \right| &\leq C\int_{\epsilon \leq |\beta-\alpha| \leq 100R} \frac{1}{|\alpha-\beta|^2} d\beta \left(\sup_{B_{100R}(\alpha)} |\nabla f| \right), \\ &\leq C' \sup_{B_{100R}(\alpha)} |\nabla f|. \end{aligned} \tag{4.183}$$

Since f has compact support, this is bounded by a quantity as in the right hand side of (4.180). For the second term composing I_1 in (4.182), we note

$$\begin{aligned} f(\alpha)\int_{\epsilon \leq |\beta-\alpha| \leq 100R} \tilde{K}^k(\alpha,\beta)d\beta &= -f(\alpha)\int_{\epsilon \leq |\beta-\alpha| \leq 100R} \partial_{\beta_k}(K(X(\alpha)-X(\beta)))d\beta, \\ &= -f(\alpha)\left(\int_{|\beta-\alpha|=\epsilon} O(\epsilon^{-2})d\sigma(\beta) + \int_{|\beta-\alpha|=100R} O(R^{-2})d\sigma(\beta) \right), \\ &= O(1)f(\alpha). \end{aligned} \tag{4.184}$$

Since f is assumed to have compact support, this is also bounded by a quantity as in the right hand side of (4.180). Now we give the bound for I_2. Observe

$$\begin{aligned} |I_2| &= \left| \int_{|\beta-\alpha|>100R} \tilde{K}^k(\alpha,\beta)f(\beta)d\beta \right|, \\ &\leq C\int_{\{|\beta-\alpha|>100R\}\cap\{|\beta|<R\}} \left| \tilde{K}^k(\alpha,\beta) \right| d\beta, \\ &\leq C'\int_{\{|\beta-\alpha|>100R\}\cap\{|\beta|<R\}} \frac{1}{|\alpha-\beta|^3} d\beta, \end{aligned} \tag{4.185}$$

where for a given α, the right hand side is 0 unless for some $|\beta| < R$, we have $|\alpha-\beta| > 100R$. This requires $|\alpha| > 99R$. For $|\beta| < R$ we would then have $|\beta| < \frac{1}{99}|\alpha|$. Thus for such α and β we have

$$\frac{1}{|\alpha-\beta|} \leq \frac{C}{|\alpha|}. \tag{4.186}$$

Using these observations in the last inequality of (4.185), we get

$$\begin{aligned} |I_2| &\leq \begin{cases} C'' \int_{|\beta|<R} \frac{1}{|\alpha|^3} d\beta, & \text{if } |\alpha| > 99R \\ 0 & \text{if } |\alpha| \leq 99R, \end{cases} \\ &\leq C(R) 1_{\mathbb{R}^3 \setminus B_R(0)}(\alpha) \frac{1}{|\alpha|^3}. \end{aligned} \tag{4.187}$$

Thus, I_2 in addition to I_1 is bounded by a quantity as in the right hand side of (4.180), and so we conclude that the bound (4.180) holds as written for an appropriate choice of C_f.

Since the right hand side of (4.180) is in L^2, we have a bound that is uniform in ϵ on $|T^{\epsilon}_{X,k} f(\alpha)|^2$ by an integrable function of α. By using the dominated convergence theorem we then see that $g(\alpha)$, which is the pointwise limit of the $T^{\epsilon}_{X,k} f(\alpha)$ as ϵ tends to zero, is in L^2. Another application of dominated convergence theorem then gives us that

$$\lim_{\epsilon \to 0} \int |T^{\epsilon}_{X,k} f(\alpha) - g(\alpha)|^2 d\alpha = 0. \tag{4.188}$$

Thus we have convergence of the limit $\lim_{\epsilon \to 0} T^{\epsilon}_{X,k} f$ in L^2 for each $f \in C^{\infty}_c$.

Now we extend this to convergence of $T^{\epsilon}_{X,k} f$ in L^2 for any $f \in L^2$. Fix such an f. Consider an arbitrary sequence $(\epsilon_l)_{l \geq 1}$ tending to zero. Given $\epsilon' > 0$, we choose $h \in C^{\infty}_c$ with $\|f - h\|_{L^2} \leq \epsilon'/C$, for C as in the right hand side of (4.166) in part (i) of the lemma. Then for $n, m \geq N$ for an N to be specified momentarily,

(4.189)

$$\begin{aligned} \|T^{\epsilon_n}_{X,k} f - T^{\epsilon_m}_{X,k} f\|_{L^2} &\leq \|T^{\epsilon_n}_{X,k} f - T^{\epsilon_n}_{X,k} h\|_{L^2} + \|T^{\epsilon_n}_{X,k} h - T^{\epsilon_m}_{X,k} h\|_{L^2} + \|T^{\epsilon_m}_{X,k} h - T^{\epsilon_m}_{X,k} f\|_{L^2}, \\ &\leq 2C \|f - h\|_{L^2} + \|T^{\epsilon_n}_{X,k} h - T^{\epsilon_m}_{X,k} h\|_{L^2}, \\ &\leq 2\epsilon' + \|T^{\epsilon_n}_{X,k} h - T^{\epsilon_m}_{X,k} h\|_{L^2}. \end{aligned}$$

Since $(T^{\epsilon_l}_{X,k} h)_{l \geq 1} \subset L^2$ is Cauchy, we can choose an N such that the second term in the last line is less than ϵ' for all $n, m \geq N$. We conclude that for $n, m \geq N$, the left hand side of (4.189) is no greater than $3\epsilon'$, and so we deduce that the sequence $(T^{\epsilon_l}_{X,k} f)_{l \geq 1}$ is Cauchy in L^2. Finally, this implies that $T^0_{X,k} f = \lim_{\epsilon \to 0} T^{\epsilon}_{X,k} f$ is a well-defined limit in L^2. By using the bound (4.166) from part (i), we get the bound (4.168) on $T^0_{X,k}$ from L^2 to L^2. □

Chapter 5

Counterexample for the compressible Euler equations

In this section, we consider the 2D compressible Euler equations, specifically in the isentropic case, given by

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{5.1}$$

$$\partial_t u + u \cdot \nabla u + \frac{1}{\rho} \nabla p = 0, \tag{5.2}$$

where $(\rho, u) = (\rho(x,t), u(x,t)) : \mathbb{R}^2 \times \mathbb{R} \to (0,\infty) \times \mathbb{R}^2$ represent the density and velocity, respectively, of a fluid or gas, and the pressure p is a known function of ρ, i.e. $p = p(\rho)$, which for the sake of concreteness, we will take to be $p(\rho) = A\rho^\gamma$ for constants $A > 0$ and $\gamma > 1$, a typical example.

Regarding questions of Lagrangian analyticity for compressible fluid equations, in [11] and [17], the question is discussed whether one has analytic trajectories for inviscid compressible fluid equations, for solutions with low spatial regularity, as one has for the incompressible Euler equations. The results in [11] do provide an example of a kind of compressible fluid model which does have this property, and we saw that the pressureless Euler-Poisson model in Section 4 also provides such an example. In this section, however, we prove that the Lagrangian trajectories for solutions to the standard compressible fluids model, the compressible Euler equations, are not in general analytic, notably distinct from previous studies, even for C^∞ initial data in our case.

To this end we provide a counterexample with initial data which is smooth and compactly

supported[1], for which we have local existence of a unique classical solution to the problem (5.1)-(5.2), and for which there exist Lagrangian trajectories that are not analytic in any small disc about $t = 0$.

The basic strategy for constructing our counterexample is to take advantage of the finite speed of propagation property exibited by systems of conservation laws such as (5.1)-(5.2). We take initial data with zero velocity outside of a ball, and so the Lagrangian trajectories of fluid parcels outside the support of u at $t = 0$ remain constant for a positive time, before the influence has propagated to these parcels. We use an elementary argument to prove that some nontrivial influence does indeed propagate away from the initial support and reach fluid parcels which were previously motionless, and at a time before the development of shocks. Thus we find that a particle moves, but was stationary for an interval of time before that moment, and so its trajectory is smooth but cannot be analytic.

We remark that for the model in [11] as well as the pressureless Euler-Poisson model considered in Section 4, the fact that the pressure term is absent is a key difference from typical fluids models, and so they are inherently different from the standard compressible Euler equations studied in this section. The presence of the pressure is an important part of the conservation law structure in the compressible Euler equations, and the conservation law structure leads to the finite speed of propagation in the system. With the method here, we show that the finite speed of propagation in the system is incompatible with automatic analyticity of Lagrangian trajectories, and one can likely use this method to show this is the case for particle trajectories in other systems of continuum mechanics with a finite speed of propagation. We make the related comment that the study [16] builds on the strategies of [11] to prove analyticity of particle trajectories in a related Newtonian cosmological model, and the authors raise the question of whether the results might extend to a relativistic cosmological model. Since having a finite speed of propagation is characteristic of relativistic models, it is suggested by the basic idea behind the results in this section that incorporating these effects could in fact break this property.

[1]For the density, we actually mean that $(\rho|_{t=0} - \overline{\rho})$ is compactly supported, for a constant $\overline{\rho} > 0$.

5.1 Non-analytic trajectories as a consequence of finite speed of propagation

The initial data we use to get non-analytic trajectories is a smooth, compactly supported perturbation of a constant state, with zero velocity outside a ball of radius 1. We give the precise definition below.

Definition 5.1.1. *Fix constants $\overline{\rho} > 0$ and $\epsilon > 0$. We define $\rho_0(x) := \overline{\rho} + \epsilon\tilde{\rho}(x)$, $u_0(x) := \epsilon\tilde{u}(x)$, with*

$$\tilde{\rho}(x) := \begin{cases} e^{\frac{1}{|x|^2-1}}, & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases} \tag{5.3}$$

and

$$\tilde{u}(x) := \begin{cases} \frac{x}{|x|} e^{q(|x|)} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases} \tag{5.4}$$

where $q(r) = \frac{1}{r^2(r^2-1)}$.

Despite the typical eventual formation of shocks for solutions to (5.1)-(5.2), it is not difficult to show that as long as the initial data is sufficiently small, the classical solution to the system exists for large times. The optimal classical existence results for the system with initial data such as ours, namely smooth, compactly supported, radial perturbations of size ϵ, are proven in [2] and [21], where classical existence is shown up to times of order $\frac{1}{\epsilon^2}$, for sufficiently small ϵ. Having classical existence for large times allows us to ensure that C^∞ smoothness is retained in an interval of time surrounding the time t at which we show that some trajectory cannot be analytic.

Proposition 5.1.2. *There exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the following holds. Let T_ϵ be the maximal time of existence for the unique classical solution (ρ, u) to the system (5.1)-(5.2) with initial data (ρ_0, u_0). Then*

$$\lim_{\epsilon \to 0} T_\epsilon = \infty. \tag{5.5}$$

Moreover, the solution (ρ, u) lies in $C^\infty([0, T_\epsilon) \times \mathbb{R}^2)$.

Proof. By the results in [2, 21], for our initial data a classical solution exists up to a time $t = C\epsilon^{-2}$ for $\epsilon < \epsilon_0$, for some constants $C > 0$, $\epsilon_0 > 0$. However, the proof of this result is a bit technical, and so here we provide a simple scaling argument that gives a lower bound on T_ϵ sufficient for (5.5).

We note that the system (5.1)-(5.2) satisfies the property that for any $\lambda > 0$, $(\rho, u)(x, t)$ is a

classical solution with initial data $(\rho^0, u^0)(x)$ if and only if

$$(\rho_\lambda, u_\lambda)(x,t) := (\rho, u)(\lambda x, \lambda t) \tag{5.6}$$

is a classical solution with initial data $(\rho^0_\lambda, u^0_\lambda)(x) = (\rho^0, u^0)(\lambda x)$. One easily checks that for a fixed $s > 0$ and large $\lambda > 0$

$$\|(\rho^0_\lambda - \overline{\rho})\|_{H^s} \leq C\lambda^{s-1}\|(\rho^0 - \overline{\rho})\|_{H^s}, \tag{5.7}$$

$$\|u^0_\lambda\|_{H^s} \leq C\lambda^{s-1}\|u^0\|_{H^s}, \tag{5.8}$$

for a constant $C = C(s)$. The standard local existence theory for symmetric hyperbolic systems of conservation laws shows the following: for $s > 2$, if for a fixed constant $\tilde{C} > 0$ we have $\|(\rho^0_\lambda - \overline{\rho}, u^0_\lambda)\|_{H^s} \leq \tilde{C}$, the initial data $(\rho^0_\lambda, u^0_\lambda)$ yield classical solutions to (5.1)-(5.2) in $C([0,T], H^s(\mathbb{R}^2)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^2))$ for $T = T(s, \tilde{C})$. Let us take $s = 3$. Now we simply note that taking $(\rho^0, u^0)(x) := (\rho_0, u_0)(x)$ as in Definition 5.1.1 and $\lambda = \epsilon^{-1/2}$, the right hand sides of (5.7) and (5.8) are uniformly bounded for all small ϵ, say $\epsilon < \epsilon_0$. Since for initial data $(\rho^0_\lambda, u^0_\lambda)(x)$ the classical solution $(\rho_\lambda, u_\lambda)(x,t)$ then extends up to a time T independent of $\epsilon < \epsilon_0$, we find that the classical solution $(\rho, u)(x,t)$ with initial data $(\rho_0, u_0)(x)$ extends up to time $T\epsilon^{-1/2}$. This is enough for (5.5), though we note that one can get an order ϵ^{-1} lower bound on T_ϵ by taking s closer to 2.

Given that (ρ, u) is a classical solution on $[0, T_\epsilon)$ with smooth initial data, one can upgrade this to a solution that is smooth in both space and time by using standard energy estimates for conservation laws. We direct the reader to Theorem 2.2 from [15] for an explicit reference. □

With the following proposition, we give a precise formulation of the finite speed of propagation property of the system (5.1)-(5.2).

Proposition 5.1.3. *If (ρ, u) is a classical solution of (5.1)-(5.2) on the time interval $[0,T]$ with initial data*

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \tag{5.9}$$

for (ρ_0, u_0) as in Definition 5.1.1, then for $0 \leq t \leq T$

$$(\rho, u) \equiv (\overline{\rho}, 0) \ \text{ on } D(t), \tag{5.10}$$

where we define $\sigma := \sqrt{p'(\overline{\rho})}$ *and*

$$D(t) := \{x \in \mathbb{R}^2 : |x| \geq 1 + \sigma t\}. \tag{5.11}$$

Proof. This follows from the proposition in [19], which applies to a class of systems of conservation laws that includes (5.1)-(5.2). □

Now we establish the main proposition, which proves that some particular Lagrangian trajectory cannot be analytic.

Proposition 5.1.4. *Let* (ρ, u) *be the solution to* (5.1)-(5.2) *as discussed in Proposition 5.1.2, and let* $X(\alpha, t)$ *denote the Lagrangian trajectory map tracking the position of the particle beginning at* $\alpha \in \mathbb{R}^2$*, defined by the equation,*

$$\begin{aligned} \frac{dX}{dt}(\alpha, t) &= u(X(\alpha, t), t), \\ X(\alpha, 0) &= \alpha. \end{aligned} \tag{5.12}$$

Then there is an $\alpha \in \mathbb{R}^2$ *and a time* $t_0 \in (0, T_\epsilon)$ *such that the trajectory* $X(\alpha, t)$ *is not analytic in* t *in any disc centered at* t_0*.*

Proof. Ultimately, all we need to prove is that, where $B := \{|x| < 1\}$,

$$B(t) := X(B, t) \neq B \quad \text{for some } t \in (0, T_\epsilon). \tag{5.13}$$

Indeed, suppose we can establish that. Then by continuity of the trajectory map, we have $X(\alpha^*, t) \neq \alpha^*$ for some α^* near the boundary of B but outside $\overline{B}$. This implies there must be a $t^* \in (0, T_\epsilon)$ such that

$$\frac{dX}{dt}(\alpha^*, t^*) \neq 0. \tag{5.14}$$

Now consider

$$t(\alpha^*) := \inf\{t \in [0, T_\epsilon) : \frac{dX}{dt}(\alpha^*, t) \neq 0\}. \tag{5.15}$$

By Proposition 5.1.3, we know that since α^* lies outside $\overline{B}$, $\frac{dX}{dt}(\alpha^*, t) = 0$ is maintained for all t in an interval $[0, \delta]$ for some $\delta > 0$, and so $t(\alpha^*) > 0$. In fact, $\frac{dX}{dt}(\alpha^*, t) = 0$ on the time interval $[0, t(\alpha^*)]$, and yet there is a sequence $\{t_n\}_{n \geq 1} \subset (0, T_\epsilon)$ with $t_n \to t(\alpha^*)$ and $\frac{dX}{dt}(\alpha^*, t_n) \neq 0$. Taking $t_0 := t(\alpha^*)$, we find that the trajectory $X(\alpha^*, t)$ is not analytic in any disc about t_0.

