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Minimal Surfaces in Hyperbolizable 3-Manifolds

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Princeton, NJ : Princeton University

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Minimal Surfaces in Hyperbolizable 3-Manifolds

Ben Lowe

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Fernando Coda Marques

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Abstract

In the first part of this thesis we study the extent to which statements about negatively curved manifolds hold when geodesics, which are one-dimensional minimal surfaces, are replaced by two-dimensional minimal surfaces. For a certain class of negatively curved 3-manifolds M we construct minimal surfaces whose tangential lifts foliate the Grassmann bundle of tangent 2-planes to M. These foliations make it possible to use homogeneous dynamics to study how minimal surfaces in variable negative curvature are distributed in the ambient space. On the other hand, we construct negatively curved M for which there cannot exist a foliation as above.

In the second part (joint with Zeno Huang), we find new estimates for several measures of the complexity of hyperbolic 3-manifolds homeomorphic to $\Sigma \times \mathbb{R}$ in terms of the geometry of the minimal surfaces that they contain. This leads to a "gap" theorem for the geometry of minimal surfaces in hyperbolic 3-manifolds that fiber over the circle.

In the final section (joint with Andre Neves), we prove new lower bounds for the areas of certain surfaces in a hyperbolizable Riemannian 3-manifold M under a lower bound on the scalar curvature of M. Our bounds are sharp and improve the best bounds known previously by a factor of three. We also prove sharp lower bounds for an entropy functional based on asymptotic counts of minimal surfaces recently introduced in [CMN] and verify a conjecture of Gromov [Gro91] on average area-distortion of maps from M to a hyperbolic 3-manifold.

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To the memory of Keith and Betty Rupple.

And should sleep and rise night and day, and the seed should spring and grow up, he knows not how. .

The Gospel of Mark

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Chapter 1

Introduction

The interplay between the geometry of a space and the minimal surfaces it contains is a fundamental topic in geometric analysis. Minimal surfaces are critical points of the area functional and are natural objects of study from a variational perspective. Despite recent advances, the space of minimal surfaces in a general Riemannian manifold remains poorly understood, and the following questions are of great interest ([CMN],[BBEN10], [IMN18],[MNS19].)

(i) What relationships are there between local curvature properties of a space and global metric quantities associated to it: for example, what is the relationship between the scalar curvature of a Riemannian manifold and the areas of its minimal surfaces?

(ii) Can one give any qualitative description of the set of minimal surfaces in a Riemannian manifold? For example, is their union dense?

The thrust of this thesis is to use results and tools from outside of the theory of minimal surfaces and the calculus of variations– Kleinian groups, homogeneous dynamics, and the Ricci flow– to give answers to questions along these lines in the case that the ambient space is homeomorphic to a hyperbolic 3-manifold. Each chapter of this thesis contains its own introduction and can be read independently of the others. We now give a brief summary of each chapter.

In Chapter 2 we study minimal surfaces in negatively curved metrics on a closed hyperbolic 3-manifold M. Let g_t be a smooth 1-parameter family of negatively curved metrics on M starting at the hyperbolic metric. We construct foliations of the Grassmann bundle $Gr_2(M)$ of tangent 2-planes whose leaves are (lifts of) minimal surfaces in (M, g_t). These foliations are deformations of the foliation of $Gr_2(M)$ by (lifts of) totally geodesic planes projected down from the universal cover

$\mathbb{H}^3$. Our construction continues to work as long as the sum of the squares of the principal curvatures of the (projections to M) of the leaves remains pointwise smaller in magnitude than the ambient Ricci curvature in the normal direction. In the second part of the chapter we give some applications and construct negatively curved metrics for which $Gr_2(M)$ cannot admit a foliation as above. This chapter is based on the paper [Low20].

In Chapter 3, we study minimal surfaces in hyperbolic 3-manifolds homeomorphic to the product of a surface S of genus greater than one with $\mathbb{R}$. An almost Fuchsian manifold is a hyperbolic three-manifold of the type $S \times \mathbb{R}$ which admits a closed minimal surface (homeomorphic to S) with the maximum principal curvature $\lambda_0 < 1$, while a weakly almost Fuchsian manifold is of the same type but it admits a closed minimal surface with $\lambda_0 \leq 1$. We first prove that any weakly almost Fuchsian manifold is geometrically finite, and we construct a Canary-Storm type compactification for the weakly almost Fuchsian space. We use this to prove uniform upper bounds on the volume of the convex core and Hausdorff dimension for the limit sets of weakly almost Fuchsian manifolds, and to show that for every g there is an ϵ depending only on g such that if a closed hyperbolic three-manifold fibers over the circle with fiber a surface of genus g, then any embedded minimal surface isotopic to the fiber has the maximum principal curvature larger than $1+\epsilon$. We also give examples of quasi-Fuchsian manifolds which admit unique stable minimal surfaces without being weakly almost Fuchsian.

In Chapter 4, we study the relationship between scalar curvature, area, and topology in three dimensions. On any closed hyperbolizable Riemannian 3-manifold, we find a sharp relation between the minimal surface entropy (introduced by Calegari-Marques-Neves) and the average area ratio (introduced by Gromov), and we show that, among metrics g with scalar curvature greater than or equal to -6, the former is maximized by the hyperbolic metric. One corollary is to solve a conjecture of Gromov regarding the average area ratio. We also prove a sharp lower bound for the area in g of a surface homotopic to a totally geodesic surface in the hyperbolic metric. Our proofs use Ricci flow with surgery and laminar measures invariant under a $\mathrm{PSL}(2,\mathbb{R})$-action.

Chapter 2

Deformations of Totally Geodesic Foliations and Minimal Surfaces in Negatively Curved 3-Manifolds

2.1 Introduction

2.1.1 Introduction

The following statement is a special case of the geodesic rigidity theorem proved by Gromov [Gro00].

Theorem 2.1.1. *Let M be a closed hyperbolic manifold, and let N be a negatively curved Riemannian manifold homeomorphic to M. Let $G(M)$ and $G(N)$ be the foliations of the unit tangent bundles $UT(M)$ and $UT(N)$ of M and N by the orbits of the geodesic flow. Then there is a homeomorphism between $UT(M)$ and $UT(N)$ sending leaves of $G(M)$ to leaves of $G(N)$.*

In this section of the thesis, we study the extent to which a version of this theorem holds when geodesics, which are one-dimensional minimal surfaces, are replaced by two-dimensional minimal surfaces. We restrict ourselves to three ambient dimensions because minimal surfaces in that dimension are better behaved and understood.

Theorem 2.1.1 implies that many properties of the geodesic flow for an arbitrary negatively curved metric on M are controlled by the constant curvature geodesic flow. Much of our interest in trying to prove a minimal surface analogue is that it will allow us to use homogeneous dynamics to study how minimal surfaces in variable negative curvature are distributed in the ambient space.

The idea to use homogeneous dynamics in this setting is recent and due to Calegari-Marques-Neves [CMN].

If M is a closed hyperbolic 3-manifold, then the Grassmann bundle of tangent 2-planes to M has a natural foliation by immersed totally geodesic planes. Denote this foliation by $\mathcal{F}$, and let g_{hyp} be a hyperbolic (constant curvature -1) metric on M (Mostow rigidity says that there is a unique such metric up to isometry.) The following theorem is the main result of this section of the thesis.

Theorem 2.1.2. *Let $\{g_t : t \in [0,1]\}$ be a smooth 1-parameter family of negatively curved metrics on M with $g_0 = g_{hyp}$. Then there exists $T \in (0,1] \cup \{\infty\}$ such that for all $t < T$ contained in the interval $[0,1]$ there is a foliation $\mathcal{F}_t$ of $Gr_2(M)$ whose leaves are immersed minimal planes in (M, g_t) lifted to $Gr_2(M)$ by their tangent planes. Moreover, there is a self-homeomorphism Φ of $Gr_2(M)$ that sends leaves of $\mathcal{F}$ to leaves of $\mathcal{F}_t$.*

If $T < \infty$, then for every sequence $t_n \nearrow T$ there exist immersed minimal planes S_n in (M, g_{t_n}) which lift to leaves of $\mathcal{F}_{t_n}$ such that the following quantity tends to zero from below for a sequence of points $p_n \in S_n$:

$$|A_n|^2 + Ric_n(\nu_n, \nu_n). \tag{2.1.1}$$

Here ν_n is the unit normal vector to S_n at p_n, A_n is the second fundamental form of S_n at p_n, and Ric_n is the Ricci curvature tensor of g_{t_n} at p_n.

Remark 2.1.1. The same theorem should actually hold for all complete hyperbolic 3-manifolds M, with the appropriate bounded geometry condition on the family g_t. The assumption that the action of $\pi_1(M)$ on $\mathbb{H}^3$ has a compact fundamental domain makes some of the proofs simpler but does not seem to be essential. Because of the applications we have in mind, though, we restrict ourselves to the closed case.

Remark 2.1.2. In Section 2.6, we construct negatively curved metrics on certain closed hyperbolic 3-manifolds M for which there cannot exist a foliation as in Theorem 2.1.2. The analogue of the geodesic rigidity of Theorem 2.1.1 therefore in general only holds in our setting in some neighborhood of the hyperbolic metric.

In words, our construction of the foliations $\mathcal{F}_t$ continues to work as long as the sum of the squares of the principal curvatures (of the projections to M) of the leaves of our foliations remains pointwise less than the absolute value of the ambient Ricci curvature in the normal direction. The proof of Theorem 2.1.2 occupies Section 2.3, where we prove a more intrinsic formulation of it (Theorem 2.3.2), and then modify that proof to give a proof of Theorem 2.1.2. The proof is loosely speaking

a method of continuity argument, where we work in the universal cover and follow the approach of Anderson [And83] to construct properly embedded minimal planes. Surfaces for which the quantity (2.1.1) is negative have small mean-convex neighborhoods, which we use to rule out the existence of minimal planes other than the ones from our construction. These would lead to gaps in the foliations we are trying to construct.

In [Gro91a], Gromov proved a stability result for the totally geodesic foliation $\mathcal{F}$ that applies to metrics g with sectional curvatures pinched close to -1. For these metrics, he constructs an immersed almost-totally-geodesic g-minimal plane in M for each leaf of $\mathcal{F}$. His construction also follows [And83] but, in contrast to the results here, works for closed hyperbolic manifolds of all dimensions and is based on Allard's regularity theorem. This section of the thesis grew out of attempts to find a more direct proof of Gromov's results in dimension 3.

2.1.2 Almost-Fuchsian Manifolds

Theorem 2.1.2 was motivated by the theory of almost-Fuchsian manifolds. A homeomorphism $f : S^2 \to S^2$ is *K-quasiconformal* if for any ball $B(x,r) \subset S^2$ there exists $r' > 0$ such that $B(f(x),r') \subset f(B(x,r)) \subset B(f(x),Kr')$. Here $B(x,s)$ denotes the ball centered at x of radius s in the round metric on S^2. The homeomorphism f is *quasiconformal* if it is K-quasiconformal for some K. For Σ a surface of genus greater than one, a hyperbolic metric on $\Sigma \times \mathbb{R}$ is given by a properly discontinuous action of $\pi_1(\Sigma)$ on $\mathbb{H}^3$ by isometries. The *limit set* of a hyperbolic metric on $\Sigma \times \mathbb{R}$ is the set of accumulation points of any orbit under this action in the boundary at infinity $\partial_\infty \mathbb{H}^3 \cong S^2$. We say that a hyperbolic metric on $\Sigma \times \mathbb{R}$ is *quasi-Fuchsian* if its limit set is the image of the equator under a quasiconformal homeomorphism of S^2. The space of all quasi-Fuchsian metrics is parametrized by a product of Teichmuller spaces which correspond to the conformal structures on the two ends.

In [Uhl83], Uhlenbeck proved a rigidity theorem for quasi-Fuchsian manifolds which admit an embedded minimal surface with principal curvatures less than 1 in magnitude: that such a quasi-Fuchsian manifold is uniquely determined by the conformal class of the induced metric on this minimal surface and a quadratic differential equivalent to its second fundamental form. These manifolds are called *almost-Fuchsian*, and have been well studied since ([HW13], [HW15], [San17a], [Sep16], [Tau04] .)

The set of metrics to which Theorem 2.1.2 applies inside the space of all negatively curved metrics on M is analogous to the set of almost-Fuchsian metrics on $\Sigma \times \mathbb{R}$ inside the space of quasi-Fuchsian

metrics, insofar as the existence of minimal surfaces with curvatures bounded by ambient curvatures allows for much greater control. This makes it possible, for instance, to prove uniqueness statements for the minimal surfaces in question. Guided by this analogy, in Section 2.6 we construct negatively curved metrics for which foliations as in the statement of Theorem 2.1.2 cannot possibly exist.

2.1.3 Applications

We now describe some applications. Principal among these is the following density result for stable properly immersed minimal surfaces in a closed Riemannian 3-manifold M which admits a foliation as in Theorem 2.1.2. Kahn and Markovic showed that for every closed hyperbolic 3-manifold M, subgroups of $\pi_1(M)$ isomorphic to the fundamental group of a closed surface, or *surface subgroups*, exist in great profusion ([KM12b],[KM12a].) (See also [Ham15], which gives a more geometric version of Kahn-Markovic's construction and generalizes their results to cocompact lattices in all rank one symmetric spaces except for hyperbolic spaces of even dimension.) Fixing a metric g on M, each of these surface subgroups gives rise by [SU82] or [SY79] to a stable properly immersed minimal surface whose fundamental group includes as a subgroup of $\pi_1(M)$ conjugate to that surface subgroup.

Let C be a circle in $\partial_\infty \mathbb{H}^3 \cong S^2$ such that the geodesic plane P in $\mathbb{H}^3$ with limit set C has dense projection to the closed hyperbolic 3-manifold M under the universal covering map. Ratner and Shah independently proved that every geodesic plane P either projects to a dense subset of M whose tangent planes are dense in $Gr_2(M)$ or a closed properly immersed surface ([Rat91],[Sha91].) (See also [MMO17] for a nice proof of this fact.) Let Γ_n be a sequence of surface subgroups of $\pi_1(M)$ with limit sets K_n-quasicircles Hausdorff converging to C with K_n tending to 1. (A K-quasicircle is the image of a round circle under a K-quasiconformal self-homeomorphism of S^2.) The existence of such a sequence of Γ_n for each C follows from [KM12b]. Let Σ_n be a sequence of stable immersed minimal surfaces in (M, g) whose fundamental groups include to the conjugacy classes of the Γ_n.

Theorem 2.1.3. *Suppose that g can be joined to g_{hyp} by a smooth family of negatively curved metrics parametrized by $[0, 1]$ to which Theorem 2.1.2 applies with $T = \infty$. Let Σ_n be a sequence as above. Then for every open set U in $Gr_2(M)$ there exists a number N so that Σ_n has a tangent plane in U for every $n > N$.*

Remark 2.1.3. This is a slightly stronger statement than simply that the tangent planes of all closed stable immersed minimal surfaces are dense in $Gr_2(M)$, which could be obtained without using the Ratner-Shah theorem mentioned above.

A natural question is whether a similar density result is true for all negatively curved metrics

on M. For negatively curved metrics on M which cannot admit foliations as in Theorem 2.1.2, like those constructed in Section 2.6 we believe it is possible that a sequence of Σ_n as above might fail to be dense in $Gr_2(M)$.

If $g = g_{hyp}$, it follows from [Sep16] that if the limit set of Γ_n is a K-quasicircle for K sufficiently close to 1, then Σ_n is the unique minimal surface whose fundamental group injectively includes as a subgroup conjugate to Γ_n. In Section 2.4, we prove a similar uniqueness result for g to which Theorem 2.1.2 applies to produce a foliation:

Theorem 2.1.4. *Suppose that g satisfies the hypotheses of the previous theorem. Then there exists $\delta > 0$ such that the following is true. Suppose the limit set of a surface subgroup Γ of $\pi_1(M)$ in $\partial_\infty \mathbb{H}^3$ is a K-quasicircle for $K < 1+\delta$. Then there is a unique stable g-minimal surface in M whose fundamental group injectively includes in $\pi_1(M)$ as a subgroup conjugate to Γ.*

2.1.4 Quantitative Density

Let Σ'_n be a sequence of minimal surfaces in (M, g) with areas tending to infinity such that the limit sets of the $\pi_1(\Sigma_n)$ in $\partial_\infty \mathbb{H}^3$ are K_n-quasicircles with K_n tending to 1. Let Σ_n be the corresponding sequence of minimal surfaces in the hyperbolic metric on M.

Let μ'_n be the probability measures on $Gr_2(M)$ that correspond to averaging over the lift of Σ'_n to $Gr_2(M)$ in the area form for the metric on Σ'_n induced by (M, g). Let μ_n be the corresponding measures for the Σ_n. We are able to prove the following quantitative version of Theorem 2.1.3.

Theorem 2.1.5. *Let g satisfy the assumptions of Theorem 2.1.3. Suppose that the μ_n weak-$*$ converge to the uniform measure on $Gr_2(M, g_{hyp})$. Then the μ'_n weak-$*$ converge to a measure μ_g on $Gr_2(M, g)$ with full support.*

If $\mathcal{F}_g$ is the foliation from Theorem 2.1.2, then the measure μ_g is the sum of a transverse invariant measure for $\mathcal{F}_g$ and the Riemannian area forms for the metrics on the leaves of $\mathcal{F}_g$ induced by their projections to (M, g)— see Section 2.5 for more details.

Remark 2.1.4. Mozes-Shah [MS95] proved that any sequence of totally geodesic Σ_n in (M, g_{hyp}) with area tending to infinity becomes uniformly distributed in $Gr_2(M)$. Theorem 2.1.5 therefore applies when the Σ_n are all totally geodesic. In this case the Σ'_n are (projections of) leaves of our foliation.

In Theorem 2.5.4 below we prove that the μ_n always converge to the uniform measure provided that M contains no properly immersed totally geodesic surfaces in its hyperbolic metric. The

condition that M have no totally geodesic surfaces is a no-closed-orbits assumption for the action of $PSL(2,\mathbb{R})$ on the frame bundle of M. It is analogous to a unique ergodicity assumption for (one-dimensional) dynamical systems. For uniquely ergodic dynamical systems on a compact space, a simple argument shows that the equidistribution ergodic theorem holds for *all* space averages, not just almost all [Esk10, Proposition 1.9]. We use a similar argument, together with Ratner's measure classification theorem, to prove that geodesic disks in $Gr_2((M, g_{hyp}))$ are becoming uniformly distributed at some rate uniform in the radii of the disks. We then locally approximate the g_{hyp}-minimal surfaces Σ_n by large totally geodesic disks to show that these surfaces are becoming uniformly distributed.

We expect that the μ_n converge to the uniform measure with no assumptions on M. It seems likely, for example, that the probability measures corresponding to a sequence of minimal surfaces Σ_n^{hyp} that injectively include to the surface subgroups that come from Hamenstadt's version [Ham15] of the Kahn-Markovic construction are becoming uniformly distributed, but we do not verify this here.

2.1.5 Non-Existence of Foliations as in Theorem 2.1.2

We now describe the construction of negatively curved metrics to which Theorem 2.1.2 cannot apply to produce a foliation. It is based on the existence of quasi-Fuchsian manifolds Q which contain several distinct embedded minimal surfaces whose inclusions are homotopy equivalences (this contrasts with the almost-Fuchsian case, where it was shown in [Uhl83] that there exists a unique such minimal surface.)

We start out with a closed hyperbolic 3-manifold M that contains an embedded totally geodesic surface. By passing to a finite cover which we also denote by M, we can make the totally geodesic surface Σ have arbitrarily large normal injectivity radius. Taking the Fuchsian cover F corresponding to Σ, which is homeomorphic to $\Sigma \times \mathbb{R}$, we modify the pulled-back metric while preserving negative curvature, so that we can cut out the middle of F and glue in the convex core of a quasi-Fuchsian Q with multiple distinct minimal surfaces. To accomplish this, we also need to modify the metric on Q near infinity, which we do using the fact that a quasi-Fuchsian metric on $\Sigma \times \mathbb{R}$, whatever disorderly behavior is happening in the middle, has a standard form near the ends. Provided the normal injectivity radius of Σ was taken large enough, the gluing can be performed inside M itself. This produces a negatively curved metric g on M for which there are multiple stable minimal surfaces isotopic to Σ, which is incompatible with the existence of a foliation as in Theorem 2.1.2.

We expect that this metric can be joined to the constant curvature metric through a smooth path of metrics with negative sectional curvature by performing the above construction on a smooth path in quasi-Fuchsian space joining F to Q. This would show that the case $T < \infty$ of Theorem 2.1.2 actually occurs. It would be good to find a more robust way of ruling out the existence of foliations as in Theorem 2.1.2— for instance, one that worked for all closed hyperbolic 3-manifolds.

2.1.6 Stability for the Foliations of Theorem 2.1.2

In the final section, we give an estimate for how fast the principal curvatures (of the projections to M) of the leaves of the foliations $\mathcal{F}_t$ are changing as the metrics g_t vary. The bound we obtain depends on the size of the principal curvatures of the (projections to M) of the leaves of the foliation compared to the ambient Ricci curvature, as well as bounds on the g_t and their derivatives in time.

2.1.7 Related Work

We now discuss some results related to this section of the thesis. Density and equidistribution theorems for minimal hypersurfaces produced by the Almgren-Pitts min-max theory have been obtained for generic metrics by Irie-Marques-Neves [IMN18] and Marques-Neves-Song [MNS19]. Recently Song-Zhou [SZ20] showed that for generic metrics sequences of minimal hypersurfaces can "scar" along stable minimal hypersurfaces, for example the ones considered in this thesis. The proofs of the above results are based on the Liokumovic-Marques-Neves Weyl law for the Almgren-Pitts volume spectrum [LMN18]. Ambrozio-Montezuma [AM18] also proved equidistribution results, by a somewhat different approach, for minimal surfaces in metrics on the round 3-sphere that are local maxima for the Simon-Smith width within their conformal class. In contrast to the minimal surfaces considered in this thesis, the minimal surfaces of most of the results mentioned in this paragraph are embedded and one expects them in general not to be local minima for the area functional.

Recent work of Calegari-Marques-Neves [CMN] considered minimal surfaces corresponding to the Kahn-Markovic surface subgroups from a dynamical perspective. Given a closed hyperbolic 3-manifold, they define a functional on Riemannian metrics on that 3-manifold with sectional curvature at most -1 based on a renormalized count of stable properly immersed minimal surfaces with limit sets close to circles, and show that the constant curvature hyperbolic metric uniquely minimizes this functional. The proof of the rigidity part of their result— that the constant curvature metric *uniquely* minimizes the counting functional— uses the Ratner-Shah theorem mentioned earlier. This part of the thesis was inspired by and draws substantially from their ideas, especially Sections 2.4

and 2.5.

In Section 2.3, we produce the leaves of the foliations of Theorem 2.1.2 by solving specific asymptotic Plateau problems in $\tilde{M}$, and arguing that the solutions are unique. The asymptotic Plateau problem in $\mathbb{H}^n$ for suitable boundary data at infinity was solved by Anderson [And83], and in simply connected Riemannian manifolds bi-Lipschitz equivalent to a metric with pinched negative sectional curvature by Bangert and Lang [BL96]. Using their results might shorten our proofs a little, but we prefer to construct the solutions to our asymptotic Plateau problems by hand. The main point for us is controlling the solutions to the relevant asymptotic Plateau problems as the metric varies and proving that the solutions are unique.

By Theorem 2.1.5 and Remark 2.1.4, for any infinite sequence of closed leaves of our foliations with areas tending to infinity (these correspond to properly immersed totally geodesic surfaces in the hyperbolic metric) the sequence of probability measures μ_n' on $Gr_2(M)$ that they determine are weak-$*$ converging to the measure μ_g of Theorem 2.1.5. One wonders what can be said about this measure in general. Its regularity depends on the regularity of the conjugating map Φ in directions transverse to the leaves.

The ergodic theory of foliations with negatively curved leaves has been studied ([Alv18], [Wal88], [Zim82].) In [Alv18], Alvarez considers certain foliations, transverse to the fibers of $\mathbb{CP}^1$ bundles over a negatively curved surface, that arise from actions of the fundamental group of the surface on $\mathbb{CP}^1$. He shows that there is a unique probability measure to which metric disks tangent to the leaves and with radii tending to infinity converge, and that this measure is singular with respect to other measures natural to the dynamics of the foliation unless the surface in the construction had constant negative curvature. It would be interesting to determine whether the story is similar for the foliations considered here.

2.2 Outline

In Section 2.3 we prove Theorem 2.1.2. In Section 2.4 we apply Theorem 2.1.2 to prove density and uniqueness results for stable properly immersed minimal surfaces in M. In Section 2.5 we prove some quantitative versions of the density results of Section 2.4 under the assumption that M has no proper totally geodesic surfaces in its hyperbolic metric. In Section 2.7 we give an estimate for how fast the principal curvatures of the leaves of the foliations of Theorem 2.1.2 are changing as the metric varies.

2.3 Construction of the Foliations

In this section we prove Theorem 2.1.2. Fix a closed hyperbolic 3-manifold M, and denote by g_{hyp} the hyperbolic metric on M. Let $\mathcal{P}$ be the set of totally geodesic planes in $\mathbb{H}^3$. By taking limit sets, there is a bijection between $\mathcal{P}$ and the set of round circles in $S^2 \cong \partial_\infty \mathbb{H}^3$. The lifts to $Gr_2(M)$ by their tangent planes of the projections of elements of $\mathcal{P}$ under the covering map are the leaves of a foliation of $Gr_2(M)$, which we denote by $\mathcal{F}$.

Let g be a metric on M. Then there is an identification between the universal cover $(\tilde{M}, \tilde{g})$ of (M, g) and the universal cover $\mathbb{H}^3$ of (M, g_{hyp}), which is well-defined up to composing with covering transformations of $\mathbb{H}^3$. Since elements of the set $\mathcal{P}$ are invariant under covering transformations, taking the images of elements of $\mathcal{P}$ under such an identification gives a well-defined set of embedded planes in $(\tilde{M}, \tilde{g})$, which we denote by $\mathcal{P}_{\tilde{g}}$.

Definition 2.3.1. Let g be a metric on M with negative sectional curvature. Consider the universal cover $(\tilde{M}, \tilde{g})$ with the metric induced by g. We say that an embedded surface Σ in $(\tilde{M}, \tilde{g})$ is ϵ-*subordinate* if it satisfies the following for every $p \in \Sigma$. Let ν be the unit normal vector to Σ at p and let A be the second fundamental form of Σ. Then

$$|A(p)|^2 < |Ric(\nu, \nu)| - \epsilon. \tag{2.3.1}$$

We say that g is in Ω_ϵ if it has negative sectional curvature and there is some $\epsilon > 0$ such that for every $P \in \mathcal{P}_{\tilde{g}}$, there is a properly embedded ϵ-subordinate minimal plane in $(\tilde{M}, \tilde{g})$ at finite Hausdorff distance from P.

Remark 2.3.1. For a plane $P \in \mathcal{P}$, it will either be the case that the (lift to $Gr_2(M)$ of the) projection of P to (M, g_{hyp}) is dense in $Gr_2(M)$, or closes up to a properly immersed surface ([Rat91], [Sha91].) Take a plane P with dense projection to $Gr_2(M)$. Then, for g a negatively curved metric on M, if there is an ϵ-subordinate minimal plane S in $(\tilde{M}, \tilde{g})$ at finite Hausdorff distance from P for some $\epsilon > 0$, then $g \in \Omega_{\epsilon'}$ for any $\epsilon' < \epsilon$. That is, it suffices to check Equation (2.3.1) on a single embedded plane corresponding to an element of $\mathcal{P}$ with dense projection to verify membership in $\Omega_{\epsilon'}$ for $\epsilon' < \epsilon$. This can be seen by approximating any $P' \in \mathcal{P}$ by orbits of P under covering transformations, taking the corresponding sequence of minimal surfaces in $(\tilde{M}, \tilde{g})$, and passing to a smooth subsequential limit as in the proof of Theorem 2.3.2 below to produce a minimal plane satisfying Equation (2.3.1) for any $\epsilon' < \epsilon$.

Presumably Ω_ϵ for ϵ close to zero contains more metrics than just those with sectional curvatures

extremely close to -1. It would be nice to have a better understanding of which metrics are contained in Ω_ϵ for ϵ small.

We now prove the main theorem of the section. It does not quite imply Theorem 2.1.2, but we will explain at the end how the proof can be modified to give a proof of Theorem 2.1.2.

We say in this thesis that a minimal surface is *absolutely minimizing* if, for every piecewise-differentiable closed curve on the surface that bounds a disk D on the surface, the area of D is less than or equal to that of any other smoothly embedded disk in the ambient space bounding ∂D. We say that a minimal surface is *κ-locally-minimizing* if the previous statement holds for every piecewise-differentiable closed curve on the surface that bounds a disk D on the surface that is contained in a ball of radius κ in the ambient metric.

Theorem 2.3.2. *Fix $\kappa > 0$, and let $\{g_t : t \in [0,1]\}$ be a smooth 1-parameter family of metrics on M^3, with $g_0 = g_{hyp}$ and $g_t \in \Omega_\epsilon$ for some fixed $\epsilon > 0$ and all t. Then there exists a constant C depending only on the family of metrics such that for all $P \in \mathcal{P}_{\tilde{g}_t}$ there exists a properly embedded minimal plane S_t in the universal cover $(\tilde{M}, \tilde{g}_t)$ at a Hausdorff distance from P of at most C, and that has the following properties:*

- *S_t is the unique properly embedded κ-locally-minimizing minimal plane at finite Hausdorff distance from P.*
- *S_t is absolutely minimizing.*
- *The lifts of the S_t to $Gr_2(\tilde{M})$ by their tangent planes are the leaves of a foliation $\tilde{\mathcal{F}}_t$ of $Gr_2(\tilde{M})$.*

The diffeomorphisms of $Gr_2(\tilde{M})$ induced by covering transformations of $\tilde{M}$ send leaves to leaves, and $\tilde{\mathcal{F}}_t$ thus descends to a foliation $\mathcal{F}_t$ of $Gr_2(M)$. Moreover, $\mathcal{F}_t$ and $\mathcal{F}$ are conjugate, in that there is a homeomorphism

$$\Phi : Gr_2((M, g_{hyp})) \to Gr_2((M, g_t))$$

that maps leaves of $\mathcal{F}$ to leaves of $\mathcal{F}_t$.

A 1-parameter family of Riemannian metrics as in the theorem gives a map

$$M \times [0,1] \to Sym^2(T^*M).$$

We say that the family of metrics is smooth if this map is smooth.

Let $\{g_t : t \in [0,1]\}$ be a smooth 1-parameter family of metrics on M^3, with $g_0 = g_{hyp}$ and $g_t \in \Omega_\epsilon$ for all t. We will prove Theorem 2.3.2 by a finite induction. Suppose that g_{t_0} satisfies the

conclusion of the theorem for some t_0. In the proof, we will use the following two properties of the S_t at $t = t_0$, the first of which we will assume at t_0 and verify in the inductive step and the second of which follows from the existence of the conjugating map Φ in the theorem and standard elliptic PDE theory.

Property 1 Suppose lifts of S_t and S'_t are leaves of $\tilde{\mathcal{F}}_t$ that correspond to totally geodesic planes S and S' in $\mathcal{P}$. If S and S' have disjoint boundary circles at infinity, then S_t and S'_t are disjoint.

Property 2 If S_n is a sequence of totally geodesic planes that converges to S on compact subsets, then the corresponding sequence of minimal planes in $(\tilde{M}, \tilde{g}_t)$ smoothly converges, uniformly on compact subsets, to the minimal plane in $(\tilde{M}, \tilde{g}_t)$ corresponding to S.

2.3.1 Outline

We carry out the induction in three steps. First we construct the minimal planes S_t in the universal cover as limits of solutions to Plateau problems for a sequence of circles going off to infinity, roughly following the approach introduced by [And83] to solving the asymptotic Plateau problem in negative curvature. Next, using the existence of mean-convex tubular neighborhoods of the S_{t_0} guaranteed by g_{t_0}'s membership in Ω_ϵ, we prove that the S_t are unique. Finally, based on the strong restrictions on how minimal surfaces can intersect in three dimensions, we prove that the lifts of the S_t to $Gr_2(\tilde{M})$ by their tangent planes give a foliation of $Gr_2(\tilde{M})$.

2.3.2 Mean-Convex Neighborhoods

Let S_t be an ϵ-subordinate properly embedded minimal disk in $(\tilde{M}, \tilde{g}_t)$ as in the statement of the theorem. Along any normal geodesic ray γ from S_t parametrized by arc-length and within the normal injectivity radius, the signed mean curvatures m of the parallel surfaces satisfy the following equation:

$$m'((\gamma(s)) = -|A(\gamma(s))|^2 - \mathrm{Ric}(\dot{\gamma}(s), \dot{\gamma}(s)), \tag{2.3.2}$$

where $A(\gamma(s))$ is the second fundamental form of the signed-distance-s parallel surface at $\gamma(s)$. This can be obtained by taking the trace of Equation (2) in Proposition 3.2.11 of [Pet16]. (See also Gromov's survey article [Gro91c] which develops some of the themes of modern geometry through this and related "tube-formulas.") The next lemma will be used at several points below.

