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Rankin-Selberg integrals in many complex variables and Rankin-Selberg integrals associated to non-unique models

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Rankin-Selberg integrals in many complex variables and Rankin-Selberg integrals associated to non-unique models

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
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Abstract

In this thesis we explore Rankin-Selberg integrals in many complex variables and Rankin-Selberg integrals associated to non-unique models. In particular we try to bring to light a possible connection between them. First, we reinterpret the integral of Kohnen-Skoruppa [26] for the Spin L-function on GSp_4 as an integral associated to a non-unique model, following Piatetski-Shapiro and Rallis [30]. Then we give two integrals on the quasisplit unitary group $\mathrm{GU}(2,2)$ in four variables. The first integral is in two complex variables, and applies to generic cusp forms on $\mathrm{GU}(2,2)$; it represents the product of the standard (8 dimensional representation) of ${}^L\mathrm{GU}(2,2)$ and the exterior square (6 dimensional repesentation) of the dual group. The second integral on $\mathrm{GU}(2,2)$ represents the exterior square L-function and is associated to a non-unique model; this integral is a reinterpretation of an integral of Gritsenko [22] who considered holomorphic Hermitian modular forms of genus two. We then give an integral representation for the Spin L-function on GSp_6, again associated to a non-unique model. As we explain below, these non-unique model integrals on $\mathrm{GU}(2,2)$ and GSp_6 may be considered as analogues of the integral of Kohnen-Skoruppa. Finally, we give two different Rankin-Selberg integrals unfolding to the Whittaker model and using many complex variables. One is a three variable integral on PGL_4, and represents the product of the L-functions of the three fundamental representations of the dual group. The other is for the exterior cube L-function on the quasisplit $\mathrm{GU}(3,3)$. This last integral is only in one complex variable, but its unramified calculation makes use of a two-variable integral on GSp_6 due to Bump-Friedberg-Ginzburg [5]. Some of the results in this thesis stem from work done jointly with Shrenik Shah.

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Many of the results below were worked out jointly with Shrenik Shah, especially those in chapters two, five, and seven. It is a pleasure to thank him for collaborating with me on this venture. Much of chaper 5 is from a working version of a preprint with Shrenik, and I thank him for permitting me to include parts of this preprint in my thesis.

This thesis has also benefited from mathematical comments from several additional people, specifically Peter Sarnak, Ila Varma, and Xiaoheng Wang.

Anyone who has spent a significant portion of time in Princeton, New Jersey will know with certainty that a human being can only be successful at his or her endeavors when receiving the love and support of family and friends. In this vein, it is with deep gratitude that I thank some of the many people whose friendship have made my life happier during the last several years, in particular Dan Collins, Shrenik Shah, Ila Varma, and especially Elisheva Weiss. Without you, this thesis would never have taken shape.

For my parents

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Chapter 1

Introduction

Suppose G is a quasisplit reductive group over a number field F, split over the extension field M of F, and $\pi = \otimes \pi_v$ is an automorphic cuspidal representation of G. Langlands has defined the dual group ${}^L G = {}^L G(\mathbf{C})^0 \rtimes Gal(M/F)$ where ${}^L G(\mathbf{C})^0$ is the complex dual group of G over $\mathbf{C}$. Outside a finite set of places S of F, G/F_v and π_v are both unramified. For such v, Langlands associates a Frobenius conjugacy class $A_v \rtimes Frob_v$ in ${}^L G$ by reinterpreting the Satake isomorphism. Now suppose $r : {}^L G \to \mathrm{GL}_N(\mathbf{C})$ is a finite dimensional algebraic representation of the dual group. To this data, he associates an Euler product

$$L^S(\pi, r, s) = \prod_{v \notin S} \det(1 - r(A_v \rtimes Frob_v) q_v^{-s})^{-1},$$

absolutely convergent when the real part of s is sufficiently large.

The conjectural meromorphic continuation of these L-functions, while known in many important examples, is still largely open. There are two techniques known for proving this continuation. The first is the so-called Langlands-Shahidi method. In this method, one finds the L-function $L(\pi, r, s)$ as the constant term of an Eisenstein series on some auxilliary group H; the known meromorphic continuation of the Eisenstein series then implies that of the L-function. The second method is that of Rankin-Selberg. Here, one identifies the L-function in question $L(\pi, r, s)$ as an integral of a cusp form ϕ in the space of π and (one or more) Eisenstein series. Again, the known continuation of the Eisenstein series then implies that of the L-function. One additional benefit of Rankin-Selberg integral representations is that they can sometimes be used to connect special values of L-functions to geometric constructions: if a Rankin-Selberg integral represents an L-function $L(\pi, r, s)$, and all the terms in the integral have geometric interpretations at some special point $s = s_0$, then one

obtains a geometric interpretation for the special value $L(\pi, r, s_0)$. The theorems in this thesis will all be Rankin-Selberg integral representations, although we will will not say anything about motivic interpretations of the special values of the resulting L-functions.

1.1 Rankin-Selberg integrals unfolding to non-unique models

Suppose now that H is an algebraic subgroup of G, π_v is an irreducible admissible representation of $G(F_v)$, and $\chi : H(\mathbf{A}) \to \mathbf{C}$ is character, trivial on $H(F)$. Denote by V_{π_v} the space of the representation π_v. By an (H, χ) model of π_v we mean a linear functional $\ell : V_{\pi_v} \to \mathbf{C}$ satisfying $\ell(\pi(h)v) = \chi(h)\ell(v)$ for all $v \in V_{\pi_v}$ and $h \in H(F_v)$. Let us say that an (H, χ) model is *unique* if the space of such functionals is at most one-dimensional, and non-unique otherwise. For example, if U is the unipotent radical of a Borel of G and χ is a nondegenerate character of U, then a (U, χ) model is a Whittaker model, and is known to always be unique. Most known Rankin-Selberg integral representations derive their Euler product structure from the uniqueness of some model at all places.

In the '70s, Andrianov [2] gave a global Rankin-Selberg integral representation for the Standard L-function of holomorphic Siegel modular forms on Sp_{2n}. As explained by Piatetski-Shapiro and Rallis [30], Andrianov's proof is global, and it is difficult to understand where the Euler product structure of the L-function comes from. In fact, the integral representation Andrianov writes down unfolds to a model that is not unique. With a desire to better understand Andrianov's result, Piatetski-Shapiro and Rallis generalized Andrianov's result to apply to arbitrary cuspidal representations on Sp_{2n}, and in the process elucidated a new method to recover a partial Euler product in situations when a Rankin-Selberg integral unfolds to a non-unique model.

In fact, there are only a few examples of global Rankin-Selberg convolutions that unfold to a model that is not unique, yet are known to represent an L-function. Besides the examples of Piatetski-Shapiro and Rallis [30], there are examples of Bump-Hoffstein [8], [9] on the higher covers of GL_n, and a few families of integrals on GL_n and classical groups by Bump-Furusawa-Ginzburg in [6] which are for Standard L-functions. There is also a preprint of Gurevich-Segal [23] which applies to cusp forms on G_2, and gives the L-function for the seven-dimensional representaion of the dual group. So far as this author is aware, this was the current exhaustive list.

1.2 Integral representations in this thesis

The first object of this thesis is to give more examples of Rankin-Selberg integral representations that unfold to models that are not unique. As explained in [6], this is an important problem, because there are many global Rankin-Selberg convolutions that unfold to non-unique models, but are not known to represent L-functions. So by giving more examples, we provide evidence that the Rankin-Selberg method may have wider applicability than was perhaps previously realized.

The examples we give are for the Spin L-functions[1] on GSp_4 and GSp_6, and the exterior square L-function on $\mathrm{GU}(2,2)$. As with the integral of Piatetski-Shapiro–Rallis [30], our integrals on GSp_4 and $\mathrm{GU}(2,2)$ have classical antecedents. The integral on GSp_4 was considered previously by Kohnen and Skoruppa in [26] for holomorphic Siegel modular forms, and the one on $\mathrm{GU}(2,2)$ by Gritsenko [22] for Hermitian modular forms. The proofs in [26] and [22], like Andrianov's in [2], are global in nature and only work for these holomorphic modular forms. By reinterpreting these integrals for arbitrary cusp forms, and giving a proof in the style of [30], we show that these clasical integrals of Kohnen-Skoruppa and Gritsenko apply to a much wider class of cuspidal representations. Our integral on GSp_6 does not have any classical predecessor.

Let us point out that, even though it is not all that difficult, it is not perfectly obvious how to intepret the integrals in [26] and [22] as integrals applying to arbitrary cusp forms. The reason is that the global convolutions in these papers use special functions that have not previously appeared in integral representations associated to unique models. The special functions that are used are perhaps best thought of as fourier coefficients of theta functions, and exist on orthogonal groups[2]. So, one interesting aspect of these integral representations is that they use special functions that perhaps have not been adequately considered in the context of Rankin-Selberg convolutions.

Let us now describe the non-unique-model integral on GSp_4 exactly. First, denote by W_4 the standard 4-dimensional representation of GSp_4, and give W_4 the symplectic basis $\{e_1, e_2, f_1, f_2\}$, so that $\langle e_i, e_j\rangle = 0 = \langle f_i, f_j\rangle$ and $\langle e_i, f_j\rangle = \delta_{ij}$. We let GSp_4 act on *the right* of W_4. Define the Klingen parabolic Q to be the stabilizer of the line $\mathbf{Q}f_1$, and take $f_Q(g;s) \in Ind(\delta_Q^s)$ a standard section, and set $E_Q(g;s) = \sum_{\gamma \in Q(\mathbf{Q})\backslash \mathrm{GSp}_4(\mathbf{Q})} f_Q(\gamma g; s)$ the Klingen Eisenstein series.

Now we define the special function to be used in the integral. Set $V_5 = \wedge_0^2(W_4) \otimes \nu^{-1}$, where ν is the similitude on GSp_4. Here $\wedge_0^2 W_4$ is by defintion the kernel of the contraction map $\wedge^2 W_4 \to \nu$.

[1] Recall that the dual group of GSp_{2n} is GSpin_{2n+1}, which has a 2^n-dimensional Spin representation.

[2] More precisely, what we mean by this is: take a theta function $\theta(g,h)$ on a dual reductive pair $G \times H$, where G is a symplectic group and H is an orthogonal group. Then, if one takes a fourier coefficient of θ with respect to G, one gets an automorphic form on H. These are the types of special functions that appear in the integral representations, although they are not defined in exactly this way in [26] or [22].

For an nonzero integer D define the element $v_D = De_1 \wedge f_2 + e_2 \wedge f_1$ of V_5. If $\alpha \in \mathcal{S}(V_5(\mathbf{A}))$ is a Schwart-Bruhat function, then we set

$$P_D^\alpha(g) = \sum_{\gamma \in Stab(v_D)(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{Q})} \alpha(v_D \gamma g). \tag{1.1}$$

As α is Schwart, the sum trivially converges and defines a function of moderate growth on GSp_4. If $L = \mathbf{Q}(\sqrt{D})$, then the group $\mathrm{GL}_{2,L}^* = \{g \in \mathrm{GL}_{2,L} : \det(g) \in \mathbf{Q}^\times\}$ embeds in GSp_4. The stabilizer of v_D is this embedded $\mathrm{GL}_{2,L}^*$. Note that there is an exceptional isomorphism $\mathrm{PGSp}_4 \simeq \mathrm{SO}(5)$, and we are really defining P via the 5-dimensional orthogonal representation of GSp_4.

For convenience we assume ϕ has trivial central character. The global integral is now

$$I(\phi; s) = \int_{[\mathrm{GSp}_4]} \phi(g) P_D^\alpha(g) E_Q(g; s)\, dg. \tag{1.2}$$

Theorem 1.2.1. *$I(\phi; s)$ represents the partial Spin L-function. More precisely, the integral unfolds to a non-unique model, and one has*

$$I(\phi; s) = I_\infty(\phi; s) \frac{L^S(\pi, Spin, 2s - \frac{1}{2})}{\zeta(4s)},$$

for an archimedean integral $I_\infty(\phi; s)$.

See the next chapter for the precise statement.

Before we describe the non-unique-model integrals on $\mathrm{GU}(2,2)$ and GSp_6, we bring in a different topic, that of Rankin-Selberg integrals in more than one complex variable. Following Bump-Friedberg [4], Bump-Friedberg-Ginzburg give in [5] some Rankin-Selberg integral representations on small rank symplectic groups that represent the product of the Standard L-function in one variable and the Spin L-function in another. One of their integrals is on GSp_4, and is as follows:

$$I(\phi; s, w) = \int_{[\mathrm{GSp}_4]} \phi(g) E_P(g; w) E_Q(g; s)\, dg,$$

where now $E_P(g; w)$ is a Siegel Eisenstein series in a new complex parameter. They show that this integral unfolds to the Whittaker model of ϕ, and that it represents the partial L-function

$$\frac{L(\pi, Std, 3w - 1)}{\zeta(6w - 2)\zeta(3w)} \frac{L(\pi, Spin, 2s - \frac{1}{2})}{\zeta(4s)}.$$

The reader will now have no doubt recognized the similarity between this two-variable integral

and the integral (1.2) above: The Siegel Eisenstein series in the variable w has become the special function P_D^α, one loses the L-function in the parameter w but one still retains the Spin L-function in the variable s.[3] And amazingly, even though the two-variable integral is only nonvanishing for generic cusp forms, the integral (1.2) can apply to nongeneric forms.

The second, and perhaps even more interesting goal of this thesis, is to give more evidence that there is a relationship between two-variable integrals on the one hand, and integrals with special functions associated to non-unique models on the other. Our integrals for the Spin L-function on GSp_6 and the exterior square L-function on $\mathrm{GU}(2,2)$, following Gritsenko, do just this. Namely, Bump-Friedberg-Ginzburg [5] also have a two variable integral for GSp_6, which uses the same Siegel Eisenstein series $E_P(g;w)$ on GSp_4 used above. Again replacing this Eisenstein series by the special function P_D^α, one gets an integral unfolding to a non-unique model but still representing the Spin L-function on GSp_6. For $\mathrm{GU}(2,2)$, we first give a new two-variable integral unfolding to the Whittaker model and representing the product of the L-functions for the standard (eight-dimensional representation) of ${}^L\mathrm{GU}(2,2)$ and the exterior square (six-dimensional representation) of the dual group. Replacing one of the Eisenstein series in this two-variable integral with an appropriate speciala function, we then get the non-unique-model integral generalizing the classical convolution of Gritsenko.

We now describe the results on $\mathrm{GU}(2,2)$ and GSp_6 more precisely. First, we give the results on $\mathrm{GU}(2,2)$. Suppose $E/\mathbf{Q}$ is a quadratic extension, and the unitary group $\mathrm{GU}(2,2)$ is defined via E. Recall that the dual group of $\mathrm{GU}(2,2)$ is ${}^L\mathrm{GU}(2,2) = (\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_4(\mathbf{C})) \rtimes Gal(E/\mathbf{Q})$. The connected component of the identity has a four-dimensional representation $\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_4(\mathbf{C}) \to \mathrm{GL}_4(\mathbf{C})$, forgetting the first factor. Inducing this four-dimensional representation to ${}^L\mathrm{GU}(2,2)$, one gets an eight-dimensional representation which we call standard. The map $\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_4(\mathbf{C}) \to \mathrm{GL}_6(\mathbf{C})$ sending $(\lambda, g) \mapsto \lambda \wedge^2 g$ extends to a six dimension representation of ${}^L\mathrm{GU}(2,2)$, which we write as $\wedge^2$. (See the main body of the text for more precise definitions.) Here is the two-variable integral for generic representations.

Theorem 1.2.2. *Assume π has trivial central character. Denote by P the standard Siegel parabolic of* $\mathrm{GU}(2,2)$*, stabilizing an isotropic two-dimensional subspace, and by Q the Klingen parabolic of* $\mathrm{GU}(2,2)$ *stabilizing an isotropic line. Then the global integral*

$$I(\phi; s, w) = \int_{[\mathrm{GU}(2,2)]} \phi(g) E_P(g; w) E_Q(g; s)\, dg$$

[3] And if one replaced the E_Q by a theta function, one would essentially get the integral of Andrianov and Piatetski-Shapiro and Rallis that gives the Standard L-function.

unfolds to the Whittaker model of ϕ, and the local unramified integrals give

$$\frac{L(\pi, Std, 2w-\frac{1}{2})}{\zeta(4w)L(\epsilon_{E/\mathbf{Q}}, 4w-1)}\frac{L(\pi, \wedge^2, 3s-1)}{\zeta_E(3s)\zeta(6s-2)}$$

where $\epsilon_{E/\mathbf{Q}}$ is the quadratic character associated to the extension E.

The unramified calculation for primes inert in E follows from that of Bump-Friedberg-Ginzburg [5] for their GSp_4 integral.

We now describe the non-unique-model integral for the exterior square L-function. As we explained above, Gritsenko [22] considered a version of this integral for Hermitian modular forms of genus two. Let us describe the integrals more exactly. To do this, note that there is a surjection $\mathrm{GU}(2,2) \to \mathrm{U}(1)$ defined by $g \mapsto \det(g)\nu(g)^{-2}$. Write H for the kernel. The group H is an orthogonal group with a six-dimensional $\mathbf{Q}$-representation, and the special function to be used in the integral representation will be defined on H. ($\mathrm{GU}(2,2)$ has a six-dimensional representation over E, namely the wedge square, but not one defined over $\mathbf{Q}$.) Similar to (1.1), the special function P_R is defined using the choice of an auxilliary non-zero integer R and a Schwartz-Bruhat function on the six-dimensional orthogonal representation of H. See the text for the precise definition of P_R. One gets

Theorem 1.2.3. *The global integral*

$$\int_{[H]} \phi(g)P_R(g)E_Q(g;s)\,dg$$

unfolds to a non-unique model, and is equal to

$$I_\infty(\phi;s)\frac{L^S(\pi,\wedge^2,3s-1)}{\zeta_E^S(3s)\zeta^S(6s-2)}$$

for an archimedean integral $I_\infty(\phi;s)$.

Here is the integral on GSp_6 associated to a non-unique model. Denote $\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4$ the set of (g_1, g_2) in $\mathrm{GL}_2 \times \mathrm{GSp}_4$ with $\det(g_1) = \nu(g_2)$, with ν always denoting the similitude. Then $\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4$ embeds in GSp_6, and for a form ϕ in the space of a cuspidal representation π of GSp_6, the global integral is

$$I(\phi;s) = \int_{[\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4]} E(g_1;s)P_D^\alpha(g_2)\phi((g_1,g_2))\,dg,$$

where $E(g_1; s)$ is the standard Eisenstein series on GL_2, and P_D^α is the special function on GSp_4 defined in (1.1).

Theorem 1.2.4. *The global integral $I(\phi; s)$ unfolds to a non-unique model, and is equal to*

$$I_\infty(\phi; s)\frac{L^S(\pi, Spin, s)}{\zeta^S(2s)},$$

for an archimedean integral $I_\infty(\phi; s)$.

The proof of the theorem makes essential use of a result of Andrianov [1], who explicitly computes the Spin L-function in terms of Hecke operators. In principle, the proof of the theorem is similar to the ones we give for Theorems 1.2.1 and 1.2.3[4], but the calculations are *substantially* more involved. Although all of the proofs of the results in this thesis are entirely elelementary, this result by far involves the lengthiest calculations.

We need to make several remarks about this result. First, all the integrals involving these special functions have degenerate versions where the special function P_D is replaced by the constant function 1 on the domain $[Stab(v_D)]$, where v_D is the element used to define P_D. In the case of GSp_6, this gives a global integral

$$I'(\phi; s) = \int_{[\mathrm{GL}_2 \boxtimes \mathrm{GL}_{2,L}^*]} E(g_1; s)\phi((g_1, g_2))\, dg.$$

The integral unfolds to the same non-unique model appearing in Theorem 1.2.4, and we prove it too represents the Spin L-function. This integral I' was in fact considered in the paper [13] of Gan and Gurevich for a special class of CAP forms on GSp_6 constructed by Ginzburg [18]. For these special forms, the model arising in the theorem actually is unique, and the Spin L-function obtained is actually a product of standard L-functions on GL_2 and $\mathrm{GL}_2 \times \mathrm{GL}_2$. The proof in [13] depends very much on these additional hypotheses, and does not carry over to a general cusp form.

We also say a bit more about the fourier coefficient ϕ_χ arising in the proof of Theorem 1.2.4. Denote by R the nonmaximal standard parabolic subgroup of GSp_6 that is the intersection of the standard Klingen and Siegel parabolics, so that the Levi of R is isomorphic to $\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_2$. Then the fourier coefficient that arises in Theorem 1.2.4 comes from a nondegenerate character χ of the unipotent radical of R. As such, it automatically vanishes for all holomorphic forms, so this integral does not apply to holomorphic Siegel modular forms. At present, we don't know much about the forms ϕ for which ϕ_χ is nonzero, other than that there are lifts from G_2 that have ϕ_χ nonzero, due to the work of Gan-Gurevich [13] and Gan-Gurevich-Jiang [14].

[4]There is one new twist, however. See the chapter on the Spin L-function on GSp_6.

When a Rankin-Selberg convolution unfolds to a model that is not unique, there is no *a priori* reason to believe it represents an L-function, let alone which exact L-function. And let us point out that the method of proof for the non-unique-model integrals only succeeds because one both

- correctly predicts that the unfolded Rankin-Selberg convolution represents an L-function;
- correctly guesses the L-function that appears.

This is especially relevant in the case of the GSp_6 integral, where the desired equality between integral representation and L-function only appears at the end of a lengthy calculation. Here one needs good prior reason to believe the calculation will work before even attempting it. In light of these remarks, the phenomenon connecting this integral to the two-variable integral of Bump-Friedberg-Ginzburg [5] is especially important, because the connection to the two-variable integral gives one hope that calculation will in fact work. Thus the evidence we give relating two-variable integrals and non-unique-model integrals is not just interesting, but quite useful, because when it applies it predicts both the existence of a new Rankin-Selberg convolution and which exact L-function it represents.

Finally, we end with two more integral representations making use of several complex variables, which although simple, appear to be missing from the literature. The first is for the exterior cube L-function of a generic π on the quasisplit unitary group $\mathrm{GU}(3,3)$. We expect this integral representation to be known to the experts[5]. To define it, note that GSp_6 embeds in $\mathrm{GU}(3,3)$ as the points fixed under the conjugation of E over $\mathbf{Q}$. If ϕ is a cusp form on $\mathrm{GU}(3,3)$, the integral is

$$\int_{[\mathrm{GSp}_6]} E_P(g;s)\phi(g)\,dg$$

where $E_P(g;s)$ is the Siegel Eisenstein series on GSp_6. When E is the split quadratic extension, $\mathrm{GU}(3,3)$ is $\mathrm{GL}_1 \times \mathrm{GL}_6$, and this is exactly the integral of Ginzburg-Rallis [21] for the exterior cube L-function of GL_6. In fact, the unfolding of the integral is essentially identical to that of [21], and the unramified calculation at places split in E is just their unramified calculation. Furthermore, the unramified calculation at places inert in E in fact follows from the unramified calculation for the two-variable integral of Bump-Friedberg-Ginzburg on GSp_6. So the result is really just the observation that these theorems of Ginzburg-Rallis and Bump-Friedberg-Ginzburg may be combined to get the exterior cube on the unitary group. But because this result dovetails well with the others of this thesis, and because we anticipate future arithmetic applications of this integral, we have decided to record it now.

[5] We have just learned of a preprint of Lei Zhang that may address this L-function

The last integral is a three variable integral on PGL_4. It represents the product of the three fundamental representations of the dual group $\mathrm{SL}_4(\mathbf{C})$. This integral is inspired by the two-variable integral on $\mathrm{GU}(2,2)$ and the three variable integral of Gan-Hundley [15] on the split orthogonal group on eight variables. Note that GL_4 may be considered a split orthogonal group on six variables. The integral we give is simply

$$\int_{[\mathrm{GL}_4]} \phi(g) E_P(g; w) E_Q(g; s_1, s_2)\, dg.$$

Here P is the maximal $(2,2)$ parabolic of GL_4, whereas Q is the nonmaximal $(1,2,1)$ parabolic.

1.3 Plan of the thesis

A quick glance at the table of contents will yield the organization of the thesis. The different chapters are essentially self-contained, so that the reader may read the integral representations in any order without fear of notational confusion. The computations throughout the main body of the text are formal. In the final chapter we explain the analytic justifications of these formal manipulations. This chapter freely uses notation from the Chapter on Spin on GSp_6. We have a couple notational conventions that we use freely thoughout the entire text; these are summarized now.

1.4 Notational Conventions

In Rankin-Selberg integral representations, domains of integration of the form $H(F)Z(\mathbf{A})\backslash H(\mathbf{A})$ are used frequently. We make this notation more compact as follows: If H is a reductive subgroup of an ambient group G, and Z is identity component of the center of G, then we write $[H]$ for $H(F)Z_H(\mathbf{A})\backslash H(\mathbf{A})$, where Z_H is the identity component of $Z \cap H$. If U is a unipotent group, we write $[U]$ for $U(F)\backslash U(\mathbf{A})$.

When defining Eisenstein series, we will often use the phrase *standard section*. By such a section f, we mean that f is factorizable, and that for almost all p, $f_p(k; s) \equiv 1$ for all k in the maximal compact. As a general rule of thumb, we use P for Siegel parabolic subgroups (meaning they stabilize a maximal istrotropic subspace of the defining representation) and Q for Klingen parabolic subgroups (meaning that Q is the stabilizer of an isotropic line). We will often use w for the complex parameter for Siegel Eisenstein series and s as the parameter for Klingen Eisenstein series.

We will always let our algebraic groups act *on the right* of their finite dimensional representations; this convention is well-suited for integral representations.

For similitude groups G, we will denote by $\nu : G \to \mathrm{GL}_1$ the similitude. Unless otherwise

specified, E_{ij} denotes the matrix with a 1 at position ij and zeroes elsewhere. We write $n_{ij}(x)$ for $1 + xE_{ij}$, for $x \in \mathbf{G}_a$.

Chapter 2

The Spin L-function on GSp_4

2.1 Introduction

The Rankin-Selberg integral representation of Kohnen-Skoruppa [26] produces the Spin L-function for holomorphic Siegel modular cusp forms on GSp_4. This integral makes use of a special Poincare series, which is in the Saito-Kurokawa subspace. The proof of Kohnen-Skoruppa is classical, and involves somewhat tricky global calculations with Fourier coefficients. In particular, they make essential use of the Maass relations, which are identities satisfied by the Fourier coefficients of Siegel modular forms that are Saito-Kurokawa lifts.

In this chapter, we reinterpret the integral in [26] as one unfolding to a non-unique model, and prove that it represents the Spin L-function, following the method of Piatetski-Shapiro and Rallis [30]. Whereas the statement and proof in [26] only applies to holomorphic Siegel modular forms, the integral representations in this chapter can apply to a much wider class of cuspidal representations. Furthermore, via the method of [30], the fact that the global integral represents the Spin L-function can be seen in a purely local and transparent fashion.

The contents of the various sections is as follows: In the next section, we define the groups and special functions involved in the integral representation. In section three, we define the global integrals, give their unfolding, and also discuss the general technique of non-unique models. In sections four and five, we give the unramified and ramified calculations, respectively.

2.2 Groups, definitions, and notations

2.2.1 Groups and embeddings

Denote by

$$J_4 := \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix},$$

so that

$$\mathrm{GSp}_4 = \{g \in \mathrm{GL}_4 \,|gJ_4g^t = \nu(g)J_4\},$$

and $\nu : \mathrm{GSp}_4 \to \mathrm{GL}_1$ is the similitude.

Let $L = \mathbf{Q}(\sqrt{D})$ be a quadratic etale $\mathbf{Q}$-algebra. Denote by $\mathrm{GL}^*_{2,L}$ the subgroup of $GL_{2,L}$ with determinant in $\mathbf{Q}$. We describe an embedding of $\mathrm{GL}^*_{2,L}$ into GSp_4. To do this, consider $(L^2, \langle\ ,\ \rangle_L)$, the two-dimensional free module over L with its standard alternating bilinear form. Define the $\mathbf{Q}$-bilinear form $\langle\ ,\ \rangle$ via $\langle x, y\rangle = \mathrm{tr}_{L/\mathbf{Q}}\left(\langle x, y\rangle_L\right)$. As $\mathrm{GL}^*_{2,L}$ is the group preserving $\langle\ ,\ \rangle_L$ up to $\mathbf{Q}^\times$ multiple, one obtains an inclusion $\mathrm{GL}^*_{2,L} \subseteq \mathrm{GSp}\left(\langle\ ,\ \rangle\right)$. If the basis of L^2 is $\{b_1, b_2\}$, then the basis

$$\frac{1}{2}b_1,\ \frac{1}{2\sqrt{D}}b_1,\ b_2,\ \sqrt{D}b_2$$

gives an isomorphism $\mathrm{GSp}\left(\langle\ ,\ \rangle\right) \cong \mathrm{GSp}_4$. We use this identification throughout. Alternatively, with this choice of basis, the map that is multiplication by $\sqrt{D}$ is given by the matrix

$$\left(\begin{array}{cc|cc} & 1 & & \\ D & & & \\ \hline & & & D \\ & & 1 & \end{array}\right),$$

and $\mathrm{GL}^*_{2,L}$ is the centralizer of this matrix inside GSp_4. One can check immediately that $\mathrm{GL}^*_{2,L}$ is the set matrices inside GSp_4 that have the following form:

$$\left(\begin{array}{cc|cc} a & b & e & f \\ Db & a & f & De \\ \hline Dc & d & x & Dy \\ d & c & y & x \end{array}\right).$$

2.2.2 The special function P_D^α

In this section we define the special Poincare series to be used in the integral representation. We denote this poincare series by P_D^α.

Recall that there is an isomorphism $\mathrm{PGSp}_4 \cong \mathrm{SO}_5$ of split groups over $\mathbf{Q}$. Denote by (V_5, q) the standard 5-dimensional representation of SO_5, together with the quadratic form q defining it, thought of as a representation of GSp_4. One may choose x_0 in $V_5(\mathbf{Q})$ such that $q(x_0) = D$, and such that the stabilizer of x_0 in GSp_4 is $\mathrm{GL}_{2,L}^*$. (See [29].) Let α be a factorizable function in $\mathcal{S}(V_5(\mathbf{A}))$, the space of Schwartz functions on $V_5(\mathbf{A})$, equal to the characteristic function of $V_5(\mathbf{Z}_p)$ almost everywhere. The special automorphic form P_D^α will be

$$P_D^\alpha(g) = \sum_{\delta \in \mathrm{GL}_{2,L}^*(F) \backslash \mathrm{GSp}_4(F)} \alpha(x_0 \delta g).$$

The function P_D is perhaps inherently an orthogonal gadget; one can make an analogous definition for any orthogonal group, and the resulting special function likely has similar properties to this one. However, it is easiest for computation to define P_D directly in the notation of GSp_4, and not go through the exceptional isomorphism $\mathrm{PGSp}_4 \cong \mathrm{SO}_5$. To do this, write $\wedge_0^2(W_4) = \ker\{\wedge^2 W_4 \to \nu\}$, and set $V_5 = \wedge_0^2(W_4) \otimes \nu^{-1}$. (The quadratic form is just the cup product.) Denote $v_D = De_1 \wedge f_2 + e_2 \wedge f_1$ in V_5, and set

$$P_D^\alpha(g) = \sum_{\delta \in \mathrm{GL}_{2,L}^*(F) \backslash \mathrm{GSp}_4(F)} \alpha(v_D \delta g).$$

This definition is sensical because the $\mathrm{GL}_{2,L}^*$ is the stabilizer of v_D in GSp_4.

2.3 Non-unique models and unfolding

We will now give two global integrals, which are closely related. The integral representations use the Klingen Eisenstein series on GSp_4. Let GSp_4 act on $W_4 = \mathbf{Q}^4$ on the right, and denote the ordered basis of W_4 by $\{e_1, e_2, f_1, f_2\}$, so that $\langle e_i, f_j \rangle = \delta_{i,j}$. The Klingen parabolic, denoted Q, is the stabilizer of the line $\mathbf{Q}f_1$ via the right action. Pick a section $f(g; s) \in Ind(\delta_Q^s)$, δ_Q the modulus character of the Klingen parabolic. One requires f to be right K finite, and $f_p(1; s) = 1$, independent of s, for almost all p. We normalize these sections by defining $f_p^*(g; s) = \zeta_p(4s) f_p(g; s)$ for almost

all p. Let us write $F = \mathbf{Q}$ for the global field. The normalized Eisenstein series is then

$$E^*(g;s) = \sum_{\gamma \in Q(F)\backslash \mathrm{GSp}_4(F)} f^*(\gamma g; s).$$

Suppose π is a cuspidal automorphic representation on GSp_4, and ϕ is a cusp form in the space of π. The global integrals are as follows:

$$I_1(\phi;s) = \int_{[\mathrm{GL}^*_{2,L}]} E^*(g;s)\phi(g)\,dg$$

$$I_2(\phi;s) = \int_{[\mathrm{GSp}_4]} P_D^{\alpha}(g) E^*(g;s)\phi(g)\,dg.$$

The integrals are very closely related. In fact, the special function P_D appears to act like a distribution for integrating over $[\mathrm{GL}^*_{2,L}]$, but we will not pursue that phenomenon further.

Recall that the standard Siegel parabolic on GSp_4 is the stabilizer of the isotropic subspace spanned by f_1, f_2. Denote by U its unipotent radical, and by N the unipotent radical of the standard Borel in $\mathrm{GL}^*_{2,L}$. Then, under the embedding $\mathrm{GL}^*_{2,L} \to \mathrm{GSp}_4$, $N \subset U$.

In fact, define a unitary character $\chi : U(\mathbf{A}) \to \mathbf{C}^\times$ as follows. As is usual, pick once and for all an additive character $\psi : F\backslash\mathbf{A} \to \mathbf{C}^\times$, with conductor equal to one almost everywhere. If

$$u = \left(\begin{array}{cc|cc} 1 & & u_{11} & u_{12} \\ & 1 & u_{12} & u_{22} \\ \hline & & 1 & \\ & & & 1 \end{array}\right),$$

then set $\chi(u) = \psi(-Du_{11} + u_{22})$. Note that χ is trivial on N. With this definition of χ, define

$$\phi_\chi(g) = \int_{[U]} \chi^{-1}(u)\phi(ug)\,du$$

and

$$\alpha_\chi(g) = \int_{N(\mathbf{A})\backslash U(\mathbf{A})} \chi(u)\alpha(v_D u g)\,du.$$

Then

Proposition 2.3.1. *The global integrals I_1 and I_2 unfold:*

$$I_1(\phi, s) = \int_{N(\mathbf{A})Z(\mathbf{A})\backslash \mathrm{GL}^*_{2,L}(\mathbf{A})} f^*(g; s)\phi_\chi(g)\, dg$$

and

$$I_2(\phi, s) = \int_{U(\mathbf{A})Z(\mathbf{A})\backslash \mathrm{GSp}_4(\mathbf{A})} f^*(g; s)\phi_\chi(g)\alpha_\chi(g)\, dg.$$

Proof. We will unfold I_2. The unfolding of I_1 is almost identical, but simpler.

Assume for now that L is a field, and not the split quadratic etale $\mathbf{Q}$ algebra. Then $\mathrm{GL}^*_{2,L}$ acts transitively on W_4, and the stabilizer B' of the line $\mathbf{Q}f_1$ is the elements of the standard Borel of $\mathrm{GL}^*_{2,L}$ whose Levi entries are diagonal when embedded in GSp_4. Hence

$$I_2(\phi; s) = \int_{B'(F)Z(\mathbf{A})\backslash \mathrm{GSp}_4(\mathbf{A})} \alpha(v_D g) f^*(g; s)\phi(g).$$

Now

$$\int_{[N]} \phi(ng)\, dn = \sum_{\gamma \in Z(F)N(F)\backslash B'(F)} \phi_\chi(\gamma g),$$

from which the proposition follows in this case.

