

**From the holdings of the Department of Special Collections , Princeton University Library,
Princeton, NJ**

These documents are either in the public domain, or can only be used for educational and research purposes (Fair use) as per U.S. Copyright law (text below). By accessing this file, all users agree that their use falls in the public domain or within fair use as defined by the copyright law. We do not charge any permission or usage fees for the publication of images of material in our collections, including those provided by Princeton University Library, via our website, catalog, or directly from staff. For more information regarding copyright, credit, and citation please review the information on the Copyright, Credit and Citation Guidelines page of our website.

<https://library.princeton.edu/special-collections/policies/copyright-credit-and-citation-guidelines>. If you have any questions, including how to order higher-quality images from the collection, contact Special Collections Public Services via our Ask Us! Form. <https://library.princeton.edu/special-collections/ask-us>

Doctoral Dissertations, 2011-2026

Mathematics

2023

Some Elliptic PDE Problems in Conformal Geometry

Lee, Sanghoon

Princeton, NJ : Princeton University

<http://arks.princeton.edu/ark:/88435/dsp017d278x338>

U.S. Copyright law: The copyright law of the United States (Title 17, §108, United States Code) governs the making of photocopies or other reproductions of copyrighted material. Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be used for any purpose other than private study, scholarship, or research. If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of fair use that user may be liable for copyright infringement. This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve a violation of copyright law. <https://www.copyright.gov/title17/title17.pdf>

SOME ELLIPTIC PDE PROBLEMS IN CONFORMAL GEOMETRY

SANGHOON LEE

A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
MATHEMATICS
ADVISER: SUN-YUNG ALICE CHANG

SEPTEMBER 2023

© Copyright by Sanghoon Lee, 2023.

All Rights Reserved

Abstract

In this thesis, divided into two parts, we explore two distinct types of elliptic partial differential equation (PDE) problems in conformal geometry. In the first part of thesis, we prove that any closed Riemannian manifold with positive Yamabe invariant and total Q -curvature can be conformally deformed into a metric with positive scalar curvature and constant Q -curvature. Moreover, we derive analogous results for the Q -curvature and T -curvature of a Riemannian manifold with umbilic boundary, subject to the condition that the first Yamabe constant and total (Q, T) -curvature are positive. The proof utilizes on the method of continuity and the Leray-Schauder degree theory. In the second part, we prove that the zero function is the sole solution to a certain degenerate PDE defined in the upper half-space under some geometric assumptions. This result implies that the Euclidean metric is the only adapted compactification of the standard half-plane model of hyperbolic space when the scalar curvature of the compactified metric has a certain sign.

Acknowledgements

First and foremost, I would like to express my deepest appreciation to my advisor, Professor Sun-Yung Alice Chang, for her unwavering support and encouragement. This thesis would never have been completed without her patient guidance. I am especially grateful to her for encouraging me to return when I was on leave from Princeton during the pandemic. I have learned innumerable lessons from her, too many to list.

It is with great honor and pleasure that I extend my sincere gratitude to Professor Paul Chien-Ping Yang and Professor Camillo De Lellis for their invaluable mathematical discussions and for serving on my thesis committee. I would also like to thank Ruobing Zhang for serving as a reader for my thesis and for the many insightful discussions we have had.

I have been very fortunate to have wonderful classmates and friends during my studies. I have learned much from them, and I truly enjoyed the time we spent together. I would like to extend my thanks to Gyujin Oh, Yangyang Li, Ben Lowe, Zhihan Wang, Vikram Giri, Seung-Yeon Ryoo, Sanghak Jeon, Joonhyun La, and Danny Nam.

I would like to express my appreciation to the staff members in the department, especially Jill LeClair and Ankit Tak, for their assistance throughout my time at Princeton.

I am grateful to the Korea Foundation for Advanced Studies for its financial support.

I am indebted to Professor Ki-Ahm Lee, Professor Seung Yeal Ha, and Professor Nam-Gyu Kang for their encouragement and support during my undergraduate studies.

I would like to extend my gratitude to Professor Junehyuk Jung, Professor Donghun Lee, Professor In-Jee Jeong, and Doctor Woon Sang Cho for their help,

friendship, and insightful conversations about life and careers.

I am very grateful to my friends Bumsoo Kim, Dongyeong Ko, Joon-Hyeok Yim, and Sung Gi Park. Thank you to Haesom Kwon for her caring and support.

Above all, I am deeply indebted to my parents for their everlasting love and support.

To my parents.

Contents

Abstract	iii
Acknowledgements	iv
 I Uniformization Theorems for the Q-curvature and the T-curvature	 1
1 Introduction	2
1.1 Background	2
1.2 Previous results	5
1.3 Main theorems	7
1.4 Outline of the proofs	8
 2 Preliminaries	 11
2.1 Functionals defined in closed manifolds	11
2.2 Functionals defined in manifolds with boundary	12
 3 Application of the continuity method: closed manifolds	 17
3.1 Existence of a minimizer at $t = \frac{2}{3}$	17
3.2 Uniform $W^{2,2}(M)$ -estimate for critical points	18
3.3 Positivity of the linearized operator	20
3.4 Proof of Theorem 1.4.1	22

4	Application of the degree theory: closed manifolds	24
4.1	Setup of PDEs	24
4.2	Some a priori estimates	25
4.3	Normal metrics on $\mathbb{R}^4$	28
4.4	Compactness of solutions	29
4.5	Proof of Theorem 1.3.1	31
4.6	A counterexample to a Andrews-type inequality	32
5	Application of the continuity method: manifolds with totally geodesic boundary	36
5.1	Existence of a minimizer at $t = \frac{2}{3}$ and uniform $W^{2,2}(M)$ -estimate . . .	36
5.2	Proof of Theorem 1.4.2	38
6	Application of the degree theory: manifolds with totally geodesic boundary	40
6.1	Setup of PDEs	40
6.2	Some a priori estimates	42
6.3	Proof of Theorem 1.3.2	44
II	Liouville-Type Theorems for Adapted Metrics of Upper Half-Plane Model of Hyperbolic Space	52
7	Introduction	53
7.1	Background	53
7.2	Previous results	55
7.3	Main theorems	56
8	Preliminaries	59
8.1	y^α -weighted Sobolev spaces	59

8.2	Liouville-type theorem	63
8.3	Scattering theory on CCE manifolds and adapted metrics	64
8.4	Conformal infinity with positive scalar curvature	66
9	Proofs of main results	67
9.1	Outline of the proofs	67
9.2	Proof for Theorem 7.3.1	68
9.3	Proofs for Liouville-type theorems	70

Part I

Uniformization Theorems for the Q -curvature and the T -curvature

Chapter 1

Introduction

1.1 Background

In conformal geometry, the Q -curvature of a 4-dimensional Riemannian manifold (M, g) is a curvature invariant associated to the Paneitz operator P_4 . They are defined respectively by

$$Q = \frac{1}{12}(-\Delta R + R^2 - 3|Ric|^2) \quad (1.1)$$

$$P_4\phi = \Delta^2\phi - \operatorname{div}\left(\frac{2}{3}Rg - 2Ric\right)d\phi \quad (1.2)$$

for any smooth function ϕ on M . Here, R is the scalar curvature, Ric is the Ricci curvature, Δ is the Laplace-Beltrami operator, and div is minus of the adjoint of the exterior derivative d . Let $g_w := g^{2w}g$ be the conformally deformed metric where w is a smooth function on M . The Paneitz operator the Q -curvature exhibit the following conformal covariance property:

$$(P_4)_w = e^{-4w}P_4, \quad P_4w + 2Q = 2Q_we^{4w}$$

where $(P_4)_w$ and Q_w are the Paneitz operator and the Q -curvature for the metric g_w . They are defined in any dimensions greater than or equal to three as discussed in [HY] and other related references. However, an especially interesting aspect in dimension four is that Q -curvature is related to the topology of M through the Chern-Gauss-Bonnet formula:

$$\int_M (Q + \frac{|W|^2}{8}) dV = 4\pi^2 \chi(M),$$

where W is the Weyl curvature and $\chi(M)$ is the Euler characteristic of M . The total Q -curvature defined by $k_p := \int_M Q dV$ is a conformal invariant.

The study of the analytic aspects and applications to geometry and topology of the Q -curvature and the Paneitz operator has been a topic of extensive research in recent decades. Notable works in this field include Chang and Yang [CY], Chang-Gursky-Yang [CGY], Djadli and Malchiodi [DM], Gursky [G1], [G2], and Malchiodi [M].

In the context of manifolds with boundary, represented as $(M, \partial M, g)$, a natural boundary operator P_3 and its associated curvature invariant, the T -curvature, have been introduced by Chang and Qing [CQ1] and [CQ2]. They are defined as follows:

$$P_3\phi = \frac{1}{2} \frac{\partial \Delta \phi}{\partial n} + \Delta_{\hat{g}} \frac{\partial \phi}{\partial n} - 2H\hat{\Delta}\phi + L(\hat{\nabla}\phi, \hat{\nabla}\phi) + \hat{\nabla}H \cdot \hat{\nabla}\phi + (F - \frac{R}{3}) \frac{\partial \phi}{\partial n} \quad (1.3)$$

$$T = -\frac{1}{12} \frac{\partial R}{\partial n} + \frac{1}{2} R \cdot H - \langle G, L \rangle + 3H^3 - \frac{1}{3} \text{Tr}(L^3) + \hat{\Delta}H \quad (1.4)$$

for any smooth function ϕ on M . Here, $\hat{\Delta}$, $\hat{\nabla}$ are the Laplace-Beltrami operator and covariant derivative with respect to the induced metric on the boundary $\hat{g} = g|_{\partial M}$. L is the second fundamental form of the boundary, H is the mean curvature, $F = R_{anan}$, $\langle G, L \rangle = R_{anbn}L_{ab}$ where n is the inward normal vector field along the boundary and a, b denote tangential components along the boundary.

They also have the conformal covariance property,

$$(P_3)_w = e^{-3w}P_3, P_3w + T = T_w e^{3w}. \quad (1.5)$$

In four dimensions, we have the following Chern-Gauss-Bonnet formula involving Q -curvature and T -curvature:

$$\int_M (Q + \frac{|W|^2}{8}) dV_g + \oint_{\partial M} (T - \mathcal{L}_4 - \mathcal{L}_5) dS_g = 4\pi^2 \chi(M) \quad (1.6)$$

where $\mathcal{L}_4$ and $\mathcal{L}_5$ are curvature quantities defined explicitly in [CQ1]. As we will be focusing on manifolds with totally geodesic boundary in this paper, the pointwise conformally invariant quantity $(\mathcal{L}_4 + \mathcal{L}_5) dS_g$ is not relevant to our discussion as it vanishes in such cases. Also note that the total (Q, T) -curvature $k_{(P_4, P_3)} := \int_M Q dV + \oint T dS$ is a conformal invariant.

It is worth noting that the Q -curvature, and T -curvature serve as a four-dimensional analogue of the Gaussian curvature and geodesic curvature of Riemannian surfaces with boundary. This is due to the conformal covariance property of the Laplace-Beltrami operator Δ , Gaussian curvature K , and geodesic curvature k of a Riemannian surface (S, g) with boundary. for a smooth function w on S , we have:

$$\begin{aligned} \Delta_w &= e^{-2w} \Delta, \quad \frac{\partial}{\partial n_w} = e^{-w} \frac{\partial}{\partial n} \text{ in } S, \\ -\Delta w + K &= K_w e^{2w}, \quad \frac{\partial w}{\partial n} + k = k_w e^w \text{ on } \partial S. \end{aligned}$$

Here, Δ_w , K_w and n_w are the Laplace-Beltrami operator, Gaussian curvature and the normal vector along the boundary of the metric $g_w := e^{2w}g$. The Chern-Gauss-Bonnet formula, incorporating the Q -curvature and T -curvature in four dimen-

sions, serves as a natural extension of the Gauss-Bonnet formula given below:

$$\int_S K dV + \int_{\partial S} k dS = 2\pi\chi(S).$$

The uniformization theorem for Q -curvature, as well as for manifolds with a boundary in terms of the pair (Q, T) , is a natural consideration. This topic was initially explored for closed manifolds in [CY], and further examined in more general cases in [DM]. Research on uniformization theorems for (Q, T) -curvature has been conducted in works by Catino and Ndiaye [CN], as well as by Ndiaye in [N1] and [N2].

In this part of the thesis, we present a stronger uniformization theorem that not only controls the Q -curvature, but also the sign of the scalar curvature under the condition that the Yamabe constant and total Q -curvature are positive.

1.2 Previous results

In this section, we review previous results regarding the uniformization theorem for Q -curvature in the case of closed manifolds and for (Q, T) -curvature in the case of compact manifolds with boundary.

In the pioneering work Chang and Yang [CY], the following theorem is proved.

Theorem 1.2.1. *[CY, Theorem 1.2] When $k_p < 8\pi^2$, assume that P_4 is a non-negative operator with the kernel of P_4 consisting of constant functions. Then, there exists w such that $Q_w = k_p \text{Vol}(g_w)^{-1}$.*

In subsequent work by [G2], it is proved then under the assumption that the Yamabe constant, which is defined as

$$Y(M, [g]) = \inf_{\tilde{g} \in [g]} \frac{\int_M \tilde{R} d\text{Vol}_{\tilde{g}}}{\text{Vol}(M, \tilde{g})^{\frac{1}{2}}},$$

and the total Q -curvature k_p are non-negative, P_4 is non-negative with $\text{Ker } P_4 = \{\text{constants}\}$ and $k_p \leq 8\pi^2$ with $k_p = 8\pi^2$ if and only if (M, g) is conformally equivalent to the standard sphere. Thus, the uniformization theorem is obtained for the case $Y(M, [g]) \geq 0$ and $0 \leq k_p < 8\pi^2$ by Theorem 1.2.1. The $Y(M, [g]) \geq 0$ and $k_p = 8\pi^2$ case is also solved, as the standard sphere has constant Q -curvature.

A uniformization theorem addressing more general cases, without requiring positivity assumptions on conformal invariants, has been proved in [DM].

Theorem 1.2.2. *[DM, Theorem 1.1] Suppose $\text{Ker } P_4 = \{\text{constants}\}$, and assume that $k_p \neq 8k\pi^2$ for $k = 1, 2, \dots$. Then (M, g) admits a conformal metric with constant Q -curvature.*

They utilize topological methods and min-max schemes in their approach, and the following compactness theorem plays a crucial role in their work.

Theorem 1.2.3. *[M, Theorem 1.1] Suppose $\text{Ker } P_4 = \{\text{constants}\}$ and that $\{u_l\}_l$ is a sequence of solutions of*

$$P_4 u_l + 2Q_l = 2k_l e^{4u_l}, \quad \int_M e^{4u_l} dV = 1$$

with $\{Q_l\}_l$ satisfying $Q_l \rightarrow Q_0$ in $C^0(M)$. Assume also that $k_p \neq 8k\pi^2$ for $k = 1, 2, \dots$. Then $\{u_l\}_l$ is bounded in $C^\alpha(M)$ for any $\alpha \in (0, 1)$.

For manifolds with umbilic boundary, Catino and Ndiaye [CN] has a result analogous to Theorem 1.2.1. Recall that the first Yamabe invariant for a manifold with boundary is defined as

$$Y(M, \partial M, [g]) = \inf_{\tilde{g} \in [g]} \frac{\int_M \tilde{R} d\text{Vol}_{\tilde{g}} + \oint_{\partial M} H dS_{\tilde{g}}}{\text{Vol}(M, \tilde{g})^{\frac{1}{2}}}.$$

When $k_{(P_4, P_3)} \geq 0$ and $Y(M, \partial M, [g]) > 0$, the operator P_4 defined on the set $\{u \in W^{2,2}(M) | \frac{\partial u}{\partial n} = 0 \text{ on } \partial M\}$ is non-negative, and $\text{Ker } P = \{\text{constants}\}$ by Proposition 5.1 of [CN]. In addition, we have $k_{(P_4, P_3)} \leq 4\pi^2$ with $k_{(P_4, P_3)} = 4\pi^2$ if and only if $(M, \partial M, g)$

is conformally equivalent to the standard hemisphere by Lemma 5.2 of [CN]. As an application of these observations, their theorem implies the following:

Theorem 1.2.4. [CN, Corollary 1.10] *Let $(M, \partial M, g)$ be a compact four-dimensional Riemannian manifold with umbilic boundary. Assume $k_{(P_4, P_3)} \geq 0$ and $Y(M, \partial M, [g]) > 0$. Then for every non-negative numbers c_1 and c_2 , M carries a metric conformal to g with $Q \equiv c_1$ and $T \equiv c_2$ and zero mean curvature.*

Ndiaye [N1] and [N2] has established the extensions of the Theorem 1.2.2 for manifolds with boundary.