So now we will show that $B(t)$ cannot be B for all times $t \in (0, T_\epsilon)$.

Assume $B(t) \equiv B$. First we claim that then $(\rho - \overline{\rho})$ and u are supported in $\overline{B}$ for all $t \in [0, T_\epsilon)$. To see this, note that if $X(B,t) \equiv B$, then for each t, u must vanish to infinite order along ∂B. From the continuity equation (5.1), one finds that this implies $(\rho - \overline{\rho})$ also must vanish to infinite order along ∂B for each t. Thus by defining

$$(\rho^\flat, u^\flat) := \begin{cases} (\rho, u) & \text{for } |x| < 1, \\ (\overline{\rho}, 0) & \text{for } |x| \geq 1, \end{cases} \tag{5.16}$$

we also get a smooth solution to the system (5.1)-(5.2) with initial data (ρ_0, u_0). By uniqueness, $(\rho^\flat, u^\flat) \equiv (\rho, u)$, and so $(\rho - \overline{\rho}, u)$ is supported in $\overline{B}$. With this in mind, we define the quantities

$$\mathcal{M}(t) := \int \rho(x,t)u(x,t) \cdot x dx, \qquad I(t) := \int (\rho(x,t) - \overline{\rho})|x|^2 dx. \tag{5.17}$$

Here $\mathcal{M}(t)$ gives a weighted average of radial momentum, and $I(t)$ represents the moment of inertia, modulo the constant state. Observe that as a consequence of the fact that ρ satisfies the continuity equation (5.1), we have the identity

$$I'(t) = \int \partial_t \rho(x,t)|x|^2 dx = -\int \operatorname{div}(\rho(x,t)u(x,t))|x|^2 dx = 2\int \rho(x,t)u(x,t) \cdot x dx = 2\mathcal{M}(t). \tag{5.18}$$

Now we claim $\mathcal{M}'(t) \geq 0$. To verify this one can use the conservation law derived from the system (5.1)-(5.2)

$$\partial_t(\rho u_i) + \sum_{j=1}^{2} \partial_{x_j}(\rho u_i u_j + \delta_{ij} p) = 0, \qquad i = 1, 2, \tag{5.19}$$

to find that

$$\mathcal{M}'(t) = \int \partial_t(\rho(x,t)u(x,t)) \cdot x dx = \int \left(\rho(x,t)|u(x,t)|^2 + 2(p(\rho) - p(\overline{\rho}))\right) dx. \tag{5.20}$$

We claim the right hand side of (5.20) is positive. To see this, note

$$\frac{1}{m(B)} \int_B p(\rho) dx = \frac{1}{m(B)} \int_B (\rho(x,t))^\gamma dx \geq \left(\frac{1}{m(B)} \int_B \rho(x,t) dx\right)^\gamma, \tag{5.21}$$

where the right hand side is conserved, and so equal to its value at $t = 0$, which is

$$\left(\frac{1}{m(B)} \int_B \rho_0(x,t) dx\right)^\gamma \geq \left(\frac{1}{m(B)} \int_B \overline{\rho} dx\right)^\gamma = \overline{\rho}^\gamma = p(\overline{\rho}). \tag{5.22}$$

This means we have

$$\int (p(\rho) - p(\overline{\rho}))dx \geq 0, \tag{5.23}$$

which, when used in (5.20), verifies that $\mathcal{M}'(t) \geq 0$. Using this with the identity (5.18), we find

$$I(t) = I(0) + \int_0^t 2\mathcal{M}(s)ds \geq I(0) + 2t\mathcal{M}(0) \geq 2t\mathcal{M}(0). \tag{5.24}$$

Meanwhile, since $\int_B \rho(x,t)dx$ must be constant,

$$I(t) = \int_B (\rho(x,t) - \overline{\rho})|x|^2 dx \leq \int_B (\rho(x,t) - \overline{\rho})dx = \int_B (\rho_0(x) - \overline{\rho})dx \leq C'\epsilon. \tag{5.25}$$

Using the definitions of $\tilde{u}$ and u_0, noting $\overline{\rho} \leq \rho_0(x)$, and combining (5.24) with (5.25), we have

$$\begin{aligned} 2\epsilon \left(\int \overline{\rho}\tilde{u}(x) \cdot x dx \right) t &\leq 2 \int \overline{\rho} u_0(x) \cdot x dx t, \\ &\leq 2 \int \rho_0(x) u_0(x) \cdot x dx t, \\ &= 2t\mathcal{M}(0), \\ &\leq I(t) \leq C'\epsilon \quad \text{for all } t \in [0, T_\epsilon). \end{aligned} \tag{5.26}$$

Note the integral in the left hand side of (5.26) is just a nonzero constant independent of ϵ that one can calculate from the initial data. Thus this implies

$$t \leq C'' \quad \text{for all } t \in [0, T_\epsilon),$$

for some constant C''. However $\lim_{\epsilon \to 0} T_\epsilon = \infty$, and so for small enough ϵ we have a contradiction. We conclude that (5.13) holds, and as we saw, that implies the desired result for the proposition. □

Typically, Lagrangian analyticity for incompressible flows is established for a brief positive time about $t = 0$. The above example explicitly contrasts this if we start time at t_0.

Corollary 5.1.5. *For the (ρ, u) given by Proposition 5.1.2 and the corresponding t_0 from Proposition 5.1.4, taking $(\rho(x,t_0), u(x,t_0))$ as smooth initial data for the system (5.1)-(5.2), one does not have analyticity of the Lagrangian trajectories for any interval of time including $t = 0$.*

Chapter 6

Counterexample for the Vlasov-Poisson system

The Vlasov-Poisson equation gives a probabilistic model of the dynamics of a plasma, or a physical system with a large number of charged particles that are governed by a long-range interaction, such as with Coulomb's law. The unknown is a probability density function $f(x, v, t)$, defined on $\mathbb{R}^{2d} \times \mathbb{R}$, describing the distribution of the positions and velocities of a large number of charged particles. Here, we deal with the Vlasov-Poisson equation in the case $d = 2$, which is

$$\partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) + E(x, t) \cdot \nabla_v f(x, v, t) = 0, \tag{6.1}$$

for the unknown probability distribution $f(x, v, t) : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ with initial data $f|_{t=0} = f_0 \in C_c^n(\mathbb{R}^4)$, for an $n \geq 10$, and where we have that the field $E(x, t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is related to the unknown f via

$$\begin{aligned} E(x, t) &= -\nabla_x \phi(x, t), & (6.2)\\ -\Delta_x \phi(x, t) &= q\rho(x, t), & (6.3)\\ \rho(x, t) &= \int f(x, w, t) dw. & (6.4) \end{aligned}$$

Here we have the potential $\phi(x, t)$ and density $\rho(x, t)$, real valued functions defined on $\mathbb{R}^2 \times \mathbb{R}$. We may take $q = \pm 1$ for the case of a repulsive or attractive force law. In what follows we just take $q = 1$, though the analysis works for either case.

We will observe that this system also has a Lagrangian formulation which can naturally be written as an ODE. The main goal of this section is to demonstrate that unlike the situation for the incompressible Euler equations, in this case the Lagrangian flow need not be analytic in time, even for highly regular initial data. We explain what we mean by this more precisely in the next subsection once we develop the Lagrangian formulation.

6.1 Lagrangian phase space formulation

Several key features are shared between the Lagrangian dynamics of the Vlasov-Poisson equation and fluid mechanics equations such as the incompressible Euler system. For example, where the curl is transported by the flow for the 2D incompressible Euler equations, for the Vlasov-Poisson equations we have that the unknown probability density f is constant when evaluated along phase space trajectories in the corresponding Hamiltonian system. We also have that the main governing force entering in the Vlasov-Poisson equation can be written as a well-behaved nonlocal operator, analogous to the operator that arises from the Biot-Savart law used for incompressible flows. This can be used to write this trajectory system, in addition to that for the incompressible Euler equations, as an ODE in an abstract function space. Now we derive the Lagrangian formulation for the system, observing these properties as they appear in the derivation.

The system (6.1)-(6.4) has a Hamiltonian system formulation, given by

$$\begin{aligned} \dot{X} &= V, \\ \dot{V} &= -(\nabla_x \phi)(X,t), \end{aligned} \tag{6.5}$$

in terms of phase space trajectories $(X(\alpha,\xi,t), V(\alpha,\xi,t)) : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}^2$, where X corresponds to spatial position, and V corresponds to momentum, or velocity, and we have the initial conditions $X(\alpha,\xi,0) = \alpha$, $V(\alpha,\xi,0) = \xi$. Additionally, we have

$$E(x,t) = \nabla_x \Delta_x^{-1} \rho(x,t) = \int \nabla_x \Delta_x^{-1} f(x,w,t) dw = \int\int K(x-y) f(y,w,t) dy dw, \tag{6.6}$$

with the integral kernel

$$K(x) = \frac{1}{2\pi} \frac{x}{|x|^2}. \tag{6.7}$$

Making the change of variables $(y,w) = (X(\beta,\eta,t), V(\beta,\eta,t))$ in the integral on the right in (6.6),

and evaluating at $x = X(\alpha, \xi, t)$, one finds that

$$E(X(\alpha,\xi,t),t) = \int\int K(X(\alpha,\xi,t) - X(\beta,\eta,t)) f(X(\beta,\eta,t), V(\beta,\eta,t), t) J_{X,V}(\beta,\eta,t) d\beta d\eta, \quad (6.8)$$

where $J_{X,V}(\alpha, \xi, t)$ is the resulting Jacobian. However, one can derive from (6.5) that $J_{X,V} = 1$. This is equivalent to the statement that volume in phase space is preserved under the natural flow for this Hamiltonian system.

Observe equation (6.1) together with the system (6.5) implies that f is constant along the trajectories in phase space, so that

$$f(X(\alpha,\xi,t), V(\alpha,\xi,t)) = f_0(\alpha,\xi). \quad (6.9)$$

Incorporating this property in (6.8) we get

$$E(X(\alpha,\xi,t),t) = \int\int K(X(\alpha,\xi,t) - X(\beta,\eta,t)) f_0(\beta,\eta) d\beta d\eta \quad (6.10)$$

and returning to (6.5), we now have the system

$$\begin{aligned} \frac{dX}{dt}(\alpha,\xi,t) &= V(\alpha,\xi,t), \\ \frac{dV}{dt}(\alpha,\xi,t) &= \int\int K(X(\alpha,\xi,t) - X(\beta,\eta,t)) f_0(\beta,\eta) d\beta d\eta, \\ (X,V)(\alpha,\xi,0) &= (\alpha,\xi). \end{aligned} \quad (6.11)$$

We claim that although the system (6.11) has certain similarities with the Lagrangian formulations for the incompressible Euler equations and the Euler-Poisson equation, it has very different analytic properties. The main result of this section is a proof of the following statement: for any $n \geq 10$ there exists C^n initial data for the Vlasov-Poisson equation so that either some Lagrangian trajectory for the solution is not analytic, or there is a compact set $K \subset \mathbb{R}^4$ such that the corresponding trajectories $(X(\alpha, \xi, t), V(\alpha, \xi, t))$ do not have radii of analyticity in time at $t = 0$ bounded below by a positive number as (α, ξ) varies in K.

In other words, we find that even if one requires the initial data to be very smooth, there are examples where the trajectory map $(X(\alpha, \xi, t), V(\alpha, \xi, t))$ either has a trajectory which is not analytic at $t = 0$, or there exists a convergent sequence (α_n, ξ_n) with the property that the radius of analyticity of $(X(\alpha_n, \xi_n, t), V(\alpha_n, \xi_n, t))$ at $t = 0$ tends to zero. This implies, for example, that

$(X(\alpha,\xi,t), V(\alpha,\xi,t))$ cannot be analytic as a function of time into the space of C^1 maps on $\mathbb{R}^4$. Although this does not prove the existence of a non-analytic trajectory, it shows the situation is quite different from that for the incompressible Euler equations and the pressureless Euler-Poisson equations, which exhibit trajectory maps analytic from time into the set of C^1 maps, when one imposes initial data with moderate spatial regularity.

We remark that the counterexample in this section is notably different from that in the section on the compressible Euler equations. The Vlasov-Poisson system has an infinite speed of propagation, so the results here provide an example of a different mechanism in a fluid equation which can cause the analyticity in the time variable of the corresponding flow maps to fail. One can verify that the natural operator associated to the formulation (6.11) in fact does not preserve analyticity in time, which is the essential property of the system that the construction in this section takes advantage of.

6.2 The setup for the counterexample

6.2.1 Construction of the initial data

We choose a smooth function ψ_1 on $\mathbb{R}^2$ supported in a compact set, satisfying

$$\psi_1 \geq 0, \quad \psi_1 = 1 \text{ in a neighborhood } V \text{ of the origin}, \quad \text{and } \|\psi_1\|_{C^n} \leq 1. \tag{6.12}$$

Let us denote

$$\chi(x,y) := \begin{cases} 1 & \text{if } y > |x| \\ 0 & \text{if } y \leq |x| \end{cases}, \tag{6.13}$$

and take

$$g(x,y) := (y^2 - x^2)^{n+1} \psi_1(x,y) \chi(x,y) \tag{6.14}$$

Note that g is C^n. We define $\mathcal{B} := \text{supp}(g)$. We also take a smooth function ψ_2 defined on $\mathbb{R}^2$, satisfying

$$\psi_2 \geq 0, \quad \mathcal{N} := \text{supp}(\psi_2) \subset B_{\frac{1}{2}}((1,0)), \quad \text{and } \|\psi_2\|_{C^n} \leq C, \tag{6.15}$$

where we have denoted the ball of radius $\frac{1}{2}$ about the point $(1,0)$ by $B_{\frac{1}{2}}((1,0))$, and we have some constant C, whose size is unimportant.

Given the above, we choose a sufficiently small $\epsilon > 0$, to be specified later.

Now, for $\eta, \beta \in \mathbb{R}^2$, we define the map

$$\varphi_\eta(\beta) := (\beta_1\eta_1 - \beta_2\eta_2, \beta_1\eta_2 + \beta_2\eta_1), \tag{6.16}$$

noting that $\varphi_\eta(\beta)$ outputs the vector in $\mathbb{R}^2$ corresponding to what one gets upon multiplying β and η, considered as complex numbers. Recall that $\mathcal{N}$, the support of ψ_2, is contained in a ball about $(1,0)$ excluding the point $(0,0)$. It follows $\varphi_\eta(\cdot)$ is invertible for each $\eta \in \mathcal{N}$, with the $C^n_{\beta,\eta}$ norm of $\varphi_\eta^{-1}(\beta)$ bounded by a universal constant.

Now we define the functions

$$f_\epsilon(\beta,\eta) := \epsilon g(\varphi_\eta^{-1}(\beta))\psi_2(\eta), \tag{6.17}$$

$$f(\beta,\eta) := g(\varphi_\eta^{-1}(\beta))\psi_2(\eta). \tag{6.18}$$

The initial data for the Vlasov-Poisson system (6.1)-(6.4) which will lead us to the desired counterexample is then $f_\epsilon(\alpha,\xi)$.

6.2.2 The uniform analyticity assumption and an analyticity test for a particular trajectory

Now we begin our discussion on the sense in which the Lagrangian trajectories for the system with initial data f_ϵ are somewhat badly behaved, with regard to analyticity in time. The way we will proceed is by assuming that the Lagrangian trajectories satisfy a kind of uniform analyticity condition, which in particular would be satisfied if $(X(\cdot,t), V(\cdot,t))$ were analytic as a function from time into the space of C^1 maps on $\mathbb{R}^4$. Under this uniform analyticity assumption, we will find that we have a formula for the following quantity for a trajectory starting at $(\alpha,\xi) = (0,0)$, where we have an analytic function of time $J(t)$, some functions $f_\pm(\beta,\eta)$ and $t_\pm(\beta,\eta)$, and a set $S \subset \mathbb{R}^4$:

$$\frac{dV_1}{dt}(0,0,t) = \sum_\pm \int_S \frac{f_\pm(\beta,\eta)}{t - t_\pm(\beta,\eta)} d\beta d\eta + J(t). \tag{6.19}$$

Once we have the formula, what follows will be a careful examination of the effects of the singularity in t in the integrals in the sum. Ultimately we will find that the first term cannot be analytic in t at zero.

Now we proceed with the motivation of providing the formula (6.19) explicitly. First we define some the quantities which will be of use in deriving the functions, such as $f_\pm(\beta,\eta)$ and $t_\pm(\beta,\eta)$,

that appear in the formula.