In case L is split, then $\mathrm{GL}^*_{2,L}$ acts on the lines in W_4 with three orbits, one represented by $\mathbf{Q}f_1$ and the other two by $\mathbf{Q}(f_1 \pm f_2)$. After a change of variables, one can check that the integrals associated to $f_1 \pm f_2$ vanish by cuspidality of ϕ. The unfolding of the integral associated to $\mathbf{Q}f_1$ proceeds as above. □

From now on, we will concentrate our attention on the integral I_2, the case of I_1 being almost identical, but simpler.

The fourier coefficient ϕ_χ does not factorize for general cusp forms ϕ. Thus, it is not at all clear that the integrals I_1 and I_2 have the structure of an Euler product. However, Piatetski-Shapiro and Rallis in [30] introduced for a technique for analyzing exactly this sort of phenomenon. Suppose (π_v, V) is an irreducible admissible representation of $\mathrm{GSp}_4(\mathbf{Q}_v)$. Define a (U, χ) model to be a linear functional $\ell : V \to \mathbf{C}$ that satisfies $\ell(\pi(u)v) = \chi(u)\ell(v)$ for all $u \in U(\mathbf{Q}_v)$ and $v \in V$. If one replaced U in this definition by U_B the unipotent radical of a Borel of GSp_4 and χ by a nondegenerate character of U_B, then one would get the definiton of the Whittaker model. However, unlike the Whittaker model, the space of (U, χ) models will usually be infinite dimensional. Consequently, the fourier coefficient ϕ_χ will not factorize. However, the following theorem allows one to recover the Euler product nature of the global integral nonetheless.

Theorem 2.3.2. *Suppose all the data are unramified: p is finite, π_p is unramified, p does not divide $2D$, f^* is right $\mathrm{GSp}_4(\mathbf{Z}_p)$ invariant, and α_p is the characteristic function of $V_5(\mathbf{Z}_p)$. Let v_0 denote a nonzero spherical vector in π_p. Then, for all (U,χ) models ℓ,*

$$I_2^p(\ell;s) := \int_{U(\mathbf{Q}_p)Z(Q_p)\backslash \mathrm{GSp}_4(\mathbf{Q}_p)} f^*(g;s)\alpha_p^\chi(g)\ell(\pi(g)v_0)\,dg = \ell(v_0)L(\pi_p, Spin, 2s-\frac{1}{2}),$$

when the left hand side is absolutely convergent.

This theorem will be proved in the next section. Let us explain how one uses it to analyze the global integral. For a finite set of places Ω, put

$$I_\Omega(s) = \int_{U(\mathbf{A}_\Omega)Z(\mathbf{A}_\Omega)\backslash \mathrm{GSp}_4(\mathbf{A}_\Omega)} f^*(g;s)\alpha_p^\chi(g)\phi_\chi(g)\,dg,$$

where for a linear algebraic group H we write $H(\mathbf{A}_\Omega) = \prod_{v\in\Omega} H(\mathbf{Q}_v)$. Then

$$I_2(\phi;s) = \varinjlim_{\Omega} I_\Omega(s).$$

Suppose S is a finite set of places containing all the bad places, i.e. all the places exlcuded in the theorem above, and $\Omega \supseteq S$. Then the theorem allows one to prove

Lemma 2.3.3. $I_{\Omega\cup\{p\}}(s) = L(\pi_p, Spin, 2s-\frac{1}{2})I_\Omega(s)$.

Proof. $I_{\Omega\cup\{p\}}(s)$

$$= \int_{U(\mathbf{A}_\Omega)Z(\mathbf{A}_\Omega)\backslash \mathrm{GSp}_4(\mathbf{A}_\Omega)} f^*(g;s)\alpha_p^\chi(g) \int_{U(\mathbf{Q}_p)Z(Q_p)\backslash \mathrm{GSp}_4(\mathbf{Q}_p)} f_p^*(g_p;s)\alpha_p^\chi(g_p)\phi_\chi(gg_p)\,dg_p\,dg.$$

But for each $g \in \mathrm{GSp}_4(\mathbf{A}_\Omega)$, the function $g_p \mapsto \phi_\chi(gg_p)$ is of the form $\ell(\pi_p(g_p)v_0)$ for some (U,χ) model ℓ and spherical vector v_0. Furthermore $\ell(v_0) = \phi_\chi(g)$. Hence the Lemma follows from the theorem. □

One concludes that $I_2(\phi;s) = I_S(s)L^S(\pi, Spin, 2s-\frac{1}{2})$. The integral I_S over the bad finite places may be controlled using the following proposition, to be proved in the fifth section.

Proposition 2.3.4. *For any v_0 in V_{π_p}, there exists a section of f_p^* of $Ind(\delta_Q^s)$, a Schwartz function α_p on $V_5(\mathbf{Q}_p)$, and a vector v in V_{π_p}, so that*

$$\int_{U(\mathbf{Q}_p)Z(Q_p)\backslash \mathrm{GSp}_4(\mathbf{Q}_p)} f^*(g;s)\alpha_p^\chi(g)\ell(\pi(g)v)\,dg = \ell(v_0).$$

From the proposition, we see that the data (f^*, ϕ, α) for the integral may be chosen so that

$$I(\phi; s) = I_\infty(s) L^S(\pi, Spin, 2s - \frac{1}{2}).$$

2.4 Unramified calculation

In this section we will prove Theorem 2.3.2. Again, the idea goes back to Piatetski-Shapiro and Rallis [30]. One starts with the explicit determination of $L(\pi_p, Spin, s)$ in terms of Hecke operators, which is codified in the following statement.

Proposition 2.4.1. *Suppose $\omega_p(g)$ is the Macdonald spherical function for the unramified representation π_p, normalized so that $\omega_p(1) = 1$. Define $\Delta^s(g) = |\nu(g)|^s \operatorname{char}(g)$, where* char *is the characteristic function of $M_4(\mathbf{Z}_p)$. Then*

$$\int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \omega_p(g)\, dg = (1 - \omega_\pi(p) p^{2-2s}) L(\pi_p, Spin, s - \frac{3}{2}),$$

where ω_π is the central character of π_p.

Proof. This is a standard result in the theory of rank two symplectic group. See [3]. □

Suppose now that ℓ is a (U, χ) model. If v_0 is a spherical vector for π_p, it follows from the proposition that

$$\begin{aligned} \int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \ell(\pi(g) v_0)\, dg &= \ell(v_0) \int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \omega_p(g)\, dg \\ &= \ell(v_0)(1 - \omega_\pi(p) p^{2-2s}) L(\pi_p, Spin, s - \frac{3}{2}). \end{aligned}$$

Thus, to prove $I_p(\ell; s) = \ell(v_0) L(\pi_p, Spin, 2s - \frac{1}{2})$, it suffices to prove

$$(1 - \omega_\pi(p) p^{2-2s}) I_p(\ell; \frac{s-1}{2}) = \int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \ell(\pi(g) v_0)\, dg.$$

This is an identity that may be simply checked: One can make explicit calculations of both sides, and compare them. That is what we will do, and we will see that the two sides are identical.

To begin, we first make the calculation of α_χ^v almost everywhere. First, a definition.

Definition 2.4.2. For a two-by-two matrix m, right K equivalent to $\left(\begin{smallmatrix} m_{11} & m_{12} \\ & m_{22} \end{smallmatrix}\right)$, define $\Delta_0'(m)$ to

be 1 if $m_{22}|m_{12}$, $m_{22}|m_{11}$, and $(\frac{m_{12}}{m_{22}})^2 \equiv D$ modulo $\frac{m_{11}}{m_{22}}$, and 0 otherwise. For a matrix

$$m = \begin{pmatrix} m_t & \\ & m_b \end{pmatrix}$$

in the Levi of the Siegel parabolic of GSp_4, so that m_t, m_b are two-by-two matrices, define

$$\Delta_0(m) = \mathrm{char}\left(\left|\frac{\det(m_t)}{\det(m_b)}\right| \leq 1\right)\left|\frac{\det(m_t)}{\det(m_b)}\right|^{1/2} \Delta_0'(m_b).$$

Proposition 2.4.3. *Suppose that p does not divide $2D$, and that α^p is the characteristic function of $V_5(\mathbf{Z}_p)$. Then if m is in the Levi of the Siegel parabolic, $\alpha_\chi^p(m) = \Delta_0(m)$.*

Proof. Suppose

$$n_{24}(z)m = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & z \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} \lambda & & & \\ & \lambda & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} a & b & & \\ c & d & & \\ \hline & & d & -c \\ & & -b & a \end{array}\right).$$

Then $v_D n_{24}(z)m$

$$= \delta^{-1}(cd - Dab)(e_1 \wedge f_1 - e_2 \wedge f_2) + \delta^{-1}(Da^2 - c^2)e_1 \wedge f_2 + \delta^{-1}(d^2 - Db^2)e_2 \wedge f_1 - \frac{z}{\lambda} f_1 \wedge f_2,$$

where $\delta = ad - bc$. To calculate $\alpha_\chi(m)$, by right K invariance we may assume $b = 0$. Then $v_D n_{24}(z)m$

$$= \frac{c}{a}(e_1 \wedge f_1 - e_2 \wedge f_2) + \frac{Da^2 - c^2}{ad} e_1 \wedge f_2 + \frac{d}{a} e_2 \wedge f_1 - \frac{z}{\lambda} f_1 \wedge f_2.$$

Thus for the integral

$$\int_z \psi(z)\alpha(v_D n_{24}(z)m)\, dz$$

to be nonzero, we need $|\lambda| \leq 1$, and the other three coefficients above to be integral. When all these conditions are met, the integral is $|\lambda|$. The conditions for the first three coefficients are exactly the conditions defining $\Delta_0'(m_b)$. Since $|\lambda| = \left|\frac{\det(m_t)}{\det(m_b)}\right|^{1/2}$, the proposition follows. □

The normalized sections $f_p^*(g; s)$ appearing in the definition of the Eisenstein series may be

constructed in the following way. Set Φ the characteristic function of $W_4(\mathbf{Z}_p)$. Then

$$f_p^*(g;s) = |\nu(g)|^{2s} \int_{\mathrm{GL}_1(\mathbf{Q}_p)} \Phi(tf_1 g)|t|^{4s}\, dt.$$

We may now compute

$$\begin{aligned}
I_p(\ell;s) &= \int_{U(\mathbf{Q}_p)Z(Q_p)\backslash \mathrm{GSp}_4(\mathbf{Q}_p)} f^*(g;s)\alpha_p^\chi(g)\ell(\pi(g)v_0)\, dg \\
&= \int_{Z(\mathbf{Q}_p)\backslash M(\mathbf{Q}_p)} \delta_P^{-1}(g) f_p^*(g;s)\alpha_\chi(g)\ell(\pi(g)v_0)\, dg \\
&= \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^{2s}\Phi(f_1 g)\alpha_\chi(g)\ell(\pi(g)v_0)\, dg \\
&= \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^{2s}\Delta_0(g)\ell(\pi(g)v_0)\, dg \\
&= \sum_{k\geq 0} (|p|^{4s}\omega_\pi(p))^k \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^{2s}\Delta_0(g)\,\mathrm{char}(p\nmid g)\ell(\pi(g)v_0)\, dg \\
&= \left(1-\omega_\pi(p)p^{-4s}\right)^{-1} \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^{2s}\Delta_0(g)\,\mathrm{char}(p\nmid g)\ell(\pi(g)v_0)\, dg.
\end{aligned}$$

Hence

$$(1-\omega_\pi(p)p^{2-2s})I_p(\ell;\frac{s-1}{2}) = \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^{s-1}\Delta_0(g)\,\mathrm{char}(p\nmid g)\ell(\pi(g)v_0)\, dg.$$

Now let us compute

$$\int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g)\ell(\pi(g)v_0)\, dg = \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^s \ell(\pi(g)v_0) \int_{U(\mathbf{Q}_p)} \chi(u)\,\mathrm{char}(ug)\, du\, dg.$$

Lemma 2.4.4. *Suppose g is in $M(\mathbf{Q}_p)$, and that g is right K_M equivalent to a matrix with bottom right 2×2 block of the form $\left(\begin{smallmatrix} p^a & b \\ & p^c\end{smallmatrix}\right)$. Then*

$$\int_{U(\mathbf{Q}_p)} \chi(u)\,\mathrm{char}(ug)\, du$$

is 0 unless $c=0$ and $b^2 \equiv D$ modulo p^a, in which case it is p^a.

Note that when p is inert in L, the only such matrix is the identity.

Proof. As a function of $g \in M(\mathbf{Q}_p)$, the integral is certainly right K_M invariant, hence we may assume g is of the special form above. Now the $U_g = \{u \in U(\mathbf{Q}_p) | \,\mathrm{char}(ug) \neq 0\}$ forms an abelian group. Hence the integral vanishes unless χ is identically on U_g. If $c > 0$ then the matrix with upper

right block $(\begin{pmatrix} & p^{-1} \end{pmatrix})$ is in U_g, so the integral vanishes. So now assume $c = 0$. Then the matrix with upper right block $\left(\begin{pmatrix} p^{-a} & -bp^{-a} \\ -bp^{-a} & b^2p^{-a} \end{pmatrix}\right)$ is in U_g. As χ of this matrix is $\psi(\frac{b^2-D}{p^a})$, we get the second condition. Finally, when both conditions are satisfied, suppose u has upper right block $\left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{smallmatrix}\right)$. Then one sees for u to be in U_g we must have $u_{11} \in p^{-a}\mathbf{Z}_p$, and then that u_{12}, u_{22} are determined modulo $\mathbf{Z}_p$. Finally, the second condition forces χ to be trivial on U_g, so the integral is the measure of U_g which is p^a. □

Lemma 2.4.5. *If $g \in M_P$, and $\ell(\pi(g)v_0) \neq 0$, then*

$$\int_{U(\mathbf{Q}_p)} \chi(u)\,\mathrm{char}(ug)\,du = |\nu(g)|^{-1}\Delta_0(g)\,\mathrm{char}(p \nmid g)$$

Proof. This is an immediate consequence of the previous lemma, except for the fact that, in the notation above, $\ell(\pi(g)v_0) \neq 0$ forces $|\det(m_t)\det(m_b)| \leq 1$. This latter fact may be seen by taking $u \in U(\mathbf{Z}_p)$, and considering the equation

$$\ell(\pi(g)v_0) = \ell(\pi(g)uv_0) = \chi(gug^{-1})\ell(\pi(g)v_0).$$

One deduces that $\ell(\pi(g)v_0) \neq 0$ implies $\chi(gug^{-1}) = 1$ for all $u \in U(\mathbf{Z}_p)$, from which one can check $|\det(m_t)\det(m_b)| \leq 1$. □

Hence

$$\begin{aligned}\int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g)\ell(\pi(g)v_0)\,dg &= \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^s\ell(\pi(g)v_0)\int_{U(\mathbf{Q}_p)}\chi(u)\,\mathrm{char}(ug)\,du\,dg \\ &= \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^s|\nu(g)|^{-1}\Delta_0(g)\,\mathrm{char}(p \nmid g)\ell(\pi(g)v_0)\,dg.\end{aligned}$$

This last line is what we obtained for $(1 - \omega_\pi(p)p^{2-2s})I_p(\ell; \frac{s-1}{2})$, and thus

$$\int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g)\ell(\pi(g)v_0)\,dg = (1 - \omega_\pi(p)p^{2-2s})I_p(\ell; \frac{s-1}{2}),$$

completing the unramified calculation.

2.5 Ramified calculation

In this section we prove Proposition 2.3.4. We first do the easier case for the integral I_1:

Proposition 2.5.1. *For any v_0 in V_{π_p}, there exists a section of f_p^* of $Ind(\delta_Q^s)$, and a vector v in V_{π_p}, so that*

$$\int_{N(\mathbf{Q}_p)Z(Q_p)\backslash \mathrm{GL}^*_{2,L}(\mathbf{Q}_p)} f^*(g;s)\ell(\pi(g)v)\,dg = \ell(v_0).$$

Proof. Denote by $K_N = \{1 + p^N M_4(\mathbf{Z}_p)\} \cap \mathrm{GSp}_4(\mathbf{Q}_p)$ the congruence subgroups of K. The vector v_0 is stabilized by some K_N, $N >> 0$. Define $\varphi_1 \in \mathcal{C}_c^\infty(\mathbf{Q}_p)$ by $\varphi_1(z) = \psi(-z)|p|^N \,\mathrm{char}(p^N z \in \mathbf{Z}_p)$. Now set

$$v_1 = \int_{\mathbf{G}_a(\mathbf{Q}_p)} \varphi_1(z)\pi(u(z))v_0\,dz,$$

where

$$u(z) = \left(\begin{array}{cc|cc} 1 & & 0 & 0 \\ & 1 & 0 & z \\ \hline & & 1 & \\ & & & 1 \end{array}\right).$$

Note that if $m \in T_L(\mathbf{Q}_p)$ the standard torus of $\mathrm{GL}^*_{2,L}(\mathbf{Q}_p)$, then

$$\ell(\pi(m)v_1) = \int_{\mathbf{G}_a(\mathbf{Q}_p)} \varphi_1(z)\chi(mu(z)m^{-1})\,dz\ell(tv_0)$$

and $\chi(mu(z)m^{-1}) = \psi(|t/N(\ell)|z)$. Since

$$\int_{\mathbf{G}_a(\mathbf{Q}_p)} \varphi_1(z)\psi(|vz)\,dz = \mathrm{char}(v \in 1 + p^N\mathbf{Z}_p),$$

we conclude that

$$\ell(\pi(m)v_1) = \mathrm{char}(|t/N(\ell)| \in 1 + p^N\mathbf{Z}_p)\ell(\pi(m)v_0).$$

Now v_1 is stable by some congruence subgroup K_{N_1}. We pick a section f^* supported on QK_{N_1}. For example, one may take

$$f^*(g;s) = \int_{Q(Q_p)} \delta_Q^{-s}(q)\,\mathrm{char}(qg \in K_{N_1})\,dq.$$

Recall that $B' = Q \cap \mathrm{GL}^*_{2,L}$, and set $K'_{N_1} = K_{N_1} \cap \mathrm{GL}^*_{2,L}$. We then get, for some positive constants

C_i

$$\begin{aligned}\int_{N(\mathbf{Q}_p)Z(Q_p)\backslash \mathrm{GL}^*_{2,L}(\mathbf{Q}_p)} f^*(g;s)\ell(\pi(g)v_1)\,dg &= \int_{N(\mathbf{Q}_p)Z(Q_p)\backslash B'(\mathbf{Q}_p)K'_{N_1}} f^*(g;s)\ell(\pi(g)v_1)\,dg \\ &= C_1\int_{N(\mathbf{Q}_p)Z(Q_p)\backslash B'(\mathbf{Q}_p)} f^*(g;s)\ell(\pi(g)v_1)\,dg \\ &= C_1\int_{Z(Q_p)\backslash T'(\mathbf{Q}_p)} \delta_B^{-1}(m)f^*(m;s)\ell(\pi(m)v_1)\,dm.\end{aligned}$$

This last line is equal to

$$C_1\int_{Z(Q_p)\backslash T'(\mathbf{Q}_p)} \delta_B^{-1}(m)\delta_Q^*(m;s)\,\mathrm{char}_N(m)\ell(\pi(m)v_0)\,dm,$$

where $\mathrm{char}_N(m) = \mathrm{char}(|t/N(\ell)| \in 1 + p^N\mathbf{Z}_p)$. But when restricted to T', the support of char_N is contained in $Z(\mathbf{Q}_p)K_N$. Furthermore, δ_Q and δ_B are identically one on this subset of T'. Hence the above is equal to $C_2\ell(v_0)$. Setting $v = C_2^{-1}v_1$ gives the proposition. □

Proof of Proposition 2.3.4. Pick α_p so that $\alpha_p(v_Dg) = \mathrm{char}(g \in \mathrm{GL}^*_{2,L}K_M)$ for M sufficiently large. Then we are reduced to the case in Proposition 2.5.1. □

Chapter 3

The two-variable integral on GU(2, 2)

3.1 Introduction

We give a Rankin-Selberg integral in two complex variables representing the product of the exterior square and standard L-functions of generic cusp forms on the quasisplit unitary group in 4 variables. See section 3.2 for more on these L-functions. This integral is inspired by the two-variable integral of Bump-Friedberg-Ginzburg [5] on GSp_4.

Denote by

$$J_4 := \left(\begin{array}{ccc|c} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right).$$

Suppose E is a quadratic field extension of F. We define the unitary group

$$G = \mathrm{GU}(2,2) = \{g \in \mathrm{GL}_4(E) : gJ_4\,{}^t\bar{g} = \nu(g)J_4\}$$

where $x \mapsto \bar{x}$ is the nontrivial automorphism of E over F and $\nu(g) \in F^\times$.

Write $\{e_1, e_2, f_2, f_1\}$ for the ordered basis of the underlying 4-dimensional vector space W_4 on which the unitary group acts. Thus, in this notation, $\langle e_i, f_j\rangle = \delta_{ij}$. We suppose that G acts on the *right* of W_4. Now define the Siegel parabolic P of G to be the stabilizer of the (isotropic)

subspace spanned by f_1, f_2, and the Klingen parabolic the stabilizer of the line spanned by f_1. If $f_P \in Ind(\delta_P^w)$ and $f_Q \in Ind(\delta_Q^s)$ are standard sections, then the Siegel and Klingen Eisenstein series are, respectively,

$$E_P(g;w) = \sum_{\gamma \in P(F)\backslash G(F)} f_P(\gamma g; w); \quad E_Q(g;s) = \sum_{\gamma \in Q(F)\backslash G(F)} f_Q(\gamma g; s).$$

We assume ϕ has trivial central character. The two-variable global integral is

$$I(\phi; s, w) = \int_{[G]} \phi(g) E_P(g;w) E_Q(g;s)\, dg.$$

Then integral unfolds to the Whittaker model of ϕ, and we will prove that, up to simple factors, the integral represents $L(\pi, \wedge^2, 3s-1)L(\pi, Std, 2w-\frac{1}{2})$ if ϕ is a cusp form in the space of π, assumed to be generic.

3.2 The exterior square and standard L-functions

The dual group of G is $(\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_4(\mathbf{C})) \rtimes Gal(E/F)$. The nontrivial element θ of $Gal(E/F)$ acts by $(\lambda, g) \mapsto (\lambda \det(g), J_4\, {}^t g^{-1} J_4^{-1})$ (for this and the contents of the next paragraph see [25] and [31]).

If a prime p of F is inert in E, then the local group $G(F_p)$ is still a unitary group. Let us describe the Satake parameters of an unramified representation π of $G(F_p)$ at such a place p. Denote by T the diagonal maximal torus of G, so that T contains T' the diagonal maximal split torus of G. Suppose α is an unramified character of $T(F_p)$ and π is an unramified subquotient of $Ind(\delta_B^{1/2}\alpha)$ with δ_B the modulus character of uppertriangular Borel of $G(F_p)$. The elements of T are of the form $t = diag(t_1, t_2, a\bar{t}_2^{-1}, a\bar{t}_1^{-1})$. Thus an unramified character α of T is given by characters $\alpha_0, \alpha_1, \alpha_2$, where α_0 is an unramified character of $F_p^\times$, the other $\alpha_i's$ are unramified characters of $E_p^\times$, and $\alpha(t) = \alpha_0(a)\alpha_1(t_1)\alpha_2(t_2)$. With these notations, the Frobenius conjugacy class of π in the dual group ${}^L G$ is then $(\alpha_0(\varpi), diag(\alpha_1(\varpi), \alpha_2(\varpi), 1, 1)) \rtimes \theta$, where ϖ is a uniformizer of F.

We will now define the exterior square representation of ${}^L G$. Set V_4 the standard 4-dimensional representation of $\mathrm{GL}_4(\mathbf{C})$. Let $\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_4(\mathbf{C})$ act on $\wedge^2 V_4$ via $(\lambda, g) \mapsto \lambda \wedge^2 (g)$. Denote this representation by ρ. Then $x \mapsto \rho(x)$ and $x \mapsto \rho(\theta^{-1} x \theta)$ are irreducible representations of $\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_4(\mathbf{C})$ with the same highest weight, and thus are isomorphic. Hence, there is an element $A \in GL(\wedge^2 V_4)$ satisfying $\rho(\theta^{-1} x \theta) = A^{-1}\rho(x)A$. Note that if A satisfies this identity, then

so does $-A$. It is important that we pick A with positive trace. (The other choice $-A$ will not give the L-function that comes out of the integral representation.) Then the map that sends $x = (\lambda, g)$ to $\rho(x)$ and $(1,1) \rtimes \theta$ to A defines a representation ${}^L G = (\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_4(\mathbf{C})) \rtimes Gal(E/F)$. This representation is what we refer to as the exterior square; we denote it by $\wedge^2$.

The following proposition computes the exterior square L-function for an unramified representation of $G(F_p)$ at an inert place p in terms of L-functions of $G' = \mathrm{GSp}_4$.

Proposition 3.2.1. *Suppose that α is an unramified character of T, and π is an unramified irreducible subquotient of $Ind_B^G(\delta_B^{1/2}\alpha)$. Denote by α' the restriction of α to T', and suppose that π' is an unramified irreducible subquotient of $Ind_{B'}^{G'}(\delta_{B'}^{1/2}\alpha')$. Then if π has central character ω_π,*

$$L(\pi, \wedge^2, s) = L(\pi', Spin, s)L(\omega_\pi, 2s).$$

Note that (the reciprocals of) both sides of the equality are of degree 6 in p^{-s}.

Proof. Let us write $e_i, 1 \leq i \leq 4$ as a basis of V_4, and $e_{ij} = e_i \wedge e_j, i < j$ as our basis of $\wedge^2 V_4$. Set $t = diag(t_1, t_2, t_3, t_4)$. Comparing $\wedge^2 t$ and $\wedge^2 J_4 \det(t)\, {}^t t^{-1} J_4^{-1}$ one finds that the permutation matrix A given by $(e_{14}e_{23})$ conjugates one to the other.

Now suppose the Frobenius conjugacy class of π_p in ${}^L G$ is represented by $(\lambda, diag(a, b, 1, 1)) \rtimes \theta$ for some $\lambda, a, b \in \mathbf{C}^\times$. Note then the that Frobenius conjugacy class of π'_p in ${}^L G'$ is represented by $((\lambda, diag(a, b, 1, 1))$. Now $L(\pi_p, \wedge^2, s)$ is the reciprocal of

$$\det(1 - p^{-s}\lambda \wedge^2 (diag(a, b, 1, 1))A| \wedge^2 (V_4)).$$

The linear transformation $\lambda \wedge^2 (diag(a, b, 1, 1))A$ has 4 diagonal entries and one two-by-two block. The four diagonal entries are for $e_{12}, e_{13}, e_{24}, e_{34}$, for which the eigenvalues are, respectively, $\lambda ab, \lambda a, \lambda b, \lambda$. Thus this part of the determinant gives $L(\pi'_p, Spin, s)^{-1}$. The two-by-two block is given by the pair of basis elements interchanged by A; namely e_{14} and e_{23}. This block contributes

$$(1 - (ab\lambda^2)p^{-2s}) = (1 - \omega_\pi(p)p^{-2s})$$

to the determinant. □

The standard representation of ${}^L G$ is defined more simply. If $x = (\lambda, g) \in \mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_4(\mathbf{C})$, set $\rho_4(x) = g \in \mathrm{GL}(V_4)$. Then the standard representation ρ_{std} is the 8-dimensional $Ind_{{}^L G^0}^{{}^L G}(\rho_4)$.

Similar to Proposition 3.2.1, the standard L-function for G at inert places may be computed in terms of the standard L-function of GSp_4.

Proposition 3.2.2. *Let the notation be as in Proposition 3.2.1. Then, if π has trivial central character,*

$$L(\pi_p, Std, w) = \frac{L(\pi'_p, Std, 2w)}{\zeta(2w)}.$$

Proof. Note that the reciprocal of the two sides are each degree 8 in p^{-w}. The underlying vector space for ρ_{std} we write as $V_4 \oplus V_4'$ and θ interchanges V_4 and V_4'. If $x = (\lambda, g)$ is in ${}^L G^0$ then x acts on V_4 via g and on V_4' via $J_4\, {}^t g^{-1} J_4^{-1}$. Write e_i' for the basis of V_4'. Then the matrix for $\rho_{std}((\lambda, diag(a,b,1,1)) \rtimes \theta)$ has four two-by-two blocks consisting of the e_i, e_i'. The block e_1, e_1' contributes $(1 - ap^{-2w})$ to the determinant. Similarly the other three blocks contribute $(1-bp^{-2w}), (1-b^{-1}p^{-2w}), (1-a^{-1}p^{-2w})$. The proposition follows. □

3.3 Global integral and unramified computation

In this section we state the unfolding for $I(\phi; s, w)$ and explain the unramified computation. The unfolding is identical to that of [5] Theorem 1.1. Using Propositions 3.2.1, 3.2.2 the unramified computation follows from that of Bump-Friedberg-Ginzburg ([5] Theorem 1.2) in the inert case. The unramified computation for p split in E is a fairly simple application of the Littlewood-Richardson rule for decomposing tensor products of representations of $\mathrm{GL}_4(\mathbf{C})$.

3.3.1 Unfolding

Following [5], set

$$\nu_P := \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right)$$

and

$$\nu_Q := \left(\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & -1 & & \\ & & & 1 \end{array}\right).$$

Define $R = (\nu_P^{-1} Q \nu_P) \cap (\nu_Q^{-1} P \nu_Q)$. Then R is the subgroup of GSp_4 consisting of matrices of the form

$$\left(\begin{array}{cc|cc} * & & * & * \\ & * & & * \\ \hline & & * & \\ & & & * \end{array}\right).$$

Lemma 3.3.1. *The double coset $Q(F)\backslash G(F)/P(F)$ is represented by $\{1, \nu_P \nu_Q^{-1}\}$.*

Proof. The coset space $Q(F)\backslash G(F)$ is naturally in bijection with the isotropic E-lines in $W_4(E)$, with the coset $Q(F)1$ representing the line Ef_1. It is easy to check every isotropic E-line is $P(F)$-equivalent to either Ef_1 or $Ee_2 = Ef_1 \nu_P \nu_Q^{-1}$. □

Proposition 3.3.2. *Denote by Z_E the center of G, which is $E^\times$ embedded diagonally, R^0 the unipotent radical of R and W_ϕ the Whittaker model of ϕ. Then*

$$I(\phi; s, w) = \int_{R^0(\mathbf{A}) Z_E(\mathbf{A}) \backslash G(\mathbf{A})} f_P(\nu_Q g; w) f_Q(\nu_P g; s) W_\phi(g)\, dg.$$

Proof. The proof is the same as that in [5], Theorem 1.1. □

Define

$$I_p(W; s, w) = \int_{R(F_p) Z_E(F_p) \backslash G(F_p)} f_P^p(\nu_Q g; w) f_Q^p(\nu_P g; s) W_p(g)\, dg.$$

Theorem 3.3.3. *Denote by $\epsilon_{E/F}$ the quadratic character associated to E/F: i.e. $\epsilon_{E/F}(p)$ is 1 if p is split in E and -1 when p is inert in E. When all the data is unramified*

$$I_p(W; s, w) = \frac{L(\pi_p, Std, 2w - \frac{1}{2}) L(\pi_p, \wedge^2, 3s - 1)}{L(\epsilon_{E/F}, 4w - 1) \zeta_F(4w) \zeta_E(3s) \zeta_F(6s - 2)}.$$

Proof. Everything here is local so we write F instead of F_p and E instead of E_p. Set $t(a, b) = diag(ab, a, 1, \bar{b}^{-1})$ where $a \in F^\times, b \in E^\times$. If p is split in E then $b = (b_1, b_2)$ with $b_i \in F$ and $\bar{b} = (b_2, b_1)$. For $x \in F$ define $n(x) = 1 + xE_{2,3}$. For $y \in E$ set $n(y) = 1 + yE_{12} - \bar{y}E_{34}$. We have

$$I_p(W; s, w) = \int_{Z_E \backslash T} \delta_B^{-1}(t) W(t) \int_{R^0 \backslash U_B} \psi(u) f_P(\nu_Q u t; w) f_Q(\nu_P u t; s)\, du\, dt.$$

Parametrize $Z\backslash T$ via $F^\times \times E^\times$ using the $t(a, b)$'s. Then $\delta_B(t(a, b)) = |a|_F^4 |b|_E^3$. Now change variable

$u \mapsto tut^{-1}$ to get that $I_p(W;s,w)$

$$= \int_{F^\times \times E^\times} \left(|a|_F^4 |b|_E^3\right)^{-1} \left(|a|_F |b|_E\right) W(t(a,b)) \int_{R^0 \backslash U_B} \psi(tut^{-1}) f_P(\nu_Q tu; w) f_Q(\nu_P tu; s)\, du\, dt.$$

Now $f_P(\nu_Q tu; w) = \delta_P^w(\nu_Q t(a,b)\nu_Q^{-1}) f_P(\nu_Q u; w) = |b|_E^{2w} f_P(\nu_Q u; w)$ and similarly $f_Q(\nu_P tu; s) = |a|_F^{3s} f_Q(\nu_P u; s)$. Also note that if $u \equiv n(x)n(y)$ modulo R^0, then $f_P(\nu_Q u; w) = f_P(\nu_Q n(x); w)$ and similarly $f_Q(\nu_P u; s) = f_Q(\nu_P n(y); s)$. Hence $I(\phi; s, w) =$

$$\int_{E^\times, F^\times} |a|_F^{3s-3} |b|_L^{2w-2} W(t(a,b)) \left(\int_F \psi(ax) f_P(\nu_Q n(x); w)\, dx\right) \left(\int_E \psi(by) f_Q(\nu_P n(y); s)\, dy\right) da\, db.$$

Now let us calculate the inner integrals. By the vanishing properties of W we may assume $|a|, |b| \leq 1$. Set $n = ord_q(a)$. If $|x| \leq 1$ then $f_P(\nu_P n(x); w) = 1$. If $|x| > 1$ then $f_P(\nu_P n(x); w) = |x|_F^{-4w}$. Hence

$$\int_F \psi(ax) f_P(\nu_Q n(x); w)\, dx = 1 + \int_{|x|>1} \psi(ax) |x|^{-4w}\, dx,$$

which is calculated to be

$$\frac{1}{\zeta_F(4w)} \left(\frac{1 - |p|_F^{(n+1)(4w-1)}}{1 - |p|_F^{4w-1}}\right).$$

Suppose p is inert in E; set $m = ord_q(b)$. One computes similarly that

$$\int_E f_Q(\nu_P n(y); s)\, dy = 1 + \int_E \psi(by) |y|_E^{-3s}\, dy = \frac{1}{\zeta_E(3s)} \frac{1 - |p|_E^{(3s-1)(m+1)}}{1 - |p|_E^{3s-1}}.$$

Now suppose p is split in E and set $m_i = ord_q(b_i)$. One can check easily that $f_Q(\nu_P n(y); s) = f_Q(\nu_P n((y_1, 1)); s) f_Q(\nu_P n((1, y_2)); s)$. Hence

$$\int_E f_Q(\nu_P n(y); s)\, dy = \left(\int_F \psi(b_1 y_1) f_Q(\nu_P n((y_1, 1)); s)\, dy_1\right) \left(\int_F \psi(b_2 y_2) f_Q(\nu_P n((1, y_2)); s)\, dy_2\right).$$

Then one computes exactly as above that this is equal to

$$\frac{1}{\zeta_E(3s)} \frac{(1 - |p|_F^{(3s-1)(m_1+1)})(1 - |p|_F^{(3s-1)(m_2+1)})}{(1 - |p|_F^{3s-1})(1 - |p|_F^{3s-1})}.$$

Set $U = q^{-(2w-\frac{1}{2})}$, $V = q^{-(3s-1)}$, and $K = \delta_B^{-1/2} W$. For p inert define $K([m,n]) = K(t(p^n, p^m))$. For p split define $K([m_1, n, m_2]) = K(t(p^n, (p^{m_1}, p^{m_2})))$. Define $I_1(W; s, w)$ by the equality $I(W; s, w) =$

$\frac{1}{\zeta_F(4w)\zeta_E(3s)} I_1(W; s, w)$. If p is inert then we get that

$$I_1(W; s, w) = \sum_{n,m \geq 0} V^n U^{2m} \left(\frac{1 - U^{2n+2}}{1 - U^2}\right) \left(\frac{1 - V^{2m+2}}{1 - V^2}\right) K([m, n]).$$

If p is split then $I_1(W; s, w)$

$$= \sum_{n,m_1,m_2} V^n U^{m_1+m_2} \left(\frac{1 - U^{2n+2}}{1 - U^2}\right) \left(\frac{1 - V^{m_1+1}}{1 - V}\right) \left(\frac{1 - V^{m_2+1}}{1 - V}\right) K([m_1, n, m_2]).$$

If p is inert, using that $K^G_{\pi_p} = K^{\mathrm{GSp}_4}_{\pi'_p}$ on the torus of $\mathrm{GSp}_4 \subseteq G$, I_1 is computed in Bump-Friedberg-Ginzburg [5] Theorem 1.2 to be

$$\frac{L(\pi'_p, Std, U^2) L(\pi'_p, Spin, V)}{\zeta(U^4)}.$$

Via Propositions 3.2.1 and 3.2.2 this is equal to

$$\frac{\zeta(U^2) L(\pi_p, Std, U) L(\pi_p, \wedge^2, V)}{\zeta(U^4)\zeta(V^2)} = \frac{L(\pi_p, Std, U) L(\pi_p, \wedge^2, V)}{L(\epsilon_{E/F}, U^2)\zeta(V^2)}.$$

Theorem 3.3.4 (proved below) plus the Casselman-Shalika formula calculates I_1 in case p is split to be

$$\frac{L(\pi_p, Std, U) L(\pi_p, \wedge^2, V)}{\zeta(U^2)\zeta(V^2)} = \frac{L(\pi_p, Std, U) L(\pi_p, \wedge^2, V)}{L(\epsilon_{E/F}, U^2)\zeta(V^2)}.$$

The theorem follows. □

Let $A[r_1, r_2, r_3]$ denote the character of the irreducible representation of $\mathrm{SL}_4(\mathbf{C})$ with highest weight $r_1\omega_1 + r_2\omega_2 + r_3\omega_3$, where ω_i are the fundamental weights.