Theorem 1.2.5. [N1, Theorem 1.1] *Suppose the kernel of the operator P_4 defined on the set $\{u \in W^{2,2}(M) | \frac{\partial u}{\partial n} = 0 \text{ on } \partial M\}$ is $\{\text{constants}\}$. Then, assuming $k_{(P_4, P_3)} \neq 4k\pi^2$ for $k = 1, 2, \dots$, we have that (M, g) admits a conformal metric with constant Q -curvature, zero T -curvature and zero mean curvature.*

Theorem 1.2.6. [N2, Theorem 1.2] *Suppose the kernel of the operator P_4 defined on the set $\{u \in W^{2,2}(M) | \frac{\partial u}{\partial n} = 0 \text{ on } \partial M\}$ is $\{\text{constants}\}$. Then, assuming $k_{(P_4, P_3)} \neq 4k\pi^2$ for $k = 1, 2, \dots$, we have that (M, g) admits a conformal metric with constant T -curvature, zero Q -curvature and zero mean curvature.*

1.3 Main theorems

We will first provide a brief explanation of the motivation behind our study, followed by a statement of the main theorems.

It is well known that for a closed Riemannian manifold, the Yamabe constant is known to be positive if and only if there exists a metric in the conformal class $[g]$ with positive scalar curvature. In dimensions greater than or equal to 6, it has been established that the existence of a conformal metric with positive scalar and Q -curvature is equivalent to the positivity of both the Yamabe invariant and the

Paneitz operator, as shown in Gursky-Han-Lin [GHL]. In dimension 4, we prove a stronger statement, namely that if the both $Y(M, [g])$ and k_p are positive, then there exists a conformal deformation of the metric with constant Q -curvature and positive scalar curvature.

Theorem 1.3.1. *Let (M, g) be a four-dimensional closed manifold with conformal invariants k_p and $Y(M, [g])$ positive. Then there exists a conformal deformation w such that Q_w is constant, and $R_w > 0$.*

Below, we state our main theorem regarding manifolds with umbilic boundary and positive conformal invariants $k_{(p_4, p_3)}$ and $Y(M, \partial M, [g])$.

Theorem 1.3.2. *Let $(M, \partial M, g)$ be a compact manifold with umbilic boundary and positive conformal invariants $k_{(p_4, p_3)}$ and $Y(M, \partial M, [g])$. Then there exist conformal deformations w_1 and w_2 such that*

$$\left\{ \begin{array}{l} Q_{w_1} \equiv \frac{k_{(p_4, p_3)}}{\text{Vol}(M, g_{w_1})}, R_{w_1} > 0 \text{ on } M \\ T_{w_1} \equiv 0, H_{w_1} \equiv 0 \text{ on } \partial M \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} Q_{w_2} \equiv 0, R_{w_2} > 0 \text{ on } M \\ T_{w_2} \equiv \frac{k_{(p_4, p_3)}}{\text{Vol}(\partial M, g_{w_2})}, H_{w_2} \equiv 0 \text{ on } \partial M. \end{array} \right.$$

Note that an umbilic boundary can always be conformally deformed to totally geodesic boundary. Therefore, we can assume that the background metric has a totally geodesic boundary.

1.4 Outline of the proofs

We shall now outline our strategy for proving the main theorems, which involves two primary steps.

The first step is to deform the background metric using the continuity method so that both the scalar curvature and the Q -curvature are pointwise positive. Specifically, we will prove the following two theorems.

Theorem 1.4.1. *Let (M, g_0) be a closed manifold with conformal invariants k_p and $Y(M, [g_0])$ positive. Then, there exists a conformal deformation w such that $Q_w > 0$ and $R_w > 0$.*

Theorem 1.4.2. *Let $(M, \partial M, g_0)$ be a compact manifold with umbilic boundary and the conformal invariants $k_{(p_4, p_3)}$ and $Y(M, \partial M, [g_0])$ positive. Then, there are conformal deformations w_1, w_2 with following properties*

$$\begin{cases} Q_{w_1} > 0, R_{w_1} > 0 \text{ on } M \\ T_{w_1} = 0, H_{w_1} = 0 \text{ on } \partial M \end{cases} \quad \text{and} \quad \begin{cases} Q_{w_2} = 0, R_{w_2} > 0 \text{ on } M \\ T_{w_2} > 0, H_{w_2} = 0 \text{ on } \partial M \end{cases}$$

The method of continuity is applied to critical points of suitable linear combinations of functionals defined in [BO] for the case of closed manifolds. These functionals appeared while computing explicit formulas for functional determinants of conformally covariant operators in 4-dimensional Riemannian manifolds and have been studied in [CY], [CGY] and [G2] to derive interesting geometric results. For manifolds with boundary, we consider linear combinations functionals appeared in [CQ1] and [CQ2]. The crucial ingredient is that the positivity of scalar curvature can be preserved throughout this process. The definitions of functionals we are considering are provided in Chapter 2, while the proofs applying the method of continuity can be found in Chapter 3 for closed manifolds and Chapter 5 for compact manifolds with boundary.

The second step in our strategy is to apply the Leray-Schauder degree theory to a one-parameter family of fourth-order equations to establish the existence of conformal metrics desired in the main theorems. As the approach for manifolds with totally geodesic boundary is similar to that of closed manifolds, we will only describe the closed manifold case. The proof of this step is detailed in Chapter 4 for closed manifolds and in Chapter 6 for compact manifolds with totally geodesic boundary.

We consider equations of the form

$$P_4 w_t = 2tk_p e^{4w_t} - 2tQ \quad (1.7)$$

with the constraint $R_{w_t} > 0$, where $t \in [0, 1]$. The compactness result is an essential part that ensures the well-definedness of the degree, and once we establish it, the remainder of the proof is straightforward. Instead of directly invoking Theorem 1.2.3, we offer an alternative approach for the compactness theorem under our assumptions. The proof employs the concept of normal metrics for $\mathbb{R}^4$, which is defined in [CQY]. A blow-up analysis shows that the limit conformal factor w_∞ on $\mathbb{R}^4$ satisfies $R_{w_\infty} \geq 0$. Then, according to the result of [CQY], w_∞ is normal, or in simpler terms, the PDE for w_∞ is transformed into an integral equation of the form

$$w_\infty(x) = \frac{3}{4\pi^2} \int_{\mathbb{R}^4} \log \frac{|y|}{|x-y|} Q_w(y) e^{4w_\infty(y)} dy + C.$$

The proof for the compactness of solutions is concluded by the classification result for solutions of the above integral equation by Xu [X]. Finally, in the last section of Chapter 4, we will discuss an example that illustrates a limitation of the method of continuity when it comes to proving Theorem 1.3.1.

Chapter 2

Preliminaries

2.1 Functionals defined in closed manifolds

In this section, we define linear combinations of some functionals studied in [BO], [CY] and [CGY]. We will also recall some basic properties for critical points of such functionals. Throughout this chapter and the next, we will assume that (M, g_0) is a closed manifold with $k_p = \int_M Q_0 dV_0 > 0$ and $Y(M, [g_0]) > 0$. According to [G2, Theorem B], we have $k_p \leq 8\pi^2$, and if $k_p = 8\pi^2$, then M is conformally equivalent to the standard sphere S^4 . Therefore, we may assume that $k_p < 8\pi^2$. Additionally, we will fix the background metric to be a Yamabe metric so that R_0 is constant.

Definition 2.1.1. Let η be a fixed $(0, 2)$ -tensor defined on M with $|\eta|_0^2 > 0$ everywhere. We define some functionals as follows:

$$\begin{aligned} I[w] &= \int 4|\eta|_0^2 w dV_0 - \left(\int |\eta|_0^2 dV_0 \right) \log \oint e^{4w} dV_0 \\ II[w] &= \int w(P_4)_0 w dV_0 + \int 4Q_0 w dV_0 - k_p \log \oint e^{4w} dV_0 \\ &= \int \left[(\Delta_0 w)^2 + \frac{2}{3} R_0 |\nabla_0 w|^2 - 2\text{Ric}_0(\nabla_0 w, \nabla_0 w) + 4Q_0 w \right] dV_0 - k_p \log \oint e^{4w} dV_0 \\ III[w] &= \frac{1}{36} \left(\int R_w^2 dV_w - \int R_0^2 dV_0 \right) = \int \left[(\Delta_0 w + |\nabla_0 w|^2)^2 - \frac{1}{3} R_0 |\nabla_0 w|^2 \right] dV_0 \end{aligned}$$

In Chapter 3, we will study critical points of the following one-parameter of family of functionals

$$F_t[w] := \gamma(t)I[w] + tII[w] + (1-t)III[w] \quad (2.1)$$

where $\gamma(t) = \frac{-tk_p}{\int |\eta|_0^2 dv_0}$. We compute the Euler-Lagrange equation of for a critical point w_t of F_t and present its weak formulation.

Proposition 2.1.2. *The Euler-Lagrange equation for a critical point w_t and its corresponding metric $g_t := e^{2w_t}g_0$ of this functional is given by:*

$$\gamma(t)|\eta|_t^2 + tQ_t + \frac{(1-t)}{12}(-\Delta_t R_t) = 0. \quad (*)_t$$

Proof. For the computation of the Euler-Lagrange equation, we refer the readers to the proof of [CY, Theorem 1.1]. $\square$

Proposition 2.1.3. *Let w_t be a critical point of the functional F_t . Then we have*

$$\begin{aligned} & \int \Delta_0 w_t \Delta_0 \phi + 2(1-t)(\nabla_0 w_t \cdot \nabla_0 \phi)(\Delta_0 w_t + |\nabla_0 w_t|^2) + (1-t)|\nabla_0 w_t|^2 \Delta_0 \phi \\ &= -(t - \frac{1}{3}) \int R_0 \nabla_0 w_t \nabla_0 \phi - 2t \text{Ric}_0(\nabla_0 w_t, \nabla_0 \phi) - \int (2tQ_0 + 2\gamma(t)|\eta|_0^2)\phi \end{aligned} \quad (2.2)$$

for any $\phi \in W^{2,2}(M)$.

2.2 Functionals defined in manifolds with boundary

In this section, we define linear combinations of some functionals studied in [CQ1] and [CQ2] and recall some basic properties for critical points of such functionals.

Throughout Chapter 2 and 5, we assume that $(M, \partial M, g_0)$ is a compact manifold with totally geodesic boundary, such that $k_{(P_4, P_3)} = \int Q + \oint T > 0$ and $Y(M, \partial M, [g_0]) >$

0. According to [CN, Lemma 5.2], we have $k_{(P_4, P_3)} \leq 4\pi^2$, with $k_p = 4\pi^2$ if and only if M is conformally equivalent to S_+^4 with the standard metric. Consequently, we may assume that $k_{(P_4, P_3)} < 4\pi^2$.

Furthermore, by [E, Theorem 6.1], we can assume that the background metric R_0 is the boundary Yamabe metric and that the boundary is totally geodesic. In particular, R_0 is constant. For manifolds with totally geodesic boundary, we have simplified expressions for the functionals considered in [CQ1] and [CQ2]. The expressions (1.3) and (1.4) are simplified to $P_3 w = \frac{1}{2} \frac{\partial \Delta w}{\partial n}$ for $\frac{\partial w}{\partial n} = 0$, and $T = -\frac{1}{12} \frac{\partial R}{\partial n}$.

Definition 2.2.1. Let η_1 be a fixed $(0, 2)$ -tensor defined on M with $|\eta_1| > 0$ everywhere. In addition, let η_2 be a fixed $(0, 2)$ -tensor defined on ∂M with $|\eta_2|_{g_0|_{\partial M}} > 0$ everywhere on ∂M . We define:

$$\begin{aligned} I[w] &= \int 4|\eta_1|_0^2 w dV_0 - \left(\int |\eta_1|_0^2 dV_0 \right) \log \oint_M e^{4w} dV_0 \\ I_b[w] &= \oint 3|\eta_2|_0^{\frac{3}{2}} w dV_0 - \left(\oint |\eta_2|_0^{\frac{3}{2}} dV_0 \right) \log \oint_{\partial M} e^{3w} dV_0 \\ II_b[w] &= \int (w P_4 w + 4Q_0 w) dV_0 + \oint_{\partial M} (2w P_3 w + 4T_0 w) dS_0 \\ &= \int \left[(\Delta_0 w)^2 + \frac{2}{3} R_0 |\nabla_0 w|^2 - 2\text{Ric}_0(\nabla_0 w, \nabla_0 w) + 4Q_0 w \right] dV_0 + \oint_{\partial M} 4T_0 w dS_0 \\ III[w] &= \frac{1}{36} \left(\int R_w^2 dV_w - \int R_0^2 dV_0 \right) = \int \left[(\Delta_0 w + |\nabla_0 w|^2)^2 - \frac{1}{3} R_0 |\nabla_0 w|^2 \right] dV_0 \end{aligned}$$

Here, we have used the same notations for $I[w]$, $III[w]$ because the expressions are the same as in the closed manifold case. Also note that $T_0 \equiv 0$ as R_0 is constant.

We recall the following useful lemma for our computations from now on. Note that R_0 is constant in our case.

Lemma 2.2.2. [CN, Lemma 2.10] Let (M, ∂, g_0) be a Riemannian manifold with totally geodesic boundary. If u is a C^2 function with $\frac{\partial u}{\partial n_0} = 0$, then $\frac{\partial |\nabla_0 u|^2}{\partial n_0} = 0$, and $\text{Ric}_0(n_0, \nabla_0 u) = 0$ on the boundary ∂M .

In the next Chapter, we will apply the method of continuity for the following two problems as a preliminary step for proving Theorem 1.3.2. The computations to derive Euler-Lagrange equations and weak formulations for critical points are identical to that of Proposition 2.1.2 and Proposition 3.2.1 except for the additional boundary terms that need to be handled.

Problem 1: We would like to find a conformal deformation w_1 such that

$$\begin{cases} Q_{w_1} > 0 \text{ on } M \\ R_{w_1} > 0 \text{ on } M \\ T_{w_1} = 0 \iff \frac{\partial \Delta_0 w_1}{\partial n_0} = 0 \text{ on } \partial M \\ H_{w_1} = 0 \iff \frac{\partial w_1}{\partial n_0} = 0 \text{ on } \partial M. \end{cases}$$

Our strategy is to find a one-parameter family of critical points w_t of functionals

$$F_t^1[w] := \gamma_1(t)I[w] + t(II_b[w] - k_{(P_4, P_3)}) \log \int e^{4w} dv_0 + (1-t)III[w]$$

defined on the set $\{w \in W^{2,2}(M) \mid \frac{\partial w}{\partial n_0} \equiv 0\}$, using the method of continuity. Here,

$$\gamma_1(t) = \frac{-tk_p}{\int |\eta_1|_0^2 dV_0}.$$

The Euler Lagrange equation for a critical point of the functional F_t^1 is

$$\begin{cases} \gamma_1(t)|\eta|_t^2 + tQ_t + \frac{(1-t)}{12}(-\Delta_t R_t) = 0 \\ T_t = 0 \\ H_t = 0 \end{cases} \quad (\star)_t$$

Obviously $H_t = 0$ is automatically satisfied since $\frac{\partial w}{\partial n_0} = 0$. The weak formulation of

the above equation is

$$\begin{aligned} & \int \Delta_0 w_t \Delta_0 \phi + 2(1-t)(\nabla_0 w_t \cdot \nabla_0 \phi)(\Delta_0 w_t + |\nabla_0 w_t|^2) + (1-t)|\nabla_0 w_t|^2 \Delta_0 \phi \\ &= -(t - \frac{1}{3}) \int R_0 \nabla_0 w_t \nabla_0 \phi - 2t \text{Ric}_0(\nabla_0 w_t, \nabla_0 \phi) - \int (2tQ_0 + 2\gamma_1(t)|\eta_1|_0^2) \phi \end{aligned} \quad (2.3)$$

for any $\phi \in W^{2,2}(M)$ with $\frac{\partial \phi}{\partial n_0} \equiv 0$.