Lemma 2.3.2. There is some ξ depending only on ϵ and the family g_t such that the parallel signed distance-r surfaces of the S_t have mean curvature greater than $\frac{\epsilon}{2}r$ if $0 < r < \xi$ and less than $\frac{\epsilon}{2}r$ if

$-\xi < r < 0$.

Proof. First note that we have uniform bounds on the L^∞ norm of the Ricci curvature tensor over all g_t. Since S_t is ϵ-subordinate, there is thus by (2.3.1) a uniform bound on the magnitude of its second fundamental form, depending only on ϵ and bounds on the g_t. The upper bound on the magnitude of the second fundamental form and the uniform bounds on $\tilde{g}_t$ and its derivatives imply a lower bound on the normal injectivity radius of S_t uniform over all S_t— i.e., a lower bound for an ϵ' such that the normal exponential map on the normal bundle to S_t is injective restricted to $S_t \times (-\epsilon', \epsilon')$.

Now suppose the statement of the lemma were false, and let $\{t_n\}$, $\{r_n\}$, and $\{x_n\}$ be sequences of times, signed-distances, and points on S_{t_n} such that:

- the distance-r_n surface to S_{t_n} has mean curvature less than $\frac{\epsilon}{2} r_n$ if $r_n > 0$ or greater than $\frac{\epsilon}{2} r_n$ if $r_n < 0$ at the point that normally projects to x_n
- $|r_n| \to 0$
- $\{t_n\}$ converges to some time t (where we've passed to a subsequence if necessary.)

Let K be a compact set containing a fundamental domain for the action of $\pi_1(M)$ on $\tilde{M}$, and for each x_n, let γ_n be a covering transformation of $\tilde{M}$ such that $\gamma_n \cdot x_n \in K$. By passing to a subsequence we can assume that $\gamma_n \cdot x_n$ converges to x. By the uniform bound on the second fundamental forms of the S_{t_n}, we can pass to a subsequence of the $\gamma_n \cdot S_{t_n}$ that graphically converges (and thus, by standard elliptic PDE theory, smoothly converges) in a neighborhood of x to a $\tilde{g}_t$-minimal disk D containing x. Since this disk inherits the property of being ϵ-subordinate from the $\gamma_n \cdot S_{t_n}$ of which it was a smooth limit, Equation (2.3.2) implies that the derivative of the signed mean curvatures of the parallel distance-r surfaces to D at $r = 0$ is greater than ϵ at every point in the interior of D.

By passing to a subsequence, we can assume that all of the r_n are either positive or negative; the argument is the same in both cases so assume that all are positive. By the mean value theorem there is a sequence of r_n' such that the derivative of the mean curvature of the parallel distance-r surfaces to $\gamma_n \cdot S_{t_n}$ along the normal geodesic to $\gamma_n \cdot x_n$ is less than $\epsilon/2$ at the distance-r_n' surface, where $0 < r_n' < r_n$. Since neighborhoods of $\gamma_n \cdot x_n$ in $\gamma_n \cdot S_{t_n}$ are smoothly converging to D, their parallel distance-r surfaces are smoothly converging to those of D. This implies that the derivative of the mean curvature of the parallel distance-r surfaces to D along the normal geodesic at x is less than or equal to $\epsilon/2$ at $r = 0$, which contradicts the previous paragraph.

□

Let $S^r_{t_0}$ be the parallel surface at signed-distance r from S_{t_0}. Then by the previous lemma for δ sufficiently small (and independent of t_0), $S^r_{t_0}$ will remain mean-convex when considered as a surface inside $(\tilde{M}, \tilde{g}_t)$, for $t \in (t_0, t_0 + \delta)$ and $\frac{\xi}{4} < |r| < \xi$. For $t \in (t_0, t_0 + \delta)$, we now construct the S_t. At several junctures below, we will put further restrictions on the size of δ that only depend on ϵ and the family of metrics.

2.3.3 Controlled Solutions to Plateau Problems

Fix a point p on S_{t_0} and let $B(s)$ be the metric disk in S_{t_0} of radius s centered at p, where S_{t_0} has the metric induced from $\tilde{g}_{t_0}$. Because S_{t_0} is minimal and $\tilde{g}_{t_0}$ has negative sectional curvature, S_{t_0} has negative curvature and the exponential map $T_pS_{t_0} \to S_{t_0}$ is a diffeomorphism. The boundaries $\partial B(s)$ are therefore embedded circles, and so we can solve the Plateau problem for $\partial B(s)$ in $(\tilde{M}, \tilde{g}_t)$ to find an embedded $\tilde{g}_t$- minimal disk $D(s)$ that bounds $\partial B(s)$, such that every other embedded disk bounding $\partial B(s)$ has area greater than or equal to that of $D(s)$ [CM11, Chapter 4].

Lemma 2.3.3. *$D(s)$ is contained in the region bounded by $S^r_{t_0}$ and $S^{-r}_{t_0}$ for $\xi/4 < r < \xi$.*

We first prove another lemma. Let S be the geodesic plane in $\mathbb{H}^3$ that corresponds to S_{t_0}. The circles in S^2 parallel to the boundary at infinity of S form a foliation of $\partial_\infty \mathbb{H}^3 \cong S^2$ minus two points. Let $\{S(x) : x \in \mathbb{R}\}$ be the foliation of $\mathbb{H}^3$ by totally geodesic planes whose limit sets are the circles in S^2 parallel to $\partial_\infty S$, parametrized so that $S(0) = S$, and let $S_{t_0}(x)$ be the corresponding minimal planes in $(\tilde{M}, \tilde{g}_{t_0})$.

Lemma 2.3.4. *The $S_{t_0}(x)$ are the leaves of a foliation of $\tilde{M}$.*

Proof. First of all by Property 1 the $S_{t_0}(x)$ are disjoint. Suppose for contradiction that there is some point q that is not contained in any $S_{t_0}(x)$. Let x^+_q be the infimum of the set of all x such that $S_{t_0}(x)$ is above q. This set is nonempty since the $S_{t_0}(x)$ are at uniformly bounded Hausdorff distance from the $S(x)$ considered as subspaces of $(\tilde{M}, \tilde{g}_t)$, so x^+_q is well-defined. If $S_{t_0}(x^+_q)$ did not contain q, then since the $S_{t_0}(x)$ vary smoothly by Property 2 above, it must be above q and we could therefore produce $S_{t_0}(x)$ above q with $x < x^+_q$ for a contradiction. It follows that q is contained in $S_{t_0}(x^+_q)$. The existence of local product charts follows from Property 2, and so the $S_{t_0}(x)$ are the leaves of a foliation.

□

We now give the proof of Lemma 2.3.3.

Proof. Suppose for contradiction that $D(s)$ is not contained in the region bounded by $S^r_{t_0}$ and $S^{-r}_{t_0}$. Then either $D(s)$ has points above $S^r_{t_0}$ or below $S^{-r}_{t_0}$— assume that the first is true since the proof in the second case is the same. Note that the parallel signed-distance-r surfaces $S^r_{t_0}(x)$ for the $S_{t_0}(x)$ in $(\tilde{M}, \tilde{g_{t_0}})$ are mean-convex in $(\tilde{M}, \tilde{g}_t)$ since $\frac{\xi}{4} < r < \xi$ and are the leaves of a foliation of $\tilde{M}$ for fixed r, since the $S_{t_0}(x)$ are the leaves of a foliation of $\tilde{M}$ by the previous lemma. It follows that the set of $x > 0$ such that $S^r_{t_0}(x)$ intersects $D(s)$ is non-empty.

Furthermore, since the $S^r_{t_0}(x)$ are at uniformly bounded Hausdorff distance from the $S(x)$ considered inside $\{\tilde{M}, \tilde{g}_t\}$, for x sufficiently large the intersection of $S^r_{t_0}(x)$ and $D(s)$ will be empty. Let x' be the largest x such that $S^r_{t_0}(x)$ intersects $D(s)$. Then since $D(s)$ is contained on one side of $S^r_{t_0}(x')$ except at the non-empty set of points where they intersect, the mean convexity of $S^r_{t_0}(x')$ gives a contradiction.

□

The next lemma follows from Schoen's curvature estimate for stable minimal surfaces [Sch83, Theorem 3].

Lemma 2.3.5. Let $\rho > 0$ be less than the injectivity radius of any (M, g_t), and let x be a point on a stable $\tilde{g}_t$-minimal embedded disk $D \subset \tilde{M}$ at a $\tilde{g}_t$-distance of at least d from the boundary of D. Then there is some constant C such that the L^∞-norm of the second fundamental form of D at p is bounded above by C. The constant C depends only on ρ, d, and bounds on the g_t (specifically, bounds on the L^∞ norms of the curvature tensor and its covariant derivative.)

We require two more lemmas to get the control on the $D(s)$ we will need to pass to a limit.

Lemma 2.3.6. Let $\eta > 0$, C, and the family g_t be given. Then there exists $\eta' > 0$ such that for any t and any two embedded surfaces S_1 and S_2 in $(\tilde{M}, \tilde{g}_t)$ with second fundamental forms bounded above by C in magnitude, the following holds. Suppose that there are points $x_1 \in S_1$ and $x_2 \in S_2$ such that $d_{\tilde{g}_t}(x_1, x_2) < \eta'$, but the distance between the unit normal vectors to S_1 and S_2 at x_1 and x_2 respectively in the unit tangent bundle to $\tilde{M}$ is at least η. Then S_1 and S_2 intersect.

Proof. Assume that the statement of the lemma fails for some $\eta > 0$, and let $\{S^n_i : i = 1, 2\}$ be a sequence of pairwise disjoint surfaces in $(\tilde{M}, \tilde{g}_{t_n})$ as above with points x^n_i such that $d(x^n_1, x^n_2) \to 0$ and the distance between the respective normal vectors at x^n_1 and x^n_2 is at least η for all n.

Fix some $\rho > 0$. For each n, identify by the exponential map the ball of radius ρ at x^n_1 with the ball of that radius in the tangent space to x^n_1, and radially dilate the pulled back metric so that the distance between the origin and dilated image of the point corresponding to x^n_2 in the dilated metric

is 1. Call the dilated metric h_n. For what follows, we fix isometric identifications of the tangent spaces $T_{x_1^n}S_1^n$ in the inner product induced by $\tilde{g}_{t_n}$, for the purpose of taking limits.

Because $d(x_1^n, x_2^n) \to 0$ and the $\tilde{g}_t$ are in a smooth family over $[0,1]$, the h_n-balls of any given radius centered at the origin are smoothly converging to Euclidean balls of that radius. Moreover, since we have a uniform bound on the second fundamental forms of the S_i^n, the intersections of their pre-images with the h_n-balls of any given radius centered at the origin are uniformly C^1-converging to planes, up to taking subsequences.

In the case of the S_1^n, this plane will simply be a subsequential limit of the tangent planes to S_1^n at x_1^n. Let Π_n be the parallel transport of the tangent plane to S_2^n at x_2^n along the geodesic joining x_1^n to x_2^n. Then in the case of S_2^n, the subsequential limit plane will be a translated copy of a subsequential limit of the Π_n. The fact that for all n the normals at x_1^n and x_2^n are at a distance of at least η implies that these two planes cannot be parallel. This means that S_1^n and S_2^n will have to intersect for some large n, a contradiction.

□

We now truncate the $D(s)$ to obtain disks better suited to taking limits. Since $B(s)$ is a disk bounding $\partial B(s)$ that intersects γ exactly once, we know that $D(s)$ intersects γ at least once, at some point p_s on γ between $\gamma(-r)$ and $\gamma(r)$ by Lemma 2.3.3. Let $\sigma(s)$ be the smallest number σ such that the $\tilde{g}_t$-ball $B_{p_s}(\sigma)$ of radius σ centered at p_s intersects $\partial D(s)$. Since $\partial D(s)$ is the boundary of an intrinsic metric disk in S_{t_0}, and S_{t_0} is properly embedded, it must be the case that $\sigma(s) \to \infty$ as $s \to \infty$.

For generic $\sigma < \sigma(s)$, the boundary of the $\tilde{g}_t$-ball $B_{p_s}(\sigma)$ of radius σ centered at p_s will intersect $D(s)$ in a union of circles by Sard's theorem. Note that the minimality of $D(s)$ implies that all connected components of $B_{p_s}(\sigma) \cap D(s)$ are disks. If there were an annuli, then its interior component in $D(s)$ would be a minimal disk D' with boundary on the boundary of $B_{p_s}(\sigma)$. Taking the largest σ' so that $B_{p_s}(\sigma')$ intersected D', the fact that metric spheres are mean-convex in negative sectional curvature gives a contradiction.

Now choose σ in the interval $(\sigma(s) - 1 - \frac{1}{s}, \sigma(s) - 1)$ so that $\partial B_{p_s}(\sigma) \cap D(s)$ is a union of circles, and let $D(s)'$ be the connected component of $B_{p_s}(\sigma) \cap D(s)$ containing p_s, which by the above paragraph is a disk. Since we took σ in the above interval, Lemma 2.3.5 applies to give an upper bound on the absolute values of the principal curvatures of the $D(s)'$.

Lemma 2.3.7. There exists δ', depending only on ϵ and the family of metrics, such that as long as δ was chosen less than δ', the nearest-point projection of $D(s)'$ to S_{t_0} is well-defined and a

diffeomorphism onto its image.

In what follows, if we refer to a quantity as $O(x)$, we mean that it tends to zero as x tends to zero.

Proof. Fix a point q on $D(s)'$. By Lemma 2.3.3, every point on $D(s)'$ is at a distance of at most $\xi/2$ from a point on S_{t_0}. For $r \in (\frac{\xi}{4}, \xi)$ the normal exponential map for S_{t_0} defines a diffeomorphism from $S_{t_0} \times (-r, r)$ to the r-neighborhood of S_{t_0}, so let q' be the point on S_{t_0} such that the normal geodesic to S_{t_0} at q'— call it ϕ— passes through q. Let $q'_{\pm r}$ be the points in $S^{\pm r}_{t_0} \cap \phi$.

The distances between the unit normal vectors to $D(s)'$ and $S^{\pm r}_{t_0}$ at q, q'_r and q'_{-r} in the appropriate orientations are pairwise $O(\xi)$ by Lemma 2.3.6. Lemma 2.3.6 applies because the $S^{\pm r}_{t_0}$ have principal curvatures bounded above in absolute value and $D(s)'$ has principal curvatures bounded above in absolute value by Lemma 2.3.5. Since the normal vectors to the $S^{\pm r}_{t_0}$ are $O(\xi)$-close to those of S_{t_0} at points that correspond under normal projection, it follows that the normal vectors to $D(s)'$ and S_{t_0} at q and q' are $O(\xi)$-close. This implies that, provided $\xi(= O(\delta))$ and δ were taken small enough, that the tangent vector to ϕ at q is very close to being perpendicular to $D(s)'$. Therefore, for points q'' on S_{t_0} in a small neighborhood U of q', the geodesic normal to S_{t_0} at q'' will intersect $D(s)'$ nearly perpendicularly at some point close to q. It follows that the normal exponential map defines a diffeomorphism from U to a neighborhood of q, so the normal projection map from $D(s)'$ to S_{t_0} is a local diffeomorphism at all points of $D(s)'$ including points on its boundary. The normal projection from $D(s)'$ to S_{t_0} is therefore a proper local diffeomorphism so, since $D(s)'$ is a disk, it must be a diffeomorphism.

□

2.3.4 Construction of Leaves

We now construct S_t as a limit of the $D(s)'$. Let B_n be the metric disk in S_{t_0} with radius n and center p in the metric on S_{t_0} induced by $\tilde{g}_t$. If s is sufficiently large, the previous lemma tells us that the normal exponential map diffeomorphically maps each B_n onto a region $B_n(s)$ in $D(s)'$. $B_n(s)$ is therefore a graph over B_n in normal exponential coordinates for S_t. Since the $D(s)'$ have uniformly bounded principal curvatures, we can pass to a subsequence of the $B_n(s)$ for a sequence of $s \to \infty$ that C^1-converges (and thus by standard elliptic PDE theory smoothly converges) over compact subsets of B_n. Doing this for each B_n and taking a diagonal subsequence we obtain an embedded minimal surface which we call S_t. Since S_t is a smooth limit of the $D(s)'$, the normal projection map defines a diffeomorphism from S_t to S_{t_0}, and S_t is a smooth properly embedded plane. The

surface S_t inherits the property of being absolutely minimizing from the $D(s)'$ of which it was a smooth limit.

2.3.5 Uniqueness

We now check that the uniqueness conditions in Theorem 2.3.2 are met. Since S_t is at a Hausdorff distance from S_{t_0} bounded by ξ, S_t is at finite Hausdorff distance from some element S of $\mathcal{P}_{\tilde{g}_t}$, since this is true for S_{t_0}. Let S'_t be the properly embedded ϵ-subordinate $\tilde{g}_t$-minimal disk at finite Hausdorff distance from S, guaranteed by g_t's membership in Ω_ϵ. We will show that $S'_t = S_t$.

Lemma 2.3.8. *S'_t is contained in the region bounded by $S^r_{t_0}$ and $S^{-r}_{t_0}$ for $r \in (\frac{\xi}{4}, \xi)$.*

Proof. The proof is similar to that of Lemma 2.3.3 but with an extra step because S'_t is not compact. Let $S_{t_0}(x)$ be the foliation given by Lemma 2.3.4, and let $S^r_{t_0}(x)$ be the signed-distance-r parallel surfaces to the $S_{t_0}(x)$, where $\frac{\xi}{4} < |r| < \xi$ and these surfaces are all mean-convex. Now assume that S'_t has points that are not contained in the region bounded by $S^r_{t_0}$ and $S^{-r}_{t_0}$— suppose for contradiction that it has points above $S^r_{t_0}$. Let $x_{\max}$ be supremal over all x such that $S^r_{t_0}(x)$ intersects S'_t. The number $x_{\max}$ is finite because S'_t is at finite Hausdorff distance from S and each $S^r_{t_0}(x)$ is at finite and uniformly bounded Hausdorff distance from the corresponding element of $\mathcal{P}_{\tilde{g}_{t_0}}$.

We can find a sequence of points p_n on S'_t and q^r_n on $S^r_{t_0}(x_{\max})$ such that $d(p_n, q^r_n)$ tends to zero. Let q_n be the points on $S_{t_0}(x_{\max})$ that are the images of the q^r_n under normal projection. Now apply covering transformations γ_n to take the p_n back to a fixed compact set containing a fundamental domain for the action of $\pi_1(M)$ on $(\tilde{M}, \tilde{g}_t)$. Let p and q be subsequential limits of the $\gamma_n \cdot p_n$ and $\gamma_n \cdot q_n$ respectively. Since S'_t and $S_{t_0}(x_{\max})$ are minimal surfaces in $\tilde{g}_t$ and $\tilde{g}_{t_0}$ with bounded principal curvatures, we can pass to a subsequence on which $\gamma_n \cdot S'_t$ and $\gamma_n \cdot S_{t_0}(x_{\max})$ are smoothly and graphically converging in small balls centered at p and q respectively. The r-neighborhood of the subsequential limit of the $\gamma_n \cdot S_{t_0}(x_{\max})$, whose boundary is strictly mean-convex, will then touch the subsequential limit of the $\gamma_n \cdot S'_t$, which is minimal, on one side at p, which is a contradiction.

□

Since S'_t is ϵ-subordinate, signed-distance-r surfaces to S'_t are strictly mean-convex for $0 < |r| < \xi$, for the ξ given by Lemma 2.3.2. By the previous lemma S'_t is contained in the $\xi/4$ $\tilde{g}_{t_0}$-neighborhood of S_{t_0}, so the $\tilde{g}_{t_0}$-normal projection from S'_t to S_{t_0} is well-defined. Since S'_t and S_{t_0} are properly embedded, the normal projection is a proper local diffeomorphism, and consequently surjective. Therefore, because S_t is contained in the $\xi/4$ $\tilde{g}_{t_0}$-neighborhood of S_{t_0}, it follows that S_t is contained

in the $\xi/2$ $\tilde{g}_{t_0}$-neighborhood of S'_t, and therefore, as long as δ was taken small enough, the $3\xi/4$ $\tilde{g}_t$-neighborhood of S'_t. If S_t were not equal to S'_t, then we could take a sequence of points on S_t approaching a supremal mean-convex parallel surface to S'_t and produce a contradiction as in the proof of the previous lemma. Therefore S_t is equal to S'_t, and in particular is ϵ-subordinate.

In a similar way, we can check that S_t is the unique properly embedded κ-locally-minimizing minimal surface at finite Hausdorff distance from S. The argument in Lemma 2.3.8 shows that any such minimal surface must be contained in the r-neighborhood of S_t, and from there we can show it must be equal to S_t by the reasoning of the previous paragraph. The κ-locally-minimizing assumption gives a uniform upper bound on the principal curvatures [Sch83] and ensures that we can pass to convergent subsequences as in the proof of Lemma 2.3.8.

2.3.6 Property 2

We now check that the surfaces S_t we have constructed satisfy Property 2. Assume for contradiction that they do not. Then there is some sequence S_n of totally geodesic planes that converges to S on compact subsets, while the corresponding sequence of minimal surfaces $S_{n,t}$ in $\tilde{g}_t$ is not smoothly converging to the minimal plane S_t corresponding to S. Since the $S_{n,t}$ are minimal, C^1 convergence of the $S_{n,t}$ on compact sets would imply smooth convergence, so we can assume that there is some $\eta > 0$ such that the lifts of the $S_{n,t}$ to the unit tangent bundle by their normal vectors all have points in some fixed compact subset of $\tilde{M}$ at a distance of at least η from the lift of S_t to the unit tangent bundle.

The sequence S_{n,t_0} of $\tilde{g}_{t_0}$ minimal surfaces corresponding to the S_n converges smoothly to the minimal surface S_{t_0} corresponding to S by assumption. For every compact set B in $(\tilde{M}, \tilde{g}_t)$ the intersection $B \cap S_{n,t}$ is therefore, in normal exponential coordinates for the ξ-neighborhood of S_{t_0}, a graph over S_{t_0} for large enough n, since $S_{n,t}$ is a graph over S_{n,t_0}. Because we have uniform bounds on the second fundamental forms of the $S_{n,t}$ by Lemma 2.3.5, we can then proceed exactly as in 2.3.4. above to pass to a subsequential limit, which is an absolutely minimizing properly embedded minimal plane at finite distance from S_t and so must equal S_t by the uniqueness properties of S_t verified in 2.3.5. But since the points on the $S_{n,t}$ where the normal vectors are at least δ from any normal vector to S_t are contained in a compact set, we can find an accumulation point of any infinite sequence of them. This contradicts equality of the limit minimal surface with S_t.

2.3.7 The S_t Give a Foliation

We now check that the S_t give a foliation of $Gr_2(\tilde{M})$ that is invariant under the action of $\pi_1(M)$. First, the set of S_t is invariant under covering transformations. This is because we've already checked that each S_t is the unique properly embedded κ-locally-minimizing minimal disk at finite distance from some element of $\mathcal{P}_{\tilde{g}_t}$— that is, some totally geodesic plane in $\mathbb{H}^3$ considered as a subspace of $(\tilde{M}, \tilde{g}_t)$— and the set $\mathcal{P}_{\tilde{g}_t}$ is invariant under covering transformations.

By our inductive hypothesis, there exists a continuous self-homeomorphism $\tilde{\Phi}_{t_0}$ of $Gr_2(\tilde{M})$ sending (lifts of) totally geodesic planes in $\mathbb{H}^3$ to the corresponding (lifts of) minimal disks S_{t_0}. Since nearest-point projection defines a diffeomorphism between the S_t and the corresponding S_{t_0}, by composing with $\tilde{\Phi}_{t_0}$ we obtain a self-map $\tilde{\Phi}$ of $Gr_2(\tilde{M})$ diffeomorphically sending (lifts of) totally geodesic planes to (lifts of) S_t. That $\tilde{\Phi}$ is continuous follows from the fact that the S_t satisfy Property 2. Note also that $\tilde{\Phi}$ commutes with diffeomorphisms of $Gr_2(\tilde{M})$ induced by covering transformations of $\tilde{M}$. This follows from the fact that this is true for $\tilde{\Phi}_{t_0}$, and that since the S_t and S_{t_0} are invariant under covering transformations, nearest-point projection from S_t to S_{t_0} commutes with covering transformations.

This shows that $\tilde{\Phi}$ descends to a continuous self-map of $Gr_2(M)$, which since this map is $O(\xi)$-close to the corresponding self-homeomorphism of $Gr_2(M)$, it must be the case, provided δ (since $\xi = O(\delta)$) was taken small enough, that the self-map of $Gr_2(M)$ induced by $\tilde{\Phi}$ is homotopic to the homeomorphism induced by $\tilde{\Phi}_{t_0}$. The map $\tilde{\Phi}$ therefore has mod-2 degree one and so is surjective. This means that every point of $Gr_2(\tilde{M})$ is the tangent plane of some S_t. The main step remaining to prove that the S_t give a foliation is to check that each point of $Gr_2(\tilde{M})$ is the tangent plane of a unique S_t, or in other words that $\tilde{\Phi}$ is injective. It will then follow that $\tilde{\Phi}$ is a homeomorphism because a continuous bijection between compact metric spaces is a homeomorphism.

The main tool for proving injectivity of $\tilde{\Phi}$ will be the following lemma, whose proof is immediate from the results in [CM11, Section 5.3]. This is one of the key places in this chapter where we use that the ambient dimension is three.

Lemma 2.3.9. Let S_1 and S_2 be properly embedded minimal planes in $(\tilde{M}, \tilde{g}_t)$. Then the intersection $S_1 \cap S_2$ is an embedded graph. At any point where the two intersect non-transversely, the intersection is locally homeomorphic to a union of $n \geq 2$ straight lines with a common point.

Let S and S' be totally geodesic disks in $\mathbb{H}^3$, and let S_t and S'_t be the corresponding minimal disks in $(\tilde{M}, \tilde{g}_t\})$.

Then by Lemma 2.3.9, S_t and S'_t intersect in a graph Γ. To show that they never intersect

non-transversely and prove injectivity of $\tilde{\Phi}$, it is enough by the previous lemma to show that Γ is either empty or a disjoint union of lines.

We claim that since S_t is absolutely-minimizing, the set difference $S_t - \Gamma$ cannot have any bounded connected components. For contradiction, assume it had such a connected component D, which by taking an innermost such component we can assume is topologically a disk. Then by taking some large circle C in S_t which bounds a disk that contains the boundary of D in S_t, we could, by cutting out D and replacing it with the bounded connected component of $S'_t - \partial D$ (which has the same area as D since all disks in S_t and S'_t minimize area over comparison disks with the same boundary), produce a non-C^1 solution to the Plateau problem for C in the ambient space. This is impossible though, since the area can be decreased by smoothing in neighborhoods of non-C^1 points of transverse intersection ([CM11, Section 5.3].) This shows that the S_t we have constructed satisfy Property 1.

In the case that S_t and S'_t have disjoint boundaries at infinity, we are done by the last paragraph, since S_t and S'_t do not intersect outside of some compact set, so if the two intersected there would have to be a compact connected component of the complement of the intersection.

Otherwise, assume S and S' intersect in a line. Assume for contradiction that S_t and S'_t intersect non-transversely at a point p, and let Γ_0 be the connected component of Γ containing p. Locally at p, Γ_0 looks like $n > 1$ lines meeting at a point. If there are more non-transverse intersections these lines might further branch, but they will never intersect each other at a point besides the initial branch point since that would create a compact connected component of their complement. Since S_t and S'_t are at finite Hausdorff distance from S and S' respectively, all points on Γ are at uniformly bounded distance from $S \cap S'$. It follows that the complement of Γ_0 in S_t has a connected component D_0 all of whose points are at uniformly bounded distance from S'_t. Since the proof of Lemma 2.3.4 only used Properties 1 and 2 which we have already verified, there is a foliation $\mathscr{F}$ of $\tilde{M}$ containing S'_t as a leaf by applying that lemma to S'_t and S'.

Without loss of generality, assume that D_0 has points above S'_t, and let d be the supremum of the set of distances from points in D_0 above S'_t to S'_t. Since the ξ-neighborhood of S'_t has a local mean-convex foliation, if d were less than $\xi/2$, we could get a contradiction by the argument of Lemma 2.3.8. Otherwise, we could choose another leaf S''_t in the foliation $\mathscr{F}$ above S'_t so that all points on D_0 above S''_t were at a distance of less than $\xi/2$ from S''_t. Since the ξ-neighborhood of S''_t also has a mean-convex foliation, this would lead to a contradiction in the same way. (A similar argument shows that S_t and S'_t intersect in a single line, although we only need to show they intersect transversely.)

The only case left to check is if S_t and S'_t intersect at a single point at infinity. The proof here is like the last case. If S_t and S'_t intersect non-transversely at some point, then we can similarly deduce the existence of some unbounded connected component of the intersection Γ all of whose points are at bounded distance from S'_t. We can then produce a contradiction as above by taking a mean-convex foliation of a neighborhood of S'_t, or else some other S''_t above or below S'_t.

2.3.8 Local Product Charts

A smooth local product chart for our foliation at any p in $\tilde{M}$ and any tangent plane P to p in $Gr_2(\tilde{M})$ can be constructed as follows. Let S_t be the surface which has P as a tangent plane. The transversal to our chart will be homeomorphic to the product of a small neighborhood U of P in the Grassmannian of the tangent space $Gr_2(T_p\tilde{M})$ with a small geodesic segment γ in $\tilde{M}$ containing p and normal to P at p. We diffeomorphically identify this product with a subspace T of $Gr_2(\tilde{M})$ by parallel transporting U along γ.

Take a small metric disk V centered at the origin in the tangent space to S_t at p. We construct a map from $V \times T$ to a small neighborhood of P in $Gr_2(\tilde{M})$ as follows.

Let $(v, (p', P')) \in V \times T$ be given. We can identify P', by parallel transport along γ, with a linear subspace P'_p of $T_p(\tilde{M})$. Provided U and γ were chosen sufficiently small, the normal projection v' to P'_p of v will have norm greater than $\frac{1}{2}|v|$.

Parallel transport of tangent vectors gives a natural identification between P'_p and P' viewed as subspaces of $T_p\tilde{M}$ and $T_{p'}\tilde{M}$ respectively. Take the vector in P' corresponding to v'— call it v''— and consider v'' inside the tangent space at p' to the surface S'_t that has P' as a tangent plane. We map v'' to its image under the exponential map of S'_t at p' in the metric on S'_t induced by $\tilde{g}_t$ and define the tangent plane to S'_t of this point to be the image of $(v, (p', P'))$ in $Gr_2(\tilde{M})$ under our map.

Knowing that the surfaces S_t vary smoothly in their tangent planes, we will know that their exponential maps vary smoothly, and smoothness of the coordinate map we have defined will follow. Suppose that a sequence $S_{t,n}$ of minimal disks we have constructed has tangent planes P_n converging to the tangent plane P to S_t at a point. Then we need to show that $S_{t,n}$ is smoothly converging to S_t on compact subsets. That the convergence is C^1 follows from the fact that $\tilde{\Phi}$ is a homeomorphism, and elliptic PDE theory implies that the convergence is smooth.

Since the differential of the coordinate map at $(0, (p, P))$ is non-singular, we can apply the inverse function theorem to restrict to a possibly smaller neighborhood of $(0, (p, P))$ in $V \times T$ on which it is

a diffeomorphism onto its image. This shows that every point in $Gr_2(\tilde{M})$ is contained in a smooth product chart for the foliation. The proof of Theorem 2.3.2 is now complete.

2.3.9 Proof of Theorem 2.1.2

We now explain how to modify the proof of Theorem 2.3.2 to give a proof of Theorem 2.1.2. Let g_t be a smooth family of metrics as in the statement of Theorem 2.1.2. Then if g_{t_0} is in Ω_ϵ, we claim that there is some δ depending only ϵ and bounds on the geometry of the g_t such that g_t is in $\Omega_{\epsilon/2}$ for $|t - t_0| < \delta$. The $\tilde{g}_t$-minimal surfaces S_t in $\tilde{M}$ can be constructed exactly as above. Each of the S_t can be made as C^1-close to the corresponding S_{t_0} as desired, uniformly in S_t, by making δ small. Elliptic PDE theory tells us that C^1-close implies C^2-close, so if we chose δ small enough, the S_t will be $\epsilon/2$ subordinate. The metric g_t will then be contained in $\Omega_{\epsilon/2}$, and the verification that the S_t give a foliation and the construction of the map $\tilde{\Phi}$ can proceed as above.

The construction of the foliations thus continues to work unless there is some time T such that for every $\epsilon > 0$ and sequence of times $t_n \nearrow T$ there is some S_{t_n} in $(\tilde{M}, \tilde{g}_{t_n})$ that fails to be ϵ-subordinate for n large enough. This proves Theorem 2.1.2.

2.4 Applications

2.4.1 Density

In this section, we prove some density results for the stable immersed minimal surfaces in M corresponding to the surface subgroups constructed by Kahn and Markovic.

Let g be a negatively curved metric on M to which Theorem 2.1.2 or Theorem 2.3.2 applies to produce a foliation. Let $\mathcal{F}_g$ be the foliation of $Gr_2((M, g))$ by (lifts of) g-minimal immersed disks, and let

$$\Phi : Gr_2((M, g_{hyp})) \to Gr_2((M, g))$$

be the conjugating homeomorphism that sends leaves of $\mathcal{F}_{g_{hyp}}$ to leaves of $\mathcal{F}_g$. Every leaf of $\mathcal{F}_{g_{hyp}}$ is either dense or properly immersed, so since Φ is a homeomorphism the leaves of $\mathcal{F}_g$ satisfy the same dichotomy.