Definition 6.2.1. *We denote by $R : \mathbb{R}^2 \to \mathbb{R}^2$ the reflection defined by $T_2(x_1, x_2) = (x_1, -x_2)$. Let us define the following functions on $\mathcal{B} \times \mathcal{N} \times [-T, T]$.*

$$r_+(\beta, \eta, t) := \tilde{\varphi}_\eta^{-1}\left[X(\varphi_\eta(\beta), \eta, t) - X(0, 0, t)\right] - [\beta_1 + i\beta_2 + t], \tag{6.20}$$

$$r_-(\beta, \eta, t) := \tilde{\varphi}_{T_2\eta}^{-1}\left[T_2X(\varphi_\eta(\beta), \eta, t) - T_2X(0, 0, t)\right] - [\beta_1 - i\beta_2 + t], \tag{6.21}$$

where $\tilde{\varphi}_\eta^{-1}(\beta)$ returns the complex number representing the vector $\varphi_\eta^{-1}(\beta) \in \mathbb{R}^2$, and

$$h_\pm(\beta, \eta, t) := (\beta_1 \pm i\beta_2) + t + r_\pm(\beta, \eta, t). \tag{6.22}$$

In the following lemma we give a precise description of the uniform analyticity assumption mentioned earlier, and take it as a hypothesis. Equipped with the above definitions, we show how the assumption leads to the definition of the $t_\pm(\beta, \eta)$ functions that appear in the formula (6.19). These functions play the most significant roles regarding singular behavior in the time variable, and the information given by the lemma will allow us to study the effects of the singularities more precisely.

Lemma 6.2.2. *Suppose the following assumption holds:*

Assumption 6.2.3. *The function $X(\alpha, \xi, t)$ is analytic in t at 0 for all $(\alpha, \xi) \in (\mathcal{B} \times \mathcal{N}) \cup \{(0, 0)\}$, with a uniform, positive lower bound on the radius of analyticity.*

Then there exists an $r > 0$ and a neighborhood U of the origin in $\mathbb{R}^2$ such that we have the following:

(i) For all $(\beta, \eta) \in \mathcal{B} \times \mathcal{N}$, $t \in \{z \in \mathbb{C} : |z| < r\}$,

$$|\partial_t h_\pm(\beta, \eta, t)| > \frac{1}{2}, \tag{6.23}$$

where we have extended the definitions of the $r_\pm(\beta, \eta, t)$ and $h_\pm(\beta, \eta, t)$ in the time variable to their complex analytic extensions.

(ii) The relations

$$h_\pm(\beta, \eta, t_\pm(\beta, \eta)) = 0 \tag{6.24}$$

define functions $t_\pm : U \times \mathcal{N} \to \mathbb{C}$, mapping into the disc $\{|z| < r\}$, with bounded $C^n_{\beta,\eta}$ norm. Sometimes we use the notation $t_\pm^\eta(\beta) := t_\pm(\beta, \eta)$.

(iii) We have

$$\|t_+^\eta(\cdot) + Id_{\mathbb{C}}\|_{C^2_\beta} \leq C\epsilon, \tag{6.25}$$

$$\|t_-^\eta(\cdot) - T_1\|_{C^2} \leq C\epsilon, \tag{6.26}$$

for all $\eta \in \mathcal{N}$*, where* $Id_{\mathbb{C}}(x_1, x_2) := x_1 + ix_2$*,* $T_1(x_1, x_2) := -x_1 + ix_2$*, and* $C > 0$ *is a universal constant.*

Proof. Regarding the proof of (i), we first note that the Taylor expansion in time for the spatial position function of a trajectory gives $X(\beta, \eta, t) = \beta + t\eta + O(t^2)$, using the initial conditions for the trajectories in the system (6.11). Now we observe that $[\beta_1 + i\beta_2 + t]$ then gives the first order expansion in t for the expression

$$\tilde{\varphi}_\eta^{-1}\left[X(\varphi_\eta(\beta), \eta, t) - X(0, 0, t)\right], \tag{6.27}$$

and so the difference $r_+(\beta, \eta, t)$ is $O(t^2)$, and $\partial_t r_+(\beta, \eta, t) = O(t)$. Using this, and a similar observation for $r_-(\beta, \eta, t)$, one can verify (i) as long as r is small enough.

The proof of (ii) now follows from an application of the implicit function theorem.

Now we consider (iii). Let us fix an $\eta \in \mathcal{N}$ and consider specifically $t_+^\eta(\cdot)$. For $\beta \in U$ we define $t(\beta)$ to be $t_+^\eta(\beta)$ regarded as a vector in $\mathbb{R}^2$ rather than a complex number, and we define $R(\beta, t_1, t_2)$ on $\mathcal{B} \times \{|t| < r\}$ to be $r_+(\beta, \eta, t_1 + it_2)$ interpreted as an $\mathbb{R}^2$-valued function. Thus we have

$$\beta + t(\beta) + R(\beta, t(\beta)) = 0 \tag{6.28}$$

for $\beta \in U$. First we claim for all $\beta \in U$

$$|(t_+^\eta + Id_{\mathbb{C}})(\beta)| = |t(\beta) + \beta| = |R(\beta, t(\beta))| = O(\epsilon). \tag{6.29}$$

The only nontrivial observation here is that the remainder $R(\beta, t_1, t_2)$ is $O(\epsilon)$, in $C^2_{\beta,(t_1,t_2)}$ norm in particular, which is good enough for the purposes of this lemma. Too see this, we note the initial data f_ϵ is $O(\epsilon)$ in C^2 norm, which can be used to estimate the size of the solution $f(x, v, t)$ to the Vlasov-Poisson system in C^2 norm, which can then be used to estimate $\frac{dV}{dt}$. One can use the estimate on $\frac{dV}{dt}$ to bound the second order Taylor expansion error $[X(\beta, \eta, t) - (\beta + t\eta)]$, and then use this to bound R in $C^2_{\beta,(t_1,t_2)}$.

Looking at the jth entry and applying ∂_{β_i} for $i = 1, 2$, gives for $j = 1, 2$,

$$\delta_{ij} + \partial_i t_j(\beta) + \partial_i R_j(\beta, t(\beta)) + \sum_{l=1}^{2} \partial_i t_l(\beta) \partial_{t_l} R_j(\beta, t(\beta)) = 0. \tag{6.30}$$

Here we have written $\partial_i R_j$ to indicate differentation of R_j with respect to β_i. Using (6.30) to solve for ∇t gives

$$\nabla t(\beta) = -(I + \nabla_\beta R(\beta, t(\beta))(I + \nabla_t R(\beta, t(\beta)))^{-1}, \tag{6.31}$$

where we denote with ∇_t the gradient with respect to (t_1, t_2). With the $O(\epsilon)$ bounds on R, we then obtain $|\nabla_\beta(t_+^\eta + Id_{\mathbb{C}})| = |\nabla t + I| \leq C\epsilon$, as desired.

Differentiating (6.30) with respect to β_k for $i, j, k = 1, 2$ gives

$$\begin{aligned} 0 \quad = \quad & \partial^2_{ki} t_j(\beta) + \partial^2_{ki} R_j(\beta, t(\beta)) + \sum_{l=1}^{2} \Big(\partial_k t_l(\beta) [\partial_{t_l} \partial_i R_j(\beta, t(\beta))] + \partial^2_{ki} t_l(\beta) \partial_{t_l} R_j(\beta, t(\beta)) \\ & + \partial_i t_l(\beta) [\partial_k \partial_{t_l} R_j(\beta, t(\beta))] \\ & + \sum_{m=1}^{2} \partial_i t_l \partial_k t_m [\partial^2_{t_m, t_l} R_j(\beta, t(\beta))] \Big). \end{aligned} \tag{6.32}$$

We define

$$\begin{aligned} [\Xi(i,k)]_j := \partial^2_{ki} R_j(\beta, t(\beta)) + \sum_{l=1}^{2} \Big(& \partial_k t_l(\beta) [\partial_{t_l} \partial_i R_j(\beta, t(\beta))] + \partial_i t_l(\beta) [\partial_k \partial_{t_l} R_j(\beta, t(\beta))] \\ & + \sum_{m=1}^{2} \partial_i t_l \partial_k t_m [\partial^2_{t_m, t_l} R_j(\beta, t(\beta))] \Big). \end{aligned} \tag{6.33}$$

Then we have

$$\partial^2_{ki} t_j(\beta) + \sum_{l=1}^{2} \partial^2_{ki} t_l(\beta) \partial_{t_l} R_j(\beta, t(\beta)) = -[\Xi(i,k)]_j. \tag{6.34}$$

Regarding $\partial^2_{ki} t(\beta)$ and $\Xi(i,k)$ as row vectors (for each fixed i, k), we find

$$\partial^2_{ki} t(\beta) = -\Xi(i,k)(I + \nabla_t R(\beta, t(\beta)))^{-1}. \tag{6.35}$$

One can check that R is $O(\epsilon)$ in C^2 norm, and thus $\Xi(i,k) = O(\epsilon)$, which then allows us to conclude the desired result for t_+^η in the statement of (iii). The proof for $t_-^\eta(\cdot)$ is similar.

□

Observe that Lemma 6.2.2 tells us that the functions $t_+^\eta(\beta)$ and $t_-^\eta(\beta)$ return approximately the complex numbers $-\beta_1 - i\beta_2$ and $-\beta_1 + i\beta_2$, respectively. Now note that in the formula (6.19) which we will derive explicitly later in this section, when the $t_\pm^\eta(\beta)$ are approximately 0, i.e. when $\beta \approx 0$, we anticipate singular behavior for $t \approx 0$. Indeed, our region of integration S in (6.19) will include β near the origin, specifically in the region described in the following definition.

Definition 6.2.4. *For any $\delta > 0$, we denote by Q_δ the open square centered at $(0, \delta/\sqrt{2})$ of sidelength δ, oriented so that one diagonal aligns with the x-axis and the other with the y-axis.*

Now, regarding the above definition, we fix a $\delta > 0$ such that

(i) Q_δ is contained in the neighborhood U given by Lemma 6.2.2,

(ii) Q_δ is contained in the neighborhood of the origin on which $\psi_1 = 1$,

(iii) and $t_\pm(Q_\delta) \subset \{z : |z| < r\}$.

Note the relationship between the singularities in the integrands in the formula (6.19), in which we will be integrating over $\beta \in Q_\delta$, and the images $t_\pm^\eta(Q_\delta)$ for each $\eta \in \mathcal{N}$. By the previous lemma, these images are approximately $\mp Q_\delta$, regarded as a subset of $\mathbb{C}$. The half-planes in which the various singularities lie is one more key piece of information we will need.

Corollary 6.2.5. *The functions $t_\pm^\eta(\beta)$ defined for $\beta \in U$, $\eta \in \mathcal{N}$ satisfy*

$$\begin{aligned} \mathrm{Im}(t_+^\eta(\beta)) &< 0 \quad \textit{for } (\beta, \eta) \in Q_\delta \times \mathcal{N}, && (6.36) \\ \mathrm{Im}(t_-^\eta(\beta)) &> 0 \quad \textit{for } (\beta, \eta) \in Q_\delta \times \mathcal{N}. && (6.37) \end{aligned}$$

Proof. First we consider (6.36). Fix $\eta \in \mathcal{N}$. Let us sweep out Q_δ with the parameterization

$$\gamma_\theta(\tau) = \tau(cos\theta, sin\theta), \quad \tau \in (0, L(\theta)), \quad \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \tag{6.38}$$

where for each θ, $L(\theta)$ is some positive value in the interval $(\delta, \sqrt{2}\delta)$. By Lemma 6.2.2, $\|t_+^\eta(\cdot) + Id_{\mathbb{C}}\|_{C^2} \leq C\epsilon$. Thus for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, the curvature of each of the curves indexed by θ given for each fixed θ by

$$\tilde{\gamma}_\theta(\tau) = t_+^\eta(\gamma_\theta(\tau)) \tag{6.39}$$

is always $O(\epsilon)$. This bounds the instantaneous rate of change of the angle of the tangent to this curve. Observe that the angle ϑ_0 of the tangent to any one of these curves at $\tau = 0$ is between $\frac{-3\pi}{4}$ and $-\frac{\pi}{4}$. It is easy to verify that the total change in angle between that at $\tau = 0$ and at $\tau = L(\theta)$ is also $O(\epsilon)$, and so for small enough ϵ, it is impossible for the angle to have deviated sufficiently from

ϑ_0 for the curve to have left the lower half-plane. It is easy to see that the curves $\tilde{\gamma}_\theta(\tau)$ sweep out the image of $t_+^\eta(\cdot)$, and so this proves (6.36). The proof of (6.37) is similar. □

Now we are ready to derive the formula (6.19). This is the purpose of the next lemma.

Lemma 6.2.6. *Suppose Assumption 6.2.3 holds. Then we have for a nonzero constant c*

$$\frac{dV_1}{dt}(0,0,t) = c\sum_{\pm} I_*^{\pm}(t) + J(t) \tag{6.40}$$

where $J(t)$ is real analytic at $t = 0$, and

$$I_*^{\pm}(t) = \int_{\mathcal{N}}\int_{Q_\delta} \frac{\tilde{f}_{\pm}(\beta,\eta)}{t - t_{\pm}(\beta,\eta)} d\beta d\eta, \tag{6.41}$$

where

$$\tilde{f}_{\pm}(\beta,\eta) = \frac{g(\beta)\psi_2(\eta)(\eta_1 \mp \eta_2)}{\partial_t h(\beta,\eta,t_{\pm}(\beta,\eta))}. \tag{6.42}$$

Proof. Given initial data f_ϵ for the Vlasov-Poisson equation, for a corresponding Lagrangian trajectory $(X(\alpha,\xi,t)), V(\alpha,\xi,t))$, we define the functions

$$X^{\pm}(\alpha,\xi,t) := X_1(\alpha,\xi,t) \pm iX_2(\alpha,\xi,t). \tag{6.43}$$

Note that the Lagrangian phase space formulation for the Vlasov-Poisson system requires that the trajectory beginning at the point $(\alpha,\xi) = (0,0)$ obeys the evolution

$$\frac{dV}{dt}(0,0,t) = \int_{\mathbb{R}^4} K(X(0,0,t) - X(\beta,\eta,t)) f_\epsilon(\beta,\eta) d\beta d\eta. \tag{6.44}$$

Meanwhile, we may write for each $j = 1, 2$,

$$\begin{aligned} K^j(y) &= \frac{1}{4\pi}\frac{y_j}{y_1^2 + y_2^2}, \\ &= \frac{1}{8\pi}\begin{cases} \frac{1}{y_1+iy_2} + \frac{1}{y_1-iy_2} & \text{if } j = 1 \\ i\left(\frac{1}{y_1+iy_2} - \frac{1}{y_1-iy_2}\right) & \text{if } j = 2 \end{cases}. \end{aligned} \tag{6.45}$$

We focus on the first component of $\frac{dV}{dt}$ in (6.44) as the function whose analyticity we will proceed

to test, which with the aid of (6.45) we can write as

$$\frac{dV_1}{dt}(0,0,t) = \frac{-\epsilon}{8\pi}\sum_{\pm}\int \frac{1}{X^{\pm}(\beta,\eta,t) - X^{\pm}(0,0,t)} f(\beta,\eta)d\beta d\eta. \tag{6.46}$$

Let us define

$$I^{\pm}(t) := \int \frac{1}{X^{\pm}(\beta,\eta,t) - X^{\pm}(0,0,t)} f(\beta,\eta)d\beta d\eta. \tag{6.47}$$

Observe that the trajectory with initial position $\beta \in \mathbb{R}^2$ and initial velocity $\eta \in \mathbb{R}^2$ can be expanded as

$$X(\beta,\eta,t) = \beta + t\eta + r(\beta,\eta,t), \tag{6.48}$$

where $r(\beta,\eta,t) = O(t^2)$.