Problem 2: We would like to find a conformal deformation w_2 such that

$$\begin{cases} Q_{w_2} = 0 \iff (P_4)_0 + 2Q_0 = 0 \text{ on } M \\ R_{w_2} > 0 \text{ on } M \\ T_{w_2} > 0 \text{ on } \partial M \\ H_{w_2} = 0 \iff \frac{\partial w_2}{\partial n_0} = 0 \text{ on } \partial M. \end{cases}$$

Likewise, our strategy is to find a 1-parameter family of critical points w_t of functionals

$$F_t^2[w] := \gamma_2(t)I_b[w] + t(II_b[w] - \frac{4}{3}k_{(P_4, P_3)} \log \oint_{\partial M} e^{3w}) + (1-t)III[w]$$

defined on the set $\{w \in W^{2,2}(M) | \frac{\partial w}{\partial n_0} \equiv 0\}$. Here, $\gamma_2(t) = \frac{-4tk_{(P_4, P_3)}}{3 \oint |\eta_2|_0^{3/2} dS_0}$.

The Euler Lagrange equation for for a critical point of the functional F_t^2 is

$$\begin{cases} tQ_t + \frac{(1-t)}{12}(-\Delta_t R_t) = 0 \\ T_t = -\frac{3}{4}\gamma_2(t)|\eta_2|_t^{\frac{3}{2}} \\ H_t = 0 \end{cases} \quad (\dagger)_t$$

The weak formulation of the above equation is

$$\begin{aligned}
& \int \Delta_0 w_t \Delta_0 \phi + 2(1-t)(\nabla_0 w_t \cdot \nabla_0 \phi)(\Delta_0 w_t + |\nabla_0 w_t|^2) + (1-t)|\nabla_0 w_t|^2 \Delta_0 \phi \\
& = -(t - \frac{1}{3}) \int R_0 \nabla_0 w_t \nabla_0 \phi - 2t \text{Ric}_0(\nabla_0 w_t, \nabla_0 \phi) - \int 2t Q_0 \phi - \frac{3}{2} \gamma_2(t) \oint |\eta_2|_0^{\frac{3}{2}} \phi \quad (2.4)
\end{aligned}$$

for any $\phi \in W^{2,2}(M)$ with $\frac{\partial \phi}{\partial n_0} \equiv 0$.

Chapter 3

Application of the continuity method: closed manifolds

3.1 Existence of a minimizer at $t = \frac{2}{3}$

In this chapter, we apply the method of continuity to a one-parameter family of functions:

$$F_t[w] := \gamma(t)I[w] + tII[w] + (1-t)III[w] \quad (3.1)$$

for $t \in [\frac{2}{3}, 1]$. Note that at $t = 1$, the Euler-Lagrange equation $(*)_t$ implies that $Q_1 = -\gamma(1)|\eta|_1^2 > 0$. The choice of $\frac{2}{3}$ as the starting point is motivated by the following lemma.

Lemma 3.1.1. *[G1, Lemma 1.2] Let (M, g) be a four-dimensional compact Riemannian manifold. Suppose that the scalar curvature R of g satisfies*

$$\Delta R - \frac{1}{6}R^2 \leq 0. \quad (3.2)$$

Then

(1) *if $Y(M, [g]) > 0$, $R > 0$ on M .*

(2) if $Y(M, [g]) = 0$, $R \equiv 0$ on M .

By using the above lemma we can prove that for any critical point of $F_{\frac{2}{3}}$, the deformed metric has positive scalar curvature. Later, we will prove that the positivity of the scalar curvature is preserved as we apply the continuity method. Let $t_0 = \frac{2}{3}$.

Lemma 3.1.2. *There exists a smooth minimizer w_{t_0} of F_{t_0} . For w_{t_0} , R_{t_0} is positive.*

Proof. In [G2, Theorem A], it is proved that $(P_4)_0$ is non-negative operator on $W^{2,2}(M)$ which also implies that $\int_M w(P_4)_0 w \geq C \|w - \int_M w\|_{W^{2,2}(M)}$ for some constant C . Furthermore, $\int R_w^2 dV_w \geq (\int R_w dV_w)^2 \geq (\int R_0 dV_0)^2 = \int R_0^2 dV_0$. Here, we used the fact that the Yamabe metric minimizes the total scalar curvature when the volume is fixed. Note that there is no $\int_M e^{4w} dV_0$ term in the functional F_{t_0} due to our choice of $\gamma(t_0)$. Therefore F_{t_0} is both bounded below and coercive, allowing us to find a minimizer w_{t_0} with the normalization $\int w_{t_0} dV_0 = 0$. w_{t_0} is smooth by [UV].

From the equation $(*)_t$, we have

$$-\Delta_{t_0} R_{t_0} + \frac{1}{6} R_{t_0}^2 = -12\gamma\left(\frac{2}{3}\right) |\eta|_{t_0}^2 + 2|E_{t_0}|^2 > 0.$$

where E_{t_0} is the traceless part of the Ric_{t_0} . By applying Lemma 3.1.1, we deduce that R_{t_0} is positive.

□

3.2 Uniform $W^{2,2}(M)$ -estimate for critical points

In this section, we observe that any critical point w_t of F_t with $R_t > 0$ satisfies $\|w_t\|_{W^{2,2}} < C$ for some constant independent of t for $\frac{2}{3} \leq t \leq 1$. This fact will be used to prove closedness part when we apply the method of continuity in Section 3.

Proposition 3.2.1. *Let w_t be a critical point of the functional F_t with normalization $\bar{w}_t := \int w_t dV_0 = 0$ for $\frac{2}{3} \leq t \leq 1$. Assume $R_t > 0$. Then, $\|w_t\|_{W^{2,2}} < C$ where C is a constant independent of t .*

Proof. Substituting $\phi = w_t$ in (2.2), we have

$$\begin{aligned} & \int (\Delta_0 w_t)^2 + (1-t)[3|\nabla_0 w_t|^2 \Delta_0 w_t + 2|\nabla_0 w_t|^4] \\ &= -(t - \frac{1}{3}) \int R_0 |\nabla_0 w_t|^2 + 2t \int Ric_0(\nabla_0 w_t, \nabla_0 w_t) - \int (2tQ_0 + 2\gamma(t)|\eta|_0^2)w_t \end{aligned}$$

Recall the scalar curvature equation:

$$\Delta_0 w_t + |\nabla_0 w_t|^2 + \frac{1}{6}R_t e^{2w_t} = \frac{1}{6}R_0.$$

As $R_t > 0$, we have $\int_M |\nabla_0 w_t|^2 = \int_M \Delta_0 w_t + |\nabla_0 w_t|^2 < \frac{1}{6} \int_M R_0$ which is an uniform bound on $\int |\nabla_0 w_t|^2$.

By Young's inequality, we have $a^2 + 3(1-t)ab + 2(1-t)b^2 \geq [1 - \frac{9(1-t)}{8}]a^2 \geq \frac{3}{8}a^2$ for any real number a, b and $t \geq \frac{2}{3}$. Applying this inequality to the left-hand side, we obtain:

$$\int (\Delta_0 w_t)^2 + (1-t)[3|\nabla_0 w_t|^2 \Delta_0 w_t + 2|\nabla_0 w_t|^4] \geq \frac{3}{8} \int_M (\Delta_0 w_t)^2.$$

For the right-hand side, we observe:

$$\begin{aligned} & -(t - \frac{1}{3}) \int R_0 |\nabla_0 w_t|^2 + 2t \int Ric_0(\nabla_0 w_t, \nabla_0 w_t) - \int (2tQ_0 + 2\gamma(t)|\eta|_0^2)w_t \\ & \leq 2t|Ric_0|_{L^\infty} \int |\nabla_0 w_t|^2 - \int (2tQ_0 + 2\gamma(t)|\eta|_0^2)(w_t - \bar{w}_t) \\ & \leq C \int |\nabla_0 w_t|^2 + \frac{1}{2} \int (2tQ_0 + 2\gamma(t)|\eta|_0^2)^2 + \frac{1}{2} \int (w_t - \bar{w}_t)^2 \\ & \leq C \int |\nabla_0 w_t|^2 + C. \end{aligned}$$

In the last line, we used the Poincaré's inequality. As $\bar{w}_t = \int w_t = 0$, this gives us a upper bound on the full $W^{2,2}(M)$ -norm of w_t . $\square$

3.3 Positivity of the linearizd operator

In this section we establish positivity of linearized operators of the equation $(*)_t$.

Let w_t be a critical point of the functional F_t defined in equation (2.1), and let $g_t = e^{2w_t}g_0$ be the corresponding metric. Denote the linearization of the equation $(*)_t$ at g_t by L_t . When L_t is restricted to the Hölder space $C^{4,\alpha}(M)$ for $0 < \alpha < 1$, $L_t : C^{4,\alpha}(M) \rightarrow C^\alpha(M)$ is a bounded linear operator. The explicit formula for L_t is derived in the proof of [CY, Theorem 2.1] and is as follows.

$$\langle L_t \phi, \phi \rangle = t \langle (P_4)_t \phi, \phi \rangle + (1-t) \left[\int (\Delta_t \phi)^2 - \frac{1}{3} \int R_t |\nabla_t \phi|^2 \right] \quad (3.3)$$

for $\phi \in W^{2,2}(M)$. To prove the openness part of the continuity method, we need to prove that $\ker L_t$ is trivial when the scalar curvature R_t is positive.

Lemma 3.3.1. *Let w_t be a critical point of the functional (2.1) with positive scalar curvature R_t where $\frac{2}{3} \leq t \leq 1$. Then, L_t is a non-negative operator on $W^{2,2}(M)$ and $\ker L_t \simeq \mathbb{R}$.*

Remark. *The proof uses the same techniques appeared in [G2, Lemma 3.1] and [CGY, Lemma 4.2]. The only difference is that the ratio between the coefficients of $II[w]$ and $III[w]$ in the functional being considered lies in a different interval.*

Proof. Denote the traceless part of the Ricci curvature of the metric g_t by E_t . Let $\phi \in W^{2,2}(M)$. The equation $(*)_t$ is rewritten as

$$-\Delta_t R_t + \frac{t}{4} R_t^2 - 3t |E_t|^2 = -12\gamma(t) |\eta_t|^2 > 0. \quad (3.4)$$

We have the following inequality using the equation (5.1)

$$\begin{aligned}
-4 \int E_t(\nabla_t \phi, \nabla_t \phi) &\geq \int -2\sqrt{3}|E_t||\nabla_t \phi|^2 \\
&\geq \int -2\epsilon \left(\frac{\sqrt{3}}{2}\right)^2 \frac{|E_t|^2}{R_t} |\nabla_t \phi|^2 - 2\epsilon^{-1} R_t |\nabla_t \phi|^2 \\
&\geq \frac{\epsilon}{2t} \int \frac{\Delta_t R_t}{R_t} |\nabla_t \phi|^2 - \left(\frac{\epsilon}{8} + \frac{2}{\epsilon}\right) \int R_t |\nabla_t \phi|^2.
\end{aligned}$$

where ϵ is a positive number to be determined later.

In addition, we have

$$\begin{aligned}
\int \frac{\Delta_t R_t}{R_t} |\nabla_t \phi|^2 &= \int -\nabla_t R_t \cdot \nabla_t (R_t^{-1}) |\nabla_t \phi|^2 - \frac{\nabla_t R_t}{R_t} \cdot \nabla_t |\nabla_t \phi|^2 \\
&= \int \frac{|\nabla_t R_t|^2}{R_t^2} |\nabla_t \phi|^2 - 2\nabla_t^2 \phi \left(\nabla_t \phi, \frac{\nabla_t R_t}{R_t} \right) \\
&\geq \int \frac{|\nabla_t R_t|^2}{R_t^2} |\nabla_t \phi|^2 - \int \left[|\nabla_t^2 \phi|^2 + \frac{|\nabla_t R_t|^2}{R_t^2} |\nabla_t \phi|^2 \right] \\
&= - \int |\nabla_t^2 \phi|^2 \\
&= \int -(\Delta_t \phi)^2 + E_t(\nabla_t \phi, \nabla_t \phi) + \frac{1}{4} R_t |\nabla_t \phi|^2.
\end{aligned}$$

In the last line, we used the Bochner formula.

Combining above inequalities, we get

$$-4 \int E_t(\nabla_t \phi, \nabla_t \phi) \geq -\frac{\epsilon}{2t} \int (\Delta_t \phi)^2 + \frac{\epsilon}{2t} \int E_t(\nabla_t \phi, \nabla_t \phi) - \left(\frac{\epsilon}{8} + \frac{2}{\epsilon} - \frac{\epsilon}{8t}\right) \int R_t |\nabla_t \phi|^2.$$

Hence we have

$$\left(-4 - \frac{\epsilon}{2t}\right) \int E_t(\nabla_t \phi, \nabla_t \phi) \geq -\frac{\epsilon}{2t} \int (\Delta_t \phi)^2 + \left(\frac{\epsilon}{8t} - \frac{\epsilon}{8} - \frac{2}{\epsilon}\right) \int R_t |\nabla_t \phi|^2. \quad (3.5)$$

By plugging the inequality (3.5) into the right hand side of the equation (3.3), we have

$$\begin{aligned}
\langle L_t \phi, \phi \rangle &= \int (\Delta_t \phi)^2 - 2t \int E_t(\nabla_t \phi, \nabla_t \phi) + \left(\frac{1}{2}t - \frac{1}{3}\right) \int |\nabla_t \phi|^2 \\
&\geq C_1(\epsilon) \int (\Delta_t \phi)^2 + C_2(\epsilon) \int R_t |\nabla_t \phi|^2
\end{aligned}$$

where $C_1(\epsilon) = (1 - \frac{\epsilon}{4+\frac{\epsilon}{2t}})$ and $C_2(\epsilon) = \frac{1}{2}t - \frac{1}{3} - (\frac{2t}{4+\frac{\epsilon}{2t}})(\frac{\epsilon}{8} + \frac{2}{\epsilon} - \frac{\epsilon}{8t})$.

Now we check that we can choose ϵ so that both of coefficients $C_1(\epsilon)$ and $C_2(\epsilon)$ is positive when $\frac{2}{3} \leq t \leq 1$.

For $C_1(\epsilon)$, we see $1 - \frac{\epsilon}{4+\frac{\epsilon}{2t}} > 0 \iff \frac{4}{1-\frac{1}{2t}} > \epsilon$ and for $C_2(\epsilon)$ we compute:
 $\frac{1}{2}t - \frac{1}{3} - (\frac{2t}{4+\frac{\epsilon}{2t}})(\frac{\epsilon}{8} + \frac{2}{\epsilon} - \frac{\epsilon}{8t}) > 0 \iff (\frac{1}{2}t - \frac{1}{3})(4 + \frac{\epsilon}{2t})\epsilon - 2t(\frac{\epsilon}{8} + \frac{2}{\epsilon} - \frac{\epsilon}{8t})\epsilon > 0 \iff$
 $(\frac{1}{2} - \frac{t}{4} - \frac{1}{6t})\epsilon^2 + (2t - \frac{4}{3})\epsilon - 4t > 0$. The later inequality is quadratic w.r.t ϵ so it is enough to show the inequality holds for $\epsilon = \frac{4}{1-\frac{1}{2t}}$.

It suffices to check $16(\frac{1}{2} - \frac{t}{4} - \frac{1}{6t}) + 4(2t - \frac{4}{3})(1 - \frac{1}{2t}) - 4t(1 - \frac{1}{2t})^2 > 0$ which is equivalent to $t > \frac{3}{8}$. Thus, we can choose ϵ so that $C_1(\epsilon)$ and $C_2(\epsilon)$ are both positive. It is trivial to see that $L_t \phi = 0$ if and only if ϕ is constant. $\square$

3.4 Proof of Theorem 1.4.1

Now we are ready to prove the main theorem of this chapter which implies that we can conformally deform the background metric so that both the scalar curvature and the Q -curvature are positive. The strategy is to apply the continuity method to equation $(*)_t$. Roughly speaking, the openness part follows from the Lemma 3.3.1 and the closedness part follows from the Proposition 3.2.1.