Fix a compact set $K_0 \subset \tilde{M}$ that contains a small neighborhood of a connected polyhedral fundamental domain for the action of $\pi_1(M)$ on $\tilde{M} \cong \mathbb{H}^3$. Let Σ_n (resp. Σ'_n) be a sequence of stable immersed g_{hyp}-minimal (resp. g-minimal) surfaces with lifts $\tilde{\Sigma}_n$ (resp. $\tilde{\Sigma}'_n$) to $\tilde{M}$. Suppose

the lifts $\tilde{\Sigma}_n$ and $\tilde{\Sigma}'_n$ were chosen so that all of the $\tilde{\Sigma}_n$ intersect K_0, and that $\tilde{\Sigma}'_n$ and $\tilde{\Sigma}_n$ are at finite Hausdorff distance from each other in $\tilde{M}$ in either (or equivalently both) of $\tilde{g}$ or $\tilde{g}_{hyp}$.

Lemma 2.4.1. Fix a circle C in $\partial_\infty \mathbb{H}^3 \cong S^2$, and suppose that the limit sets of the $\pi_1(\tilde{\Sigma}_n)$ are Hausdorff converging to C in $\partial_\infty \mathbb{H}^3$. Let L' be the minimal disk in $(\tilde{M}, \tilde{g})$ whose lift to $Gr_2(\tilde{M})$ is the image under $\tilde{\Phi}$ of the lift of the totally geodesic plane L with limit set C. Then the $\tilde{\Sigma}'_n$ converge smoothly to L' uniformly on compact sets.

Proof. Let $L'(t)$ be the foliation of $\tilde{M}$ given by Lemma 2.3.4 with $\tilde{g}$-minimal leaves and $L'(0) = L'$. Let $L(t)$ be the corresponding foliation of $\mathbb{H}^3$ by geodesic planes with $L(0) = L$, so that $\tilde{\Phi}$ sends lifts of the $L(t)$ to lifts of the $L'(t)$. For every $\alpha > 0$ and large enough n, $\tilde{\Sigma}_n$, which is contained in the convex hull of its limit set, will be contained between $L(\alpha)$ and $L(-\alpha)$. We claim that $\tilde{\Sigma}'_n$ is contained between $L'(\alpha)$ and $L'(-\alpha)$.

Recall that the ξ-neighborhood of every $L'(t)$ has a foliation by mean-convex parallel surfaces, where ξ depends on ϵ and g. Now, if $\tilde{\Sigma}'_n$ were not contained between $L'(\alpha)$ and $L'(-\alpha)$, then using the mean-convex parallel surfaces of the $L'(t)$ and the fact that $\tilde{\Sigma}'_n$ and the $L'(t)$ are at uniformly bounded Hausdorff distance from respectively $\tilde{\Sigma}_n$ and the corresponding $L(t)$, one could produce a contradiction by arguments similar to those of the last section. One can also show by reasoning similar to subsection 2.3.6. of the last section that $\tilde{\Sigma}'_n$ is C^1-converging and thus smoothly converging to $L' = L'(0)$ on compact subsets.

□

Kahn and Markovic showed that for every circle C at infinity in $\partial_\infty \mathbb{H}^3$ there is a sequence of surface subgroups Γ_n of $\pi_1(M)$ whose limit sets C_n are Hausdorff converging to C [KM12b]. The C_n are the images of round circles under K_n-quasiconformal homeomorphisms of S^2— or K_n-quasicircles— with K_n tending to 1. By [SU82] or [SY79], there exists a sequence Σ_n (resp. Σ'_n) of stable properly immersed g_{hyp}-minimal (resp. g-minimal) surfaces in M whose fundamental groups injectively include in $\pi_1(M)$ to subgroups conjugate to Γ_n.

Theorem 2.4.1. *Let C be a circle in $\partial_\infty \mathbb{H}^3$ bounded by a geodesic plane L in $\mathbb{H}^3$ that does not project to a properly immersed surface in M. Then for any sequence Σ_n of stable properly immersed minimal surfaces with lifts $\tilde{\Sigma}_n$ to $\mathbb{H}^3$ whose boundaries at infinity are K_n-quasicircles with K_n tending to 1 and Hausdorff converging to C, the following is true. Let U be any open set in $Gr_2(M)$. Then there exists N so that for $n > N$, the intersection of the lift of Σ'_n to $Gr_2(M)$ with U is nonempty.*

Proof. By [Rat91] or [Sha91], the lift to $Gr_2(M)$ of the covering projection of L to M is dense. Let

(the lift of) L' be the image under $\tilde{\Phi}$ of the (lift of) L. Then as observed earlier in the section, L' is also dense in $Gr_2(M)$. It is enough to prove the theorem for U a small ball in $Gr_2(M)$. For any such U, let $\tilde{U}$ be a lift of U to $Gr_2(\tilde{M})$. Then we can find $\gamma \in \pi_1(M)$ so that the image $\gamma \cdot L'$ of (the lift to $Gr_2(M)$ of) L' under the covering transformation corresponding to γ intersects $\tilde{U}$.

By Lemma 2.4.1 the $\tilde{\Sigma}'_n$ are smoothly converging to L' on compact sets, so for all n sufficiently large $\gamma \cdot \tilde{\Sigma}'_n$ will intersect $\tilde{U}$. Therefore for all sufficiently large n, Σ'_n will intersect U.

□

2.4.2 Uniqueness

We now prove uniqueness for properly immersed minimal surfaces whose fundamental groups injectively include to the conjugacy class of a given surface subgroup of $\pi_1(M)$, under the assumption that the limit set of the surface subgroup is close to a circle.

Let Σ be a stable properly immersed minimal surface in (M, g_{hyp}) whose fundamental group injectively includes in $\pi_1(M)$ as a subgroup conjugate to a surface subgroup Γ of $\pi_1(M)$. Then if the limit set of Γ is a K-quasicircle for K sufficiently close to 1, the main result of [Sep16] implies that Σ will be the unique such surface. The next theorem is an analogous uniqueness result for (M, g). The proof occupies the remainder of the section.

Theorem 2.4.2. *Fix (M, g) to which Theorem 2.1.2 or 2.3.2 applies to produce a foliation. Then there exists $\eta > 0$ such that the following is true. Suppose the limit set $\partial_\infty \tilde{\Sigma}$ of $\tilde{\Sigma}$ is a K-quasicircle for $K < 1 + \eta$. Then there is a unique stable g-minimal surface in M whose fundamental group injectively includes in $\pi_1(M)$ as a subgroup conjugate to Γ.*

Construction of the Maps $\tilde{f}_\Sigma$

By [Sep16][Proposition 4.1], there exist, for every point p on a minimal disc D with limit set a $1+\eta$ quasicircle, planes $L_1(p)$ and $L_2(p)$ respectively above and below D such that the quantity

$$\max(d(p, L_1(p)), d(p, L_2(p))) \tag{2.4.1}$$

tends to zero uniformly in p as η tends to zero. This builds on work of [KS08] and [Eps84]. Taking $D = \tilde{\Sigma}$, we can choose $L_1(p)$ and $L_2(p)$ such that the arc-length-parametrized geodesic γ_p normal to $\tilde{\Sigma}$ at p perpendicularly intersects $L_1(p)$ and $L_2(p)$ at $\gamma_p(\delta)$ and $\gamma_p(-\delta)$ respectively, where δ is independent of p and can be made as small as desired by making η small. In addition, Lemma 2.3.6

implies that $L_1(p)$, $L_2(p)$, and δ can be chosen so that the following is true: the distance from the tangent plane to $\tilde{\Sigma}$ at p from the tangent planes to $L_1(p)$ and $L_2(p)$ at $\gamma_p(\delta)$ and $\gamma_p(-\delta)$ tends to zero as η tends to zero, uniformly in p.

Let Σ' be a stable π_1-injective g-minimal surface the inclusion of whose fundamental group in $\pi_1(M)$ is conjugate to that of $\pi_1(\Sigma)$. Fix some $p \in \tilde{\Sigma}$. Let P_1 and P_2 be the tangent planes to $L_1(p)$ and $L_2(p)$ at $\gamma_p(\delta)$ and $\gamma_p(-\delta)$. Then if $\tilde{\Sigma}$ is contained between $L_1(p)$ and $L_2(p)$, we assume that the lift $\tilde{\Sigma}'$ is chosen so that it is contained between the $\tilde{g}$-minimal planes $L_1'(p)$ and $L_2'(p)$ which correspond under $\tilde{\Phi}$ to $L_1(p)$ and $L_2(p)$. We take p' to be a point on the intersection of $\tilde{\Sigma}'$ with the geodesic segment γ_p' joining the projections to $\tilde{M}$ of $\tilde{\Phi}(P_1)$ and $\tilde{\Phi}(P_2)$.

We claim that p' is the unique such point of intersection provided η and δ were chosen small enough. Assume there were some other point of intersection p'', and let ϕ' be the unique geodesic segment on $\tilde{\Sigma}'$ joining p' to p''.

On the one hand, given any $R > 0$ and small $\epsilon > 0$, we claim that we can ensure that the intrinsic ball $B_R(p')$ of radius R in $\tilde{\Sigma}'$ centered at p' will be at a C^1 distance of less than ϵ from the ball of that radius in either $L_1'(p)$ or $L_2'(p)$ centered at the intersection with $\gamma_{p'}$, provided that η and δ were chosen sufficiently small.

This follows from the same argument by contradiction as the proof of Lemma 2.4.1. As a consequence of this, provided $\epsilon > 0$ and $\delta > 0$ were taken small enough that $B_R(p')$ is sufficiently C^1 close to $L_1'(p)$ and $L_2'(p)$, the length of ϕ' must be at least R.

On the other hand, the fact that $\tilde{\Phi}$ is a uniformly bi-Lipschitz homeomorphism from $(\tilde{M}, \tilde{g}_{hyp}) \to (\tilde{M}, \tilde{g})$ implies that for R large enough (and independent of $\tilde{\Sigma}'$), all points in $\tilde{\Sigma}'$ at intrinsic $\tilde{\Sigma}'$-distance of at least R from p' are at $(\tilde{M}, \tilde{g})$-distance from p' of at least 1. This is a contradiction.

Therefore γ_p' and $\tilde{\Sigma}'$ intersect at a single point provided η and δ were chosen sufficiently small. The map $\tilde{f}_\Sigma$ that sends a point p to the point p' defined as above is therefore a well-defined map between $\tilde{\Sigma}$ and $\tilde{\Sigma}'$. Because $\tilde{f}_\Sigma$ can locally be made arbitrarily C^1-close to $\tilde{\Phi}$ by making η and δ small, we can assume that $\tilde{f}_\Sigma$ is a local diffeomorphism.

Because $\tilde{\Phi}$ commutes with the action of $\pi_1(M)$ on $\tilde{M}$ by isometries of $\tilde{g}_{hyp}$ and $\tilde{g}$, $\tilde{f}_\Sigma$ commutes with the action of $\pi_1(\Sigma)$ and descends to a map $f_\Sigma : \Sigma \to \Sigma'$. Since f_Σ is a local diffeomorphism and Σ and Σ' are homeomorphic, it must be a diffeomorphism, and in particular $\tilde{f}_\Sigma$ is surjective.

Proof of Theorem 2.4.2

Suppose that the projections of the leaves of $\mathcal{F}_g$ to M are ϵ-subordinate. Then $\tilde{\Sigma}'$ is $\frac{\epsilon}{2}$-subordinate provided η and δ were taken sufficiently small. This is because, as noted above, $\tilde{\Sigma}'$ is locally C^1-

converging and thus smoothly converging to (projections of) leaves of $\mathcal{F}_g$.

Now let η and δ be small enough to satisfy all of the restrictions above, as well as one further restriction we will make below in a moment. Assume for contradiction that Σ' and Σ'' are distinct stable g-minimal surfaces the inclusions of whose fundamental groups in $\pi_1(M)$ are both injective and conjugate to the same surface subgroup Γ whose limit set in $\partial_\infty \mathbb{H}^3$ is a $K < 1 + \eta$ quasicircle.

Let $\tilde{\Sigma}'$ and $\tilde{\Sigma}''$ be lifts to $\tilde{M}$ at finite distance from $\tilde{\Sigma}$ considered inside $(\tilde{M}, \tilde{g})$, and note that since these are $\frac{\epsilon}{2}$-subordinate, there exists a uniform ξ such that the ξ-neighborhood of each has a mean-convex foliation by parallel surfaces. This implies that $\tilde{\Sigma}'$ and $\tilde{\Sigma}''$ are at a Hausdorff distance of at least ξ from each other.

For each of $\tilde{\Sigma}'$ and $\tilde{\Sigma}''$ we have a map $\tilde{f}_\Sigma$ defined as above. Since the definition of the two maps is the same up until taking the intersection with a geodesic segment in $(\tilde{M}, \tilde{g})$ of length that tends to zero as η and δ tend to zero, the two images of each point on $\tilde{\Sigma}$ under the two maps will be will be at a distance that tends to zero as η and δ tend to zero. Taking η and δ small enough to make this distance less than $\xi/2$, the fact that both maps $\tilde{f}_\Sigma$ are surjective gives a contradiction. This completes the proof of Theorem 2.4.2.

2.5 Quantitative Density

2.5.1 Constant Curvature

We begin with the constant curvature case. We assume throughout this subsection that M contains no properly immersed totally geodesic surfaces in its constant curvature metric.

Definition 2.5.1. For a tangent plane P in $Gr_2(M)$ based at $p \in M$, we define $C_{P,r} \subset Gr_2(M)$ as follows. Lift (p, P) to a point $(\tilde{p}, \tilde{P})$ in $Gr_2(\tilde{M}) \cong Gr_2(\mathbb{H}^3)$, and let $\Pi \subset \tilde{M}$ be the geodesic plane tangent to $\tilde{p}$ at $\tilde{P}$. Now take the circle $\tilde{C}_{P,r}$ in Π of radius r centered at $\tilde{p}$, lift it to $Gr_2(\tilde{M})$ by planes tangent to Π, and let $\tilde{D}_{P,r}$ be the totally geodesic disk that $\tilde{C}_{P,r}$ bounds. We define $C_{P,r}$ and $D_{P,r}$ to be the projections of $\tilde{C}_{P,r}$ and $\tilde{D}_{P,r}$ to $Gr_2(M)$, and we define $\mu_{P,r}$ to be the probability measure that corresponds to averaging over $C_{P,r}$ parametrized by arc-length in the metric induced by $Gr_2(M)$.

Proposition 2.5.2. *Let*

$$f : Gr_2(M) \to \mathbb{R}$$

be a continuous function. Then for every $\epsilon > 0$ there exists R such that for all $P \in Gr_2(M)$ and all $r > R$,

$$\left| \int_{Gr_2(M)} f d\mu_{P,r} - avg(f) \right| < \epsilon, \tag{2.5.1}$$

where $avg(f)$ is the average of f over $Gr_2(M)$ in its volume form for the metric induced by the hyperbolic metric on M.

Proof. We are going to prove this by applying Ratner's measure classification theorem ([Esk10, Theorem 1.11].) Fix an orientation for $C_{P,r}$ and let $\hat{C}_{P,r}$ be its natural lift to the frame bundle $F \cong PSL(2,\mathbb{C})/\pi_1(M)$ of M by, for a point $p \in C_{P,r}$, taking the first vector in the frame to be the outward unit normal vector to the projection of $C_{P,r}$ to M tangent to (the projection of) Π, and the second to be tangent to $C_{P,r}$ in the direction determined by its orientation. The third vector is then determined by the orientation of M. Let $\hat{\mu}_{P,r}$ be the probability measure on F given by averaging over $\hat{C}_{P,r}$.

Assume that the statement is false, and that for some f and ϵ there existed a sequence μ_{P_n,r_n} such that (2.5.1) fails for all n. We can pull back f to a function $\hat{f}$ on F so that

$$\left| \int_F \hat{f} d\hat{\mu}_{P_n,r_n} - avg(f) \right| \geq \epsilon \tag{2.5.2}$$

for all n. Since F is compact, we can take a weak-$*$ limit of the $\hat{\mu}_{P_n,r_n}$ to obtain a probability measure $\hat{\mu}$ for which the $\hat{\mu}$-average and the Haar-measure-average of $\hat{f}$ differ by at least ϵ.

Let U be the projection to $PSL(2,\mathbb{R})$ of the unipotent subgroup of $SL(2,\mathbb{R})$

$$\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Claim: $\hat{\mu}$ is U-invariant

Fix an element $u \in U$. Then for every $\delta > 0$, there exists N such that for $n > N$, there are arc-length parametrizations ϕ_1 and ϕ_2 of respectively $u \cdot \hat{C}_{P_n,r_n}$ and $\hat{C}_{P_n,r_n}$ such that

$$d(\phi_1(s), \phi_2(s)) < \delta \tag{2.5.3}$$

for all s, where d is the distance in F. This follows from the fact that metric circles of large radius in $\mathbb{H}^2$ can be approximated in large neighborhoods of each point by horocyles, which are preserved by U.

The inequality (2.5.3) implies that for any continuous $g : F \to \mathbb{R}$,

$$\left| \int_F g d(u_* \hat{\mu}_{P_n, r_n}) - \int_F g d\hat{\mu}_{P_n, r_n} \right|$$

tends to zero as $n \to \infty$. Therefore $\hat{\mu}$ is U-invariant.

Claim: Any U-invariant measure $\hat{\mu}$ on F must be the volume measure

By Ratner's measure classification theorem, μ is supported on a union of closed orbits of subgroups H of $PSL(2, \mathbb{C})$ containing U. We claim that any such H must be equal to $PSL(2, \mathbb{C})$, which we check by ruling out intermediate candidates for H one by one.

First of all, U has no closed orbit in F. This is because no point on a U-orbit in $PSL(2, \mathbb{C})$ is mapped to another point on the same orbit by the action of a matrix corresponding to a hyperbolic isometry of $\mathbb{H}^3$. To see this, let T be a hyperbolic element of $PSL(2, \mathbb{C})$. If T mapped some frame to another frame on the same U-orbit, then for some $u \in U$, $T \circ u$ would fix some frame, and therefore have to be the identity, a contradiction. Therefore, no points on a U-orbit can be identified when modding out by Γ, and H must properly contain U. Similar reasoning shows that H cannot be the (projection to $PSL(2, \mathbb{C})$ of) the group of matrices of the form

$$\left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\},$$

whose orbits are horospheres.

Our assumption on the absence of properly immersed totally geodesic surfaces rules out $PSL(2, \mathbb{R})$ as a possibility for H. It also rules out the group of real upper triangular matrices in $PSL(2, \mathbb{C})$. An orbit of this group corresponds to a pair consisting of a totally geodesic plane Π and a point $p \in \partial_\infty \Pi$ on the sphere at infinity of $\mathbb{H}^3$. Such an orbit consists of all frames lying over the plane Π, such that the first vector of the frame is tangent to a geodesic ray with endpoint p on the sphere at infinity when lifted to the universal cover. Because this set lies over a totally geodesic plane in $\mathbb{H}^3$, it cannot have a closed projection to F. One can rule out the subgroup of complex upper triangular matrices on similar grounds.

Since we've exhausted all conjugacy classes of intermediate closed subgroups (see [Kap09, Section 4.6]), H must be equal to $PSL(2, \mathbb{C})$. This implies that $\hat{\mu}$ equals the Haar measure on F, which contradicts (2.5.2.) □

Let $\mu_{D,P,r}$ be the measure obtained by averaging over $D_{P,r}$. The next corollary follows by

integrating in polar coordinates.

Corollary 2.5.3. *For any continuous f and fixed $\epsilon > 0$ there exists R_0 such that for all $P \in Gr_2(M)$ and all $R > R_0$,*

$$\left| \int_{Gr_2(M)} f d\mu_{D,P,R} - avg(f) \right| < \epsilon, \tag{2.5.4}$$

We can now prove equidistribution for certain sequences of minimal surfaces in M. First we prove a lemma.

Lemma 2.5.1. Let $R > 1$ be given. Then for every ϵ there is some δ so that the following is true. Let Σ be a closed Riemannian surface with Gauss curvature everywhere in the interval $(-1-\delta, -1+\delta)$. Let f be a function on Σ, and f_R the function that at each point is equal to the average of f over the disk of radius R at that point. If the injectivity radius of Σ is less than R at that point, we define this average by lifting the disk of radius R at that point to the universal cover of Σ and taking the average of the pullback of f over this disk.

Then the averages $\text{avg}(f)$ and $\text{avg}(f_R)$ of these two functions over Σ satisfy

$$|\text{avg}(f) - \text{avg}(f_R)| < \epsilon \max(|f|). \tag{2.5.5}$$

Proof. Let Σ_0 be the complement of the 1-skeleton of the standard $4g$-gon cell structure on Σ, over which the tangent bundle to Σ is trivial. Define a metric on the product of a disk D and Σ_0 as follows. For fixed $p \in \Sigma_0$, we identify D with the disk of radius R in $T_p\Sigma$, and pull back the metric on Σ under the exponential map to get a metric on D, whose area form we denote by $dV_{D,p}$. This metric varies smoothly in p, so we can define a smooth metric on $D \times \Sigma_0$ such that the induced metric on each $\{d\} \times \Sigma_0 \subset D \times \Sigma_0$ is isometric to the metric on Σ_0, whose area form we denote by dV_{Σ_0}. The resulting volume form on $D \times \Sigma_0$ at (x, y) splits as

$$dV_{D,y}(x) \wedge dV_{\Sigma_0}(y).$$

For fixed y, we identify $D \times \{y\}$ with the disk of radius R in the hyperbolic plane via the exponential map of the centerpoint of D, and we write

$$dV_{D,y}(x) = \phi(x, y) dV_{\mathbb{H}^2}(x),$$

where $dV_{\mathbb{H}^2}(x)$ is the hyperbolic area form on $D \times \{y\}$ under this identification. Given f as in the

theorem, we define

$$\hat{f} : D \times \Sigma_0 \to \mathbb{R}$$

by setting $\hat{f}(x, y)$ to be the value of f at the point on Σ that is the image of x under the natural map from $D \times \{y\}$ to Σ. We have that,

$$\int_\Sigma f_R = \int_{\Sigma_0} \frac{1}{\text{Area}(D(y, R))} \int_D \hat{f}(x, y)\phi(x, y) dV_{\mathbb{H}^2}((x)) dV_{\Sigma_0}((y)), \tag{2.5.6}$$

where $\text{Area}(D(y, R))$ is the area of the disk of radius R centered at a lift of y to the universal cover of Σ. By taking δ small enough we can make $\phi(x, y)$ pointwise arbitrarily close to 1, uniformly over all Σ satisfying the hypotheses of the theorem. We can thus also make $\text{Area}(D(y, R))$ arbitrarily close to the area of the disk of radius R in $\mathbb{H}^2$.

Define the function $m_x(y)$ to be the number of distinct geodesic segments of length less than R joining x to y. By partitioning D in its hyperbolic metric and Σ_0 into small almost-Euclidean rectangles and taking Riemann sums for the double integral in (2.5.6), we see that the contribution of $f(x)$ to the integral is weighted by the quantity

$$\int_\Sigma m_x,$$

which can be made as close as desired to the area of the disk of radius R in the hyperbolic plane, since the integral of m_x over Σ is just the area of the lift of the disk of radius R at x to the universal cover of Σ. It follows that the quantity in (2.5.6) can be made $\epsilon \max |f|$-close to the integral of f over Σ by taking δ small.

□

Theorem 2.5.4. *Let M be a closed hyperbolic 3-manifold with no properly immersed totally geodesic surfaces, and let f be a continuous function on $Gr_2(M)$ with average $avg(f)$. Then for every $\epsilon > 0$ we can find δ such that the following holds. Take any surface subgroup of $\pi_1(M)$ realized by a properly immersed minimal surface Σ. Assume also that the limit set of a lift of Σ to the universal cover is a K-quasicircle for $K < 1 + \delta$. The surface Σ includes in $Gr_2(M)$ by its tangent planes, and we define $avg(f, \Sigma)$ to be the average of the pullback of f over Σ in the metric on Σ induced by M. Then*

$$|avg(f) - avg(f, \Sigma)| < \epsilon.$$

Proof. For δ small enough, [Sep16] implies that Σ is unique and has principal curvatures pointwise of magnitude $O(\log(1+\delta))$. Seppi proves his result by, for each point x in the universal cover $\tilde{\Sigma}$ of Σ in $\mathbb{H}^3$, producing nearby totally geodesic planes above and below $\tilde{\Sigma}$ that are orthogonal to a geodesic through x. We point out that the conclusion of [Sep16][Lemma 4.11] in Seppi's paper also follows in our setting by Lemma 2.3.5, since the minimal discs we consider are lifts of stable minimal surfaces in (M, g_{hyp}) with a uniform lower bound on their injectivity radii.

Let R be larger than the R_0 given by Corollary 2.5.3 applied to f and $\epsilon/4$. By making δ small enough, we can ensure that lifts D of intrinsic disks of radius R in Σ to the universal cover are as C^1 close as desired to totally geodesic disks of that radius in the universal cover. In particular, for any lift of an intrinsic disk D in Σ we can find a totally geodesic disk D' in $\mathbb{H}^3$ such that the averages of the pullback of f over D and D' differ by at most $\epsilon/4$, and the average over D therefore differs from the average of f over $Gr_2(M)$ by at most $\epsilon/2$.

Now if M is the maximum of $|f|$ over $Gr_2(M)$, taking the ϵ of the previous lemma to be $\epsilon/(2M)$ and making δ small enough finishes the proof.

□

2.5.2 Variable Curvature

We now prove Theorem 2.1.5. Let g be a metric on M to which Theorems 2.1.2 or 2.3.2 apply to construct a foliation $\mathcal{F}_g$ conjugate to the totally geodesic foliation in constant curvature via

$$\Phi : Gr_2((M, g_{hyp})) \to Gr_2((M, g)).$$

Let Σ_n be a sequence of properly immersed minimal surfaces in (M, g_{hyp}) that satisfy the hypotheses of Theorem 2.1.5: i.e., with areas tending to infinity such that the limit sets of the $\pi_1(\Sigma_n)$ in $\partial_\infty \mathbb{H}^3$ are K_n-quasicircles with K_n tending to 1. Assume that the Σ_n become uniformly distributed in $Gr_2(M)$. In the previous subsection, we checked that this is always the case if M contains no closed totally geodesic surfaces. We will show that sequences of g-minimal surfaces Σ_n' corresponding to the Σ_n converges to a measure μ_g with full support.

First we describe the measure μ_g. To do so, we need to introduce some terminology. A foliation $\mathcal{G}$ with total space E is given by a collection of charts of the form

$$\phi_\alpha : U_\alpha \times T_\alpha \to E,$$

where ϕ_α is a homeomorphism onto its image, $\phi_\alpha|_{U_\alpha \times \{t_\alpha\}}$ maps into a leaf of $\mathcal{G}$ for any fixed $t_\alpha \in T_\alpha$, and U_α and T_α are both homeomorphic to balls. We call T_α the *transversal* for the local chart defined by ϕ_α. Any closed loop in the total space beginning and ending at a point in the chart defines a *transverse holonomy map* $T_\alpha \to T_\alpha$.

Definition 2.5.5. A *transverse invariant measure* for a foliation is a choice of measure on the transversal for each local chart for the foliation, well-defined with respect to transition maps between overlapping charts, that is invariant under transverse holonomy maps.

The model foliation $\mathcal{F} = \mathcal{F}_{g_{hyp}}$ of $Gr_2(M, g_{hyp})$ has a natural tranverse invariant measure μ_T because its leaves are the orbits of an $SL(2,\mathbb{R})$ action, and $SL(2,\mathbb{R})$ is unimodular. The natural volume form $\mu_{g_{hyp}}$ for $Gr_2(M, g_{hyp})$, normalized to have unit volume, decomposes in a local chart for $\mathcal{F}$ as a transverse sum of μ_T and the area forms for the leaves, up to scaling by a constant. Taking the pushforward of μ_T by the map Φ that conjugates $\mathcal{F}$ to $\mathcal{F}_g$ and summing this transverse invariant measure with the area forms for the leaves of $\mathcal{F}_g$ in the induced metrics for their projections to (M, g) gives a measure on $Gr_2(M, g)$. We define μ_g to be this measure, normalized to have unit volume.

It is clear from the construction of Φ in terms of normal geodesic projections that Φ restricted to any leaf of $\mathcal{F}$ is smooth, and the leafwise differential of Φ varies continuously in the total space of $\mathcal{F}$. We can thus define a continuous function $f_\Phi : Gr_2(M, g_{hyp}) \to \mathbb{R}$ by setting $f_\Phi(x)$ equal to the determinant of the differential at x of Φ restricted to the leaf of $\mathcal{F}$ through x. We can write this as

$$\Phi_* \mu_{g_{hyp}} = f_\Phi \circ \Phi^{-1} \cdot \mu_g. \tag{2.5.7}$$

Now let Σ_n be the sequence of closed immersed surfaces in (M, g_{hyp}) from the start of the section. Let Σ'_n be the corresponding minimal surfaces in (M, g). Recall the maps $f_{\Sigma_n} : \Sigma_n \to \Sigma'_n$ from the proof of 2.1.4, that exist for n sufficiently large. As n tends to infinity, Σ_n and Σ'_n locally converge to (projections of) leaves of $\mathcal{F}$ and $\mathcal{F}_g$, and the maps f_Σ locally converge to the restriction of Φ to the Σ_n.

Let μ'_n and μ_n be the probability measures on $Gr_2(M, g)$ and $Gr_2(M, g_{hyp})$ corresponding to the Σ'_n and the Σ_n. We claim that the measures μ'_n weak-$*$ converge to μ_g. This is almost immediate from how everything has been set up. Since the f_{Σ_n} are locally converging to Φ, the pushforwards of the μ_n by the f_{Σ_n} are locally converging to the μ'_n multiplied by $f_\Phi \circ \Phi^{-1}$. By assumption the μ_n are converging to the volume measure $\mu_{g_{hyp}}$ on $Gr_2(M, g_{hyp})$. Therefore by (2.5.7), the μ'_n are

converging to μ_g. This finishes the proof of Theorem 2.1.5.

We note that the only place where we used the fact that the Σ'_n and the projections to (M, g) of the leaves of $\mathcal{F}_g$ were minimal surfaces was in showing that the Σ'_n locally converged to leaves of $\mathcal{F}_g$ and that the f_{Σ_n} locally converged to Φ.

2.6 Examples where Theorem 2.1.2 Cannot Apply

2.6.1 Introduction

We now present examples of negatively curved metrics g on manifolds M diffeomorphic to hy for which $Gr_2(M)$ cannot possibly admit a foliation as in Theorem 2.1.2 or Theorem 2.3.2. Recall that $\mathcal{P}_{\tilde{g}}$ is the set of totally geodesic planes in $\mathbb{H}^3$ considered as subspaces of $(\tilde{M}, \tilde{g})$. In the examples we construct, there will be multiple properly embedded minimal planes in $(\tilde{M}, \tilde{g})$ at finite Hausdorff distance from the same element of $\mathcal{P}_{\tilde{g}}$, and so Theorem 2.1.2 could not apply to produce a foliation. The starting point for our construction will be a closed hyperbolic 3-manifold M with a proper embedded totally geodesic surface Σ in its hyperbolic metric (see [MR03] for examples.)

2.6.2 Modification of the Quasi-Fuchsian Metric Near Infinity

In [HW15], examples of quasi-Fuchsian hyperbolic 3-manifolds were constructed with arbitrarily many stable embedded minimal surfaces in their convex core whose inclusions are homotopy equivalences. Fix such a quasi-Fuchsian manifold Q.

Let $2I_1^*$ and $2I_2^*$ be hyperbolic metrics in the conformal classes of the two ends of Q. In [KS08, Theorem 5.8] it is shown that there are neighborhoods of infinity of each of the ends of Q, homeomorphic to $\Sigma \times (0, \infty)$, on which there are coordinates such that the metric on Q can be written, for t larger than some $T >> 1$, in the form

$$\frac{1}{2}(e^{2t} I_j^* + 2\mathrm{II}_j^* + e^{-2t}\mathrm{III}_j^*) + dt^2 \quad j = 1, 2, \tag{2.6.1}$$

where the symmetric 2-tensors II_j^* and III_j^* are determined by the I_j^*. There thus exist neighborhoods of each of the ends that have foliations by negatively curved equidistant surfaces, which are given by taking T large enough and setting t to be constant in the above coordinates. These foliations were first constructed by Epstein [Eps84].

Lemma 2.6.1. Let g_t, $t \in [0, 1]$, be a smooth family of metrics on the surface Σ. Then there exists

T such that the warped product metric

$$\frac{1}{4}e^{2t}g_t + dt^2$$

for $t \in (T, T+1)$ has negative sectional curvature. The number T depend only on norms of the family g_t and its time derivatives relative to some fixed background metric.

This can be checked from the formula for the sectional curvature in terms of the Christoffel symbols, and the formula for the Christoffel symbols in terms of derivatives of the metric. The terms that contain t-derivatives of the metric are much larger than the other terms and control the sign. (Compare [FJ89][Lemma 3.5].)