Theorem 3.3.4. *The sum*

$$\sum_{n,m_1,m_2} V^n U^{m_1+m_2} \left(\frac{1 - U^{2n+2}}{1 - U^2}\right) \left(\frac{1 - V^{m_1+1}}{1 - V}\right) \left(\frac{1 - V^{m_2+1}}{1 - V}\right) A[m_1, n, m_2]$$

is equal to

$$\frac{L(\pi_p, Std, U) L(\pi_p, \wedge^2, V)}{\zeta(U^2)\zeta(V^2)}.$$

Proof. Note that "Std" here means the 8-dimensional representation that is the sum of the two irreducible 4-dimensional representations. We sketch the proof. Here are the system of claims that prove the theorem.

Claim 3.3.5. $A[u,0,v] \times A[0,m,0]$

$$= \sum A[a_1 - b_1 + u - a_2, b_1 + a_2 - b_2, b_2 + v - b_3]$$

where the sum is over integers $a_i, b_i \geq 0$ *satisfying*

- $0 \leq b_2 \leq a_2 \leq u$
- $0 \leq b_1 \leq a_1$
- $a_1 + a_2 = m \leq v + b_1 + b_2$

Proof. Apply Littlewood-Richardson. See, for example, [11], formula $(A.8)$. □

Claim 3.3.6. *The above sum is equal to*

$$\sum A[u - m + 2\ell + k, k + j, v + m - 2j - 2\ell - k]$$

where the sum is over integers $j, k, \ell \geq 0$ *satisfying*

- $\ell + j \leq v$
- $j + k + \ell \leq m \leq \ell + k + u$

Proof. Use $j = a_2 - b_2, \ell = a_1 - b_1, k = b_1$. □

Claim 3.3.7. *In order to for* $A[n_1, n_2, n_3]U^s V^m$ *to appear in*

$$\left(\sum_{u,v \geq 0} A[u,0,v] U^{u+v} \right) \left(\sum_{m \geq 0} A[0,m,0] V^m \right)$$

need

- $n_1 + n_2 + n_3 \geq m \geq n_2$
- $n_1 + 2n_2 + n_3 \geq s \geq n_1 + n_3$

and $s - (n_1 + n_3)$ *to be even.*

Consider $m, n_i, s = u+v$ fixed. Call a nonnegative integer ℓ *allowable* if there exists nonnegative integers j, k, u, v satisfying the conditions of Claim 3.3.6 so that $s = u+v$, $n_1 = u - m + 2\ell + k$ etc. Then

Claim 3.3.8. *The allowable ℓ satisfy*

$$0, m - n_2 - n_3 \leq \ell \leq m - n_2, n_1.$$

Claim 3.3.9. *Assume m, s, n_i satisfy the conditions of Claim 4.2. Then, the coefficient of $A[n_1, n_2, n_3]U^s V^m$ is the number of ℓ satisfying the condition in Claim 3.3.8.*

Claim 3.3.10. *Consider the expression*

$$\sum_{n_2, n_1, n_3} V^{n_2} U^{n_1+n_3} \left(\frac{1-U^{2n_2+2}}{1-U^2}\right)\left(\frac{1-V^{n_1+1}}{1-V}\right)\left(\frac{1-V^{n_3+1}}{1-V}\right) A[n_1, n_2, n_3].$$

Then, in this expression $A[n_1, n_2, n_3]$ appears only if the conditions of Claim 4.2 are satisfied, and then its coefficient is the number of ℓ satisfying the conditions of Claim 3.3.8.

We conclude that

$$\sum_{n_2, n_1, n_3} V^{n_2} U^{n_1+n_3} \left(\frac{1-U^{2n_2+2}}{1-U^2}\right)\left(\frac{1-V^{n_1+1}}{1-V}\right)\left(\frac{1-V^{n_3+1}}{1-V}\right) A[n_1, n_2, n_3]$$

is equal to

$$\left(\sum_m A[0, m, 0]V^m\right)\left(\sum_{u,v} A[u, 0, v]U^{u+v}\right).$$

As

$$\sum_m A[0, m, 0]V^m = \frac{L(\wedge^2, V)}{\zeta(V^2)}$$

and

$$\sum_{u,v} A[u, 0, v]U^{u+v} = (1-U^2)\left(\sum_u A[u, 0, 0]U^u\right)\left(\sum_v A[0, 0, v]U^v\right) = \frac{L(Std, U)}{\zeta(U^2)}$$

we get the theorem. □

Chapter 4

The exterior square L-function on $\mathrm{GU}(2,2)$

4.1 Introduction

We give a set of Rankin-Selberg integrals representing the exterior square L-function on $\mathrm{GU}(2,2)$, the quasisplit unitary group on 4 variables. Gritsenko [22] has considered a version of this integral for Hermitian modular forms of genus two. As with the classical integrals of Andrianov [2] and Kohnen-Skoruppa [26], Gritsenko's proof only works for Hermitian modular forms. The integrals here apply to a much wider class of cuspidal representations, and are associated to a non-unique model. Let us also note that in [12] Furusawa and Morimoto have given an integral representation for generic cusp forms on $\mathrm{GU}(2,2)$ for the exterior square L-function. Also we must remark that $\mathrm{GU}(2,2)$ is closely related to the quasisplit orthogonal group $O(4,2)$, and Sugano [32] and Ginzburg–Piatetski-Shapiro–Rallis [16] have considered the standard L-functions for these orthogonal groups, via different integral representations.

Denote by

$$J_4 := \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right).$$

Suppose E is a quadratic field extension of $F = \mathbf{Q}$. We define the unitary group

$$G = \mathrm{GU}(2,2) = \{g \in \mathrm{GL}_4(E) : gJ_4\,{}^t\bar{g} = \nu(g)J_4\}$$

where $x \mapsto \bar{x}$ is the nontrivial automorphism of E over F and $\nu(g) \in F^\times$. We write W_4 for the underlying vector space on which G acts, and give W_4 the ordered basis $\{e_1, e_2, f_1, f_2\}$.

The integrals we give may be considered analogues of the integral of Kohnen-Skoruppa [26] on GSp_4. We will now define the integrals. First, note that there is a map $G = \mathrm{GU}(2,2) \to \mathrm{U}(1)$ sending $g \mapsto \det(g)\nu(g)^{-2}$, where by $U(1)$ we mean $x \in \mathrm{GL}_1(E) : x\bar{x} = 1$. This map is surjective, and we denote by H its kernel. Now, the group G has a representation on a 6-dimensional E vector space, namely $\wedge^2 W_4(E)$, but it does not have a (nontrivial) representation on a 6-dimensional F vector space. However, the group H does have a such a representation. Indeed, define V_6 to be the F vector subspace of $\wedge^2 W_4(E) \otimes \nu^{-1}$ consisting of elements of the form

$$ae_1 \wedge e_2 + xf_1 \wedge e_2 + wf_2 \wedge e_2 + \bar{w}e_1 \wedge f_1 + ye_1 \wedge f_2 + bf_1 \wedge f_2$$

where $a, b, x, y \in F$ and $w \in E$. One can check by explicit computation with generators that H preserves this space V_6. The effect of the twist by ν^{-1} is that $F^\times$ embedded diagonally in (the center of) H acts trivally on V_6.

Now, for $R \in F^\times$, we set $v_R = e_1 \wedge f_2 + Re_2 \wedge f_1 \in V_6$. Define $G_R \subset H$ to be the stabilizer of v_R. If $R = 1$, then G_R is conjugate inside G to GSp_4. In general, G_R is a quaternion unitary group on two variables; the quaternion algebra for which it is defined is the set of

$$\begin{pmatrix} x & R\bar{y} \\ y & \bar{x} \end{pmatrix} \in M_2(E).$$

We will really only be concerned with the class of R in $F^\times / N_{E/F}(E^\times)$, so for R a norm from E we take $R = 1$ for simplicity.

Define E_Q^* to be the normalized Klingen Eisenstein series on G, where the Klingen parabolic is the stabilizer of the E-line Ef_1. For a cuspidal automorphic representation π of G, and a cusp form ϕ in the space of π, the first set of integrals is

$$I_R(\phi; s) = \int_{[G_R]} \phi(g) E_Q^*(g)\, dg.$$

We will show that this integral represents the exterior square L-function of π, in the sense that one may write it as a product of the partial L-function times an archimedean integral.

The next set of integrals is nearly identical to the I_R, but uses a type of Poincare series instead of the subgroup G_R. For a Schwartz function α on $V_6(\mathbf{A})$, we set

$$P_R^\alpha(g) = \sum_{\gamma \in G_R(F)\backslash H(F)} \alpha(v_R \gamma g),$$

an automorphic form on H. Then we define

$$I'_R(\phi; s) = \int_{[H]} \phi(g) P_R(g) E_Q^*(g)\, dg.$$

Again, we will see that this integral too represents the exterior square L-function of π, in same sense as above. As in all of these results, the unramified computation for I'_R reduces to that for I_R.

The contents of the sections is as follows. In section 4.2 we give some global preliminaries and the unfoldings of the integrals. In section 4.3 we give the unramified computations, assuming the determination of exterior square L-function in terms of Hecke operators. This Hecke operator calculation is done in section 4.4. In section 4.5 we control the local integral for I_R at the bad finite places.

4.2 Global arguments

It will be convenient for us later to understand the group G_R explicitly in terms of matrices. To this end, let us note now that G_R is generated by elements of the form

$$\left(\begin{array}{cc|cc} x & R\bar{y} & & \\ y & \bar{x} & & \\ \hline & & x & -\bar{y} \\ & & -Ry & \bar{x} \end{array}\right); \qquad \left(\begin{array}{cc|cc} \lambda & & & \\ & \lambda & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right);$$

$$\left(\begin{array}{cc|cc} 1 & & Ry & w \\ & 1 & \bar{z} & y \\ \hline & & 1 & \\ & & & 1 \end{array}\right); \qquad \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline x & w & 1 & \\ \bar{w} & Rx & & 1 \end{array}\right).$$

The first two types of elements generate the Levi of the Siegel parabolic P_R of G_R, and the third type makes up the unipotent radical of P_R. As remarked in the introduction, there is $\gamma_0 \in G(F)$ so that $G_1 = \gamma_0^{-1} \mathrm{GSp}_4 \gamma_0$. Suppose $E = F(\sqrt{D})$. Then take

$$\gamma_0 = \left(\begin{array}{cc|cc} \sqrt{D} & -\sqrt{D} & & \\ D & D & & \\ \hline & & \sqrt{D} & -\sqrt{D} \\ & & -1 & -1 \end{array}\right).$$

Indeed, inside $\wedge^2 W_4(E) \otimes \nu^{-1}$,

$$(e_1 \wedge f_1 + e_2 \wedge f_2)\gamma_0 = e_1 \wedge f_2 + e_2 \wedge f_1$$

from which the equality $G_1 = \gamma_0^{-1} \mathrm{GSp}_4 \gamma_0$ follows.

Lemma 4.2.1. *If R is not a norm from E, the double coset space $Q(F)\backslash G(F)/G_R(F)$ is a singleton. If $R = 1$, $Q(F)\backslash G(F)/G_R(F)$ is represented by $\{1, \gamma_0\}$.*

Proof. The coset space $Q(F)\backslash G(F)$ is in bijection with the isotropic E lines in $W_4(E)$. When R is not a norm from E it is easy to check via explicit matrix computations that all the isotropic lines are in one orbit under G_R. When $R = 1$, one checks similarly that there are two orbits of isotropic lines, given by $Ef_1G_1(F)$ and $E(f_1 - f_2)G_1(F)$. As $Ef_1\gamma_0 = E(f_1 - f_2)$ we are done. □

We now define the normalized Klingen Eisenstein series and the Fourier coefficient that will appear in the unfolded integrals. For the Eisenstein series, pick a Schwartz function Φ on $W_4(\mathbf{A}_E)$. Set

$$f^\Phi(g; s) = |\nu(g)|_F^{3s} \int_{\mathrm{GL}_1(\mathbf{A}_E)} \Phi(tf_1 g)|t|_E^{3s}\, dt$$

Then $f^\Phi \in Ind(\delta_Q^s)$, and $f^\Phi(1; s) = \zeta_E(3s)$. We set $E_Q^\Phi(g; s) = \sum_{\gamma \in Q(F)\backslash G(F)} f^\Phi(\gamma g; s)$. Define the subgroup D_E of G_R to be the set of $diag(x, \bar{x}, x, \bar{x})$ with $x \in E^\times$, so that $D_E \cong \mathrm{GL}_1(E)$. Denote by U_P the unipotent radical of the Siegel parabolic of G. We define $\chi_R : U_P(\mathbf{A}) \to \mathbf{C}^\times$ via $\chi_R(u) = \psi(u_{13} - Ru_{24})$ where u_{ij} denotes the ij entry of the matrix u. Note that D_E centralizes χ_R for all R. We set

$$\phi_{\chi_R, N}(g) = \int_{[U_P]} \chi_R^{-1}(u)\phi(ug)\, du$$

and

$$\phi_{\chi_R}(g) = \int_{D_E(F)Z_F(\mathbf{A})\backslash D_E(\mathbf{A})} \phi_{\chi_R,N}(xg)\,dx.$$

Proposition 4.2.2. *Denote by N_R the unipotent radical of the Seigel parabolic of G_R. Then*

$$I_R(\phi;s) = \int_{N_R(\mathbf{A})\backslash G_R(\mathbf{A})} |\nu(g)|_F^{3s}\Phi(f_1 g)\phi_{\chi_R}(g)\,dg.$$

Proof. Observe that

$$\int_{[N_R]} \phi(ng)\,dn = \sum_{\lambda\in F^\times} \phi_{\chi,N}(d_\lambda g) \tag{4.1}$$

where $d_\lambda = diag(\lambda,\lambda,1,1)$.

We first suppose R is not a norm from E. Unfolding the Eisenstein series using Lemma 4.2.1, integrating over N_R, and then using (4.1) we obtain

$$\begin{aligned} I_R(\phi;s) &= \int_{D_E(F)N_R(\mathbf{A})Z_F(\mathbf{A})\backslash G_R(\mathbf{A})} f_Q^\Phi(g;s)\phi_{\chi,N}(g)\,dg \\ &= \int_{D_E(\mathbf{A})N_R(\mathbf{A})\backslash G_R(\mathbf{A})} f_Q^\Phi(g;s)\phi_\chi(g)\,dg. \end{aligned}$$

Performing an inner integral over D_E in the expression

$$\int_{N_R(\mathbf{A})\backslash G_R(\mathbf{A})} |\nu(g)|_F^{3s}\Phi(f_1 g)\phi_{\chi_R}(g)\,dg$$

we get the above, proving the Proposition in this case. Now suppose $R = 1$. Unfolding the Eisenstein series, we get $I = I_1 + I_{\gamma_0}$, corresponding to the two element set $\{1,\gamma_0\} = Q(F)\backslash G(F)/G_1(F)$. We will see that the integral I_{γ_0} vanishes by the cuspidality of ϕ. Thus $I = I_1$ and one proceeds exactly as above to get the Proposition. To see that I_{γ_0} vanishes, set S_{γ_0} the stabilizer of $Q(F)\gamma_0$ inside $G_1(F)$. Then

$$I_{\gamma_0}(\phi;s) = \int_{S_{\gamma_0}(F)Z(\mathbf{A})\backslash G_1(\mathbf{A})} f_Q^\Phi(\gamma_0 g;s)\phi(g)\,dg.$$

Then, since $G_1 = \gamma_0^{-1}GSp_4\gamma_0$, $S_{\gamma_0} = \gamma_0^{-1}Q'(F)\gamma_0$, where Q' is the Klingen parabolic of GSp_4. Hence, changing variables,

$$I_{\gamma_0}(\phi;s) = \int_{Q'(F)Z(\mathbf{A})\backslash \mathrm{GSp}_4(\mathbf{A})} f_Q(h\gamma_0;s)\phi(h\gamma_0)\,dh.$$

It is now easy to check I_{γ_0} vanishes by cuspidality of ϕ. □

Define

$$\alpha_\chi(v_R g) = \int_{N_R(\mathbf{A})\backslash U_P(\mathbf{A})} \chi(u)\alpha(v_R u g)\, du.$$

Proposition 4.2.3.

$$I'_R(\phi; s) = \int_{U_P(\mathbf{A})\backslash H(\mathbf{A})} \alpha_\chi(v_R g)|\nu(g)|_F^{3s}\Phi(f_1 g)\phi_{\chi_R}(g)\, dg.$$

Proof. This is nearly identical to the unfolding for I_R. The only thing to pay attention to is that when $R = 1$, γ_0 is not in H, but it does normalize it. □

4.3 Unramified computation

In this section we give the unramified computation, assuming Theorem 4.4.1 of the next section which determines the L-functions in terms of Hecke operators. In this and the next section everything is local, so we abuse notation and denote by F the local field F_p and by E a quadratic etale extension of F, which is thus either $F \times F$ or the unique unramified quadratic field extension of F. Assume furthermore that $p \nmid 2R$.

If E is a field, $G(F)$ is still a unitary group.

Definition 4.3.1. Suppose E is a field. Define $\Delta^s(g) = |\nu(g)|_F^s \operatorname{char}(g)$ where the characteristic function is that of $M_4(\mathcal{O}_E)$.

Suppose now $E = F \times F$. For $g \in G(F)$, the two maps $E \to F$ induce maps $G(F) \to \mathrm{GL}_4(F)$. Write g_1, g_2 for the image of g under these two maps. Then $g \mapsto (\nu(g), g_1)$ defines an isomorphism $G \cong \mathrm{GL}_1 \times \mathrm{GL}_4$. In this notation, one recovers g_2 as $\nu J_4\, {}^t g_1^{-1} J_4^{-1}$. So if $g = (g_1, g_2)$, $\bar{g} = (g_2, g_1)$.

Definition 4.3.2. Suppose $E = F \times F$. Define $\Delta^s(g) = |\nu(g)|_F^s \operatorname{char}(g) \operatorname{char}(|\det(g_1)|_F = |\nu(g)|_F^2 = |\det(g_2)|_F)$. Here the first characteristic function is again of $M_4(\mathcal{O}_E)$, which is really the simultaneous integrality of g_1 and g_2. That is, $\operatorname{char}(g) = \operatorname{char}(g_1 \in M_4(\mathcal{O}_F)) \operatorname{char}(g_2 \in M_4(\mathcal{O}_F))$.

Note that the analogue of the condition $\operatorname{char}(|\det(g_1)|_F = |\nu(g)|_F^2 = |\det(g_2)|_F)$ is automatic in the case E is a field. For ease of notation we define $\Delta^0(g) = \operatorname{char}(g) \operatorname{char}(|\det(g_1)|_F = |\nu(g)|_F^2 = |\det(g_2)|_F)$ the product of these characteristic functions.

The functions $\Delta^s(g)$ are bi-K-invariant. Theorem 4.4.1 of section 4.4 relates these Δ^s's to the exterior square L-function. Using that theorem, we now give the unramified computation. Let $\ell : \pi \to \mathbf{C}$ be a (U_P, χ) functional. (We do not need to hypothesize the extra invariance under D_E).

Let v_0 be the spherical vector of π_p, and assume Φ is the characteristic function of $W_4(\mathcal{O}_E)$. Set

$$I(\ell; s) = \int_{N_R(F)\backslash G_R(F)} |\nu(g)|_F^{3s} \Phi(f_1 g) \ell(\pi(g)v_0)\, dg.$$

Theorem 4.3.3. *Assume trivial central character.*

$$I(\ell; s) = \frac{L(\pi, \wedge^2, 3s-1)}{\zeta(6s-2)} \ell(v_0).$$

Assume moreover α is the characteristic function of $V_6(\mathcal{O}_F)$. Set

$$I'(\ell; s) = \int_{U_P(F)\backslash H(F)} \alpha_\chi(v_R g) |\nu(g)|_F^{3s} \Phi(f_1 g) \ell(\pi(g)v_0)\, dg.$$

Theorem 4.3.4. *Assume trivial central character.*

$$I'(\ell; s) = \frac{L(\pi, \wedge^2, 3s-1)}{\zeta(6s-2)} \ell(v_0).$$

The second theorem will follow from the first together with an explicit calculation of $\alpha_\chi(v_R g)$. Define

$$I^\Delta(\ell; s) = \int_{G(F)} \Delta^s(g) \ell(\pi(g)v_0)\, dg.$$

We will show $I(\ell; s) = \zeta(6s) I^\Delta(\ell; 3s+1)$. The theorem then follows from Theorem 4.4.1. There are two cases: E a field and $E = F \times F$. The case E is a field is easier, so we do that first.

Proof of Theorem 4.3.3 when E is a field. For ease of notation we set $\lambda(g) = \ell(\pi(g)v_0)$. We have

$$I^\Delta(\lambda; s) = \int_M \delta_P^{-1}(m) |\nu(m)|_F^s \lambda(m) \left(\int_U \chi(u) \operatorname{char}(um)\, du \right) dm.$$

We compute this inner integral as a function of m. Define $U_m = \{u \in U_P(F) : um \in M_4(\mathcal{O}_E)\}$. Then U_m is an abelian group, so the inner integral vanishes unless

$$\chi \text{ is trivial when restricted to } U_m; \tag{4.2}$$

if condition (4.2) is satisfied then the inner integral is equal to the measure of U_m.

Now as a function of m, the inner integral is right K invariant, so we may assume the assume the bottom two-by-two block m_2 of m is of the form $\left(\begin{smallmatrix} p^a & b \\ & p^c \end{smallmatrix}\right)$ with $a, c \geq 0$ integers and $b \in \mathcal{O}_E$. Let us assume (4.2) is satisfied, and see what conditions this imposes on a, b, c. First, we find $c = 0$.

Indeed, if not, then the matrix with upper right 2×2 block $\left(\begin{smallmatrix} 0 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)$ is in U_m.

Now consider the general element u with top right two-by-two block equal to $h = \left(\begin{smallmatrix} x & w \\ \bar{w} & y \end{smallmatrix}\right)$. Suppose

$$m = \begin{pmatrix} m_t & \\ & m_b \end{pmatrix}.$$

Then

$$hm_b = \begin{pmatrix} p^a x & bx + w \\ p^a \bar{w} & b\bar{w} + y \end{pmatrix}$$

Hence $u \in U_m$ if and only if $x \in p^{-a}\mathbf{Z}_p$, $w \equiv -bx \bmod \mathbf{Z}_p$, and $y \equiv -b\bar{w} \equiv N(b)x \bmod \mathbf{Z}_p$. It follows that the measure of U_m is $p^a = |\det(m_b)|_E^{-1/2}$. We also find that $\psi((p^{-a}(1 - RN(b)))x')$ is trivial for all $x' \in \mathbf{Z}_p$. Hence $p^{-a}(1 - RN(b)) \in \mathbf{Z}_p$; and if this condition is satisfied then indeed (4.2) is satisfied too. Proposition 4.3.6 below proves that this is so if and only if $m \in M_R K_M$, $K_M := K \cap M$.

Hence

$$\int_U \chi(u) \operatorname{char}(um)\, du = |\det(m_b)|_E^{-1/2} \operatorname{char}(p \nmid m_b) \operatorname{char}(m) \operatorname{char}(m \in M_R K_M).$$

Now $\delta_P^{-1}(m)|\det(m_2)|_E^{-1/2} = \delta_{P_R}^{-1}(m)|\nu(m)|_F^{-1}$. Hence

$$I^\Delta(\lambda; s) = \int_{M_R} \delta_{P_R}^{-1}(m)|\nu(m)|^{s-1} \operatorname{char}(p \nmid m_b) \operatorname{char}(m)\lambda(m)\, dm.$$

On the other hand,

$$\begin{aligned} I(\lambda; s) &= \int_{N_R \backslash G_R} |\nu(g)|_F^{3s} \Phi(f_1 g)\lambda(g)\, dg \\ &= \int_{M_R} \delta_{P_R}^{-1}(m)|\nu(g)|_F^{3s} \Phi(f_1 g)\lambda(g)\, dg \\ &= \int_{M_R} \delta_{P_R}^{-1}(m)|\nu(m)|_F^{3s} \operatorname{char}(m_b) \operatorname{char}(m)\lambda(m)\, dm \\ &= \zeta(6s) \int_{M_R} \delta_{P_R}^{-1}(m)|\nu(m)|_F^{3s} \operatorname{char}(p \nmid m_b) \operatorname{char}(m)\lambda(m)\, dm \end{aligned}$$

where we have used that $m_b \in M_2(\mathcal{O}_E)$ and $\lambda(m) \neq 0$ implies $m \in M_4(\mathcal{O}_E)$. Thus $I(\lambda; s) = \zeta(6s) I^\Delta(\lambda; 3s + 1)$ and the theorem follows in this case. □

Proof of Theorem 4.3.3 when $E = F \times F$. The calculation of $I(\lambda; s)$ above is the same when E is

split. So we must just calculate $I^{\Delta}(\lambda; s)$. This is

$$\int_M \delta_P^{-1}(m)|\nu(m)|_F^s \lambda(m) \int_U \chi(u)\Delta^0(um)\, du\, dm.$$

We will again find that the inner integral is

$$|\det(m_b)|_E^{-1/2} \operatorname{char}(m \in M_R K_M) \operatorname{char}(m) \operatorname{char}(p \nmid m_b),$$

from which the theorem follows in this case. So let us compute this unipotent integral.

First, $m_b \in \mathrm{GL}_2(E)$, and we denote $m_1, m_2 \in \mathrm{GL}_2(F)$ its two projections. The condition $\Delta^0(m) \neq 0$ implies $|det(m_1)|_F = |\det(m_2)|_F$. Furthermore if h is the upper right two-by-two block of u, then $h_2 = {}^t h_1$ where h_i are its two projections. The condition $\Delta^0(um) \neq 0$ is then m integral, $|\det(m_1)|_F = |\det(m_2)|_F$, $h_1 m_1$ integral, and ${}^t h_1 m_2$ integral. For $m \in M$ with $\Delta^0(m) \neq 0$, we set U_m the u with $\Delta^0(um) \neq 0$. This is an abelian group, and so again we again must analyze the m with χ trivial on U_m.

So assume m is such that χ is trivial on U_m. By right K_M invariance, we may assume $m_1 = \left(\begin{smallmatrix} p^{a_1} & b_1 \\ 0 & p^{c_1} \end{smallmatrix}\right)$ and $m_2 = \left(\begin{smallmatrix} p^{a_2} & b_2 \\ 0 & p^{c_2} \end{smallmatrix}\right)$.

Claim 4.3.5. *p doesn't divide either m_1 or m_2.*

Proof. Assume p divides m_1. We will show χ is nontrivial U_m. If $c_2 > 0$, then take $h = \left(\begin{smallmatrix} 0 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)$. So assume $c_2 = 0$. Since $|\det(m_1)| = |\det(m_2)|$, $a_2 > 0$. Now we have two cases: $p|b_2$ and $p \nmid b_2$. If $p|b_2$ then take $h = \left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. If $p \nmid b_2$ then take ${}^t h = \left(\begin{smallmatrix} 0 & 0 \\ p^{-1}b_2^{-1} & -p^{-1} \end{smallmatrix}\right)$. The case p divides m_2 is the same. □

Now if both $c_i > 0$, then $h = \left(\begin{smallmatrix} 0 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right) \in U_m$. Hence one of the $c_i = 0$. Assume, without loss of generality, that $c_1 = 0$. Then $a_1 = a_2 + c_2$. Taking $h = \left(\begin{smallmatrix} 0 & 0 \\ p^{-c_2} & -b_1 p^{-c_2} \end{smallmatrix}\right)$, we find that p^{c_2} divides b_1. For ease of notation, set $a = a_2, c = c_2$, and $b_1' = b_1/p^c$. We then have that

$$m_1 = \begin{pmatrix} p^c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^a & b_1' \\ 0 & 1 \end{pmatrix}; \quad m_2 = \begin{pmatrix} p^a & b_2 \\ 0 & p^c \end{pmatrix}.$$

Let us consider a general two-by-two matrix h. Such a matrix may be written

$$h = \begin{pmatrix} x & w_1' - p^c x b_1' \\ w_2' - p^{-c} x b_2 & y'' + x b_1' b_2 - p^c w_2' b_1' \end{pmatrix}.$$

With these coordinates,

$$hm_1 = \begin{pmatrix} p^{a+c}x & w_1' \\ p^{a+c}w_2' - p^a x b_2 & y'' \end{pmatrix}$$

and

$${}^t hm_2 = \begin{pmatrix} p^a x & p^c w_2' \\ p^a w_1' - p^{a+c} x b_1' & w_1' b_2 + p^c y'' - p^{2c} w_2' b_1' \end{pmatrix}.$$

Hence h corresponds to an element of U_m if and only if $x \in p^{-a}\mathbf{Z}_p$, $w_2' \in p^{-c}\mathbf{Z}_p$, and $w_1', y'' \in \mathbf{Z}_p$. Applying χ we see $\psi(x(1 - Rb_1'b_2)) \equiv 1$, from which we conclude $1 - Rb_1'b_2 \in p^a\mathbf{Z}_p$. And conversely if $1 - Rb_1b_2' \in p^a\mathbf{Z}_p$ we see χ is trivial on U_m. Finally the measure of U_m is $p^{a+c} = |\det(m_b)|_E^{-1/2}$. The theorem now follows from Proposition 4.3.7 below, which says that elements of these form are precisely the $M_R K_M$. □

Proposition 4.3.6. *Suppose E is a field, and $m \in M_2(\mathcal{O}_E) \cap GL(E)$. Then $m \in M_R K_M$ if and only if m is right K equivalent to a matrix of the form $p^c \left(\begin{smallmatrix} p^a & b \\ 0 & 1 \end{smallmatrix}\right)$ with $1 - RN(b) \in p^a\mathbf{Z}_p$.*

Proof. First suppose $m = \left(\begin{smallmatrix} x & \bar{y} \\ Ry & \bar{x} \end{smallmatrix}\right)$ is the bottom two-by-two part of an element of M_R. We may assume, without loss of generality, that $p \nmid m$. Hence p doesn't divide both x and y. Furthermore, m is right K equivalent to a matrix with x and $\bar{y}$ interchanged. Hence we may assume without loss of generality that $p \nmid x$. Then one finds m is right K equivalent to

$$\begin{pmatrix} \delta & -\frac{\bar{y}}{\bar{x}} \\ 0 & 1 \end{pmatrix}$$

with $\delta = x\bar{x} - Ry\bar{y}$ the determinant of m. Hence $1 - RN(\bar{y}/\bar{x}) \in \delta\mathbf{Z}_p$, as desired.

Conversely, assume $m = \left(\begin{smallmatrix} \delta & b \\ 0 & 1 \end{smallmatrix}\right)$ with $1 - RN(b) \in \delta\mathbf{Z}_p$. By right K action, we may change b by a multiple of δ so that $1 - RN(b) = \delta u$ for $u \in \mathbf{Z}_p^\times$. Now m is equivalent to $\left(\begin{smallmatrix} \delta u & b \\ 0 & 1 \end{smallmatrix}\right)$, which is equivalent to $\begin{pmatrix} \delta u + RN(b) & b \\ R\bar{b} & 1 \end{pmatrix}$. □

Proposition 4.3.7. *Suppose $E = F \times F$. Define X_R to be the set of pairs of matrices of the form*

$$\left(\begin{pmatrix} p^c & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^a & b_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p^a & b_2 \\ 0 & p^c \end{pmatrix}\right) \text{ or } \left(\begin{pmatrix} p^a & b_1 \\ 0 & p^c \end{pmatrix}, \begin{pmatrix} p^c & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^a & b_2 \\ 0 & 1 \end{pmatrix}\right)$$

with $1 - Rb_1b_2 \in p^a\mathbf{Z}_p$. Then $M_R K_M = p^{\mathbf{Z}} X_R K_M$.

Proof. We first prove the inclusion $M_R K_M \subseteq p^{\mathbf{Z}} X_R K_M$. Note that the elements in M_R correspond

to pairs

$$(m_1, m_2) = \left(\begin{pmatrix} x_1 & y_2 \\ Ry_1 & x_2 \end{pmatrix}, \begin{pmatrix} x_2 & y_1 \\ Ry_2 & x_1 \end{pmatrix} \right).$$

Define $\epsilon_R = \left(\begin{smallmatrix} 0 & 1 \\ R & 0 \end{smallmatrix}\right)$. Then these are the pairs with $m_2 = \epsilon_R^{-1} m_1 \epsilon_R$. We have seen in the proof of Theorem 4.3.3 that elements $m \in M$ with χ trivial on U_m (in the notation of that proof) are in $p^{\mathbf{Z}} X_R K_M$. Take $m \in M_R$. By multiplying m by an appropriate power of p, we may assume m_b is integral and $p \nmid m_b$. Without loss of generality we may assume then that $p \nmid m_1$. Thus for this direction it suffices to prove that χ is trivial on U_m. Now if $h = \left(\begin{smallmatrix} x & w_1 \\ w_2 & y \end{smallmatrix}\right)$ is the two-by-two matrix corresponding to an element of U_m, we have hm_1 integral and ${}^t h m_2$ integral. Hence

$$\begin{pmatrix} 0 & R \\ 1 & 0 \end{pmatrix} {}^t h \epsilon_R m_1 = \begin{pmatrix} Ry & w_1 \\ w_2 & R^{-1}x \end{pmatrix} m_1$$

is integral. Subtracting, we get

$$(x - Ry) \begin{pmatrix} 1 & 0 \\ 0 & -R^{-1} \end{pmatrix} m_1$$

integral. Since $p \nmid m_1$ it follows that $x - yR$ is integral, and hence χ is trivial on U_m, as desired.