So

$$X^+(\varphi_\eta(\beta),\eta,t) - X^+(0,0,t) = \varphi_\eta^1(\beta) + i\varphi_\eta^2(\beta) + t(\eta_1 + i\eta_2) + \tilde{r}(\beta,\eta,t), \tag{6.49}$$

where, given Assumption 6.2.3 holds, $\tilde{r}(\beta,\eta,t)$ satisfies the same kind of uniform analyticity in t as $X(\beta,\eta,t)$ does, as described in Assumption 6.2.3, and $\tilde{r}(\beta,\eta,t) = O(t^2)$. Moreover

$$X^+(\varphi_\eta(\beta),\eta,t) - X^+(0,0,t) = (\beta_1 + i\beta_2)(\eta_1 + i\eta_2) + t(\eta_1 + i\eta_2) + \tilde{r}(\beta,\eta,t). \tag{6.50}$$

Observe that the function $r_+(\beta,\eta,t)$ defined in (6.20) can be expressed the following way:

$$r_+(\beta,\eta,t) = \frac{1}{\eta_1 + i\eta_2}\tilde{r}(\beta,\eta,t), \quad \text{for } \beta \in \mathcal{B},\ \eta \in \mathcal{N}. \tag{6.51}$$

Now we use a change of variables and the definition of f, in (6.18), to write

$$\begin{aligned} I^+(t) &= \int \frac{f(\varphi_\eta(\beta),\eta)}{X^+(\varphi_\eta(\beta),\eta,t) - X^+(0,0,t)}|\eta|^2 d\beta d\eta, \\ &= \int \frac{g(\beta)\psi_2(\eta)}{(\beta_1 + i\beta_2)(\eta_1 + i\eta_2) + t(\eta_1 + i\eta_2) + r_+(\beta,\eta,t)(\eta_1 + i\eta_2)}|\eta|^2 d\beta d\eta, \\ &= \int \frac{g(\beta)\psi_2(\eta)(\eta_1 - i\eta_2)}{(\beta_1 + i\beta_2) + t + r_+(\beta,\eta,t)} d\beta d\eta, \\ &= \int \frac{g(\beta)\psi_2(\eta)(\eta_1 - i\eta_2)}{h_+(\beta,\eta,t)} d\beta d\eta. \end{aligned} \tag{6.52}$$

Meanwhile, one similarly finds

$$\begin{aligned} I^-(t) &= \int \frac{g(\beta)\psi_2(\eta)(\eta_1 + i\eta_2)}{(\beta_1 - i\beta_2) + t + r_-(\beta,\eta,t)} d\beta d\eta, \\ &= \int \frac{g(\beta)\psi_2(\eta)(\eta_1 + i\eta_2)}{h_-(\beta,\eta,t)} d\beta d\eta. \end{aligned} \tag{6.53}$$

Let us define $r'_\pm(\beta,\eta,t) := t^{-2} r_\pm(\beta,\eta,t)$. Recall from the construction given in (6.12)-(6.14) that $\mathcal{B} = \text{supp}(g)$ is contained in the set $\{(x,y) : y \geq |x|\}$. Observe that $\beta \in \mathcal{B} \setminus Q_\delta$ then implies $|\beta| \geq \delta$. Thus there is an $r_0 > 0$ such that for all $\beta \in \mathcal{B} \setminus Q_\delta$, $\eta \in \mathcal{N}$,

$$|h_\pm(\beta,\eta,t)| = |(\beta_1 \pm i\beta_2) + t + t^2 r'_\pm(\beta,\eta,t)| \geq \frac{1}{10}\delta \quad \text{for } |t| < r_0, \tag{6.54}$$

meaning all complex t in the disc of radius r_0. For such t, we write

$$I^\pm(t) = I^\pm_\flat(t) + I^\pm_\sharp(t), \tag{6.55}$$

where

$$I^\pm_\flat(t) := \int_{\mathcal{N}} \int_{\mathcal{B}\setminus Q_\delta} \frac{g(\beta)\psi_2(\eta)(\eta_1 \mp i\eta_2)}{h_\pm(\beta,\eta,t)} d\beta d\eta, \tag{6.56}$$

and

$$I^\pm_\sharp(t) := \int_{\mathcal{N}} \int_{Q_\delta} \frac{g(\beta)\psi_2(\eta)(\eta_1 \mp i\eta_2)}{h_\pm(\beta,\eta,t)} d\beta d\eta, \tag{6.57}$$

In view of (6.54), by using Assumption 6.2.3, which gives a positive, uniform lower bound on the radii of analyticity of $r_\pm(\beta,\eta,t)$ as we let (β,η) vary in $\mathcal{B}\times\mathcal{N}$, we see that the $I^\pm_\flat(t)$ must be analytic in t in a disc about zero.

Now we consider the quantities $I^\pm_\sharp(t)$. Again using Assumption 6.2.3, note that we may write for $(\beta,\eta) \in Q_\delta \times \mathcal{N}$, and t in a disc about zero,

$$\frac{1}{h_\pm(\beta,\eta,t)} = \frac{1}{c_\pm(\beta,\eta)(t - t_\pm(\beta,\eta))} + R_\pm(\beta,\eta,t), \tag{6.58}$$

where $R_\pm(\beta,\eta,t)$ is analytic in t, and

$$c_\pm(\beta,\eta) := \partial_t h(\beta,\eta,t_\pm(\beta,\eta)). \tag{6.59}$$

We then find $|c_\pm(\beta,\eta)| \geq \frac{1}{2}$ in $Q_\delta \times \mathcal{N}$, recalling (i) of Lemma 6.2.2, along with item (iii) following Definition 6.2.4.

So we have

$$\begin{aligned} I_\sharp^\pm(t) &= \int_{\mathcal{N}}\int_{Q_\delta} \frac{1}{t - t_\pm(\beta,\eta)} \cdot \frac{g(\beta)\psi_2(\eta)(\eta_1 \mp i\eta_2)}{c_\pm(\beta,\eta)} d\beta d\eta \\ &\quad + \int_{\mathcal{N}}\int_{Q_\delta} R_\pm(\beta,\eta,t) g(\beta)\psi_2(\eta)(\eta_1 \mp i\eta_2) d\beta d\eta, \\ &= I_*^\pm(t) + I_0^\pm(t). \end{aligned} \tag{6.60}$$

Using the lower bound on the radii of analyticity for $X(\beta,\eta,t)$ due to Assumption 6.2.3, it is not hard to show the $I_0^\pm(t)$ are analytic. Collecting all the terms, we now find

$$\frac{dV_1}{dt}(0,0,t) = -\frac{\epsilon}{8\pi}\sum_\pm \left(I_*^\pm(t) + I_0^\pm(t) + I_\flat^\pm(t)\right), \tag{6.61}$$

where the functions $I_0^\pm(t)$ and $I_\flat^\pm(t)$ are analytic at $t = 0$, giving us the result in the statement of the lemma. □

For the rest of this section, the goal is to show that under Assumption 6.2.3, in the decomposition given by Lemma 6.2.6, $\sum_\pm I_*^\pm(t)$ cannot be analytic at $t = 0$. This implies that the function we started with, $\frac{dV_1}{dt}(0,0,t)$, cannot be analytic at 0. In fact this contradicts Assumption 6.2.3, and so once it is shown, we can conclude that the assumption cannot hold.

One of the key tools in our proof is a certain function transform, which is essentially the FBI transform with some minor modifications. We recall the following fundamental fact arising in the study of real-analytic functions: a function is real-analytic if and only if its FBI transform satisfies a particular decay condition. In the following lemma, we describe a simplified version of the FBI transform, sufficient for our purposes, and we show that it also must satisfy a certain decay condition if the original function is analytic. Eventually we will verify that this test fails for $\sum_\pm I_*^\pm(t)$, and thus conclude that the function cannot be analytic.

Lemma 6.2.7. *Consider a continuous function $\mathfrak{a}(t) : [-1,1] \to \mathbb{C}$. Suppose $\mathfrak{a}(t)$ is analytic at $t = 0$. We define for $s > 0$, $\lambda > 0$,*

$$\tilde{\mathfrak{a}}(s,\lambda) := \int_{-1}^{1} e^{-2i\lambda st} e^{-st^2} \mathfrak{a}(t) dt. \tag{6.62}$$

Then there exist constants $A > 0$, $\gamma > 0$, $C > 0$, such that for any $s > 0$, $\lambda > A$,

$$|\tilde{\mathfrak{a}}(s,\lambda)| \le Ce^{-\gamma s}. \tag{6.63}$$

Proof. We pick $\varepsilon > 0$ less than $\frac{1}{2}$, and small enough that the rectangle $\{\mu + i\nu : |\mu| < 2\varepsilon,\ |\nu| < \varepsilon\}$ is

contained in a complex neighborhood of $t = 0$ on which $\mathfrak{a}(t)$ has an analytic extension. We choose a real-valued, continuously differentiable function $h(t)$ on $[-1, 1]$ satisfying $|h| \leq 1$ on $[-1, 1]$, $h(t) = 1$ for $|t| < \varepsilon$, and $h(t) = 0$ for $|t| > 2\varepsilon$.

We replace the contour traced out in the integral for $\tilde{\mathfrak{a}}(s, \lambda)$ with the contour traced out by $t \mapsto (t - i\varepsilon h(t))$, noting that this does not change the value of $\tilde{\mathfrak{a}}(s, \lambda)$, given the analyticity of $\mathfrak{a}(t)$. We then have

$$\begin{aligned} \tilde{\mathfrak{a}}(s, \lambda) &= \int_{-1}^{1} e^{-2i\lambda s(t - i\varepsilon h(t))} e^{-s(t - i\varepsilon h(t))^2} \mathfrak{a}(t - i\varepsilon h(t))(1 - i\varepsilon h'(t))dt, \\ &= \int_{-1}^{1} e^{-2i\lambda st - 2\lambda s\varepsilon h(t) - st^2 + 2ist\varepsilon h(t) + s\varepsilon^2 (h(t))^2} \mathfrak{a}(t - i\varepsilon h(t))(1 - i\varepsilon h'(t))dt. \end{aligned} \tag{6.64}$$

So

$$|\tilde{\mathfrak{a}}(s, \lambda)| \leq c \sup |h'| \int_{-1}^{1} e^{-s(2\varepsilon\lambda h(t) + t^2 - \varepsilon^2 (h(t))^2))} |\mathfrak{a}(t - i\varepsilon h(t))|dt. \tag{6.65}$$

Fix $\lambda \geq \varepsilon$. Consider $|t| \leq \varepsilon$. Then we have

$$2\varepsilon\lambda h(t) + t^2 - \varepsilon^2 (h(t))^2 = 2\varepsilon\lambda + t^2 - \varepsilon^2 \geq 2\varepsilon^2 - \varepsilon^2 + t^2 \geq \varepsilon^2. \tag{6.66}$$

On the other hand, in the case $|t| > \varepsilon$ we have

$$2\varepsilon\lambda h(t) + t^2 - \varepsilon^2 (h(t))^2 = \varepsilon h(t)(2\lambda - \varepsilon h(t)) + t^2 \geq \varepsilon h(t)(2\varepsilon - \varepsilon) + t^2 = \varepsilon^2 h(t) + t^2 > \varepsilon^2. \tag{6.67}$$

So in both cases,

$$e^{-s(2\varepsilon\lambda h(t) + t^2 - \varepsilon^2 (h(t))^2)} \leq e^{-\varepsilon^2 s}, \tag{6.68}$$

and we can conclude from (6.65) that for $\lambda > \varepsilon$ and any $s > 0$

$$|\tilde{\mathfrak{a}}(s, \lambda)| \leq Ce^{-\varepsilon^2 s}. \tag{6.69}$$

□

Now that we have established the necessary relationship between analyticity and the decay of the transform, in the next proposition we consider the result of applying the transform to the function $\Sigma(t) := \sum_{\pm} I_*^{\pm}(t)$ to get a function $\tilde{\Sigma}(s, \lambda)$. We show that $\tilde{\Sigma}(s, \lambda)$ can be written as the sum of two terms, one of which is in a form prepared for a computation of asymptotics in the parameters $s > 0$

and $\lambda > 0$, and the second of which satisfies an exponential decay estimate. It is the first term that we will see contributes a main, lower order part which cannot satisfy the exponential decay estimate of Lemma 6.2.7 that is required for analytic functions.

Proposition 6.2.8. *For the function $\Sigma(t) := \sum_{\pm} I_*^{\pm}(t)$, the transform $\tilde{\Sigma}(s,\lambda)$, defined by*

$$\tilde{\Sigma}(s,\lambda) := \int_{-1}^{1} e^{-2i\lambda st} e^{-st^2} \Sigma(t) dt, \tag{6.70}$$

can be represented by the following formula:

$$\tilde{\Sigma}(s,\lambda) = \int_{\mathcal{N}} I_\eta(s,\lambda) d\eta + \tilde{r}(s,\lambda), \tag{6.71}$$

where for a nonzero constant $c \in \mathbb{C}$ we have

$$I_\eta(s,\lambda) = c e^{-s\lambda^2} \int_0^\delta \int_0^\delta e^{-s(t_\eta(x,y)+i\lambda)^2} x^{n+1} y^{n+1} b_\eta(x,y) dx dy, \tag{6.72}$$

where for each fixed $\eta \in \mathcal{N}$ we have some functions $t_\eta : [0,\delta]^2 \to \mathbb{C}$ and $b_\eta : [0,\delta]^2 \to \mathbb{C}$. Moreover,

(i) for some $C' > 0$, $\gamma > 0$, for all sufficiently large $\lambda, s > 0$,

$$|\tilde{r}(s,\lambda)| \leq C' e^{-\gamma s}. \tag{6.73}$$

(ii) for each $\eta \in \mathcal{N}$, the function $t_\eta : [0,\delta]^2 \to \mathbb{C}$ satisfies $\|t_\eta + (R_{\frac{\pi}{4}})_{\mathbb{C}}\|_{C^2} \leq C\tilde{\epsilon}$ for a constant $C > 0$, $(R_{\frac{\pi}{4}})_{\mathbb{C}}$ denoting counterclockwise rotation through an angle of $\pi/4$, with the output considered as a complex number, and

$$\begin{aligned} t_\eta(0) &= 0, \\ \partial_x t_\eta(0) &= -\frac{1+i}{\sqrt{2}}, \\ \partial_y t_\eta(0) &= \frac{1-i}{\sqrt{2}}, \end{aligned} \tag{6.74}$$

(iii) and for $\eta \in \mathcal{N}$ the function $b_\eta(\cdot)$ has bounded C^1 norm and satisfies

$$\int_{\mathcal{N}} b_\eta(0) d\eta \neq 0. \tag{6.75}$$

Proof. We define $\tilde{I}_*^{\pm}(s,\lambda)$ by applying our simplified FBI transform to $I_*^{\pm}(t)$ (the transform applied

to $\mathfrak{a}(t)$ in the statement of Lemma 6.2.7 and to $\Sigma(t) = \sum_{\pm} I_*^{\pm}(t)$ in (6.70)). Recall

$$I_*^{\pm}(t) = \int_{\mathcal{N}} \int_{Q_\delta} \frac{1}{t - t_{\pm}(\beta, \eta)} \cdot \frac{g(\beta)\psi_2(\eta)(\eta_1 \mp i\eta_2)}{c_{\pm}(\beta, \eta)} d\beta d\eta, \tag{6.76}$$

where $c_{\pm}(\beta, \eta)$ is as defined in (6.59). Let us define for $(\beta, \eta) \in \overline{Q_\delta} \times \mathcal{N}$

$$b_{\pm}^{\eta}(\beta) := \frac{g(\beta)\psi_2(\eta)(\eta_1 \mp i\eta_2)}{c_{\pm}(\beta, \eta)}. \tag{6.77}$$

We then find that

$$\begin{aligned} \tilde{I}_*^{\pm}(s, \lambda) &= \int_{-1}^{1} e^{-2i\lambda st} e^{-st^2} \int_{\mathcal{N}} \int_{Q_\delta} \frac{1}{t - t_{\pm}^{\eta}(\beta} b_{\pm}^{\eta}(\beta) d\beta d\eta dt, & (6.78) \\ &= \int_{\mathcal{N}} \int_{Q_\delta} \left(\int_{-1}^{1} \frac{1}{t - t_{\pm}^{\eta}(\beta)} e^{-2i\lambda st} e^{-st^2} dt \right) b_{\pm}^{\eta}(\beta) d\beta d\eta. & (6.79) \end{aligned}$$

Now we define the quantity

$$B := \sup_{(\beta,\eta) \in Q_\delta \times \mathcal{N}} \left| t_+^{\eta}(\beta) \right|. \tag{6.80}$$

Let us take the segment $[-1, 1]$ over which we are integrating in the innermost integral in (6.79) and deform it in the complex plane to form a new contour Γ. The contour Γ begins at -1, goes directly downward to $-1 - 2iB$, then across to $1 - 2iB$, and then up to 1. We write this as the union of three contours Γ_1, Γ_2, and Γ_3, with

$$\Gamma_1 = [-1, -1 - 2iB], \quad \Gamma_2 = [-1 - 2iB, 1 - 2iB], \quad \Gamma_3 = [1 - 2iB, 1], \tag{6.81}$$

We claim that for each fixed $(\beta, \eta) \in Q_\delta \times \mathcal{N}$, making this change of contour in the expression in parentheses in (6.79) has the following effect.

$$\begin{aligned} \int_{-1}^{1} \frac{1}{t - t_+^{\eta}(\beta)} e^{-2i\lambda st} e^{-st^2} dt &= \int_{\Gamma} \frac{1}{t - t_+^{\eta}(\beta)} e^{-2i\lambda st} e^{-st^2} dt - 2\pi i e^{-2i\lambda s t_+^{\eta}(\beta)} e^{-s(t_+^{\eta}(\beta))^2}, & (6.82) \\ \int_{-1}^{1} \frac{1}{t - t_-^{\eta}(\beta)} e^{-2i\lambda st} e^{-st^2} dt &= \int_{\Gamma} \frac{1}{t - t_-^{\eta}(\beta)} e^{-2i\lambda st} e^{-st^2} dt. & (6.83) \end{aligned}$$

Here we have assumed that δ is much smaller than 1, which has the effect that in particular $t_+^{\eta}(Q_\delta)$ is completely contained in the rectangle with boundary $[-1, 1] \cup \Gamma$. This also uses Lemma 6.2.2 and Corollary 6.2.5 which guarantee $t_+^{\eta}(\cdot)$ very closely approximates $-Id_{\mathbb{C}}$, that $\operatorname{Im}\left(t_+^{\eta}(\beta)\right) < 0$, which is used for (6.82), and that $\operatorname{Im}\left(t_-^{\eta}(\beta)\right) > 0$, which is used for (6.83).