Now, for t_0 large, take a smooth path of metrics γ_t^j, $t \in [0,1]$, joining

$$2(I_j^* + 2e^{-2t_0}II_j^* + e^{-4t_0}III_j^*) \tag{2.6.2}$$

to

$$(1 + 2e^{-2(t_0+1)} + e^{-4(t_0+1)})g_\Sigma = 4e^{-2(t_0+1)}\cosh^2(t_0+1)g_\Sigma,$$

where g_Σ is the induced hyperbolic metric on the totally geodesic surface Σ in M. We also require that for all ϵ small,

$$\gamma_\epsilon^j = 2(I_j^* + (2e^{-2(t_0+\epsilon)}II_j^* + e^{-4(t_0+\epsilon)}III_j^*))$$

and

$$\gamma_{1-\epsilon}^j = (1 + 2e^{-2((t_0+1-\epsilon)} + e^{-4(t_0+1-\epsilon)})g_\Sigma.$$

Clearly the paths γ_t^j can be chosen so that the norms of the metrics γ_t^j and their derivatives in time are uniformly bounded, independent of t_0, relative to a fixed background metric. Then applying Lemma 2.6.1, provided we chose t_0 large enough, the metric

$$\frac{1}{4}e^{2t}(\gamma_{(t-t_0)}^j) + dt^2 \tag{2.6.3}$$

on $\Sigma \times (t_0, t_0+1)$ has negative sectional curvature.

2.6.3 Modification of the Metric on a Fuchsian Cover

Now take M to be any closed hyperbolic 3-manifold that admits a closed embedded totally geodesic surface Σ. Let $F \cong \Sigma \times \mathbb{R}$ be the cover of M corresponding to Σ. If g_Σ is the metric on Σ induced

by the hyperbolic metric on M, the metric on F can be written as a warped product

$$(\cosh^2 t)g_\Sigma + dt^2. \tag{2.6.4}$$

We can define a new negatively curved metric on F by cutting out the $t_0 + 1$-neighborhood of Σ and gluing in the metric (2.6.3) from the previous subsection, which interpolates between the metrics on the ends of F and the middle of Q. This new metric on F contains a region isometric to the convex core of Q. Denote this new metric on F by g'.

2.6.4 Passage to Finite Covers to Facilitate Gluing

We claim that by passing to finite covers, we can make the normal injectivity radius of Σ in M arbitrarily large. It is a fact that for every $g \in \pi_1(M) \setminus \pi_1(\Sigma)$, there exists a finite index subgroup G of $\pi_1(M)$ such that $\pi_1(\Sigma) \subset G$ but $g \notin G$ [MR03, Lemma 5.3.6].

Fix a basepoint for $\pi_1(M)$, and let F be the Fuchsian cover of M corresponding to $\pi_1(\Sigma)$. Fix a connected polyhedral fundamental domain P for the action of $\pi_1(M)$ on $\mathbb{H}^3$. Then F is tessellated by copies of P which are fundamental domains for the covering map from F to M, and any fixed normal neighborhood N of the central totally geodesic copy of Σ in F is contained in a finite number copies of P in F. By choosing elements of $\pi_1(M)$ representing the cosets of $\pi_1(\Sigma)$ corresponding to these finitely many fundamental domains, we can find a finite index subgroup of $\pi_1(M)$ containing $\pi_1(\Sigma)$ but not containing any of these coset representatives. The finite cover M' corresponding to this subgroup is then also covered by F, and the projection of the normal neighborhood N to M' is injective. This shows that the normal injectivity radius of Σ can be made arbitrarily large by passing to finite covers of M.

2.6.5 Construction of the Metric

Outside of a large central neighborhood N in F, the metric g' agrees with the Fuchsian metric. By 2.6.4, we can find a finite cover M' of M such that every point on N is at a distance from Σ in the Fuchsian metric less than the normal injectivity radius of Σ in M'. Then N projects injectively to M' under the covering map, so we can use g' to define a new negatively curved metric on M' which we also call g'. Outside of the projection of N to M', g' is equal to the original hyperbolic metric on M'.

In this new metric g' on M', there are several properly embedded π_1-injective stable minimal surfaces whose fundamental groups include as subgroups in the conjugacy class of $\pi_1(\Sigma)$ in $\pi_1(M')$.

These lift to the universal cover of M' to give several distinct properly embedded minimal planes at finite Hausdorff distance from the same element of $\mathcal{P}_{\tilde{g}'}$.

Remark 2.6.2. It seems likely that the metric on M' we constructed can be joined to the constant curvature metric through a smooth path of negatively curved metrics by performing the above construction on a smooth path in quasi-Fuchsian space joining F to Q. Theorem 2.1.2 would then apply with $T < \infty$ to this path of metrics.

2.7 A Stability Estimate for the Foliations of Theorem 2.1.2

Fix a smooth family of metrics g_t, $t \in [0,1]$, to which Theorem 2.1.2 applies with $T = \infty$ to produce foliations. Let S be a totally geodesic plane in $\tilde{M} \cong \mathbb{H}^3$, and let S_t be the $\tilde{g}_t$-minimal plane in $\tilde{M}$ corresponding to S. For fixed $t_0 \in [0,1)$, elliptic PDE theory implies that S_t converges smoothly uniformly to S_{t_0}. For t close to t_0, S_t is a graph over S_{t_0} in normal coordinates for a tubular neighborhood of S_{t_0}, and so differentiating in t at t_0 we obtain a vector field v normal to S_{t_0}.

Theorem 2.7.1. *Let $\epsilon_0 < \epsilon$ be given, and suppose S_{t_0} is ϵ-subordinate. Then there exists δ depending only on ϵ and bounds on the g_t and the time derivatives g_t' of the g_t such that if $\delta > t - t_0 > 0$, then S_t is ϵ_0-subordinate. (The same δ works for every ϵ-subordinate S_{t_0}.)*

By bounds on g_t and g_t', we mean bounds on these tensors and their covariant derivatives up to second order in the L^∞ norms induced by the hyperbolic metric g_0.

Proof. The minimal surface equation for a graph u in $\mathbb{R}^3$ over a region in the xy-plane can be written

$$(1 + u_{x_2}^2)u_{x_1x_1} + (1 + u_{x_1}^2)u_{x_2x_2} - 2u_{x_1}u_{x_2}u_{x_1x_2} = 0. \tag{2.7.1}$$

If we dilate the metric and zoom in at a point x_0 on S_{t_0} at a scale where the metrics on both $\tilde{M}$ and S_{t_0} are almost Euclidean to second order, then the coefficients A^{ij},B^i,C of the second order equation

$$A^{ij}u_{x_ix_j} + B^i u_{x_i} + C = 0$$

S_{t_0} satisfies, writing it locally as a graph u over a coordinate plane, can be made arbitrarily C^0 close to those of Equation (2.7.1). The amount we need to dilate the metric at x_0 to obtain a given degree of closeness is determined by bounds on g_{t_0} and the principal curvatures of S_{t_0}, and the latter can be bounded in terms of bounds on g_{t_0} by the fact that S_{t_0} is ϵ-subordinate. For t close

to t_0 and x close to x_0, we can write the surfaces S_t as graphs $u(x,t)$, where $u(x,t_0) = u(x)$. Then the derivative vector v in these coordinates equals $u_t(x,t_0)$. Differentiating the minimal surface equations the $u(x,t)$ satisfy in time at t_0, we obtain a second order equation of the form

$$a^{ij} v_{x_i x_j} + b^i v_{x_i} + c = 0. \tag{2.7.2}$$

The a^{ij} can be made arbitrarily C^0 close to the coefficients of $u_{x_i x_j}$ in Equation (2.7.1) provided we dilated the metric enough at x_0, and so we can take (2.7.2) to be uniformly elliptic with ellipticity constant $1/2$. The amount we need to dilate the metric to ensure this again just depends on bounds on g_{t_0} and the principal curvatures of S_{t_0}. The b^i and c can be bounded in terms of the first two derivatives of u— which in turn are bounded by the absolute values of the principal curvatures of S_{t_0} and bounds on g_{t_0}, bounds on g_{t_0}, and bounds on g'_{t_0}. It follows that we can estimate the C^2-norm of v if we have a C^0 estimate for v [GT01, Theorem 6.2, Schauder Interior Estimates].

By Lemma 2.3.2, the mean curvatures of the parallel distance-s surfaces to S_{t_0} are greater than $\frac{\epsilon}{2}s$ for s small and positive and less than $\frac{\epsilon}{2}s$ for s small and negative. Taking a coordinate chart where the parallel distance-s surface is a graph y^s over a coordinate plane, we can write the mean curvature of the parallel distance-s surface in the metric g_t as

$$a^{ij}(t,s) y^s_{x_i x_j} + b^i(t,s) y^s_{x_i} + c(t,s),$$

for t close to t_0. Since

$$\max(|a^{ij}(t,s) - a^{ij}(t_0,s)|, |b^i(t,s) - b^i(t_0,s)|, |c(t,s) - c(t_0,s)) < M|t - t_0|$$

for some M just depending on the norm of g_t and its derivative in time and ϵ, we can find some small ξ such that if

$$|t - t_0| < \xi |s|, \tag{2.7.3}$$

then the distance-s parallel surfaces are mean-convex in the metric g_t. The constant ξ can be chosen so that this statement also holds for all of the other ϵ-subordinate (projections of) leaves of the foliation, not just the given S_{t_0} we are considering. One can then show by essentially the same argument as the proof of Lemma 2.3.8 that S_t is contained between the signed distance $\pm s$ parallel surfaces to S_{t_0} if t and s satisfy (2.7.3.) By sending t to t_0, this implies that the magnitude of v is bounded above by $1/\xi$ in the coordinates we chose. This gives the desired C^0 estimate for v.

In this way, we can obtain an upper bound for the C^2 norm of the normal derivative vector field $v(t)$ for S_t which are ϵ-subordinate. We thus obtain an upper bound for the C^2 norm of $v(t)$ for S_t that are ϵ_0-subordinate, since such S_t are also ϵ-subordinate. The surface S_{t_0} is ϵ_0-subordinate, and S_t remains ϵ_0-subordinate for $t - t_0$ less than some δ. One can obtain a lower bound for δ in terms of the difference $\epsilon - \epsilon_0$, bounds on g_t and g_t', and the bound on the C^2 norm of $v(t)$ by differentiating the formula for the principal curvatures of S_t with respect to time in a local coordinate chart, and using the C^2 bound on $v(t)$. This completes the proof.

□

Chapter 3

Beyond Almost Fuchsian Space

3.1 Introduction

3.1.1 Motivating Questions

Closed incompressible surfaces are fundamental in three-manifold theory. Thurston observed that a closed surface of principal curvatures less than 1 in magnitude is incompressible in a hyperbolic three-manifold and this was proved in [Lei06]. In the setting of complete hyperbolic three-manifolds which are diffeomorphic to $S \times \mathbb{R}$ (S a closed surface of genus at least two), closed surfaces of small curvatures, especially when they are also minimal, play an important role (see for instance [Uhl83, Rub05, KS07, CMN]).

There is a well-developed deformation theory for complete hyperbolic three-manifolds of the type $S \times \mathbb{R}$ (see for instance [Thu86, BB04, Min10, BCM12] and many others). For this class of hyperbolic three-manifolds without accidental parabolics, non-degenerate ones are quasi-Fuchsian. We denote the quasi-Fuchsian space by $\mathcal{QF}$, and the almost Fuchsian space, consisting of elements of quasi-Fuchsian that admit a closed *minimal surface* homeomorphic to S of principal curvatures less than one, by $\mathcal{AF}$. An almost Fuchsian manifold has many favorable properties. For instance it admits an equidistant foliation by closed surfaces: if $M \in \mathcal{AF}$ and Σ is the unique minimal surface in M, then $M = \bigcup_{r\in\mathbb{R}} \Sigma(r)$, where $\Sigma(r)$ is the surface with signed distance r from Σ (see calculations in [Uhl83, Eps84]). Many geometrical quantities associated to M are controlled in terms of λ_0, the maximum of the principal curvature over Σ: quasi-isometry constant between M and the corresponding Fuchsian manifold ([Uhl83]), Teichmüller distance between the conformal ends of M ([GHW10, Sep16]), volume of the convex core ([HW13]), Hausdorff dimension of the limit

set ([HW13]). Despite these estimates, some basic questions remain unanswered, for instance, how bad can an almost Fuchsian manifold be? or can it degenerate? This leads to some very interesting rigidity questions. More specifically, it is natural to ask:

Question 3.1.1. If $M \in \mathcal{QF}$, is there a constant $L > 0$, depending only on the genus of S, such that whenever the volume of the convex core is greater than L, then M is not almost Fuchsian?

and similarly,

Question 3.1.2. If $M = \mathbb{H}^3/\Gamma \in \mathcal{QF}$, is there a constant $\epsilon > 0$, depending only on the genus of S, such that whenever the Hausdorff dimension of the limit set for the group Γ is greater than $2 - \epsilon$, then M is not almost Fuchsian?

Furthermore, it is well-known that any almost Fuchsian manifold admits a unique closed minimal surface, while there are examples of quasi-Fuchsian manifolds which admit multiple or even arbitrarily many closed stable minimal surfaces ([And06, HW19]). This naturally leads to the question of whether having a unique stable minimal surface characterizes the closure $\overline{\mathcal{AF}}$ of almost Fuchsian space within quasi-Fuchsian space:

Question 3.1.3. Does there exist $M \in \mathcal{QF} \backslash \overline{\mathcal{AF}}$ such that M admits a unique stable closed incompressible minimal surface?

In this section of the thesis, we work to answer these questions and other related questions. Our approach is to construct a compactification of a wider class (we call *weakly almost Fuchsian space*) of complete hyperbolic three-manifolds of the type $S \times \mathbb{R}$ where each element admits a closed minimal surface of the maximum principal curvature $\lambda_0 \leq 1$. This class was also considered by Uhlenbeck ([Uhl83]).

3.1.2 Notation and Terminology

We list a few notations we will frequently refer to in this section of the thesis. Throughout this chapter S is a closed surface of genus $g \geq 2$.

(1) The Teichmüller space $\mathcal{T}_g(S)$ is the space of conformal structures on S, modulo biholomorphisms in the homotopy class of the identity. Every conformal structure $\sigma \in \mathcal{T}_g(S)$ on S admits a unique hyperbolic metric denoted by g_σ.

(2) $\mathcal{M}^3$ is the class of complete hyperbolic three-manifolds diffeomorphic to $S \times \mathbb{R}$.

(3) For a closed incompressible surface $\Sigma \subset M \in \mathcal{M}^3$, we always denote by $\lambda(\Sigma)$ its principal curvatures, and λ_0 the maximum absolute value of a principal curvature over Σ.

(4) $\mathcal{B} \subset \mathcal{M}^3$ is the subclass of $\mathcal{M}^3$ such that each $B \in \mathcal{B}$ admits a closed incompressible surface Σ' (diffeomorphic to S) with $|\lambda(\Sigma')| \leq 1$, and there exists at least one point $p \in \Sigma'$ such that $|\lambda(p)| < 1$. Note that we do not require Σ' to be minimal here.

(5) $\mathcal{QF} \subset \mathcal{M}^3$ is the space of quasi-Fuchsian manifolds.

(6) $\mathcal{AF} \subset \mathcal{QF}$ is the space of almost Fuchsian manifolds. Each $M \in \mathcal{AF}$ admits a closed incompressible minimal surface whose principal curvatures are strictly less than one in magnitude, namely, $\lambda_0 < 1$. As a consequence of the maximum principle, every almost Fuchsian manifold admits a unique closed minimal surface ([Uhl83]).

(7) $\mathcal{B}_0$ is the subclass of $\mathcal{M}^3$ such that each $B \in \mathcal{B}_0$ admits a closed incompressible minimal surface Σ (diffeomorphic to S) with $\lambda_0 \leq 1$. We call such B *weakly almost Fuchsian* and $\mathcal{B}_0$ the *weakly almost Fuchsian space*. Note that by this definition, $\mathcal{AF} \subset \mathcal{B}_0$.

We know that $\mathcal{B}_0 \subset \mathcal{B}$ because there must be some point $p \in \Sigma$ such that $\lambda(p) = 0$. This is due to the fact that the second fundamental form of a minimal immersion in a manifold of constant curvature is the real part of a holomorphic quadratic differential ([Hop89]), and any holomorphic quadratic differential on a closed Riemann surface of genus $g \geq 2$ has exactly $4g - 4$ zeros, counting multiplicity.

3.1.3 Main results

It has been an open question ([Theorem 3.3, [Uhl83]]) whether M must be quasi-Fuchsian if $M \in \mathcal{M}^3$ admits a closed incompressible minimal surface Σ such that $|\lambda(\Sigma)| \leq 1$ and $\lambda_0 = 1$. Note that while a quasi-Fuchsian manifold does not have accidental parabolics, apriori a weakly almost Fuchsian manifold may. A partial answer was given in ([San17b]) where the author showed there are no doubly degenerate limits of almost Fuchsian groups. Our first result is the following more general statement:

Theorem A. If B is a complete hyperbolic three-manifold of the type $S \times \mathbb{R}$ and it admits a closed incompressible surface Σ' (diffeomorphic to S but not necessarily minimal) with $|\lambda(\Sigma')| \leq 1$, there exists at least one point $p \in \Sigma'$ such that $|\lambda(p)| < 1$, and if the corresponding Kleinian group has no accidental parabolics then B is quasi-Fuchsian.

See Section 4.2 for definitions. As an immediate consequence, we have

Corollary 3.1.1. *Any weakly almost Fuchsian manifold that contains no accidental parabolics is quasi-Fuchsian.*

When we do not assume away accidental parabolics, we are able to prove the following statement:

Theorem A1. Every element of $\mathcal{B}_0$ is geometrically finite, i.e., no element of $\mathcal{B}_0$ has a degenerate end.

Rubinstein [Rub05] gave an example that suggests that there might actually be elements of $\mathcal{B}_0$ that contain accidental parabolics– see Remark 3.3.1. It remains unclear whether $\mathcal{B}_0 = \mathcal{B}$. This is similar to the following open question ([Rub05]): if $M \in \mathcal{QF}$ admits a closed incompressible surface (not necessarily minimal) of principal curvatures less than 1 in magnitude, then is $M \in \mathcal{AF}$?

To derive rigidity properties for the weakly almost Fuchsian space, we construct a compactification $\overline{\mathcal{B}_0}$ of $\mathcal{B}_0$. This compactification is analogous to the compactification of the space of Kleinian surface groups constructed by Canary-Storm ([CS12]). The difference with their approach is that our compactification is defined in terms of data associated to the unique minimal surface in each element of $\mathcal{B}_0$.

Theorem B. There exists a compactification $\overline{\mathcal{B}_0}$ of the (unmarked) weakly almost Fuchsian space $\mathcal{B}_0$ that extends the Deligne-Mumford compactification of moduli space of Riemann surfaces. Moreover, $\overline{\mathcal{B}_0}$ can be topologized so that the volume of the convex core extends to a continuous function on $\overline{\mathcal{B}_0}$.

The construction of our compactification is inspired by Canary-Storm's approach. The points at infinity that we add correspond to disjoint unions of cusped weakly almost Fuchsian manifolds. By working with minimal surfaces our approach is adapted to the applications below, and seems to require less in the way of structural results about Kleinian surface groups, although we work in a more specialized setting. We comment that since $\mathcal{B}_0$ stays away from singly or doubly degenerate Kleinian surface groups, our compactification is better behaved than the Canary-Storm compactification, which has non-closed points.

Theorem B allows us to answer Question 3.1.1 and Question 3.1.2:

Corollary 3.1.2. *There is an $\epsilon > 0$ and $L > 0$ such that for every element $M \in \mathcal{B}_0$ such that the Hausdorff dimension of the limit set corresponding to M is at most $2 - \epsilon$, and the volume of the convex core of M is at most L.*

In Corollary 3.1.2 the constants ϵ and L depend on the genus g of the surface S. It would be interesting to determine how these constants depend on the genus.

Another application is to prove a gap theorem for the maximum principal curvature of minimal surfaces in fibered hyperbolic three-manifold:

Theorem C. For every $g \geq 2$ there is an ϵ depending only on g such that if a closed hyperbolic three-manifold fibers over the circle with fiber a surface of genus g, then any embedded minimal surface isotopic to the fiber has the maximum principal curvature larger than $1 + \epsilon$.

Farre-Vargas-Pallete recently obtained results similar to Theorem C for certain sequences of hyperbolic mapping tori [FVP21]. Their approach is more direct than that of this paper and is based on an analysis of how minimal surfaces interact with curves of short length in the ambient hyperbolic 3-manifold.

We also answer Question 3.1.3 by constructing examples of quasi-Fuchsian manifolds which are not weakly almost Fuchsian but that each admit a unique *stable* minimal surface.

Theorem D. There exist $M \in \mathcal{QF} \backslash \mathcal{B}_0$ that contain a unique closed stable minimal surface with the maximum principal curvature $\lambda_0 > 1$ at some point.

3.1.4 Outline

After reviewing some relevant preliminary facts about Kleinian surface groups and almost Fuchsian manifolds in §3.2, we will prove our main results in Sections §3.3, §3.4 and §3.5. In particular Theorems A and A1 are proved in §3.3 and the compactification of the weakly almost Fuchsian space is constructed in §3.4, which proves Theorem B and deduces some applications of the compactness result, and finally in §3.5 we prove Theorems C and D.

3.2 Preliminaries

3.2.1 Kleinian Surface Groups

A Kleinian group Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$, the orientation preserving isometry group of $\mathbb{H}^3$. Any complete hyperbolic three-manifold can be written as some $\mathbb{H}^3/\Gamma$. Since we work with the Poincaré ball model for $\mathbb{H}^3$, we denote by S^2_∞ the sphere at infinity, and $\Lambda_\Gamma \subset S^2_\infty$ the limit set of Γ as the set of accumulation points on S^2_∞ of the orbits Γx of $x \in \mathbb{H}^3$. The domain of discontinuity for Γ is defined to be $\Omega_\Gamma = S^2_\infty \backslash \Lambda_\Gamma$. We restrict ourselves to the case when Γ is a Kleinian surface group, namely, Γ is isomorphic to the fundamental group of a hyperbolic surface of finite area. Equivalently Γ can be viewed as the image of a discrete and faithful representation from the fundamental group

of such a surface S to $\mathrm{PSL}(2,\mathbb{C})$ (we always assume $\pi_1(S)$ contains no parabolics). It's a basic result of Thurston and Bonahon ([Bon86]) that the quotient $\mathbb{H}^3/\Gamma$ is diffeomorphic to $S \times \mathbb{R}$.

The convex hull of Γ is the smallest non-empty closed convex subset of $\mathbb{H}^3$ invariant under Γ, and its quotient, $C(M)$, by Γ is the *convex core* of the hyperbolic three-manifold $M = \mathbb{H}^3/\Gamma$. We call Γ *geometrically finite* if the volume of $C(M)$ is finite. Otherwise we call it *geometrically infinite*.

When the limit set of Γ is a round circle, it spans a totally geodesic hyperbolic disk inside $\mathbb{H}^3$, and Γ is called a Fuchsian group and $M = \mathbb{H}^3/\Gamma$ a Fuchsian manifold. In fact it is a warped product of a hyperbolic surface with the real line, of the following explicit metric expression: $dt^2 + \cosh^2(t) g_\sigma$, where g_σ is the hyperbolic metric on S. When Λ_Γ is homeomorphic to a circle, we call Γ (and the quotient manifold) *quasi-Fuchsian*. Analytically a quasi-Fuchsian manifold is a quasi-conformal deformation of a Fuchsian manifold, and hence geometrically finite and the convex core $C(M)$ is homeomorphic to $S \times [0,1]$.

A parabolic element $\gamma \in \Gamma$ is *accidental parabolic* if it is conjugated to a hyperbolic element by a conformal map between a simply connected invariant component of Ω_Γ and the unit disc. When a Kleinian surface group Γ does not contain any accidental parabolics, then ([Bon86, Thu86]) $\mathbb{H}^3/\Gamma$ is either quasi-Fuchsian, or $C(M)$ is homeomorphic to $S \times [0,\infty)$ (*simply degenerate*) or $C(M)$ is homeomorphic to $S \times (-\infty,\infty)$ (*doubly degenerate*).

A sequence of Kleinian groups Γ_i converges to Γ *algebraically* if there are isomorphisms $\Gamma \to \Gamma_i$ that converge to the identity. The sequence Γ_i converges *geometrically* to Γ if there exists a sequence of balls $B_i \subset \mathbb{H}^3$ that exhaust $\mathbb{H}^3$ such that B_i/Γ_i can be mapped k_i-quasi-isometrically onto its image in $\mathbb{H}^3/\Gamma$ and $k_i \to 1$ as $i \to \infty$. Finally Γ_n converges *strongly* to Γ if it converges both algebraically and geometrically to Γ.

3.2.2 Almost Fuchsian Manifolds

A quasi-Fuchsian manifold is topologically $S \times \mathbb{R}$, but it can be very complicated geometrically. Uhlenbeck and others introduced techniques of minimal surfaces of small principal curvatures to study Kleinian surface groups. In particular Uhlenbeck ([Uhl83]) defined a subclass: a quasi-Fuchsian manifold M is *almost Fuchsian* if it admits a closed *minimal surface* S whose maximum principal curvature satisfies $\lambda_0 < 1$. A natural question is how far it is to the boundary of the quasi-Fuchsian space in the deformation space of Kleinian surface groups. She was able to derive an explicit formula for the hyperbolic metric of an almost Fuchsian manifold M in terms of the conformal structure of the unique minimal surface S and its second fundamental form. She accomplished this by showing

that the normal exponential map at the unique minimal surface gave global coordinates on M. This enabled her to prove that the quasi-isometric constant for a quasi-isometry between an almost Fuchsian manifold and a Fuchsian manifold is bounded above by $\frac{1+\lambda_0}{1-\lambda_0}$. Note that if λ_0 is allowed to be 1, this estimate yields no information.

Among the invariants of a quasi-Fuchsian manifold, the volume of the convex core and the Hausdorff dimension of Λ_Γ are particularly important. For a quasi-Fuchsian manifold, it is well-known that the Hausdorff dimension is an analytic function defined on quasi-Fuchsian space ([Rue82]), and is valued between 1 (exactly when Γ is Fuchsian) and 2 ([Bow79]). When Γ is in the class of almost Fuchsian, we know the Hausdorff dimension is bounded from above by $1+\lambda_0^2$, and the volume of the convex core is bounded from above by $4\pi(g-1)(\frac{\lambda_0}{1-\lambda_0^2}+\frac{1}{2}\ln\frac{1+\lambda_0}{1-\lambda_0})$ ([HW13]). Neither estimate gives any information if λ_0 is allowed to tend to 1.

Both [Uhl83] and [Eps86] considered the case of *weakly almost Fuchsian*, namely, allowing $\lambda_0 = 1$. Because the explicit hyperbolic three-manifold for the three-manifold is non-singular even when $\lambda_0 \leq 1$ (see Page 161 of [Uhl83]), there exist global coordinates for a weakly almost Fuchsian manifold via the hyperbolic Gauss map. This is the key geometric property of weakly almost Fuchsian manifolds that we will use.

3.3 Weakly Almost Fuchsian is Geometrically Finite

In this section we will prove Theorem A and Theorem A1. Theorem A1 is the first and most important step in constructing the compactification of Theorem C. Our arguments build on ideas of Sanders [San17b].

Proof of Theorem A:

Let $M \in \mathcal{B}$, then M contains a surface Σ diffeomorphic to S such that $|\lambda(\Sigma)| \leq 1$. For contradiction, assume that one of the ends of M is simply degenerate. We choose a properly embedded disk $\tilde{\Sigma}$ lifting Σ to the universal cover $\tilde{B} \cong \mathbb{H}^3$. Then the normal exponential map

$$\eta : \tilde{\Sigma} \times \mathbb{R} \to \mathbb{H}^3$$

is a diffeomorphism by, for example, the arguments in [Eps86]. The forward and backwards Gauss maps

$$\mathcal{G}^{\pm}_{\tilde{\Sigma}} : \tilde{\Sigma} \to \partial_\infty \mathbb{H}^3$$

associated to $\tilde{\Sigma}$ are defined by

$$\mathcal{G}^{\pm}_{\tilde{\Sigma}}(p) = \lim_{t \to \pm\infty} \eta(p,t),$$

where we understand these limits to lie in the sphere at infinity in the Poincaré ball model for $\mathbb{H}^3$.

Let U be a small disk in $\tilde{\Sigma}$ such that $|\lambda(\tilde{\Sigma})| < 1$ on the closure of U. It follows from Epstein [Equation (5.5) in [Eps86]] that $\mathcal{G}^{\pm}_{\tilde{\Sigma}}|_U$ has quasi-conformal dilatation bounded from above by $\frac{2}{\epsilon}$, where

$$\epsilon = \min_{i,j=1,2}(|1 + (-1)^j \kappa_i|), \tag{3.3.1}$$

and κ_1 and κ_2 are the supremal and infimal principal curvatures of $\tilde{\Sigma}$ on U. The generalization of the Koebe 1/4 Theorem proved by Astala and Gehring ([AG85]) implies that the image of U under $\mathcal{G}^{\pm}_{\tilde{\Sigma}}$ contains a disk $D^{\pm}$ in the sphere at infinity (in its spherical metric) of radius bounded below by a constant that only depends on the quasiconformal dilatation. The disks $D^{\pm}$ are disjoint from the limit set of $\tilde{\Sigma}$ by the existence of global normal exponential coordinates: The convex hulls of $\overline{D}^{\pm}$ define solid hemisphere regions $H^{\pm}$ in $\mathbb{H}^3$ disjoint from $\tilde{\Sigma}$, and the fact that η is a diffeomorphism implies that $\tilde{\Sigma}$ is disjoint from $H^{\pm}$. The regions $H^{\pm}$ both project to open regions outside of the convex core of M. Since they project to different ends of M, this shows that M has no non-degenerate ends. Bonahon ([Bon86]) proved that both ends of M are geometrically tame. This means that their ends are either geometrically finite or simply degenerate. Since both ends of M are geometrically finite, and M contains no accidental parabolics, it follows that M is quasi-Fuchsian.
□

Remark 3.3.1. Rubinstein [Rub05] constructed examples of essential immersed surfaces Σ in the figure eight knot complement with principal curvatures less than or equal to 1 in magnitude, for which the associated Kleinian surface group has an accidental parabolic. The accidental parabolics correspond to embedded horocycles that lie on the surfaces Σ. Although the Σ are not minimal, it seems possible that they are homotopic to minimal surfaces with $\lambda_0 \leq 1$. If this is the case, then there would be elements of $\mathcal{B}_0$ for which the corresponding Kleinian groups had accidental parabolics.

If there are accidental parabolics in the corresponding Kleinian surface group of $M \in \mathcal{B}_0$, we will take advantage the explicit metric (see (3.3.2)) on M to prove that M is geometrically finite.

Proof of Theorem A1:

The convex core of M is homeomorphic to $\Sigma \times [0,1]$, and is determined by the following information [BCM12]. There is a multicurve C on $S \times \{1\}$ each component of which corresponds to an

upward cusp in M and an accidental parabolic in the Kleinian surface group. Each component of the complement of the multicurve in $S \times \{1\}$ either corresponds to an upward geometrically finite end or an upward degenerate end. The downward end is described in a similar way. The only compatibility condition is that the multicurve for the downward end not contain any of the curves in C .

We first show that every component of the multicurve C is realized by an embedded horocycle in M that lies on Σ. For an alternative approach to constructing embedded horocyles on Σ see Proposition 3.4 and the proof of Theorem 4.2 in [FVP21]. These horocycles are preserved under the normal exponential map, and thus serve as barriers guaranteeing that geodesics normal to Σ are eventually contained in the end corresponding to the connected component of the complement of C from which they began. This allows us to apply the argument in the proof of Theorem A above to rule out degenerate ends.

Assume for contradiction that M has a degenerate end, which we can assume is upward and corresponds to a complementary region Σ_0 of C in Σ. Let γ be a connected component of the boundary of Σ_0. The curve γ corresponds to an accidental parabolic, and so we can find a sequence of horocycles γ_n homotopic to γ exiting the end with lengths tending to zero.

Recall that the normal exponential map $\eta : \Sigma \times \mathbb{R} \to M$ for Σ is a diffeomorphism and gives global coordinates for M. In these coordinates Uhlenbeck ([Uhl83]) showed that the metric can be written as:

$$e^{2v(x)}(\cosh(r)\mathrm{I} + \sinh(r)e^{-2v(x)}A(x))^2 + dr^2, \tag{3.3.2}$$

where $e^{2v(x)}$ is the conformal factor of the induced metric on Σ in conformally flat coordinates, I is the hyperbolic metric in these coordinates, and $A(x)$ is the second fundamental form of Σ at x.

It is clear from this formula that any tangent vector to Σ gets exponential expanded as $r \to \infty$ unless it is tangent to a principal direction with principal curvature 1. Note that the function $s = |A|^2$ attains its maxima precisely at the points where the principal curvatures are 1 and that s is real analytic on Σ (see e.g. [Proof of Lemma 4.1, [WW20]]). Therefore the set of points Γ where Σ has principal curvature equal to 1 has connected components that are either isolated points or embedded graphs.