Proof of Theorem 1.4.1. We use the method of continuity to solve the one-parameter family of equations $(*)_t$. We define

$$S = \{t \in [2/3, 1] \mid (*)_t \text{ has a smooth solution with positive scalar curvature}\}.$$

For $t = \frac{2}{3}$, we find a minimizer $w_{\frac{2}{3}}$ for the functional $\gamma_1(2/3)I + 8II + \frac{1}{3}III$. Then by Lemma 3.1.2, $g_{\frac{2}{3}}$ has positive scalar curvature. Hence $\frac{2}{3} \in S$. We will show that S is both open and closed to conclude $S = [2/3, 1]$.

First, we prove that S is open. Assume $t_1 \in S$. Since the linearization of the equation $(*)_t$ is positive by Lemma 3.3.1, there is a unique smooth solution w_t of $(*)_t$ for all t sufficiently close to t_1 , provided that we normalize by $\int w_t dV_0 = 0$ according to the perturbation theorem. By taking a sufficiently small $C^{4,\alpha}(M)$ -neighborhood of w_{t_1} , we can ensure that these solutions w_t and $g_t := e^{2w_t}g_0$ also have positive scalar curvature. This demonstrates that S is open.

Now we prove that S is closed. Suppose $t_n \in S$, and $t_n \rightarrow t'$. Let w_{t_n} denote the corresponding solutions of $(*)_{t_n}$. By Proposition 3.2.1, there exists $w_{t'} \in W^{2,2}(M)$, such that $w_{t_n} \rightharpoonup w_{t'}$ in $W^{2,2}(M)$. It is easy to check that $w_{t'}$ is a weak solution to the $(*)_{t'}$ for $t = t'$. The regularity theorem of [UV] shows $w_{t'}$ is smooth. Next, we prove that $R_{t'} > 0$.

Since the scalar curvature R_{t_n} is always positive, for any non-negative smooth test function ϕ , the following inequality holds:

$$\int \phi(\Delta_0 w_{t_n} + |\nabla_0 w_{t_n}|^2) = \int \frac{1}{6}(R_0 - R_{t_n}e^{2w_{t_n}})\phi \leq \int \frac{1}{6}R_0\phi.$$

As $w_{t_n} \rightharpoonup w_{t'}$, we also have

$$\int \frac{1}{6}(R_0 - R_{t'}e^{2w_{t'}})\phi = \int \phi(\Delta_0 w_{t'} + |\nabla_0 w_{t'}|^2) \leq \int \frac{1}{6}R_0\phi$$

Thus $R_{t'}$ is non-negative in the weak sense, which also implies that $R_{t'}$ is non-negative pointwisely. If $R_{t'}$ achieves 0 at some point, it is a contradiction by the strong maximum principle applied to the equation $-\Delta_t R_t + \frac{t}{4}R_t^2 = 3t|E_t|^2 - 12\gamma(t)|\eta|_t^2 > 0$. Hence $t' \in S$ and S is closed. $\square$

Chapter 4

Application of the degree theory: closed manifolds

4.1 Setup of PDEs

Let (M, g) be a closed compact Riemannian manifold with $k_p, Y(M, [g]) > 0$. By Theorem 1.4.1, we can deform the metric so that both the Q -curvature and the scalar curvature are positive. Hence, without loss of generality, we may assume that the background metric satisfies $Q, R > 0$ to prove Theorem 1.3.1. Also, we normalize the volume of the background metric to be 1.

We will consider the following 1-parameter family of fourth order equations with constraint for $t \in [0, 1]$.

$$\begin{cases} Q_t = tk_p + (1-t)Qe^{-4w_t} \iff P_4 w_t = 2tk_p e^{4w_t} - 2tQ \\ \text{with } R_t > 0, \text{ and normalization } Vol(M, g_t) = \int_M e^{4w_t} dV = 1 \end{cases} \quad (**)_t$$

where $g_t := e^{2w_t} g$ and R_t, Q_t denote the scalar curvature and the Q -curvature of the

metric g_t , respectively. Observe that

$$\int_M Q_t dV_t = \int_M (tk_p + (1-t)Qe^{-4w_t}) dV_t = tk_p + (1-t) \int_M Q dV = k_p$$

which implies that the equation is consistent with the fact that k_p is a conformal invariant. In this chapter, w_t will denote a solution of the elliptic PDE $(**)_t$ instead of the equation $(*)_t$.

The majority of this chapter is devoted to establishing the compactness of the set

$$\{w_t \mid w_t \text{ is a solution of the equation } (**)_t \text{ for some } t \in [0, 1] \}$$

in $C^{4,\alpha}(M)$ -topology for any $0 < \alpha < 1$. This compactness result will be used to prove that the equation $(**)_t$ indeed has a solution at $t = 1$ when we apply the Leray-Schauder degree theory to the one-parameter of equations $(**)_t$ for $0 \leq t \leq 1$. Since P_4 is a non-negative operator with trivial kernel, $w_0 \equiv 0$ is the only solution to the equation $(**)_t$ at $t = 0$, and $g_0 = g$.

4.2 Some a priori estimates

We first establish some a priori estimates for the equation $(**)_t$. Throughout this section we will denote constants independent of w_t and t by C .

Proposition 4.2.1. *Let w_t be a solution of the PDE $(**)_t$ for some $0 \leq t \leq 1$. Let $p \in M$ and $B_r(p)$ be a geodesic ball centered at p with radius r . Then, $\int_M |\nabla w_t|^2, \int_M (w_t - \bar{w}_t)^2 < C$, and $\int_{B_r(p)} |\nabla w_t|^2 < \frac{C}{r^2}$ for sufficiently small r , where C is a constant independent of t and r , and $\bar{w}_t := \int w_t dV$.*

Proof. We have $\Delta w_t + |\nabla w_t|^2 \leq \frac{1}{6}R$ from the scalar curvature equation. If we integrate both sides and apply Poincaré's inequality, we get the first two inequalities.

For the second inequality, suppose r is sufficiently smaller than the injectivity radius. We multiply a cut-off function η_r^2 on the scalar curvature equation. η_r is a smooth test function satisfying $\eta_r \equiv 1$ on $B_r(p)$, $\eta_r \equiv 0$ on $B_{2r}(p)$, and $|\nabla \eta_r| \leq \frac{C}{r}$ for some constant C . We have the following estimate

$$\begin{aligned} \int_{B_{2r}(p)} \eta_r^2 |\nabla w_t|^2 &\leq \int_M \frac{1}{6} R \eta_r^2 + 2 \int_{B_{2r}(p)} \eta_r \nabla \eta_r \cdot \nabla w_t \\ &\leq \int_M \frac{1}{6} R \eta_r^2 + \frac{1}{2} \int_{B_{2r}(p)} \eta_r^2 |\nabla w_t|^2 + 2 \int_{B_{2r}(p)} |\nabla \eta_r|^2 \\ &\leq \frac{1}{2} \int_{B_{2r}(p)} \eta_r^2 |\nabla w_t|^2 + C \|R\|_{L^\infty} r^4 + C r^2. \end{aligned}$$

This gives us the desired estimate. $\square$

Next, we will prove the following energy estimate for solutions of fourth order PDEs with bi-Laplacian as a leading order term.

Proposition 4.2.2. *(Energy estimate) Suppose w is a weak solution to a fourth order PDE $\Delta^2 w + \delta(A)dw + f = 0$ where A is a smooth symmetric 2-tensor and f is a function in $L^2(M)$. In other words,*

$$\int_M \Delta w \Delta \phi + A(\nabla w, \nabla \phi) + f \phi = 0$$

for every $\phi \in W^{2,2}(M)$. Then, for all sufficiently small $r > 0$,

$$\|\nabla^2 w\|_{L^2(B_r)} \leq C_r (\|w\|_{W^{1,2}(B_{2r})} + \|f\|_{L^2(B_{2r})})$$

where B_r and B_{2r} are two concentric geodesic balls and C_r is a constant depending on r and $\|A\|_{L^\infty(B_{2r})}$.

Proof. We test with $\phi = \eta_r^2 w$. η_r is a cut-off function defined in the proof of Lemma 4.2.1 with an additional property $|\nabla^2 \eta_r| \leq \frac{C}{r^2}$. We have the following series of

integral identities,

$$\begin{aligned}
\int_M \Delta w \Delta(\eta_r^2 w) &= \int_M \Delta w [\eta_r^2 \Delta w + 4\eta_r \nabla \eta_r \cdot w + (2\eta_r \Delta \eta_r + 2|\nabla \eta_r|^2)w] \\
&= \int_M (\Delta(\eta_r w))^2 - 4|\nabla \eta_r \cdot \nabla w|^2 - (\Delta \eta_r \cdot w)^2 + 2|\nabla \eta_r|^2 \Delta w \cdot w \\
&\quad - 4\nabla \eta_r \cdot \nabla w \Delta \eta_r w \\
&= \int_M (\Delta(\eta_r w))^2 - 2|\nabla \eta_r|^2 |\nabla w|^2 - 2(\nabla |\nabla \eta_r|^2 \cdot \nabla w)w \\
&\quad - 4|\nabla \eta_r \cdot \nabla w|^2 - (\Delta \eta_r \cdot w)^2 - 4(\nabla \eta_r \cdot \nabla w) \Delta \eta_r \cdot w \\
&\geq \int_{B_{2r}} (\Delta(\eta_r w))^2 - C \int_{B_{2r}} \left(\frac{|\nabla w|^2}{r^2} + \frac{w^2}{r^4} \right)
\end{aligned}$$

where at the last line, we use Young's inequality. For the lower order terms, we follow the standard argument:

$$\begin{aligned}
\int_M A(\nabla w, \nabla(\eta_r^2 w)) &= \int_M \eta_r^2 A(\nabla w, \nabla w) + 2w \eta_r A(\nabla w, \nabla \eta_r) \\
&\leq C \int_{B_{2r}} \|A\|_{L^\infty(B_{2r})} (|\nabla w|^2 + \frac{w^2}{r^2})
\end{aligned}$$

and

$$\int_M f \eta_r^2 w \geq - \int_{B_{2r}} |f| |w| \geq - \|f\|_{L^2(B_{2r})} \|w\|_{L^2(B_{2r})}.$$

As $\nabla(\eta_r w)|_{\partial B_{2r}} = 0$, integral by parts formula gives us $\int_{B_{2r}} (\Delta(\eta_r w))^2 = \int_{B_{2r}} |\nabla^2(\eta_r w)|^2 \geq \int_{B_r} |\nabla^2 w|^2$. This completes the proof. $\square$

Following lemma establishes an a priori estimate for a strong solution of PDEs considered in Proposition 4.2.1.

Lemma 4.2.3. *Suppose $w \in W^{4,2}(M)$ is a strong solution to the equation $\Delta^2 w + \delta(A)dw +$*

$f = 0$ where A is a smooth 2-tensor and $f \in L^2(M)$. Then, for sufficiently small r ,

$$\|\nabla^2 w\|_{W^{2,2}(B_r)} \leq C_r(\|f\|_{L^2(B_{2r})} + \|w\|_{W^{1,2}(B_{2r})}).$$

where B_r and B_{2r} are two concentric geodesic balls and C_r is a constant depending on r and $\|A\|_{L^\infty(B_{2r})}, \|\nabla A\|_{L^\infty(B_{2r})}$.

Proof. We start from the identity $\Delta(\Delta w) = -f + A_{ij}w_{ij} + A_{iji}w_j$ By the L^2 - theory of second order elliptic PDE, and Proposition 4.2.1,

$$\begin{aligned} \|\Delta w\|_{W^{2,2}(B_r)} &\leq C_r(\|f\|_{L^2(B_{2r})} + \|A\|_{L^\infty(B_{2r})}\|\nabla^2 w\|_{L^2(B_{2r})} + \|\nabla A\|_{L^\infty(B_{2r})}\|\nabla w\|_{L^2(B_{2r})} + \|\Delta w\|_{L^2(B_r)}) \\ &\leq C_r(\|f\|_{L^2(B_{2r})} + \|w\|_{W^{2,2}(B_{2r})}) \\ &\leq C_r(\|f\|_{L^2(B_{4r})} + \|w\|_{L^2(B_{4r})}). \end{aligned}$$

An application of integral by parts formula gives us the full control of $W^{4,2}(B_r)$ -norm of w . □

4.3 Normal metrics on $\mathbb{R}^4$

We briefly review necessary materials from [CQY] and [X].

Definition 4.3.1. [CQY, Definition 3.1] Let $(\mathbb{R}^4, g_0)$ be the standard Euclidean space. A conformal metric $e^{2w}g_0$ satisfying $\int_{\mathbb{R}^4} |Q_w|e^{4w} < \infty$ is defined to be normal if

$$w(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log \frac{|y|}{|x-y|} Q_w(y) e^{4w(y)} dy + C. \quad (4.1)$$

We will utilize the following criterion to determine if a metric is normal, which will allow us to transform the Q -curvature equation in $\mathbb{R}^4$ into an integral equation.

Theorem 4.3.2. [CQY, Theorem 1.4] Suppose that the Q -curvature of a metric $e^{2w}|dx|^2$ on

$\mathbb{R}^4$ is absolutely integrable and suppose that its scalar curvature is nonnegative at infinity. Then the metric is normal.

We have the following classification result from [X], which provide us the blow up profile of the Q-curvature equation.

Theorem 4.3.3. [X, Theorem 1.2] Suppose $w(x) \in C^1(\mathbb{R}^m)$ is a solution of the equation

$$w(x) = \alpha_m \int_{\mathbb{R}^m} \left[\log \left(\frac{|y|}{|x - y|} \right) \right] e^{mw(y)} dy + C_0 \quad (4.2)$$

for a dimensional constant α_m such that $e^{mw(x)}$ is integrable over $\mathbb{R}^m$. Then there exist a positive constant λ and a point $x_0 \in \mathbb{R}^m$ such that

$$w(x) = \log \left[\frac{2\lambda}{\lambda^2 + |x - x_0|^2} \right].$$

Note that the dimensional constant α_m is explicitly computed by substituting the specific solution defined above into the equation (4.2). It is easy to see that α_m is independent of λ and x_0 .

4.4 Compactness of solutions

The following compactness result for solutions of $(**)_{\alpha}$ can now be proven. Let $0 < \alpha < 1$.

Theorem 4.4.1. There exists $C_{\alpha} > 0$ such that $\|w_t\|_{C^{4,\alpha}} < C_{\alpha}$ for every solution w_t of the equation $(**)_{\alpha}$ where C_{α} independent of $0 \leq t \leq 1$.

Proof. First we claim that it suffices to show $\sup w_t < C$ to prove the theorem. Assume $\sup w_t < C$. Then the right-hand side of $(**)_{\alpha}$ is bounded above pointwisely. Note that $w_t - \bar{w}_t$ satisfies the same equation $(**)_{\alpha}$ with different normalization. We apply Lemma 4.2.3 to $w_t - \bar{w}_t$. This gives us $\|w_t - \bar{w}_t\|_{W^{4,2}(M)} < C + C\|w - \bar{w}_t\|_{W^{1,2}(M)}$.

Lemma 4.2.3 is a local estimate, but we can patch those estimates to get a global estimate. Proposition 4.2.1 gives us a uniform upper bounded for $\|w - \bar{w}_t\|_{W^{1,2}(M)}$. We need to prove that $\bar{w}_t$ is uniformly bounded. w_t is bounded above as $\sup w_t < C$. From the Moser-Trudinger's inequality [CY, Theorem 1.2],

$$k_p \log \int e^{4(w_t - \bar{w}_t)} \leq \int w_t P w_t = \int (w_t - \bar{w}_t) P_4 (w_t - \bar{w}_t) \leq C.$$

As $\int e^{4w_t} = 1$, we have $0 \leq C + 4k_p \log \bar{w}_t$ from the above inequality. Hence we have a uniform bound of $\|w_t\|_{W^{4,2}(M)}$. By Sobolev mbedding theorem, $\|\nabla^2 w\|_{L^p(M)} < C_p$ for any $2 < p < \infty$. Applying L^p estimate to w_t as we applied L^2 -estimate in Lemma 4.2.3, we see that $\|\Delta w\|_{W^{2,p}(M)} < C_p$. Again by the L^p -estimate, $\|w\|_{W^{4,2}(M)} < C_p$. By the Morrey's inequality, $\|w\|_{C^{2,\alpha}(M)} < C_\alpha$. We repeat the argument of Lemma 4.2.3 by applying the Schauder's estimate instead of the L^2 -estimate, to see $\|\Delta w\|_{C^{2,\alpha}(M)} < C_\alpha$. The Schauder's estimate gives us $\|w\|_{C^{4,\alpha}(M)} < C_\alpha$.