Now we record the fact that we have a lower bound on $|t - t_{\pm}^{\eta}(\beta)|$ as t varies along Γ, uniform

in $(\beta, \eta) \in Q_\delta \times \mathcal{N}$.

Given $t \in \mathbb{C}$, we write $t = \mu + i\nu$, for $\mu, \nu \in \mathbb{R}$, and note

$$-2i\lambda st - st^2 = -s(t^2 + 2i\lambda t) = -s((\mu + i\nu)^2 + 2i\lambda(\mu + i\nu)) = -s(\mu^2 - \nu^2 - 2\lambda\nu + i[2\mu\nu + 2\lambda\mu]), \tag{6.84}$$

so that when we take the real part of this quantity we get

$$-s(\mu^2 - \nu^2 - 2\lambda\nu) = s(\nu^2 + 2\lambda\nu - \mu^2). \tag{6.85}$$

Let us now impose $\lambda \geq 3B$. Consider t varying along Γ, first with $t = \mu + i\nu \in \Gamma_1 \cup \Gamma_3$, the union of the vertical segments. Then we have $\mu^2 = 1$, and we find for each fixed $s > 0$ the quantity in (6.85) is maximized in ν over $[-2B, 0]$ when $\nu = 0$. So

$$\left| \int_{\Gamma_1 \cup \Gamma_3} \frac{1}{t - t^\eta_\pm(\beta)} e^{-2i\lambda st} e^{-st^2} \right| \leq Ce^{-s}. \tag{6.86}$$

On the other hand, when we have $t \in \Gamma_2$, we have that $\nu = -2B$, and the quantity in (6.85) is maximized when $\mu = 0$. Recalling $\lambda \geq 3B$, we find

$$\left| \int_{\Gamma_2} \frac{1}{t - t^\eta_\pm(\beta)} e^{-2i\lambda st} e^{-st^2} \right| \leq Ce^{(-4\lambda B + 4B^2)s} \leq Ce^{-8B^2 s}. \tag{6.87}$$

Combining (6.86) and (6.87), and defining $\gamma = \min(8B^2, 1)$, we obtain

$$\left| \int_{\Gamma} \frac{1}{t - t^\eta_\pm(\beta)} e^{-2i\lambda st} e^{-st^2} \right| \leq C' e^{-\gamma s} \tag{6.88}$$

Using this with (6.78)-(6.79), (6.82), and (6.83) gives us

$$\tilde{\Sigma}(s, \lambda) = \sum_\pm \tilde{I}^\pm_*(s, \lambda) = O(e^{-\gamma s}) + \int_{\mathcal{N}} \int_{Q_\delta} (-2\pi i) e^{-2i\lambda s t^\eta_+(\beta)} e^{-s(t^\eta_+(\beta))^2} b^\eta_+(\beta) d\beta d\eta, \tag{6.89}$$

where we have also used that the $|b^\eta_\pm(\beta)|$ are uniformly bounded over $Q_\delta \times \mathcal{N}$.

The $\tilde{r}(s, \lambda)$ in the statement of (i) of the Proposition is defined to be the $O(e^{-\gamma s})$ term in the right hand side of (6.89). Thus, we just need to show the remaining integral in the right hand side of (6.89) has the form $\int_{\mathcal{N}} I_\eta(s, \lambda) d\eta$, with $I_\eta(s, \lambda)$ as in (6.72), and with (ii) and (iii) of the proposition

satisfied. We denote the remaining integral in the right hand side of (6.89) by

$$I^*(s,\lambda) := -2\pi i \int_{\mathcal{N}} \int_{Q_\delta} e^{-2i\lambda s t_+^\eta(\beta)} e^{-s(t_+^\eta(\beta))^2} b_+^\eta(\beta) d\beta d\eta \tag{6.90}$$

Let us define for $(\beta,\eta) \in \overline{Q_\delta} \times \mathcal{N}$

$$\tilde{b}_\eta(\beta) := \frac{\psi_2(\eta)(\eta_1 - i\eta_2)}{c_+(\beta,\eta)} = \frac{\psi_2(\eta)(\eta_1 - i\eta_2)}{\partial_t h(\beta,\eta,t_+(\beta,\eta))}, \tag{6.91}$$

noting in particular for each fixed $\eta \in \mathcal{N}$ that $\tilde{b}_\eta(\cdot)$ is then a C^1 function with bounded first order partial derivatives on Q_δ. Also note from the definition of b_+^η, given in (6.77), that

$$b_+^\eta(\beta) = g(\beta)\tilde{b}_\eta(\beta). \tag{6.92}$$

The next step is to replace the variable β in the integral in the right hand side of (6.91) with rotated coordinates (x,y), with $R_{\pi/4}(x,y) = \beta$ (where $R_{\pi/4}$ acts as a counterclockwise rotation by $\pi/4$). We define for $(x,y) \in [0,\delta]^2$, $\eta \in \mathcal{N}$,

$$\begin{aligned} t_\eta(x,y) &:= t_+^\eta(R_{\pi/4}(x,y)), \\ b_\eta(x,y) &:= \tilde{b}_\eta(R_{\pi/4}(x,y)). \end{aligned} \tag{6.93}$$

We remark that for each $\eta \in \mathcal{N}$, $b_\eta(\cdot)$ has bounded C^1 norm since $\tilde{b}_\eta(\cdot)$ does.

One can easily calculate from the definition of t_+^η, using (6.22) and (6.24), that for each $\eta \in \mathcal{N}$, (6.74) holds. Moreover, since Lemma 6.2.2 gives us that

$$\|t_+^\eta(\cdot) + Id_{\mathbb{C}}\|_{C^2} \leq C\epsilon, \tag{6.94}$$

we find that

$$\|t_\eta(\cdot) + (R_{\frac{\pi}{4}})_{\mathbb{C}}\|_{C^2} \leq C\epsilon, \tag{6.95}$$

so that t_η satisfies the properties claimed in (ii) of the proposition. Now observe from the definition of g, given by (6.14), and item (ii) immediately following Definition 6.2.4, that for $\beta \in Q_\delta$, we have $g(\beta) = (\beta_2^2 - \beta_1^2)^{n+1}$. Thus we have

$$g(R_{\pi/4}(x,y)) = C' x^{n+1} y^{n+1}, \quad \text{for } (x,y) \in [0,\delta]^2 \tag{6.96}$$

for some constant C'. So making changing to the rotated coordinates (x, y), with $R_{\pi/4}(x, y) = \beta$, in the integral in (6.90), we have

$$I^*(s,\lambda) := c\int_{\mathcal{N}}\int_0^\delta\int_0^\delta e^{-2i\lambda s t_\eta(x,y)}e^{-s(t_\eta(x,y))^2}x^{n+1}y^{n+1}b_\eta(x,y)dxdyd\eta, \tag{6.97}$$

for a nonzero constant c. Note that $t_\eta^2 + 2i\lambda t_\eta = (t_\eta + i\lambda)^2 + \lambda^2$, so

$$e^{-2i\lambda s t_\eta}e^{-st_\eta^2} = e^{-s(t_\eta^2+2i\lambda t_\eta)^2} = e^{-s\lambda^2}e^{-s(t_\eta+i\lambda)^2}, \tag{6.98}$$

which yields

$$I^*(s,\lambda) = ce^{-s\lambda^2}\left(\int_{\mathcal{N}}\int_0^\delta\int_0^\delta e^{-s(t_\eta(x,y)+i\lambda)^2}x^{n+1}y^{n+1}b_\eta(x,y)dxdyd\eta\right). \tag{6.99}$$

Recall that $I^*(s, \lambda)$ is the integral in the right hand side of (6.89). Thus if we define for each $\eta \in \mathcal{N}$

$$I_\eta(s,\lambda) := c\int_0^\delta\int_0^\delta e^{-s(t_\eta(x,y)+i\lambda)^2}x^{n+1}y^{n+1}b_\eta(x,y)dxdy, \tag{6.100}$$

we have the decomposition proposed in the statement of the proposition. To complete the proof, it remains only to show that

$$\int_{\mathcal{N}} b_\eta(0)d\eta \neq 0. \tag{6.101}$$

To see this, we note

$$b_\eta(0) = \tilde{b}_\eta(0) = \frac{\psi_2(\eta)(\eta_1 - i\eta_2)}{c_+(0,\eta)}. \tag{6.102}$$

It is not difficult to verify that in fact for any $\eta \in \mathcal{N}$ we have

$$c_+(0,\eta) = \partial_t h_+(0,\eta,t_+(0,\eta)) = \partial_t h(0,\eta,0) = 1. \tag{6.103}$$

Moreover, recall ψ_2 is real and nonnegative, and $\mathcal{N}$, the support of ψ_2, is contained in $B_{\frac{1}{2}}((1,0))$. Thus we can guarantee

$$\int_{\mathcal{N}} b_\eta(0)d\eta = \int_{\mathcal{N}} \psi_2(\eta)(\eta_1 - i\eta_2)d\eta \neq 0. \tag{6.104}$$

This shows that the function b_η satisfies the properties stated in part (iii) of the Proposition, concluding the proof. □

6.3 Proof of the impossibility of uniform analyticity

Let us summarize what has been established so far. We have found that given Assumption 6.2.3, which is a kind of uniform analyticity assumption on the Lagrangian trajectories for our solution to the Vlasov-Poisson system, Lemma 6.2.6 implies that we have the decomposition of $\frac{dV_1}{dt}(0,0,t)$ into an analytic piece and a constant multiple of $\Sigma(t) = \sum_{\pm} I_*^{\pm}(t)$. Meanwhile, Proposition 6.2.8 implies that the transform of $\Sigma(t)$, denoted $\tilde{\Sigma}(s,\lambda)$, can be written as a sum of two quantities, one of which is a multiple of an integral over η of the $I_\eta(s,\lambda)$ function, given by (6.72), and the other of which is a function that decays exponentially in s, for large s, λ. The majority of the remaining work consists of computing the asymptotics in terms of large $s, \lambda > 0$ of $I_\eta(s,\lambda)$.

Looking to Lemma 6.2.7, we see that the analyticity of $\Sigma(t)$, and thus that of $\frac{dV_1}{dt}(0,0,t)$, depends critically on the asymptotics in large $s, \lambda > 0$ of $\tilde{\Sigma}(s,\lambda)$, and thus on those of $I_\eta(s,\lambda)$. Once we have developed the necessary integral asymptotics, we find that $|I_\eta(s,\lambda)|$ can only possibly decay algebraically in s and λ. From this we can deduce that, for large s, λ, the decay of $\tilde{\Sigma}(s,\lambda)$ in s does not pass the test of Lemma 6.2.7, and so $\Sigma(t)$ cannot be analytic at $t = 0$, and so Assumption 6.2.3 must be false. This will conclude the proof that trajectories to Vlasov-Poisson arising from the initial data f_ϵ fail to satisfy the kind of uniform analyticity condition described by Assumption 6.2.3.

6.3.1 Integral asymptotics

Throughout this subsection, we assume that we are given a complex-valued function $\sigma(x,y)$ defined on the square $[0,\delta]^2$, for the $\delta > 0$ chosen in our construction of the initial data f_ϵ for the Vlasov-Poisson system. Moreover, we assume $\sigma(0) = 0$, $\partial_x\sigma(0) = -\frac{1+i}{\sqrt{2}}$, $\partial_y\sigma(0) = \frac{1-i}{\sqrt{2}}$, and that

$$\|\sigma(\cdot) + (R_{\frac{\pi}{4}})_{\mathbb{C}}\|_{C^2} \leq C\epsilon, \tag{6.105}$$

for a universal constant C, where $(R_{\frac{\pi}{4}})_{\mathbb{C}}$ denotes a counterclockwise rotation by $\frac{\pi}{4}$, followed by identification of the vector in $\mathbb{R}^2$ as a complex number.

Given such a function $\sigma(x,y)$, we seek to give asymptotics in terms of large $s, \lambda > 0$ for integrals of the form

$$\int_0^\delta \int_0^\delta e^{-s(\sigma(x,y)+i\lambda)^2} x^{n+1} y^{n+1} a(x,y)\,dx\,dy, \tag{6.106}$$

for a complex-valued C^1 function $a(x,y)$. Compare this with the integral in the expression $I_\eta(s,\lambda)$ in (6.72), for each fixed $\eta \in \mathcal{N}$, in which $t_\eta(x,y)$ plays the role of $\sigma(x,y)$. Note that part (ii)

of Proposition 6.2.8 asserts that the $t_\eta(x,y)$ satisfy that which we require of $\sigma(x,y)$ above. Once we develop the asymptotics for integrals of the form (6.106), sufficient information on the decay properties of $I_\eta(s,\lambda)$ and $\tilde{\Sigma}(s,\lambda)$ in large s,λ are readily obtained, through Proposition 6.2.8.

Definition 6.3.1. *Given $\sigma : [0,\delta]^2 \to \mathbb{C}$ satisfying $\sigma(0) = 0$, $\partial_x\sigma(0) = -\frac{1+i}{\sqrt{2}}$, $\partial_y\sigma(0) = \frac{1-i}{\sqrt{2}}$, and* (6.105)*, we define the quantities*

$$\begin{aligned} \phi(x,y) &:= (\sigma(x,y)+i\lambda)^2, \\ \zeta_j(x,y) &:= (\partial_j\phi(x,y))^{-1} = \frac{1}{2}\left[(\sigma(x,y)+i\lambda)\partial_j\sigma(x,y)\right]^{-1} \quad \text{for } j=1,2. \end{aligned} \tag{6.107}$$

We decompose $\sigma(x,y)$ into its real and imaginary parts, with

$$\sigma(x,y) = \mu(x,y) + i\nu(x,y), \tag{6.108}$$

defining $\mu(x,y)$ and $\nu(x,y)$ accordingly, and we define

$$\begin{aligned} \rho(x,y) &:= Re(\phi(x,y)) = (\mu(x,y))^2 - (\nu(x,y)+\lambda)^2, \\ \varphi_j(x,y) &:= (\partial_j\rho(x,y))^{-1} = \frac{1}{2}\left[\mu(x,y)\partial_j\mu(x,y) - (\nu(x,y)+\lambda)\partial_j\nu(x,y)\right]^{-1} \quad \text{for } j=1,2. \end{aligned} \tag{6.109}$$

We remark that in the above definitions we have suppressed the dependence of the functions on the parameter λ with the above notation. This is primarily done for greater ease of reading, though it should be kept in mind that these functions do indeed depend on λ.

Now we outline some simple preliminary estimates for the functions given in Definition 6.3.1, which are just straightforward consequences of computations relying on (6.105) and the other requirements for $\sigma(x,y)$. We remind the reader that the n appearing in the next lemma is the fixed integer greater than or equal to 10 that was chosen in the construction of our initial data.

Lemma 6.3.2. *If $\epsilon > 0$ is small enough, for the quantities defined in Definition 6.3.1, for each $j,k = 1,2$, we have the estimates*

$$\begin{aligned} |\zeta_j(x,y)| &\leq a_1\lambda^{-1}, \\ |\partial_k\zeta_j(x,y)| &\leq a_2 n^{-1}\lambda^{-1}, \\ |\varphi_j(x,y)| &\leq a_1\lambda^{-1}, \\ |\partial_k\varphi_j(x,y)| &\leq a_2 n^{-1}\lambda^{-1}, \end{aligned} \tag{6.110}$$

for all $(x,y) \in [0,\delta]^2$*, and sufficiently large* $s, \lambda > 0$*, where* $a_1 = 3/4$*,* $a_2 = 10^{-3}$*.*

We have one more preliminary estimate to make, which gives a crude bound on the quantity $e^{-s\rho}$ (which we remark is the magnitude of the exponential in (6.106)) evaluated along the two edges making up the boundary of the square $[0,\delta]^2$ not intersecting the origin.