Now we prove the other inclusion. By symmetry it suffices to consider the first case. By changing b_1 by a multiple of p^a we may assume $1 - Rb_1b_2 = p^a u'$ with $u' \in \mathbf{Z}_p^\times$. Now

$$\begin{pmatrix} p^a & b_1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} p^a u' & b_1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} p^a u' + Rb_1b_2 & b_1 \\ Rb_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b_1 \\ Rb_2 & 1 \end{pmatrix},$$

where $\sim$ denotes right-K equivalence. Thus $\left(\begin{smallmatrix} p^c & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^a & b_1 \\ 0 & 1 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} p^c & p^c b_1 \\ Rb_2 & 1 \end{smallmatrix}\right)$ and

$$\begin{pmatrix} p^a & b_2 \\ 0 & p^c \end{pmatrix} \sim \begin{pmatrix} p^a u' & b_2 \\ 0 & p^c \end{pmatrix} \sim \begin{pmatrix} p^a u' + Rb_1b_2 & b_2 \\ Rb_1 p^c & p^c \end{pmatrix} = \begin{pmatrix} 1 & b_2 \\ Rp^c b_1 & p^c \end{pmatrix}$$

which is the desired form. □

The proof of Theorem 4.3.4 will follow from that of Theorem 4.3.3 once we calculate $\alpha_\chi(v_R g)$ explicitly, which we now do.

Proposition 4.3.8. *Suppose $m \in M_H = M \cap H$. Define $Y(m) = \mathrm{char}(|\nu(m)|_F |\det(m_b)|_E^{-1/2} \leq 1)$.*

Then $\alpha_\chi(v_R m) = Y(m)|\nu(m)|_F|\det(m_b)|_E^{-1/2}\,\mathrm{char}(m \in M_R K_M)$.

Remark 4.3.9. An easy computation, which we omit, shows that $\ell(\pi(m)v_0) \neq 0$ implies

$$|\nu(m)|_F|\det(m_b)|_E^{-1/2} \leq 1.$$

So this condition is natural, and will go away in the presence of integral $I'_R(\ell; s)$.

Proof. Recall $v_R = e_1 \wedge f_2 + Re_2 \wedge f_1$. First, for $u \in U_P$ with upper right corner equal to $h = \left(\begin{smallmatrix} x & w \\ \bar{w} & y \end{smallmatrix}\right)$, we find

$$v_R u = v_R + (x - yR) f_1 \wedge f_2.$$

Since $m \in M_H$, we may write

$$m = \left(\begin{array}{cc|cc} \lambda\bar{d} & -\lambda\bar{c} & & \\ -\lambda\bar{b} & \lambda\bar{a} & & \\ \hline & & a & b \\ & & c & d \end{array}\right)$$

with $ad - bc, \lambda \in F^\times$. Note $|\lambda|_F = |\nu(m)|_F|\det(m_b)|_E^{-1/2}$, which is where this quantity will come from. Now by right K_M invariance, we may assume m_b is upper triangular, i.e. that $c = 0$. We calculate $(v_R + z f_1 \wedge f_2)m$

$$= -R\frac{\bar{b}}{d} e_1 \wedge f_1 + R\frac{\bar{a}b}{ad} e_2 \wedge f_2 + \frac{1}{ad}(N(d) - RN(b)) e_1 \wedge f_2 + R\frac{\bar{a}}{d} e_2 \wedge f_1 + \frac{z}{\lambda} f_1 \wedge f_2. \tag{4.3}$$

Now, the form of this expression immediately implies that the set of $u \in N_R \backslash U_P$ for which $\alpha(v_R u m) \neq 0$ is an abelian group. Hence, the integral defining α_χ will vanish if unless χ is trivial on this set. It follows that for $\alpha_\chi(v_R m)$ to not vanish, we need $|\lambda|_F \leq 1$, and that furthermore that the integral

$$\int_{N_R \backslash U_P} \chi(u)\alpha(v_R u m)\, du$$

in this case is $|\lambda|$. This gives the first couple terms of the proposition. The only thing left to do is to determine for which m can the first four coefficients in (4.3) be integral.

Note that since $\det(m_b) = ad$ is in $F^\times$, $ad = \bar{a}\bar{d}$, so the second coefficient of (4.3) is $Rb/\bar{d}$. Hence these coefficients are integral precisely when $d|\bar{b}, d|\bar{a}$ and $\left(1 - RN\left(\frac{\bar{b}}{\bar{d}}\right)\right)$ is in $\frac{\bar{a}}{d}\mathbf{Z}_p$. When E is a field, these are exactly the conditions of Proposition 4.3.6 and when $E = F \times F$ these are exactly the conditions of Proposition 4.3.7. Hence we have integrality if and only if $m \in M_R K_M$, proving

the proposition. □

Proof of Theorem 4.3.4. We again must show

$$I'(\lambda;s)=\int_{U_P\backslash H}\alpha_\chi(v_Rg)|\nu(g)|_F^{3s}\Phi(f_1g)\lambda(g)\,dg$$

is equal to $\zeta(6s)I^\Delta(\lambda;3s+1)$. But

$$\begin{aligned}I'(\lambda;s)&=\int_{M_H}\delta_{P_H}^{-1}(m)|\nu(m)|_F|\det(m_b)|_E^{-1/2}\,\mathrm{char}(m\in M_RK_M)|\nu(m)|_F^{3s}\,\mathrm{char}(m)\lambda(m)\,dm\\&=\int_{M_R}\delta_{P_R}^{-1}(m)|\nu(m)|_F^{3s}\,\mathrm{char}(m)\lambda(m)\,dm\end{aligned}$$

as $\delta_{P,H}=\delta_P$, and we computed above $\delta_P^{-1}(m)|\nu(m)|_F|\det(m_b)|_E^{-1/2}=\delta_{P_R}^{-1}(m)$. This last integral is the one that appeared in the proof of Theorem 4.3.3, and was shown there to be $\zeta(6s)I^\Delta(\lambda;3s+1)$, proving the theorem. □

4.4 Hecke operators

Suppose ω_α is the MacDonald spherical function associated to $Ind(\alpha\delta_B^{1/2})$ where α is a character of the diagonal torus T of $G(F)$, normalized so that $\omega_\alpha(1)=1$. Let π be an unramified subqoutient of $Ind(\alpha\delta_B^{1/2})$, and let $\Delta^s(g)$ be the functions defined in Definitions 4.3.1 and 4.3.2. In this section we prove the following theorem.

Theorem 4.4.1. *Denote by ω_π the central character of π.*

$$I^\Delta(\alpha;s):=\int_{G(F)}\omega_\alpha(g)\Delta^s(g)\,dg=\frac{L(\pi,\wedge^2,s-2)}{L(\omega_\pi,2s-4)L(\omega_\pi,2s-2)}$$

We treat this theorem as two theorems, one for E a field and one for E split. One can likely give an argument that applies to both cases simultaneously, but for clarity we treat the two cases separately.

Proof of Theorem 4.4.1 when E is a field. Denote by ϕ_α the unique spherical vector in $Ind(\delta_B^{1/2}\alpha)$ that is 1 at the identity. Then

$$\begin{aligned}I^\Delta(\alpha;s)&=\int_G\phi_\alpha(g)|\nu(g)|_F^s\,\mathrm{char}(g)\,dg\\&=\int_M\delta_P^{-1}(m)|\nu(m)|_F^s\phi_\alpha(m)\left(\int_U\mathrm{char}(um)\,du\right)dm.\end{aligned}$$

We compute this inner integral. For an $m = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(E) \cap M_2(\mathcal{O}_E)$, set $\Delta_1(m) = \max\{|a|_E, |b|_E, |c|_E, |d|_E\}^{-1}$.

Claim 4.4.2.

$$\int_U \mathrm{char}(um)\, du = |\det(m_b)|_E^{-1/2} \Delta_1(m_b)\, \mathrm{char}(m).$$

Proof. Both sides of the proposed equality are bi-K-invariant: if $m \mapsto km$, then, in the notation of Theorem 4.3.3, $U_m \mapsto {}^t\bar{k}^{-1} U_m k^{-1}$; the right K invariance is clear and so is the bi-K-invariance of the right-hand-side. So, it suffices to check the claim on matrices m with $m_b = diag(p^j, p^k)$ with $j \geq k$. If m_b is of this form, then the left hand side is immediately seen to be $p^{j+k}p^{2k}$, which is the right hand side. □

Thus

$$\begin{aligned} I^{\Delta}(\alpha; s) &= \int_T \delta_B^{-1}(t) |\nu(t)|_F^s |\det(t_b)|_E^{-1/2} \phi_\alpha(t) \left(\int_{U_P \backslash U_B} \Delta_1(ut)\, \mathrm{char}(ut)\, du \right) dt \\ &= \int_T \delta_B^{-1/2}(t) |\nu(t)|_F^s |\det(t_b)|_E^{-1/2} \alpha(t) \left(\int_{U_P \backslash U_B} \Delta_1(ut)\, \mathrm{char}(ut)\, du \right) dt \end{aligned}$$

We will now compute the inner integral. First, the following

Claim 4.4.3. *For $\alpha, \beta \in E^\times \cap \mathcal{O}_E$,*

$$\int_{\beta^{-1}\mathcal{O}_E} \Delta_1 \left(\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \right) \right) dx = |\beta|_E^{-1} \left(m + 1 - m|p|_E \right)$$

where $m = \min\{val_p(\alpha), val_p(\beta)\}$.

Proof. The integral is equal to

$$\int_{\beta^{-1}\mathcal{O}_E} \max\{|p^m|_E, |\beta x|_E\}^{-1}\, dx$$

which is

$$|\beta|_E^{-1} \int_{\mathcal{O}_E} \max\{|p^m|_E, |y|_E\}^{-1}\, dy$$

by changing variables. The integral is now computed to be as in the claim. □

Set now $m = \min\{val_p(t_i)\}$ where $t = diag(t_1, t_2, t_3, t_4)$. Up to right K equivalence, we may

assume $\det(t) = \det(\bar{t})$. Thus, $t = diag(s\bar{\beta}, s\bar{\alpha}, \alpha, \beta)$ where $\alpha\beta \in F^\times$ and $s \in F^\times$. Then

$$ut = \left(\begin{array}{cc|cc} 1 & & & \\ -\bar{x} & 1 & & \\ \hline & & 1 & x \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} s\bar{\beta} & & & \\ & s\bar{\alpha} & & \\ \hline & & \alpha & \\ & & & \beta \end{array}\right) = \left(\begin{array}{cc|cc} s\bar{\beta} & & & \\ -s\bar{\beta}\bar{x} & s\bar{\alpha} & & \\ \hline & & \alpha & \beta x \\ & & & \beta \end{array}\right). \tag{4.4}$$

We separate into two cases: $|s| \leq 1$ and $|s| > 1$. If $|s| \leq 1$, then $m = \min\{val(\alpha), val(\beta)\}$, and ut is integral if and only if $x\beta$ is integral. Hence

$$\int_{U_P \backslash U_B} \Delta_1(ut)\,\mathrm{char}(ut)\,du = |\beta|_E^{-1}\,(m + 1 - |p|_E m)$$

by the claim. If $|s| > 1$, then $m = \min\{val(s\alpha), val(s\beta)\}$ and

$$\begin{aligned} \int_{U_P \backslash U_B} \Delta_1(ut)\,\mathrm{char}(ut)\,du &= \int_E \mathrm{char}(s\beta x)\max\{|\alpha|_E, |\beta|_E, |x\beta|_E\}^{-1}\,dx & (4.5)\\ &= |s|_E \int_E \mathrm{char}(s\beta x)\max\{|s\alpha|_E, |s\beta|_E, |xs\beta|_E\}^{-1}\,dx & (4.6)\\ &= |s|_E |s\beta|_E^{-1}\,(m + 1 - |p|_E m) & (4.7) \end{aligned}$$

by the claim. Hence, we get that the unipotent integral is

$$|\beta|_E^{-1}\,(m + 1 - |p|_E m) = |t_4|_E^{-1}\,(m + 1 - |p|_E m)$$

in both cases. Since $\delta_B^{-1/2}(t)|t_3 t_4|_E^{-1/2}|t_4|_E^{-1} = |\nu(t)|_F^{-2}$, we get that

$$I^\Delta(\alpha; s) = \int_T \alpha(t)|\nu(t)|_F^{s-2}\,(m(t) + 1 - |p|_E m(t))\,\mathrm{char}(t)\,dt.$$

Set

$$I_m(s) = \int_T \alpha(t)|\nu(t)|_F^{s-2}\,(m(t))\,\mathrm{char}(t)\,dt \tag{4.8}$$

and

$$I_{m'}(s) = \int_T \alpha(t)|\nu(t)|_F^{s-2}\,(m(t) + 1)\,\mathrm{char}(t)\,dt. \tag{4.9}$$

Then

$$\begin{aligned} I_m(s) &= \int_T \alpha(pt)|\nu(pt)|_F^{s-2}\,(m(pt))\,\mathrm{char}(pt)\,dt \\ &= \omega_\pi(p)|p|_F^{2s-4}\int_T \alpha(t)|\nu(t)|_F^{s-2}\,(m(t)+1)\,\mathrm{char}(pt)\,dt \\ &= \omega_\pi(p)|p|_F^{2s-4}\int_T \alpha(t)|\nu(t)|_F^{s-2}\,(m(t)+1)\,\mathrm{char}(t)\,dt \\ &= \omega_\pi(p)|p|_F^{2s-4} I_{m'}(s) \end{aligned}$$

since $m(t)+1$ is zero if $\mathrm{char}(pt)\neq 0$ but $\mathrm{char}(t)=0$. Thus

$$I^\Delta(\alpha;s) = I_{m'}(s) - |p|_F^2 I_m(s) = \left(1-\omega_\pi(p)|p|_F^{2s-2}\right) I_{m'}(s) = \frac{I_{m'}(s)}{L(\omega_\pi, 2s-2)}.$$

Define T_k the subset of elements of T with $m(t)=k$. Consider the following four elements of T_0: $t_A = diag(p,p,1,1), t_B = diag(1,1,p,p), t_C = diag(p,1,1,p), t_D = diag(1,p,p,1)$. Note that every element of T_0/K_T can be written uniquely as a product of these four elements. More generally, it follows from this that every element of T_k/K_T can be written as a product of these four elements in $k+1 = m(t)+1$ ways. Hence we find

$$\begin{aligned} I_{m'}(s+2) &= \int_T \alpha(t)|\nu(t)|_F^s\,\mathrm{char}(t)\,(m(t)+1)\;dt \\ &= \left(\sum_j (\alpha(t_A)|p|_F^s)^j\right)\left(\sum_k (\alpha(t_B)|p|_F^s)^k\right)\left(\sum_\ell (\alpha(t_C)|p|_F^s)^\ell\right)\left(\sum_m (\alpha(t_D)|p|_F^s)^m\right) \\ &= \left(1-ab\lambda p^{-s}\right)^{-1}\left(1-\lambda p^{-s}\right)^{-1}\left(1-a\lambda p^{-s}\right)^{-1}\left(1-b\lambda p^{-s}\right)^{-1} \\ &= \frac{L(\pi,\wedge^2,s)}{L(\omega_\pi,2s)} \end{aligned}$$

where λ, a, b are the Satake parameters of π. The theorem follows. □

Proof of Theorem 4.4.1 when E is split. As above, we have

$$I^\Delta(\alpha;s) = \int_M \delta_P^{-1}(m)|\nu(m)|_F^s \phi_\alpha(m)|\nu(m)|_F^s \left(\int_U \Delta^0(um)\,du\right) dm.$$

The same argument as before proves that the inner integral is bi-K-invariant as a function of m, hence we may calculate it assuming m is diagonal. We obtain

Claim 4.4.4. *For $m=(m_1,m_2)\in \mathrm{GL}_2(E)\cap M_2(\mathcal{O}_E)$, set $\Delta_1(m) = \max\{|m_i^{jk}|_E\}^{-1}$ where $i=1,2$*

and m_i^{jk} refers to the jk entry of the matrix m_i. Then

$$\int_U \Delta^0(um)\,du = |\det(m_b)|_E^{-1/2}\Delta_1(m_b)\Delta^0(m).$$

Proof. The condition $\Delta^0(um) \neq 0$ is equivalent to $\Delta^0(m) \neq 0$, hm_1 integral, and ${}^thm_2$ integral, where h is the upper right two-by-two block of u_1. The condition $\Delta^0(m) \neq 0$ is that $|\det(m_1)|_F = |\det(m_2)|_F$. By bi-$K$-invariance, we may suppose that $m_1 = diag(p^a, p^b)$, $m_2 = (p^c, p^d)$ with $a \leq b$, $c \leq d$. Since the determinants are equal $p^{a+b} = p^{c+d} =: \delta$. The integral $\int_U \Delta^0(um)\,du$ is thus the measure of the set of two-by-two matrices h such that hm_1 and ${}^thm_2$ is integral, which is immediately computed to be

$$\min\{p^a, p^c\}\min\{p^a, p^d\}\min\{p^b, p^c\}\min\{p^b, p^d\} = \delta p^{2a}.$$

The right hand side is $|\det(m_b)|_E^{-1/2}\Delta_1(m_b)$. □

We now compute the integral $\int_{U_P\backslash U_B} \Delta_1(ut)\Delta^0(ut)\,du$. This integral is 0 if $\Delta^0(t) = 0$, so assume $\Delta^0(t) \neq 0$. Since $\Delta^0(t) \neq 0$, by right-K-invariance we may assume $\det(t) = \det(\bar{t})$, and then let the notation be as in (4.4). Here is the analogue of Claim 4.4.3.

Claim 4.4.5. *Suppose $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in E^\times \cap \mathcal{O}_E$. Then*

$$\int_{E=F\times F} \mathrm{char}(x\beta \in \mathcal{O}_E)\max\{|\alpha_1|_F, |\alpha_2|_F, |\beta_1|_F, |\beta_2|_F, |x\beta_1|_F, |x\beta_2|_F\}^{-2}\,dx = |\beta|_E^{-1}\left(m + 1 - |p|_E m\right)$$

where $|\beta|_E = |\beta_1|_F|\beta_2|_F$ and $m = \min\{val(\alpha_1), val(\alpha_2), val(\beta_1), val(\beta_2)\}$.

Proof. The proof is similar to the proof of 4.4.3: One changes variables to obtain that the integal is equal to

$$|\beta|_E^{-1}\int_{\mathcal{O}_E=\mathcal{O}_F\times\mathcal{O}_F} \max\{|p^m|_F, |y_1|_F, |y_2|_F\}^{-2}\,dy_1\,dy_2$$

and this latter integral is easily seen to be equal to $m(1 - |p|_E) + 1$ by breaking it into a sum over the pieces with $\min\{val(y_1), val(y_2)\}$ being constant. □

Define $m(t)$ to be the minimum valuation of all entries of the two projections of t. Equivalently, $m(t)$ is the greatest integer k so that $p^k t' = t$ for some integral element t'. If $|s| \leq 1$, then $m(t) = \min\{val(\alpha_1), val(\alpha_2), val(\beta_1), val(\beta_2)\}$. If $|s| \leq 1$ Claim 4.4.5 gives

$$\int_{U_P\backslash U_B} \Delta_1(ut)\Delta^0(ut)\,du = |\beta|_E^{-1}\left(m(t) + 1 - |p|_E m(t)\right).$$

If $|s| > 1$, as in (4.7), this integral is again found to be $|\beta|_E^{-1}(m(t)+1-|p|_E m(t))$. Using again that $\delta_B^{-1/2}(t)|\det(t_3 t_4)|_E^{-1/2}|t_4|_E^{-1} = |\nu(t)|_F^{-2}$ we find

$$I^\Delta(\alpha;s) = \int_T \alpha(t)|\nu(t)|_F^{s-2}\Delta^0(t)\left(m(t)+1-|p|_E m(t)\right)\,dt.$$

Let $I_m(s), I_{m'}(s)$ be exactly as in (4.8), (4.9), except with $\Delta^0(t)$ replacing $\mathrm{char}(t)$. Then we again have

$$I_m(s) = \omega_\pi(p)|p|_F^{2s-4} I_{m'}(s)$$

and thus

$$I^\Delta(\alpha;s) = \frac{I_{m'}(s)}{L(\omega_\pi, 2s-2)} = \frac{1}{L(\omega_\pi,2s-2)L(\omega_\pi,2s-4)}\left(L(\omega_\pi,2s-4)I_{m'}(s)\right).$$

Now

$$\begin{aligned}
L(\omega_\pi,2s-4)I_{m'}(s) &= \left(\sum_j \left(\omega_\pi(p)|p|_F^{2s-4}\right)^j\right)\int_T \alpha(t)|\nu(t)|_F^{s-2}\Delta^0(t)\left(m(t)+1\right)\,dt \\
&= \sum_j \int_T \alpha(p^j t)|\nu(p^j t)|_F^{s-2}\Delta^0(t)\left(m(t)+1\right)\,dt \\
&= \int_T \alpha(t)|\nu(t)|_F^{s-2}\Delta^0(t)\{(m(t)+1)+(m(t))+(m(t)-1)+\cdots+1\}\,dt \\
&= \int_T \alpha(t)|\nu(t)|_F^{s-2}\Delta^0(t)\binom{m(t)+2}{2}\,dt
\end{aligned}$$

Define $t_1^{12} = diag(p,p,1,1)$, $t_1^{13} = diag(p,1,p,1)$ etc with t_1^{IJ} the diagonal matrix in GL_4 with p at the i and j spots and 1 at the other two spots. Similar to the case where E is a field, consider the 6 elements $t^{IJ} = (p, t_1^{IJ}) = (\nu, g_1) \in \mathrm{GL}_1 \times \mathrm{GL}_4$, with $1 \le I < J \le 4$. Define T_k to be the subset of T with $\Delta^0(t) \neq 0$ and $m(t) = k$. Then one can check that every element of T_0/K_T can be written uniquely as a product of the t^{IJ}. It follows that on T_k/K_T, every element can be written as a product of the t^{IJ} in $\binom{k+2}{2}$ ways, since $\binom{k+2}{2}$ is the number of ways to write k as the sum of three nonnegative integers. Hence we conclude

$$\begin{aligned}
L(\omega_\pi(p),2s)I_{m'}(s+2) &= \prod_{1\le I<J\le 4}(1-\alpha(t^{IJ})|p|_F^s)^{-1} \\
&= L(\pi,\wedge^2,s).
\end{aligned}$$

The theorem follows. □

4.5 Ramified computation

In this section we control the local integral $I_R(\ell; s)$ at bad finite places.

Proposition 4.5.1. *Let v_0 in the space of π_p be given. Then, there exist v in the space of π_p and a Schwartz function Φ on $W_4(E)$, depending only on v_0, so that for all (U_P, χ) models ℓ,*

$$I_R(\ell; v, s) := \int_{N_R \backslash G_R} |\nu(g)|_F^{3s} \Phi(f_1 g) \ell(\pi(g) v_0)\, dg = \ell(v_0).$$

Proof. For $z \in F$, define $u(z) = 1 + zE_{24} \in U_P$. Write K_N for the congruence subgroups of K. There is K_{N_0} stabilizing v_0. For a function $\varphi_1 \in \mathcal{C}_c^\infty(F)$ to be determined later, define

$$v_1 = \int_F \varphi_1(z) \pi(u(z)) v_0\, dz.$$

Suppose $m \in M_R$ is

$$\left(\begin{array}{cc|cc} \lambda x & \lambda R\bar{y} & & \\ \lambda y & \lambda \bar{x} & & \\ \hline & & x & -\bar{y} \\ & & -Ry & \bar{x} \end{array}\right).$$

Then $\ell(\pi(m)\pi(u)v_0) = \psi(-R\lambda z)\ell(\pi(m)v_0)$. We may and do pick φ_1 locally constant of compact support so that

$$\int_F \varphi_1(z)\psi(-R\lambda z)\, dz = \mathrm{char}(\lambda \in 1 + p^{N_0}).$$

Hence with this choice $\ell(\pi(m)v_1) = \mathrm{char}(\lambda \in 1 + p^{N_0})\ell(v_0)$.

Now v_1 is stable by some K_N for some $N \geq N_0$. We define $\Phi = \mathrm{char}(f_1 + p^N W_4(\mathcal{O}_E))$. Define $B_{R,N}$ the subgroup of the Siegel parabolic of G_R with the bottom right block $\equiv 1$ modulo p^N, and $K_{N,R} = K_N \cap G_R$. Then for $g \in G_R$, $\Phi(g) \neq 0$ implies $g \in B_{N,R} K_{N,R}$, and Φ is identically one on this set. Hence, for some positive constants C_i,

$$\begin{aligned} I(\lambda; v_1; s) &= \int_{N_R \backslash G_R} |\nu(g)|_F^{3s} \Phi(f_1 g) \ell(\pi(g) v_1)\, dg \\ &= \int_{N_R \backslash B_{R,N} K_{R,N}} |\nu(g)|_F^{3s} \ell(\pi(g) v_1)\, dg \\ &= C_1 \int_{M_{R,N} := M_R \cap B_{R,N}} \delta_{B_R}^{-1}(m) |\nu(m)|_F^{3s} \ell(\pi(m) v_1)\, dm \end{aligned}$$

But now $\ell(\pi(m)v_1) = \mathrm{char}(\lambda \in 1 + p^{N_0})\ell(\pi(m)v_0)$. The integrand above is thus $\ell(v_0)$ where it is

supported, and its support is an open compact subgroup of G_R. Hence $I(\ell; v_1, s) = C_2\ell(v_0)$, and setting $v = C_2^{-1}v_1$ gives the proposition. $\square$

Chapter 5

The Spin L-function on GSp_6

5.1 Introduction

We give new Rankin-Selberg integral representations for the Spin L-function of cuspidal automorphic representations π on GSp_6. Previously, Bump-Ginzburg [7] and Vo [34] had considered the case when π is generic. The integral representations in this paper do not assume genericity. Instead, the global integrals we write down unfold to a Fourier coefficient that does not appear to factorize for general cusp forms. We will analyze the integrals via the method of Piatetski-Shapiro and Rallis, except with a twist (see below).

We give two global integrals, which are similar, and inherit their meromorphicity from an Eisenstein series on GL_2. The first integral we give has been considered previously by Gan-Gurevich [13], but when restricted to a special class of CAP forms on GSp_6 constructed by Ginzburg [19]. For these special forms, Gan-Gurevich [13] show that the Fourier coefficient alluded to above factorizes, and hence the global integral is an Euler product.

Let us describe the global integrals more precisely. Fix an auxillary etale quadratic extension $L = \mathbf{Q}(\sqrt{D})$ of $\mathbf{Q}$. Denote by $\mathrm{GL}_{2,L}^*$ the algebraic group of elements of $\mathrm{GL}_{2,L}$ with determinant in $\mathbf{Q}$. Set H the subsgroup of elements (g_1, g_2) in $\mathrm{GL}_2 \times \mathrm{GL}_{2,L}^*$ with $\det g_1 = \det g_2$. There is an embedding $H \to \mathrm{GSp}_6$. Set $E^*(g_1, s) = \zeta(2s)E(g_1, s)$, where $E(g_1, s)$ is the usual Eisenstein series on GL_2. Let ϕ be a cusp form in the space of π. We set

$$I(\phi, s) = \int_{[H]} E^*(g_1, s)\phi(g)dg.$$

As is shown in Gan-Gurevich [13], this integral unfolds. Denote by R the parabolic subgroup

of GSp_6 with Levi $\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_2$, and by U_R its unipotent radical. Associated to the quadratic extension L is a nondegenerate unitary character $\chi : U_R(\mathbf{Q})\backslash U_R(\mathbf{A}) \to \mathbf{C}^\times$. Define the Fourier coefficient

$$\phi_\chi(g) = \int_{[U_R]} \chi^{-1}(u)\phi(ug)du.$$

Then one has

$$I(\phi, s) = \int_{N(\mathbf{A})Z(\mathbf{A})\backslash H(\mathbf{A})} f^*(g_1, s)\phi_\chi(g)dg,$$

where $f^*(g_1, s)$ is the (normalized) section defining the Eisenstein series, and N is the unipotent radical of a Borel subgroup of H.

Recall that a (U_R, χ) model on an irreducible admissible representation (π_p, V) is a linear functional $\ell : V \to \mathbf{C}$ satisfying $\ell(\pi(u)v) = \chi(u)\ell(v)$ for all $u \in U_R, v \in V$. The following is our first main result:

Theorem 5.1.1. *Suppose $\pi = \otimes\pi_v$, p is finite and does not divide $2D$, and π_p is unramified. Let ℓ by any (U_R, χ) model on π_p, and v_0 a spherical vector. Set $\lambda(g) = \ell(\pi(g)v_0)$. Then*

$$\int_{N(\mathbf{Q}_p)Z(\mathbf{Q}_p)\backslash H(\mathbf{Q}_p)} f^*(g_1, s)\lambda(g)dg,$$

is $\lambda(1)$ times the local spin L-function, $L(\pi_p, \mathrm{Spin}, s)$.

This theorem, plus the ramified computation in section 5.5, shows that $I(\phi; s)$ is equal to $I_\infty(\phi; s)\frac{L^S(\pi, Spin, s)}{\zeta^S(2s)}$ for an archimedean integral $I_\infty(\phi; s)$.

For the next integral, we first define a special automorphic form (see section 5.2) P_D on GSp_4. Denote by $\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4$ the set of elements (g_1, g_2) in $\mathrm{GL}_2 \times \mathrm{GSp}_4$ with $\det g_1 = \nu(g_2)$, where ν is the similitude. Then $\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4$ embeds in $G = \mathrm{GSp}_6$. We consider the global integral

$$I'(\phi, s) = \int_{[\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4]} E^*(g_1, s)\phi(g)P_D(g_2)dg,$$

for a cusp form ϕ on G.

This integral also unfolds to the Fourier coefficient ϕ_χ, and we show that it too represents the partial Spin L-function, in the same sense as that for the integral $I(\phi; s)$ above. In fact, the two integrals yield the same unramified calculation, which is the main theorem

Theorem 5.1.2. *Denote by λ any function as in the theorem above. Then*

$$\zeta_p(2s)\sum |t|^{-3}|\ell|^2\lambda(\iota(t, \ell))|t|^s = \lambda(1)L(\pi_p, \mathrm{Spin}, s),$$

where the sum extends over all $t \in \mathbf{Q}_p^\times$, $\ell \in L_p^\times$, *satisfying* $|t| \leq |\ell| \leq 1$, *modulo the action of* $\mathbf{Z}_p^\times \times \mathcal{O}_L^\times$.

To prove this theorem, we use a result of Andrianov [1], who explicitly computes $L(\pi_p, \mathrm{Spin}, s)$ in terms of Hecke operators. Denote by $\Delta^s(g) = |\nu(g)|^s \,\mathrm{char}(g)$ the absolute value of the similitude to the s times the characteristic function of $\mathrm{GSp}_6(\mathbf{Z}_p)$. Let $\omega(g)$ be the spherical function for π_p, normalized so that $\omega(1) = 1$. Then Andrianov proves

$$\int_{\mathrm{GSp}_6(\mathbf{Q}_p)} \omega(g)\Delta^s(g)dg = N(\omega, s)L(\pi_p, \mathrm{Spin}, s-3)$$

where $N(\omega, s) = (N(s) * \omega)(1)$, and

$$\begin{aligned} N(s) &= 1 - p^2(T_{2,3} + (p^4+p^2+1)T_{3,3})p^{-2s} + p^4(1+p)T_{0,3}T_{3,3}p^{-3s} \\ &\quad -p^7T_{3,3}(T_{2,3} + (p^4+p^2+1)T_{3,3})p^{-4s} + p^{15}T_{3,3}^3p^{-6s}. \end{aligned}$$

Here the $T_{i,3}$ are the usual Hecke operators on GSp_6. (See Section 5.4.) Due to the bi-invariance of Δ by $\mathrm{GSp}_6(\mathbf{Z}_p)$, one has the identity

$$\int_{\mathrm{GSp}_6(\mathbf{Q}_p)} \lambda(g)\Delta^s(g)dg = \lambda(1)\int_{\mathrm{GSp}_6(\mathbf{Q}_p)} \omega(g)\Delta^s(g)dg.$$

Thus, to prove the theorem, it suffices to show that

$$\int_{\mathrm{GSp}_6(\mathbf{Q}_p)} \lambda(g)\Delta^s(g)dg = N(s) * \left(\zeta_p(\omega, 2s-6)\sum |t|^{-6}|\ell|^2\lambda(\iota(t,\ell))|t|^s\right).$$

This is what we actually do. We explicitly compute both sides, and see that they are identical.

Note that in other integral representations involving non-unique models, there is an operator N as above, but those N act via the central character, and so are essentially trivial. Here, the operator N is highly nontrivial. This is the "twist" we referred to above. Let us also point out the preprint [23] of Gurevich-Segal which uses this same type of argument.

The outline of the paper is as follows. In Section 5.2, we give the unfolding of the global integrals. In Section 5.3 we explicitly compute the left hand side of the above equality and in Section 5.4 we compute the right-hand side. The ramified integral is controlled in section 5.5.

5.2 Unfolding of global integrals

5.2.1 Groups and embeddings

For our symplectic form, we use the nonstandard matrix

$$\begin{pmatrix} & & & 1 \\ & & 1_2 & \\ & -1_2 & & \\ -1 & & & \end{pmatrix}.$$

We denote by

$$J_4 := \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix}$$

the standard matrix defining the group GSp_4, and embed $\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4$ into GSp_6 in the obvious way:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \mapsto \begin{pmatrix} a & & & b \\ & a' & b' & \\ & c' & d' & \\ c & & & d \end{pmatrix}.$$

Here a', b', c', d' are 2×2 matrices. If W_1, W_2 are subgroups of GL_2, GSp_4, respectively, we denote by $W_1 \boxtimes W_2$ the subgroup of elements (w_1, w_2) of $\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4$ with $w_1 \in W_1, w_2 \in W_2$.

With this choice of form, elements of the Levi of R look as follows:

$$\begin{pmatrix} w & & & \\ & x & & \\ & & y & \\ & & & z \end{pmatrix}, \tag{5.1}$$

with $x = wzy^{-t}$, and where x and y are 2×2 matrices. Elements of the unipotent radical U_R of R have the form

$$\begin{pmatrix} 1 & v & r & * \\ & 1_2 & u & r^t - uv^t \\ & & 1_2 & -v^t \\ & & & 1 \end{pmatrix}$$

with $u = u^t$, and no condition on the element $*$.