Now we prove that $\sup w_t < C$. Assume the contrary. We choose $t_n \rightarrow t_\infty$, w_n solutions of $(**)_{t_n}$ for t_n , $w_n(p_n) \rightarrow +\infty$, $p_n \rightarrow p$.

For δ smaller than the injectivity radius, we use the exponential map and the dilation map to define a normalized sequence $\tilde{w}_n$. Specifically, let $\tilde{w}_n(x) = w_{t_n}(r_n x) + \log r_n$ where r_n is chosen to satisfy $1 = w_n(p_n) + \log r_n = w_n(0) + \log r_n$. Obviously, $r_n \rightarrow 0$ and $\tilde{w}_n \leq 1$. The PDE for $\tilde{w}_n$ defined on a Euclidean ball $B_{\frac{\delta}{r_n}}(0)$ is as follows:

$$P_{\tilde{g}_n} \tilde{w}_n(x) + 2r_n^4 t_n Q(r_n x) = 2t_n k_p e^{4\tilde{w}_n(x)} \quad (4.3)$$

where $\tilde{g}_n$ is the rescaled metric converging to the Euclidean metric. Note that we have $\int_{B_{\frac{\delta}{r_n}}(0)} e^{4\tilde{w}_n} = \int_{B_\delta(x_n)} e^{4w_n} \leq 1$.

Let $\rho_n = \frac{\delta}{r_n}$. Obviously, $\rho_n \rightarrow +\infty$. For a fixed $\rho > 0$ choose suitably large n such that $\rho_n > 2\rho$. By Proposition 4.2.1 and the scaling argument, $\int_{B_{2\rho}(0)} |\nabla \tilde{w}_n|^2 = \frac{\int_{B_{2\rho r_n}(0)} |\nabla w_n|^2}{r_n^2} \leq C_\rho \rho^2$. From Poincaré's inequality, we have $\int_{B_{2\rho}(0)} |\tilde{w}_n - \fint_{B_\rho(0)} \tilde{w}_n|^2 \leq$

$C_\rho \rho^4$. Now by applying Proposition 4.2 and Lemma 4.2.3 to $\tilde{w}_n - \oint_{B_\rho(0)} \tilde{w}_n$, we have $\|\tilde{w}_n - \oint_{B_\rho(0)} \tilde{w}_n\|_{W^{4,2}(B_\rho(0))} < C_\rho$. In particular this inequality gives us an uniform $C^{1,\beta}(B_\rho(0))$ -norm bound on $\tilde{w}_n - \oint_{B_\rho(0)} \tilde{w}_n$ for some $\beta > 0$. Since $\sup \tilde{w}_n = \tilde{w}_n(0) = 1$, and $\|\nabla \tilde{w}_n\|_{L^\infty(B_\rho(0))} = \|\nabla(\tilde{w}_n - \oint_{B_\rho(0)} \tilde{w}_n)\|_{L^\infty(B_\rho(0))}$ is bounded, this yields the uniform bound on $W^{4,2}(B_\rho(0))$ norm for $\tilde{w}_n$ on $B_\rho(0)$.

From the above paragraph, we immediately see that there exists $w_\infty \in W_{\text{loc}}^{4,2}(\mathbb{R}^n)$ s.t. $w_n \rightarrow w_\infty$ in $W^{3,2}(B_\rho(0))$ and $w_n \rightharpoonup w_\infty$ in $W^{4,2}(B_\rho(0))$ for every $\rho > 0$. It is easy to see that w_∞ is a weak solution of the PDE $\Delta^2 w_\infty = 2t_\infty k_p e^{4w_\infty}$ with $\int e^{4w_\infty} \leq 1$, $\sup w_\infty = 1$, or

$$\Delta^2 \left(w_\infty - \frac{\log 3/(t_\infty k_p)}{4} \right) = 6e^{4[w_\infty - (\frac{\log 3/(t_\infty k_p)}{4})]}.$$

Let $\hat{w}_\infty = w_\infty - \frac{\log 3/(t_\infty k_p)}{4}$. As R_∞ is non-negative and $\int_{\mathbb{R}^4} |Q_\infty| e^{4w_\infty} < \infty$, by Theorem 4.3.2, $\hat{w}_\infty$ is normal. Hence we have the following integral representation of $\hat{w}_\infty$

$$\hat{w}_\infty(x) = \frac{3}{4\pi^2} \int_{\mathbb{R}^4} \log \left(\frac{|y|}{|x-y|} \right) e^{4\hat{w}_\infty(y)} dy + C_0.$$

By Theorem 4.3.3, we know that the solution to the above integral equation is $\hat{w}_\infty(x) = \log \left(\frac{2\lambda}{\lambda^2 + |x|^2} \right)$ for some $\lambda > 0$. Then, we have $\frac{8\pi^2}{3} = \int_{\mathbb{R}^4} e^{4\hat{w}_\infty} = \int_{\mathbb{R}^4} e^{4w_\infty} \cdot \left(\frac{t_\infty k_p}{3} \right)$, or $\int_{\mathbb{R}^4} e^{4w_\infty} = \frac{8\pi^2}{t_\infty k_p} > 1$ as $t \leq 1$ and $k_p < 8\pi^2$. This contradicts the fact that $\int_{\mathbb{R}^4} e^{4w_\infty} \leq 1$. $\square$

Remark. The conclusion of Theorem 4.4.1 remains valid under the normalization $\int w_t dV = 0$. This can be easily observed by the shifting solutions by appropriate constants.

4.5 Proof of Theorem 1.3.1

Now we are ready to apply the Leray-Schauder degree theory to finish the proof of Theorem 1.3.1.

Proof of Theorem 1.3.1. By Theorem 1.4.1, we may assume that the background metric satisfies $Q, R > 0$. We will apply Leray-Schauder degree theory to a 1-

parameter family of equations $(**) _t$ but with different normalization $\int w_t dV_g = 0$. Let $\mathcal{O} = \{u \in C^{4,\alpha}(M) | R_u > 0, \|u\|_{C^{4,\alpha}} < C_\alpha, \int_M u dV_g = 0\}$ where R_u denotes the scalar curvature of the metric $g_u := e^{2u}g$, and C_α is the constant appearing in the remark after Theorem 4.4.1. $\mathcal{O}$ is a bounded open subset of the Banach space $C_0^{4,\alpha}(M) := \{f \in C^{4,\alpha}(M) | \int f = 0\}$.

We define operators $F_t : C_0^{4,\alpha}(M) \rightarrow C_0^{4,\alpha}(M)$ by $F_t(u) = 2tP^{-1}(k_p e^{4u} - Q)$ where t runs from 0 to 1. F_t is a continuous 1-parameter family of compact operators by the regularity argument discussed in the proof of Theorem 4.4.1. Solving $(**) _t$ is equivalent to solving equations $(Id - F_t)(w_t) = 0$. At $t = 0$, $F_0 \equiv 0$ and $w_0 \equiv 0$ is the unique solution. Thus, the degree of $(Id - F_t)$ in $\overline{\mathcal{O}}$ at the point $0 \in C_0^{4,\alpha}(M)$ is 1.

The degree of maps $(Id - F_t)$ in $\overline{\mathcal{O}}$ at the point 0 is well-defined and homotopy invariant if $0 \notin (Id - F_t)(\partial\mathcal{O})$. Suppose $w_t \in \partial\mathcal{O}$ is a solution to the equation $(Id - F_t)(w_t) = 0$. $w_t \in \partial\mathcal{O}$ implies either $\|w_t\|_{C^{4,\alpha}} = C_\alpha$, or $R_{w_t} \geq 0$ and $R_{w_t}(p_0) = 0$ for some $p_0 \in M$. The first case is obviously a contradiction to the Theorem 4.4.1. The second case contradicts the strong maximum principle applied to $Q_{w_t} > 0$. Hence the degree of a map $(Id - F_1)$ in $\overline{\mathcal{O}}$ at the point 0 is also 1, and there exists a solution to the equation $(**) _t$ at $t = 1$.

□

4.6 A counterexample to a Andrews-type inequality

In order to apply the method of continuity to prove Theorem 1.3.1, we need to show that the kernel of the linearized operator $\tilde{L}_t$ of $(**) _t$ is $\{0\}$. For smooth ϕ with $\int_M \phi dV_t = 0$, $\tilde{L}_t$ at a solution w_t is given by

$$\tilde{L}_t(\phi) = (P_4)_t \phi - 8tk_p \phi.$$

As $\tilde{L}_t$ is non-negative operator at $t = 0$, a reasonable approach is to prove that $\tilde{L}_t$ is non-negative for $t \in [0, 1]$. This is equivalent to showing that the first non-zero eigenvalue of $\tilde{L}_t$ is greater or equal to $8tk_p$. Note that $Q_t = tk_p + (1 - t)Qe^{-4w_t} \geq tk_p$. Hence one would try to estimate a lower bound on the first eigenvalue of when the Q -curvature is bounded below by a positive constant, and the scalar curvature is positive.

When (M, g) is a 2-dimensional Riemannian surface, the Gaussian curvature is a curvature quantity analogous to the Q -curvature, and such estimate is true by Lichnerowicz and Obata's theorem. For σ_2 -curvature, which is another type of conformal curvature that is widely studied, there is a similar result by Gursky and Streets [GS].

For a four-dimensional Riemannian manifold (M^4, g) , we denote the Schouten tensor by $A = \frac{1}{2}(\text{Ric} - \frac{1}{6}Rg)$. Then the σ_2 -curvature is defined as $\sigma_2(A) = -\frac{1}{2}|E|^2 + \frac{1}{24}R^2$. Observe that $Q = -\frac{1}{12}\Delta R + \frac{1}{2}\sigma_2(A)$.

Proposition 4.6.1. [GS, Corollary 3.15] *Let (M^4, g) be a closed Riemannian manifold such that the scalar curvature R and the σ_2 -curvature is positive. Given $\phi \in C^\infty(M)$ such that $\int_M \phi dV = 0$, then*

$$\int_M \frac{1}{\sigma_2(A_g)} T_1(A_g)^{ij} \nabla_i \phi \nabla_j \phi \geq 4 \int_M \phi^2 dV_g.$$

where T_1 is the Newton transform. The equality holds if and only if $\phi \equiv 0$ or (M^4, g) is isometric the round sphere.

The above inequality is inspired by Andrews' inequality, which is proved in his unpublished work.

Proposition 4.6.2. [CLN, pg. 517] *Let (M^m, g) be a closed riemannian manifold with positive Ricci curvature. Given $\phi \in C^\infty(M)$ such that $\int_M \phi dV = 0$, then*

$$\frac{m}{m-1} \int_M \phi^2 dV \leq \int_M (\text{Ric}^{-1})^{ij} \nabla_i \phi \nabla_j \phi.$$

One may consider an inequality similar to Andrews' inequality for a four-dimensional Riemannian manifold (M, g) with positive scalar curvature and Q -curvature.

$$\int P_4 \phi \cdot \phi \geq 8 \int Q \phi^2 \text{ or, } \int \left(\frac{P_4 \phi \cdot \phi}{Q} \right) \geq 8 \int \phi^2 \quad (4.4)$$

for smooth ϕ such that $\int_M \phi dV = 0$. While those two inequalities are true if (M, g) is an Einstein manifold, they are not true in general.

Example 4.6.1. (4.4) is false for some manifold (S^4, g) where g is a perturbed metric of the standard metric g_c .

Proof. We embed S^4 in $\mathbb{R}^5$ as usual. Let x_1 be a coordinate function. Recall that $f_1 = x_1$ and $f_2 = 5x_1^2 - 1$ are the first and second eigenfunctions of the Laplace-Beltrami operator, respectively. Let $\bar{g} = e^{2f_2 t} g_c$ for some sufficiently small t . We will show that for some small t , $\int \bar{P}_4 f_1 \cdot f_1 dVol_{\bar{g}} < 8 \int \bar{Q} f_1^2 dVol_{\bar{g}}$.

From the conformal covariance of the Paneitz operator,

$$\int \bar{P}_4 f_1 \cdot f_1 dVol_{\bar{g}} = \int P_4 f_1 \cdot f_1 dVol_{g_c} = 8Q \int f_1^2 dVol_{g_c}$$

$$8 \int \bar{Q} f_1^2 dVol_{\bar{g}} = \int (4t P_4 f_2 + 8Q) f_1^2 dVol_{g_c}$$

Note that $\int f_2 dVol_{g_c} = \int f_1 e^{4t f_2} dVol_{g_c} = 0$ as f_1 is an odd function while f_2 is an even function. Now we compute :

$$\begin{aligned} \int \bar{P}_4 f_1 \cdot f_1 dVol_{\bar{g}} - 8 \int \bar{Q} f_1^2 dVol_{\bar{g}} &= 8Q \int f_1^2 dVol_{g_c} - \int (4t P_4 f_2 + 8Q) f_1^2 dVol_{g_c} \\ &= -4t \int P_4 f_2 \cdot f_1^2 dVol_{g_c} \\ &= -4t \int P_4 (5x_1^2 - 1) \cdot x_1^2 dVol_{g_c} \\ &= -\frac{4t}{5} \int P_4 (5x_1^2 - 1) \cdot (5x_1^2 - 1) \end{aligned}$$

The value of the last line is positive as f_2 is a eigenfuction of the laplacian. If we choose t to be positive, the assertion is proved. Also note that we can also perturb $\bar{g}$ further to make it stay out of the conformal class $[g_c]$. The same method gives a counterexample to the second inequality of (4.4).

□

Remark. *If one can prove either of (4.4) for (M, g) with positive scalar curvature and constant positive Q -curvature, then the solution to $(**)_{\mathbf{t}}$ at $t = 1$ is unique by degree theory.*

Chapter 5

Application of the continuity method: manifolds with totally geodesic boundary

5.1 Existence of a minimizer at $t = \frac{2}{3}$ and uniform $W^{2,2}(M)$ -estimate

In this chapter, we apply the continuity method to the following one-parameter families of functionals defined in Chapter 2.

$$F_t^1[w] := \gamma_1(t)I[w] + t(II_b[w] - k_{(P_4, P_3)}) \log \oint e^{4w} dv_0 + (1-t)III[w]$$

$$F_t^2[w] := \gamma_2(t)I_b[w] + t(II_b[w] - \frac{4}{3}k_{(P_4, P_3)}) \log \oint_{\partial M} e^{3w} + (1-t)III[w]$$

In this section, we prove that for $t = \frac{2}{3}$, there exist minimizers with positive scalar curvature for both functionals. After that we will establish the uniform $W^{2,2}(M)$ -estimate for critical points of both functionals as we vary the parameter t from $\frac{2}{3}$ to 1. Let $t_0 = \frac{2}{3}$.

Lemma 5.1.1. *There exist minimizers for both $F_{t_0}^1$ and $F_{t_0}^2$. For a minimizer w_{t_0} of $F_{t_0}^1$ or $F_{t_0}^2$, the scalar curvature R_{t_0} is positive.*

Proof. According to [CN, Theorem 1.9], $(P_4)_0$ is non-negative operator defined on the set $\{u \in W^{2,2}(M) | \frac{\partial u}{\partial n} = 0\}$ and $\ker(P_4)_0 \cong \mathbb{R}$. Therefore, we can find minimizers of $F_{\frac{2}{3}}^1$ or $F_{\frac{2}{3}}^2$ as in Lemma 3.1.2. These minimizers are smooth on the interior by [UV]. On boundary points, the same method as in [UV] is applied, except that we have to use $W^{2,p}$ -estimates for Neumann boundary problems.

By [E, Lemma 1.1 and Proposition 1.3], the first non zero-Neumann eigenvalue λ of the linear elliptic operator $(-\Delta_{t_0} + \frac{1}{6}R_{t_0})$ is positive. The eigenfunction f corresponding to λ , is positive on $\overline{M}$. We have $(-\Delta_{t_0} + \frac{1}{6}R_{t_0})f = \lambda f$, $\frac{\partial f}{\partial n_{t_0}} = 0$.