Lemma 6.3.3. *For* $\sigma(x,y)$ *and* $\rho(x,y)$ *as above, and large enough* s*,* λ*, we have the estimates*

$$\begin{aligned} e^{-s\rho(x,\delta)} &\leq (s\lambda)^{-3n} e^{s\lambda^2} \quad \textit{for } 0 \leq x \leq \delta, \\ e^{-s\rho(\delta,y)} &\leq (s\lambda)^{-3n} e^{s\lambda^2} \quad \textit{for } 0 \leq y \leq \delta. \end{aligned} \tag{6.111}$$

Proof. First we note that given the bound (6.105), we can ensure that for any $0 \leq \tau \leq \delta$, we have $-10\delta < \nu(\tau,\delta) < -\delta/10$ and $-10\delta < \nu(\delta,\tau) < -\delta/10$. Let us write $a := \delta/10$ and $A := 10\delta$. So for any $0 \leq \tau \leq \delta$, as long as s and λ are sufficiently large,

$$\begin{aligned} e^{-s\rho(\tau,\delta)} &= e^{s(\lambda^2 + 2\nu(\tau,\delta)\lambda + (\nu(\tau,\delta))^2 - (\mu(\tau,\delta))^2)}, \qquad (6.112) \\ &\leq e^{s(\lambda^2 - 2a\lambda + A^2)}, \\ &= e^{s\lambda^2} e^{(-2a + A^2\lambda^{-1})(s\lambda)}, \\ &\leq e^{s\lambda^2} e^{-as\lambda}, \qquad (6.113) \\ &\leq (s\lambda)^{-3n} e^{s\lambda^2}. \end{aligned}$$

In particular, to get (6.113), we assume $\lambda > A^2/a$.

Similarly, one can check

$$e^{-s\rho(\delta,\tau)} \leq (s\lambda)^{-3n} e^{s\lambda^2} \tag{6.114}$$

for sufficiently large s, λ. With these bounds, the lemma is proven. □

Now we are ready to begin with the integral asymptotics. We will eventually prove Lemma 6.3.8, which provides the leading order term for integrals of the form (6.106), along with appropriate error bounds. We build up to this by first proving some simpler asymptotic integral estimates, beginning with a one-dimensional version, with a real phase function. Also, rather than explicitly calculating a leading term for an expansion, we simply provide an upper bound on the size of the quantity.

Lemma 6.3.4. *Let us denote* $\underline{\rho}(y) := \rho(0,y)$ *for* $0 \leq y \leq \delta$*. Then for sufficiently large* s, λ*, we have for* $0 \leq j \leq n+2$*,*

$$\int_0^\delta e^{-s\underline{\rho}(y)} y^j dy \leq \frac{j!}{(s\lambda)^{j+1}} e^{s\lambda^2}. \tag{6.115}$$

Proof. First we show this for $j = 0$. Define $\theta(y) := \left(\underline{\rho}'(y)\right)^{-1}$. Then we have

$$\begin{aligned}\int_0^\delta e^{-s\underline{\rho}(y)}dy &= -s^{-1}\int_0^\delta \theta(y)\left(e^{-s\underline{\rho}(y)}\right)' dy, \\ &= s^{-1}\int_0^\delta \theta'(y)e^{-s\underline{\rho}(y)}dy - s^{-1}\theta(\delta)e^{-s\underline{\rho}(\delta)} + s^{-1}\theta(0)e^{-s\underline{\rho}(0)}, \\ &= A + B + C.\end{aligned} \tag{6.116}$$

Note

$$|A| \leq s^{-1}\max|\theta'(y)|\int_0^\delta e^{-s\underline{\rho}(y)}dy \leq (s\lambda)^{-1}\int_0^\delta e^{-s\underline{\rho}(y)}dy, \tag{6.117}$$

where we have used Lemma 6.3.2, using that $\theta'(y) = \partial_y\varphi_2(0, y)$. Also, by Lemmas 6.3.2 and 6.3.3,

$$|B| \leq s^{-1}\max|\theta(y)|e^{-s\underline{\rho}(\delta)} \leq (s\lambda)^{-3n}e^{s\lambda^2}. \tag{6.118}$$

Noting $\underline{\rho}(0) = -\lambda^2$, and that $\theta(0) = \varphi_2(0, 0)$, and using Lemma 6.3.2,

$$|C| \leq s^{-1}|\theta(0)|e^{s\lambda^2} \leq a_1(s\lambda)^{-1}e^{s\lambda^2}. \tag{6.119}$$

So incorporating bounds (6.117)-(6.119) into (6.116), we find

$$\left(1 - (s\lambda)^{-1}\right)\int_0^\delta e^{-s\underline{\rho}(y)}dy \leq (s\lambda)^{-3n}e^{s\lambda^2} + a_1(s\lambda)^{-1}e^{s\lambda^2}. \tag{6.120}$$

Recalling $a_1 = 3/4$, we find this then implies

$$\int_0^\delta e^{-s\underline{\rho}(y)}dy \leq (s\lambda)^{-1}e^{s\lambda^2}, \tag{6.121}$$

for sufficiently large s, λ. This gives us the base case.

Fix $1 \leq j \leq n + 2$ and assume (6.115) holds for $j' < j$. Then

$$\begin{aligned}\int_0^\delta e^{-s\underline{\rho}(y)}y^j dy &= -s^{-1}\int_0^\delta y^j\theta(y)\left(e^{-s\underline{\rho}(y)}\right)dy \\ &= s^{-1}\int_0^\delta \left(jy^{j-1}\theta(y) + y^j\theta'(y)\right)e^{-s\underline{\rho}(y)}dy - s^{-1}\delta^j\theta(\delta)e^{-s\underline{\rho}(\delta)}, \\ &= A' + B'.\end{aligned} \tag{6.122}$$

Regarding A', we may write $\theta(y) = \theta(0) + y\tilde{\theta}(y)$, where $\tilde{\theta}(y) = \frac{1}{y}\int_0^y \theta'(\tilde{y})d\tilde{y}$. With this,

$$A' = s^{-1}\int_0^\delta \left[jy^{j-1}\left(\theta(0) + y\tilde{\theta}(y)\right) + y^j\theta'(y)\right] e^{-s\underline{\rho}(y)}dy, \tag{6.123}$$

and so we obtain

$$|A'| \leq s^{-1}j|\theta(0)|\int_0^\delta e^{-s\underline{\rho}(y)}y^{j-1}dy + s^{-1}(j+1)\max|\theta'(y)|\int_0^\delta e^{-s\underline{\rho}(y)}y^j dy. \tag{6.124}$$

Using Lemma 6.3.2 to bound $|\theta(0)| = |\varphi_2(0,0)|$ and $|\theta'(y)| = |\partial_y\varphi_2(0,y)|$, we get

$$|A'| \leq a_1 j(s\lambda)^{-1}\int_0^\delta e^{-s\underline{\rho}(y)}y^{j-1}dy + a_2(j+1)n^{-1}(s\lambda)^{-1}\int_0^\delta e^{-s\underline{\rho}(y)}y^j dy \tag{6.125}$$

Now we use the induction step on the first term in the right hand side of (6.125), and for the second term we note that $a_2(j+1)n^{-1} = 10^{-3}(j+1)n^{-1} < 1$, and we find

$$|A'| \leq a_1\frac{j!}{(s\lambda)^{j+1}}e^{s\lambda^2} + (s\lambda)^{-1}\int_0^\delta e^{-s\underline{\rho}(y)}y^j dy. \tag{6.126}$$

Additionally, Lemma 6.3.3 in particular gives us that the term B' from (6.122) satisfies the bound

$$|B'| = |s^{-1}\delta^j\theta(\delta)e^{-s\underline{\rho}(\delta)}| \leq ce^{-s\underline{\rho}(\delta)} \leq c(s\lambda)^{-3n}e^{s\lambda^2}. \tag{6.127}$$

Using this with the bound (6.126) on A' in (6.122), we obtain

$$(1-(s\lambda)^{-1})\int_0^\delta e^{-s\underline{\rho}(y)}y^j dy \leq a_1\frac{j!}{(s\lambda)^{j+1}}e^{s\lambda^2} + c(s\lambda)^{-3n}e^{s\lambda^2}. \tag{6.128}$$

This implies that for large enough s, λ, we have

$$\int_0^\delta e^{-s\underline{\rho}(y)}y^j dy \leq \frac{j!}{(s\lambda)^{j+1}}e^{s\lambda^2}, \tag{6.129}$$

finishing the inductive step. □

Now we give an estimate for essentially the two-dimensional version of the integral handled in Lemma 6.3.4.

Lemma 6.3.5. *For sufficiently large s, λ, we have for $0 \le j \le n+2$, $0 \le k \le n+2$,*

$$\int_0^\delta \int_0^\delta e^{-s\rho(x,y)} x^j y^k dxdy \le \frac{j!k!}{(s\lambda)^{j+k+2}} e^{s\lambda^2}. \tag{6.130}$$

Proof. Define $\Theta(x,y) := (\partial_x \rho(x,y))^{-1}$. First we consider the base case. We have

$$\begin{aligned} \int_0^\delta \int_0^\delta e^{-s\rho(x,y)} dxdy &= -s^{-1} \int_0^\delta \int_0^\delta \Theta(x,y) \partial_x \left(e^{-s\rho(x,y)} \right) dxdy, \\ &= s^{-1} \int_0^\delta \int_0^\delta \partial_x \Theta(x,y) e^{-s\rho(x,y)} dxdy - s^{-1} \int_0^\delta \Theta(\delta,y) e^{-s\rho(\delta,y)} dy \\ &\quad + s^{-1} \int_0^\delta \Theta(0,y) e^{-s\rho(0,y)} dy, \\ &= A + B + C. \end{aligned} \tag{6.131}$$

First we handle A. Notice $\Theta(x,y) = \varphi_1(x,y)$. So we may use Lemma 6.3.2 to bound $\partial_x \Theta(x,y)$ in the following:

$$|A| \le s^{-1} \max |\partial_x \Theta(x,y)| \int_0^\delta \int_0^\delta e^{-s\rho(x,y)} dxdy \le (s\lambda)^{-1} \int_0^\delta \int_0^\delta e^{-s\rho(x,y)} dxdy. \tag{6.132}$$

For B, by using Lemma 6.3.3, we find that

$$|B| \le c \int_0^\delta e^{-s\rho(\delta,y)} dy \le c'(s\lambda)^{-3n} e^{s\lambda^2}. \tag{6.133}$$

Now we consider C. Note that we have $\Theta(0,y) = \Theta(0,0) + y\tilde{\Theta}(y)$ where $\tilde{\Theta}(y) = \frac{1}{y} \int_0^y \partial_y \Theta(0,\tilde{y}) d\tilde{y}$. Then

$$\begin{aligned} C = s^{-1} \int_0^\delta \left(\Theta(0,0) + y\tilde{\Theta}(y) \right) e^{-s\underline{\rho}(y)} dy &= s^{-1} \Theta(0,0) \int_0^\delta e^{-s\underline{\rho}(y)} dy + s^{-1} \int_0^\delta y\tilde{\Theta}(y) e^{-s\underline{\rho}(y)} dy \\ &= C_1 + C_2. \end{aligned} \tag{6.134}$$

Using Lemmas 6.3.2 and 6.3.4 we bound C_1:

$$|C_1| \le a_1 (s\lambda)^{-2} e^{s\lambda^2}. \tag{6.135}$$

For C_2, we use Lemmas 6.3.2 and 6.3.4 to find

$$|C_2| \leq s^{-1} \max |\partial_y \Theta| \int_0^\delta e^{-s\underline{\rho}(y)} y dy \leq a_2 (s\lambda)^{-3} e^{s\lambda^2}. \tag{6.136}$$

In view of (6.131)-(6.136), we find

$$(1 - (s\lambda)^{-1}) \int_0^\delta \int_0^\delta e^{-s\rho(x,y)} dx dy \leq (s\lambda)^{-3n} e^{s\lambda^2} + (a_1 + a_2 (s\lambda)^{-1})(s\lambda)^{-2} e^{s\lambda^2}. \tag{6.137}$$

Thus for large enough s, λ, we have

$$\int_0^\delta \int_0^\delta e^{-s\rho(x,y)} dx dy \leq \frac{1}{(s\lambda)^2} e^{s\lambda^2}, \tag{6.138}$$

which gives us the base case.

Assume (6.130) holds for all $j', k' \geq 0$, with $j' + k' < j + k$ for some pair j, k with $j + k \geq 1$, $0 \leq j, k \leq n + 2$.

(6.139)

$$\begin{aligned}
\int_0^\delta \int_0^\delta e^{-s\rho(x,y)} x^j y^k dx dy &= \int_0^\delta y^k \left[-s^{-1} \int_0^\delta x^j \Theta(x,y) \partial_x \left(e^{-s\rho(x,y)} \right) dx \right] dy, \\
&= \int_0^\delta y^k \Big[s^{-1} \int_0^\delta \left(j x^{j-1} \Theta(x,y) + x^j \partial_x \Theta(x,y) \right) e^{-s\rho(x,y)} dx \\
&\qquad - s^{-1} \delta^j \Theta(\delta, y) e^{-s\rho(\delta,y)} + \begin{cases} 0 & \text{if } j \geq 1 \\ s^{-1} \Theta(0,y) e^{-s\underline{\rho}(y)} & \text{if } j = 0 \end{cases} \Big] dy, \\
&= A + B + \begin{cases} 0 & \text{if } j \geq 1 \\ C & \text{if } j = 0, \end{cases}
\end{aligned}$$

where $C := s^{-1} \int_0^\delta y^k \Theta(0,y) e^{-s\underline{\rho}(y)} dy$ and A and B are the first two integrals one gets upon splitting up the expression in the obvious way. First let us handle A. In the following we make use of Lemma

6.3.2 and the induction hypothesis.

$$
\begin{aligned}
|A| &\leq js^{-1}\max|\Theta|\int_0^\delta\int_0^\delta e^{-s\rho(x,y)}x^{j-1}y^k dxdy + s^{-1}\max|\partial_x\Theta|\int_0^\delta\int_0^\delta e^{-s\rho(x,y)}x^j y^k dxdy, \\
&\leq a_1 j(s\lambda)^{-1}\left(\frac{(j-1)!k!}{(s\lambda)^{j+k+1}}e^{s\lambda^2}\right) + (s\lambda)^{-1}\int_0^\delta\int_0^\delta e^{-s\rho(x,y)}x^j y^k dxdy, \\
&= \left(\begin{cases} A' & \text{if } j\geq 1 \\ 0 & \text{if } j=0 \end{cases}\right) + (s\lambda)^{-1}\int_0^\delta\int_0^\delta e^{-s\rho(x,y)}x^j y^k dxdy,
\end{aligned}
\tag{6.140}
$$

where

$$
A' = a_1\frac{j!k!}{(s\lambda)^{j+k+2}}e^{s\lambda^2}. \tag{6.141}
$$

Meanwhile, for B we observe with applications of Lemmas 6.3.2 and 6.3.3 that

$$
|B| \leq (s\lambda)^{-3n}e^{s\lambda^2}. \tag{6.142}
$$

For C, by applying Lemmas 6.3.2 and 6.3.4 we get

$$
|C| \leq s^{-1}\max|\Theta|\int_0^\delta e^{-s\underline{\rho}(y)}y^k dy \leq a_1\frac{k!}{(s\lambda)^{k+2}}e^{s\lambda^2}. \tag{6.143}
$$

By combining the estimates for A, B, and C, we arrive at

$$
(1-(s\lambda)^{-1})\int_0^\delta\int_0^\delta e^{-s\rho(x,y)}x^j y^k dxdy \leq (s\lambda)^{-3n}e^{s\lambda^2} + a_1\frac{j!k!}{(s\lambda)^{j+k+2}}e^{s\lambda^2}, \tag{6.144}
$$

which implies for large enough s,λ, that

$$
\int_0^\delta\int_0^\delta e^{-s\rho(x,y)}x^j y^k dxdy \leq \frac{j!k!}{(s\lambda)^{j+k+2}}e^{s\lambda^2}. \tag{6.145}
$$

This completes the inductive step. □

Now that we have Lemmas 6.3.4 and 6.3.5, which provide bounds for integrals with ρ, i.e. the real part of $\phi=(\sigma+i\lambda)^2$, as the phase function in the exponential, we are ready to calculate more precise asymptotics for the corresponding integrals with ϕ itself as the phase function. In the next lemma, we calculate the leading order term along with bounds on the error for a one-dimensional integral in which the phase function ϕ appears.