In the normal exponential coordinates η and since the length of γ_n tends to zero as n tends to infinity, for each γ_n we can choose some r_n and ϵ_n so that $\gamma_n \subset \eta(\Sigma \times [r_n, r_n + \epsilon_n])$ and $\epsilon_n \to 0$, as $n \to \infty$. Let γ_n' be the normal projection of γ_n to $\eta(\Sigma \times \{r_n\})$, whose length in M also tends to zero as $n \to \infty$, and which we can identify with a curve $\hat{\gamma}_n$ in $\Sigma \times \{0\}$ homotopic to γ. By the exponential expansion of tangent vectors not tangent to a principal direction with principal curvature -1, we

know that for every $\epsilon > 0$ the ϵ-neighborhood of Γ will contain $\hat{\gamma}_n$ for sufficiently large n. Since the $\hat{\gamma}_n$ are essential, we know that for n large enough they must be ϵ-close to the components of Γ that are embedded graphs (as opposed to the components that are isolated points.)

For n large enough there is a well-defined projection of the $\hat{\gamma}_n$ to Γ. We thus obtain curves $\overline{\gamma}_n$ in the same homotopy class as $\hat{\gamma}_n$ whose lengths considered as subsets of $\eta(\Sigma \times \{r_n\})$ tend to zero as n tends to infinity. This is because we can guarantee that $\hat{\gamma}_n$ stays away from any edge of Γ that at some point fails to travel tangent to a principal curvature -1 principal direction by taking n large enough. Moreover, up to modifications by homotopies that do not increase the length inside $\eta(\Sigma \times \{r_n\})$, we can take $\overline{\gamma}_n$ to be a concatenation of edges of Γ, which we call $\Gamma_0(n)$.

If $\Gamma_0(n)$ either intersected itself transversely or contained a point of piecewise non-differentiability, then $\Gamma_0(n)$ would contain a segment of a definite length none of whose vectors were tangent to a principal direction with principal curvature -1, which is impossible provided n was taken large enough. Consequently $\Gamma_0(n)$ is a smoothly embedded horocycle for n large. Since $\Gamma_0(n)$ is homotopic to γ, this proves that γ is homotopic in Σ to a horocycle in M.

Running this argument for every connected component of the boundary of Σ_0 determines a surface Σ_0' in Σ homeomorphic to Σ_0 whose boundary components are horocycles embedded in Σ. Choose a small neighborhood U of a point in the interior of Σ_0' on which the principal curvatures are strictly less than 1. Then for some $T > 0$ all endpoints of geodesic segments that begin normal to U and have length larger than T are contained in the degenerate end. We can then lift U to the universal cover and apply the argument from the proof of Theorem A above to produce points in normal geodesic rays from U that are not contained in the convex core of M, in contradiction with the assumption that the end corresponding to Σ_0 was non-degenerate. It follows that all ends are non-degenerate, and that M is geometrically finite. □

An immediate consequence is that elements of $\mathcal{B}_0$ which have no accidental parabolics are quasi-Fuchsian, which is Corollary 3.1.1.

Although the statements of Theorems A and A1 are for closed surfaces S, the same proof applies to prove them for punctured surfaces S' of finite type and negative Euler characteristic. This will be important for the construction of the compactification in the next section, and we write it as a corollary.

Corollary 3.3.1. *Let M_1 be a hyperbolic three-manifold diffeomorphic to $S' \times \mathbb{R}$, where S' is a complete surface of genus $g \geq 2$ with finitely many punctures. If M_1 admits an incompressible minimal surface Σ' (diffeomorphic to S') with $|\lambda(\Sigma')| \leq 1$ then M_1 is geometrically finite.*

There is also a notion of a closed surface being quasi-Fuchsian. A closed surface S of genus $g \geq 2$ in a complete hyperbolic three-manifold N is called quasi-Fuchsian if a lift of the inclusion of the universal covers is a quasi-isometry. This is equivalent to S being π_1-injective and the cover of N corresponding to $\pi_1(S)$ being quasi-Fuchsian. A result of Thurston (proved in [Lei06]) states that a closed surface (of genus at least 2) in a complete hyperbolic three-manifold is quasi-Fuchsian if its principal curvatures are strictly less than 1 in magnitude. The proof of Theorem A generalizes this to the following:

Corollary 3.3.2. *If N is a complete hyperbolic three-manifold, and S is a closed surface in N such that the principal curvatures are less than or equal to one in magnitude and strictly less than one in magnitude at some point, and if the Kleinian group corresponding to S contains no accidental parabolics, then S is quasi-Fuchsian.*

Proof. We consider the lift $\tilde{S}$ of S in $\mathbb{H}^3$. Then the principal curvatures $\lambda(\tilde{S})$ of $\tilde{S}$ satisfy that $|\lambda(\tilde{S})| \leq 1$ and there exists some point $\tilde{p} \in \tilde{S}$ such that $|\lambda(p)| < 1$. If $\tilde{S}$ were not homeomorphic to a disk, then taking a closed geodesic in $\tilde{S}$ in its induced metric and applying the argument in the proof of [Eps84][Theorem 3.4] would give a contradiction. Epstein shows that the hyperbolic cosine of the distance from the starting point of a curve in $\mathbb{H}^3$ with geodesic curvature less than or equal to 1 in absolute value (such as a geodesic on $\tilde{S}$ in its induced metric) is convex along that curve, and that therefore such a curve cannot return to its starting point. It follows that $\tilde{S}$ must be a disk. The argument can then proceed as in the proof of Theorem A above to show that S is quasi-Fuchsian. □

3.4 Compactifying Weakly Almost Fuchsian Space

In this section, we construct the compactification $\overline{\mathcal{B}}_0$ of the space of unmarked weakly almost Fuchsian manifolds $\mathcal{B}_0$. Our compactification extends the Deligne-Mumford compactification of the moduli space of Riemann surfaces, and is analogous to the compactification defined by Canary-Storm [CS12] of the space of unmarked Kleinian surface groups.

Proof of Theorem B:

Our construction is to utilize a triple of data on the unique minimal surface on a weakly almost Fuchsian manifold. Taking a sequence of $M_k \in \mathcal{B}_0$, we let Σ_k be the unique closed embedded minimal surface in M_k, σ_k be the corresponding conformal structure, and α_k be the holomorphic quadratic differential in (Σ_k, σ_k) that encodes the second fundamental form of the minimal immersion. We

write the induced metric on Σ_k as the hyperbolic metric g_{σ_k} multiplied by a conformal factor e^{2u_k}.

Recall that Uhlenbeck [Uhl83] showed that any triple $(g_\sigma, e^{2u}, \alpha)$ of a hyperbolic metric, conformal factor, and holomorphic quadratic differential on (Σ, g_σ) that satisfies the Gauss equation and has principal curvatures less than or equal to one gives a unique hyperbolic structure M on $\Sigma \times \mathbb{R}$ such that there is a minimal surface in M with second fundamental form given by the real part $\Re(\alpha)$ of α, principal curvatures no more one in magnitude, and induced metric $e^{2u}g_\sigma$. This is the unique minimal surface in M. Furthermore, under the principal curvature condition $\lambda_0 \leq 1$, the solution u for the Gauss equation is unique. That α be holomorphic is equivalent to the second fundamental form it defines satisfying the Codazzi equations, provided the surface is minimal. When we say that a triple $(g_\sigma, e^{2u}, \alpha)$ satisfies the Gauss equation and has principal curvatures less than or equal to one, we mean that a minimal surface in a hyperbolic three-manifold with induced metric $e^{2u}g_\sigma$ and second fundamental form given by $\Re(\alpha)$ has this property if it exists. We know a posteriori that a minimal surface with this data exists by [Uhl83].

Returning to the sequence M_k, which by the last paragraph is determined by the sequence of triples $(g_{\sigma_k}, e^{2u_k}, \alpha_k)$, we can pass to a subsequence, which by abuse of notation we also denote by $(g_{\sigma_k}, e^{2u_k}, \alpha_k)$, of the unmarked hyperbolic structures g_{σ_k} that converge to a point in the Deligne-Mumford compactification of the moduli space. This point is given by a disjoint union of cusped surfaces $\overline{\Sigma}_1, ..., \overline{\Sigma}_n$.

We can pass to a subsequential limit of the u_k, because they are uniformly bounded in L^∞ and satisfy an elliptic equation. In fact we see first that $u_k \leq 0$ by the maximum principle as in [Uhl83]. Furthermore the principal curvature condition $\lambda_0 \leq 1$ implies that the Gaussian curvatures are $-1 - \lambda_0^2 \geq -2$. By the conformal change equation the Gaussian curvature is given by $e^{-2u_k}(-1 - \Delta_{\sigma_k} u_k)$, where Δ_{σ_k} is the Laplace operator for the hyperbolic metric g_{σ_k}. Therefore we have

$$e^{-2u_k}(-1 - \Delta_{\sigma_k} u_k) \geq -2,$$

and we deduce by the maximum principle that $u_k \geq \frac{-\ln(2)}{2}$.

Therefore by passing to a subsequence of the α_k and conformal factors e^{2u_k}, we get holomorphic quadratic differentials $\overline{\alpha}_1, ..., \overline{\alpha}_n$ and smooth functions $e^{2\bar{u}_i}$ on each of the $\overline{\Sigma}_1, ..., \overline{\Sigma}_n$. For each cusped surface $(\overline{\Sigma}_i, e^{2\bar{u}_i}, \overline{\alpha}_i)$ we can then construct a cusped weakly almost Fuchsian manifold $\overline{M}_i$, by Corollary 3.3.1. We define $\overline{\mathcal{B}}_0$ to be the set of all disjoint unions of $\overline{M}_i$ that can be obtained as subsequential limits in this way.

The set $\overline{\mathcal{B}}_0$ has a topology, extending that of the Deligne-Mumford compactification, for which

the total space is compact. This topology is defined in terms of the data of the triple $(g_\sigma, e^{2u}, \alpha)$ of conformal structure, conformal factor, and holomorphic quadratic differential on the minimal surfaces. The first part of this data gives a point in the Deligne-Mumford compactification. Fix a metric d_{DM} on the Deligne-Mumford compactification. For points $p = (g_{\sigma_p}, e^{2u_p}, \alpha_p)$ and $p' = (g_{\sigma_{p'}}, e^{2u_{p'}}, \alpha_{p'})$ in $\overline{\mathcal{B}}_0$, we define $d_\epsilon(p, p')$ as follows: the points p and p' correspond to complete hyperbolic manifolds M_p and $M_{p'}$ diffeomorphic to the product of a (possibly disconnected) surface with $\mathbb{R}$. For boundary points of the compactification these complete hyperbolic manifolds will be the disjoint unions of the $\overline{M}_i$ from the previous paragraph. Let Φ_ϵ be a map, diffeomorphic onto its image, from the ϵ-thick part Σ_ϵ of the unique (possibly disconnected) minimal surface in M_p in its induced metric $e^{2u_p} g_{\sigma_p}$, to the unique minimal surface for $M_{p'}$ (when M_p has multiple connected components by the unique minimal surface we mean the union of the unique minimal surface in each component). Let Distortion(Φ_ϵ) equal the maximum of the Lipschitz constants of Φ_ϵ and Φ_ϵ^{-1}, where Φ_ϵ^{-1} is defined on the image of Φ_ϵ. Set

$$d_{\Phi_\epsilon}(p, p') = ||\Phi_\epsilon^* e^{2u_{p'}} - e^{2u_p}||_\infty + \log(\text{Distortion}(\Phi_\epsilon)),$$

Then we set

$$d_\epsilon(p, p') = d_{DM}(g_{\sigma_p}, g_{\sigma_{p'}}) + \inf_{\Phi_\epsilon} d_{\Phi_\epsilon}(p_1, p_2),$$

where the infimum is taken over all Φ_ϵ as above. Finally, we define a metric d on $\overline{\mathcal{B}}_0$ by

$$d(p, p') = \sum_{n=0}^{\infty} \frac{1}{2^n} d_{\frac{1}{2^n}}(p, p'),$$

and take the topology on $\overline{\mathcal{B}}_0$ induced by this metric. Note that the subspace consisting of p for which $u_p = 0$ and $\alpha_p = 0$ is homeomorphic to the Deligne-Mumford compactification. We also point out that since the holomorphic quadratic differential α_p for $p \in \overline{\mathcal{B}}_0$ is equivalent to the second fundamental form of the unique minimal surface in M_p, which is determined by the induced metric $e^{2u_p} g_{\sigma_p}$, convergence in the metric d implies convergence of the corresponding sequence of holomorphic quadratic differentials.

To show that $\overline{\mathcal{B}}_0$ is compact in the metric d, take a sequence p_k in $\overline{\mathcal{B}}_0$ corresponding to a sequence M_k of possibly disconnected hyperbolic 3-manifolds. First, by the compactness of the Deligne-Mumford compactification we can pass to a convergent subsequence of the conformal structures. Then since the conformal factors have uniform L^∞ bounds and satisfy the second order elliptic PDE given by the Gauss equation, we can pass to a smoothly convergent subsequence of the conformal

factors. This uniquely determines a holomorphic quadratic differential, and we thus obtain a triple in $\overline{\mathcal{B}}_0$ to which a subsequence of the p_k converge.

To complete the proof of Theorem B, we now further analyze the convergence of the M_k to the disjoint union $\sqcup_{i=1}^m \overline{M}_i$. In the proof of the next proposition we assume for simplicity that the M_k are almost-Fuchsian interior points of the compactification $\overline{\mathcal{B}}_0$; the proof in the general case is very similar.

Proposition 3.4.1. The volumes of the convex cores of the M_k converge to the sum of the volumes of the convex cores of the $\overline{M}_i$.

Proof. Let $c_1, .., c_\ell$ be the simple closed curves on the minimal surface $\Sigma_k \subset M_k$ which become nodes in the limit. We claim that normal neighborhoods in M_k of each connected component $C_i(k)$ of the complement of the disjoint union of curves homotopic to the c_j are converging to the $\overline{M}_i$ on compact subsets, $i = 1, .., m$ (we implicitly choose some consistent marking and identification of all of the Σ_k so that this makes sense.) By normal neighborhood we mean the image of some subset of the form $C_i(k) \times (-L, L)$ of the normal bundle to $C_i(k)$ under the normal exponential map.

More precisely, for each $\overline{M}_i$ there exist a map $h_k^i : \overline{M}_i \to M_k$ whose image is a normal neighborhood of the complementary region $C_i(k)$, which is a homotopy equivalence onto its image, and which restricted to any compact subset of $\overline{M}_i$ is a diffeomorphism onto its images for large enough k. Furthermore, on each compact subset of $\overline{M}_i$ the maps h_k^i are smoothly converging to isometries as k tends to infinity. This follows from the explicit formula (3.3.2) for the metric on the normal neighborhood of the minimal surfaces Σ_k with $\lambda_0(\Sigma_k) \leq 1$, and the fact that the metrics on the $C_i(k)$ are smoothly converging to $e^{2\overline{u}_i}$ times the hyperbolic metric on $\overline{\Sigma}_i$. Fix some i. It follows, for a choice of basepoints $q_k = q_k(i)$ of M_k, each the h_k^i-image of some fixed point q in $\overline{M}_i$, that (M_k, q_k) converges geometrically to $(\overline{M}_i, q)$. Here the choice of basepoints amounts to, in the limit, throwing out the complement of $\pi_1(C_i(k))$ in $\pi_1(M_k)$. It also follows that, for all i, the Kleinian group $\Gamma_{C_i(k)}$ obtained by restricting the Kleinian group for M_k to $C_i(k)$ converges strongly as $k \to \infty$ to a Kleinian group $\overline{\Gamma}_i$ such that $\overline{M}_i = \mathbb{H}^3/\overline{\Gamma}_i$.

Identify all of the universal covers of the M_k and the $\overline{M}_i$ with a fixed $\mathbb{H}^3$, such that $\mathbb{H}^3$ has a basepoint 0 that projects to the q_k and q. We think of 0 as the origin in the Poincare ball model for $\mathbb{H}^3$.

The strong convergence of the $\Gamma_{C_i(k)}$ to $(\overline{M}_i, q)$ implies that for each ϵ the ϵ-thick parts of the convex cores of the $\mathbb{H}^3/\Gamma_{C_i(k)}$ (thought of as subsets of (M_k, q_k)) converge to the ϵ-thick part of the convex core of $\overline{M}_i$ [McM99][Theorem 4.1] (see also [Tay97].) Here we identify compact subsets of

$\overline{M}_i$ with compact subsets of (M_k, q_k)) by means of the h_k^i. Recall that the ϵ-thick part of the convex core is the set of points of the convex core with injectivity radius at least ϵ, and the ϵ-thick part of any geometrically finite manifold is compact. We now finish the proof assuming the following lemma:

Lemma 3.4.2. For each i and $q_k = q_k(i)$ as above the convex cores of (M_k, q_k) Hausdorff converge to the convex core of $(\overline{M}_i, q)$ on compact sets.

By repeating the arguments above for $i = 1, .., m$, together with the lemma, we get that for large enough k the ϵ-thick part of the convex core of M_k has at least m components. Each connected component of the ϵ-thin part of the convex core has volume bounded above by some constant that tends to zero as $\epsilon \to 0$. To show convergence of the volumes of the convex cores of the M_k to the sum of the volumes of the convex cores of the $\overline{M}_i$, it is therefore enough to show for each fixed small ϵ that there are not pieces of the ϵ-thick part of the convex core that go off to infinity and result in the convex core losing volume in the limit. Put more precisely, it is enough to show that for k sufficiently large, the ϵ-thick part of the convex core of M_k has exactly m connected components, one for each of the $\overline{M}_i$, and that they converge to the ϵ-thick part of the convex core of the corresponding $\overline{M}_i$. There will then be a uniformly bounded number of connected components of the ϵ-thin part of the convex core of M_k– these come from elements of $\pi_1(M_k)$ that become nodes or accidental parabolics in the limit. Call the m connected components M_k^i, $i = 1, .., m$, of the convex core of M_k, that correspond to the ϵ-thick parts of the convex cores of the $\overline{M}_i$, the ϵ-*permanent* part of the convex core of M_k. To finish the proof, it is enough to rule out other ϵ-thick connected components.

There are two kinds of complementary regions to the ϵ-permanent part of the convex core: regions that contain closed geodesics that become nodes in the limit and regions that contain closed geodesics that become accidental parabolics in the limit. Each such region contains a unique closed geodesic γ_k whose length tends to zero as $k \to \infty$. Provided ϵ was taken sufficiently small and k was taken sufficiently large, the closure of the connected component $C(\gamma_k)$ of the ϵ-thin part of the convex core containing γ_k contains the components of the boundaries of the region or regions $M_k^{i_1}$ and $M_k^{i_2}$ that meet $C(\gamma_k)$. If γ_k corresponds to to an accidental parabolic or a nodal curve both sides of which are contained in the same $C_i(k)$, then $i_1 = i_2$.

Geodesics joining any two points in a component of the ϵ-thin part of a hyperbolic 3-manifold, and that are homotopic relative to their endpoints to a curve in that component of the ϵ-thin part, stay in the ϵ-thin part their whole length. Recall also that geodesic segments in negative curvature are unique in their relative homotopy class. Consequently a subsegment of a closed geodesic which

begins in $M_k^{i_1}$, enters a component of the ϵ-thin part, and then exits at either $M_k^{i_1}$ or $M_k^{i_2}$, must be contained in that component of the ϵ-thin part. Since closed geodesics are dense in the convex core, this shows that there cannot be ϵ-thick points of the convex core contained in any of the complementary regions and completes the proof, assuming Lemma 3.4.2 above.

To prove Lemma 3.4.2, denote by C_k^i the convex core of $\mathbb{H}^3/\Gamma_{C_i(k)}$ considered as a subset of M_k. We will show that for any i and the corresponding choice of basepoints q_k of M_k and q of $\overline{M}_i$ as above, that C_k^i converges to the convex core of M_k on compact sets (where as above we use h_k^i to identify compact subsets of $(\overline{M}_i, q)$ with compact subsets of (M_k, q_k).) Since as we already noted (C_k^i, q_k) converges to the convex core of $(\overline{M}_i, q)$ on compact sets, this will prove the lemma.

Because closed geodesics are dense in the convex core, it is enough to show that each homotopy class of loop γ in M_k can be represented by a loop in the union of the C_k^i that is $\delta(k)$-close to a geodesic, where $\delta(k)$ is independent of γ and tends to zero as $k \to \infty$. We will prove this using a small modification of the McMullen-Taylor curve-straightening argument, following [McM99][Section 4].

Decompose γ minimally as a composition of homotopy classes of segments relative to their endpoints $[\xi]$ contained in the $\epsilon(k)$-thick part of C_k^i together with the regions of the thin part of C_k^i corresponding to accidental parabolics, and $[\delta]$ that traverse a region of the thin part containing a geodesic that becomes a node in the limit. We choose $\epsilon(k)$ so that it tends to zero as $k \to \infty$ and so that for each of the finitely many homotopy classes of primitive loops c_j that become nodes in the limit the length of c_j in M_k divided by $\epsilon(k)$ tends to zero as $k \to \infty$.

We can take each ξ to have endpoints on the boundary of the $\epsilon(k)$-thick part of the corresponding C_k^i. The segment ξ is then homotopic relative to its endpoints to a unique geodesic segment $\overline{\xi}$ in C_k^i. Each δ is homotopic relative to its endpoints to a unique geodesic segment $\overline{\delta}$ joining two C_k^i (that are possibly the same.) The geodesic segment $\overline{\delta}$ is contained in the thin part of M_k corresponding to some short geodesic loop homotopic to one of the c_j. The fact that the length of the geodesic loop in the homotopy class of c_j divided by $\epsilon(k)$ tends to zero as $k \to \infty$ implies that for any two $\overline{\xi}$ and $\overline{\delta}$ that share an endpoint p the following is true: there is a sequence $n(k)$ tending to infinity as $k \to \infty$ such that $n(k)\epsilon(k)$ tends to zero, and so that the length of the connected component of p in the intersection of $\overline{\xi}$ with the $n(k)\epsilon(k)$-thin part of M_k and the length of $\overline{\delta}$ both tend to infinity as $k \to \infty$. Both $\overline{\xi}$ and $\overline{\delta}$ are thus almost perpendicular at p to the boundary of the $\epsilon(k)$-thin part, and they consequently meet at p at an angle that tends to zero as $k \to \infty$ independent of γ (see [McM99][pg. 14].) Here we are using the fact that the geometry of the connected component of the thin part of M_k containing p approaches that of a cuspidal region as $k \to \infty$. The composition of the $\overline{\xi}$ and the $\overline{\delta}$ is thus a loop in the homotopy class of γ that is at a distance from the unique

geodesic loop in that homotopy class tending to zero independent of γ as $k \to \infty$, which finishes the proof of the lemma.

□

We remark that the c_j's in the proof are analogous to the shattering set considered in [CS12]. If we define the Hausdorff dimension of the limit set and the volume of the convex core of an element of $\overline{\mathcal{B}}_0$ that corresponds to a hyperbolic 3-manifold with multiple connected components to be the maximum Hausdorff dimension over the limit sets of all connected components and the sum of the volumes of the convex cores over all connected components, respectively, then Proposition 3.4.1 implies the Corollary 3.1.2 in the introduction, which we restate here.

Corollary 3.4.1. *There exist L and $\epsilon > 0$ such that the volumes of the convex core and the Hausdorff dimension of the limit set of any element of $\overline{\mathcal{B}}_0$ are bounded above by respectively L and $2 - \epsilon$.*

Proof. The uniform bound on the volume of the convex core follows from the fact that the volume of the convex core defines a continuous function on $\overline{\mathcal{B}}_0$ by Proposition 3.4.1, and the fact that $\overline{\mathcal{B}}_0$ is compact.

It follows from [Corollary A of [BC94]] that for any L, there exists $\epsilon > 0$ such that if the volume of the unit neighborhood of the convex core of a geometrically finite infinite volume hyperbolic 3-manifold M is bounded above by L, then the Hausdorff dimension of the limit set of M is bounded above by $2 - \epsilon$. The same argument as in the proof of Proposition 3.4.1 shows that the volume of the unit neighborhood of the convex core defines a continuous function on $\overline{\mathcal{B}}_0$, and there is thus a uniform bound on this quantity over all of $\overline{\mathcal{B}}_0$. □

We have thus proved Theorem B and Corollary 3.1.2, which answers Question 3.1.1 and Question 3.1.2 from the introduction. □

3.5 Beyond weakly almost Fuchsian Space

We now explore further applications and related results in Kleinian surface groups outside of the case of weakly almost Fuchsian. As before we let S be a closed surface and let M be a complete hyperbolic three-manifold diffeomorphic to $S \times \mathbb{R}$. We first prove the following theorem.

Theorem 3.5.1. If M is doubly degenerate, then there exists $\epsilon > 0$, depending only on S, such that any embedded minimal surface (homeomorphic to S) in M has principal curvatures larger in magnitude than $1 + \epsilon$ at some point.

Proof. We argue by contradiction. Suppose not, and that there is a sequence of doubly degenerate hyperbolic three-manifolds M_k with minimal surfaces Σ_k as in the statement, such that the supremum of the principal curvatures of the Σ_k are tending to 1. As above we can pass to a convergent subsequence of the Σ_k and their associated holomorphic quadratic differentials and conformal factors to get a disjoint union $\sqcup_{i=1}^{n}(\overline{\Sigma}_i, \overline{\alpha}_i)$. Since the principal curvatures are no more than 1 in magnitude, we can construct hyperbolic structures $\overline{M}_i$ on $\overline{\Sigma}_i \times \mathbb{R}$ in which the $\overline{\Sigma}_i$ are the unique minimal surfaces. We can then define maps h_k^i as in the previous section, to show that on the complement of a multicurve the M_k are strongly converging to the disjoint union of the $\overline{M}_i$. This is because the normal exponential maps of the Σ_k restricted to subsets $\Sigma_k^i \times [-T_k, T_k]$ of the normal bundle to Σ_k are diffeomorphisms onto their images, for T_k tending to infinity and Σ_k^i converging to $\overline{\Sigma}_i$. Since each of the $\overline{M}_i$ is geometrically finite by Theorem A1, the fact that the h_k^i are locally C^∞ converging to isometries implies that the M_k contains points of arbitrarily large injectivity radius as k tends to infinity. But this is a contradiction because there is a uniform upper bound on the injectivity radius of a doubly degenerate Kleinian surface group depending only on genus (see for instance [Can96]). □

Remark 3.5.2. We also note here, when M is singly or doubly degenerate, one expects it contains a large number of closed minimal surfaces. The existence of some closed minimal surfaces is studied in for instance [Cos21], but it is still open if it admits a closed *incompressible* minimal surface.

We have the following corollary of Theorem 3.5.1, which is Theorem C in the introduction and which is similar to a result of Breslin [Bre11] which assumed a lower bound on the injectivity radius.

Corollary 3.5.1. *There is some $\epsilon > 0$ depending only on S such that any closed minimal surface in F isotopic to the fiber must have principal curvatures greater than $1 + \epsilon$ at some point.*

Proof. This follows by applying Theorem 3.5.1 to the $\mathbb{Z}$-cover of F homeomorphic to $S \times \mathbb{R}$. □

Remark 3.5.3. Theorem 3.5.1 implies that hyperbolic structures M on $S \times \mathbb{R}$ that contain embedded minimal surfaces with principal curvatures smaller than $1 + \epsilon$ for $\epsilon = \epsilon(S)$ have to be geometrically finite, and quasi-Fuchsian if there are no accidental parabolics.

We now prove Theorem D, restated below. We note here that the proof of this theorem does not depend on the other theorems proved in this section.

Theorem 3.5.4. There exist quasi-Fuchsian manifolds M which contain a unique stable minimal surface Σ with principal curvatures strictly greater than 1 in absolute value at some point.

Proof. Take a path $\{M_t\}$ joining a Fuchsian manifold to a quasi-Fuchsian manifold with multiple stable incompressible minimal surfaces. Such examples were constructed in for instance [HW15]. Let t' be the greatest t such that M_t contains an incompressible minimal surface $\Sigma_{t'}$ with principal curvatures less than or equal to 1. By [HLT21], we know that $\Sigma_{t'}$ is strictly stable, namely the bottom eigenvalue of the second variation operator

$$L = -\Delta_{\Sigma_{t'}} - |A|^2 + 2$$

of $\Sigma_{t'}$ is positive. The argument in appendix A of [CG18], which we reproduce in abridged form here, then shows that a neighborhood of $\Sigma_{t'}$ has a mean-convex foliation. Let $\phi \in C^\infty(\Sigma_{t'})$ be a corresponding eigenfunction with bottom eigenvalue, which we can take to be strictly positive, and let N be the unit normal vector field to $\Sigma_{t'}$. Then if $F(x,t)$ is a variation of $\Sigma_{t'}$ with $F_t(x,0) = \phi \cdot N$ and $\Sigma_{t'}(\tau) = F(\Sigma_{t'}, \tau)$, then

$$\frac{d}{d\tau} H_{\Sigma_{t'}(\tau)}|_{\tau=0} = L\phi = \lambda\phi > 0.$$

The $\Sigma_{t'}(\tau)$ for τ in some small interval about 0 therefore give a mean-convex foliation (with respect to the outward normal vector) of a neighborhood of $\Sigma_{t'}$ in M_t.

We now claim that for small enough ϵ, M_t has a unique stable minimal surface for $t \in [t', t' + \epsilon]$. For contradiction suppose not, and that there is a sequence of $t_n \searrow t'$ such that each M_{t_n} has multiple stable minimal surfaces. For n greater than some large N, the implicit function theorem implies that we can choose minimal surfaces Σ_{t_n} in M_{t_n} converging to $\Sigma_{t'}$. Since the δ-neighborhood of each of the Σ_{t_n} has a mean-convex foliation for $n > N$ and δ independent of n. The argument we gave above applies for Σ_t with t sufficiently close to t'. We know that any other stable minimal surface S_{t_n} in M_{t_n} must be at a distance of at least δ from Σ_{t_n}. Passing to a convergent subsequence of the S_{t_n}, which is possible by uniform upper bound on the norm of the second fundamental form of a stable minimal surface ([Sch83]) and the fact that the ends of any quasi-Fuchsian manifold have mean-convex foliations that serve as barriers ([MP11]), we obtain a stable minimal surface $S_{t'}$ in $M_{t'}$ at a distance of at least δ from $\Sigma_{t'}$. This is a contradiction because $M_{t'} \in \mathcal{B}_0$ and hence it admits a unique stable minimal surface. □

Chapter 4

Area, Scalar Curvature, and Hyperbolizable 3-Manifolds

4.1 Introduction

The interplay between scalar curvature, area, and topology is a beautiful chapter in mathematics. For an extended overview of the subject containing the most recent developments, the reader can consult Gromov's Four Lectures on Scalar Curvature [Gro21].

We are interested in studying the area functional on closed manifolds admitting a hyperbolic metric. Variational methods, which were pioneered by Schoen and Yau, are less effective in this setting because the restrictions imposed by the second variation of area are not sharp in the hyperbolic case. On the other hand, the study of length, curvature, and topology has a very rich literature in the negative curvature case, partly because the fact that the geodesic flow is Anosov brings an extra structure to the problem. In the same vein, there is an extra structure for minimal surfaces coming from a "natural" $\mathrm{PSL}(2,\mathbb{R})$-action on the space of minimal immersions (formalized very clearly by Labourie in [Lab]). The general principle we follow, initiated by Calegari–Marques–Neves in [CMN], is to combine the rigidity of that action (due to Ratner and Shah) with geometric methods to obtain sharp relations between area, scalar curvature, and minimal surfaces. One consequence is to answer a conjecture of Gromov regarding the least possible value for the average area ratio.

Using spin methods, Min-Oo [MO89] proved a positive mass Theorem for asymptotically hyperbolic manifolds and Ono [Ono88], Davaux [Dav03] proved sharp spectral inequalities for hyperbolizable manifolds. Andersson, Cai, and Galloway [ACG08] proved a positive mass Theorem for

asymptotically hyperbolic manifolds using variational methods.

Let (M, g) be a closed Riemannian 3-manifold admitting a hyperbolic metric g_0. We refer the reader to Section 4.2 for all the definitions.

4.1.1 Minimal Surface Entropy

Let $S_\varepsilon(M)$ denote the set of all homotopy class Π of essential surfaces whose limit set is a $(1+\varepsilon)$-quasicircle. We define

$$\text{area}_g(\Pi) := \inf\{\text{area}_g(S) : S \in \Pi\}.$$

Inspired by the following expression for the volume entropy $E_{vol}(g)$ on negatively curved manifolds

$$E_{vol}(g) = \lim_{L\to\infty} \frac{\ln \#\{\text{length}_g(\gamma) \leq L : \gamma \text{ closed geodesic in } (M,g)\}}{L}$$

Calegari, Marques, and Neves defined in [CMN] the *minimal surface entropy*[1]

$$E(g) := \lim_{\varepsilon\to 0} \limsup_{L\to\infty} \frac{\ln \#\{\text{area}_g(\Pi) \leq 4\pi(L-1) : \Pi \in S_\varepsilon(M)\}}{L \ln L}. \tag{4.1.1}$$

The authors showed in [CMN] that $E(g_0) = 2$ and that, among metrics g with sectional curvature less than or equal to -1, $E(g) \geq E(g_0) = 2$ and equality implies $g = g_0$ (up to isometry). The inequality follows almost directly from Gauss equation. On the other hand, to show that the hyperbolic metric is the unique maximizer the authors had to use the rigidity of $\text{PSL}(2,\mathbb{R})$-orbits in the frame bundle (due to Ratner [Rat91] and Shah [Sha91]).

4.1.2 Average Area Ratio

Consider another closed hyperbolic 3-manifold N with an hyperbolic metric h_0. The Grassmanian bundle over N (or M) of unoriented tangent 2-planes to N (or M) is denoted by $Gr_2(N)$ (or $Gr_2(M)$). The metric h_0 induces a natural metric on $Gr_2(N)$.