If w_{t_0} is a solution of either problem 1 or problem 2, the Q -curvature equation gives us $(-\Delta_{t_0} + \frac{1}{6}R_{t_0})R_{t_0} \geq 0$ and the T -curvature equation gives us $\frac{\partial R_{t_0}}{\partial n_{t_0}} \leq 0$.

Choose p_0 such that $\min_{\overline{M}} \frac{R_{t_0}}{f} = \frac{R_{t_0}}{f}(p_0)$. If p_0 is in the interior, we compute $\Delta_{t_0}(\frac{R_{t_0}}{f})(p_0) = [\frac{\Delta_{t_0}R_{t_0} - \frac{1}{6}R_{t_0}^2}{f} + \frac{\lambda R_{t_0}}{f}](p_0) \leq \frac{\lambda R_{t_0}}{f}(p_0)$. As $f > 0$, we get $R_{t_0}(p_0) \geq 0$. If $R_{t_0}(p_0) = 0$, then by the strong maximum principle $R_{t_0} \equiv 0$, which contradicts $Y(M, \partial M, [g_0]) > 0$. Hence $R_{t_0} > 0$ in this case. Next, suppose p_0 is on the boundary. If $\frac{R_{t_0}}{f}(p_0) \leq 0$, then by Hopf's lemma we have $\frac{\partial(R_{t_0}/f)}{\partial n_{t_0}} > 0$. This contradicts the T -curvature equation.

□

Next, we observe that any critical point w_t of F_t^1 or F_t^2 satisfies $\|w_t\|_{W^{2,2}} \leq C$ for some constant independent of t for $\frac{2}{3} \leq t \leq 1$. This fact will be used to prove closedness part when we apply the method of continuity in the next section.

Proposition 5.1.2. *Let w_t be a critical point of F_t^1 or F_t^2 with normalization $\int_M w_t = 0$ for F_t^1 and $\oint_{\partial M} w_t = 0$ for F_t^2 . Assume $R_t > 0$ for both cases. Then $\|w_t\|_{W^{2,2}(M)} \leq C$ for some C independent of t and w_t .*

Proof. The proof is similar to that of Proposition 3.2.1.

□

5.2 Proof of Theorem 1.4.2

In this section, we prove that the linearized equations of $(\star)_t$ and $(\dagger)_t$ have only the trivial solution. The basic strategy follows that of the Proposition 3.3.1 with only minor differences. We then use it to prove Theorem 1.4.2.

Lemma 5.2.1. *Let $\frac{2}{3} \leq t \leq 1$. Linearized problems for $(\star)_t$ and $(\dagger)_t$ only have trivial solution under normalization $\int_M w_t = 0$ or $\oint_{\partial M} w_t = 0$ when $R_t > 0$.*

Proof. By direct computation, both linearized equations are as follows:

$$\begin{cases} L_t(\phi) := \Delta_t^2 \phi + t \operatorname{div}_t(\frac{2}{3} R_t g_t - 2 \operatorname{Ric}_t) d\phi + (\frac{1-t}{3}) \nabla_t(R_t \nabla_t \phi) = 0 \text{ on } M \\ \frac{\partial \Delta_t \phi}{\partial n_t} = 0 \text{ on } \partial M \\ \frac{\partial \phi}{\partial n_t} = 0 \text{ on } \partial M \end{cases}$$

Let ϕ be a function with $\frac{\partial \Delta_t \phi}{\partial n_t} = 0$ and $\frac{\partial \phi}{\partial n_t} = 0$ on the boundary. We shall prove that

$$\int_M \langle L_t(\phi), \phi \rangle dV_t \geq 0.$$

Since the boundary is totally geodesic, $\operatorname{Ric}_t(n_t, \nabla_t \phi) = 0$ and $\frac{\partial |\nabla_t \phi|^2}{\partial n_t} = 0$ by Lemma 2.2.2.

Denote the traceless part of the Ricci curvature of the metric g_t by E_t . Let $\phi \in W^{2,2}(M)$. The equation $(\star)_t$ is rewritten as

$$-\Delta_t R_t + \frac{t}{4} R_t^2 - 3t |E_t|^2 = -12\gamma(t) |\eta_t|^2 > 0. \quad (5.1)$$

There are two identities in the proof of Lemma 3.3.1 that include additional bound-

any integral terms. The first identity is

$$\begin{aligned}
\langle L_t \phi, \phi \rangle &= \int_M (\Delta_t \phi)^2 - 2t \int_M E_t(\nabla_t \phi, \nabla_t \phi) + \left(\frac{1}{2}t - \frac{1}{3}\right) \int_M |\nabla_t \phi|^2 \\
&\quad - \oint_{\partial M} \frac{\partial \Delta_t \phi}{\partial n_t} \phi + \oint_{\partial M} \Delta_t \phi \frac{\partial \phi}{\partial n_t} + \oint_{\partial M} \left(-\frac{2R_t}{3}\right) \frac{\partial \phi}{\partial n_t} + 2\text{Ric}_t(\nabla_t \phi, n_t) \\
&= \int_M (\Delta_t \phi)^2 - 2t \int_M E_t(\nabla_t \phi, \nabla_t \phi) + \left(\frac{1}{2}t - \frac{1}{3}\right) \int_M |\nabla_t \phi|^2,
\end{aligned}$$

and the second identity is

$$\begin{aligned}
\int \frac{\Delta_t R_t}{R_t} |\nabla_t \phi|^2 &= \int_M -\nabla_t R_t \cdot \nabla_t (R_t^{-1}) |\nabla_t \phi|^2 - \frac{\nabla_t R_t}{R_t} \cdot \nabla_t |\nabla_t \phi|^2 - \oint_{\partial M} \frac{1}{R_t} \frac{\partial R_t}{\partial n_t} |\nabla_t \phi|^2 \\
&= \int_M -\nabla_t R_t \cdot \nabla_t (R_t^{-1}) |\nabla_t \phi|^2 - \frac{\nabla_t R_t}{R_t} \cdot \nabla_t |\nabla_t \phi|^2 + \oint_{\partial M} 12 \left(\frac{T_t}{R_t}\right) |\nabla_t \phi|^2.
\end{aligned}$$

The rest of the proof is exactly the same as the closed manifold case. We have the inequality

$$\langle L_t \phi, \phi \rangle \geq C_1(t) \int (\Delta_t \phi)^2 + C_2(t) \int R_t |\nabla_t \phi|^2 + C_3(t) \oint_{\partial M} \left(\frac{T_t}{R_t}\right) |\nabla_t \phi|^2.$$

for some positive constants $C_1(t)$, $C_2(t)$, and $C_3(t)$. As both of $(\star)_t$ and $(\dagger)_t$ imply $T_t \geq 0$, we obtain the result. $\square$

Proof of Theorem 1.4.2. The proof is identical to that of Proposition 1.4.1 using Lemma 5.1.1, Proposition 5.1.2, and Lemma 5.2.1, except for that we have to use Hopf maximum principle to show the scalar curvature is strictly positive. $\square$

Chapter 6

Application of the degree theory: manifolds with totally geodesic boundary

6.1 Setup of PDEs

We aim to prove a compactness result for solutions of two one-parameter families of equations, which is a key step in establishing Theorem 1.3.2. Let $(M, \partial M, g)$ be a compact Riemannian manifold with totally geodesic boundary. We assume $k_{(P_4, P_3)}, Y(M, \partial M, [g]) > 0$.

Firstly, we describe an one-parameter family of equations that will be used to find w_1 in Theorem 1.3.2. In this case, we conformally deform the metric g into g_1 so that the boundary is totally geodesic, $T_1 \equiv 0$, and both Q_1 and R_1 are positive pointwisely, according to the first part of Theorem 1.4.2. Under these assumptions, we show that there exists an uniform $C^{4,\alpha}$ bound on solutions to the following equations:

$$\left\{ \begin{array}{l} Q_t = tk_p e^{-4w_t} + (1-t)Q_1 \iff (P_4)_1 w_t = 2tk_{(P_4, P_3)} e^{4w_t} - 2tQ_1 \text{ on } M \\ R_t > 0 \text{ on } M \\ T_t = 0 \iff \frac{\partial \Delta_1 w_t}{\partial n_1} = 0 \text{ on } \partial M \\ H_t = 0 \iff \frac{\partial w_t}{\partial n_1} = 0 \text{ on } \partial M \\ \int_M e^{4w_t} = 1 \end{array} \right. \quad (\star\star)_t$$

where Q_t , R_t , T_t , and H_t denote the Q -curvature, scalar curvature, T -curvature, and mean curvature of the metric $g_t := e^{2w_t} g_1$ respectively.

Secondly, we describe an one-parameter family of equations that will be used for finding w_2 in Theorem 1.3.2. In this case, we conformally deform the metric g into g_2 so that the boundary is totally geodesic, $Q_2 \equiv 0$, and both of T_2 and R_2 are positive pointwisely according to the second part of Theorem 1.4.2. Under these assumptions, we show that there exists an uniform $C^{4,\alpha}$ bound on solutions of following equations:

$$\left\{ \begin{array}{l} Q_t = 0 \iff (P_4)_2 w_t = 0 \text{ on } M \\ R_t > 0 \text{ on } M \\ T_t = tk_{(P_4, P_4)} e^{-3w_t} + (1-t)T_2 \iff (P_3)_2 w_t = tk_{(P_4, P_4)} e^{3w_t} - tT_2 \text{ on } \partial M \\ H_t = 0 \iff \frac{\partial w}{\partial n_2} = 0 \text{ on } \partial M \\ \int_{\partial M} e^{3w_t} = 1 \end{array} \right. \quad (\dagger\dagger)_t$$

where Q_t , R_t , T_t , and H_t denote the Q -curvature, scalar curvature, T -curvature, and mean curvature of the metric $g_t := e^{2w_t} g_2$ respectively.

Throughout this chapter, we will suppress subscripts 1 and 2 whenever the argument is independent of whether background metric is g_1 or g_2 for notational

convenience.

6.2 Some a priori estimates

In this section, we prove basic a priori estimates for solutions of $(\star\star)_t$ and $(\dagger\dagger)_t$. Through out this section we will denote constants independent of w_t and t by C .

Proposition 6.2.1. *Let w_t be a solution of the PDE $(\star\star)_t$ or $(\dagger\dagger)_t$ for some $0 \leq t \leq 1$. Let $p \in M$ and $B_r(p)$ be a geodesic ball centered at p with radius r . Then, $\int_M |\nabla w_t|^2$, $\int_M (w_t - \oint_{\partial M} w_t)^2$, $\int_M (w_t - \oint_{\partial M} w_t)^2 < C$, and $\oint_{B_r(p) \cap M} |\nabla w_t|^2 < \frac{C}{r^2}$ for sufficiently small r , where C is a constant independent of t and r .*

Proof. Taking into account of the fact $\frac{\partial w_t}{\partial n} = 0$, there is no difference in the computation with the closed manifold case Proposition 4.2.1. $\square$

Next, we prove the following energy estimate for solutions of Neumann boundary valued fourth order PDEs having bi-Laplacian as a leading-order term.

Proposition 6.2.2. *(Boundary energy estimate) Suppose $w \in W^{2,2}(M)$ is a weak solution to the following fourth-order PDE with Neumann boundary condition.*

$$\begin{cases} \Delta^2 w + \delta(A)dw + f = 0 \\ \frac{\partial \Delta w}{\partial n} = -g \\ \frac{\partial w}{\partial n} = 0 \end{cases} \quad (6.1)$$

where f, g are bounded functions, A is a smooth symmetric 2-tensor and such that $A(\nabla w, n) = 0$ (See Lemma 2.2.2). In other words,

$$\int_M \Delta w \Delta \phi + E(\nabla w, \nabla \phi) + f\phi + \int_{\partial M} g\phi = 0$$

for every $\phi \in W^{2,2}(M)$ with $\frac{\partial \phi}{\partial n} \equiv 0$. Then, for all sufficiently small $r > 0$,

$$\|\nabla^2 w\|_{L^2(B_r)} \leq C_r(\|w\|_{W^{1,2}(B_{2r})} + \|f\|_{L^2(B_{2r})} + \|g\|_{L^2(B_{2r} \cap \partial M)})$$

where B_r and B_{2r} are two concentric geodesic balls and C_r is a constant depending on r and $\|A\|_{L^\infty(B_{2r})}$.

Proof. We test with $\phi = \eta_r^2 w$. The only difference is that we have to choose η_r with $\frac{\partial \eta_r}{\partial n} \equiv 0$ on ∂M . This can be achieved by choosing η_r depending only on the distance from a point on the boundary. As the boundary is totally geodesic, it is easy to see that $\frac{\partial \eta_r}{\partial n} \equiv 0$ is achieved. The rest is the same as that of Proposition 4.2.2.

□

We also note a fundamental fact regarding the boundary regularity.

Lemma 6.2.3. *Suppose $w \in W^{4,2}(M)$ is a strong solution to the equation (6.1) with the same conditions on A and f, g . Then, we have*

$$\|\nabla^2 w\|_{W^{2,2}(B_r)} \leq C_r(\|f\|_{L^2(B_{2r})} + \|w\|_{W^{1,2}(B_{2r})} + \|g\|_{W^{1/2,2}(B_{2r} \cap \partial M)}).$$

where B_r and B_{2r} are two concentric geodesic balls and C_r is a constant depending on r and $\|A\|_{L^\infty(B_{2r})}, \|\nabla A\|_{L^\infty(B_{2r})}$.

Proof. The only difference with Lemma 4.2.3 is that we use the L^2 -estimate for Neumann boundary problem:

$$\|\nabla^2 v\|_{L^2(B_r)} \leq C_r(\|\Delta v\|_{L^2(B_{2r})} + \|v\|_{L^2(B_{2r})} + \|\frac{\partial v}{\partial n}\|_{W^{1/2,2}(B_{2r} \cap \partial M)}).$$

□

6.3 Proof of Theorem 1.3.2

We will prove the compactness results for solutions of $(\star\star)_t$ and $(\dagger\dagger)_t$. Then the proof of Theorem 1.3.2 is almost identical to that of Theorem 1.3.1. Let $0 < \alpha < 1$.

Theorem 6.3.1. *There exists $C_\alpha > 0$ such that $\|w_t\|_{C^{4,\alpha}} < C_\alpha$ for every solution w_t of the equation $(\star\star)_t$ where C_α independent of $0 \leq t \leq 1$.*

Proof. We use the notation g instead of g_1 for convenience. It is enough to show that $\sup_t w_t < C$ uniformly. Suppose there exists $t_n \rightarrow t_\infty$, w_n solution at t_n , $w_n(p_n) \rightarrow +\infty$, $p_n \rightarrow p$. Let $d_n = \text{dist}(p_n, \partial M)$, and choose $r_n > 0$ such that $w_n(p_n) + \log r_n = 1$. Clearly $r_n > 0$ and $r_n \rightarrow 0$. Denote $\epsilon_n = \frac{d_n}{r_n}$ and let δ be the injectivity radius of M . Depending on whether $\lim d_n = 0$ or $\limsup \epsilon_n < \infty$, we have separate arguments.

Case 1: $\liminf d_n > 0$. This is the case when p_n uniformly stays away from the boundary. Let $d_n > \epsilon$ for some $\epsilon > 0$. We define $\tilde{w}_n(x) = w_{t_n}(r_n x) + \log(r_n)$ for $x \in B_{\min\{\epsilon/r_n, \delta/r_n\}}(0)$ using the exponential map. In this case, the blow up argument is the same as the closed manifold case.

Case 2: $\lim d_n = 0$, and $\limsup \epsilon_n = \infty$. In this case, p_n approaches the boundary at a relatively slow pace so we do not observe the boundary when we blow-up. We define $\tilde{w}_n(x) = w_n(r_n x) + \log(r_n)$ for $x \in B_{\epsilon_n/2}(0)$. Again, the blow up argument is the same as the closed manifold case.