We will need several abelian subgroups of U_R. Define

$$N_V = \left\{ n_v \in \mathrm{GSp}_6 \,\middle|\, n_v = \left(\begin{array}{ccc|ccc} 1 & * & * & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & 1 & & * \\ & & & & 1 & * \\ & & & & & 1 \end{array}\right) \right\},$$

$$N_U = \left\{ n_u \in \mathrm{GSp}_6 \,\middle|\, n_u = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & * & * & \\ & & 1 & * & * & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \right\},$$

$$\text{and } N_{U'} = \left\{ n_{u'} \in \mathrm{GSp}_6 \,\middle|\, n_{u'} = \left(\begin{array}{ccc|ccc} 1 & & & * & * & * \\ & 1 & & & & * \\ & & 1 & & & * \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \right\}.$$

Typical elements of these groups are written n_v, n_u, and $n_{u'}$, respectively. If n_v is an element of N_V, we write v for the corresponding 1×2 matrix, and V for the abelian group of 1×2 matrices. Similarly we write u for the 2×2 symmetric matrix corresponding to an element of n_u of N_U, and U for the group of these. Sometimes we write $n(u)$ in place of n_u. If $u = \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{smallmatrix}\right)$, $v = (v_1, v_2)$, and $n \in U_R$ with $n \equiv n_v n_u \pmod{[U_R, U_R]}$, we define $\chi(n) = \psi(v_1 - Du_{11} + u_{22})$.

We describe the embedding of $\mathrm{GL}_{2,L}^*$ into GSp_4. To do this, consider $(L^2, \langle\ ,\ \rangle_L)$, the two-dimensional vector space over L with its standard alternating bilinear form. Define the $\mathbf{Q}$-bilinear form $\langle\ ,\ \rangle$ via $\langle x, y\rangle = \mathrm{tr}_{L/\mathbf{Q}}(\langle x, y\rangle_L)$. As $\mathrm{GL}_{2,L}^*$ is the group preserving $\langle\ ,\ \rangle_L$ up to $\mathbf{Q}^\times$ multiple,

one obtains an inclusion $\mathrm{GL}_{2,L}^* \subseteq \mathrm{GSp}(\langle\ ,\ \rangle)$. If the basis of L^2 is $\{b_1, b_2\}$, then the basis

$$\frac{1}{2}b_1,\ \frac{1}{2\sqrt{D}}b_1,\ b_2,\ \sqrt{D}b_2$$

gives an isomorphism $\mathrm{GSp}(\langle\ ,\ \rangle) \cong \mathrm{GSp}_4$. We use this identification throughout. Alternatively, with this choice of basis, the map that is multiplication by $\sqrt{D}$ is given by the matrix

$$\left(\begin{array}{cc|cc} & 1 & & \\ D & & & \\ \hline & & & D \\ & & 1 & \end{array}\right),$$

and $\mathrm{GL}_{2,L}^*$ is the centralizer of this matrix inside GSp_4. The group H is embedded in GSp_6 via the composite of the embeddings $\mathrm{GL}_{2,L}^* \subset \mathrm{GSp}_4$ and $\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4 \subset \mathrm{GSp}_6$.

The Levi of the upper triangular Borel of $\mathrm{GL}_{2,L}^*$ is isomorphic to $\mathrm{GL}_1 \times \mathrm{GL}_{1,L}$. Suppose $\ell = x + y\sqrt{D} \in L$. Then the embedding of $\mathrm{GL}_{2,L}^*$ into GSp_4 sends

$$\left(\begin{array}{cc|cc} t\ell^{-1} & & & \\ & & & \\ \hline & & & \\ & & & \ell \end{array}\right) \mapsto \left(\begin{array}{cc|cc} tm_\ell^{-t} & & & \\ & & & \\ \hline & & & \\ & & & m_\ell \end{array}\right), \tag{5.2}$$

with $m_\ell = \begin{pmatrix} x & Dy \\ y & x \end{pmatrix}$, and

$$\left(\begin{array}{cc|cc} 1 & & \ell & \\ & & & \\ \hline & & 1 & \\ & & & \end{array}\right) \mapsto \left(\begin{array}{cc|cc} 1 & & n_\ell & \\ & & & \\ \hline & & 1 & \\ & & & \end{array}\right),$$

with $n_\ell = 2\begin{pmatrix} x & Dy \\ Dy & Dx \end{pmatrix}$.

Definition 5.2.1. We define the map $\iota : \mathrm{GL}_1 \times \mathrm{GL}_{1,L} \to \mathrm{GSp}_6$ via

$$(t, \ell) \mapsto \begin{pmatrix} t & & & & & \\ & tm_\ell^{-t} & & & & \\ & & & & m_\ell & \\ & & & & & 1 \end{pmatrix}.$$

Sometimes we will drop the ι from the notation, and just write (t, ℓ) for $\iota(t, \ell)$ if no confusion is

possible. Denote by N_L the unipotent radical of the Borel of $\mathrm{GL}^*_{2,L}$. Note that χ is trivial on the image of N_L in GSp_6 (embedded via $n \mapsto (1,n) \in H \subset \mathrm{GSp}_6$).

We record for future use some modulus characters:

Lemma 5.2.2. *In the notation above, $\delta_R = |\nu|^6 |z|^{-6} |\det(y)|^{-3}$, where ν is the similitude. The modulus character for the Borel of $\mathrm{GL}^*_{2,L}$ is $|t|^3 |\ell|_L^{-2}$. If P_4 denotes the Siegel parabolic on GSp_4, then $\delta_{P_4} = |\nu|^3 |\det(y)|^{-3}$.*

Proof. Left to the reader. □

5.2.2 Results on P_D^α

In this section we give some results related to the special function P_D.

Recall that there is an isomorphism $\mathrm{PGSp}_4 \cong \mathrm{SO}_5$ of split groups over $\mathbf{Q}$. Denote by (V_5, q) the standard 5-dimensional representation of SO_5, together with the quadratic form q defining it, thought of as a representation of GSp_4. One may choose x_0 in $V_5(\mathbf{Q})$ such that $q(x_0) = D$, and such that the stabilizer of x_0 in GSp_4 is $\mathrm{GL}^*_{2,L}$. (See [29].) Let α be a factorizable function in $\mathcal{S}(V_5(\mathbf{A}))$, the space of Schwartz functions on $V_5(\mathbf{A})$, equal to the characteristic function of $V_5(\mathbf{Z}_p)$ almost everywhere. The special automorphic form P_D^α will be

$$P_D^\alpha(g) = \sum_{\delta \in \mathrm{GL}^*_{2,L}(F) \backslash \mathrm{GSp}_4(F)} \alpha(x_0 \delta g).$$

The function P_D is perhaps inherently an orthogonal gadget; one can make an analogous definition for any orthogonal group, and the resulting special function likely has similar properties to this one. However, it is easiest for computation to define P_D directly in the notation of GSp_4, and not go through the exceptional isomorphism $\mathrm{PGSp}_4 \cong \mathrm{SO}_5$. To do this, write $\wedge^2_0(W_4) = \ker\{\wedge^2 W_4 \to \nu\}$, and set $V_5 = \wedge^2_0(W_4) \otimes \nu^{-1}$. (The quadratic form is just the cup product.) Denote $v_D = De_1 \wedge f_2 + e_2 \wedge f_1$ in V_5, and set

$$P_D^\alpha(g) = \sum_{\delta \in \mathrm{GL}^*_{2,L}(F) \backslash \mathrm{GSp}_4(F)} \alpha(v_D \delta g).$$

This definition is sensical because the $\mathrm{GL}^*_{2,L}$ is the stabilizer of v_D in GSp_4.

When p is split in L, we fix two uniformizers π_1, π_2 above p such that $p = \pi_1 \pi_2$, as follows. Set h a fixed squareroot of D inside $\mathbf{Z}_p$. We define π_1 to be the element of $L_p^\times$ that maps to p under the map $L_p \to \mathbf{Q}_p$ determined by $\sqrt{D} \mapsto h$, and that maps to 1 under the map determined

by $\sqrt{D} \mapsto -h$. The element π_2 is defined in the same way, with h and $-h$ reversed. The following lemma will be useful in the sequel.

Lemma 5.2.3. *Suppose p is split in L. Then π_1^k is represented by the matrix*

$$\begin{pmatrix} \frac{p^k+1}{2} & \frac{p^k-1}{2}h \\ \frac{p^k-1}{2}h^{-1} & \frac{p^k+1}{2} \end{pmatrix}$$

and this matrix is right $\mathrm{GL}_2(\mathbf{Z}_p)$ equivalent to

$$\begin{pmatrix} p^k & -h \\ & 1 \end{pmatrix}.$$

Switching h with $-h$ gives the analogous statement for π_2.

Proof. For π_1^k, we must solve the equations $x + hy = p^k$, $x - hy = 1$ for x, y, giving the first part. The second part follows by simple computation. Switching h and $-h$ gives the lemma for π_2. □

Set U_4 the unipotent radical of the Siegel parabolic on GSp_4, and let χ denote by abuse of notation the restriction of χ to U_4. Define

$$\alpha_\chi(g) := \int_{N_L(\mathbf{A})\backslash U_4(\mathbf{A})} \chi(u)\alpha(v_D u g)\, du = \prod_v \alpha_\chi^v(g).$$

The following proposition computes α_χ^v almost everywhere

Proposition 5.2.4. *Denote by T_L the torus of $\mathrm{GL}_{2,L}^*$, and K_M the intersection of the Levi of the Siegel parabolic with $\mathrm{GSp}_4(\mathbf{Z}_p)$. Suppose that p does not divide $2D$, and that α^p is the characteristic function of $V_5(\mathbf{Z}_p)$. If m is in the Levi of the Siegel parabolic, then $\alpha_\chi^p(m) = 0$ if m is not in $T_L K_M$. If $m \in T_L$, then, in notation of (5.2), $\alpha_\chi(m)^v = |t|_F|\ell|^{-1}$ when $|t| \le |\ell|$, and 0 if $|t| > |\ell|$.*

Proof. Suppose

$$n_{24}(z)m = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & z \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} \lambda & & & \\ & \lambda & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} a & b & & \\ c & d & & \\ \hline & & d & -c \\ & & -b & a \end{array}\right).$$

Then $v_D n_{24}(z)m$

$$= \delta^{-1}(cd - Dab)(e_1 \wedge f_1 - e_2 \wedge f_2) + \delta^{-1}(Da^2 - c^2)e_1 \wedge f_2 + \delta^{-1}(d^2 - Db^2)e_2 \wedge f_1 - \frac{z}{\lambda} f_1 \wedge f_2,$$

where $\delta = ad - bc$. To calculate $\alpha_\chi(m)$, by right K invariance we may assume $b = 0$. Then $v_D n_{24}(z)m$

$$= \frac{c}{a}(e_1 \wedge f_1 - e_2 \wedge f_2) + \frac{Da^2 - c^2}{ad} e_1 \wedge f_2 + \frac{d}{a} e_2 \wedge f_1 - \frac{z}{\lambda} f_1 \wedge f_2.$$

Thus for the integral

$$\int_z \psi(z)\alpha(v_D n_{24}(z)m)\, dz$$

to be nonzero, we need $|\lambda| \leq 1$, and the other three coefficients above to be integral. When all these conditions are met, the integral is $|\lambda|$. The conditions for the first three coefficients are exactly the conditions for $m \in T_L K_M$, by Lemma 5.2.3. Since $|\lambda| = |t||\ell|^{-1}$, the proposition follows. □

5.2.3 Unfolding

We define $E^*(g, s) = \zeta(2s)E(g, s)$,

$$E(g, s) = \sum_{\gamma \in B(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbf{Q})} f(\gamma g, s),$$

where $f \in \mathrm{Ind}(\delta_B^s)$, δ_B the modulus character of the Borel, and $f = \prod f_p$ is factorizable. (We are *not* using normalized induction.) At almost all places, f_p is normalized so that $f_p(bk, s) = \delta_p(b)^s$, for b upper triangular in $\mathrm{GL}_2(\mathbf{Q}_p)$ and $k \in \mathrm{GL}_2(\mathbf{Z}_p)$. Set $f_p^*(g, s) = \zeta_p(2s)f_p(g, s)$.

Proposition 5.2.5. $I'(\phi, s) =$

$$\int_{[\mathrm{GL}_2 \boxtimes \mathrm{GSp}(4)]} E^*(g_1, s)\phi(g)P_D(g_2)dg = \int_{N_2(\mathbf{A})U_{P,4}(\mathbf{A})\backslash(\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4)(\mathbf{A})} f^*(g_1, s)\phi_\chi(g)\alpha_\chi(g_2)dg,$$

where N_2 is the unipotent radical of the Borel of GL_2*, $U_{P,4}$ is the unipotent radical of the Siegel parabolic on GSp_4, f, ϕ_χ and α_χ are as above. Applying the Iwasawa decomposition, the local unramified integral becomes*

$$\zeta_p(2s) \int_{\mathrm{GL}_1(\mathbf{Q}_p) \times \mathrm{GL}_1(L_p)} |t|^s (|t|^4 |\ell|^{-3})^{-1} (|t|/|\ell|) \lambda(\iota(t, \ell)) dt d\ell.$$

Proof. Unfolding the Eisenstein series and the function P_D gives

$$\int_{(T_2\boxtimes \mathrm{GL}_{2,L}^*)(F)N_2(\mathbf{A})Z(\mathbf{A})\backslash(\mathrm{GL}_2\boxtimes \mathrm{GSp}_4)(\mathbf{A})} f^*(g_1,s)\phi_{00}(g)\alpha(g_2^{-1}x_0)dg.$$

Here T_2 is the torus of GL_2 and

$$\phi_{00}(g)=\int_{[N_2]}\phi(ng)dn.$$

Note that N_2 is the one-dimensional center of $U_{Q,6}$, the unipotent radical of the standard Klingen parabolic. Define

$$\phi_1(g)=\int_{[N_V N_{U'}/N_2=U_{Q,6}/N_2]}\psi^{-1}(v_1)\phi_{00}(n_v n_{u'}g)dn_v dn_{u'}.$$

Fourier expanding along the abelian group $U_{Q,6}/N_2$ and using the cuspidality of ϕ gives

$$\phi_{00}(g)=\sum_{\gamma\in B'(F)\backslash(T_2\boxtimes \mathrm{GL}_{2,L}^*)(F)}\phi_1(\gamma g),$$

where B' is the group of upper triangular matrices in $\mathrm{GL}_{2,L}$ both of whose diagonal entries are in $F^\times$. We have embedded B' in $T_2\boxtimes \mathrm{GL}_{2,L}^*$ so that, in the notation (5.1), $x=w1_2$ and $y=z1_2$.

We now obtain

$$I'(\phi,s)=\int_{T'(F)N_2(\mathbf{A})N_L(\mathbf{A})Z(\mathbf{A})\backslash(\mathrm{GL}_2\boxtimes \mathrm{GSp}_4)(\mathbf{A})} f^*(g_1,s)\phi_1^{N_L}(g)\alpha(g_2^{-1}x_0)dg,$$

where $T'=B'\cap T_L$ (embedded the same way) and

$$\phi_1^{N_L}(g)=\int_{[N_L]}\phi_1(ng)dn.$$

One may now Fourier expand $\phi_1^{N_L}$ along $U_{P,4}/N_L$ to obtain

$$\phi_1^{N_L}(g)=\sum_{\delta\in Z(F)\backslash T'(F)}\phi_\chi(\delta g),$$

again using the cuspidality of ϕ. Hence

$$I'(\phi,s)=\int_{N_2(\mathbf{A})N_L(\mathbf{A})Z(\mathbf{A})\backslash(\mathrm{GL}_2\boxtimes \mathrm{GSp}_4)(\mathbf{A})} f^*(g_1,s)\phi_\chi(g)\alpha(g_2^{-1}x_0)dg.$$

Integrating over $N_L(\mathbf{A})\backslash U_{P,4}(\mathbf{A})$ gives the proposition. □

Proposition 5.2.6.

$$I(\phi, s) = \int_{H(\mathbf{Q})Z(\mathbf{A})\backslash H(\mathbf{A})} E^*(g_1, s)\phi(g)dg = \int_{N(\mathbf{A})Z(\mathbf{A})\backslash H(\mathbf{A})} f^*(g_1, s)\phi_\chi(g)dg,$$

where $f^*(g_1, s) = \zeta(2s)f(g_1, s)$ is the (normalized) section defining the Eisenstein series, and N is the unipotent radical of the Borel subgroup of H. Applying the Iwasawa decomposition, the local unramified integral becomes

$$\zeta_p(2s) \int_{\mathrm{GL}_1(\mathbf{Q}_p)\times \mathrm{GL}_1(L_p)} |t|^s (|t|^3|\ell|^{-2})^{-1} \lambda(\iota(t, \ell))dtd\ell.$$

Proof. The unfolding is almost identical to, but a bit simpler than, the one above. Or see Gan-Gurevich [13]. □

One turns the local unramified integrals into a sum via the following

Lemma 5.2.7. *If $g \in R$ has Levi part m and $\lambda(g) \neq 0$, then in the notation* 5.1*, z divides the top row of y. If $\lambda(\iota(t, \ell)) \neq 0$, then $|t| \leq |\ell| \leq 1$.*

Proof. Take n_v with $v \in V(\mathbf{Z}_p)$, and suppose $g = mn$, with $m \in M_R$ and $n \in U_R$. Then

$$\lambda(g) = \lambda(mnn_v) = \lambda(mn'n_vn) = \chi(mn'n_vm^{-1})\lambda(mn) = \chi(mn_vm^{-1})\lambda(g),$$

where $n' \in [U_R, U_R]$. Hence $\lambda(g) \neq 0$ implies $\chi(mn_vm^{-1})$ is 1 for all n_v with $v \in V(\mathbf{Z}_p)$. It follows that z divides the top row of y.

For the second claim, note that the part above gives $\ell \in \mathcal{O}_L$, i.e. $|\ell| \leq 1$. Suppose $u \in U(\mathbf{Z}_p)$. Then

$$\lambda(\iota(t, \ell)) = \lambda(\iota(t, \ell)n_u) = \chi(\iota(t, \ell)n_u\iota(t, \ell)^{-1})\lambda(\iota(t, \ell)).$$

As was computed in Proposition 5.2.4,

$$\iota(t, \ell)n_u\iota(t, \ell)^{-1} = n\left(\frac{t}{N(\ell)}u\right) + n'$$

with $n' \in N_L$. Thus $\lambda(\iota(t, \ell)) \neq 0$ implies $|\frac{t}{N(\ell)}| \leq 1$, as desired. □

5.3 Calculation with Δ

In this section and the next, we assume $p \nmid 2D$, and π_p is unramified. Denote by $\mathbf{p}$ the element $p \cdot 1_6$ of the center of GSp_6. When p is split in L, pick π_1, π_2 two uniformizers above p such that $p = \pi_1\pi_2$. We denote by τ the matrix

$$\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ & & p & & & \\ \hline & & & p & & \\ & & & & 1 & \\ & & & & & p \end{array}\right).$$

The purpose of this section is to prove the following two theorems.

Theorem 5.3.1. *Suppose that p is inert in L. Then*

$$\int_{\mathrm{GSp}_6(\mathbf{Q}_p)} \lambda(g)\Delta(g)^s dg = \sum_{r\geq 0} \lambda(\iota(p^r,1))p^{6r-rs} - \sum_{r\geq 2} \lambda(\iota(p^{r-1},1)\tau)p^{6r-6-rs}.$$

Theorem 5.3.2. *Suppose that p is split in L. Then*

$$\begin{aligned}\int_{\mathrm{GSp}_6(\mathbf{Q}_p)} \lambda(g)\Delta(g)^s dg = &\sum_{\substack{a\geq 0\\ r\geq a\\ i\in\{1,2\}}} \lambda(\iota(p^r,\pi_i^a))p^{6r-2a-rs} - \sum_{r\geq 2}\lambda(\iota(p^{r-1},1)\tau)p^{6r-6-rs} \\ &- 2\sum_{r\geq 2}\lambda(\iota(p^{r-2},1)\mathbf{p})p^{6r-8-rs} - \sum_{\substack{a\geq 2\\ r\geq a+1\\ i\in\{1,2\}}} \lambda(\iota(p^{r-2},\pi_1^{a-1})\mathbf{p})p^{6r-2a-6-rs}.\end{aligned}$$

Recall the definitions of R, N_V, N_U, and $N_{U'}$ from Section 5.2.1. Then

$$\begin{aligned}&\int_{\mathrm{GSp}_6(\mathbf{Q}_p)} \lambda(g)\Delta^s(g)dg = \int_{R(\mathbf{Q}_p)} \delta_R^{-1}(g)\lambda(g)\Delta^s(g)dg \\ &= \int_{M_R(\mathbf{Q}_p)} \delta_R^{-1}(g)|\nu(g)|^s\lambda(g) \int_{N_V(\mathbf{Q}_p)} \chi(n_v)\,\mathrm{char}(n_v g) \int_{N_U(\mathbf{Q}_p)} \chi(n_u) \\ &\qquad\qquad \cdot \int_{N_{U'}(\mathbf{Q}_p)} \mathrm{char}(n_{u'}n_u n_v g)\,du'\,du\,dv\,dg. \end{aligned} \tag{5.3}$$

We evaluate these integrals from the inside out.

Definition 5.3.3. Observe that any element of $(\mathrm{GL}_3(\mathbf{Q}_p) \cap M_3(\mathbf{Z}_p)) \,/\, \mathrm{GL}_3(\mathbf{Z}_p)$ may be represented

uniquely by a matrix

$$m = \begin{pmatrix} p^a & \beta & \gamma_1 \\ 0 & p^b & \gamma_2 \\ 0 & 0 & p^c \end{pmatrix}, \beta, \gamma_1 \in [0, p^a - 1] \text{ and } \gamma_2 \in [0, p^b - 1]. \tag{5.4}$$

Such m will arise as representatives of the right $\mathrm{GL}_3(\mathbf{Z}_p)$-equivalence class of the lower right hand corner of $g \in M_R(\mathbf{Q}_p)$.

We need only compute the inner integrals in (5.3) when $\lambda(g) \neq 0$. By Lemma 5.2.7, $\lambda(g) \neq 0$ implies that $a \geq c$ and $p^c | \beta$. We define $\mathbf{M}$ to be the set of matrices m of the form (5.4) meeting these two additional conditions.

For calculations, it will be more useful to also consider a larger, redundant set of representatives. We define

$$\mathbf{M}' = \left\{ \left. \begin{pmatrix} p^a & \beta & \gamma_1 \\ 0 & p^b & \gamma_2 \\ 0 & 0 & p^c \end{pmatrix} \right| \beta, \gamma_1, \gamma_2 \in \mathbf{Z}_p, a \geq c, p^c | \beta \right\}.$$

There is a natural quotient map $\iota : \mathbf{M}' \to \mathbf{M}$ defined by replacing above-diagonal entries with representatives in (5.4).

We will also write y for the minor $\left(\begin{smallmatrix} p^a & \beta \\ 0 & p^b \end{smallmatrix}\right)$ of $m \in \mathbf{M}'$ or $\mathbf{M}$.

We will be able to compute the two outer integrals in (5.3) in terms of a sum over the entries of $\mathbf{M}$.

Definition 5.3.4. For $m \in \mathbf{M}'$, denote by U_m the set of $u \in M_2(\mathbf{Q}_p)$ satisfying

- $u^t = u$
- $uy \in M_2(\mathbf{Z}_p)$
- $(\gamma_1, \gamma_2)uy \in p^c \mathbf{Z}_p \oplus p^{\min(b,c)} \mathbf{Z}_p$.

Observe that U_m is a subgroup of $M_2(\mathbf{Q}_p)$.

Remark 5.3.5. One can check directly that if $m \in \mathbf{M}'$, U_m depends only on its image $\iota(m)$ in $\mathbf{M}$.

Proposition 5.3.6. *Suppose $n_v g \in M_6(\mathbf{Z}_p)$, and suppose the lower right 3×3 block of $n_v g$ is right*

$\mathrm{GL}_3(\mathbf{Z}_p)$-*equivalent to* $m \in \mathbf{M}'$. *Then*

$$\int_{N_{U'}(\mathbf{Q}_p)} \mathrm{char}(n_{u'} n_u n_v g)\, du' = \begin{cases} p^{2c+\min(b,c)} & \text{if } u \in U_m \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Proof. Since the integrand and domain are right $\mathrm{GL}_3(\mathbf{Z}_p)$-invariant, we may assume the lower right 3×3 block of $n_v g$ is exactly m.

A matrix multiplication shows that the integrality of $n_{u'} n_u n_v g$ with $u' = \left(\begin{smallmatrix} u'_1 & u'_2 & u'_3 \\ 0 & 0 & u'_1 \\ 0 & 0 & u'_2 \end{smallmatrix}\right)$ is equivalent to

$$u \cdot y \in M_2(\mathbf{Z}_p), u \cdot \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} + \begin{pmatrix} u'_1 p^c \\ u'_2 p^c \end{pmatrix} \in \mathbf{Z}_p^2, (u'_1, u'_2) \cdot y \in \mathbf{Z}_p^2, \text{ and } \gamma_1 u'_1 + \gamma_2 u'_2 + p^c u'_3 \in \mathbf{Z}_p. \tag{5.6}$$

Since u'_3 can always be chosen to satisfy the last condition, the existence of an element u' making $\mathrm{char}(n_{u'} n_u n_v g) = 1$ is equivalent to $u \cdot y \in M_2(\mathbf{Z}_p)$ together with the existence of $u'' = (u'_1, u'_2)$ with $p^c \cdot {}^t u'' + u \cdot \left(\begin{smallmatrix} \gamma_1 \\ \gamma_2 \end{smallmatrix}\right)$ and $u''y$ integral. The condition on uy is independent of u', and we can rewrite the other two conditions as follows.

Replacing the conditions with their transposes, we need $p^c u'' + (\gamma_1, \gamma_2)u$ and $u''y$ integral. Replacing u'' with $u''' = u''y$, the condition is now $u''' \in \mathbf{Z}_p^2$ and $p^c u''' y^{-1} + (\gamma_1, \gamma_2)u$ integral. The latter expression can be rewritten as $p^c u''' + (\gamma_1, \gamma_2)uy \in \mathbf{Z}_p^2 y$, so the two conditions can be replaced by $(\gamma_1, \gamma_2)uy \in p^c \mathbf{Z}_p^2 + \mathbf{Z}_p^2 y$. Using the explicit form of y and Lemma 5.2.7, one rewrites this as $(\gamma_1, \gamma_2)uy \in p^c \mathbf{Z}_p \oplus p^{\min(b,c)} \mathbf{Z}_p$, which now depends only on u.

We have now shown that if $\int_{N_{U'}(\mathbf{Q}_p)} \mathrm{char}(n_{u'} n_u n_v g)\, du' \neq 0$, we must have $u \in U_m$. Assuming that u is in U_m, we now determine the measure of the $u' \in U'$ with $\mathrm{char}(n_{u'} n_u n_v g) = 1$. There exists one such u', call it u'_{st}, by the calculation above. Let $u'_{\mathrm{sh}} = u' - u'_{\mathrm{st}} = \left(\begin{smallmatrix} u'_{\mathrm{sh},1} & u'_{\mathrm{sh},2} & u'_{\mathrm{sh},3} \\ 0 & 0 & u'_{\mathrm{sh},1} \\ 0 & 0 & u'_{\mathrm{sh},2} \end{smallmatrix}\right)$. Using (5.6), we need $(u'_{\mathrm{sh},1}, u'_{\mathrm{sh},2})y \in \mathbf{Z}_p^2$, $p^c \cdot \left(\begin{smallmatrix} u'_{\mathrm{sh},1} \\ u'_{\mathrm{sh},2} \end{smallmatrix}\right) \in \mathbf{Z}_p^2$, and $\gamma_1 u'_{\mathrm{sh},1} + \gamma_2 u'_{\mathrm{sh},2} + p^c u'_{\mathrm{sh},3} \in \mathbf{Z}_p$. For any fixed $u'_{\mathrm{sh},1}$ and $u'_{\mathrm{sh},2}$ meeting the first two conditions, the measure of $u'_{\mathrm{sh},3}$ meeting the third is p^c. The first two conditions are satisfied when $p^c u'_{\mathrm{sh},1}$, $p^c u'_{\mathrm{sh},2}$, $p^a u'_{\mathrm{sh},1}$, and $\beta u'_{\mathrm{sh},1} + p^b u'_{\mathrm{sh},2}$ are integral. Since $a \geq c$, we see that $u'_{\mathrm{sh},1}$ can be any element of $p^{-c}\mathbf{Z}_p$, which (using $p^c | \beta$) gives $u'_{\mathrm{sh},2} \in p^{-\min(b,c)} \mathbf{Z}_p$. So the measure of possible u'_{sh} is $p^{2c+\min(b,c)}$, giving (5.5) by left-invariance of the Haar measure du'.

□

For each $m \in \mathbf{M}$, we must calculate

$$\int_{N_U} \chi(n_u)\,\mathrm{char}_{U_m}(u)\,du.$$

As U_m is a group, this is nonzero only when $U_m \subseteq \ker\chi$.

Definition 5.3.7. We say that an element $m' \in M_3(\mathbf{Z}_p)$ is called *admissible* if it is right $\mathrm{GL}_3(\mathbf{Z}_p)$-equivalent to an element $m \in \mathbf{M}$ with $U_m \subseteq \ker\chi$. Define the *cumulative measure* $B(m')$ associated to an admissible m' to be $p^{2c+\min(b,c)}meas(U_m)$, and define $B(m') = 0$ for any $m' \in M_3(\mathbf{Z}_p)$ that is not admissible.

For the purpose of evaluating the iterated integral (5.3), we may replace the two innermost integrals by the cumulative measure $B(m')$. Most of the effort in this section is devoted to finding the admissible $m \in \mathbf{M}$. The answer depends on whether p is split or inert in L. We will first prove some claims that apply in both cases. For convenience, we will use the larger set $\mathbf{M}'$.

The proofs of these claims will follow from careful selection of a matrix u to check that if certain conditions do not hold, we have $U_m \not\subseteq \ker\chi$. It will be useful to write out the matrices in the defining conditions for U_m explicitly: if $u = \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{smallmatrix}\right)$, then

$$uy = \begin{pmatrix} u_{11}p^a & u_{11}\beta + u_{12}p^b \\ u_{12}p^a & u_{12}\beta + u_{22}p^b \end{pmatrix} \tag{5.7}$$

$$\text{and } (\gamma_1,\gamma_2)uy = (u_{11}p^a\gamma_1 + u_{12}p^a\gamma_2, (u_{11}\beta + u_{12}p^b)\gamma_1 + (u_{12}\beta + u_{22}p^b)\gamma_2).$$

Claim 5.3.8. *If $m \in \mathbf{M}'$ is admissible, then $c \geq b$.*

Proof. If $b > c$, then $\left(\begin{smallmatrix} 0 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right) \in U_m$, so $U_m \not\subseteq \ker\chi$. □

Claim 5.3.9. *If $m \in \mathbf{M}'$ is admissible, then one of p^a, β, or γ_1 is not divisible by p. Similarly, if m is allowable, one of p^b or γ_2 is not divisible by p.*

Proof. For the first case, we observe that if $a \geq 1$, $p|\beta$, and $p|\gamma_1$, then by the conditions $c \leq a$ and $p^c|\beta$ together with Claim 5.3.8, we have $p^c|p^{a-1}\gamma_1$ and $p^b|p^{-1}\beta\gamma_1$, so $\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right) \in U_m$ and thus $U_m \not\subseteq \ker\chi$.

For the second case, we similarly find $\left(\begin{smallmatrix} 0 & 0 \\ 0 & p^{-1} \end{smallmatrix}\right) \in U_m$, so $U_m \not\subseteq \ker\chi$. □

Claim 5.3.10. *If $m \in \mathbf{M}'$ is admissible, then we have either $a = c$ or $b = c$.*

Proof. Assume that $a > c > b$. Since $a \geq c+1 \geq 1$, $p|\beta$, and $c \geq b+1$, it follows that $\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right) \in U_m$. So either $a = c$ or $c = b$. □

We may now state the classification of the allowable $m \in \mathbf{M}$ and corresponding measures $B(m)$.

Remark 5.3.11. Due to the right $\mathrm{GL}_3(\mathbf{Z}_p)$-action, for an element $m \in \mathbf{M}$, the matrix entries β and γ_1 of m can be thought of as elements of $\mathbf{Z}/p^a\mathbf{Z}$, and the entry γ_2 of m can be thought of as an element of $\mathbf{Z}/p^b\mathbf{Z}$. It makes sense, therefore, to write $\beta^{-1}, \gamma_1^{-1}$, and γ_2^{-1} for the inverses in these rings, taken to lie in $[0, p^a - 1]$ or $[0, p^b - 1]$, whenever these entries are invertible. We will use this notation in the following classification result.

Theorem 5.3.12 (Inert case)**.** *Suppose p inert, and $m \in \mathbf{M}$ is admissible. Then m is of the form*

$$\begin{pmatrix} p^a & 0 & \gamma_1 \\ & 1 & 0 \\ & & p^a \end{pmatrix},$$

where $p \nmid \gamma_1$ if $a \geq 1$. The cumulative measure for such an m is $B(m) = p^{2a}$.

Theorem 5.3.13 (Split case)**.** *Suppose p is split. There are several cases:*

- *($b = 0, c = 0$.) The admissible $m \in \mathbf{M}$ are then of the form*

$$\begin{pmatrix} p^a & \beta & \gamma_1 \\ & 1 & 0 \\ & & 1 \end{pmatrix},$$

  where $\beta^2 \equiv D \pmod{p^a}$. The cumulative measure for such an m is $B(m) = p^a$.

- *($b = 0, a = c, a \geq 1$.) The admissible $m \in \mathbf{M}$ are then of the form*

$$\begin{pmatrix} p^a & 0 & \gamma_1 \\ & 1 & 0 \\ & & p^a \end{pmatrix},$$

  where $p \nmid \gamma_1$. The cumulative measure for such an m is $B(m) = p^{2a}$.

- *($a = c \geq b \geq 1$.) The admissible $m \in \mathbf{M}$ are then of the form*

$$\begin{pmatrix} p^a & 0 & \gamma_1 \\ & p^b & \gamma_2 \\ & & p^a \end{pmatrix},$$

where $(\gamma_1, p) = (\gamma_2, p) = 1$, and $(\gamma_1 \cdot \gamma_2^{-1})^2 \equiv D \pmod{p^b}$. The cumulative measure for such an m is $B(m) = p^{2a+2b}$.

- *($a > c = b \geq 1$.) Since in this case $p^c | \beta$, we may define $\beta' = p^{-c}\beta$. The admissible $m \in \mathbf{M}$ are then of the form*

$$\begin{pmatrix} p^a & \beta & \gamma_1 \\ & p^c & \gamma_2 \\ & & p^c \end{pmatrix},$$

where $p^c | \beta$, $(\gamma_1, p) = (\gamma_2, p) = 1$, $(\gamma_1 \cdot \gamma_2^{-1})^2 \equiv D \pmod{p^c}$, $(\beta')^2 \equiv D \pmod{p^{a-c}}$, and $\beta' + (\gamma_1 \cdot \gamma_2^{-1}) \equiv 0 \pmod{p^{\min(a-c,c)}}$. The cumulative measure for such an m is $B(m) = p^{a+3c}$.

Remark 5.3.14. The cumulative measure is p^{a+2b+c} in all cases.

Remark 5.3.15. These conditions are all invariant under adding a multiple of p^c to either γ_1 or γ_2.

We now prove a succession of claims that will establish Theorems 5.3.12 and 5.3.13.