Given $T : M \to N$ a smooth map, the following function is defined for almost all $(x, P) \in Gr_2(N)$:

$$(x,P) \in Gr_2(N) \mapsto |\wedge^2 T^{-1}|(x,P) := \lim_{\delta\to 0} \frac{\text{area}_g(T^{-1}(P_\delta))}{\delta},$$

where $P_\delta \subset N$ is a disc transversal to T, tangent to P at $x \in N$, and area δ.

[1] In [CMN] the definition used lim inf instead of lim sup but all the results proven there are also true with this definition.

Gromov [Gro91a, page 73] defined $\mathrm{Area}_T(g/h_0)$ the *average area ratio of* T to be

$$\mathrm{Area}_T(g/h_0) := \frac{1}{\mathrm{vol}_{h_0}(Gr_2(N))} \int_{Gr_2(N)} |\wedge^2 T^{-1}|(x,P) dV_{h_0}(x,P).$$

A more detailed definition is in (4.2.1).

In the same paper, Gromov used a second variation argument to show that $\mathrm{Area}_T(g/h_0) \geq \mathrm{degree}\, T/3$ if the scalar curvature of g satisfies $R(g) \geq -6$. The reason he got the factor $1/3$ is because the second variation of area is not sharp on hyperbolic manifolds. He conjectured in Remark 2.4.C' of [Gro91a] that assuming $R(g) \geq -6$

$$\mathrm{Area}_T(g/h_0) \geq \mathrm{degree}\, T \textit{ holds true and it is sharp.}$$

This question was also addressed in Section 3.H of [Gro91b] and in the more recent [Gro21] at the end of Section 5.9.

4.1.3 Motivation

We follow the spirit of [Gro91a, Gro91b] and explain, informally, one of the motivations behind $\mathrm{Area}_{\mathrm{Id}}(g/g_0)$. An important object in the study of negatively curved manifolds is the 1-dimensional foliation of the unit tangent bundle where each leaf is an orbit for the geodesic flow.

Gromov proposed the following surface analogue: A 2-dimensional foliation $\mathcal{L}$ of $Gr_2(M)$ is a family $\{S_\lambda\}_{\lambda\in\Lambda}$ of complete surfaces of M so that through every point in $(p,\tau) \in Gr_2(M)$ there is a unique λ so that S_λ passes through p with tangent plane τ. Laminations where each leaf is a stable minimal surface are analogous to the geodesic flow foliation of the unit tangent bundle. There is a canonical foliation $\mathcal{L}_0$ of $Gr_2(M)$ whose leafs are totally geodesic planes.

Given a metric g on M we obtain an area form on each leaf of $\mathcal{L}$. Assuming $\mathcal{L}$ has a transversal measure we can integrate the area of each leaf with respect to this measure and obtain $vol_g(\mathcal{L})$.

The general question Gromov asks is how low can one make $vol_g(\mathcal{L})$ subject to the constraint that $R(g) \geq -6$. There is a "natural" transversal measure on $\mathcal{L}_0$ such that $\mathrm{Area}_{\mathrm{Id}}(g/g_0)$ coincides with $vol_g(\mathcal{L}_0)/vol_{g_0}(\mathcal{L}_0)$. Therefore, Gromov's conjecture when $T = \mathrm{Id}$ can be rephrased as

$$vol_g(\mathcal{L}_0) \geq vol_{g_0}(\mathcal{L}_0) \quad \text{if} \quad R(g) \geq -6.$$

The following comparison is also interesting: if we consider a 3-dimensional foliation of $Gr_3(M) \simeq M$, there is only one leaf, the (unit) transverse measure is trivial, and given a metric on g, the volume

of this foliation is $vol_g(M)$. Schoen conjectured that $vol_g(M) \geq vol_{g_0}(M)$ among all metrics with $R(g) \geq -6$ and this was proven by Perelman using Ricci flow (see [And06] for the argument).

4.1.4 Main Theorems

The next theorem uses Ricci flow in the spirit of [BBEN10, MN12] and $\mathrm{PSL}(2,\mathbb{R})$-invariant laminar measures.

Theorem 4.1.1. *Suppose (M,g) has $R(g) \geq -6$. Then*

$$E(g) \leq 2$$

and equality occurs if and only if g is isometric to the hyperbolic metric.

This result was shown by the author of this thesis in [Low] under the condition that the (normalized) Ricci flow starting at g converges to the hyperbolic metric faster than e^{-ct} for some $c > 1$. As it is pointed out in [Low], there are closed manifolds $\mathbb{H}^3/\Gamma$ for which that will happen for all metrics and closed manifolds $\mathbb{H}^3/\Gamma$ with metrics for which that will not happen. Manifolds containing closed embedded totally geodesic surfaces fall into the second category.

We note also that Agol, Storm, and Thurston [AST07] conjectured that $E_{vol}(g)$ is maximized by the hyperbolic metric among all metrics with $R(g) \geq -6$.

The next theorem relates the average area ratio and the minimal surface entropy in a sharp way.

Theorem 4.1.2. *For every Riemannian metric (M,g) we have*

$$\mathrm{Area}_{\mathrm{Id}}(g/g_0)E(g) \geq 2$$

and equality holds if and only if $g = cg_0$ for some constant $c > 0$.

Appealing to the previous interpretation of $\mathrm{Area}_{\mathrm{Id}}(g/g_0)$, we can restate Theorem 4.1.2 as saying that for every metric g

$$vol_g(\mathcal{L}_0)E(g) \geq 2vol_{g_0}(\mathcal{L}_0)$$

with equality if and only if $g = cg_0$ for some $c > 0$. This is reminiscent of Besson, Courtois, and Gallot [BCG95] which says that

$$vol_g(M)^{1/3}E_{vol}(g) \geq 2vol_{g_0}(M)^{1/3}$$

for every metric g with equality if and only if g is isometric to g_0. It is a conjecture of Calegari-Marques-Neves [DCN] that

$$vol_g(M)^{2/3}E(g) \geq 2vol_{g_0}(M)^{2/3}$$

for every metric g with equality occurring only for g_0 (up to isometry).

We now deduce two corollaries. The first one follows from combining Theorem 4.1.1 and Theorem 4.1.2.

Corollary 4.1.3. *Consider $T : M \to N$ a local diffeomorphism with degree d. If (M, g) has scalar curvature $R(g) \geq -6$ then*

$$\mathrm{Area}_T(g/h_0) \geq d$$

and equality holds if and only if T is a local isometry between g and h_0.

This result confirms Gromov's conjecture for local diffeomorphisms. If we just assume that T has degree d there may exist no local isometry between M and N and thus no obvious optimal map.

Proof of Corollary 4.1.3. We have

$$\mathrm{Area}_T(g/h_0) = d\,\mathrm{Area}_{\mathrm{Id}}(g/T^*(h_0)).$$

Hence, with $g_0 := T^*(h_0)$, we have from Theorem 4.1.2 and Theorem 4.1.1 that

$$2\mathrm{Area}_T(g/h_0) \geq E(g)\mathrm{Area}_T(g/h_0) = dE(g)\mathrm{Area}_{\mathrm{Id}}(g/g_0) \geq 2d$$

and equality holds if and only if $g = g_0 = T^*(h_0)$. □

The second corollary is below.

Corollary 4.1.4. *Suppose $\Sigma \subset M$ is a totally geodesic closed immersion. If (M, g) has scalar curvature $R(g) \geq -6$ then, with Π denoting the homotopy class of Σ,*

$$\mathrm{area}_g(\Pi) \geq -2\pi\chi(\Sigma)$$

with equality if and only if g is an hyperbolic metric.

Proof. This follows from the proof of Theorem 4.1.1, namely Theorem 4.5.1 applied with $\Pi_m = \Pi$ for all $m \in \mathbb{N}$. □

Without the condition that Π contains a totally geodesic immersion in the hyperbolic metric the result fails already for $g = g_0$.

4.1.5 Sketch of proofs

We describe succinctly the main ideas behind the proofs of Theorem 4.1.1 and Theorem 4.1.2.

Assume that (M, g) has $R(g) \geq -6$. To prove Theorem 4.1.1 we need to show that for every sequence of homotopy classes $\Pi_m \in S_{1/m}(M)$ we have

$$\lim_{m\to\infty} \text{area}_g(\Pi_m)/\text{area}_{g_0}(\Pi_m) \geq 1.$$

Let $(g_t)_{t\geq 0}$ be a solution to normalized Ricci flow, which we first assume exists for all time and converges to g_0. We show that if the inequality above fails then for all m sufficiently large we have for some $\delta > 0$ and all $t \geq 0$

$$\text{area}_{g_t}(\Pi_m)/\text{area}_{g_0}(\Pi_m) \leq 1 - \delta e^{-t}.$$

Stability analysis shows that $g_t \simeq g_0 + e^{-t}\bar{h}$ as $t \to \infty$, where $\bar{h}$ is an eigentensor for the linearization of the trace-free Ricci tensor at the hyperbolic metric. Letting $m \to \infty$ we obtain measures μ_t on the frame bundle of M so that

$$\mu_t(1) = \lim_{m\to\infty} \text{area}_{g_t}(\Pi_m)/\text{area}_{g_0}(\Pi_m).$$

Necessarily $\mu_t(1) \leq 1 - \delta e^{-t}$. Using a form of Gauss equation (see (4.3.9)) for these measures we have an identity of the type

$$1 = \mu_t(1) + \mu_t(\text{curvature terms}).$$

Combining with the asymptotics $g_t \simeq g_0 + e^{-t}\bar{h}$ we show

$$1 = \mu_t(1) + \mu_t(\text{curvature terms}) = \mu_t(1) + e^{-t}\mu_{+\infty}(\text{terms with } \bar{h}) + o(e^{-t}).$$

The measure $\mu_{+\infty}$ is $\text{PSL}(2, \mathbb{R})$-invariant. Hence, using Ratner's classification theorem [Rat91] and the fact that $\bar{h}$ is an eigentensor, we show that

$$\mu_{+\infty}(\text{terms with } \bar{h}) = 0.$$

Thus $1 = \mu_t(1) + o(e^{-t})$, which contradicts $\mu_t(1) \leq 1 - \delta e^{-t}$.

For the general case, we proceed as above but use Perelman's Ricci flow with surgery [Per03]. Because M has an hyperbolic metric all surgeries correspond to removing capped horns and replacing them by standard caps. The key point to check is that essential surfaces minimizing area stay away from the capped horns. We achieve this via area comparison.

We now sketch the argument behind Theorem 4.1.2. We start by improving a construction of Labourie in [Lab] and find a sequence $\Sigma_m \subset M$ of connected immersed minimal surfaces with respect to g_0 which becomes equidistributed in the frame bundle. We show in Proposition 4.6.3 that

$$\mathrm{Area}_{\mathrm{Id}}(g/g_0) = \lim_{m\to\infty} \mathrm{area}_g(\Sigma_m)/\mathrm{area}_{g_0}(\Sigma_m).$$

A counting argument implies that

$$E(g) \geq E(g_0) \lim_{m\to\infty} \mathrm{area}_{g_0}(\Sigma_m)/\mathrm{area}_g(\Sigma_m)$$

and these two expressions give that $E(g)\mathrm{Area}_{\mathrm{Id}}(g/g_0) \geq 2$. If equality holds we show first that the metric g is Zoll, i.e., every totally geodesic disc with respect to g_0 is minimal with respect to g, and then we show that g must be a multiple of g_0. This proves Theorem 4.1.2.

4.2 Preliminaries

Assume (M, g) is a closed Riemannian 3-manifold admitting a hyperbolic metric g_0 and N another closed hyperbolic 3-manifold N with an hyperbolic metric h_0.

Suppose $T : M \to N$ is smooth map. Given $(x, P) \in Gr_2(M)$ we denote by $|\wedge^2 T|_g(x, P)$ the Jacobian of

$$dT_x : P \to T_{T(x)}M,$$

meaning that if e_1, e_2 is an orthonormal basis of P and $u_i := dT_x(e_i)$, $i = 1, 2$,

$$|\wedge^2 T|_g(x, P) := \sqrt{h_0(u_1, u_1)h_0(u_2, u_2) - h_0^2(u_1, u_2)}.$$

Given $y \in N$ a regular value and $\tau = (y, V) \in Gr_2(N)$ set

$$|\wedge^2 T^{-1}|_g(y, V) = \sum_{x\in T^{-1}(y)} \frac{1}{|\wedge^2 T|_g(x, (dT_x)^{-1}(V))}$$

The function $\tau \mapsto |\wedge^2 T^{-1}|_g(\tau)$ is defined almost everywhere on $Gr_2(N)$.

For context, if T is transverse to a closed surface $S \subset M$, then $\Sigma := T^{-1}(S)$ is a surface of N with area

$$\text{area}_g(\Sigma) = \int_S |\wedge^2 T^{-1}|_g(y, T_y S) dA_{h_0}(y).$$

Gromov [Gro91a, page 73] defined the *average area ratio of* T as

$$\text{Area}_T(g/h_0) := \int_{Gr_2(N)} |\wedge^2 T^{-1}|_g(\tau) d\mu_{h_0}(\tau), \tag{4.2.1}$$

where integration is with respect to the *unit* volume measure μ_{h_0} on $Gr_2(N)$ induced by h_0. In particular, $\text{Area}_{\text{Id}}(h_0/h_0) = 1$. In the definition above it is implicitly assumed that $|\wedge^2 T^{-1}|_g$ is in L^1. If that is not the case we define $\text{Area}_T(g/h_0) = \infty$.

We use $\pi_1(M)$ to denote as well its representation into $\text{PSL}(2, \mathbb{C})$. A closed immersed surface $\Sigma \subset M$ is *essential* if the immersion $\iota : \Sigma \to M$ is π_1-injective. Essential surfaces lift to discs in $\mathbb{H}^3$. Using the representation of $\pi_1(M)$ into $\text{PSL}(2, \mathbb{C})$ we have $\iota_*(\pi_1(\Sigma)) < \text{PSL}(2, \mathbb{C})$ and so we can associate its limit set $\Lambda(\Sigma) \subset \partial_\infty \mathbb{H}^3 \simeq S^2$ (for the definition see for instance [CMN, Section 2.2]).

Set $\mathcal{C}_\varepsilon$ to be the space of $(1+\varepsilon)$-quasicircles in $\partial_\infty \mathbb{H}^3$ (see [CMN, Definition 2.3] for precise definition). The group $\pi_1(M)$ acts on $\mathcal{C}_\varepsilon$ and preserves $\mathcal{C}_0$ (which is the space of all geodesic circles in $S^2 \simeq \partial_\infty \mathbb{H}^3$).

Let $S_\varepsilon(M)$ denote the set of all homotopy class Π of essential surfaces with limit set in $\mathcal{C}_\varepsilon$. Recall that we defined

$$\text{area}_g(\Pi) := \inf\{\text{area}_g(S) : S \in \Pi\}.$$

From Schoen-Yau [SY79] there is an immersed minimal surface (with respect to g) $\mathbf{\Sigma}_g(\Pi) \in \Pi$ which realizes $\text{area}_g(\Pi)$.

From [Sep16], assuming ε is sufficiently small, for all $\gamma \in \mathcal{C}_\varepsilon$ there is a unique embedded area-minimizing disc $D(\gamma) \subset \mathbb{H}^3$ (with respect to g_0) with $\partial_\infty D(\gamma) = \gamma \subset \partial_\infty \mathbb{H}^3$ and principal curvatures that can be made arbitrarily small by choosing ε small enough.

The same argument as in Theorem 4.3 of [Low20] (adapted to the case where the minimal discs are not necessarily preserved by some surface group), shows that we can find a neighborhood $\mathcal{U}$ of g_0 and $\bar{\varepsilon}$ small so that for each $\gamma \in \mathcal{C}_{\bar{\varepsilon}}$ there is a unique non-degenerate area-minimizing disc $\Sigma_g(\gamma)$ with respect to the metric g so that $\partial_\infty \Sigma_g(\gamma) = \gamma$. The discs $\Sigma_g(\gamma)$ and $D(\gamma)$ are at a bounded Hausdorff distance from each other (independent of g) and if $g \to g_0$ then $\Sigma_g(\gamma)$ converges to $D(\gamma)$ uniformly in $C^{2,\alpha}$. Therefore we can decrease $\mathcal{U}$ and $\bar{\varepsilon}$ so that for all $\gamma \in \mathcal{C}_{\bar{\varepsilon}}$ there is $f_\gamma \in C^\infty(D(\gamma))$

(depending on g) such that its graph over $D(\gamma)$ is identical to $\Sigma_g(\gamma)$ and $|f_\gamma|_{2,\alpha} < 1$. There is an ambiguity on the sign of f_γ but the quantities we consider, like $|f_\gamma|$, will be sign independent.

If $\Pi \in S_{\bar{\varepsilon}}(M)$ and $g \in \mathcal{U}$ then, with $\gamma := \Lambda(\mathbf{\Sigma}_g(\Pi))$, uniqueness implies that $\Sigma_g(\gamma)$ covers a minimal surface $\mathbf{\Sigma}_g(\gamma)$ in Π which must coincide with $\mathbf{\Sigma}_g(\Pi)$ and thus satisfy

$$\text{area}_g(\Pi) = \text{area}_g(\mathbf{\Sigma}_g(\gamma)).$$

Given $\gamma \in \mathcal{C}_\varepsilon$, let $n(\gamma)$ denote a continuous unit normal vector field along $D(\gamma)$ with respect to g_0. Consider the diffeomorphism (using the hyperboloid model)

$$F_\gamma : D(\gamma) \to \Sigma_g(\gamma), \quad x \mapsto \cosh(f_\gamma(x))x + \sinh(f_\gamma(x))n(\gamma)(x). \tag{4.2.2}$$

We omit the dependence of g on F_γ to avoid too much notation. When $g = g_0$, $f_\gamma = 0$ and thus F_γ is the identity.

4.3 Laminations and Laminar measures

We follow the presentation of Labourie in [Lab] and add some auxiliary results.

4.3.1 Laminations

Consider the space of *stable minimal conformal immersions* $\mathcal{F}(\mathbb{H}^3, \varepsilon)$ (with $\varepsilon \leq \bar{\varepsilon}$) defined in Definition 5.1 of [Lab], i.e., the space of conformal minimal immersions

$$\phi : \mathbb{H}^2 \to (\mathbb{H}^3, g_0) \quad \text{with } \partial\phi := \phi(\partial_\infty \mathbb{H}^2) \in \mathcal{C}_\varepsilon.$$

Because $\varepsilon \leq \bar{\varepsilon}$, $D(\partial\phi) = \phi(\mathbb{H}^2)$, ϕ is an embedding and stable for the second variation of area. The topology we choose is the same as the one considered in Definition 5.1 of [Lab] and it makes the map $\phi \mapsto \partial\phi$ continuous (Theorem 5.2 of [Lab]). Thus

$$\cap_{k\in\mathbb{N}} \mathcal{F}(\mathbb{H}^3, 1/k) = \mathcal{F}(\mathbb{H}^3, 0).$$

Similar to [Lab] we also consider $\mathcal{F}(M, \varepsilon) := \mathcal{F}(\mathbb{H}^3, \varepsilon)/\pi_1(M)$ with the quotient topology. The

group $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ and thus it acts on $\mathcal{F}(M,\bar{\varepsilon})$ in the following way:

$$\tau \in \mathrm{PSL}(2,\mathbb{R}), \quad R_\tau : \mathcal{F}(M,\varepsilon) \to \mathcal{F}(M,\varepsilon) \quad R_\tau(\phi) := \phi \circ \tau^{-1}. \tag{4.3.1}$$

The space $\mathcal{F}(M,\bar{\varepsilon})$ together with the $\mathrm{PSL}(2,\mathbb{R})$-action is called the *conformal minimal laminations* of M.

Fix a fundamental domain $\Delta \subset \mathbb{H}^3$ of M. Given $\phi \in \mathcal{F}(M,\varepsilon)$, there is a unique lift to an element of $\mathcal{F}(\mathbb{H}^3,\varepsilon)$, denoted by ϕ as well, that is uniquely determined by the requirement that $\phi(i) \in \Delta$. Thus for each $\phi \in \mathcal{F}(M,\varepsilon)$ we obtain $\partial\phi \in \mathcal{C}_\varepsilon$ but this map is not necessarily continuous. Theorem 5.2 (i) of [Lab] says that the evaluation map which sends $\phi \in \mathcal{F}(\mathbb{H}^3,\bar{\varepsilon})$ to $\phi(i) \in \mathbb{H}^3$ is proper. As a result we deduce at once the lemma below.

Lemma 4.3.1. *The space $\mathcal{F}(M,\bar{\varepsilon})$ is sequentially compact.*

Given $\phi \in \mathcal{F}(M,\bar{\varepsilon})$, let $C(\phi) > 0$ be the conformal factor of $\phi^*(g_0)$. Denote the Gaussian curvature of $D(\partial\phi)$ by $K(\phi)$. From Gauss equation and [Sep16] we have, after making $\bar{\varepsilon}$ smaller if necessary, $-2 \leq K(\phi) \leq -1$. The maximum principle applied to the equation satisfied by $C(\phi)$ implies that

$$\frac{1}{2} \leq \frac{1}{\sup_{D(\partial\phi)} |K(\phi)|} \leq C(\phi) \leq 1. \tag{4.3.2}$$

Let $F(M)$ denote the frame bundle of M, i.e., $F(M) = \mathrm{PSL}(2,\mathbb{C})/\pi_1(M)$. Fix $\{e_1, e_2\}$ an orthonormal basis of $\mathbb{H}^2$ and, given $\phi \in \mathcal{F}(M,\bar{\varepsilon})$, set

$$e_1(\phi) := d\phi_i(e_1)C(\phi)^{-1/2} \quad \text{and} \quad e_2(\phi) := d\phi_i(e_2)C(\phi)^{-1/2}.$$

There is a unique $n(\phi) \in T_{\phi(i)}M$ so that $\{e_1(\phi), e_2(\phi), n(\phi)\}$ is a positive frame. Consider the continuous map

$$\Omega : \mathcal{F}(M,\bar{\varepsilon}) \to F(M), \quad \Omega(\phi) = (\phi(i), \{e_1(\phi), e_2(\phi), n(\phi)\}). \tag{4.3.3}$$

Given $\phi \in \mathcal{F}(M,\bar{\varepsilon})$, set $\gamma := \partial\phi$ and denote the Jacobian of (see (4.2.2))

$$F_\gamma \circ \phi : \mathbb{H}^2 \to (\Sigma_g(\gamma), g)$$

by $|Jac_g(F_\gamma \circ \phi)|$. Consider the function

$$\Lambda_g : \mathcal{F}(M, \bar{\varepsilon}) \to \mathbb{R}, \quad \Lambda_g(\phi) := |Jac_g(F_\gamma \circ \phi)|(i). \tag{4.3.4}$$

If dA_g is the area element of $\Sigma_g(\gamma)$, then $(F_\gamma \circ \phi)_i^*(dA_g) = \Lambda_g(\phi)dx \wedge dy$, in isothermal coordinates. This function is continuous because it is independent of the particular lift of ϕ that was chosen.

Let $\nu_g(\gamma)$ denote a continuous unit normal vector field along $\Sigma_g(\gamma)$ with respect to g, where $\gamma \in \mathcal{C}_{\bar{\varepsilon}}$. Consider the following functions

$$|A|_g^2 : \mathcal{F}(M, \varepsilon) \to \mathbb{R}, \quad \phi \mapsto |A|_g^2(\Sigma_g(\partial\phi))(F_{\partial\phi} \circ \phi(i)), \tag{4.3.5}$$

$$Ric(g)(\nu, \nu) : \mathcal{F}(M, \varepsilon) \to \mathbb{R}, \quad \phi \mapsto Ric(g)_{|F_{\partial\phi} \circ \phi(i)}(\nu_g(\partial\phi), \nu_g(\partial\phi)), \tag{4.3.6}$$

$$R(g) : \mathcal{F}(M, \varepsilon) \to \mathbb{R}, \quad \phi \mapsto R(g)(F_{\partial\phi} \circ \phi(i)). \tag{4.3.7}$$

The definition of all these functions is independent of the particular lift of $\phi \in \mathcal{F}(M, \varepsilon)$ that was chosen and thus they are continuous.

4.3.2 Laminar measures

A *laminar measure* μ on $\mathcal{F}(M, \bar{\varepsilon})$ is a probability measure that is invariant under the $\mathrm{PSL}(2, \mathbb{R})$-action given by (4.3.1).

A laminar measure μ and the map Ω defined in (4.3.3) induces a probability measure $\Omega_*\mu$ on $F(M)$. That measure is invariant under a $\mathrm{PSL}(2, \mathbb{R})$-action which will not coincide in general with the homogeneous action of $\mathrm{PSL}(2, \mathbb{R})$ as a subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Another issue that needs to be addressed is the fact that the space of laminar measures is not necessarily weakly compact (a related problem is put as an open question in [Lab02]).

Lemma 4.3.2. *Let μ_k be a sequence of laminar measures on $\mathcal{F}(M, 1/k)$ so that $\Omega_*\mu_k$ converges weakly to a probability measure $\bar{\mu}$ on $F(M)$. Then $\bar{\mu}$ is invariant under the homogeneous action of $PSL(2, \mathbb{R})$.*

Proof. Let $\Omega_0 : \mathcal{F}(M, 0) \to F(M)$ be the restriction of Ω to $\mathcal{F}(M, 0)$. Every $\phi \in \mathcal{F}(M, 0)$ has the property that $D(\partial\phi)$ is a totally geodesic disc and so $\phi : \mathbb{H}^2 \to M$ is an isometric immersion. Thus ϕ is uniquely determined by $\phi(i), e_1(\phi)$, and $e_2(\phi)$. Hence Ω_0 is bijective and a homeomorphism.

Recall the $\mathrm{PSL}(2, \mathbb{R})$-action on $\mathcal{F}(M, \bar{\varepsilon})$ defined in (4.3.1). Ω_0 induces a $\mathrm{PSL}(2, \mathbb{R})$-action on

$F(M)$ in the following way:

$$\tau \in \mathrm{PSL}(2,\mathbb{R}), \quad F_\tau : F(M) \to F(M), \quad F_\tau(x) = \Omega_0(R_\tau \circ \Omega_0^{-1}(x)).$$

This action corresponds to the homogeneous action of $\mathrm{PSL}(2,\mathbb{R})$. Thus, given $f \in C^0(F(M))$, we need to check that

$$\bar{\mu}(f \circ F_\tau) = \bar{\mu}(f) \quad \text{for all } \tau \in \mathrm{PSL}(2,\mathbb{R}).$$

Consider the "projection" of $\mathcal{F}(M,\varepsilon)$ onto $\mathcal{F}(M,0)$ given by

$$P := \Omega_0^{-1} \circ \Omega : \mathcal{F}(M,\varepsilon) \to \mathcal{F}(M,0).$$

Set $\eta := f \circ \Omega \circ R_\tau$. We have $f \circ F_\tau \circ \Omega = \eta \circ P$ and thus

$$\bar{\mu}(f \circ F_\tau) = \lim_{k\to\infty} \Omega_* \mu_k(f \circ F_\tau) = \lim_{k\to\infty} \mu_k(\eta \circ P).$$

We also have $\Omega_* \mu_k(f) = \mu_k(\eta)$ for all $k \in \mathbb{N}$ by $\mathrm{PSL}(2,\mathbb{R})$-invariance and thus

$$\bar{\mu}(f) = \lim_{k\to\infty} \Omega_* \mu_k(f) = \lim_{k\to\infty} \mu_k(\eta).$$

In light of these last two identities it suffices to check that

$$\lim_{k\to\infty} |\mu_k(\eta \circ P) - \mu_k(\eta)| = 0$$

and this follows at once if we show that

$$\lim_{k\to\infty} \sup_{\phi \in \mathcal{F}(M,1/k)} |\eta \circ P(\phi) - \eta(\phi)| = 0.$$

If this identity does not hold we find $\delta > 0$ and $\phi_k \in \mathcal{F}(M,1/k)$ so that $|\eta \circ P(\phi_k) - \eta(\phi_k)| \geq \delta$ for all $k \in \mathbb{N}$. From Lemma 4.3.1 we know that, after passing to a subsequence, ϕ_k converges to some $\phi \in \mathcal{F}(M,0)$ which must satisfy $|\eta \circ P(\phi) - \eta(\phi)| \geq \delta$. This is impossible because $P(\phi) = \phi$. □

Let Γ be a Fuchsian subgroup of $\mathrm{PSL}(2,\mathbb{R})$ so that $\mathbb{H}^2/\Gamma$ is a closed hyperbolic surface with genus l. All Fuchsian groups we consider will have this property with no need for further mentioning.

$\mathrm{PSL}(2,\mathbb{R})/\Gamma$ is the frame bundle of $\mathbb{H}^2/\Gamma$. With respect to the invariant metric on $\mathrm{PSL}(2,\mathbb{R})/\Gamma$ we have $vol(\mathrm{PSL}(2,\mathbb{R})/\Gamma) = \alpha_0 4\pi(l-1)$ for some universal constant α_0.

Suppose $\phi \in \mathcal{F}(M, \bar{\varepsilon})$ is equivariant with respect to a representation of $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ in $\pi_1(M) < \mathrm{PSL}(2,\mathbb{C})$. Consider $U \subset \mathrm{PSL}(2,\mathbb{R})$ a fundamental domain of $\mathrm{PSL}(2,\mathbb{R})/\Gamma$. Following Proposition 5.5 of [Lab] we define δ_ϕ a laminar measure on $\mathcal{F}(M, \bar{\varepsilon})$

$$\delta_\phi(f) := \frac{1}{vol(U)} \int_U f(\phi \circ \tau) d\bar{\nu}(\tau), \quad f \in C^0(\mathcal{F}(M, \bar{\varepsilon}))$$

where $\bar{\nu}$ is the bi-invariant measure on $\mathrm{PSL}(2,\mathbb{R})$.

Equivariance implies that $D(\partial\phi)$ projects to a closed surface $\mathbf{D}(\partial\phi)$ in M and that ϕ descends to an immersion from $\mathbb{H}^2/\Gamma$ to $\mathbf{D}(\partial\phi)$ that we still denote by ϕ. The uniqueness property implies that $\Sigma_g(\partial\phi)$ projects to a closed surface $\mathbf{\Sigma}_g(\partial\phi)$ on M that is homotopic to $\mathbf{D}(\partial\phi)$. The map $F_{\partial\phi} \circ \phi$ is also equivariant and so descends to a map from $\mathbb{H}^2/\Gamma$ to $\mathbf{\Sigma}(\partial\phi)$ that we denote by $\mathbf{F}_{\partial\phi} \circ \phi$. The unit normal vector field $\nu_g(\gamma)$ induces a unit normal vector field along $\mathbf{\Sigma}_g(\gamma)$ that we also denote by $\nu_g(\gamma)$.

For context, suppose f is a continuous function in $Gr_2(M)$ and set

$$\hat{f} : \mathcal{F}(M, \bar{\varepsilon}) \to \mathbb{R}, \quad \phi \mapsto f(\pi \circ F_{\partial\phi} \circ \phi(i), d(\pi \circ F_{\partial\phi})_{|\phi(i)}(d\phi_{|i}(T_i\mathbb{H}^2))).$$

The function $\hat{f}$ is continuous. With dA_{hyp} denoting the hyperbolic volume form on $\mathbb{H}^2$ we have

$$\begin{aligned}
&\int_{\mathbf{\Sigma}_g(\partial\phi)} f(x, T_x\mathbf{\Sigma}_g(\partial\phi)) dA_g(x) \\
&\qquad = \int_{\mathbb{H}^2/\Gamma} f(\mathbf{F}_{\partial\phi} \circ \phi(z), d\mathbf{F}_{\partial\phi_{|\phi(z)}}(d_z\phi(\mathbb{H}^2)) |Jac_g(\mathbf{F}_{\partial\phi} \circ \phi)|(z) dA_{hyp}(z) \\
&\qquad\qquad = \frac{1}{\alpha_0} \int_U \hat{f}(\phi \circ \tau) \Lambda_g(\phi \circ \tau) d\bar{\nu}(\tau) = 4\pi(l-1)\delta_\phi(\hat{f}\Lambda_g). \quad (4.3.8)
\end{aligned}$$

Gauss identity for laminar measures

From Gauss equation we have

$$\begin{aligned}
4\pi(l-1) = \mathrm{area}_g(\mathbf{\Sigma}_g(\partial\phi)) \\
+ \int_{\mathbf{\Sigma}_g(\partial\phi)} Ric(g)(\nu_g(\partial\phi), \nu_g(\partial\phi)) - \frac{1}{3}R(g) + \frac{|A|^2}{2} - \frac{R(g)+6}{6} dA_g.
\end{aligned}$$

When interpreted in terms of laminar measures this identity becomes

$$1 = \frac{\mathrm{area}_g(\mathbf{\Sigma}_g(\partial\phi))}{4\pi(l-1)} + \delta_\phi\left(\left[Ric(g)(\nu,\nu) - R(g)/3 + |A|_g^2/2\right]\Lambda_g\right)$$

$$- \delta_\phi \left(\frac{R(g)+6}{6} \Lambda_g \right), \quad (4.3.9)$$

where the functions $|A|_g^2$, $Ric(g)(\nu,\nu)$, and $R(g)$ are as defined in (4.3.5), (4.3.6), and (4.3.7).