Case 3: $\lim d_n = 0$, and $\epsilon_n \rightarrow \epsilon$ for some $\epsilon \geq 0$. This is the only case we take the boundary into account after the blow-up. Let $q_n \in \partial M$ such that $\text{dist}(p_n, q_n) = \text{dist}(p_n, \partial M)$. Using the exponential map centered at q_n , and since the boundary is totally geodesic, we can define $\tilde{w}_n(x) = w(r_n x) + \log(r_n)$ for $x \in B_{\delta/(2r_n)}^+(0)$. $\tilde{w}_n(0, \epsilon_n) = w(p_n) = 1$. By Proposition 6.2.2, Lemma 6.2.3, there exists $w_\infty \in W_{\text{loc}}^{4,2}(\mathbb{R}_+^4)$ s.t. $\tilde{w}_n \rightarrow w_\infty$ in $W^{3,2}(B_R^+(0))$. $\tilde{w}_n \rightharpoonup w_\infty$ in $W^{4,2}(B_R^+(0))$.

It is straightforward to observe that w_∞ is a weak solution of the equation $\Delta^2 w_\infty = 2t_\infty k_{(P_4, P_3)} e^{4w_\infty}$ with boundary conditions $\frac{\partial w_\infty}{\partial y} = 0$, $\frac{\partial \Delta w_\infty}{\partial y} = 0$ on $\mathbb{R}^3$. Clearly,

we have $\int e^{4w_\infty} \leq 1$, $\sup w_\infty = 1$.

Define $\bar{w}_\infty(x, y) = w_\infty(x, y)$ for $y \geq 0$, $\bar{w}_\infty(x, y) = w_\infty(x, -y)$ for $y \leq 0$. Since $\frac{\partial \Delta w_\infty}{\partial y} = 0$, $\bar{w}_\infty \in W_{loc}^{4,2}(\mathbb{R}^4)$. We check below that $\bar{w}_\infty$ is a weak solution of the equation $\Delta^2 \bar{w}_\infty = 2t_\infty k_{(P_4, P_3)} e^{4\bar{w}_\infty}$ on $\mathbb{R}^4$.

$$\begin{aligned}
\int_{\mathbb{R}^4} \bar{w}_\infty \Delta^2 \phi &= \int_{\mathbb{R}_+^4} w_\infty \Delta^2 \phi + \int_{\mathbb{R}_-^4} w_\infty(x, -y) \Delta^2 \phi \\
&= \int_{\mathbb{R}_+^4} w_\infty \Delta^2 \phi + w_\infty \Delta^2 \phi(x, -y) \\
&= \int_{\mathbb{R}_+^4} \phi \Delta^2 w_\infty + \phi(x, -y) \Delta^2 w_\infty \\
&\quad + \int_{\mathbb{R}^3} w_\infty \frac{\partial \Delta \phi}{\partial y} + \Delta w_\infty \frac{\partial \phi}{\partial y} + w_\infty \frac{\partial \Delta \phi}{\partial y}(x, -y) + \Delta w_\infty \frac{\partial \phi}{\partial y}(x, -y) \\
&= \int_{\mathbb{R}_+^4} 2t_\infty k_{(P_4, P_3)} e^{4w_\infty} \phi + 2t_\infty k_{(P_4, P_3)} e^{4w_\infty} \phi(x, -y) \\
&= \int_{\mathbb{R}^4} 2t_\infty k_{(P_4, P_3)} e^{4\bar{w}_\infty} \phi.
\end{aligned}$$

Now the rest of the proof is now the same as in the closed manifold case. Since $\int_{\mathbb{R}^4} e^{4\bar{w}_\infty} \leq 2$ and $k_{(P_4, P_3)} < 4\pi^2$, the argument is applied to $\bar{w}_\infty - \frac{\log 2}{4}$. $\square$

Theorem 6.3.2. *There exists $C_\alpha > 0$ such that $\|w_t\|_{C^{4,\alpha}} < C_\alpha$ for every solution w_t of the equation $(\dagger\dagger)_t$ where C_α independent of $0 \leq t \leq 1$.*

Proof. Again, we abuse the notation g with g_2 for convenience. Assume that $\sup_t w_t < C$ uniformly. We apply L^2 -estimate for Neumann boundary problem to $w_t - (w_t)_{\partial M}$. We have

$$\|\nabla^2 w_t\|_{W^{2,2}(M)} \leq C(\|w_t - (w_t)_{\partial M}\|_{W^{1,2}(M)} + \|k_{(P_4, P_3)} e^{3w_t}\|_{W^{1/2,2}(\partial M)} + \|T\|_{W^{1/2,2}(\partial M)}).$$

As $\|w_t - (w_t)_{\partial M}\|_{W^{1,2}(M)}$ is bounded by Lemma 6.2.1, we have the estimate

$$\|e^{3w_t}\|_{W^{1/2,2}(\partial M)} \leq Ce^{3M}(1 + \|w_t - (w_t)_{\partial M}\|_{W^{1/2,2}(\partial M)}) \leq Ce^{3M}(1 + \|w_t - (w_t)_{\partial M}\|_{W^{1,2}(M)})$$

by the trace Sobolev inequality. Thus, we have uniform bound on $\|w_t - (w_t)_{\partial M}\|_{W^{4,2}(M)}$. The upperbound on $(w_t)_{\partial M}$ is given by the Jensen's inequality, and the lower bound is derived from the boundary Moser-Trudinger inequality [N2, Lemma 2.4]. Therefore, it is sufficient to show that $\sup_t w_t < C$ to prove the theorem.

Suppose there exists $t_n \rightarrow t_\infty$, w_n solution for t_n , $w_n(p_n) \rightarrow +\infty$, $p_n \rightarrow p$. Let $d_n = \text{dist}(p_n, \partial M)$, and choose $r_n > 0$ such that $w_n(p_n) + \log r_n = 1$. Clearly $r_n > 0$ and $r_n \rightarrow 0$. Denote $\epsilon_n = \frac{d_n}{r_n}$ and let δ be the injectivity radius of M . Case 1 and Case 2, as in the proof of Theorem 6.3.1, are dealt with by the exactly same argument. We only need to take care of Case 3: $\lim d_n = 0$, and $\epsilon_n \rightarrow \epsilon$ for some $\epsilon \geq 0$.

Let $q_n \in \partial M$ such that $\text{dist}(p_n, q_n) = \text{dist}(p_n, \partial M)$. Using the exponential map centered at q_n , and since the boundary is totally geodesic, we can define $\tilde{w}_n(x) = w_n(r_n x) + \log(r_n)$ for $x \in B_{\delta/(2r_n)}^+(0)$. $\tilde{w}_n(0, \epsilon_n) = w_n(p_n) = 1$. By Proposition 6.2.2, Lemma 6.2.3, there exists $w_\infty \in W_{\text{loc}}^{4,2}(\mathbb{R}_+^4)$ s.t. $\tilde{w}_n \rightarrow w_\infty$ in $W^{3,2}(B_R^+(0))$. $w_n \rightarrow w_\infty$ in $W^{4,2}(B_R^+(0))$.

It is easy to show that w_∞ satisfies the equation $\Delta^2 w_\infty = 0$ with the boundary conditions $\frac{\partial w_\infty}{\partial y} = 0$ and $\frac{\partial \Delta w_\infty}{\partial y} = 2t_\infty k_{(P_4, P_3)} e^{3w_\infty}$ on the boundary $\mathbb{R}^3$. Clearly, we have $\Delta w_\infty + |\nabla w_\infty|^2 \leq 0$, $\int_{\mathbb{R}^3} e^{3w_\infty} \leq 1$, $\sup w_\infty = 1$.

Our next goal is to prove that w_∞ is normal for our T -curvature problem, as defined in a manner similar to Definition 4.3.1. We imitate the proof of Theorem 4.3.2.

Lemma 6.3.3. $w_\infty(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \log\left(\frac{|y|}{|x-y|}\right) t_\infty k_{(P_4, P_3)} e^{3w_\infty(y)} dy + C_0$.

Proof. Let $v = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \log\left(\frac{|y|}{|x-y|}\right) t_\infty k_{(P_4, P_3)} e^{3w_\infty(y)} dy$ and $h = w_\infty - v$. Then, we have $\Delta^2 h = 0$ on the upper half plane and $\frac{\partial h}{\partial y} = 0$ and $\frac{\partial \Delta h}{\partial y} = 0$ on the boundary. We will

show that h is a constant.

Δh is a harmonic function with $\frac{\partial \Delta h}{\partial y} = 0$ allowing us to consider its even reflection, which yields a smooth, entire, harmonic function. We can also extend h , w_∞ and v to be C^2 functions on the entire plane using even reflection. We abuse notations h , w_∞ , and v to refer to these extended function. By the mean value theorem, we have

$$\begin{aligned} \Delta h(x_0) &= \oint_{\partial B_r(x_0)} \Delta h \\ &\leq - \oint_{\partial B_r(x_0)} |\nabla w_\infty|^2 - \oint_{\partial B_r(x_0)} \Delta v \\ &\leq -C \int_{\mathbb{R}^3} \left[\oint_{S^3} \frac{1}{|r\sigma + x_0 - y|^2} \right] t k_{(P_4, P_3)} e^{3w_\infty(y)} \\ &\leq -\frac{C}{\pi r^2}. \end{aligned}$$

If we take $r \rightarrow \infty$, we obtain $\Delta h(x_0) \leq 0$ for all $x_0 \in \mathbb{R}^4$. Therefore, by Liouville's theorem, we have $\Delta h \equiv c_0$ for some $c_0 \leq 0$.

Again by applying the mean value theorem and Cauchy Schwartz inequality, we obtain

$$|\nabla h|^2(x_0) \leq C \oint_{\partial B_r(x_0)} |\nabla h|^2.$$

Next, we observe that

$$\begin{aligned} |\nabla h|^2(x_0) &\leq 2|\nabla w_\infty|^2(x_0) + 2|\nabla v|^2(x_0) \\ &\leq -2\Delta w_\infty(x_0) + 2|\nabla v|^2(x_0) \\ &\leq -2c_0 - 2\Delta v(x_0) + 2|\nabla v|^2(x_0). \end{aligned}$$

We handle the Δv term as shown previously. To estimate $|\nabla v|^2(x_0)$, we have

$$|\nabla v|^2(x_0) \leq C \left(\int_{\mathbb{R}^3} \frac{1}{|x_0 - y|^2} k_{(P_4, P_3)} e^{3w_\infty(y)} \right) \left(\int_{\mathbb{R}^3} k_{(P_4, P_3)} e^{3w_\infty(y)} \right) \leq C \int_{\mathbb{R}^3} \frac{1}{|x_0 - y|^2} e^{3w_\infty(y)}.$$

This can also be estimated as Δv using the same technique. Hence, taking $r \rightarrow \infty$, we see that $|\nabla h|^2$ is bounded. Since every partial derivative of h is a harmonic function, it is constant by Liouville theorem. Therefore, $\Delta h = c_0 = 0$, and we can conclude that h is a constant. $\square$

Now by the above lemma, we have

$$w_\infty(x) - \left(\frac{\log 2/(t_\infty k_{(P_4, P_3)})}{3}\right) = \frac{1}{\pi^2} \int_{\mathbb{R}^3} \log\left(\frac{|y|}{|x-y|}\right) e^{3(w_\infty(y) - (\frac{\log 2/(t_\infty k_{(P_4, P_3)})}{3}))} dy + C_0$$

Let $\hat{w}_\infty(x) = w_\infty(x) - (\frac{\log 2/(t_\infty k_{(P_4, P_3)})}{3})$. The following integral equation holds.

$$\hat{w}_\infty(x) = \frac{1}{\pi^2} \int_{\mathbb{R}^3} \log\left(\frac{|y|}{|x-y|}\right) e^{3\hat{w}_\infty(y)} dy + C_0. \quad (6.2)$$

By Theorem 4.3.3, we have $\hat{w}_\infty(x) = \log(\frac{2\lambda}{\lambda^2 + |x|^2})$ for some $\lambda > 0$. Then, we see that $2\pi^2 = \int_{\mathbb{R}^3} e^{3\hat{w}_\infty} = \int_{\mathbb{R}^4} e^{3w_\infty} \cdot (\frac{t_\infty k_{(P_4, P_3)}}{2})$, or $\int_{\mathbb{R}^3} e^{3w_\infty} = \frac{4\pi^2}{t_\infty k_{(P_4, P_3)}} > 1$. This contradicts $\int_{\mathbb{R}^3} e^{3w_\infty} \leq 1$. $\square$

The proof of Theorem 1.3.2 is straightforward using Theorem 6.3.1 and Theorem 6.3.2.

Proof of Theorem 1.3.2. To apply the degree theory, we use Theorem 6.3.1 and Theorem 6.3.2. However, unlike the proof of Theorem 1.3.1, we need to use the Hopf's strong maximum principle to demonstrate that the degree is well-defined and invariant with respect to the parameter t . $\square$

Bibliography

- [BO] T. Branson and B. Ørsted, Explicit functional determinants in four dimensions, *Proc. A. M. S.* 113 (1991), 669–682.
- [CGY] S.-Y. A. Chang, M. J. Gursky, and P. C. Yang, An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature, *Ann. Math.* 155 (2002), 709-787.
- [CLN] B. Chow, P. Lu, L. Ni, Hamilton’s Ricci flow, *Graduate Studies in Mathematics* 77, Science, Beijing (2006).
- [CN] G. Catino and C. B. Ndiaye, Integral pinching results for manifolds with boundary. *Ann. Sci. Norm. Super Pisa Cl. Sci.* 9(4), 785–813 (2010)
- [CQ1] S.-Y. A. Chang and J. Qing, The zeta functional determinants on manifolds with boundary. I. The formula, *J. Funct. Anal.* 147 (1997), no. 2, 327-362.
- [CQ2] S.-Y. A. Chang and J. Qing, The zeta functional determinants on manifolds with boundary. II. Extremal metrics and compactness of isospectral set, *J. Funct. Anal.* 147 (1997), no. 2, 363-399.
- [CQY] S.-Y. A. Chang, J. Qing and P. Yang, On the Chern-Gauss-Bonnet integral for conformal metrics on $\mathbb{R}^4$, *Duke Math. Jour.* 103 (2000), pp 523-544.
- [CY] S.-Y. A. Chang and P. C. Yang. Extremal metrics of zeta function determinants on 4- manifolds. *Ann. of Math.* (2) 142 (1995), no. 1, 171–212.

- [DM] Z. Djadli and A. Malchiodi, Existence of conformal metrics with constant Q-curvature, *Ann of Math.* 168 (2008), 813–858
- [E] J. Escobar. The Yamabe problem on manifolds with boundary, *J. Differ. Geom.*, 35 (1992), 21-84.
- [G1] M. J. Gursky, The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics. *Ann. Math.* 148 (1998), 315–337.
- [G2] The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE, *Comm. Math. Phys.* 207 (1999), 131–143.
- [GS] M.J. Gursky and J. Streets. A formal Riemannian structure on conformal classes and uniqueness for the σ_2 -Yamabe problem, *Geom. Topol.*, 22 (2018), 3501–3573.
- [GHL] M. J. Gursky, F. B. Hang and Y.-J. Lin. Riemannian manifolds with positive Yamabe invariant and Paneitz operator. *Int. Math. Res. Not.* 2016 (2016), 1348–1367.
- [HY] F. B. Hang and P. C. Yang. Lectures on the fourth-order Q curvature equation. *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, 31 (2016), 1–33.
- [L] C. S. Lin, A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$. *Comment. Math. Helv.* 73, (1998) 206–231.
- [M] A. Malchiodi; Compactness of solutions to some geometric fourth-order equations, *J. Reine Angew. Math.*, 594 (2006), 137-174.
- [N1] C. B. Ndiaye, Conformal metrics with constant Q-curvature for manifolds with boundary, *Comm. Anal. Geom.* 16 (2008), no. 5, 1049–1124.

- [N2] Ndiaye C.B., Constant T -curvature conformal metric on 4-manifolds with boundary, *Pacific J. Math.* 240 (2009), no. 1, 151–184.
- [UV] K. K. Uhlenbeck and J. A. Viaclovsky, Regularity of weak solutions to critical exponent variational equations, *Math. Res. Lett.* 7 (2000), 651–656.
- [X] Xu X. Uniqueness and non-existence theorems for conformally invariant equations. *J Funct Anal* 222 (2005) 1-28.