Claim 5.3.16. *Suppose that $m \in \mathbf{M}'$ has $b \geq 1$. If p is inert, m is not admissible. If p is split and m is admissible, then $(\gamma_1 \cdot \gamma_2^{-1})^2 \equiv D \pmod{p^b}$.*

Proof. If $b \geq 1$, then by Claim 5.3.8, $a \geq c \geq 1$, so by Claim 5.3.9 and Lemma 5.2.7, p cannot divide either γ_1 or γ_2, so $\gamma = \gamma_1 \cdot \gamma_2^{-1}$ is in $\mathbf{Z}_p^\times$. Observe that $u_0 = p^{-b} \left(\begin{smallmatrix} -\gamma^{-1} & 1 \\ 1 & -\gamma \end{smallmatrix} \right) \in U_m$. Moreover, $\chi(u_0) = \psi(-p^{-b}\gamma^{-1}(\gamma^2 - D))$. To have $u_0 \in \ker \chi$, we need $\gamma^2 \equiv D \pmod{p^b}$. In the inert case, this is not possible. □

We will maintain this notation: for $m \in \mathbf{M}'$, if $b \geq 1$, we set $\gamma = \gamma_1 \cdot \gamma_2^{-1} \in \mathbf{Z}_p^\times$.

Claim 5.3.17. *For $m \in \mathbf{M}'$, suppose that $b \geq 1$, p splits in L, and $c = a$. Then $(\gamma_1 \cdot \gamma_2^{-1})^2 \equiv D \pmod{p^b}$ is a necessary and sufficient condition for m to be admissible and, in this case, we have $B(m) = p^{2c+2b}$.*

Proof. Let $u = \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{smallmatrix} \right)$, and suppose that $u \in U_m$. By the integrality condition on uy, we have $p^a u_{11} = p^c u_{11}$ integral, so βu_{11} is also integral, and (looking at the upper-right entry of uy) $p^b u_{12}$

is integral. So since $c \geq b$, βu_{12} is integral, and (looking at the bottom-right entry of uy) $p^b u_{22}$ is integral. By the condition on $(\gamma_1, \gamma_2)uy$ and the assumption $a = c$, we need $u_{11}p^a\gamma_1 + u_{12}p^a\gamma_2 \in p^a\mathbf{Z}_p$, or $u_{11}\gamma + u_{12} \in \mathbf{Z}_p$. In particular, since $p^b u_{12}$ is integral, $p^b u_{11}$ is also integral. Therefore, $p^b u$ is integral.

We also need $(u_{11}\beta + u_{12}p^b)\gamma_1 + (u_{12}\beta + u_{22}p^b)\gamma_2 \in p^b\mathbf{Z}_p$, or $(u_{11}\gamma + u_{12})\beta + (u_{12}\gamma + u_{22})p^b \in p^b\mathbf{Z}_p$. By the preceding calculation, this reduces to $u_{12}\gamma + u_{22} \in \mathbf{Z}_p$. By setting $u_{12} = p^{-b}\alpha$ and solving for the other entries, we deduce that up to addition of a diagonal integral matrix, u has the form $p^{-b}\alpha \left(\begin{smallmatrix} -\gamma^{-1} & 1 \\ 1 & -\gamma \end{smallmatrix}\right)$ for $\alpha \in \mathbf{Z}_p$. We checked in Claim 5.3 that $\gamma^2 \equiv D \pmod{p^b}$ is a necessary condition. When γ satisfies this congruence, any u of the form just calculated has $\chi(u) = 1$, so m is admissible under this condition.

Once the choice of $u_{12} \in p^{-b}\mathbf{Z}_p$ has been made, the choices of u_{11} and u_{22} have measure 1. So using Proposition 5.3.6, we obtain a cumulative measure of p^{2c+2b}.

□

For $m \in \mathbf{M}'$, we set $\beta_0 = p^{-c}\beta \in \mathbf{Z}_p$.

Claim 5.3.18. *For $m \in \mathbf{M}'$, assume that $a > c = b \geq 1$ and that p is split in L. Then if m is admissible, we must have $(\beta_0, p) = 1$ and $\beta_0^2 \equiv D \pmod{p^{a-b}}$.*

Proof. If $p|\beta_0$, we observe that $\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right) \in U_m$, so we must have $(\beta_0, p) = 1$. For the second claim, observe that $u = p^{b-a}\left(\begin{smallmatrix} 1 & -\beta_0 \\ -\beta_0 & \beta_0^2 \end{smallmatrix}\right) \in U_m$ and $\chi(u) = \psi(\frac{\beta_0^2 - D}{p^{a-b}})$. If m is admissible, we must have $\beta_0^2 \equiv D \pmod{p^{a-b}}$. □

Claim 5.3.19. *For $m \in \mathbf{M}'$, assume that $a > c = b \geq 1$ and that p is split in L. Then if m is admissible, we must have $-\beta_0(\gamma_1 \cdot \gamma_2^{-1}) \equiv D \pmod{p^{\min(a-b,b)}}$.*

Proof. Define $b' = \min(a - b, b)$. Observe that $u = p^{-b'}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -\beta_0\gamma \end{smallmatrix}\right) \in U_m$. We must have $\chi(u) = \psi(p^{-b'}(-D - \beta_0\gamma)) = 1$, so $-\beta_0(\gamma_1 \cdot \gamma_2^{-1}) \equiv D \pmod{p^{\min(a-b,b)}}$. □

Claim 5.3.20. *For $m \in \mathbf{M}'$, assume that $a > c = b \geq 1$ and that p is split in L. If*

$$(\beta_0, p) = 1, \beta_0^2 \equiv D \pmod{p^{a-b}}, \gamma^2 \equiv D \pmod{p^c}, \text{ and } -\beta_0(\gamma_1 \cdot \gamma_2^{-1}) \equiv D \pmod{p^{\min(a-b,b)}},$$

then m is admissible and $B(m) = p^{a+3c}$.

Proof. If we add the congruences $\beta_0^2 \equiv D \pmod{p^{a-b}}$ and $-\beta_0(\gamma_1 \cdot \gamma_2^{-1}) \equiv D \pmod{p^{\min(a-b,b)}}$, we obtain $\beta_0(\beta_0 - \gamma) \equiv 2D \pmod{p^{\min(b,a-b)}}$. Since β_0 and $2D$ are both p-adic units (the latter by

hypothesis) and $\min(a-b,b) > 1$, we deduce that $\beta_0 - \gamma$ is also a p-adic unit. We can also subtract the congruences to similarly obtain $\beta_0 + \gamma \equiv 0 \pmod{p^{\min(a,b)}}$.

From here on, we will handle the cases $a - b \geq b$ and $b \geq a - b$ separately.

Case $a - b \geq b$: We may write any symmetric 2×2 matrix in the form

$$u = \begin{pmatrix} u_{11} & -\beta_0 u_{11} + u'_{12} \\ -\beta_0 u_{11} + u'_{12} & \beta_0^2 u_{11} - (\beta_0 + \gamma)u'_{12} + u'_{22} \end{pmatrix}.$$

The integrality conditions become

$$uy = \begin{pmatrix} p^a u_{11} & p^b u'_{12} \\ -p^a \beta_0 u_{11} + p^a u'_{12} & -\gamma p^b u'_{12} + p^b u'_{22} \end{pmatrix} \in M_2(\mathbf{Z}_p) \text{ and } \gamma_2(p^a(\gamma - \beta_0)u_{11} + p^a u'_{12}, p^b u'_{22}) \in p^b \mathbf{Z}_p^2.$$

These conditions imply $u'_{12} \in p^{-b}\mathbf{Z}_p$ and $u'_{22} \in \mathbf{Z}_p$. The inequality $a - b \geq b$ implies $p^a u'_{12} \in p^b \mathbf{Z}_p$. It follows from the second integrality condition that $(\gamma - \beta_0)p^a u_{11} \in p^b \mathbf{Z}_p$ as well. We showed above that $\gamma - \beta_0 \in \mathbf{Z}_p^\times$, so $p^a u_{11} \in p^b \mathbf{Z}_p$.

Conversely, if $u_{11} \in p^{b-a}\mathbf{Z}_p$, $u'_{12} \in p^{-b}\mathbf{Z}_p$, and $u'_{22} \in \mathbf{Z}_p$, the integrality conditions hold. Moreover, we have

$$\chi(u) = \psi(-Du_{11} + \beta_0^2 u_{11} - (\beta_0 + \gamma)u'_{12} + u'_{22}) = \psi((\beta_0^2 - D)u_{11} - (\beta_0 + \gamma)u'_{12}) = 1$$

since $\beta_0^2 - D \in p^{a-b}\mathbf{Z}_p$ and $\beta_0 + \gamma \in p^b \mathbf{Z}_p$. It follows that the cumulative measure is $p^{a-b+b} \cdot p^{2c+\min(b,c)} = p^{a+3c}$.

Case $b \geq a - b$: We may write any symmetric 2×2 matrix in the form

$$u = \begin{pmatrix} u_{11} & -\gamma u_{11} + u'_{12} \\ -\gamma u_{11} + u'_{12} & \gamma^2 u_{11} - (\beta_0 + \gamma)u'_{12} + u'_{22} \end{pmatrix}.$$

The integrality conditions become

$$uy = \begin{pmatrix} p^a u_{11} & p^b(\beta_0 - \gamma)u_{11} + p^b u'_{12} \\ -p^a \gamma u_{11} + p^a u'_{12} & p^b \gamma(\gamma - \beta_0)u_{11} - p^b \gamma u'_{12} + p^b u'_{22} \end{pmatrix} \in M_2(\mathbf{Z}_p)$$
$$\text{and } \gamma_2(p^a u'_{12}, p^b u'_{22}) \in p^b \mathbf{Z}_p^2.$$

For these to hold, we must have $u'_{22} \in \mathbf{Z}_p$ and $u'_{12} \in p^{b-a}\mathbf{Z}_p$. In particular, since $b \geq a - b$,

$p^b u'_{12} \in \mathbf{Z}_p$. Since $\gamma - \beta_0 \in \mathbf{Z}_p^\times$, we obtain $p^b u_{11} \in \mathbf{Z}_p$ from the condition on $p^b(\beta_0 - \gamma)u_{11} + p^b u'_{12}$. Conversely, if $u'_{22} \in \mathbf{Z}_p$, $u'_{12} \in p^{b-a}\mathbf{Z}_p$, and $p^b u_{11} \in \mathbf{Z}_p$, the integrality conditions are met.

We calculate that under these conditions,

$$\chi(u) = \psi((\gamma^2 - D)u_{11} - (\gamma + \beta_0)u'_{12} + u'_{22}) = 1$$

since $u'_{22} \in \mathbf{Z}_p$, $p^{a-b}|\gamma + \beta_0$, and $p^b|\gamma^2 - D$. It follows that the cumulative measure is $p^{a-b+b} \cdot p^{2c+\min(b,c)} = p^{a+3c}$.

□

This finishes the cases with $b \geq 1$.

Claim 5.3.21. *Assume that $b = c = 0$ and that $m \in \mathbf{M}'$ is admissible. If p is inert, then $U_m \subseteq \ker \chi$ implies $a = 0$ (so $a = c$). If p is split, $U_m \subseteq \ker \chi$ implies that $\beta^2 \equiv D \pmod{p^a}$. These are sufficient conditions too, and we have $B(m) = p^a$ in both cases.*

Proof. First observe that if $b = c = 0$, the conditions $uy \in \mathbf{M}_2(\mathbf{Z}_p)$ and $u^t = u$ are sufficient to verify that $u \in U_m$. Observe that $p^{-a}\left(\begin{smallmatrix} 1 & -\beta \\ -\beta & \beta^2 \end{smallmatrix}\right) \in U_m$. For m to be admissible, we need $\chi(u) = \psi(p^{-a}(\beta^2 - D)) = 1$ or $\beta^2 \equiv D \pmod{p^a}$. If p is inert, this forces $a = 0$ as claimed, and the calculation of $B(m)$ is clear.

So assume that p is split and $\beta^2 \equiv D \pmod{p^a}$. For $u = \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{smallmatrix}\right)$, we have $uy = \left(\begin{smallmatrix} p^a u_{11} & \beta u_{11} + u_{12} \\ p^a u_{12} & \beta u_{12} + u_{22} \end{smallmatrix}\right)$. The condition $u \in U_m$ is equivalent to $u_{11} \in p^{-a}\mathbf{Z}_p$, $u_{12} = -\beta u_{11} + u'_{12}$ for some $u'_{12} \in \mathbf{Z}_p$, and $u_{22} = -\beta u_{12} + u'_{22}$ for some $u'_{22} \in \mathbf{Z}_p$. For u of this form,

$$\chi(u) = \psi(-Du_{11} + u_{12}) = \psi(u_{11}(\beta^2 - D)) = 1.$$

The choice of u_{11} gives a measure of p^a, the elements u'_{12} and u'_{22} are integral, and $b = c = 0$, so the cumulative measure is $B(m) = p^a$. □

Claim 5.3.22. *Assume that $b = 0$ and $a = c > 0$. Then $m \in \mathbf{M}'$ is admissible if and only if $p^a|\beta$ and $(p, \gamma_1) = 1$. The cumulative measure in this case is $B(m) = p^{2a}$.*

Proof. If m is admissible, then $p^a|\beta$ by definition of $\mathbf{M}'$ and $(pm\gamma_1) = 1$ by Claim 5.3.9. Now suppose that these conditions hold and that $u = \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{smallmatrix}\right) \in U_m$. Using $p^a|\beta$, the condition $uy \in M_2(\mathbf{Z}_p)$ simplifies to $u_{11} \in p^{-a}\mathbf{Z}_p$, $u_{12} \in \mathbf{Z}_p$, and $u_{22} \in \mathbf{Z}_p$. Using $u_{12} \in \mathbf{Z}_p$, the condition on $(\gamma_1, \gamma_2)uy$ simplifies to $\gamma_1 p^a u_{11} \in p^a \mathbf{Z}_p$. Since $(p, \gamma_1) = 1$, we require $u_{11} \in \mathbf{Z}_p$. It is clear that $U_m \subseteq \ker \chi$ and that the cumulative measure is $B(m) = p^{2a}$. □

The element $g \in M_R$ is determined by its lower right hand 3×3 minor and upper left hand entry w. By right K-invariance, to calculate (5.3), we may assume that $w = p^r$ for some $r \in \mathbf{Z}_{\geq 0}$. For $m \in \mathbf{M}'$ and $r \in \mathbf{Z}_{\geq 0}$, write $g_{m,r}$ for the element of M_R determined by m and r. We must next evaluate

$$\int_{V(\mathbf{Q}_p)} \chi(v) \operatorname{char}_{M_6(\mathbf{Z}_p)}(n_v g_{m,r}) B(m)\, dv.$$

Observe that integrality of $n_v g_{m,r}$ is equivalent to integrality of vx and vz. If $r \geq a$ and m is admissible, then wy^{-t} is integral. Hence, as $vx = vzwy^{-t}$, vz integral implies vx integral for admissible m. Thus, calculating the integral above is simply a sum over $\mathbf{M}$ when $r \geq a$. In fact, the integral is 0 when $r < a$.

Proposition 5.3.23. *Suppose that $m \in \mathbf{M}$ is admissible. If $r < a$ then*

$$\int_{V(\mathbf{Q}_p)} \chi(v) \operatorname{char}_{M_6(\mathbf{Z}_p)}(n_v g_{m,r}) B(m)\, dv = 0.$$

Proof. Using the notation in (5.4), we calculate

$$vx = vzwy^{-t} = (p^{-c}\gamma_1, p^{-c}\gamma_2) \begin{pmatrix} p^{r+c-a} & \\ -\beta p^{r+c-a-b} & p^{r+c-b} \end{pmatrix} = (p^{r-a}\gamma_1 - p^{r-a-b}\beta\gamma_2, p^{r-b}\gamma_2).$$

In the inert case, we have $\beta = 0$, so we require $p^{r-a}\gamma_1 \in \mathbf{Z}_p$. However, if $a = 0$ then $r < a$ is impossible, while if $a \geq 1$, $p \nmid \gamma_1$, so $p^{r-a}\gamma_1$ can never be integral if $r < a$.

We now handle the split cases; note that we combine the second and third cases of Theorem 5.3.13 below. We write $v = (v_1, v_2)$ below.

- $(b = 0, c = 0.)$ We have $\gamma_2 = 0$, so the integrality condition simplifies to $p^{r-a}\gamma_1 \in \mathbf{Z}_p$. Since $c = 0$, we have $v = (\gamma_1, \gamma_2)$, so $\chi(v) \operatorname{char}_{M_6(\mathbf{Z}_p)}(n_v g_{m,r}) B(m) = \psi(v_1) B(m)$ on the domain where $v_1 \in p^{a-r}\mathbf{Z}_p$ and $v_2 \in \mathbf{Z}_p$. The desired vanishing follows.

- $(a = c \geq b.)$ The argument is identical to the one in the inert case.

- $(a > c = b \geq 1.)$ Note that the conditions $p \nmid \gamma_2$ and $p^{r-b}\gamma_2 \in \mathbf{Z}_p$ force $b \leq r < a$. Write $\beta' = p^{-c}\beta$. We require $p \nmid \gamma_1$ and $b = c$, so if $b \leq r < a$, the integrality condition reduces to $p^{r-a}(\gamma_1 - \beta_0\gamma_2) \in \mathbf{Z}_p$. Since γ_2 is a unit, this is equivalent to $\gamma - \beta_0 \in p^{a-r}\mathbf{Z}_p$. Finally, using the condition $p \nmid 2D$, we showed in the proof of Claim 5.3.20 that $\gamma - \beta_0$ is a unit, a contradiction if $r < a$.

□

Calculating the integrals in the case $r \geq a$ gives Theorems 5.3.1 and 5.3.2. In the split case, one uses Lemma 5.2.3 to rewrite the result in the desired form.

5.4 Calculation with N

Here we give the calculation of $N(s) * \zeta_p(2s-6) \sum |t|^{s-6}|l|^2 \lambda(\iota(t,l))$.

5.4.1 Hecke operators

In the definition of N above, $T_{0,3}$ is the Hecke operator that is the characteristic function of

$$GSp_6(\mathbf{Z}_p) \left(\begin{array}{c|ccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ \hline & & & p \\ & & & & p \\ & & & & & p \end{array} \right) GSp_6(\mathbf{Z}_p),$$

$T_{3,3}$ is the Hecke operator that is the characteristic function of

$$GSp_6(\mathbf{Z}_p) \left(\begin{array}{c|ccc} p & & & \\ & p & & \\ & & p & \\ \hline & & & p \\ & & & & p \\ & & & & & p \end{array} \right) GSp_6(\mathbf{Z}_p),$$

and $T_{2,3}$ is the Hecke operator that is the characteristic function of

$$GSp_6(\mathbf{Z}_p) \left(\begin{array}{c|ccc} 1 & & & \\ & p & & \\ & & p & \\ \hline & & & p \\ & & & & p \\ & & & & & p^2 \end{array} \right) GSp_6(\mathbf{Z}_p).$$

We first reduce these Hecke operators to GL_3 Hecke operators. To this end, suppose m is an element of $(\mathrm{GL}_3(\mathbf{Q}_p) \cap M_3(\mathbf{Z}_p))$, such that $pm^{-1} \in M_3(\mathbf{Z}_p)$. Consider the double coset

$$\mathrm{GL}_3(\mathbf{Z}_p) m \, \mathrm{GL}_3(\mathbf{Z}_p),$$

and suppose it has a coset decomposition

$$\coprod_\beta v_\beta \, \mathrm{GL}_3(\mathbf{Z}_p).$$

We use the shorthand $\lambda(g[m]_p)$ to mean

$$\sum_\beta \lambda(g \tilde{v}_\beta),$$

where $\tilde{v}_\beta$ is the element of the Levi of the Siegel parabolic whose top left 3×3 block is v_β, and which has similitude p. Write M_P for the Levi of the Siegel parabolic. The first result is

Proposition 5.4.1. *Suppose $g \in M_P$, $g = \iota(t, \ell)$, with $|t| \leq |\ell| \leq 1$. Then $T_{0,3}\lambda(g) =$*

$$\lambda\left(g \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}_p\right) + p\lambda\left(g \begin{bmatrix} p & & \\ & 1 & \\ & & 1 \end{bmatrix}_p\right) + p^3\lambda\left(g \begin{bmatrix} p & & \\ & p & \\ & & 1 \end{bmatrix}_p\right) + p^6\lambda\left(g \begin{bmatrix} p & & \\ & p & \\ & & p \end{bmatrix}_p\right).$$

Proof. The decomposition into cosets of Hecke operators for the symplectic group is standard; see, for example, Andrianov [3, Lemma 3.49]. However, the proof of the proposition is facilitated by a good choice of coset representatives, and its truth depends on the hypothesis $g = \iota(t, \ell)$, with $|t| \leq |\ell| \leq 1$, so we briefly explain.

For an element $m \in M_P$ define

$$U(m) := \{u \in U_P(\mathbf{Q}_p) : mu \in M_6(\mathbf{Z}_p)\}.$$

Also, for ease of notation, denote by X_0 the set of representatives appearing in the sum for the

operator $[1_3]_p$. Similarly, denote by X_1, X_2, X_3 the sets of representatives appearing in the operators

$$\begin{bmatrix} p & & \\ & 1 & \\ & & 1 \end{bmatrix}_p, \quad \begin{bmatrix} p & & \\ & p & \\ & & 1 \end{bmatrix}_p, \quad \text{and} \quad \begin{bmatrix} p & & \\ & p & \\ & & p \end{bmatrix}_p,$$

respectively.

It follows from the Lemma in Andrianov above that for general g one has

$$T_{0,3}\lambda(g) = \sum_{0 \le i \le 3} \sum_{x \in X_i} \sum_{\substack{u \in U(x) \\ \bmod U_P(\mathbf{Z}_p)}} \lambda(gxu).$$

Now,

$$\lambda(gxu) = \chi(gxux^{-1}g^{-1})\lambda(gx).$$

One may check that for the x appearing above, representatives $u \in U(x)/U_P(\mathbf{Z}_p)$ may be chosen so that xux^{-1} is still integral, i.e. in $M_6(\mathbf{Z}_p)$. (This is special to the similitude p case.) Now, if $g = \iota(t, \ell)$, with $|t| \le |\ell| \le 1$, and $n \in U_R(\mathbf{Z}_p)$, then $\chi(gng^{-1}) = 1$. Hence, to compute $T_{0,3}\lambda(g)$, one must just count the sizes of the sets $U(x)/U_P(\mathbf{Z}_p)$. These sizes are $1, p, p^3, p^6$ if x is in X_0, X_1, X_2, X_3, respectively. The proposition follows. □

Set $\epsilon_L(p)$ to be 1 if p is split in L, and -1 if p is inert. Denote by ω the central character of π. Then

Proposition 5.4.2. *Suppose $g = \iota(t, \ell)$, with $|t| \le |\ell| \le 1$. Then $T_{2,3}\lambda(g) =$*

$$\lambda\left(g\left(\begin{array}{c|c} 1_3 & \\ \hline & p1_3 \end{array}\right)\begin{bmatrix} 1 & & \\ & p & \\ & & p \end{bmatrix}_p\right) + p^4\lambda\left(g\left(\begin{array}{c|c} p1_3 & \\ \hline & 1_3 \end{array}\right)\begin{bmatrix} p & & \\ & 1 & \\ & & 1 \end{bmatrix}_p\right)$$

$$+\omega(p)(p^3 - 1)\lambda(\iota(t, \ell)) + \delta_{|t|,|\ell|}\omega(p)(-p^3 + \epsilon_L(p)p^2).$$

Here $\delta_{|t|,|\ell|}$ is 1 when $|t| = |\ell|$ and is 0 otherwise.

Proof. This is similar to the above; again see Andrianov [3, Lemma 3.49]. We briefly explain.

Set $U(p)'$ to be the elements u of $U(p1_6)$ such that $(p1_6)u$ has rank one over $\mathbf{F}_p$. Then one has

$T_{2,3}\lambda(g)$

$$= \lambda\left(g\left(\begin{array}{c|c} 1_3 & \\ \hline & p1_3 \end{array}\right)\begin{bmatrix} 1 & & \\ & p & \\ & & p \end{bmatrix}_p\right) + p^4\lambda\left(g\left(\begin{array}{c|c} p1_3 & \\ \hline & 1_3 \end{array}\right)\begin{bmatrix} p & & \\ & 1 & \\ & & 1 \end{bmatrix}_p\right) + \sum_{u\in U(p)'}\omega(p)\lambda(gu).$$

The term with the coefficient p^4, as above, uses that $g = \iota(t,\ell)$ with $|t| \le |\ell| \le 1$, and integrality properties of a nice choice of representatives. Now, $\lambda(gu) = \chi(gug^{-1})\lambda(g)$, and

$$\sum_{u\in U(p)'}\chi(gug^{-1}) = p^3 - 1 + \sum_{u\in U(p)'}\big(\chi(gug^{-1}) - 1\big).$$

If $|t| < |\ell|$, then $\chi(gug^{-1}) = 1$, and the sum vanishes. If $|t| = |\ell|$, then $\chi(gug^{-1}) = \chi(u)$, and one evaluates the sum to be $-p^3 + \epsilon_L(p)p^2$. □

We will reduce the GL_3 Hecke operators above to the following GL_2 Hecke operators.

Definition 5.4.3. Define

$$T(u) = \left(\begin{array}{c|c|c|c} 1 & & & \\ \hline & u & & \\ \hline & & pu^{-t} & \\ \hline & & & p \end{array}\right).$$

For $g \in G$, we denote $(T_p\lambda)(g)$ to be the sum $\sum_u \lambda\left(gT(u)\right)$ where u in the sum vary over the set of size $p+1$ containing $\begin{pmatrix} 1 & \\ & p \end{pmatrix}$ and $\begin{pmatrix} p & a \\ & 1 \end{pmatrix}$ for distinct representatives a of the integers modulo p. Similarly, we define

$$T'(u) = \left(\begin{array}{c|c|c|c} p & & & \\ \hline & u & & \\ \hline & & pu^{-t} & \\ \hline & & & 1 \end{array}\right)$$

and denote $(T'_p\lambda)(g)$ to be the sum $\sum_u \lambda\left(gT'(u)\right)$ where u varies over the same set.

We will relate T_p and T'_p shortly, and then reduce all the Hecke operators down to T_p. To do this, we first need a lemma.

Lemma 5.4.4.

$$\iota(t,\ell)\begin{pmatrix} 1_3 & \\ \hline & p1_3 \end{pmatrix} = \iota(t/p,\ell)(p1_6);\ \iota(t,\ell)\begin{pmatrix} p1_3 & \\ \hline & 1_3 \end{pmatrix} = \iota(pt,\ell);$$

$$\iota(t,\ell)\left(\begin{array}{ccc|ccc} p & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & p & & \\ & & & & p & \\ & & & & & 1 \end{array}\right) = \iota(pt,p\ell);\ \iota(t,\ell)\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & p & & & & \\ & & p & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & p \end{array}\right) = \iota(t/p,\ell/p)(p1_6);$$

From the lemma, it follows that

Lemma 5.4.5. $(T'_p\lambda)((t,\ell)) = \omega(p)^{-1}(T_p\lambda)((p^2t,p\ell))$.

Proof.

$$T'(u) = (p^{-1}1_6)\begin{pmatrix} p1_3 & \\ \hline & 1_3 \end{pmatrix}\left(\begin{array}{ccc|ccc} p & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & p & & \\ & & & & p & \\ & & & & & 1 \end{array}\right) T(u).$$

□

Now we have

Proposition 5.4.6. $T_{0,3}\lambda((t,\ell)) =$

$$\omega(p)\lambda((t/p,\ell)) + p^3\lambda((pt,p\ell)) + p(T_p\lambda)((t,\ell)) + p^4\omega(p)^{-1}(T_p\lambda)((p^2t,p\ell))$$

$$+p^3\omega(p)\lambda((t/p,\ell/p)) + p^6\lambda((pt,\ell)).$$

Proof. One applies the above lemmas to Proposition 5.4.1, and uses the well-known coset decompo-

sitions for Hecke operators on GL_3. For example,

$$\mathrm{GL}_3(\mathbf{Z}_p)\begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix}\mathrm{GL}_3(\mathbf{Z}_p) =$$

$$\coprod_{a,b}\begin{pmatrix} p & a & b \\ & 1 & \\ & & 1 \end{pmatrix}\mathrm{GL}_3(\mathbf{Z}_p)\coprod\coprod_{a}\begin{pmatrix} 1 & & \\ & p & a \\ & & 1 \end{pmatrix}\mathrm{GL}_3(\mathbf{Z}_p)\coprod\begin{pmatrix} 1 & & \\ & 1 & \\ & & p \end{pmatrix}\mathrm{GL}_3(\mathbf{Z}_p).$$

Here a and b range over distinct representatives of the integers modulo p.

Using that $\chi(gng^{-1}) = 1$ for $n \in U_R(\mathbf{Z}_p)$ and $g = \iota(t,\ell)$ with $|t| \le |\ell| \le 1$, as in the proof of Proposition 5.4.1, and using Lemma 5.4.4, the first term above leads to $p^2\lambda((pt,p\ell))$. The second and third terms combine to give $T_p\lambda((t,\ell))$.

One handles the other terms in Proposition 5.4.1 similarly. The first and fourth terms are immediately handled; the third term (with the p^3 coefficient) gives a T'_p, which one converts to a T_p via Lemma 5.4.5. □

Proposition 5.4.7. $T_{2,3}\lambda((t,\ell)) =$

$$p(T_p\lambda)((pt,p\ell)) + \omega(p)^2\lambda((t/p^2,\ell/p)) + p^6\lambda((p^2t,p\ell)) + p^4(T_p\lambda)((pt,\ell))$$

$$+\omega(p)(p^3-1)\lambda((t,\ell)) + \delta(|t| = |\ell|)(-p^3 + \epsilon_L(p)p^2).$$

Proof. This is completely analagous to Proposition 5.4.6. □

5.4.2 Some lemmas

We will soon use the two propositions above to compute $N(s)$ applied to the local dirichlet series. Before doing so, we need some lemmas concerning λ.

Lemma 5.4.8. *If $|\ell| = |p^{r-2}|$, then $(T_p\lambda)((p^{r-1},p\ell)) = 0$.*

Proof. Observe that

$$\frac{1}{p}\begin{pmatrix} p & a \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} p & \\ a & 1 \end{pmatrix} = \begin{pmatrix} 2a & 1 \\ 1 & \end{pmatrix}.$$

Set

$$u = \begin{pmatrix} p & a & \\ & & 1 \end{pmatrix}, \ w = \begin{pmatrix} & & 1 \\ & 1 & \end{pmatrix}.$$

Since $\iota(p^{r-1}, p\ell)n(w)\iota(p^{r-1}, p\ell)^{-1} \equiv n(w/p) \mod N_L$,

$$\lambda((p^{r-1}, p\ell)T(u)) = \lambda((p^{r-1}, p\ell)T(u)n(w)) = \psi(-D\frac{2a}{p})\lambda((p^{r-1}, p\ell)T(u)).$$

One concludes $\lambda((p^{r-1}, p\ell)T(u)) = 0$ when $a \neq 0$. This argument handles $p-1$ of the $p+1$ terms in the sum defining T_p; the other two terms are similar. □

Proposition 5.4.9. $(T_p\lambda)((p^r, 1)) = \lambda((p^r, 1)\tau)$. *If* $|\ell| = |p^r|$, *then*

$$(T_p\lambda)((p^r, \ell)) = \begin{cases} 0 & p \text{ inert} \\ \omega(p)\lambda((p^{r-1}, \ell\pi_1/p)) + \omega(p)\lambda((p^{r-1}, \ell\pi_2/p)) & p \text{ split.} \end{cases}$$

Proof. If $u = \begin{pmatrix} p & a & \\ & & 1 \end{pmatrix}$, then $(p^r, 1)T(u)$ does not satisfy the conditions of Lemma 5.2.7: p does not divide the top row y. Thus only one term contributes to the sum defining $T_p\lambda((p^r, 1))$, giving the first part of the proposition.

Now suppose $|\ell| = |p^r|$. If $u = \begin{pmatrix} 1 & & \\ & & p \end{pmatrix}$, then by taking $w = \begin{pmatrix} 1 & & \\ & & \end{pmatrix}$, one shows as in Lemma 5.4.8 that $\lambda((p^r, \ell)T(u)) = 0$. Similarly, if $u = \begin{pmatrix} p & a & \\ & & 1 \end{pmatrix}$, then by taking $w = \begin{pmatrix} & & \\ & 1 & \end{pmatrix}$, one shows as in Lemma 5.4.8 that $\lambda((p^r, \ell)T(u)) = \psi(\frac{-Da^2+1}{p})\lambda((p^r, \ell)T(u))$.

It follows that $\lambda((p^r, \ell)T(u)) = 0$ when p is inert, giving the proposition in this case. If p is split, then the only two terms that may contribute to the sum defining $T_p\lambda((p^r, \ell))$ are the ones with a a squareroot of D^{-1} modulo p. These two terms are right $M_R \cap \mathrm{GSp}_6(\mathbf{Z}_p)$ equivalent to the two terms $(p^{-1}, \pi_i/p)(p1_6)$, $i = 1, 2$, giving the proposition in the split case. □

Proposition 5.4.10. *If* $k \geq 1$ *and* p *is split, then*

$$(T_p\lambda)((p^r, \pi_i^k)) = \omega(p)\lambda((p^{r-1}, \pi_i^{k-1}))$$

for $i = 1, 2$.

Proof. Using Lemma 5.2.3, one sees that $(p^r, \pi_1^k)T(u)$ does not satisfy the nonvanishing criterion Lemma 5.2.7 unless $p^t u^{-1} = \begin{pmatrix} p & a \\ & 1 \end{pmatrix}$ with $a \equiv h \mod p$. Again from Lemma 5.2.3, this $T(u)$ is right $M_R \cap \mathrm{GSp}_6(\mathbf{Z}_p)$ equivalent $(p^{-1}, \pi_2/p)(p1_6)$. This gives the lemma for $i = 1$, and of course the $i = 2$ case is the same. □

5.4.3 Final computation

The results of the previous two subsections give us enough tools to compute $N(s)$ applied to the Dirichlet series

$$D(s) = \zeta_p(\omega, 2s-6) \sum |t|^{-6} |\ell|^2 \lambda(\iota(t,\ell))|t|^s$$

directly. The lengthy calculation is summarized in the following two theorems.

Theorem 5.4.11 (Inert case)**.** *Suppose p is inert in L. The coefficient of p^{-rs} in $N(s)D(s)$ is*

$$\lambda((p^r,1))p^{6r} - \lambda(\iota(p^{r-1},1)\tau)p^{6r-6}.$$

Theorem 5.4.12 (Split case)**.** *Suppose p is split in L. The coefficient of p^{-rs} in $N(s)D(s)$ is*

$$\lambda(\iota(p^r,1))p^{6r} - \lambda(\iota(p^{r-1},1)\tau)p^{6r-6}$$

$$+ \sum_{i=1,2,1\le k\le r} \lambda(\iota(p^r,\pi_i^k))|\pi_i^k|^2 p^{6r} - \sum_{0\le j\le r-2} p^4\omega(p)|\pi_i^j|^2\lambda(\iota(p^{r-2},\pi_i^j))p^{6r-12}.$$

Proof. This is a direct, but perhaps tedious, calculation, so we omit it. One applies Propositions 5.4.6 and 5.4.7 to the sum in $D(s)$, except without the zeta factor $\zeta_p(\omega, 2s-6)$. Expanding all the sums, one finds much cancellation. Then, one applies Lemma 5.4.8 and Propositions 5.4.9, 5.4.10 to get more simplification. Finally, multiplying by the zeta factor $\zeta_p(\omega, 2s-6)$ there is telescoping, giving the theorem. □

Comparing this theorem with the main theorem of the previous section, one sees that they are identical, hence the main result follows.

5.5 Ramified integral

In this section we show that the data for the integral I can be chosen to trivialize the integral at the bad finite places.

Proposition 5.5.1. *Given v_0 in π_p, there exists $v \in \pi_p$ and a local section f_p for the Eisenstein series, the two depending only on v_0, so that for all (U_R, χ)-linear functionals $\ell : \pi_p \to \mathbf{C}$*

$$\int_{U_B Z \backslash \mathrm{GL}_2 \boxtimes \mathrm{GL}_{2,L}^*} f(g;s)\ell(\pi(g)v)\,dg = \ell(v_0).$$

Here is a simple lemma which we do first.