Lemma 6.3.6. *We define* $\underline{\phi}(y) := \phi(0,y)$ *and* $\omega := \frac{i-1}{2\sqrt{2}}$. *For sufficiently large* $s, \lambda > 0$, *we have the following. For* $0 \leq j \leq n+1$,

$$\int_0^\delta e^{-s\underline{\phi}(y)} y^j dy = \omega^{j+1} \frac{j!}{(s\lambda)^{j+1}} e^{s\lambda^2} + \mathcal{R}_j(s,\lambda), \tag{6.146}$$

where

$$|\mathcal{R}_j(s,\lambda)| \leq \frac{j!}{(s\lambda)^{j+2}} e^{s\lambda^2}. \tag{6.147}$$

Proof. First we show this for $j = 0$. Define $\psi(y) := \left(\underline{\phi}(y)'\right)^{-1}$. Then

$$\begin{aligned} \int_0^\delta e^{-s\underline{\phi}(y)} dy &= -s^{-1} \int_0^\delta \psi(y) \left(e^{-s\underline{\phi}(y)}\right)' dy, \\ &= s^{-1} \int_0^\delta \psi'(y) e^{-s\underline{\phi}(y)} dy - s^{-1}\psi(\delta) e^{-s\underline{\phi}(\delta)} + s^{-1}\psi(0) e^{-s\underline{\phi}(0)}, \\ &= A + B + C. \end{aligned} \tag{6.148}$$

In handling A, we note that $\psi(y) = \zeta_2(0,y)$ and use Lemmas 6.3.2 and 6.3.4 to get the bound

$$|A| \leq s^{-1} \max|\psi'| \int_0^\delta e^{-s\underline{\rho}(y)} dy \leq a_2 (s\lambda)^{-2} e^{s\lambda^2}. \tag{6.149}$$

We apply Lemmas 6.3.2 and 6.3.3 to bound

$$|B| \leq s^{-1} \max|\psi| e^{-s\underline{\rho}(\delta)} \leq (s\lambda)^{-3n} e^{s\lambda^2}. \tag{6.150}$$

For C, evaluating $\psi(0)$ and $\underline{\phi}(0)$, we find

$$C = \left(\frac{1-i}{2\sqrt{2}}\right) \frac{1}{s\lambda} e^{s\lambda^2}. \tag{6.151}$$

Using this and the estimates on A and B, we find that we have

$$\int_0^\delta e^{-s\underline{\phi}(y)} dy = \omega \frac{1}{s\lambda} e^{s\lambda^2} + \mathcal{R}_0(s,\lambda), \tag{6.152}$$

where

$$|\mathcal{R}_0(s,\lambda)| \leq (s\lambda)^{-3n} e^{s\lambda^2} + a_2 (s\lambda)^{-2} \leq (s\lambda)^{-2} e^{s\lambda^2}. \tag{6.153}$$

Now we verify the inductive step. Fix $1 \leq j \leq n+1$ and assume we have (6.146) and (6.147) for

$j' < j$. Then

$$\begin{aligned}\int_0^\delta e^{-s\underline{\phi}(y)}y^j dy &= -s^{-1}\int_0^\delta y^j\psi(y)\left(e^{-s\underline{\phi}(y)}\right)' dy, \\ &= s^{-1}\int_0^\delta \left(jy^{j-1}\psi(y)+y^j\psi'(y)\right)e^{-s\underline{\phi}(y)}dy - s^{-1}\delta^j e^{-s\underline{\phi}(\delta)}\psi(\delta), \\ &= A'+B'.\end{aligned} \tag{6.154}$$

Regarding A', we define $\tilde{\psi}(y) := \frac{1}{y}\int_0^y \psi'(\tilde{y})d\tilde{y}$, and find

$$\begin{aligned}A' &= s^{-1}\int_0^\delta \left(jy^{j-1}\left[\psi(0)+y\tilde{\psi}(y)\right]+y^j\psi'(y)\right)e^{-s\underline{\phi}(y)}dy, \\ &= js^{-1}\psi(0)\int_0^\delta e^{-s\underline{\phi}(y)}y^{j-1}dy + s^{-1}\int_0^\delta \left(j\tilde{\psi}(y)+\psi'(y)\right)e^{-s\underline{\phi}(y)}y^j dy, \\ &= A_1'+A_2'.\end{aligned} \tag{6.155}$$

For A_1', we find by evaluating $\psi(0)$ and using the induction hypothesis

$$A_1' = j(s\lambda)^{-1}\omega\left(\omega^j\frac{(j-1)!}{(s\lambda)^j}e^{s\lambda^2}+\mathcal{R}_{j-1}(s,\lambda)\right), \tag{6.156}$$

where

$$|\mathcal{R}_{j-1}(s,\lambda)| \le \frac{(j-1)!}{(s\lambda)^{j+1}}e^{s\lambda^2}. \tag{6.157}$$

Continuing from (6.156), we find

$$A_1' = \omega^{j+1}\frac{j!}{(s\lambda)^{j+1}}e^{s\lambda^2}+\omega\frac{j}{s\lambda}\mathcal{R}_{j-1}(s,\lambda). \tag{6.158}$$

On the other hand, for A_2', by applying Lemmas 6.3.2 and 6.3.4 we find

$$|A_2'| \le s^{-1}(j+1)\max|\psi'|\int_0^\delta e^{-s\underline{\rho}(y)}y^j dy \le 2a_2(s\lambda)^{-1}\left(\frac{j!}{(s\lambda)^{j+1}}e^{s\lambda^2}\right) \le 2a_2\frac{j!}{(s\lambda)^{j+2}}e^{s\lambda^2}. \tag{6.159}$$

For B', from Lemmas 6.3.2 and 6.3.3 we have

$$|B'| \le e^{-s\underline{\rho}(\delta)} \le (s\lambda)^{-3n}e^{s\lambda^2}. \tag{6.160}$$

In summary, we find

$$\begin{aligned}\int_0^\delta e^{-s\underline{+}\phi(y)}y^j dy &= \omega^{j+1}\frac{j!}{(s\lambda)^{j+1}}e^{s\lambda^2}+\omega\frac{j}{s\lambda}\mathcal{R}_{j-1}(s,\lambda)+A_2'+B', \\ &= \omega^{j+1}\frac{j!}{(s\lambda)^{j+1}}e^{s\lambda^2}+\mathcal{R}_j(s,\lambda),\end{aligned} \tag{6.161}$$

having defined $\mathcal{R}_j(s,\lambda)$ in the obvious way, with

$$\begin{aligned}|\mathcal{R}_j(s,\lambda)| &\leq |\omega|\frac{j}{s\lambda}|\mathcal{R}_{j-1}(s,\lambda)|+|A_2'|+|B'|, \\ &\leq \frac{1}{2}\frac{j!}{(s\lambda)^{j+2}}e^{s\lambda^2}+2a_2\frac{j!}{(s\lambda)^{j+2}}e^{s\lambda^2}+\frac{1}{(s\lambda)^{3n}}e^{s\lambda^2}, \\ &\leq \frac{j!}{(s\lambda)^{j+2}}e^{s\lambda^2},\end{aligned} \tag{6.162}$$

for sufficiently large s,λ. Thus, we conclude (6.146) together with (6.147) holds for all $j=0,\ldots,n+1$, for large s,λ. □

Now we are able to use the above lemmas to get the key asymptotic calculation necessary for handling (6.106). With this next lemma, we describe the leading order term for the two-dimensional analogue of the quantity considered in Lemma 6.3.6. Notice the integral we deal with is the same as the integral (6.106) in the case $a(x,y)=1$.

Lemma 6.3.7. *For ω as defined in Lemma 6.3.6 and sufficiently large $s,\lambda>0$ we have the following. For $0\leq j,k\leq n+1$,*

$$\int_0^\delta\int_0^\delta e^{-s\phi(x,y)}x^j y^k dxdy = (-i)^{j+1}\omega^{j+k+2}\frac{j!k!}{(s\lambda)^{j+k+2}}e^{s\lambda^2}+\mathcal{R}'_{j,k}(s,\lambda), \tag{6.163}$$

where

$$|\mathcal{R}'_{j,k}(s,\lambda)|\leq\frac{j!k!}{(s\lambda)^{j+k+3}}e^{s\lambda^2}. \tag{6.164}$$

Proof. First we handle the case $j=k=0$. Define $\Psi(x,y):=(\partial_x\phi(x,y))^{-1}$. Then we have

$$\begin{aligned}\int_0^\delta\int_0^\delta e^{-s\phi(x,y)}dxdy &= -\int_0^\delta s^{-1}\int_0^\delta\Psi(x,y)\partial_x\left(e^{-s\phi(x,y)}\right)dxdy, \\ &= \int_0^\delta s^{-1}\bigg(\int_0^\delta\partial_x\Psi(x,y)e^{-s\phi(x,y)}dx \\ &\qquad -\Psi(\delta,y)e^{-s\phi(\delta,y)}+\Psi(0,y)e^{-s\phi(0,y)}\bigg)dy, \\ &= A+B+C.\end{aligned} \tag{6.165}$$

Using Lemmas 6.3.2 and 6.3.5, noting that $\Psi(x,y) = \zeta_1(x,y)$, we find

$$|A| \leq s^{-1} \max |\partial_x \Psi| \int_0^\delta \int_0^\delta e^{-s\rho(x,y)} dxdy \leq a_2 \frac{1}{(s\lambda)^3} e^{s\lambda^2}. \tag{6.166}$$

Applying Lemmas 6.3.2 and 6.3.3, for B we get

$$|B| \leq (s\lambda)^{-3n} e^{s\lambda^2}. \tag{6.167}$$

Now we write C the following way, defining $\underline{\tilde{\Psi}}(y) := \frac{1}{y}\int_0^y \Psi(0,\tilde{y})d\tilde{y}$ and evaluating $\Psi(0,0)$:

$$\begin{aligned} C &= s^{-1} \int_0^\delta e^{-s\underline{\phi}(y)} \left(\Psi(0,0) + y\underline{\tilde{\Psi}}(y) \right) dy, \\ &= s^{-1} \frac{-i\omega}{\lambda} \int_0^\delta e^{-s\underline{\phi}(y)} dy + s^{-1} \int_0^\delta \underline{\tilde{\Psi}}(y) e^{-s\underline{\phi}(y)} dy, \\ &= C_1 + C_2. \end{aligned} \tag{6.168}$$

With the use of Lemma 6.3.6 we get

$$C_1 = \frac{-i\omega}{s\lambda} \left(\frac{\omega}{s\lambda} e^{s\lambda^2} + \mathcal{R}_0(s,\lambda) \right) = -i\omega^2 \frac{1}{(s\lambda)^2} e^{s\lambda^2} - \frac{i\omega}{s\lambda} \mathcal{R}_0(s,\lambda), \tag{6.169}$$

where $|\mathcal{R}_0(s,\lambda)| \leq (s\lambda)^{-2} e^{s\lambda^2}$. Meanwhile, for C_2, we use Lemmas 6.3.2 and 6.3.4.

$$|C_2| \leq s^{-1} \max |\partial_y \Psi| \int_0^\delta e^{-s\underline{\rho}(y)} y dy \leq a_2 \frac{1}{(s\lambda)^3} e^{s\lambda^2}. \tag{6.170}$$

Now we separate the apparent leading term from the error terms, which we collect into $\mathcal{R}_0'(s,\lambda)$ in the following:

$$\begin{aligned} \int_0^\delta \int_0^\delta e^{-s\phi(x,y)} dxdy &= A + B - i\omega^2 \frac{1}{(s\lambda)^2} e^{s\lambda^2} - \frac{i\omega}{s\lambda} \mathcal{R}_0(s,\lambda) + C_2, \\ &= -i\omega^2 \frac{1}{(s\lambda)^2} e^{s\lambda^2} + \mathcal{R}_0'(s,\lambda), \end{aligned} \tag{6.171}$$

where for large s, λ,

$$\begin{aligned} |\mathcal{R}_0'(s,\lambda)| &\leq |A| + |B| + \frac{1}{2s\lambda} |\mathcal{R}_0(s,\lambda)| + |C_2|, \\ &\leq a_2 \frac{1}{(s\lambda)^3} + \frac{1}{(s\lambda)^{3n}} + \frac{1}{2} \frac{1}{(s\lambda)^3} e^{s\lambda^2} + a_2 \frac{1}{(s\lambda)^3} e^{s\lambda^2}, \\ &\leq \frac{1}{(s\lambda)^3} e^{s\lambda^2}. \end{aligned} \tag{6.172}$$

This takes care of the base case. Now we suppose that we have (6.163) and (6.164) for all pairs j', k' with $j' + k' < j + k$, for a pair j, k with $j \neq 0$ or $k \neq 0$ and $0 \leq j, k \leq n + 1$.

$$\begin{aligned}
\int_0^\delta \int_0^\delta e^{-s\phi(x,y)} x^j y^k dx dy &= \int_0^\delta y^k \Big[- s^{-1} \int_0^\delta \left(\int_0^\delta x^j \Psi(x,y) \partial_x \left(e^{-s\phi(x,y)} \right) dx \right) dy \Big], \\
&= \int_0^\delta y^k s^{-1} \Big[\int_0^\delta \left(j x^{j-1} \Psi(x,y) + x^j \partial_x \Psi(x,y) \right) e^{-s\phi(x,y)} dx \\
&\qquad - \delta^j \Psi(\delta, y) e^{-s\phi(\delta,y)} + \begin{cases} 0 & \text{if } j \geq 1 \\ \Psi(0,y) e^{-s\underline{\phi}(y)} & \text{if } j = 0 \end{cases} \Big] dy, \\
&= A + B + \begin{cases} 0 & \text{if } j \geq 1, \\ C & \text{if } j = 0, \end{cases}
\end{aligned} \tag{6.173}$$

where $C = s^{-1} \int_0^\delta \Psi(0,y) e^{-s\underline{\phi}(y)} dy$, and A and B give the other two main terms arising from splitting up the above expression in the obvious way.

First we handle B, with applications of Lemmas 6.3.2 and 6.3.3. This gives

$$|B| \leq (s\lambda)^{-3n} e^{s\lambda^2}. \tag{6.174}$$

Now we handle C. We use the notation $\underline{\Psi}(y) := \Psi(0,y)$ and recall $\underline{\tilde{\Psi}}(y) = \frac{1}{y} \int_0^y \Psi(0,\tilde{y}) d\tilde{y}$, obtaining

$$\begin{aligned}
C &= s^{-1} \int_0^\delta \underline{\Psi}(y) y^k e^{-s\underline{\phi}(y)} dy, &(6.175)\\
&= s^{-1} \int_0^\delta \left(\underline{\Psi}(0) + y \underline{\tilde{\Psi}}(y) \right) y^k e^{-s\underline{\phi}(y)} dy, \\
&= \frac{-i\omega}{s\lambda} \left(\omega^{k+1} \frac{k!}{(s\lambda)^{k+1}} e^{s\lambda^2} + \mathcal{R}_k(s,\lambda) \right) + s^{-1} \int_0^\delta y^{k+1} \underline{\tilde{\Psi}}(y) e^{-s\underline{\phi}(y)} dy, &(6.176)\\
&= -i\omega^{k+2} \frac{k!}{(s\lambda)^{k+2}} e^{s\lambda^2} - \frac{i\omega}{s\lambda} \mathcal{R}_k(s,\lambda) + s^{-1} \int_0^\delta y^{k+1} \underline{\tilde{\Psi}}(y) e^{-s\underline{\phi}(y)} dy, \\
&= -i\omega^{k+2} \frac{k!}{(s\lambda)^{k+2}} e^{s\lambda^2} + C_1' + C_2'.
\end{aligned}$$

In line (6.176), we have applied Lemma 6.3.6, which gives us that $|\mathcal{R}_k(s,\lambda)| \leq \frac{k!}{(s\lambda)^{k+2}} e^{s\lambda^2}$. Observe

$$|C_1'| \leq \frac{1}{2} \frac{k!}{(s\lambda)^{k+3}} e^{s\lambda^2}, \tag{6.177}$$

and note with use of Lemmas 6.3.2 and 6.3.4 we get

$$\begin{aligned} |C_2'| &\leq s^{-1}\max|\partial_x\Psi|\int_0^\delta e^{-s\underline{\rho}(y)}y^{k+1}dy, \\ &\leq a_2 n^{-1}(s\lambda)^{-1}\frac{(k+1)!}{(s\lambda)^{k+2}}e^{s\lambda^2}, \\ &= 10^{-2}\frac{(k+1)}{10n}\frac{k!}{(s\lambda)^{k+3}}e^{s\lambda^2}, \\ &\leq 10^{-2}\frac{k!}{(s\lambda)^{k+3}}e^{s\lambda^2}. \end{aligned} \tag{6.178}$$

Now we handle the term A, from (6.173). Observe that we may write for some continuous functions $\tilde{\Psi}_1(x,y),\ \tilde{\Psi}_2(x,y)$, with $|\tilde{\Psi}_1|, |\tilde{\Psi}_1| \leq \max|\nabla\Psi|$,

$$\Psi(x,y) = \Psi(0,0) + x\tilde{\Psi}_1(x,y) + y\tilde{\Psi}_2(x,y), \quad \text{for } 0 \leq x,y \leq \delta. \tag{6.179}$$