4.4 Proof of Theorem 4.1.1: Part I

Recall that M is a closed manifold with an hyperbolic metric g_0. Throughout this section of the thesis we refer to normalized Ricci flow as a one-parameter family of metrics $(\bar{g}_t)_{t\in I}$ which solve

$$\frac{d\bar{g}_t}{dt} = -2Ric(\bar{g}_t) - 4\bar{g}_t. \quad (4.4.1)$$

Theorem 4.4.1. *There is a neighborhood $\mathcal{V}$ of $\bar{g}$ so that for all $g \in \mathcal{V}$ with $R(g) \geq -6$ the following holds: For any sequence $\Pi_m \in S_{1/m}(M)$ we have*

$$\liminf_{m\to\infty} \frac{\mathrm{area}_g(\Pi_m)}{4\pi(l_m - 1)} \geq 1,$$

where l_m is the genus of an essential surface in Π_m.

If equality holds then g is isometric to g_0.

Proof. Consider the neighborhood $\mathcal{U}$ of g_0 described in Section 4.2. From [KY09, Appendix A] we see that we can find a neighborhood of g_0 in the $C^{2,\alpha}$-topology so that for every initial condition in that neighborhood, the normalized Ricci flow exists for all time and converges exponentially fast in the $C^{2,\alpha}$-topology to an hyperbolic metric in $\mathcal{U}$. Reasoning like in [Ham82, Section 17] we can upgrade the convergence and find a smaller open neighborhood $\mathcal{V} \subset \mathcal{U}$ of g_0 so that for every $g \in \mathcal{V}$ the normalized Ricci flow $(\bar{g}_t)_{t\geq 0}$ starting at g exists for all time, does not leave $\mathcal{U}$, and converges exponentially fast to an Einstein metric in $\mathcal{U}$, which must be isometric to g_0 from Mostow rigidity. Furthermore, in [KY09] it is also constructed a family of diffeomorphisms $\{\Phi_t\}_{t\geq 0}$ converging strongly to some diffeomorphism so that $g_t := \Phi_t^* \bar{g}_t$ solves the DeTurck-modified Ricci flow (which is strictly parabolic) and g_t converges to g_0 as $t \to \infty$.

The maximum principle implies that the condition $R(g) \geq -6$ is preserved by the normalized Ricci flow because

$$\frac{d}{dt} R(\bar{g}_t) \geq \Delta_{g_t} R(\bar{g}_t) + \frac{2}{3} R(\bar{g}_t)(R(\bar{g}_t) + 6).$$

4.4.1 Proof of inequality

Suppose for contradiction that the inequality fails for some metric $g \in \mathcal{V}$. Thus we can find $\Pi_m \in S_{1/m}(M) \cap S_{\bar{\varepsilon}}(M)$ and $\delta > 0$ so that for all $m \in \mathbb{N}$

$$\frac{\text{area}_g(\Pi_m)}{4\pi(l_m - 1)} \leq 1 - \delta.$$

Because $\bar{g}_t \in \mathcal{U}$ for all t, there is $\phi_m \in \mathcal{F}(M, 1/m)$ equivariant with respect to a representation of a Fuchsian subgroup of $\text{PSL}(2, \mathbb{R})$ in $\pi_1(M)$ so that $\mathbf{\Sigma}_{g_t}(\partial\phi_m) \in \Pi_m$, $\mathbf{\Sigma}_{g_t}(\partial\phi_m)$ depends smoothly on t, and $\text{area}_{g_t}(\Pi_m) = \text{area}_{g_t}(\mathbf{\Sigma}_{g_t}(\partial\phi_m))$ for all $t \geq 0$. Fix $m \in \mathbb{N}$ and denote by ν_t the normal vector to $\mathbf{\Sigma}_{g_t}(\partial\phi_m)$ with respect to g_t. Using $R(g_t) \geq -6$ and Gauss equation

$$\begin{aligned}\frac{d}{dt}\text{area}_{g_t}(\Pi_m) &= -\int_{\mathbf{\Sigma}_{g_t}(\partial\phi_m)} R(g_t) - Ric(g_t)(\nu_t, \nu_t) + 4\, dA_{g_t} \\ &= 4\pi(l_m - 1) - \text{area}_{g_t}(\Pi_m) - \int_{\mathbf{\Sigma}_{g_t}(\partial\phi_m)} \frac{|A|^2}{2} + \frac{R(g_t) + 6}{2} dA_{g_t} \\ &\leq 4\pi(l_m - 1) - \text{area}_{g_t}(\Pi_m). \quad (4.4.2)\end{aligned}$$

Solving the ODE obtained by replacing the inequality sign above by an equality sign we get that for all $t \geq 0$

$$\frac{\text{area}_{g_t}(\Pi_m)}{4\pi(l_m - 1)} \leq 1 - e^{-t}\left(1 - \frac{\text{area}_g(\Pi_m)}{4\pi(l_m - 1)}\right) \leq 1 - \delta e^{-t}. \quad (4.4.3)$$

Combining (4.3.9) with (4.4.3) and using the fact that $R(g_t) + 6 \geq 0$ we get

$$\delta e^{-t} \leq \delta_{\phi_m}\left(\left[Ric(g_t)(\nu, \nu) - R(g_t)/3 + |A|^2_{g_t}/2\right]\Lambda_{g_t}\right). \quad (4.4.4)$$

After passing to a subsequence, $\Omega_*\delta_{\phi_m}$ converges weakly to a unit measure $\bar{\mu}$ on $F(M)$. From Lemma 4.3.2 we have that $\bar{\mu}$ is invariant under the homogeneous action of $\text{PSL}(2, \mathbb{R})$.

Let L denote the linearization of the traceless Ricci tensor at g_0. For every 2-tensor h on M set $\theta(h) : F(M) \to \mathbb{R}$ to be the continuous function given by

$$\theta(h)(x, \{u_1, u_2, n\}) := L(h)_x(n, n). \quad (4.4.5)$$

Set $h_t := g_t - g_0$. We use $O(e^{at})$ to denote a term bounded by Ce^{at}, where C does not depend on m or t.

Lemma 4.4.2. $\delta \leq \bar{\mu}(e^t\theta(h_t)) + O(e^{-1/3t})$.

Proof. Write

- $E_1 := \delta_{\phi_m}(|A|^2_{g_t}/2\,\Lambda_{g_t})$;
- $E_2 := \delta_{\phi_m}([Ric(g_t)(\nu,\nu) - R(g_t)/3](\Lambda_{g_t} - 1))$;
- $E_3 := \delta_{\phi_m}([Ric(g_t)(\nu,\nu) - R(g_t)/3] - \theta(h_t)\circ\Omega)$.

We now estimate these terms. We know from Proposition 4.7.2 that

$$|h_t|_{C^4} = O(e^{-2/3t}).$$

Fix $m \in \mathbb{N}$ and set $\gamma_m := \partial\phi_m$. The surface $\Sigma_{g_t}(\gamma_m)$ has zero mean curvature with respect to g_t. Thus the mean curvature $H_{g_0}(\Sigma_{g_t}(\gamma_m))$ of $\Sigma_{g_t}(\gamma_m)$ with respect to the hyperbolic metric has $C^{0,\alpha}$-norm bounded by $O(e^{-2/3t})$. (This follows from the fact that h_t and ∇h_t have both that order).

Using the mean convex foliation of $\mathbb{H}^3$ coming from the discs equidistant to $D(\gamma_m)$ that was described by Uhlenbeck in [Uhl83, Theorem 3.3], the maximum principle implies that for some constant c_0

$$\sup_{m\in\mathbb{N}} |f_{\gamma_m}|_{L^\infty} \leq c_0 \sup_{m\in\mathbb{N}} |H_{g_0}(\Sigma_{g_t}(\gamma_m))|_{L^\infty} = O(e^{-2/3t}).$$

Elliptic regularity implies the existence of a constant $c_1 > 0$ such that

$$\sup_{m\in\mathbb{N}} |f_{\gamma_m}|_{C^{2,\alpha}} \leq c_1 \sup_{m\in\mathbb{N}} (|f_{\gamma_m}|_{L^\infty} + |H_{g_0}(\Sigma_{g_t}(\gamma_m))|_{C^{0,\alpha}}) = O(e^{-2/3t}).$$

Thus $|A|^2_{g_0}(\Sigma_{g_t}(\gamma_m)) = O(e^{-4/3t})$. If $\lambda_j(g_t)$ and $\lambda_j(g_0)$, $j = 1, 2$ denote, respectively, the principal curvatures of $\Sigma(\gamma_m)$ with respect to g_t and g_0, we have

$$\lambda_j(g_t) = \lambda_j(g_0) + O(|h_t|_{C^1}), \quad j = 1, 2.$$

Hence $|A|^2_{g_t}(\Sigma_{g_t}(\gamma_m)) = O(e^{-4/3t})$ and thus

$$2E_1 = \frac{1}{4\pi(l_k - 1)} \int_{\mathbf{\Sigma}_{g_t}(\gamma_m)} |A|^2_{g_t} dA_{g_t} = O(e^{-4/3t}). \tag{4.4.6}$$

From Gauss equation, (4.3.2), and $|A|^2_{g_t}(\Sigma_{g_t}(\gamma_m)) = O(e^{-4/3t})$ we obtain that

$$|C(\phi_m) - 1| = O(e^{-4/3t}).$$

Moreover, from Proposition 4.7.1 we obtain

$$|Jac_{g_t}(F_{\gamma_m})| = 1 + O(|h_t|_{C^1}) + O(|f_{\gamma_m}|_{C^1}) = 1 + O(e^{-2/3t}).$$

Thus $|Jac_{g_t}(F_{\gamma_m} \circ \phi_m)| = |Jac_{g_t}(F_{\gamma_m})|C(\phi_m) = 1 + O(e^{-2/3t})$ and this means (see (4.3.4)) that

$$\Lambda_{g_t}(\phi_m) = 1 + O(e^{-2/3t}). \tag{4.4.7}$$

Recall the definitions in (4.3.3) and (4.4.5). From Proposition 4.7.1 and the fact that $|h_t|_{C^4}$ and $\sup_{m\in\mathbb{N}} |f_{\gamma_m}|_{C^{2,\alpha}}$ have order $O(e^{-2/3t})$, we see that

$$\begin{aligned}[Ric(g_t)(\nu,\nu) - R(g_t)/3](\phi_m) \\ &= L(h_t)_{\phi_m(i)}(n(\gamma_m), n(\gamma_m)) + O(e^{-4/3t}) \\ &= \theta(h_t) \circ \Omega(\phi_m) + O(e^{-4/3t}). \end{aligned} \tag{4.4.8}$$

Note that $\theta(h_t) = O(|h_t|_{C^2}) = O(e^{-2/3t})$ and so we obtain from (4.4.8) and (4.4.7) that both

$$E_2 = \delta_{\phi_m}([\theta(h_t) \circ \Omega + O(e^{-4/3t})](\Lambda_{g_t} - 1)) = O(e^{-4/3t})$$

and $E_3 = O(e^{-4/3t})$. As a result,

$$\begin{aligned}\delta_{\phi_m}\left(\left[Ric(g_t)(\nu,\nu) - R(g_t)/3 + |A|^2_{g_t}/2\right]\Lambda_{g_t}\right) \\ &= \delta_{\phi_m}(\theta(h_t) \circ \Omega) + E_1 + E_2 + E_3 = \Omega^* \delta_{\phi_m}(\theta(h_t)) + O(e^{-4/3t}) \end{aligned}$$

and so we have from (4.4.4)

$$\delta e^{-t} \leq \Omega^* \delta_{\phi_m}(\theta(h_t)) + O(e^{-4/3t}).$$

Making $m \to \infty$ the result follows. □

Consider the operator on symmetric 2-tensors

$$\mathcal{A}(h) = \Delta_{g_0} h - 2(tr_{g_0} h)g_0 + 2h. \tag{4.4.9}$$

We know from Proposition 4.7.2 that $e^t h_t$ converges in $W^{2,2}$ to $\bar{h}$, where $\bar{h}$ satisfies $\mathcal{A}(\bar{h}) = -\bar{h}$.

Thus $e^t\theta(h_t) \to \theta(\bar{h})$ in L^2 and the previous lemma implies

$$\delta \leq \bar{\mu}(\theta(\bar{h})).$$

We now show that $\bar{\mu}(\theta(\bar{h})) = 0$, which gives us a contradiction and thus implies that for any sequence $\Pi_m \in S_{1/m}(M)$ we have

$$\liminf_{m\to\infty} \frac{\text{area}_g(\Pi_m)}{\text{area}_{g_0}(\Pi_m)} \geq 1.$$

Proposition 4.4.3. *Let η be an ergodic $PSL(2,\mathbb{R})$-invariant probability measure on $F(M)$. Then $\eta(\theta(\bar{h})) = 0$.*

Proof. Because $\bar{h}$ is the lowest eigenfunction for $\mathcal{A}$, it follows from Koiso Bochner formula (see [KY09]) that $\bar{h}$ is trace free and divergence free. Hence from the formula for the linearization of the traceless Ricci tensor [Bes08, Theorem 1.174] we have

$$L(\bar{h}) = -\frac{1}{2}\mathcal{A}(\bar{h}) = \frac{\bar{h}}{2}.$$

Thus for all $(x, \{u_1, u_2, n\}) \in F(M)$ we have

$$\theta(\bar{h})((x, \{u_1, u_2, n\})) = \frac{1}{2}\bar{h}_x(n, n). \tag{4.4.10}$$

From Ratner's classification theorem [Rat91] we have that η is either the homogeneous Lebesgue measure μ_{Leb} on $F(M)$ or there is a totally geodesic surface $S \subset M$ so that η is the homogeneous Lebesgue measure on $F(S)$, the frame bundle of S.

Suppose the first case occurs. We have $tr_{g_0}\bar{h} = 0$ and thus

$$\eta(\theta(\bar{h})) = \mu_{Leb}(\theta(\bar{h})) = \frac{1}{\text{vol}(M)} \int_M \frac{1}{6} tr_{g_0}\bar{h}\, dV_{g_0} = 0.$$

Suppose the second case occurs and choose n a normal vector field along S. Consider the vector field X tangent to S such that

$$g_0(X, Y) = \nabla_n \bar{h}(Y, n) \quad \text{for every vector field } Y \text{ tangent to } S.$$

Set $f : S \to \mathbb{R}$, where $f(x) := \bar{h}_x(n(x), n(x))$.

Lemma 4.4.4. *With Δ_S and div_S denoting, respectively, the Laplacian and divergence on S, we have $\Delta_S f + 6f = div_S X$.*

Proof. Fix $x \in S$ and an orthonormal basis $\{e_1, e_2\}$ of $T_x S$.

By commuting derivatives and using the fact that $tr_{g_0}\bar{h} = 0$ we deduce

$$\begin{aligned}\sum_{i=1}^{2}(\nabla_n\nabla_{e_i}\bar{h})(e_i,n) =& \sum_{i=1}^{2}(\nabla_{e_i}\nabla_n\bar{h})(e_i,n) + \sum_{i=1}^{2}\bar{h}(R(e_i,n)e_i,n)\\ &+\sum_{i=1}^{2}\bar{h}(e_i,R(e_i,n)n)\\ =&\sum_{i=1}^{2}(\nabla_{e_i}\nabla_n\bar{h})(e_i,n) - 2\bar{h}(n,n) + \sum_{i=1}^{2}\bar{h}(e_i,e_i)\\ =&div_S X(x) - 3f(x).\end{aligned}$$

Differentiating $\operatorname{div}\bar{h} = 0$ in the direction of n we obtain that at $x \in S$

$$(\nabla_n\nabla_{e_1}\bar{h})(e_1,n) + (\nabla_n\nabla_{e_2}\bar{h})(e_2,n) + (\nabla_n\nabla_n\bar{h})(n,n) = 0.$$

From (4.4.9) and $\mathcal{A}(\bar{h}) = -\bar{h}$ we have that $\Delta_{g_0}\bar{h} = -3\bar{h}$. Thus the last two identities we derived and the fact that S is totally geodesic imply

$$\begin{aligned}\Delta_S f(x) &= (\Delta_{g_0}\bar{h})(n,n) - (\nabla_n\nabla_n\bar{h})(n,n) = -3f(x) - (\nabla_n\nabla_n\bar{h})(n,n)\\ &= -3f(x) + \sum_{i=1}^{2}(\nabla_n\nabla_{e_i}\bar{h})(e_i,n) = -6f(x) + div_S X(x).\end{aligned}$$

□

From (4.4.10) we see that $\theta(\bar{h}) = f/2$ when restricted to $F(S) \subset F(M)$. Because η is the unit Lebesgue measure on $F(S)$ we have from Lemma 4.4.4 that

$$\eta(\theta(\bar{h})) = \frac{1}{\operatorname{area}_{g_0}(S)}\int_S \frac{f}{2}dA_{g_0} = \frac{1}{12\operatorname{area}_{g_0}(S)}\int_S div_S X - \Delta_S f dA_{g_0} = 0.$$

□

The ergodic decomposition theorem for $\mathrm{PSL}(2,\mathbb{R})$-actions (Theorem 4.2.6 in [HK02]) gives the existence of a space B with a probability measure ν on B so that for all $b \in B$ there is an ergodic $\mathrm{PSL}(2,\mathbb{R})$-invariant probability measure μ_b on $F(M)$ so that $\bar{\mu}(f) = \int_B \mu_b(f)d\nu(b)$ for all $f \in$

$C^0(F(M))$. Proposition 4.4.3 implies that $\bar{\mu}(\theta(\bar{h})) = 0$.

4.4.2 Proof of rigidity

Suppose that for some metric $g \in \mathcal{V}$ with $R(g) \geq -6$ and some sequence $\Pi_m \in S_{1/m}(M)$ we have

$$\liminf_{m\to\infty} \frac{\text{area}_g(\Pi_m)}{4\pi(l_m - 1)} = 1,$$

where l_m is the genus of an essential surface in Π_m. Run normalized Ricci flow $(g_t)_{0\leq t\leq \bar{t}}$ starting at g for a short time interval and set

$$a(t) := \liminf_{m\to\infty} \frac{\text{area}_{g_t}(\Pi_m)}{4\pi(l_m - 1)}.$$

From (4.4.3) we see we see that $a(0) = 1$ implies that $a(t) \leq 1$ for all $0 \leq t \leq \bar{t}$ and thus $a(t) = 1$ for all $0 \leq t \leq \bar{t}$.

Suppose that g is not Einstein. From the strong maximum principle applied to the evolution equation of $R(g_t)$ we obtain the existence of δ so that $R(g_t) \geq -6 + 2\delta$ for all $\bar{t}/2 \leq t \leq \bar{t}$. Thus we see from (4.4.2) that for all $\bar{t}/2 \leq t \leq \bar{t}$ and all $m \in \mathbb{N}$

$$\frac{d}{dt}\text{area}_{g_t}(\Pi_m) \leq 4\pi(l_m - 1) - (1 + \delta)\text{area}_{g_t}(\Pi_m).$$

ODE comparison gives us a contradiction because

$$a(\bar{t}) \leq a(\bar{t}/2)e^{-(1+\delta)\bar{t}/2} + \frac{1 - e^{-(1+\delta)\bar{t}/2}}{1 + \delta} = e^{-(1+\delta)\bar{t}/2} + \frac{1 - e^{-(1+\delta)\bar{t}/2}}{1 + \delta} < 1.$$

□

4.5 Proof of Theorem 4.1.1: Part II

We use Perelman's Ricci flow with surgery [Per03] to remove the local condition on Theorem 4.4.1.

Theorem 4.5.1. *Assume g is a metric on M such that $R(g) \geq -6$. For any sequence $\Pi_m \in S_{1/m}(M)$ we have*

$$\liminf_{m\to\infty} \frac{\text{area}_g(\Pi_m)}{4\pi(l_m - 1)} \geq 1,$$

where l_m is the genus of an essential surface in Π_m.

If equality holds then g is isometric to g_0.

Proof. We will use the notation and results of [Per03]. For a more detailed treatment the reader can consult [Bam17].

Note that Ricci flow and normalized Ricci flow (4.4.1) differ only by scaling. With ε_0 small, Perelman finds in [Per03, Section 4] a sequence of manifolds M_k with $M_0 = M$, a discrete set of times $\{t_k\}_{k\in\mathbb{N}_0}$ with $t_0 = 0$, and a sequence of smooth solutions to normalized Ricci flow $(\bar{g}_t)_{t_k\leq t<t_{k+1}}$ on M_k with $\bar{g}_0 = g$ so that the following holds: each M_{k+1} is obtained from M_k by surgery as described in [Per03, Section 4.4] but, because M is irreducible and contains no embedded projective planes, the only possible surgeries remove ε_0-caps and glue in (perturbed) standard caps. Hence $M_k = M$ for all $k \in \mathbb{N}_0$ (after discarding 3-spheres) and, still following [Per03, Section 4], there is a sequence of compact sets $\Lambda_k \subset M$ such that $M \setminus \Lambda_k$ is diffeomorphic to a union of open balls, $\bar{g}_t$ converges smoothly to $\bar{g}_{t_{k+1}}$ on Λ_k as $t \to t_{k+1}$, and for all $t < t_{k+1}$ close to t_{k+1}, the metric $\bar{g}_t$ is such that every p in the boundary of $M \setminus \Lambda_k$ is the center slice of some ε_0-neck.

In Section 7.1 of [Per03] Perelman argues that $R(\bar{g}_t) \geq -6$ for all $t \geq 0$ (for the normalized Ricci flow with surgery). In Section 7.3 and Section 7.4 of [Per03], Perelman shows that $(M, \bar{g}_t)$ admits, for all t sufficiently large, a thick-thin decomposition, where the thick part meets the thin part along incompressible tori. Because M is closed and admits an hyperbolic metric, it has no incompressible tori and thus, for all t sufficiently large, $(M, \bar{g}_t)$ coincides with the thick part. Therefore, we obtain from Lemma 7.2 of [Per03] that $|Ric(\bar{g}_t) + 2\bar{g}_t|_{C^0}$ is small for all t sufficiently large. Thus we see from [Ye93] that for all t sufficiently large $(\bar{g}_t)_{t\geq 0}$ is a smooth solution to normalized Ricci flow and converges to an hyperbolic metric on M. In particular there are only finitely many surgery times $t_1 < t_2 < \ldots < t_K$. Using Mostow rigidity we can assume that, after applying a diffeomorphism to g_0, $(\bar{g}_t)_{t\geq 0}$ converges smoothly to g_0 as $t \to \infty$.

Given a homotopy class Π of essential surfaces, there is $\Sigma_t \in \Pi$, a minimal surface with respect to $\bar{g}_t$, such that $\mathrm{area}_{\bar{g}_t}(\Pi) = \mathrm{area}_{\bar{g}_t}(\Sigma_t)$.

Lemma 4.5.2. *We can find ε_0 small so that for every homotopy class Π of essential surfaces and every $t < t_{k+1}$ close to t_{k+1}, $\Sigma_t \subset \Lambda_k$.*

Proof. For all $t < t_{k+1}$ close to t_{k+1}, the metric $\bar{g}_t$ is such that every p in the boundary of $M \setminus \Lambda_k$ is the center slice of some ε_0-neck. This means that there exists a neighborhood $N \subset M$ and a diffeomorphism (depending on t and p)

$$\Phi : S^2 \times [-{\varepsilon_0}^{-1}, {\varepsilon_0}^{-1}] \to N$$

such that $p \in \Phi(S^2 \times \{0\})$ and for some $\lambda > 0$ (depending on x and t)

$$|\lambda^{-2}\Phi^* \bar{g}_t - g_{S^2 \times [-\varepsilon_0{}^{-1}, \varepsilon_0{}^{-1}]}|_{C^{\lceil 1/\varepsilon_0 \rceil}} < \varepsilon_0. \tag{4.5.1}$$

Moreover, $\Phi(S^2 \times \{0\})$ is homotopic to a boundary component of Λ_k because $M \setminus \Lambda_k$ consists of ε_0-caps.

Assume for contradiction that Σ_t passes through p. The region $M \setminus \Lambda_k$ is diffeomorphic to a disjoint union of balls and so there is a ball B_k containing N so that $\partial B_k = \Phi(S^2 \times \{-1/\varepsilon_0\})$. It cannot be the case that Σ_t is contained in B_k and thus Σ_t must intersect ∂B_k. We can perturb B_k slightly so that ∂B_k intersects Σ_t transversely in a union of circles. Let D_p be the connected region in $\Sigma_t \cap B_k$ that passes through p.

On the one hand, ∂D_p can be filled in by a region in ∂B_k with area at most roughly $4\pi\lambda^2$. On the other hand, D_p must intersect every cross-section $\Phi(S^2 \times \{y\})$ for $-1/\varepsilon_0 < y < 0$. By the monotonicity formula, there is a universal constant c such that for $y \in (-1/\varepsilon_0 + 1/2, -1/2)$,

$$\mathrm{area}_{\bar{g}_t}(D_p \cap \Phi(S^2 \times (y - 1/2, y + 1/2)) > c\lambda^2.$$

It follows by choosing disjoint unit intervals in $(-1/\varepsilon_0 + 1/2, -1/2)$ that if we chose ε_0 such that $1/\varepsilon_0$ is greater than $5\pi/c$, then the area of D_p will be larger than $(4\pi + 1)\lambda^2$. By cutting out D_p and gluing in a region of ∂B_k we could then produce a surface homotopic to Σ_t but with smaller area, which is a contradiction. □

This lemma implies that we can use [BBEN10, Lemma 9] and conclude that $\bar{A}(t) := \mathrm{area}_{\bar{g}_t}(\Pi)$ is a Lipschitz function and thus differentiable almost everywhere. Let $\bar{t}$ be a point of differentiability of $\bar{A}$ and consider the function $a(t) := \mathrm{area}_{\bar{g}_t}(\Sigma_{\bar{t}})$. We have $\bar{A}(t) \leq a(t)$ for all t and $\bar{A}(\bar{t}) = a(\bar{t})$. Hence, arguing like in (4.4.2) we deduce

$$\bar{A}'(t) \leq a'(\bar{t}) \leq -2\pi\chi(\Sigma_{\bar{t}}) - \bar{A}(t).$$

From this ODE we can argue like in the proof of Theorem 4.4.1 and conclude that if for some sequence $\Pi_m \in S_{1/m}(M)$ we have

$$\frac{\mathrm{area}_g(\Pi_m)}{4\pi(l_m - 1)} \leq 1 - \delta,$$

where l_m is the genus of an essential surface in Π_m, then for all $t \geq 0$

$$\frac{\text{area}_{\bar{g}_t}(\Pi_m)}{4\pi(l_m - 1)} \leq 1 - \delta e^{-t}.$$

We have that $\bar{g}_t$ converges smoothly to g_0 and so this contradicts Theorem 4.4.1. Thus the inequality in Theorem 4.5.1 must hold.

If equality holds in Theorem 4.5.1, the very same argument used as in Theorem 4.4.1 shows that g_0 must be Einstein. □

We can now derive Theorem 4.1.1 from Theorem 4.5.1.

Theorem 4.5.3. *Suppose (M, g) has $R(g) \geq -6$. Then $E(g) \leq 2$ and equality occurs if and only if g is hyperbolic.*

Proof. In what follows we will use the fact (from [Sep16]) that for any $\Pi_m \in S_{1/m}(M)$, if l_m is the genus of an essential surface in Π_m, then we have $\text{area}_{g_0}(\Pi_m)/4\pi(l_m - 1) \to 1$ as $m \to \infty$.

Choose $\delta > 0$. Theorem 4.5.1 implies the the existence of ε_0 so that for all $\Pi \in S_{\varepsilon_0}(M)$ we have

$$\text{area}_{g_0}(\Pi) \leq (1 + \delta)\text{area}_g(\Pi).$$

Hence for all $\varepsilon \leq \varepsilon_0$ we have

$$\begin{aligned}\#\{\text{area}_g(\Pi) \leq 4\pi(L-1) : \Pi \in S_\varepsilon(M)\} \\ \leq \#\{\text{area}_{g_0}(\Pi) \leq (1+\delta)4\pi(L-1) : \Pi \in S_\varepsilon(M)\}.\end{aligned}$$

This inequality and the expression (4.1.1) for the minimal surface entropy imply

$$E(g) \leq E(g_0)(1 + \delta) = 2(1 + \delta).$$

The inequality follows from making $\delta \to 0$.

Suppose now that $E(g) = 2$ for some metric with $R(g) \geq -6$. Reasoning as above we see that if we could find $\delta > 0$ and ε_0 so that for all $\Pi \in S_{\varepsilon_0}(M)$

$$\text{area}_{g_0}(\Pi) \leq (1 - \delta)\text{area}_g(\Pi),$$

then $E(g) \leq 2(1-\delta)$. Hence there is $\Pi_m \in S_{1/m}(M)$ so that

$$\liminf_{m\to\infty} \frac{\text{area}_g(\Pi_m)}{\text{area}_{g_0}(\Pi_m)} \leq 1$$

and Theorem 4.5.1 implies that g is hyperbolic. □

4.6 Proof of Theorem 4.1.2

For each $m \in \mathbb{N}$, Labourie [Lab, Theorem 5.7] found $N_m \in \mathbb{N}$, $\{\phi_m^i\}_{i=1}^{N_m}$ in $\mathcal{F}(M, 1/m)$, and $0 < a_m^1, \ldots, a_m^{N_m} < 1$ with $a_m^1 + \ldots + a_m^{N_m} = 1$, so that: ϕ_m^i is equivariant with respect to a representation of a Fuchsian subgroup of $\text{PSL}(2,\mathbb{R})$ in $\pi_1(M) < \text{PSL}(2,\mathbb{C})$ and the laminar measure

$$\delta_m := \sum_{i=1}^{N_m} a_m^i \delta_{\phi_m^i} \tag{4.6.1}$$

is such that $\Omega_* \delta_m$ converges to μ_{Leb}, the unit Lebesgue homogeneous measure on $F(M)$.

Proposition 4.6.1. *There is ϕ_m in $\mathcal{F}(M, 1/m)$ equivariant with respect to a representation of a Fuchsian group Γ_m of $\text{PSL}(2,\mathbb{R})$ in $\pi_1(M) < PSL(2,\mathbb{C})$ such that $\Omega_* \delta_{\phi_m}$ converges to μ_{Leb} as $m \to \infty$.*

Proof. The space of all closed totally geodesic immersions in (M, g_0) is countable and so

$$\mathcal{T} := \{F(S) \subset F(M) : \text{S is a closed totally geodesic immersion in } (M, g_0)\}$$

is also countable, where $F(S)$ denotes the frame bundle of S which injects naturally in $F(M)$.

Considering tubular neighborhoods, we can find a decreasing sequence of open sets $\{B_k\}_{k\in\mathbb{N}} \subset F(M)$ so that for all $k \in \mathbb{N}$

$$\cup_{T\in\mathcal{T}} T \subset B_k, \quad \mu_{Leb}(\partial B_k) = 0, \quad \text{and} \quad \mu_{Leb}(B_k) \leq 2^{-2k-3}.$$

Lemma 4.6.2. *For each $j \in \mathbb{N}$, there is $j \leq m_j \in \mathbb{N}$ and $\phi_j \in \{\phi_{m_j}^1, \ldots, \phi_{m_j}^{N_{m_j}}\}$ so that*

$$\Omega_* \delta_{\phi_j}(B_k) \leq 2^{-(k+1)} \quad \textit{for all } k \leq j.$$

Proof. For all $k \in \mathbb{N}$, $\Omega_* \delta_m(B_k) \to \mu_{Leb}(B_k)$ as $m \to \infty$. Thus we can find a strictly increasing

sequence of integers $\{m_j\}_{j\in\mathbb{N}}$ so that

$$\Omega_*\delta_{m_j}(B_k) \leq 2\mu_{Leb}(B_k) \leq 2^{-2(k+1)} \quad \text{for all } k \leq j. \tag{4.6.2}$$

Relabel m_j to be j so that $\delta_{m_j}, \phi^i_{m_j}, N_{m_j}$, and $a^i_{m_j}$ become δ_j, ϕ^i_j, N_j, and a^i_j, respectively.

Consider μ_j a unit measure on $\{1,\dots,N_j\}$ so that $\mu_j(i) = a^i_j$ and set

$$J_{j,k} := \{i \in \{1,\dots,N_j\} : \Omega_*\delta_{\phi^i_j}(B_k) \geq 2^{-(k+1)}\}.$$

From (4.6.2) and the definition of $J_{j,k}$ we have that for all $j \geq k$

$$\mu_j(J_{j,k}) = \sum_{i\in J_{j,k}} a^i_j \leq 2^{k+1} \sum_{i\in J_{j,k}} a^i_j \Omega_*\delta_{\phi^i_j}(B_k) \leq 2^{k+1}\Omega_*\delta_j(B_k) \leq 2^{-(k+1)}.$$

Thus, for all $j \in \mathbb{N}$ and $k \leq j$, $\mu_j(\cup_{k=1}^j J_{j,k}) \leq \sum_{k=1}^j 2^{-(k+1)} \leq 1/2$. Hence

$$A_j := \{1,\dots,N_j\} \setminus \cup_{k=1}^j J_{j,k} \neq \emptyset$$

and we pick $l_j \in A_j$. The maps $\phi_j := \phi^{l_j}_{m_j}$ satisfy the desired conditions. □

After passing to a subsequence, $\Omega_*\delta_{\phi_j}$ converges weakly to a unit measure ν on $F(M)$. From Lemma 4.3.2 we have that ν is invariant under the homogeneous action of $\mathrm{PSL}(2,\mathbb{R})$. The proof will be completed if we show that $\nu = \mu_{Leb}$.