Lemma 5.5.2. *Let M be F or L (split or inert) as a locally compact abelian group. Denote by $||$ the $\mathbf{R}_{>0}^{\times}$-valued function on $M^{\times}$ satisfying $d(mz) = |m|dz$, dz the Haar measure on M giving $\mathcal{O}_M$ measure 1. Then*

$$\int_M \psi(vz)|p|^n \,\mathrm{char}(p^n z \in \mathcal{O}_M)\,dz = \begin{cases} 1 & v \in p^n\mathcal{O}_M \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\int_M \psi(vz)\psi(-z)|p|^n \,\mathrm{char}(p^n z \in \mathcal{O}_M)\,dz = \begin{cases} 1 & v \in 1 + p^n\mathcal{O}_M \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. The second statement follows from the first. The first statement is simple. □

Proof. Define the congruence subgroup $K_N = \left(1 + p^N M_6(\mathbf{Z}_p)\right) \cap \mathrm{GSp}_6(\mathbf{Q}_p)$. There is a K_{n_0} stabilizing v_0. Define $\varphi_1 \in \mathcal{C}_c^{\infty}(\mathbf{Q}_p)$ by $\varphi_1(z) = \psi(-z)|p|^{n_0} \,\mathrm{char}(p^{n_0} z \in \mathbf{Z}_p)$. Now set

$$v_1 = \int_{\mathbf{G}_a(\mathbf{Q}_p)} \varphi_1(z)\pi(u(z))v_0\,dz,$$

where

$$u(z) = \left(\begin{array}{c|cc|cc} 1 & & & & \\ \hline & 1 & & 0 & 0 \\ & & 1 & 0 & z \\ \hline & & & 1 & \\ & & & & 1 \\ \hline & & & & 1 \end{array}\right).$$

Then v_1 is stabilized some congruence subgroup K_n. Now set $\varphi_{up,n}^L(\gamma) = |p|_L^n \,\mathrm{char}(p^n\gamma \in \mathcal{O}_L)$, $\varphi_{low,n}^L(\gamma) = \psi_L(-\gamma)|p|_L^n \,\mathrm{char}(p^n\gamma \in \mathcal{O}_L)$, where $\psi_L(\gamma) = \psi(Re(\gamma))$. For an element γ of L we abuse

notation and also let γ denote the column vector ${}^t(Re(\gamma), Im(\gamma))$. Define

$$u_{up}(\gamma) = \begin{pmatrix} 1 & & {}^t\gamma & \\ & 1_2 & & \gamma \\ & & 1_2 & \\ & & & 1 \end{pmatrix},$$

$$u_{low}(\gamma) = \begin{pmatrix} 1 & -{}^t\gamma & & \\ & 1_2 & & \\ & & 1_2 & \gamma \\ & & & 1 \end{pmatrix}.$$

Now set

$$v_2 = \int_{\mathbf{G}_a(L_p)} \varphi^L_{low,n}(\gamma)\pi(u_{low}(\gamma))v_1\, d\gamma,$$

$$v_3 = \int_{\mathbf{G}_a(L_p)} \varphi^L_{up,n}(\gamma)\pi(u_{up}(\gamma))v_2\, d\gamma.$$

Suppose

$$g = \begin{pmatrix} \nu & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}$$

with the four-by-four matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}^*_{2,L}$. Then

$$\begin{aligned} \ell(\pi(g)v_3) &= \int_L \varphi^L_{up,n}(\gamma)\ell(\pi(\pi(g)u_{up}(\gamma))v_2)\, d\gamma \\ &= \left(\int_L \varphi^L_{up,n}(\gamma)\chi(gu_{up}(\gamma)g^{-1})\, d\gamma\right)\ell(\pi(g)v_2) \\ &= \left(\int_L \varphi^L_{up,n}(\gamma)\psi(c_{11}\gamma_1 + c_{12}\gamma_2)\, d\gamma\right)\ell(\pi(g)v_2) \\ &= \mathrm{char}(c \in p^n M_2(\mathbf{Z}_p))\ell(\pi(g)v_2). \end{aligned}$$

Here we have used Lemma 5.5.2 and the fact the the elements of $\mathrm{GL}^*_{2,L}$ are those matrices in GSp_4

that have the form

$$\left(\begin{array}{cc|cc} a_{11} & a_{12} & b_{11} & b_{12} \\ Da_{12} & a_{11} & b_{12} & Db_{11} \\ \hline Dc_{22} & c_{12} & d_{11} & Dd_{21} \\ c_{12} & c_{22} & d_{21} & d_{11} \end{array}\right).$$

Similarly $\ell(\pi(g)v_2) = \mathrm{char}(d \in 1 + p^n\mathcal{O}_L)\ell(\pi(g)v_1)$. Set $K_{n,L} = \mathrm{GL}_{2,L}^* \cap (1 + p^n M_4(\mathbf{Z}_p))$ and define $B_{L,n} = \{\left(\begin{smallmatrix} b_1 & * \\ & b_2 \end{smallmatrix}\right) : b_2 \in 1 + p^n\mathcal{O}_L\}$. Note that $c \in p^n\mathcal{O}_L$, $d \in 1 + p^n\mathcal{O}_L$ imply $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in B_{L,n}K_{n,L}$. Now there is $N \geq n$ so that v_3 is stable by K_N. Set $K_{N,2} = \mathrm{GL}_2(\mathbf{Q}_p) \cap \left(1 + p^N M_2(\mathbf{Z}_p)\right)$. Pick a nonzero section f_N^* supported on $B_2K_{N,2}$, for example

$$f_N^*(g;s) = \int_{B_2} \delta_{B_2}^{-s}(b)\,\mathrm{char}(bg \in K_{N,2})\,db$$

where db is right Haar measure. Then

$$\int_{U_B Z \backslash \mathrm{GL}_2 \boxtimes \mathrm{GL}_{2,L}^*} f(g_1;s)\ell(\pi(g)v_3)\,dg = \int_{U_B Z \backslash B_2K_{N,2} \boxtimes \mathrm{GL}_{2,L}^*} f(g_1;s)\ell(\pi(g)v_3)\,dg$$

and this is, up to positive constants,

$$\begin{aligned} &= \int_{U_B Z \backslash B_2 \boxtimes \mathrm{GL}_{2,L}^*} \delta_{B_2}^{-1}(g_1) f(g_1;s)\ell(\pi(g)v_3)\,dg \\ &= \int_{N_L \backslash \mathrm{GL}_{2,L}^*} |\nu(g_2)|^{s-1}\ell(\pi(g_2)v_3)\,dg \end{aligned}$$

where we have embedded $GL_{2,L}^*$ in GSp_6 via the elements

$$g_2 = \begin{pmatrix} \nu & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}$$

with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_{2,L}^*$. Set $T_{L,n} = T_L \cap B_{L,n}$. Then again up to a positive constant this is

$$\begin{aligned} &= \int_{N_L \backslash B_{L,n}K_{n,L}} |\nu(g_2)|^{s-1}\ell(\pi(g_2)v_1)\,dg \\ &= \int_{T_{L,n}} \delta_{B_L}^{-1}(t)|\nu(t)|^{s-1}\ell(\pi(t)v_1)\,dt \end{aligned}$$

Now, applying Lemma 5.5.2 again as above,

$$\ell(\pi(t)v_1) = \text{char}\left(\frac{\nu}{N(\ell)} \in 1 + p^{n_0}\mathbf{Z}_p\right)\ell(\pi(t)v_0)$$

where, in block diagonals, $t = diag(\nu, \nu\,{}^t\ell^{-1}, \ell, 1)$. Under the conditions $\frac{\nu}{N(\ell)} \in 1 + p^{n_0}\mathbf{Z}_p$, the support of this last integral becomes open compact, and the integrand becomes the constant $\ell(v_0)$ where the integrand is supported. Hence, for some positive constant C,

$$\int_{U_B Z \backslash \,\mathrm{GL}_2 \boxtimes \mathrm{GL}^*_{2,L}} f(g;s)\ell(\pi(g)v_3)\,dg = C\ell(v_0).$$

Setting $v = C^{-1}v_3$ gives the proposition. □

Chapter 6

The exterior cube L-function on $\mathrm{GU}(3,3)$

6.1 Introduction

The purpose of this document is to record a statement and proof of a Rankin-Selberg integral for the exterior cube L-function on the quasisplit unitary group in 6 variables. If one takes the *split* unitary group, which is isomorphic to $\mathrm{GL}_1 \times \mathrm{GL}_6$, then Ginzburg-Rallis have addressed this case in [21]. The integral we write down for $\mathrm{GU}(3,3)$ is essentially identical to the one considered in [21], with the difference being that $\mathrm{GU}(3,3)$ replaces GL_6. Just as in [21], the integral unfolds to the Whittaker model of the cusp form on $\mathrm{GU}(3,3)$. At the split places for the unitary group, the unramified calculation is the same as in Ginzburg-Rallis. At the inert places for the unitary group, it turns out that the unramified calculation in fact follows from the unramified calculation for the *two-variable* Rankin-Selberg integral in Bump-Friedberg-Ginzburg [5] for generic cusp forms on GSp_6. (Note that the rational root system of $\mathrm{GU}(3,3)$ is of type C_3.) For these reasons, we expect this integral must be known to the experts, although it does not seem to appear in the literature. Because this result dovetails strongly with some of the other results in this thesis, and because we anticipate future arithmetic applications of this integral, we have chosen to record its statement and proof at this time.

6.2 The exterior cube L-function

Denote by

$$J_6 := \left(\begin{array}{cccc|cc} & & & & & 1 \\ & & & & 1 & \\ & & & & 1 & \\ \hline & & & -1 & & \\ & & -1 & & & \\ & -1 & & & & \end{array}\right).$$

Suppose E is a quadratic etale extension of F. We define the unitary group

$$G = \mathrm{GU}(3,3) = \{g \in \mathrm{GL}_6(E) : gJ_6\,{}^t\bar{g} = \nu(g)J_6\}$$

where $x \mapsto \bar{x}$ is the nontrivial automorphism of E over F and $\nu(g) \in F^\times$. We set $G' = \mathrm{GSp}_6 \subseteq G$ to be $G \cap \mathrm{GL}_6(F)$.

The dual group of G is $(\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_6(\mathbf{C})) \rtimes Gal(E/F)$. The nontrivial element θ of $Gal(E/F)$ acts by $(\lambda, g) \mapsto (\lambda \det(g), J_6\,{}^t g^{-1} J_6^{-1})$ (for this and the contents of the next paragraph see [25] and [31]).

If a prime p of F is inert in E, then the local group $G(F_p)$ is still a unitary group. Let us describe the Satake parameters of an unramified representation π of $G(F_p)$ at such a place p. Denote by T the diagonal maximal torus of G, so that T contains T' the diagonal maximal split torus of G. Suppose α is an unramified character of $T(F_p)$ and π is an unramified irreducible subquotient of $Ind(\delta_B^{1/2}\alpha)$ with δ_B the modulus character of uppertriangular Borel of $G(F_p)$. The elements of T are of the form $t = diag(t_1, t_2, t_3, a\bar{t_3}^{-1}, a\bar{t_2}^{-1}, a\bar{t_1}^{-1})$. Thus an unramified character α of T is given by characters $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, where α_0 is an unramified character of $F_p^\times$, the other $\alpha_i's$ are unramified characters of $E_p^\times$, and $\alpha(t) = \alpha_0(a)\alpha_1(t_1)\alpha_2(t_2)\alpha_3(t_3)$. With these notations, the Frobenius conjugacy class of π in the dual group ${}^L G$ is then $(\alpha_0(\varpi), diag(\alpha_1(\varpi), \alpha_2(\varpi), \alpha_3(\varpi), 1, 1, 1)) \rtimes \theta$, where ϖ is a uniformizer of F.

We will now define the exterior cube representation of ${}^L G$. Set V_6 the standard 6-dimensional representation of $\mathrm{GL}_6(\mathbf{C})$. Let $\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_6(\mathbf{C})$ act on $\wedge^3 V_6$ via $(\lambda, g) \mapsto \lambda \wedge^3 (g)$. Denote this representation by ρ. Then $x \mapsto \rho(x)$ and $x \mapsto \rho(\theta^{-1} x \theta)$ are irreducible representations of $\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_6(\mathbf{C})$ with the same highest weight, and thus are isomorphic. Hence, there is an element $A \in GL(\wedge^3 V_6)$ satisfying $\rho(\theta^{-1} x \theta) = A^{-1}\rho(x)A$. Note that if A satisfies this identity, then

so does $-A$. It is important that we pick A with positive trace. (The other choice $-A$ will not give the L-function which comes out of the integral representation below.) Then the map which sends $x = (\lambda, g)$ to $\rho(x)$ and $(1,1) \rtimes \theta$ to A defines a representation ${}^L G = (\mathrm{GL}_1(\mathbf{C}) \times \mathrm{GL}_6(\mathbf{C})) \rtimes Gal(E/F)$. This representation is what we refer to as the exterior cube; we denote it by $\wedge^3$.

The following proposition computes the exterior cube L-function for an unramified representation of $G(F_p)$ at an inert place p in terms of L-functions of $G' = \mathrm{GSp}_6$.

Proposition 6.2.1. *Suppose that α is an unramified character of T, and π is an unramified irreducible subquotient of $Ind_B^G(\delta_B^{1/2}\alpha)$. Denote by α' the restriction of α to T', and suppose that π' is an unramified irreducible subquotient of $Ind_{B'}^{G'}(\delta_{B'}^{1/2}\alpha')$. Then if π has trivial central character,*

$$L(\pi, \wedge^3, s) = \frac{L(\pi', Spin, s)L(\pi', Std, 2s)}{\zeta(2s)}.$$

Note that (the reciprocals of) both sides of the equality are of degree 20 in p^{-s}.

Proof. Let us write $e_i, 1 \leq i \leq 6$ as a basis of V_6, and $e_{ijk} = e_i \wedge e_j \wedge e_k, i < j < k$ as our basis of $\wedge^3 V_6$. Set $t = diag(t_1, t_2, t_3, t_4, t_5, t_6)$. Comparing $\wedge^3 t$ and $\wedge^3 J_6 \det(t)\, {}^t t^{-1} J_6^{-1}$ one finds that the permutation matrix A given by

$$(e_{125}e_{134})(e_{126}e_{234})(e_{136}e_{235})(e_{146}e_{245})(e_{156}e_{345})(e_{256}e_{346})$$

conjugates one to the other.

Now suppose the Frobenius conjugacy class of π_p in ${}^L G$ is represented by

$$((\lambda, diag(a, b, c, 1, 1, 1)) \rtimes \theta)$$

for some $\lambda, a, b, c \in \mathbf{C}^\times$. Note then the that Frobenius conjugacy class of π'_p in ${}^L G'$ is represented by $((\lambda, diag(a, b, c, 1, 1, 1))$. Now $L(\pi_p, \wedge^3, s)$ is the reciprocal of

$$\det(1 - p^{-s}\lambda \wedge^3 (diag(a, b, c, 1, 1, 1))A| \wedge^3 (V_6)).$$

The linear transformation $\lambda \wedge^3 (diag(a, b, c, 1, 1, 1))A$ has 8 diagonal entries and 6 pairs of two-by-two blocks. The eight diagonal entries are for $e_{123}, e_{124}, e_{135}, e_{145}, e_{236}, e_{246}, e_{356}, e_{456}$, for which the eigenvalues are, respectively, $\lambda abc, \lambda ab, \lambda ac, \lambda a, \lambda bc, \lambda b, \lambda c, \lambda$. Thus this part of the determinant gives $L(\pi'_p, Spin, s)^{-1}$. The six two-by-two blocks are given by the six pairs of basis elements interchanged

by A. The block determined by e_{126}, e_{234} contributes

$$(1 - b(abc\lambda^2)p^{-2s}) = (1 - b\omega_\pi(p)p^{-2s})$$

to the determinant. Similarly the other 5 blocks contribute $(1 - a\omega_\pi(p)p^{-2s})$, $(1 - c\omega_\pi(p)p^{-2s})$, $(1-a^{-1}\omega_\pi(p)p^{-2s})$, $(1-b^{-1}\omega_\pi(p)p^{-2s})$, $(1-c^{-1}\omega_\pi(p)p^{-2s})$. Hence the two-by-two blocks contribute $\big(L(\pi'_p, Std, 2s)/\zeta(2s)\big)^{-1}$ to the determinant. □

6.3 Global integrals and unfolding

In this section we state the global integral for G, and unfold it. The unfolding is essentially identical to that of Ginzburg-Rallis [21]. The unramified calculation in the next section will make critical use of the one used in Bump-Friedberg-Ginzburg [5] for their two-variable integral for cusp forms on GSp_6. However, the casual reader might have some difficulty in figuring out what exactly goes into this unramified computation of [5]: the calculation for that integral is omitted. Furthermore, the statement of the unfolding for that integral (Theorem 2.1 of [5]) is not correct (and the proof is omitted), so the reader would not be able to immediately deduce the contents of the unramified calculation from Theorem 2.1 of that paper. For this reason, we will also take this opportunity to give the correctly unfolded two-variable integral of Bump-Friedberg-Ginzburg. It is interesting to note that the unfolding for this two-variable integral is quite similar to the single variable one for G.

Let us write $\{e_1, e_2, e_3, f_3, f_2, f_1\}$ for the ordered basis of W_6, the 6-dimensional F or E vector space that is the defining representation of G' and G, respectively. We think of these groups as acting on the right of W_6. The Siegel parabolic P' of G' is the stabilizer of the (isotropic) subspace spanned by f_3, f_2, f_1. The Siegel Eisenstein on G' is, as usual, given by

$$\sum_{\gamma \in P'(F)\backslash G'(F)} f(\gamma g; s)$$

where f is a right K' finite standard section in $Ind(\delta_{P'}^s)$. Set

$$w = \left(\begin{array}{cc|cc|cc} 1 & & & & & \\ & 1 & & & & \\ \hline & & & 1 & & \\ & & -1 & & & \\ \hline & & & & 1 & \\ & & & & & 1 \end{array}\right).$$

If π is a cuspidal automorphic representation of G, and ϕ is a cusp form in the space of π, the global integral is then

$$I(\phi; s) = \int_{[G']} E(g; s)\phi(gw^{-1})\, dg.$$

For x in G, define $j(x) = wxw^{-1}$. Finally, define $X : \mathbf{G}_a \to G' \subseteq G$ by $X(r) = 1_6 + rE_{4,3}$, where $E_{4,3}$ is the 6×6 matrix with 1 at the $(4, 3)$ spot and zeroes elsewhere. Then

Theorem 6.3.1. *Denote by $U_{B'}$ the unipotent radical of the uppertriangular Borel of G'.*

$$I(\phi; s) = \int_{U_{B'}(\mathbf{A})Z(\mathbf{A})\backslash G'(\mathbf{A})} f(g; s) \int_{\mathbf{A}} W_\phi(X(r)j(g))\, dr\, dg$$

where W_ϕ is the Whittaker model of ϕ.

We stress that the proof is essentially due to Ginzburg-Rallis [21].

Proof. Let us first make some notation. Pick $D \in F$ so that $E = F(\sqrt{D})$. Define $\psi_E : \mathbf{A}_E \to \mathbf{C}^\times$ via $\psi_E(x) = \psi(Im(x))$. If $x = a + b\sqrt{D}$ is in E with $a, b \in F$ then $Im(x)$ is defined to be b. Now, if $U_{P'}$ is the unipotent radical of P', then

$$\int_{[U_{P'}]} \phi(ug)\, du = \sum_{\gamma \in M'(F)\backslash M_{P'}(F)} \phi_{(24)-(35)}(\gamma g),$$

where we now explain the notation: $M_{P'}$ denotes the Levi of the Siegel parabolic P'; M' is the

subgroup of $M_{P'}$ of the form

$$\left(\begin{array}{ccc|ccc} * & * & * & & & \\ & * & * & & & \\ & * & * & & & \\ \hline & & & * & * & * \\ & & & * & * & * \\ & & & & & * \end{array}\right)$$

where the upper and lower two-by-two blocks have the same determinant. The fourier coefficient $\phi_{(24)-(35)}$ is given by

$$\phi_{(24)-(35)}(g) = \int_{[U_P]} \psi_E^{-1}(u_{24} - u_{35})\phi(ug)\, du,$$

where U_P is the unipotent radical of the Siegel parabolic of G, and u_{ij} denotes the entry of u in the i, j position.

Some more notation: define U_B^F the subgroup of U_B with (23) entry in F and U_B^0 the subgroup with (34) entry equal to 0. For $\alpha \in F$ we define a function $\chi_\alpha : U_B(\mathbf{A}) \to \mathbf{C}^\times$ by $\chi_\alpha(u) = \psi((u_{12} + u_{56}) + \alpha(u_{23} + u_{45}) + (u_{24} - u_{35}))$. Then χ_α is a character on both U_B^F and U_B^0. Indeed, on U_B^0 each of the entries $(12), (23), (24), (35), (45), (56)$ is additive, and on U_B^F of course the entries $(12), (56)$ are additive, and $(uv)_{24} - (uv)_{35} = (u_{24} - u_{35}) + (v_{24} - v_{35}) + u_{23}v_{34} - u_{34}v_{45}$. Since $u_{34}, v_{34} \in F$ by virtue of u, v being in G, and u_{23}, v_{45} are in F since $u, v \in U_B^F$, ψ_E is 1 on $u_{23}v_{34} - u_{34}v_{45}$.

Finally then, with this behind us, unfolding the Eisenstein series and using the above we obtain easily

$$I(\phi; s) = \int_{T''(F)U_{B'}(\mathbf{A})Z(\mathbf{A})\backslash G'(\mathbf{A})} f(g; s)\phi_{\chi_0}(gw^{-1})\, dg$$

where $T''(F)$ is the set of $diag(\alpha, \alpha, 1, \alpha, 1, 1)$ with $\alpha \in F^\times$ and

$$\phi_{\chi_0}(g) = \int_{[U_B^F]} \chi_0^{-1}(u)\phi(ug)\, du.$$

Lemma 6.3.2.

$$\phi_{\chi_0}(g) = \int_{\mathbf{A}} \int_{[U_B^0]} \chi_0^{-1}(v)\phi(vu_{34}(r)g)\, dv\, dr$$

where $u_{34}(r) = 1 + rE_{3,4}$.

Proof. Set $U_B^{F,0} = U_B^F \cap U_B^0$. Then

$$\phi_{\chi_0}(g) = \int_{[U_B^{F,0}]} \int_{[\mathbf{G}_a]} \chi_0^{-1}(u)\phi(uu_{34}(r)g)\,dr\,du.$$

Fourier expanding along the $(2,3)$ position, this is

$$\phi_{\chi_0}(g) = \int_{[U_B^0]} \sum_{\alpha \in F} \int_{[\mathbf{G}_a]} \chi_\alpha^{-1}(v)\phi(vu_{34}(r)g)\,dr\,dv.$$

Now $u_{34}(r)$ normalizes U_B^0 and $\chi_\alpha(v) = \chi_0(u_{34}(-\alpha)vu_{34}(\alpha))$. Thus, changing variables on U_B^0 by a conjugation by $u_{34}(\alpha)$ we obtain

$$\phi_{\chi_0}(g) = \int_{[U_B^0]} \sum_{\alpha \in F} \int_{[\mathbf{G}_a]} \chi_0^{-1}(v)\phi(vu_{34}(r-\alpha)g)\,dr\,dv.$$

Unfolding the sum over α in F gives the lemma. □

Via the lemma we get

$$\begin{aligned} \phi_{\chi_0}(gw^{-1}) &= \int_{\mathbf{A}} \int_{[U_B^0]} \chi_0^{-1}(v)\phi(wvw^{-1}X(r)j(g))\,dv\,dr \\ &= \int_{\mathbf{A}} \int_{[U_B^0]} \chi_0^{-1}(w^{-1}vw)\phi(vX(r)j(g))\,dv\,dr \end{aligned}$$

But now $\chi_0(w^{-1}vw) = \psi_E(v_{12} + v_{23} + v_{45} + v_{56})$, from which the theorem follows. □

Let us now state the two-variable integral of Bump-Friedberg-Ginzburg [5]. For this, set H to be the subgroup of elements (g_1, g_2) of $\mathrm{GL}_2 \times \mathrm{GSp}_4$ with $\det(g_1) = \nu(g_2)$. Identifying GL_2 with the automorphism group of the symplectic subspace of W_6 spanned by e_1, f_1 and GSp_4 with the automorphism group of the subspace spanned by e_2, e_3, f_3, f_2 gives an embedding $H \subseteq G' = \mathrm{GSp}_6$. Define $E_B(g_1; s)$ to be the standard Eisenstein series on GL_2 (from the parabolic B stabilizing Ff_1) and $E_P(g_2; w)$ the standard Siegel Eisenstein series on GSp_4 (from the parabolic stabilizing $Ff_3 + Ff_2$). The two-variable global integral is then

$$I'(\phi; s, w) = \int_{[H]} E_B(g_1, ; s)E_P(g_2; w)\phi(gw^{-1})\,dg.$$

Let w, j, and $X(r)$ be as above, and set

$$w_2 = \left(\begin{array}{ccc|c|c} & 1 & & & \\ 1 & & & & \\ \hline & & 1 & & \\ & & & 1 & \\ \hline & & & & 1 \\ & & & 1 & \end{array}\right).$$

Then

Theorem 6.3.3 (Bump-Friedberg-Ginzburg)**.** *Denote by $U_{B,H}$ the unipotent radical of the upper-triangular Borel of H. Then*

$$I'(\phi; s, w) = \int_{U_{B,H}(\mathbf{A})Z(\mathbf{A})\backslash H(\mathbf{A})} f_B(g_1; s) f_P(g_2; w) \int_{\mathbf{A}} W_\phi(w_2 X(r) j(g))\, dr\, dg$$

where W_ϕ is the Whittaker model of ϕ.

Proof. Again, this is due to Bump-Friedberg-Ginzburg; we sketch the proof for the convenience of the reader. Unfolding the Eisenstein series and doing the usual Fourier expansions one obtains

$$I'(\phi; s, w) = \int_{T''(F)U_{B,H}(\mathbf{A})Z(\mathbf{A})\backslash H(\mathbf{A})} f_B(g_1; s) f_P(g_2; w) \phi_1(gw^{-1})\, dg$$

where

$$\phi_1(g) = \int_{[U''_B]} \psi^{-1}(u_{21} + u_{65} + u_{14} + u_{36}) \phi(ug)\, du$$

and U''_B is the unipotent subgroup of G' consisting of elements of the form

$$\left(\begin{array}{ccc|ccc} 1 & & & * & * & * \\ * & 1 & * & * & * & * \\ & & 1 & * & * & * \\ \hline & & & 1 & * & \\ & & & & 1 & \\ & & & & * & 1 \end{array}\right).$$

Set U'_B the subgroup of U_B with $(2,3)$ entry equal to 0. Define ϕ_{χ_0} analogously to the proof above, i.e. via taking the χ_0 fourier coeffient over U'_B. Then $U'_B = w_2 U''_B w_2^{-1}$, and for $u'' \in U''_B$,

$\chi_0(w_2u''w_2^{-1}) = \psi(u''_{21} + u''_{65} + u''_{14} + u''_{36})$. It follows that $\phi_1(g) = \phi_{\phi_0}(w_2g)$. Hence

$$I'(\phi; s, w) = \int_{T''(F)U_{B,H}(\mathbf{A})Z(\mathbf{A})\backslash H(\mathbf{A})} f_B(g_1; s) f_P(g_2; w) \phi_{\chi_0}(w_2 g w^{-1})\, dg.$$

At this point, one proceeds exactly as in the proof above to obtain the theorem. □

6.4 Unramified computations

Here are the unramified computations. Denote by I_p and I'_p the p-local analogues of I and I' above that come from the unfoldings. That is,

$$I_p(\phi; s) = \int_{U_{B'}(F_p)Z(F_p)\backslash G'(F_p)} f^p(g; s) \int_{F_p} W_\phi(X(r)j(g))\, dr\, dg$$

and

$$I'_p(\phi; s, w) = \int_{U_{B,H}(F_p)Z(F_p)\backslash H(F_p)} f_B^p(g_1; s) f_P^p(g_2; w) \int_{F_p} W_\phi(w_2 X(r) j(g))\, dr\, dg.$$

Theorem 6.4.1. *Assume trivial central character. If all the data is unramified, then*

$$I_p(\phi; s) = \frac{L(\pi_p, \wedge^3, 2s - \frac{1}{2})}{\zeta(4s)\zeta(8s-2)}.$$

Theorem 6.4.2 (Bump-Friedberg-Ginzburg)**.** *If all the data is unramified, then*

$$I'_p(\phi; s) = \frac{L(\pi_p, Spin, s) L(\pi_p, Std, 3w-1)}{\zeta(2s)\zeta(3w)\zeta(6w-2)}.$$

The calculation of I_p for p split in E is exactly the one from Ginzburg-Rallis [21]. The calculation of I_p for p inert in E follows from the key step in the proof of Theorem 6.4.2, which we now state.

Assume π_p is generic, and has trivial central character. For g in GSp_6, set, as is standard, $K_{\pi_p}(g) = \delta_B^{-1/2}(g) W_{\pi_p}(g)$, where W_{π_p} is the Whittaker model of π_p normalized so that $W_{\pi_p}(1) = 1$. Denote

$$[n_1, n_2, n_3] = diag(p^{n_1+n_2+n_3}, p^{n_2+n_3}, p^{n_3}, 1, p^{-n_2}, p^{-n_1-n_2}).$$

For indeterminates Y, Z define

$$L(\pi_p, Std, Y) = \sum_{j \geq 0} tr(Sym^j Std(A_p)) Y^j$$

and

$$L(\pi_p, Spin, Z) = \sum_{k\geq 0} tr(Sym^k Spin(A_p))Z^k$$

where A_p is the conjugacy class in ${}^L\mathrm{GSp}_6$ corresponding to π_p. Similarly define

$$\zeta(X) = (1-X)^{-1}.$$

Then

Theorem 6.4.3 (Bump-Friedberg-Ginzburg)**.**

$$\sum_{n_1,n_2,n_3\geq 0} \left(\frac{1-Y^{n_3+1}}{1-Y}\right) Z^{n_3+2n_2} Y^{n_1+n_2} K_{\pi_p}([n_1,n_2,n_3]) = \frac{L(\pi_p, Std, Y)L(\pi_p, Spin, Z)}{\zeta(Y^2)\zeta(Z^2)}.$$

Proof of Theorem 6.4.2. The usual manipulations give

$$I'(\phi; s, w) = \frac{1}{\zeta(3w)} \sum_{n_1,n_2,n_3\geq 0} \left(\frac{1-Y^{n_3+1}}{1-Y}\right) Z^{n_3+2n_2} Y^{n_1+n_2} K_{\pi_p}([n_1,n_2,n_3])$$

with $Y = q^{1-3w}, Z = q^{-s}$. The theorem then follows from Theorem 6.4.3. □

The group GSp_6 becomes relevant for the unramified calculation of I_p for inert p because the Casselman-Shalika formula for $G(F_p)$ is most cleanly stated in terms of G'. Indeed, denote by π'_p the representation of $G'(F_p)$ described in Proposition 6.2.1, and $K^G_{\pi_p}, K^{G'}_{\pi'_p}$ as above. Then, if λ is an unramified cocharachter of $T'(F_p)$, $K^G_{\pi_p}(\lambda(\varpi)) = K^{G'}_{\pi'_p}(\lambda(\varpi))$. Hence, the Casselman-Shalika formula for G reduces to that of G'. This nice formulation of the Casselman-Shalika formula in the quasisplit case is due to Tamir [33] and explained in Gan-Hundley [15].

Of Theorem 6.4.1. One proceeds identically as in the unramified calculation Proposition 2.1 of [21] to obtain

$$I(\phi; s) = \frac{1}{\zeta(4s)} \sum_{n_1,n_2,n_3\geq 0} \left(\frac{1-X^{2n_3+2}}{1-X^2}\right) X^{n_3+2n_1+4n_2} K^{G(F_p)}_{\pi_p}([n_1,n_2,n_3])$$

where $X = q^{-(2s-\frac{1}{2})}$. In the case p is split in E that this is equal to

$$\frac{L(\pi_p, \wedge^3, 2s-\frac{1}{2})}{\zeta(4s)\zeta(8s-2)}$$

is of course in Ginzburg-Rallis. In the case p is inert in E, we have that $K^G_{\pi_p}([n_1,n_2,n_3]) =$

$K_{\pi'_p}^{G'}([n_1, n_2, n_3])$ as explained above. Now apply the Theorem 6.4.3 of Bump-Friedberg-Ginzburg with $Y = X^2, Z = X$ to obtain

$$I(\phi; s) = \frac{L(\pi'_p, Std, 2(2s - \frac{1}{2}))L(\pi'_p, Spin, 2s - \frac{1}{2})}{\zeta(4s)\zeta(4s-1)\zeta(8s-2)}.$$

One now concludes by applying Proposition 6.2.1. □

Chapter 7

The three-variable integral on PGL_4

7.1 Introduction

We give a Rankin-Selberg integral for cusp forms on PGL_4 in three complex variables representing the product of the three fundamental representations of the dual group $\mathrm{SL}_4(\mathbf{C})$. This integral is inspired by the two-variable integral on $\mathrm{GU}(2,2)$ (itself inspired by the two-variable integral of Bump-Friedberg-Ginzburg [5] on GSp_4) and the three variable integral of Gan-Hundley [15] on the split orthogonal group in eight variables. Note that GL_4 may be considered a split orthogonal group on six variables. Also note that Ginzburg-Hundley [20] have given Rankin-Selberg integrals in more than one complex variable on split orthogonal groups of small rank. However, the integral we give here is not the one suggested in [20]. Those authors instead suggest a global integral on GL_4 using the so-called mirabolic Eisenstein series.

Denote by P the maximal standard parabolic of GL_4 that has Levi isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_2$. Denote by Q the nonmaximal standard $(1,2,1)$ parabolic; i.e. if b_1, b_2, b_3, b_4 is an ordered basis of V_4 the standard representation of GL_4 with the *right* action, then Q stabilizes the flag $\langle b_4 \rangle \subseteq \langle b_2, b_3, b_4 \rangle$. In this notation, P is the stabilizer of $\langle b_3, b_4 \rangle$.

We define Eisenstein series for P and Q with degenerate data. For P, pick a standard section $f_P(g;w) \in Ind(\delta_P^w)$. For Q, first denote by $P_{3,1}$ the maximal parabolic stabiling the line $\langle b_4 \rangle$ via the right action, so that the elements of $P_{3,1}$ have 0's for their $(4,1)$, $(4,2)$, and $(4,3)$ entries. Similary define $P_{1,3}$ the parabolic with 0's in the spots $(2,1), (3,1)$, and $(4,1)$, so that $Q = P_{3,1} \cap P_{1,3}$. Then we pick a standard section $f_Q(g;s_1,s_2)$ in $Ind(\delta_{P_{3,1}}^{s_1}\delta_{P_{1,3}}^{s_2})$. The Eisenstein series are then $E_P(g;w) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_4(F)} f_P(\gamma g;w)$ and $E_Q(g;s_1,s_2) = \sum_{\gamma \in Q(F)\backslash \mathrm{GL}_4(F)} f_Q(\gamma g;s_1,s_2)$, abso-

lutely convergent when the real parts of w, s_1, s_2 are sufficiently large. If π is a cuspidal representation, and ϕ is in the space of π, the global integral is

$$I(\phi; s_1, s_2, w) = \int_{\mathrm{GL}_4(F)Z(\mathbf{A})\backslash \mathrm{GL}_4(\mathbf{A})} \phi(g) E_P(g; w) E_Q(g; s_1, s_2)\, dg.$$

We will show that the integral unfolds to the Whittaker model of ϕ, and that it represents the product of the L-functions of the three fundamental representations of SL_4.