Incorporating this in the expression for A, we obtain

$$\begin{aligned} A &= \int_0^\delta y^k s^{-1}\int_0^\delta \left(jx^{j-1}\left[\Psi(0,0)+x\tilde{\Psi}_1(x,y)+y\tilde{\Psi}_2(x,y)\right]+x^j\partial_x\Psi(x,y)\right)e^{-s\phi(x,y)}dxdy, \\ &= js^{-1}\Psi(0,0)\int_0^\delta\int_0^\delta e^{-s\phi(x,y)}x^{j-1}y^k dxdy \\ &\quad +s^{-1}\int_0^\delta\int_0^\delta\left(j\tilde{\Psi}_1(x,y)+\partial_x\Psi(x,y)\right)e^{-s\phi(x,y)}x^j y^k dxdy \\ &\quad +js^{-1}\int_0^\delta\int_0^\delta\tilde{\Psi}_2(x,y)e^{-s\phi(x,y)}x^{j-1}y^{k+1}dxdy, \\ &= A_0+A_1+A_2. \end{aligned} \tag{6.180}$$

Now, regarding A_0, we evaluate $\Psi(0,0)$, and apply the induction hypothesis.

$$\begin{aligned} A_0 &= js^{-1}\frac{-i\omega}{\lambda}\int_0^\delta\int_0^\delta e^{-s\phi(x,y)}x^{j-1}y^k dxdy, \\ &= \frac{j}{s\lambda}(-i\omega)\left((-i)^j\omega^{j+k+1}\frac{(j-1)!k!}{(s\lambda)^{j+k+1}}e^{s\lambda^2}+\mathcal{R}'_{j-1,k}(s,\lambda)\right), \\ &= \left(\begin{cases}(-i)^{j+1}\omega^{j+k+2}\frac{j!k!}{(s\lambda)^{j+k+2}}e^{s\lambda^2} & \text{if } j\geq 1\\ 0 & \text{if } j=0\end{cases}\right)-i\omega\frac{j}{s\lambda}\mathcal{R}'_{j-1,k}(s,\lambda), \\ &= \left(\begin{cases}A_0' & \text{if } j\geq 1\\ 0 & \text{if } j=0\end{cases}\right)+R'(s,\lambda), \end{aligned} \tag{6.181}$$

where we have defined A_0' in the obvious way, and

$$R'(s,\lambda) = -i\omega \frac{j}{s\lambda} \mathcal{R}'_{j-1,k}(s,\lambda), \tag{6.182}$$

where

$$|\mathcal{R}'_{j-1,k}(s,\lambda)| \leq \frac{(j-1)!k!}{(s\lambda)^{j+k+2}} e^{s\lambda^2}, \tag{6.183}$$

so

$$|R'(s,\lambda)| \leq \begin{cases} \frac{1}{2}\frac{j!k!}{(s\lambda)^{j+k+3}} e^{s\lambda^2} & \text{if } j \geq 1, \\ 0 & \text{if } j = 0. \end{cases} \tag{6.184}$$

For A_1, we recall $|\tilde{\Psi}_1| \leq \max|\nabla\Psi|$, and apply Lemmas 6.3.2 and 6.3.5 to find

$$\begin{aligned} |A_1| &\leq s^{-1}(j+1)\max|\nabla\Psi| \int_0^\delta \int_0^\delta e^{-s\rho(x,y)} x^j y^k dxdy, \\ &\leq s^{-1} a_2 \frac{(j+1)}{n\lambda} \left(\frac{j!k!}{(s\lambda)^{j+k+2}} e^{s\lambda^2} \right), \\ &\leq 10^{-2} \frac{j!k!}{(s\lambda)^{j+k+3}} e^{s\lambda^2}. \end{aligned} \tag{6.185}$$

For A_2, using the fact $|\tilde{\Psi}_2| \leq \max|\nabla\Psi|$ and Lemmas 6.3.2 and 6.3.5 gives

$$\begin{aligned} |A_2| &\leq s^{-1} j \max|\nabla\Psi| \int_0^\delta \int_0^\delta e^{-s\rho(x,y)} x^{j-1} y^{k+1} dxdy, \\ &\leq s^{-1} j a_2 n^{-1} \lambda^{-1} \int_0^\delta \int_0^\delta e^{-s\rho(x,y)} x^{j-1} y^{k+1} dxdy, \\ &\leq \begin{cases} 10^{-2} \frac{j!k!}{(s\lambda)^{j+k+3}} e^{s\lambda^2} & \text{if } j \geq 1, \\ 0 & \text{if } j = 0. \end{cases} \end{aligned} \tag{6.186}$$

Now we take the original expression we started with in (6.173), and write it out in terms of the

quantities we have discussed above.

$$
\begin{aligned}
\int_0^\delta \int_0^\delta e^{-s\phi(x,y)} x^j y^k dxdy &= A + B + \begin{cases} 0 & \text{if } j \geq 1 \\ C & \text{if } j = 0 \end{cases}, \\
&= A_0 + A_1 + A_2 + B + \begin{cases} 0 & \text{if } j \geq 1 \\ C & \text{if } j = 0 \end{cases}, \\
&= R'(s,\lambda) + A_1 + A_2 + B + \begin{cases} A_0' & \text{if } j \geq 1 \\ C & \text{if } j = 0 \end{cases}, \\
&= R'(s,\lambda) + A_1 + A_2 + B \qquad (6.188) \\
&\quad + (-i)^{j+1}\omega^{j+k+2}\frac{j!k!}{(s\lambda)^{j+k+2}}e^{s\lambda^2} + \begin{cases} 0 & \text{if } j \geq 1 \\ C_1' + C_2' & \text{if } j = 0 \end{cases}, \\
&= (-i)^{j+1}\omega^{j+k+2}\frac{j!k!}{(s\lambda)^{j+k+2}}e^{s\lambda^2} + \mathcal{R}'_{j,k}(s,\lambda),
\end{aligned}
\tag{6.187}
$$

where

$$
\mathcal{R}'_{j,k}(s,\lambda) := A_1 + A_2 + B + R'(s,\lambda) + \begin{cases} 0 & \text{if } j \geq 1 \\ C_1' + C_2' & \text{if } j = 0 \end{cases}. \tag{6.189}
$$

We comment that in (6.188) we have used (6.181) and (6.175), in particular. Now we collect the bounds on the various quantities composing the error term $\mathcal{R}'_{j,k}(s,\lambda)$. We have (6.185) for A_1, (6.186) for A_2, (6.174) for B, (6.184) for $R'(s,\lambda)$, (6.177) for C_1', and (6.178) for C_2', which give us

$$
\begin{aligned}
|\mathcal{R}'_{j,k}(s,\lambda)| &\leq |A_1| + |A_2| + |B| + |R'(s,\lambda)| + \begin{cases} 0 & \text{if } j \geq 1 \\ |C_1'| + |C_2'| & \text{if } j = 0 \end{cases}, \qquad (6.190) \\
&\leq 2 \cdot 10^{-2}\frac{j!k!}{(s\lambda)^{j+k+3}}e^{s\lambda^2} + \frac{1}{(s\lambda)^{3n}}e^{s\lambda^2} + \begin{cases} \frac{1}{2}\frac{j!k!}{(s\lambda)^{j+k+3}}e^{s\lambda^2} & \text{if } j \geq 1 \\ \left(\frac{1}{2} + 10^{-2}\right)\frac{k!}{(s\lambda)^{k+3}}e^{s\lambda^2} & \text{if } j = 0 \end{cases}, \\
&\leq \frac{j!k!}{(s\lambda)^{j+k+3}}e^{s\lambda^2},
\end{aligned}
$$

finishing the proof. □

Finally, with the following lemma, we get the desired asymptotics for the integral in (6.106). The

proof requires little more than an application of Lemma 6.3.7.

Lemma 6.3.8. *Consider a C^1 function $a(x,y)$ on $\mathbb{R}^2$ taking values in $\mathbb{C}$. Then for sufficiently large $s,\lambda>0$ we have*

$$\int_0^\delta\int_0^\delta e^{-s\phi(x,y)}x^{n+1}y^{n+1}a(x,y)dxdy=\frac{c_n}{(s\lambda)^{2n+4}}e^{s\lambda^2}a(0)+\mathcal{R}(s,\lambda), \tag{6.191}$$

where

$$c_n=(-1)^n\frac{((n+1)!)^2}{4^{n+2}}, \tag{6.192}$$

and we have the bound

$$|\mathcal{R}(s,\lambda)|\leq\frac{d_n}{(s\lambda)^{2n+5}}e^{s\lambda^2}\|a\|_{C^1}, \tag{6.193}$$

for some constant d_n depending only on n.

Proof. Observe that we can write $a(x,y)=a(0)+x\tilde{a}_1(x,y)+y\tilde{a}_2(x,y)$ for continuous functions $\tilde{a}_1(x,y)$ and $\tilde{a}_1(x,y)$ with $|\tilde{a}_1|,|\tilde{a}_2|\leq\max|\nabla a|$. This gives us

$$\begin{aligned}\int_0^\delta\int_0^\delta e^{-s\phi(x,y)}x^{n+1}y^{n+1}a(x,y)dxdy &= \int_0^\delta\int_0^\delta e^{-s\phi(x,y)}x^{n+1}y^{n+1} \\ &\qquad\times(a(0)+x\tilde{a}_1(x,y)+y\tilde{a}_2(x,y))\,dxdy, \\ &= a(0)\int_0^\delta\int_0^\delta e^{-s\phi(x,y)}x^{n+1}y^{n+1}dxdy+A_1+A_2.\end{aligned} \tag{6.194}$$

We then find with an application of Lemma 6.3.5 that

$$|A_1|\leq\max|\tilde{a}_1|\int_0^\delta\int_0^\delta e^{-s\rho(x,y)}x^{n+2}y^{n+1}dxdy\leq\frac{(n+2)!(n+1)!}{(s\lambda)^{2n+5}}e^{s\lambda^2}\|a\|_{C^1}. \tag{6.195}$$

Similarly

$$|A_2|\leq\frac{(n+2)!(n+1)!}{(s\lambda)^{2n+5}}e^{s\lambda^2}\|a\|_{C^1}. \tag{6.196}$$

Now, using Lemma 6.3.7 we get

$$a(0)\int_0^\delta\int_0^\delta e^{-s\phi(x,y)}x^{n+1}y^{n+1}dxdy=a(0)(-i)^{n+2}\omega^{2n+4}\frac{((n+1)!)^2}{(s\lambda)^{2n+4}}e^{s\lambda^2}+a(0)\mathcal{R}'_{n+1,n+1}(s,\lambda), \tag{6.197}$$

where

$$|a(0)\mathcal{R}'_{n+1,n+1}(s,\lambda)| \leq \|a\|_{C^1} \frac{((n+1)!)^2}{(s\lambda)^{2n+5}} e^{s\lambda^2}. \tag{6.198}$$

Adding the upper bounds for $|A_1|$ and $|A_2|$ in (6.195) and (6.196) with the bound (6.198) gives a bound on the total error as claimed in the statement of the lemma. Using (6.197) in (6.194), and recalling $\omega = \frac{i-1}{2\sqrt{2}}$, we find the coefficient of the leading term $(s\lambda)^{-2n-4}e^{s\lambda^2}$ is indeed given by

$$(-i)^{n+2}\omega^{2n+4}((n+1)!)^2 a(0) = (-i\omega)^{n+2}\omega^{n+2}((n+1)!)^2 a(0) = (-1)^n \frac{((n+1)!)^2}{4^{n+2}} a(0). \tag{6.199}$$

□

6.3.2 Proof of the main theorem

Now that we have the necessary asymptotics, we can show that the uniform analyticity assumption, Assumption 6.2.3, cannot hold for our choice of initial data. The idea is to assume it does hold, recall the estimate that must be satisfied for the transform of an analytic function discussed in Lemma 6.2.7, and show that from the decomposition provided in Proposition 6.2.8 for the transform of $\Sigma(t) := \sum_{\pm} I_*^{\pm}(t)$, i.e. $\tilde{\Sigma}(s,\lambda)$, one finds that such an estimate is impossible. This will be made evident with the use of the asymptotic formulas established in the previous subsection. Lemma 6.2.2 taken together with Lemma 6.2.7 then clearly shows $\frac{dV_1}{dt}(0,0,t)$ cannot be analytic, which means the assumption has to be false.

Theorem 6.3.9. *Consider the Vlasov-Poisson equation with initial data f_ϵ. Then the corresponding Lagrangian trajectory map $(X(\alpha,\xi,t), V(\alpha,\xi,t))$ solving the system* (6.11) *cannot satisfy Assumption 6.2.3. That is, either $X(\alpha,\xi,t)$ is not analytic at $t=0$ for some $(\alpha,\xi) \in (\mathcal{B}\times\mathcal{N}) \cup \{(0,0)\}$, or there is no positive, uniform lower bound on the radii of analyticity of $X(\alpha,\xi,t)$ as (α,ξ) varies in $(\mathcal{B}\times\mathcal{N}) \cup \{(0,0)\}$.*

Proof. Assume the statement of Assumption 6.2.3 holds when we take f_ϵ as our initial data. Observe that if we can deduce that the function $\Sigma(t) := \sum_{\pm} I_*^{\pm}(t)$ is not analytic at $t=0$, then we arrive at a contradiction by using Lemma 6.2.2, which allows us to conclude Assumption 6.2.3 does not hold. This is how we proceed with the proof. Note that if $\Sigma(t)$ were analytic at $t=0$, there would exist constants $A>0$, $\gamma>0$, and $C>0$, such that for any $s>0$, $\lambda > A$,

$$\left|\tilde{\Sigma}(s,\lambda)\right| \leq Ce^{-\gamma s}. \tag{6.200}$$

We will show that $\tilde{\Sigma}(s,\lambda)$ cannot satisfy such an estimate.

Recall that from Proposition 6.2.8 we have the decomposition

$$\tilde{\Sigma}(s,\lambda) = \int_{\mathcal{N}} I_\eta(s,\lambda)d\eta + \tilde{r}(s,\lambda), \tag{6.201}$$

with

$$I_\eta(s,\lambda) = ce^{-s\lambda^2}\int_0^\delta\int_0^\delta e^{-s(t_\eta(x,y)+i\lambda)^2}x^{n+1}y^{n+1}b_\eta(x,y)dxdy, \tag{6.202}$$

and where for some $C'>0$, $\gamma'>0$, for all sufficiently large $s,\lambda>0$,

$$|\tilde{r}(s,\lambda)| \leq Ce^{-\gamma s}. \tag{6.203}$$

Regarding the right hand side of (6.202), by using (i), (ii), and (iii) of Proposition 6.2.8, for each $\eta\in\mathcal{N}$ we can apply Lemma 6.3.7 to the integral in the expression for $I_\eta(s,\lambda)$ in the right hand side of (6.202), taking $\sigma = t_\eta$, and recalling $\phi = (\sigma+i\lambda)^2$. Thus we conclude that for each $\eta\in\mathcal{N}$, we have

$$I_\eta(s,\lambda) = ce^{-s\lambda^2}\left(\frac{c_n}{(s\lambda)^{2n+4}}e^{s\lambda^2}b_\eta(0) + \mathcal{R}_\eta(s,\lambda)\right), \tag{6.204}$$

for nonzero constants c and c_n independent of η, s, and λ, and where

$$|\mathcal{R}_\eta(s,\lambda)| \leq A\frac{d_n}{(s\lambda)^{2n+5}}e^{s\lambda^2} \tag{6.205}$$

for d_n depending only on n and $A := \sup_{\eta\in\mathcal{N}}\|b_\eta(\cdot)\|_{C^1}$.

Hence we find

$$\begin{aligned} |\tilde{\Sigma}(s,\lambda)| &= \left|\int_{\mathcal{N}} I_\eta(s,\lambda)d\eta + \tilde{r}(s,\lambda)\right|, \\ &= c'\left|\frac{c_n}{(s\lambda)^{2n+4}}\int_{\mathcal{N}} b_\eta(0)d\eta + O\left(\frac{1}{(s\lambda)^{2n+5}}\right)\right|. \end{aligned} \tag{6.206}$$

Using (iii) from Proposition 6.2.8 in the right hand side of (6.206), we conclude that we have for some constant $\tilde{c}>0$,

$$|\tilde{\Sigma}(s,\lambda)| \geq \frac{\tilde{c}}{(s\lambda)^{2n+4}}, \tag{6.207}$$

for sufficiently large s,λ. Therefore an estimate of the form (6.200) is not possible for $\tilde{\Sigma}(s,\lambda)$, and by Lemma 6.2.7, $\Sigma(t)$ cannot be analytic at $t=0$, finishing the proof. □

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