Ratner's classification theorem [Rat91] implies that every ergodic probability measures on $F(M)$ invariant under the homogeneous action of $\mathrm{PSL}(2,\mathbb{R})$ is either μ_{Leb} or is supported in some $T \in \mathcal{T}$. Thus, from the ergodic decomposition theorem for $\mathrm{PSL}(2,\mathbb{R})$-actions (Theorem 4.2.6 in [HK02]), there is $0 \leq \theta \leq 1$ so that the measure ν decomposes as $\nu = \theta\mu_{Leb} + (1-\theta)\mu_{\mathcal{T}}$, where $\mu_{\mathcal{T}}$ is some probability measure on $F(M)$ with support in $\cup_{T\in\mathcal{T}}T$. Moreover we have from Lemma 4.6.2 that $\nu(B_k) \leq 2^{-(k+1)}$ for all $k \in \mathbb{N}$.

Recall that $\cup_{T\in\mathcal{T}}T \subset B_k$ for all $k \in \mathbb{N}$ and so

$$1-\theta = (1-\theta)\mu_{\mathcal{T}}(B_k) \leq \nu(B_k) \leq 2^{-(k+1)}.$$

Hence $\theta = 1$, which means that $\nu = \mu_{Leb}$. □

Proposition 4.6.3. $\mathrm{Area}_{\mathrm{Id}}(g/g_0) = \lim_{m\to\infty} \frac{\mathrm{area}_g(\mathbf{D}(\partial\phi_m))}{2\pi|\chi(\mathbf{D}(\partial\phi_m))|}$.

Proof. We have for all $(x, P) \in Gr_2(M)$

$$|\wedge^2 \mathrm{Id}^{-1}|_g(x, P) = |\wedge^2 \mathrm{Id}|_g^{-1}(x, P).$$

We abuse notation and also denote by $|\wedge^2 \mathrm{Id}|_g$ the following smooth positive function on the frame bundle $F(M)$

$$|\wedge^2 \mathrm{Id}|_g : F(M) \to \mathbb{R}, \quad (y, \{e_1, e_2, n\}) \mapsto |\wedge^2 \mathrm{Id}|_g(y, \mathrm{span}\{e_1, e_2\}).$$

We have

$$\mathrm{Area}_{\mathrm{Id}}(g/g_0) = \mu_{Leb}(|\wedge^2 \mathrm{Id}|_g^{-1})$$

and thus the fact that $\Omega_* \delta_{\phi_m} \to \mu_{Leb}$ implies

$$\mathrm{Area}_{\mathrm{Id}}(g/g_0) = \mu_{Leb}(|\wedge^2 \mathrm{Id}|_g^{-1}) = \lim_{m\to\infty} \Omega_* \delta_{\phi_m}(|\wedge^2 \mathrm{Id}|_g^{-1}). \tag{4.6.3}$$

Consider $U_m \subset \mathrm{PSL}(2, \mathbb{R})$ a fundamental domain of $\mathrm{PSL}(2, \mathbb{R})/\Gamma_m$. Then

$$\Omega_* \delta_{\phi_m}(|\wedge^2 \mathrm{Id}|_g^{-1}) = \frac{1}{vol(U_m)} \int_{U_m} |\wedge^2 \mathrm{Id}|_g^{-1} \circ \Omega(\phi_m \circ \tau) d\bar{\nu}(\tau).$$

Note that for all $\tau \in \mathrm{PSL}(2, \mathbb{R})$ we have, with $z = \tau(i)$,

$$|\wedge^2 \mathrm{Id}|_g \circ \Omega(\phi_m \circ \tau) = |\wedge^2 \mathrm{Id}|_g(\phi_m(z), (d\phi_m)_z(T_z\mathbb{H}^2)).$$

Thus, denoting the hyperbolic volume form on $\mathbb{H}^2$ by dA_{hyp},

$$\begin{aligned}
&\Omega_* \delta_{\phi_m}(|\wedge^2 \mathrm{Id}|_g^{-1}) \\
&\qquad = \frac{1}{2\pi|\chi(\mathbf{D}(\partial\phi_m))|} \int_{\mathbb{H}^2/\Gamma_m} |\wedge^2 \mathrm{Id}|_g^{-1}(\phi_m(z), (d\phi_m)_z(T_z\mathbb{H}^2)) dA_{hyp}(z) \\
&\qquad\qquad = \frac{1}{2\pi|\chi(\mathbf{D}(\partial\phi_m))|} \int_{\mathbf{D}(\partial\phi_m)} \frac{|\wedge^2 \mathrm{Id}|_g^{-1}(y, T_y\mathbf{D}(\partial\phi_m))}{C(\phi_m) \circ \phi_m^{-1}(y)} dA_{g_0}(y).
\end{aligned} \tag{4.6.4}$$

Set $A_m := ||A||_{L^\infty(\mathbf{D}(\partial\phi_m))}$. From Gauss equation and (4.3.2) we have

$$1 \leq \frac{1}{C(\phi_m)} \leq 1 + A_m^2. \tag{4.6.5}$$

The co-area formula says that

$$\int_{\mathbf{D}(\partial\phi_m)} |\wedge^2 \mathrm{Id}|_g^{-1}(y, T_y\mathbf{D}(\partial\psi_m))dA_{g_0} = \mathrm{area}_g(\mathbf{D}(\partial\phi_m))$$

and thus, combining with (4.6.4) and (4.6.5), we obtain

$$\frac{\mathrm{area}_g(\mathbf{D}(\partial\phi_m))}{2\pi|\chi(\mathbf{D}(\partial\phi_m))|} \leq \Omega_*\delta_{\phi_m}(|\wedge^2 \mathrm{Id}|_g^{-1}) \leq (1+A_m^2)\frac{\mathrm{area}_g(\mathbf{D}(\partial\phi_m))}{2\pi|\chi(\mathbf{D}(\partial\phi_m))|}.$$

We have that $A_m \to 0$ as $m \to \infty$ and hence we deduce from this inequality and (4.6.3) the desired result. □

4.6.1 Proof of inequality

The closed surfaces $\mathbf{D}(\partial\phi_m)$ define a homotopy class $\Pi_m \in S_{1/m}(M)$ for all $m \in \mathbb{N}$. We denote the genus of $\mathbf{D}(\partial\phi_m)$ by l_m.

Proposition 4.6.4. $\mathrm{Area}_{\mathrm{Id}}(g/g_0)E(g) \geq 2$.

Proof. After passing to a subsequence, set

$$\alpha := \lim_{m\to\infty} \frac{\mathrm{area}_g(\Pi_m)}{4\pi(l_m-1)}.$$

Given δ we have that for all m sufficiently large

$$\mathrm{area}_g(\Pi_m) \leq (\alpha+\delta)4\pi(l_m-1).$$

Let $\mathbf{D}_m^k$, l_m^k, and $\Pi_m^k \in S_{1/m}(M)$ denote, respectively, a k-cover of $\mathbf{D}(\partial\phi_m)$, its genus, and its homotopy class. From the inequality above we have for all $k \in \mathbb{N}$ and all $m \in \mathbb{N}$ sufficiently large

$$\frac{\mathrm{area}_g(\Pi_m^k)}{4\pi(l_m^k-1)} \leq \frac{k\,\mathrm{area}_g(\Pi_m)}{4\pi(l_m^k-1)} = \frac{\mathrm{area}_g(\Pi_m)}{4\pi(l_m-1)} \leq \alpha+\delta. \tag{4.6.6}$$

From the Müller-Puchta's formula (see [CMN, Section 4]) there is $c(m) > 0$ so that $\mathbf{D}(\partial\phi_m)$ has at least $(c(m)l_m^k)^{2l_m^k}$ distinct covers of degree less than or equal to k. Thus, if we choose L_m^k so that $4\pi(L_m^k-1) = (\alpha+\delta)4\pi(l_m^k-1)$, we have from (4.6.6) that for all $k \in \mathbb{N}$ and all $m \in \mathbb{N}$ sufficiently large

$$\#\{\mathrm{area}_g(\Pi) \leq 4\pi(L_m^k-1) : \Pi \in S_{1/m}(M)\} \geq (c(m)l_m^k)^{2l_m^k}.$$

Hence for all m sufficiently large

$$\lim_{k\to\infty} \frac{\ln \#\{\mathrm{area}_g(\Pi) \leq 4\pi(L_m^k - 1) : \Pi \in S_{1/m}(M)\}}{L_m^k \ln L_m^k} \geq \lim_{k\to\infty} \frac{2l_m^k \ln(c(m)l_m^k)}{L_m^k \ln L_m^k} = \frac{2}{\alpha+\delta}.$$

From the expression for $E(g)$ in (4.1.1) we obtain that $(\alpha+\delta)E(g) \geq 2$. Making $\delta \to 0$ we deduce $\alpha E(g) \geq 2$. From Proposition 4.6.3 we have

$$\mathrm{Area}_{\mathrm{Id}}(g/g_0) = \lim_{m\to\infty} \frac{\mathrm{area}_g(\mathbf{D}(\partial\phi_m))}{2\pi|\chi(\mathbf{D}(\partial\phi_m))|} \geq \lim_{m\to\infty} \frac{\mathrm{area}_g(\Pi_m)}{4\pi(l_m - 1)} = \alpha$$

and thus $\mathrm{Area}_{\mathrm{Id}}(g/g_0)E(g) \geq 2$. □

4.6.2 Proof of rigidity

Suppose that $\mathrm{Area}_{\mathrm{Id}}(g/g_0)E(g) = 2$. In this case, the proof of the preceding proposition implies that

$$\lim_{m\to\infty} \frac{\mathrm{area}_g(\mathbf{D}(\partial\phi_m))}{\mathrm{area}_g(\Pi_m)} = 1. \tag{4.6.7}$$

The mean curvature of a surface in (M, g) is denoted by H_g.

Lemma 4.6.5. $\lim_{m\to\infty} \frac{1}{\mathrm{area}_g(\mathbf{D}(\partial\phi_m))} \int_{\mathbf{D}(\partial\phi_m)} |H_g|^2 dA_g = 0.$

Proof. Suppose that, after passing to a subsequence, there is $\delta > 0$ so that

$$\int_{\mathbf{D}(\partial\phi_m)} |H_g|^2 dA_g \geq 2\delta\, \mathrm{area}_g(\mathbf{D}(\partial\phi_m)) \tag{4.6.8}$$

for all $m \in \mathbb{N}$.

The second fundamental form of $\mathbf{D}(\partial\phi_m)$ (with respect to g) and any of its derivatives are uniformly bounded independently of m. Arguing like in [EH91, Corollary 4.4] we find $t_0 > 0$, $C_0 > 0$, and $\{\mathbf{D}_m(t)\}_{0\leq t\leq t_0}$ a solution to mean curvature flow with initial condition $\mathbf{D}_m(0) := \mathbf{D}(\partial\phi_m)$ so that both the mean curvature of $\mathbf{D}_m(t)$ and its derivative is bounded uniformly by C_0.

With $H_g(t)$, $A_g(t)$, and ν_t denoting, respectively, the mean curvature, second fundamental form, and normal vector of $\mathbf{D}_m(t)$, we have the evolution equation

$$\frac{d}{dt} H_g(t) = \Delta H_g(t) + (|A_g(t)|^2 + Rc(g)(\nu_t, \nu_t))H_g(t).$$

Thus there is a constant $C_1 > 0$ (depending only on C_0 and g) so that for all $m \in \mathbb{N}$ and $0 \leq t \leq t_0$

$$\frac{d}{dt}\int_{\mathbf{D}_m(t)} |H_g(t)|^2 dA_g \geq -C_1 \text{area}_g(\mathbf{D}_m(t)) \geq -C_1 \text{area}_g(\mathbf{D}(\partial\phi_m)).$$

Choose $t_1 < \min\{\delta/C_1, t_0\}$. From (4.6.8) and the inequality above we have that for all $m \in \mathbb{N}$ and $0 \leq t \leq t_1$

$$\int_{\mathbf{D}_m(t)} |H_g|^2 dA_g \geq \delta \,\text{area}_g(\mathbf{D}(\partial\phi_m)) \geq \delta \,\text{area}_g(\mathbf{D}_m(t)).$$

Therefore

$$\frac{d}{dt}\text{area}_g(\mathbf{D}_m(t)) = -\int_{\mathbf{D}_m(t)} |H_g|^2 dA_g \leq -\delta \,\text{area}_g(\mathbf{D}_m(t))$$

and thus for all $m \in \mathbb{N}$ we have

$$\text{area}_g(\Pi_m) \leq \text{area}_g(\mathbf{D}_m(t_1)) \leq e^{-\delta t_1}\text{area}_g(\mathbf{D}(\partial\phi_m)).$$

This contradicts (4.6.7). □

Recall that $\Omega^*\delta_{\phi_m}$ converges to μ_{Leb} as $m \to \infty$. The previous lemma allows us to apply Theorem 4.7.3 to conclude the following:

For every $\gamma \in \mathcal{C}_0$, there is a lift $D_m \subset \mathbb{H}^3$ of $\mathbf{D}(\partial\phi_m) \subset M$ so that, after passing to a subsequence, D_m converges on compact sets to the totally geodesic disc $D(\gamma) \subset \mathbb{H}^3$ and

$$\lim_{m\to\infty} \int_{D_m \cap B_R(0)} |H_g|^2 dA_{g_0} = 0 \quad \text{for all } R > 0.$$

Thus we obtain that $D(\gamma)$ is a minimal disc for the metric g. The arbitrariness of γ implies that every totally geodesic disc of $\mathbb{H}^3$ is minimal with respect to g.

The next theorem follows from adapting some of the arguments in [LAN21].

Theorem 4.6.6. *For some $c > 0$, $g = cg_0$.*

Proof. We denote the space of Killing symmetric 2-tensors on a Riemannian manifold (X, h) by $K_2(X, h)$. They are characterized by the property (see [Tak83, Section 1]) that a symmetric 2-tensor k lies in $K_2(X, h)$ if and only if for every geodesic $\gamma \subset X$ the function below is constant

$$t \mapsto k(\gamma'(t), \gamma'(t)).$$

Given a symmetric 2-tensor k on X, we denote the absolute value of its determinant with respect

to h by $|k|_h$.

Let δ denote the Euclidean metric in $\mathbb{R}^3$ and $B \subset \mathbb{H}^3$ denote the open unit ball. Consider the Beltrami-Klein model (B, g_{hyp}) for $(\mathbb{H}^3, g_0)$. The important property of this model is that the image of geodesics and totally geodesic discs in (B, g_{hyp}) is the same as affine lines and affine planes in (B, δ).

In [LAN21, Theorem 9.8] the authors classified all metric h on S^n for which every totally geodesic hypersphere is minimal with respect to h. They are those for which

$$|h|_{g_{\text{round}}}^{-2/(n+1)} h \in K_2(S^n, g_{\text{round}}).$$

A similar reasoning, which we repeat for the sake of completeness, gives

Proposition 4.6.7. *A positive definite symmetric 2-tensor h on B has the property that every affine plane is minimal with respect to h if and only if*

$$|h|_{g_{\text{hyp}}}^{-1/2} h \in K_2(B, g_{\text{hyp}}).$$

Proof. We will use the following theorem of Hangan [Han96] : A metric h on B is such that all affine planes have zero mean curvature with respect to h if and only if

$$|h|_{\delta}^{-1/2} h \in K_2(B, \delta).$$

Lemma 4.6.8. *With k a 2-tensor on B then*

$$k \in K_2(B, g_{\text{hyp}}) \iff |g_{\text{hyp}}|_{\delta}^{-1/2} k \in K_2(B, \delta)$$

Proof. From Hangan's theorem we have that $|g_{\text{hyp}}|_{\delta}^{-1/2} g_{\text{hyp}} \in K_2(B, \delta)$ and hence, for every Euclidean geodesic $\gamma \subset B$, we have some constant $c_0 = c_0(\gamma)$ such that for all t

$$g_{\text{hyp}}(\gamma'(t), \gamma'(t)) = |g_{\text{hyp}}|_{\delta}^{1/2}(\gamma(t)) c_0.$$

Note that geodesics with respect to g_{hyp} are also straight lines. Hence if we perform a change of variable according to

$$\frac{dt}{ds}(s) = |g_{\text{hyp}}|_{\delta}^{-1/4}(\gamma(t(s)))$$

we have that $s \mapsto \sigma(s) := \gamma(t(s))$ is a geodesic for g_{hyp}. Therefore

$$\frac{d}{ds}k(\sigma'(s),\sigma'(s)) = \frac{dt}{ds}(s)\frac{d}{dt}_{t=t(s)}\left(\frac{k(\gamma'(t),\gamma'(t))}{|g_{\text{hyp}}|_\delta^{1/2}(\gamma(t))}\right)$$

and so

$$\frac{d}{ds}k(\sigma'(s),\sigma'(s)) = 0 \iff \frac{d}{dt}_{t=t(s)}\left(\frac{k(\gamma'(t),\gamma'(t))}{|g_{\text{hyp}}|_\delta^{1/2}(\gamma(t))}\right) = 0.$$

Thus k is constant along hyperbolic geodesics if and only if $|g_{\text{hyp}}|_\delta^{-1/2}k$ is constant along Euclidean geodesics, which implies the result. □

Using this lemma and the identity $|h|_\delta = |h|_{g_{\text{hyp}}}|g_{\text{hyp}}|_\delta$ we have

$$|h|_\delta^{-1/2}h \in K_2(B,\delta) \iff |h|_{g_{\text{hyp}}}^{-1/2}h \in K_2(B,g_{\text{hyp}}).$$

Hangan's theorem [Han96] implies the desired result. □

Denote the lift of the metric g to $\mathbb{H}^3$ by g as well. We are assuming that every totally geodesic disc in $\mathbb{H}^3$ is minimal with respect to g and so the previous proposition implies that $G := |g|_{g_0}^{-1/2}g \in K_2(M,g_0)$.

The geodesic flow in (M,g_0) is ergodic and so we can choose a geodesic $\sigma \subset M$ which is dense in the unit tangent bundle. From the fact that $t \mapsto G(\sigma'(t),\sigma'(t))$ is constant we deduce the existence of a constant α so that $G(Y,Y) = \alpha g_0(Y,Y)$ for every vector field Y. This implies that $G = \alpha g_0$. Using the fact that $G = |g|_{g_0}^{-1/2}g$ we deduce that g is a positive multiple of g_0.

□

4.7 Auxiliary Results

4.7.1 Asymptotic Expansion

With $g \in \mathcal{U}$, set $h := g - g_0$. Without loss of generality we assume that $|h|_{C^4} \leq 1$.

Given $\gamma \in \mathcal{C}_{\bar{\varepsilon}}$, Recall that $\nu_g(\gamma)$ denotes a continuous unit normal vector field along $\Sigma_g(\gamma)$ with respect to g and $n(\gamma)$ denotes a continuous unit normal vector field along $D(\gamma)$ with respect to g_0.

Recall the diffeomorphism (using the hyperboloid model)

$$F_\gamma : D(\gamma) \to \Sigma_g(\gamma), \quad x \mapsto \cosh(f_\gamma(x))x + \sinh(f_\gamma(x))n(\gamma)(x).$$

Given $p \in M$ and $k \in \mathbb{N}_0$, $l \in \mathbb{N}$, we denote by $O_p^k(|h|^l)$ any quantity for which there is a constant $\alpha_{k,l}$ (independent of p and $g \in \mathcal{U}$) so that its absolute value at p is bounded by $\alpha_{k,l} \sum_{j=0}^k |\nabla^k h|^l(p)$. Likewise, given $\gamma \in \mathcal{C}_{\bar{\varepsilon}}$ and $x \in D(\gamma)$, we denote by $O_x^k(|f_\gamma|^l)$ any quantity for which there is a constant $\beta_{k,l}$ (independent of x, γ, and $g \in \mathcal{U}$) so that its absolute value at x is bounded by $\alpha_{k,l} \sum_{j=0}^k |\nabla^k f_\gamma|^l(x)$.

Let L denote the linearization at g_0 of

$$h \mapsto \mathring{Ric}(g_0 + h) := Ric(g_0 + h) - \frac{R(g_0 + h)}{3}(g_0 + h).$$

Proposition 4.7.1. *With $\gamma \in \mathcal{C}_\varepsilon$, $x \in D(\gamma)$ and $y = F_\gamma(x)$ we have*

$$Ric(g)_y(\nu_g(\gamma), \nu_g(\gamma)) - \frac{1}{3}R(g)(y) = L(h)_x(n(\gamma), n(\gamma)) + O_x^3(|h|^2) + O_x^1(|f_\gamma|^2).$$

With $|Jac_g F_\gamma|$ the Jacobian of $F_\gamma : (D(\gamma), g_0) \to (\Sigma_g(\gamma), g)$ we have

$$|Jac_g F_\gamma|(x) = 1 + O_x^1(|h|) + O_x^1(|f_\gamma|).$$

Proof. Set

$$X := \{(\gamma, f) : \gamma \in \mathcal{C}_{\bar{\varepsilon}}, f \in C^2(D(\gamma)), |f|_{C^2} \leq 1\}.$$

Given $(\gamma, f) \in X$ set

$$F(\gamma, f) : D(\gamma) \to \Sigma_g(\gamma), \quad x \mapsto \cosh(f(x))x + \sinh(f(x))n(\gamma)(x)$$

and $\Sigma(\gamma, f) := F(\gamma, f)(D(\gamma))$. Let $n(\gamma, f)$ be the unit normal vector field along $\Sigma(\gamma, f)$, defined so that $n(\gamma, f)$ depends smoothly on (γ, f) and $n(\gamma, 0) = n(\gamma)$. We have for all $0 \leq t \leq 1$

$$\left| \frac{d^k}{(dt)^k}(n(\gamma, tf)(F(\gamma, tf)(x)) \right| = O_x^1(|f|^k), \quad k = 1, 2.$$

Hence, if G is a 2-tensor on M with $|G|_{C^2} \leq 1$, then

$$\alpha(t) := G(n(\gamma, tf)(F(\gamma, tf)(x)), n(\gamma, tf)(F(\gamma, tf)(x)))$$

satisfies

$$|\alpha'(0)| \leq (|\nabla G|(x) + |G|(x))O^1_x(|f|) \quad \text{and} \quad \sup_{0\leq t\leq 1} |\alpha''(t)| = O^1_x(|f|^2).$$

Therefore, with $y = F(\gamma, f)(x)$, we obtain from Taylor's expansion

$$|G(n(\gamma, f)(y), n(\gamma, f)(y)) - G(n(\gamma)(x), n(\gamma)(x))| \\ \leq |\nabla G|^2(x) + |G(x)|^2 + O^1_x(|f|^2). \quad (4.7.1)$$

Setting $G = |\nabla^k h|^2 g_0$ in this inequality we deduce that for $k = 0, 1, 2$.

$$|\nabla^k h|(y) = O^{k+1}_x(|h|) + O^1_x(|f|). \quad (4.7.2)$$

Setting $G = L(h)$ in (4.7.1) and using the fact that L is a second order differential operator we deduce

$$|L(h)_x(n(\gamma)(x), n(\gamma)(x)) - L(h)_y(n(\gamma, f)(y), n(\gamma, f)(y))| \\ = O^3_x(|h|^2) + O^1_x(|f|^2). \quad (4.7.3)$$

Let $\nu_g(\gamma, f)$ denote the unit normal vector field along $\Sigma(\gamma, f)$ with respect to $g = g_0 + h$ so that $\nu_{g_0}(\gamma, f) = n(\gamma, f)$ and ν_g depends smoothly on its parameters. Then

$$|\nu_g(\gamma, f)(y) - n(\gamma, f)(y)| = O^0_y(|h|) = O^1_x(|h|) + O^1_x(|f|),$$

where in the last identity we used (4.7.2). Using this identity and (4.7.2) we have

$$L(h)_y(\nu_g(\gamma, f)(y), \nu_g(\gamma, f)(y)) \\ = L(h)_y(n(\gamma, f)(y), n(\gamma, f)(y)) + O^3_x(|h|^2) + O^1_x(|f|^2).$$

Combining with (4.7.3) we deduce

$$L(h)_y(\nu_g(\gamma, f)(y), \nu_g(\gamma, f)(y)) \\ = L(h)_y(n(\gamma)(x), n(\gamma)(x)) + O^3_x(|h|^2) + O^1_x(|f|^2).$$

Using Taylor's expansion we have that for every $y \in M$, every unit vector field $Y \in T_yM$ and

$g_0 + h \in \mathcal{U}$

$$|\mathring{Ric}(g_0+h)_y(Y,Y) - L(h)_y(Y,Y)| = O_y^2(|h|^2) = O_x^3(|h|^2) + O_x^1(|f|^2).$$

Therefore

$$\mathring{Ric}(g_0+h)_y(\nu_g(\gamma,f)(y),\nu_g(\gamma,f)(y)) \\ = L(h)_y(n(\gamma)(x),n(\gamma)(x)) + O_x^3(|h|^2) + O_x^1(|f|^2).$$

The first statement in the proposition follows from choosing $f = f_\gamma$ in the identity above.

We now prove the statement regarding $|Jac_g F_\gamma|$. With $(\gamma, f) \in X$, denote by $|Jac_g F(\gamma,f)|(g)$ the Jacobian of $F(\gamma,f) : (D(\gamma), g_0) \to (\Sigma(\gamma,f), g)$. With $g = g_0 + h \in \mathcal{U}$ and $y = F(\gamma,f)(x)$ we have

$$\begin{aligned} |Jac_g F(\gamma,f)|(x) &= |Jac_{g_0} F(\gamma,f)|(x) + O_y^0(|h|) \\ &= |Jac_{g_0} F(\gamma,f)|(x) + O_x^1(|h|) + O_x^1(|f|) \\ &= |Jac_{g_0} F(\gamma,0)|(x) + O_x^1(|h|) + O_x^1(|f|) \\ &= 1 + O_x^1(|h|) + O_x^1(|f|). \end{aligned}$$

□

4.7.2 Stability of Ricci flow

We assume we have a solution to normalized Ricci flow $(\bar{g}_t)_{t\geq 0}$ (4.4.1) and a smooth family of diffeomorphisms $\{\Phi_t\}_{t\geq 0}$ converging strongly to some diffeomorphism so that $g_t := \Phi_t^* \bar{g}_t$ solves the DeTurck-modified Ricci flow (which is strictly parabolic) and converges to g_0 as $t \to \infty$.

Recall the operator $\mathcal{A}$ defined in (4.4.9). It has discrete spectrum $1 \leq \lambda_1 \leq \lambda_2 \leq \dots$

Proposition 4.7.2. *Set $h_t := \Phi_t^* \bar{g}_t - g_0$, $t \geq 0$. Then*

- $|h_t|_{C^4} \leq O(e^{-2t/3})$;
- *$e^t h_t$ converges in $W^{2,2}$ as $t \to \infty$ to $\bar{h}$, where $\bar{h}$ is a smooth 2-tensor with $\mathcal{A}(\bar{h}) = -\bar{h}$.*

Proof. In [KY09] it is shown that $|h_t|_{C^2} \leq O(e^{-2t/3})$. Standard estimates (similar to [Has12, Lemma 5.3]) show that for all $k \in \mathbb{N}$, $|Ric(\bar{g}_t) + 2\bar{g}_t|_{C^k} \leq O(e^{-2t/3})$ and thus $|\bar{g}_t - \bar{g}_{+\infty}|_{C^k} \leq O(e^{-2t/3})$ as well. The diffeomorphisms $\{\Phi_t\}_{t\geq 0}$ depend only on g_0 and $\{\bar{g}_t\}_{t\geq 0}$, and one can check that for all $k \in \mathbb{N}$, $|\Phi_t - \Phi_{+\infty}|_{C^k} \leq O(e^{-2t/3})$. In particular, $|h_t|_{C^4} \leq O(e^{-2t/3})$.

In [KY09] it is also show that the tensors h_t satisfy an equation of the form

$$\frac{dh_t}{dt} = \mathcal{A}(h_t) + Q_t,$$

where Q_t is a non-linear term depending on $g_0, h_t, \nabla h_t, \nabla^2 h_t$. The important property we need is that $|Q_t|_{C^k} \leq O(|h_t|^2_{C^{k+2}})$. From here on we consider Q_t as being a fixed non-homogeneous term where $|Q_t|_{C^2} \leq O(e^{-4t/3})$.

Consider an L^2-orthonormal basis $\{u_k\}_{k\in\mathbb{N}}$ for the space of symmetric 2-tensors on M made of eigentensors for $\mathcal{A}$. Necessarily

$$h_t = \sum_{k=0}^{\infty} \left(e^{-\lambda_k t}\langle h_0, u_k\rangle + \int_0^t e^{\lambda_k(\tau-t)}\langle Q_\tau, u_k\rangle d\tau \right) u_k.$$

Set $\eta_t := h_t - f(t)$, where

$$f(t) := \sum_{\lambda_k=1} \left(e^{-t}\langle h_0, u_k\rangle + \int_0^t e^{(\tau-t)}\langle Q_\tau, u_k\rangle d\tau \right) u_k.$$

Because η_t is orthogonal to the 1-eigentensors there is some δ so that

$$\int_M \mathcal{A}(\eta_t)\eta_t dV_{g_0} \leq -(1+2\delta)\int_M \eta_t^2 dV_{g_0}.$$

Moreover we have $\partial_t \eta_t = \mathcal{A}(\eta_t) + O(e^{-4t/3})$, which when combined with the inequality above implies that

$$\begin{aligned}
\frac{d}{dt}\int_M \eta_t^2 dV_{g_0} &\leq -2(1+2\delta)\int_M \eta_t^2 dV_{g_0} + O(e^{-4t/3})\int_M \eta_t dV_{g_0} \\
&\leq -2(1+2\delta)\int_M \eta_t^2 dV_{g_0} + O(e^{-4t/3})\left(\int_M \eta_t^2 dV_{g_0}\right)^{1/2} \\
&\leq -2(1+\delta)\int_M \eta_t^2 dV_{g_0} + O(e^{-8t/3}).
\end{aligned}$$

Hence $\int_M (e^t\eta_t)^2 dV_{g_0} \to 0$ as $t \to \infty$. Likewise, $\eta_t^1 := \mathcal{A}(\eta_t)$ is also orthogonal to all 1-eigentensors, $\partial_t \eta_t^1 = \mathcal{A}(\eta_t^1) + O(e^{-4t/3})$, and the same argument gives $\int_M (e^t\eta_t^1)^2 dV_{g_0} \to 0$. This implies that $e^t\eta_t$ tends to zero in $W^{2,2}$ as $t \to \infty$. The result follows because the 1-eigentensor $e^t f(t)$ converges in $W^{2,2}$ as $t \to \infty$ to some smooth tensor $\bar{h}$. □

4.7.3 A theorem of Calegari-Marques-Neves

Consider a sequence $\phi_i \in \mathcal{F}(M, 1/i)$ equivariant with respect to a representation of a Fuchsian subgroup of $\mathrm{PSL}(2,\mathbb{R})$ in $\pi_1(M) < \mathrm{PSL}(2,\mathbb{C})$. Let $G_i < \pi_1(M)$ be the image of that representation. The group G_i preserves $D(\partial\phi_i) \subset \mathbb{H}^3$ and recall that $\mathbf{D}(\partial\phi_i) = D(\partial\phi_i)/G_i$. We assume

- $\Omega_*\phi_i$ converges weakly to a measure ν on $F(M)$ as $i \to \infty$, where the measure ν is such that so that $\nu(O) > 0$ for every open set O;

- there is a sequence of immersed surfaces $\Sigma_i \subset M$ homotopic to $\mathbf{D}(\partial\phi_i)$ and $f_i \in C^0(\Sigma_i)$ so that
$$\lim_{i\to\infty} \frac{1}{\mathrm{area}_{g_0}(\Sigma_i)} \int_{\Sigma_i} |f_i| dA_{g_0} = 0.$$

The Hausdorff distance between two sets in $\mathbb{H}^3$ is denoted by d_H.

The following theorem corresponds to Theorem 6.1 of [CMN], where it is assumed that Σ_i is area-minimizing with respect to some metric. An inspection of the proof shows that one only needs the areas of Σ_i and $\mathbf{D}(\partial\phi_i)$ to be comparable and their universal covers to be at a uniform Hausdorff distance from each other.

Theorem 4.7.3. *Assume the existence of $C > 0$ and, for all $i \in \mathbb{N}$, a covering $\Omega_i \subset \mathbb{H}^3$ of Σ_i so that*
$$\mathrm{area}_{g_0}(\Sigma_i) \leq C\mathrm{area}_{g_0}(\mathbf{D}(\partial\phi_i)) \quad \textit{and} \quad C^{-1} \leq d_H(\Omega_i, D(\partial\phi_i)) \leq C.$$
For every $\gamma \in \mathcal{C}_0$ there is $\eta_i \in \pi_1(M) < Isom(\mathbb{H}^3)$ such that, after passing to a subsequence, $\eta_i(D(\partial\phi_i))$ converges on compact sets to $D(\gamma)$ and
$$\lim_{i\to\infty} \int_{\eta_i(\Omega_i)\cap B_R(0)} |f_i \circ \eta_i^{-1}| dA_{g_0} = 0 \quad \textit{for all } R > 0.$$

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