7.2 Unfolding and unramified computation

Set

$$\nu_P := \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right)$$

and

$$\nu_Q := \left(\begin{array}{c|c|c} 1 & & \\ \hline & & 1 \\ & 1 & \\ \hline & & 1 \end{array}\right).$$

The following lemma trivializes the unfolding.

Lemma 7.2.1. *The double coset space $Q(F)\backslash \mathrm{GL}_4(F)/P(F)$ is represented by the four permutation matrices corresponding to the permutations $\{\nu_P \nu_Q^{-1} = (1243), 1, \gamma_1 = (123), \gamma_2 = (243)\}$. The stabilizers of the cosets $Q(F)\gamma_i$ inside $P(F)$ contain the unipotent radical of a parabolic subgroup of* GL_4.

Proof. The Bruhat decomposition gives $Q(F)\backslash \mathrm{GL}_4(F)/P(F) = \langle(23)\rangle\backslash S_4/\langle(12),(34)\rangle$. The coset space $S_4/\langle(12),(34)\rangle$ is represented by the six permutations $\sigma \in S_4$ with $\sigma(1) < \sigma(2)$ and $\sigma(3) < \sigma(4)$. Letting $\langle(23)\rangle$ act on this set, one finds 4 distinct lists of the form $(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$, namely $(1,2,3,4)$, $(1,4,2,3)$, $(2,3,1,4)$ and $(2,4,1,3)$. These correspond to the permutations listed. The stabilizer for γ_1 contains the unipotent radical of the parabolic $P_{3,1}$, and the stabilizer for γ_2 contains the unipotent radical of $P_{1,3}$. □

Define $R = (\nu_P^{-1} Q \nu_P) \cap (\nu_Q^{-1} P \nu_Q)$. Then R is the subgroup of GL_4 consisting of matrices of the

form

$$\left(\begin{array}{c|cc} * & * & * \\ & * & * \\ \hline & * & \\ & & * \end{array}\right).$$

Define R^0 to be the unipotent radical of R.

Proposition 7.2.2. *Denote by W_ϕ the Whittaker model of ϕ. Then*

$$I(\phi; s_1, s_2, w) = \int_{R^0(\mathbf{A})Z(\mathbf{A})\backslash \mathrm{GL}_4(\mathbf{A})} W_\phi(g) f_P(\nu_Q g; w) f_Q(\nu_P g; s_1, s_2)\, dg.$$

Proof. Unfolding the Eisentstein series E_P and then E_Q gives, using the Lemma and the cuspidality of ϕ,

$$I(\phi; s_1, s_2, w) = \int_{\left(\nu_Q \nu_P^{-1} Q(F) \nu_P \nu_Q^{-1}\right) \cap P(F)Z(\mathbf{A})\backslash \mathrm{GL}_4(\mathbf{A})} \phi(g) f_P(g; w) f_Q(\nu_P \nu_Q^{-1} g; s_1, s_2)\, dg.$$

Changing variables, this is then

$$\int_{R(F)Z(\mathbf{A})\backslash \mathrm{GL}_4(\mathbf{A})} \phi(g) f_P(\nu_Q g; w) f_Q(\nu_P g; s_1, s_2)\, dg.$$

Integrating over $R^0(\mathbf{A})$ and expanding again gives the result. □

Define

$$I_p(W; s_1, s_2, w) = \int_{R^0(F_p)Z(F_p)\backslash G(F_p)} W_\phi^p(g) f_P(\nu_Q g; w) f_Q(\nu_P g; s_1, s_2)\, dg.$$

Theorem 7.2.3. *When all the data is unramified,*

$$I_p(W; s_1, s_2, w) = \frac{L(\pi_p, Std, 2w + s_1 - s_2 - \frac{1}{2}) L(\pi_p, \wedge^2, 2s_1 + 2s_2 - 1) L(\pi_p, \wedge^3, 2w - s_1 + s_2 - \frac{1}{2})}{\zeta_F(4w)\zeta_F(4s_1)\zeta_F(4s_2)\zeta_F(4w-1)\zeta_F(4s_1 + 4s_2 - 2)}$$

Proof. Denote by T the torus of GL_4, a typical element of which is $t = diag(t_1, t_2, t_3, t_4)$, and U_B the unipotent radical of the uppertriangular Borel of GL_4. Since everything is local, we write F for F_v, p for a uniformizer of $\mathcal{O}_F$ and $|\ |$ the absolute value on F so that $|p|$ is the size of $\mathcal{O}_F$ modulo

its maximal ideal. Then $I(W; s_1, s_2, w)$

$$= \int_{Z\backslash T} \delta_B^{-1}(t) W(t) \int_{R^0\backslash U_B} \chi(u) f_P(\nu_Q ut; w) f_Q(\nu_P ut; s_1, s_2)\, du\, dt.$$
$$= \int_{Z\backslash T} \delta_B^{-1}(t) W(t) \left|\frac{t_1}{t_4}\right| \int_{R^0\backslash U_B} \chi(tut^{-1}) f_P(\nu_Q tu; w) f_Q(\nu_P tu; s_1, s_2)\, du\, dt$$
$$= \int_{Z\backslash T} \delta_B^{-1}(t) W(t) \left|\frac{t_1}{t_4}\right| \left|\frac{t_1 t_3}{t_2 t_4}\right|^{2w} \left|\frac{t_2 t_1 t_4}{t_3^3}\right|^{s_1} \left|\frac{t_2^3}{t_1 t_4 t_3}\right|^{s_2} \int_{R^0\backslash U_B} \chi(tut^{-1}) f_P(\nu_Q u; w) f_Q(\nu_P u; s_1, s_2)\, du\, dt$$

Parametrize $Z\backslash T$ by $(a, b, c) \in F^\times \times F^\times \times F^\times$ with (a, b, c) mapping to the diagonal element $t(a, b, c) = diag(abc, bc, c, 1)$. As usual, set $K = \delta_B^{-1/2} W$. With $t = t(a, b, c)$, then

$$\delta_B^{-1}(t) W(t) \left|\frac{t_1}{t_4}\right| \left|\frac{t_1 t_3}{t_2 t_4}\right|^{2w} \left|\frac{t_2 t_1 t_4}{t_3^3}\right|^{s_1} \left|\frac{t_2^3}{t_1 t_4 t_3}\right|^{s_2} = K(t(a, b, c))|ac|^{2w-\frac{1}{2}} \left|\frac{ab^2}{c}\right|^{s_1-\frac{1}{4}} \left|\frac{b^2 c}{a}\right|^{s_2-\frac{1}{4}},$$

and the vanishing properties of W imply this expression is 0 unless $a, b, c \in \mathcal{O}_F$.

With $t = t(a, b, c)$ and $a, b, c \in \mathcal{O}_F$, we compute the inner integral. For $r \in F$, set $n_{ij}(r) = 1 + rE_{ij}$, E_{ij} the matrix with 1 at the (i, j) position and 0's elsewhere. Then $R^0\backslash U_B$ is isomorphic to $F \times F \times F$ via the map sending (x, y, z) to $n_{12}(x)n_{23}(y)n_{34}(z)$. Observe that the two-by-two matrix $\left(\begin{smallmatrix} 1 & \\ r & 1 \end{smallmatrix}\right)$ is right K equivalent to $\left(\begin{smallmatrix} 1 & r^{-1} \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} r^{-1} & \\ & r \end{smallmatrix}\right)$ when $|r| > 1$. It follows that $f_P(\nu_Q u; w) = f_P(\nu_Q n_{23}(y)) = |y|^{-4w}$ if $|y| > 1$ and is 1 if $|y| \leq 1$. Similary, $f_Q(\nu_P u; s_1, s_2) = f_Q(\nu_P n_{12}(x); s_1, s_2) f_Q(\nu_P n_{34}(z); s_1, s_2)$. One has $f_Q(\nu_P n_{12}(x); s_1, s_2)$ is 1 when $|x| \leq 1$ and $|x|^{-4s_2}$ when $|x| > 1$, and $f_Q(\nu_P n_{34}(z); s_1, s_2)$ is 1 when $|z| \leq 1$ and $|z|^{-4s_1}$ when $|z| > 1$.

The inner integral is thus

$$= \left(1 + \int_{|y|>1} \psi(by)|y|^{-4w}\, dy\right)\left(1 + \int_{|x|>1} \psi(ax)|x|^{-4s_2}\, dx\right)\left(1 + \int_{|z|>1} \psi(cz)|z|^{-4s_1}\, dz\right)$$
$$= \frac{1}{\zeta(4w)\zeta(4s_1)\zeta(4s_2)} \left(\frac{1 - |p|^{(n_b+1)(4w-1)}}{1 - |p|^{4w-1}}\right)\left(\frac{1 - |p|^{(n_a+1)(4s_2-1)}}{1 - |p|^{4s_2-1}}\right)\left(\frac{1 - |p|^{(n_c+1)(4s_1-1)}}{1 - |p|^{4s_1-1}}\right)$$

where n_a, n_b, n_c are the p valuations of a, b, c.

Let us write $K[\ell, m, n]$ for $K(t(p^\ell, p^m, p^n))$. We get that $I(W; s_1, s_2, w)$ is thus

$$\frac{1}{\zeta(4w)\zeta(4s_1)\zeta(4s_2)} \sum_{\ell,m,n\geq 0} K[\ell, m, n]|p|^{(\ell+n)(2w-\frac{1}{2})}|p|^{(\ell+2m-n)(s_1-\frac{1}{4})}|p|^{(2m+n-\ell)(s_2-\frac{1}{4})}$$
$$\times \left(\frac{1 - |p|^{(m+1)(4w-1)}}{1 - |p|^{4w-1}}\right)\left(\frac{1 - |p|^{(\ell+1)(4s_2-1)}}{1 - |p|^{4s_2-1}}\right)\left(\frac{1 - |p|^{(n+1)(4s_1-1)}}{1 - |p|^{4s_1-1}}\right).$$

Set $X = |p|^{(2w-\frac{1}{2})+(s_1-\frac{1}{4})-(s_2-\frac{1}{4})}$, $Y = |p|^{2[(s_1-\frac{1}{4})+(s_2-\frac{1}{4})]}$, $Z = |p|^{(2w-\frac{1}{2})-(s_1-\frac{1}{4})+(s_2-\frac{1}{4})}$. Then

$XZ = |p|^{4w-1}$, $\frac{YZ}{X} = |p|^{4s_2-1}$, $\frac{XY}{Z} = |p|^{4s_1-1}$, and the above is the reciprocal of the zeta factors times

$$\sum_{\ell,m,n\geq 0} K[\ell,m,n]Y^m \left(\frac{1-(XZ)^{m+1}}{1-XZ}\right)\left(\frac{X^{\ell+1}-(YZ)^{\ell+1}}{X-YZ}\right)\left(\frac{Z^{n+1}-(XY)^{n+1}}{Z-XY}\right).$$

By Theorem 7.2.4 below and the Casselman-Shalika formula this is

$$\left(\sum_{u,v\geq 0} A_\pi[u,0,v]X^uZ^v\right)\left(\sum_{t\geq 0} A_\pi[0,t,0]Y^t\right)$$

where $A_\pi[n_1,n_2,n_3]$ is the character $A[n_1,n_2,n_3]$ (see below) evaluated on the conjugacy class associated to π_v in $\mathrm{SL}_4(\mathbf{C})$. This in turn is

$$\frac{L(Std,X)L(\wedge^3,Z)}{\zeta(XZ)}\frac{L(\wedge^2,Y)}{\zeta(Y^2)},$$

and the theorem follows. □

Denote by $\omega_1,\omega_2,\omega_3$ the fundamental weights of SL_4, so that ω_1 is the highest weight of the standard representation, ω_2 the highest weight of $\wedge^2$, and ω_3 the highest weight of $\wedge^3$ or the dual representation.

Theorem 7.2.4. *Write $A[n_1,n_2,n_3]$ for the character of the irreducible representation of SL_4 with highest weight $n_1\omega_1+n_2\omega_2+n_3\omega_3$. Then one has the identity of power series*

$$\sum_{\ell,m,n\geq 0} A[\ell,m,n]Y^m \left(\frac{1-(XZ)^{m+1}}{1-XZ}\right)\left(\frac{X^{\ell+1}-(YZ)^{\ell+1}}{X-YZ}\right)\left(\frac{Z^{n+1}-(XY)^{n+1}}{Z-XY}\right)$$

$$= \left(\sum_{u,v\geq 0} A[u,0,v]X^uZ^v\right)\left(\sum_{t\geq 0} A[0,t,0]Y^t\right)$$

Proof. Omitted. □

Chapter 8

Analytic Justifications

Many of the manipulations in the text are formal. I explain here the standard, well-known analytic justifications of these manipulations.

8.1 Lemmas needed in unfolding and unramified calculations

As our reference for analytic properties of cusp forms useful for integral representations, we will use the very convenient [27] of Miller and Schmid and also Moeglin-Waldspurger [28]. It should suffice to give some detail in the case of one sufficiently complicated integral; all the other integrals are similar. So, let us consider the integral for Spin on GSp_6 that is

$$I(\phi; s) = \int_{[H]} E(g_1; s)\phi(g)\, dg$$

where $H = \mathrm{GL}_2 \boxtimes \mathrm{GL}^*_{2,L}$.

Lemma 8.1.1. *The integral converges absolutely for all s for which the Eisenstein series is finite, and thus defines a meromorphic function of s.*

Proof. The point is that the cusp form ϕ decreases rapidly on Siegel sets, and that the Eisenstein series is slowly increasing wherever it is finite.

To be more precise, first pick an operator norm $||\cdot||$ on $\mathrm{GSp}_6(\mathbf{R})$. Let T be the diagonal torus of GSp_6; it is contained in H. Set T^1 to be elements of T with final entry equal to 1. Parametrize T^1 by elements $a_1, \ldots, a_5$ with the set of a_i corresponding to the element of T^1 that is $diag(a_1a_2a_3a_4a_5, a_1a_2a_3a_4, \ldots, a_1, 1)$. For a constant $C > 0$, define $T^1_C(\mathbf{R})$ to be the elements of

$T^1(\mathbf{R})$ with $|a_i| \geq C$ for all i. Now there is a compact set Ω in $H(\mathbf{A})$ such that a fundamental domain for $H(F)Z(\mathbf{A})\backslash H(\mathbf{A})$ is contained in $T^1_C(\mathbf{R})\Omega$ for some $C > 0$.

Assume s is such that $E(g;s)$ is finite. Then by virtue of its moderate growth, there is some $N > 0$ and some constant $A > 0$ so that $|E(t\omega;s)| \leq A||t||^N$ for all $t \in T^1_C(\mathbf{R})$ and all $\omega \in \Omega$. Now, the cuspidality of ϕ implies that for any integer M there is a constant $B > 0$ so that $|\phi(t\omega)| \leq B||t||^{-M}$ for all $t \in T^1_C(\mathbf{R}), \omega \in \Omega$ (for example, [27] Theorem B). Hence we may bound $|\phi(t\omega)E((t\omega)_1;s)|$ by $D||t||^{-L}$ on $T^1_C(\mathbf{R})\Omega$ for some large integer L and some positive constant D.

Note that $||t|| \geq |a_1|\cdots|a_5|$. Hence $||t||^{-L} \leq (|a_1|\cdots|a_5|)^{-L}$. As the integral of $(|a_1|\cdots|a_5|)^{-L}$ over $T^1_C(\mathbf{R})\Omega$ is finite, the lemma follows. □

We will now prove the manipulations in the unfolding are justified. Let us denote by H the linear functional $\phi \mapsto \phi_\chi(1)$ on the space of smooth vectors of π. Set

$$\phi^1 = \int_{K_H} f^*(k_1;s)\pi(k)\phi\, dk$$

where K is a maximal compact of $\mathrm{GSp}_6(\mathbf{A})$ and $K_H = K \cap H(\mathbf{A})$.

Proposition 8.1.2. *When $Re(s)$ is sufficiently large,*

$$I(\phi,s) = \int_{Z(\mathbf{A})\backslash T_H(\mathbf{A})} \delta_{B_H}^{-1}(t)\delta_{B_2}(t_1)^s H(\pi(t)\phi^1)\, dt,$$

and the right hand side of the above equality converges absolutely.

In the body of the text, we state that

$$I(\phi;s) = \int_{N(\mathbf{A})Z(\mathbf{A})\backslash H(\mathbf{A})} f^*(g_1,s)\phi_\chi(g)\, dg,$$

and then in the unramified computation we apply the Iwasawa decomposition to H and integrate over K_H, to get the local analogue of the right hand side of the Proposition. However, an intermediate equality such as above is never used, and so we only need to justify the equality in this proposition. This is what we will do.

Lemma 8.1.3. *The right hand side converges absolutely for $Re(s)$ sufficiently large.*

Proof. This is an application of the result of Dixmier-Malliavin [10]. Recall the abelian unipotent group N_V, and set $X \simeq \mathbf{G}_a$ the unipotent group consisting of the $n_{35}(z)$'s for $z \in \mathbf{G}_a$. Since ϕ^1 is a smooth vector, applying Dixmier-Malliavin to $N_V(\mathbf{A})$, we have that ϕ^1 is a finite sum of terms of

the form $\Phi_N(n_v) * \xi$ for a Schwartz-Bruhat function Φ_N on N_V. Applying Dixmier-Malliavin again, this time to the group $X(\mathbf{A})$ and the vectors ξ, we have that ϕ^1 is a finite sum of terms of the form $\Phi_N(n_v) * \Phi_X(x) * \xi$.

We parametrize $j : \mathrm{GL}_1(\mathbf{A}) \times \mathrm{GL}_{1,L}(\mathbf{A}) \simeq Z(\mathbf{A}) \backslash T_H(\mathbf{A})$ by $j(r, \ell) = diag(rN(\ell), rN(\ell)\, {}^t\ell^{-1}, \ell, 1)$. We have now $H(j(r,\ell)\Phi_N * \Phi_X * \xi) = \hat{\Phi}_N(\ell)\hat{\Phi}_X(r)H(j(r,\ell)\xi)$ where the $\hat{\Phi}$'s are the Fourier transforms. Hence we only must check the absolute convergence of the integrals

$$\int_{\mathrm{GL}_1(\mathbf{A}) \times \mathrm{GL}_{1,L}(\mathbf{A})} |r^3 N(\ell)|^{-1} |rN(\ell)|^s \hat{\Phi}_N(\ell)\hat{\Phi}_X(r) H(j(r,\ell)\xi)\, d^\times r\, d^\times \ell.$$

But the functional H comes from a Fourier coefficient of a cusp form, so $|H(j(r,\ell)\xi)|$ is bounded independently of r, ℓ, and the absolute convergence follows. □

Proof of Proposition. Take $Re(s)$ sufficiently large so that the sum defining the Eisenstein series converges absolutely. The first step is just the unfolding of the Eisenstein series. This is immediately justified if we can show the absolute convergence of the unfolded integral

$$\int_{(P_2 \boxtimes \mathrm{GL}^*_{2,L})(F) Z(\mathbf{A}) \backslash H(\mathbf{A})} f(g_1; s)\phi(g)\, dg.$$

By Theorem D of [27] we have $|\phi(pk)| \leq C \sup\{1, \delta_Q(p)\}^{-M}$ where Q is the Klingen parabolic of GSp_6, $k \in K_H$, and $p \in P_2 \boxtimes \mathrm{GL}^*_{2,L}$. Write $g_1 = u_1 t_1 k_1$. We also have $|f(g_1; s)| \leq C' \delta^s_{B_2}(t_1)$. Hence we must bound

$$\int_{(P_2 \boxtimes \mathrm{GL}^*_{2,L})(F) Z(\mathbf{A}) \backslash H(\mathbf{A})} \delta^s_{B_2}(t_1) \sup\{1, \delta_Q(p)\}^{-M}\, dg.$$

By Fubini and the Iwasawa decomposition, it is thus sufficient to check the finiteness of

$$\int_{[\mathrm{GL}^*_{2,L}]} |\det(x)|^{s-1} \sup\{1, |\det(x)|\}^{-M}\, dx, \tag{8.1}$$

where we embed $\mathrm{GL}^*_{2,L}$ into H as elements of the form

$$\begin{pmatrix} \nu & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}. \tag{8.2}$$

The finiteness of (8.1) follows from that of

$$\int_{[\mathrm{SL}_{2,L}]} dx$$

and

$$\int_{[\mathrm{GL}_1]} |r|^s \sup\{1, |r|\}^{-M} \, dr \tag{8.3}$$

for M sufficiently large.

Set $\phi_0(g) = \int_{[N_2]} \phi(ng)\, dn$. Then by Fubini's theorem

$$\int_{(P_2 \boxtimes \mathrm{GL}^*_{2,L})(F)Z(\mathbf{A})\backslash H(\mathbf{A})} f(g_1; s)\phi(g)\, dg = \int_{(T_2 \boxtimes \mathrm{GL}^*_{2,L})(F)N_2(\mathbf{A})Z(\mathbf{A})\backslash H(\mathbf{A})} f(g_1; s)\phi_0(g)\, dg.$$

Now we want to replace ϕ_0 by its Fourier expansion along U_{Q_6}/N_2, and unfold further. First we will apply the Iwasawa decomposition to GL_2 and integrate over $\mathrm{GL}_2 \cap K =: K_1$, with GL_2 embedded in H as elements

$$\begin{pmatrix} a & & & b \\ & \nu & & \\ & & 1 & \\ c & & & d \end{pmatrix}.$$

Set

$$\phi' = \int_{K_1} f(k_1; s)\pi(k_1)\phi \, dk_1.$$

We obtain

$$\int_{[\mathrm{GL}^*_{2,L}]} |\det(x)|^{s-1}\phi'_0(x)\, dx.$$

The Fourier expasion is $\phi' = \sum_{v \in W_4, v \neq 0} \phi_v(g)$, which converges absolutely by [28], pages 32-33, since ϕ' is smooth. The unfolded integral would be

$$\int_{(\mathrm{GL}_1 \ltimes N_L)(F)\backslash \mathrm{GL}^*_{2,L}(\mathbf{A})} |\det(x)|^{s-1}\phi'_{f_1}(x)\, dx, \tag{8.4}$$

so the unfolding is justified by the absolute convergence of (8.4). To see this absolute convergence, first apply Dixmier-Malliavin to the abelian groups N_V and $N_{U'}/N_2$ to get that (8.4) is a finite sum of terms of the form

$$\int_{(\mathrm{GL}_1 \ltimes N_L)(F)\backslash \mathrm{GL}^*_{2,L}(\mathbf{A})} |\det(x)|^{s-1}\hat{\Phi}_1(c(x))\hat{\Phi}_2(d(x))H_{f_1}(\pi(x)\xi)\, dx$$

where H_{f_1} is the linear functional $\phi \mapsto \phi_{f_1}(1)$ on the smooth vectors of π, ξ is a smooth vector for π, and $c(x)$, respectively $d(x)$, are the c, d in the notation (8.2). Here Φ_i are Schwartz-Bruhat functions on $\mathbf{G}_a$ over L. Now by [28], pages 32-33, $H_{f_1}(\pi(x)\xi)$ is bounded by $|\xi'(x)|$ for some other ξ' in the space of π, and this in turn is bounded by $\sup\{1, |\det(x)|\}^{-M}$ by Theorem D of [27] applied to the Klingen parabolic Q. So, to check the absolute convergence of (8.4), it suffices to check that of

$$\int_{(\mathrm{GL}_1 \ltimes N_L)(F)\backslash \mathrm{GL}^*_{2,L}(\mathbf{A})} |\det(x)|^{s-1} \hat{\Phi}_1(c(x))\hat{\Phi}_2(d(x)) \sup\{1, |\det(x)|\}^{-M}\, dx.$$

This last integral converges absolutely since

$$\int_{N_L(F)\backslash \mathrm{SL}_{2,L}(\mathbf{A})} |\hat{\Phi}_1(c(x))||\hat{\Phi}_2(d(x))|\, dx$$

is finite and (8.3) is finite for $M >> 0$.

Thus we have that

$$I(\phi; s) = \int_{(\mathrm{GL}_1 \ltimes N_L)(F)\backslash \mathrm{GL}^*_{2,L}(\mathbf{A})} |\det(x)|^{s-1} \phi'_{f_1}(x)\, dx,$$

an equality of absolutely convergent integrals when $Re(s)$ sufficiently large. Set $(\phi')^0_{f_1}(g) = \int_{[N_L]} \phi'_{f_1}(ng)\, dn$. By Fubini, we have

$$I(\phi; s) = \int_{\mathrm{GL}_1(F)N_L(\mathbf{A})\backslash \mathrm{GL}^*_{2,L}(\mathbf{A})} |\det(x)|^{s-1} (\phi')^0_{f_1}(x)\, dx.$$

Now apply Fubini again and the Iwasawa decomposition to get that this is equal to

$$\int_{\mathrm{GL}_1(F)\backslash(\mathrm{GL}_1 \times \mathrm{GL}_{1,L})(\mathbf{A})} \delta^{-1}_{B_{\mathrm{GL}^*_{2,L}}}(t)|t_1|^{s-1} \int_{K \cap \mathrm{GL}^*_{2,L}(\mathbf{A})} (\phi')^0_{f_1}(xk)\, dk\, dt.$$

This inner integral is equal to $(\phi^1)_{f_1}(g)$, and thus this is

$$\int_{\mathrm{GL}_1(F)\backslash(\mathrm{GL}_1 \times \mathrm{GL}_{1,L})(\mathbf{A})} \delta^{-1}_{B_H}(t)|t_1|^{s} (\phi^1)^0_{f_1}(t)\, dt.$$

Hence the final step of the unfolding is justfied by the absolute convergence of the unfolded integral, which is

$$\int_{(\mathrm{GL}_1 \times \mathrm{GL}_{1,L})(\mathbf{A})} \delta^{-1}_{B_H}(t)\delta^{s}_{B_2}(t_1) H(\pi(t)\phi^1)\, dt. \tag{8.5}$$

This absolute convergence was established in the previous lemma. The proposition follows. □

Let Ω denote a finite set of places of F. The absolute convergence gives us $I(\phi;s) = \varinjlim_{\Omega} I_\Omega(\phi;s)$ where

$$I_\Omega(\phi;s) = \int_{Z(\mathbf{A}_\Omega)\backslash T_H(\mathbf{A}_\Omega)} \delta_{B_H}^{-1}(t)\delta_{B_2}^{s}(t_1)H(\pi(t)\phi^1)\,dt.$$

The direct limit is analyzed via the unramified (resp. ramified) calculations, which computes (resp. controls) the local integrals

$$I_v(\ell,s) = \int_{Z(F_v)\backslash T_H(F_v)} \delta_{B_H}^{-1}(t)\delta_{B_2}^{s}(t_1)\ell(\pi(t)v)\,dt$$

when they are absolutely convergent.

Lemma 8.1.4. *The local integrals $I_v(\ell;s)$ needed to compute the limit $\varinjlim_{\Omega}(s)$ are absolutely convergent for $Re(s)$ sufficiently large.*

Proof. The proof is identical to that of Lemma 8.1.3, with F_v replacing $\mathbf{A}$, since the functionals ℓ needed come from Fourier coefficients of cusp forms and thus $\ell(\pi(g)v)$ is bounded as a function of g. □

Lemma 8.1.5. *There is an equality $I(\phi,s) = I_S(s)L^S(\pi,Spin,s) = I_\infty(s)L^S(\pi,Spin,s)$ for $Re(s)$ sufficiently large.*

Proof. We borrow the idea from the proof of [23, Thm. 3.1]. For Ω containing the bad places S, we must evaluate $I_{\Omega\cup\{v\}}(s)$. Take σ_0 so that for $Re(s) > \sigma_0$ the global integral (8.5) converges absolutely, and $\sigma_1 > \sigma_0$ so that for $Re(s) > \sigma_1$ the conclusion of Lemma 8.1.4 is satisfied. Then $I_{\Omega\cup\{v\}}(s)$ is absolutely convergent for $Re(s) > \sigma_0$ and defines a holomporphic function there. On the other hand, for $Re(s) > \sigma_1$, $I_{\Omega\cup\{v\}}(s) = I_\Omega(s)L(\pi_v,Spin,s)$. It follows that $I_{\Omega\cup\{v\}}(s) = I_\Omega(s)L(\pi_v,Spin,s)$ for $Re(s) > \sigma_0$. We conclude $I(\phi,s) = I_S(s)L^S(\pi,Spin,s)$ for $Re(s)$ sufficiently large. □

8.2 Archimedean integral

We now prove the nonvanishing of the archimedean integral

$$I_\infty(s) = \int_{(\mathrm{GL}_1 \times \mathrm{GL}_{1,L})(\mathbf{R})} \delta_{B_H}^{-1}(t)\delta_{B_2}^{s}(t_1)H(\pi(t)\phi^1)\,dt,$$

given that the cuspidal representation supports the fourier coefficient. To do this, we will need to change the order of integration used in the integral, so we first prove

Lemma 8.2.1. *The integral*

$$\int_{U_B^H(\mathbf{R})Z(\mathbf{R})\backslash H(\mathbf{R})} f(g_1;s)\phi_\chi(g)\,dg$$

converges absolutely for $Re(s) >> 0$, and thus is equal to $I_\infty(s)$.

Proof. We follow the style of proof in Ginzburg [17] pgs 762-765 and Jacquet-Shalika [24], section 5.4, who prove the analogous result with $\mathbf{R}$ replaced by $\mathbf{A}$. First, note that we must prove the finiteness of

$$\int_{K_H} f(k_1;s)\int_{T^1}\delta_{B_H}^{-1}(t)\delta_{B_2}^{s}(t_1)|H(\pi(tk)\phi)|\,dt\,dk$$

where T^1 is the set of elements of T_H with final entry equal to 1. We will bound $|H(\pi(t)\phi^1)|$ by a nonnegative Schwartz function of t, independent of k.

To do this, by Dixmier-Malliavin [10] applied to the group $H(\mathbf{R})$, $H(\pi(tk)\phi)$ is a finite sum of terms of the form

$$\int_H H(\pi(tk)\pi(h)\xi)f(h)\,dh$$

for a smooth, compactly supported function f. Hence, changing variables, it suffices to bound

$$\begin{aligned}\int_H f(k^{-1}h)H(\pi(th)\xi)\,dh &= \int_{U_B\backslash H}\int_{U_B} f(k^{-1}uh)H(\pi(tuh)\xi)\,du\,dh \\ &= \int_{U_B\backslash H}\left\{\int_{U_B} f(k^{-1}uh)\chi(tut^{-1})\,du\right\}H(\pi(th)\xi)\,dh.\end{aligned}$$

Consider the function of h,

$$h\mapsto\left|\int_{U_B} f(k^{-1}uh)\chi(tut^{-1})\,du\right|.$$

Since f has compact support, this function has compact support on $U_B\backslash H$. Furthermore, since $H(\pi(th)\xi)$ is a fourier coefficient of a cusp form, it is bounded independently of t, h. It follows that we have

$$|H(\pi(tk)\phi)| \le C\left|\int_{U_B} f(k^{-1}uh)\chi(tut^{-1})\,du\right|$$

for some $h\in H$ and positive constant C, independent of t, k. Now, since k varies in a compact set, the Schwartz functions $u\mapsto f(k^{-1}uh)$ lie in a bounded set in Schwartz space, and thus their fourier transforms

$$\int_{U_B} f(k^{-1}uh)\chi(tut^{-1})\,du$$

(note U_B is abelian) may be bounded by a positive Schwartz function $\Phi(t)$, independent of k. This

is what we wanted to prove. We thus have

$$\int_{U_B^H(\mathbf{R})Z(\mathbf{R})\backslash H(\mathbf{R})} |f(g_1;s)||\phi_\chi(g)|\,dg \leq \int_{K_H} |f(k_1;s)| \left\{ \int_{T^1} \delta_{B_2}^s(t_1)\delta_{B_H}^{-1}(t)\Phi(t)\,dt \right\} dk.$$

Since the inner integral is

$$\int_{(\mathrm{GL}_1 \times \mathrm{GL}_{1,L})(\mathbf{R})} |rN(\ell)|^s |r^3 N(\ell)|^{-1} \Phi(r,\ell)\,d^\times r\,d^\times \ell,$$

this converges for $Re(s)$ sufficiently large. □

Now we prove the nonvanishing.

Proposition 8.2.2. *If the cuspidal representation π supports the fourier coefficient ϕ_χ, then $I_\infty(s)$ is not identically zero for some choice of smooth data $f(g_1;s), \phi$.*

Proof. We work in the range of absolute convergence. By the lemma just proved, we have

$$\begin{aligned}\int_{U_B^H(\mathbf{R})Z(\mathbf{R})\backslash H(\mathbf{R})} f(g_1;s)\phi_\chi(g)\,dg &= \int_{N_L\backslash \mathrm{GL}_{2,L}^* K_2} |\det(g)|^s f(k_1,s)\phi_\chi(gk_1)\,dg\,dk_1 \\ &= \int_{K_1} f(k_1;s) \left\{ \int_{N_L\backslash \mathrm{GL}_{2,L}^*} |\det(g)|^s H(\pi(g)\pi(k)\phi)\,dg \right\} dk_1.\end{aligned}$$

Since we may pick $f(k_1;s)$, we are reduced to showing that we may pick ξ so that

$$\int_{N_L\backslash \mathrm{GL}_{2,L}^*} |\det(g)|^s H(\pi(g)\xi)\,dg$$

is not zero. Now, every $g \in N_L\backslash \mathrm{GL}_{2,L}^*$ may be written uniquely as xr, for $x \in N_L\backslash \mathrm{SL}_{2,L}$ and $r = j(r,1)$, slightly abusing notation. Thus we must check the nonvanishing of

$$\int_{N_L\backslash \mathrm{SL}_{2,L}} \int_{\mathrm{GL}_1} |r|^s H(\pi(x)\pi(r)\xi)\,dr\,dx.$$

Now, note that we may pick ξ_0 so that

$$\int_{\mathrm{GL}_1} |r|^s H(\pi(r)\xi_0)\,dr \neq 0.$$

Indeed, this is simple: take ϕ with $H(\phi) \neq 0$, as hypothesized, and set $\xi_0 = \int_{\mathbf{G}_a} \varphi(z)\pi(n_{35}(z))\phi\,dz$ for an appropriate Schwartz function φ. Now, take Schwartz functions Φ_1, Φ_2 supported on N_V and N_U/N_2, and set $\xi = \Phi_1(v) * \Phi_2(u) * \xi_0$. Then $H(\pi(x)\pi(r)\xi) = \hat{\Phi}_1(c(x))\hat{\Phi}_2(d(x))H(\pi(xr)\xi_0)$. With

the appropriate choice of Φ_1, Φ_2, we can make $\hat{\Phi}_1$ supported only near 0, and $\hat{\Phi}_2$ supported only near 1. Thus $\hat{\Phi}_1(c(x))\hat{\Phi}_2(d(x))$ is supported only near $x = 1$ in $N_L \backslash \mathrm{SL}_{2,L}$. It follows that we may make

$$\int_{N_L \backslash \mathrm{SL}_{2,L}} \hat{\Phi}_1(c(x))\hat{\Phi}_2(d(x)) \int_{\mathrm{GL}_1} |r|^s H(\pi(xr)\xi_0)\, dr\, dx = \int_{N_L \backslash \mathrm{SL}_{2,L}} \int_{\mathrm{GL}_1} |r|^s H(\pi(x)\pi(r)\xi)\, dr\, dx$$

nonzero. □

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