Part II

Liouville-Type Theorems for Adapted Metrics of Upper Half-Plane Model of Hyperbolic Space

Chapter 7

Introduction

7.1 Background

Let $(X^{n+1}, \partial X = M, g_+)$ be a $(n+1)$ -dimensional smooth manifold of dimension with boundary M . Denote the interior of $\bar{X}$ as X . Let ρ be a smooth boundary defining function. A smooth conformally compact metric g^+ on X is a Riemannian metric such that $g = \rho^2 g_+$ extends smoothly to the closure $\bar{X}$ for some defining function ρ of the boundary M . By definition, ρ is a function defined on $\bar{X}$ such that

$$0 \leq \rho \in C^\infty(\bar{X}), \rho > 0 \text{ in } X, \rho = 0 \text{ on } M, D\rho \neq 0 \text{ on } M.$$

A conformally compact metric g_+ on X is said to be conformally compact Einstein (CCE) with conformal infinity $(M, [\hat{g}])$, if g_+ is a complete metric in X such that

$$\begin{cases} Ric_{g_+} = -n g_+ \\ g|_M = (\rho^2 g_+)|_M \in [\hat{g}]. \end{cases}$$

for a defining function ρ .

The most commonly used compactifications of the metric g_+ are the adapted

compactification $g_s^* := \rho_s^2 g_+$ where s is a real parameter, and the conformal factor ρ_s is defined through the scattering theory on conformally compact asymptotically Einstein manifolds. Roughly speaking, when $\frac{n}{2} < s < n$, $\rho_s := v_s^{\frac{1}{n-s}}$ where v_s is a positive function satisfying

$$\begin{cases} -\Delta_+ v_s - s(n-s)v_s = 0 \text{ on } X \\ v_s = F\rho^{n-s} + G\rho^s, F|_{\partial M} = 10 \text{ on } \bar{X} \end{cases}$$

A rigorous definition of functions F, G defined on $\bar{X}$ is given in the paper Graham and Zworski [GZ]. When $s = n$, v_n has slightly different definition and the compactification in this case is called the Fefferman-Graham compactification. As the scattering theory is an extremely abundant and extensive field of study, we will only review facts necessary for our study in this thesis.

We aim to prove Liouville-type theorems for adapted metrics on the upper half-plane model of standard hyperbolic space $(\mathbb{R}_+^{n+1}, \mathbb{R}^n, g_+)$ with a designated sign on the scalar curvature. Here, $\mathbb{R}_+^{n+1} := \{z = (x, y) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n, y \in \mathbb{R}, y > 0\}$, $\mathbb{R}^n := \{z = (x, 0) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n\}$ and $g_+ := \frac{1}{y^2} |dz|^2$. As scattering theory is not defined in the non-compact manifolds case, the adapted metric for $(\mathbb{R}_+^{n+1}, \mathbb{R}^n, g_+)$ is defined by $g_s^* := \rho_s^2 g_+$, where $\rho_s := v_s^{\frac{1}{n-s}}$ for a positive solution v_s of the following PDE:

$$\begin{cases} -\Delta_+ v_s - s(n-s)v_s = 0 \text{ in } \mathbb{R}_+^{n+1} \\ \frac{v_s}{y^{n-s}} = 1 \text{ on } \mathbb{R}^n \iff g_s^*|_{\mathbb{R}^n} \equiv |dx|^2 \end{cases} \quad (7.1)$$

We prove that $g_s^* \equiv |dz|^2$ under the assumption on the sign of the scalar curvature of the metric g_s^* . Our Liouville-type theorems are expected to handle the boundary curvature blow-up to prove compactness results of CCE manifolds with positive scalar on the conformal infinity. Hence, the assumption on the sign of the scalar curvature is justified, as it follows from the positivity of scalar curvature of the

conformal infinity in the CCE case.

7.2 Previous results

Let $s := \frac{n}{2} + \gamma$, and $a := 1 - 2\gamma$ for $0 < \gamma < \frac{n}{2}$. Define $w_s := 1 - \frac{v_s}{y^{n-s}}$. The equation $-\Delta_+ v_s - s(n-s)v_s = 0$ on $(\mathbb{R}_+^{n+1}, \mathbb{R}^n, g_+)$ can be rewritten with respect to the Euclidean metric as follows:

$$\Delta_a w_s = 0 \tag{7.2}$$

The degenerate elliptic operator Δ_a is defined as $\Delta_a u := \operatorname{div}(y^a \nabla u)$ for a function u . We assume that $w_s \equiv 0$ on $\mathbb{R}^n$, which is equivalent to saying that $g_s^* := \rho_s^2 g_+ = v_s^{\frac{2}{n-s}} g_+$ satisfies $g_s|_{\mathbb{R}^n} \equiv |dx|^2$.

The equation (7.2) appeared in the celebrated paper Caffarelli and Silvestre [CS] while studying extension problems for fractional Laplacians on the Euclidean space in the case $0 < \gamma < 1$. Later, Yang [Y] generalized the extension problem to $\gamma > 1$. We also note that a connection of such extension problems to the conformal fractional Laplacian and fractional $Q_{2\gamma}$ -curvature has been studied in Case and Chang [CC].

When $\gamma = \frac{3}{2}$, the Liouville-type theorems for the equation (7.2) with the positivity assumption on the scalar curvature are proved and used in Chang and Ge [CG], and Chang-Ge-Jin-Qing [CGJQ] to handle the boundary curvature blow-up while establishing compactness results of CCE manifolds with positive scalar curvature on the conformal infinity. The readers are referred to [CG, Proposition 4.8] and [CGJQ, Lemma 5.2] for these special cases.

For more general γ , the following Liouville-type theorems are proved in the work Sire-Terracini-Vita [STV2].

Theorem 7.2.1. [STV2, Theorem 3.1] Let $a \in (-1, 1)$, and w be a solution to

$$\begin{cases} \Delta_a w = 0 \text{ in } \mathbb{R}_+^{n+1} \\ w = 0 \text{ on } \mathbb{R}^n, \end{cases}$$

and let us suppose that for some $d \in [0, 1 - a)$, $C > 0$ it holds

$$|w(z)| \leq C(1 + |z|^d)$$

for every z . Then w is identically zero.

Theorem 7.2.2. [STV2, Theorem 3.2] Let $a \in (-\infty, 1)$, and w be a solution to

$$\begin{cases} \Delta_a w = 0 \text{ in } \mathbb{R}_+^{n+1} \\ w = 0 \text{ on } \mathbb{R}^n, \end{cases}$$

and let us suppose that for some $d \in [0, 1)$, $C > 0$ it holds

$$|w(z)| \leq Cy^{2-a}(1 + |z|^d)$$

for every z . Then w is identically zero.

7.3 Main theorems

We now present our main theorems of the second part of the thesis.

The first main result examines the sign of the scalar curvature of adapted metrics of CCE manifolds in the case $\gamma \in (0, 1)$. We prove that the sign of the scalar curvature is the same as the sign of $\gamma - \frac{1}{2}$ when the conformal infinity has positive scalar curvature. It is shown in [CC, Proposition 6.4] that the scalar curvature of the adapted metric is positive if the scalar curvature of the conformal infinity is

positive for $\gamma \in [1, \frac{n}{2} + 1]$. Our result extends the result of [CC] from $s \in [\frac{n}{2} + 1, n + 1]$ to $s \in (\frac{n}{2}, n + 1]$. We denote the covariant derivative and the scalar curvature of the adapted metric g_s^* by ∇_s^* and R_s^* .

Theorem 7.3.1. *Let $(X^{n+1}, \partial X = M^n, g_+)$ be a CCE manifold with the conformal infinity $(M, [\hat{g}])$. Suppose the scalar curvature $\hat{R}$ of $(M^n, \hat{g})$ is positive. Let $0 < \gamma < 1$. Then, $|\nabla_s^* \rho_s|^2 < 1$ holds in $\bar{X}$. In particular, we have $\text{sign}(R_s^*) = \text{sign}(\gamma - \frac{1}{2})$.*

The above theorem justifies the assumption $|\nabla_s^* \rho_s| \leq 1$ for the adapted metrics g_s^* on $(\mathbb{R}_+^{n+1}, \mathbb{R}^n, g_+)$, and the sign of R_s^* follows from the identity $R_s^* = (s - \frac{n+1}{2}) \frac{f_s}{\rho_s^2}$. Assuming $|\nabla_s^* \rho_s| \leq 1$, we have the following Liouville-type theorem for $\gamma \in (0, 1)$ which follows directly from Theorem 7.2.2.

Theorem 7.3.2. *Let $\gamma \in (0, 1)$. Suppose v_s is a positive solution of the PDE $-\Delta_+ v_s - s(n - s)v_s = 0$ on $\mathbb{R}_+^{n+1}$. Let $w_s := 1 - \frac{v_s}{y^{n-s}}$. Assume:*

- (1) $w_s \equiv 0$ on $\mathbb{R}^n \iff g_s^*|_{\mathbb{R}^n} \equiv |dx|^2$.
- (2) $|\nabla_s^* \rho_s| \leq 1$. This is equivalent to $\text{sgn}(R_s^*) = \text{sgn}(\gamma - \frac{1}{2})$ when $\gamma \neq \frac{1}{2}$.
- (3) $w_s \in \tilde{H}^{1,\alpha}(\mathbb{R}_+^{n+1})$.

Then, $v_s = y^{n-s}$ or equivalently, $g_s^* \equiv |dz|^2$.

The precise definition of the functional space $\tilde{H}^{1,\alpha}(\mathbb{R}_+^{n+1})$ is given in the Definition 8.1.1. Roughly speaking, $w \in \tilde{H}^{1,\alpha}(\mathbb{R}_+^{n+1})$ if the y^α -weighted Sobolev norm of w is locally bounded and the trace of w_s on $\mathbb{R}^n$ vanishes. For $\gamma \in [1, \frac{n}{2})$, we prove the following Liouville-type theorem when the scalar curvature of the adapted metric is non-negative.

Theorem 7.3.3. *Let $s := \frac{n}{2} + \gamma$ for some $\gamma \in [1, \frac{n}{2})$. Suppose v_s is a positive solution of the PDE $-\Delta_+ v_s - s(n - s)v_s = 0$ on $\mathbb{R}_+^{n+1}$. Let $w_s := 1 - \frac{v_s}{y^{n-s}}$. Assume:*

- (1) R_s^* is non-negative.
- (2) $w_s \in \tilde{H}^{1,0}(\mathbb{R}_+^{n+1})$.

Then, $v_s = y^{n-s}$ or equivalently, $g_s^* \equiv |dz|^2$.

For $\gamma = \frac{n}{2}$, corresponding to the Fefferman-Graham compactification, we have:

Theorem 7.3.4. *Suppose v is a positive solution of the PDE $-\Delta_+ v - n = 0$ on $\mathbb{R}_+^{n+1}$. Let $w := v - \log y$. Assume:*

- (1) R_n^* is non-negative.
- (2) $w \in \tilde{H}^{1,0}(\mathbb{R}_+^{n+1})$.
- (3) There exists $C, y_0 > 0$ such that $|w| < C$ for $y < y_0$.

Without the assumption on the scalar curvature, $v_s = y^{n-s} + y^s$ is another solution for $s < n$.

Chapter 8

Preliminaries

8.1 y^a -weighted Sobolev spaces

Let a be a real number. In this section, we review some analytical preliminaries of y^a -weighted Sobolev spaces on $\mathbb{R}_+^{n+1}$. Denote $B_r^+ := \{z = (x, y) \in \mathbb{R}_+^{n+1} \mid |z| < r\}$, and $(\partial B_r)^+ := \{z = (x, y) \in \mathbb{R}_+^{n+1} \mid |z| = r\}$ for $r > 0$.

Definition 8.1.1. For a real number a , we define

$$H^{1,a}(\mathbb{R}_+^{n+1}) := \{u \in W_{\text{loc}}^{1,1}(\mathbb{R}_+^{n+1}) \mid \int_{B_r^+} (|\nabla u|^2 + u^2) y^a dz < \infty \text{ for every } r > 0\}$$

$$\begin{aligned} \tilde{H}^{1,a}(\mathbb{R}_+^{n+1}) := & \{u \in H^{1,a}(\mathbb{R}_+^{n+1}) \mid \text{For every } r > 0, \exists \{\phi_k\}_{k \geq 1}, \text{ a sequence of functions} \\ & \text{in } C_c^1(\overline{B_r^+} \setminus \mathbb{R}^n) \text{ s.t. } \int_{B_r^+} (|\nabla(u - \phi_k)|^2 + |u - \phi_k|^2) y^a dz \rightarrow 0\}. \end{aligned}$$

We have the following inclusion property for $\tilde{H}^{1,a}(\mathbb{R}_+^{n+1})$.

Proposition 8.1.2. Let $a \leq b$ be real numbers. Then, we have

$$\tilde{H}^{1,a}(\mathbb{R}_+^{n+1}) \subset \tilde{H}^{1,b}(\mathbb{R}_+^{n+1}).$$

Proof. Let $u \in \tilde{H}^{1,a}(\mathbb{R}_+^{n+1})$. For a $r > 0$, there exists a sequence of functions $\{\phi_k \in$

$C_c^1(\overline{B_r^+} \setminus \mathbb{R}^n)\}_{k \geq 1}$ such that

$$\int_{B_r^+} (|\nabla(u - \phi_k)|^2 + |u - \phi_k|^2) y^a dz \rightarrow 0$$

by definition. We now check that $\int_{B_r^+} (|\nabla u|^2 + u^2) y^b dz < \infty$, and $\int_{B_r^+} (|\nabla(u - \phi_k)|^2 + |u - \phi_k|^2) y^b dz \rightarrow 0$. This will confirm that $u \in \tilde{H}^{1,b}(\mathbb{R}_+^{n+1})$.

$$\begin{aligned} \int_{B_r^+} (|\nabla u|^2 + u^2) y^b dz &= \int_{B_r^+ \cap \{y \leq 1\}} (|\nabla u|^2 + u^2) y^b dz + \int_{B_r^+ \cap \{1 \leq y \leq r\}} (|\nabla u|^2 + u^2) y^b dz \\ &\leq \int_{B_r^+ \cap \{y \leq 1\}} (|\nabla u|^2 + u^2) y^a dz + r^{b-a} \int_{B_r^+ \cap \{1 \leq y \leq r\}} (|\nabla u|^2 + u^2) y^a dz \\ &\leq \max(1, r^{b-a}) \int_{B_r^+} (|\nabla u|^2 + u^2) y^a dz < \infty \end{aligned}$$

Likewise, we observe

$$\int_{B_r^+} (|\nabla(u - \phi_k)|^2 + |u - \phi_k|^2) y^b dz \leq \max(1, r^{b-a}) \int_{B_r^+} (|\nabla(u - \phi_k)|^2 + |u - \phi_k|^2) y^a dz.$$

As the right-hand side converges to 0, this concludes the proof. $\square$

For $a \in (-1, 1)$ or $\gamma \in (0, 1)$, the weight y^a belongs to the $\mathcal{A}_2$ -Muckenhoupt class, and $\mathcal{A}_2$ -weighted inequalities such as Sobolev embedding, Hardy's inequality are well-known. In particular, we recall some Hardy-type inequalities and corresponding Sobolev trace inequality for $a \in (-1, 1)$. We include proofs for readers' convenience and refer to [STV1] and [STV2] for statements for more general a .

Proposition 8.1.3. (*Hardy's inequality*) Let $r > 0$. For $w \in C^1(\bar{B}_r^+)$ with $w(x, 0) = 0$, we have following inequalities:

- (1) $\frac{1}{4} \int_{B_r^+} \frac{w^2}{y^2} \leq \int_{B_r^+} |\nabla w|^2.$
- (2) $\int_{(\partial B_r)^+} \frac{w^2}{y} \leq \int_{B_r^+} |\nabla w|^2.$

Proof. Let $f \in C^1([0, t])$ for some $t > 0$ with $f(0) = 0$. (1) follows from the classical 1-dimensional Hardy's inequality

$$\frac{1}{4} \int_0^t \frac{f(y)^2}{y^2} dy \leq \int_0^t |f'(y)|^2 dy, \quad (8.1)$$

and the Fubini's theorem,

$$\begin{aligned} \frac{1}{4} \int_{B_r^+} \frac{w^2}{y^2} dz &= \frac{1}{4} \int_{|x| < r} \left(\int_{0 < y < \sqrt{r^2 - |x|^2}} \frac{w(x, y)^2}{y^2} dy \right) dx \\ &\leq \int_{|x| < r} \left(\int_{0 < y < \sqrt{r^2 - |x|^2}} |\partial_y w(x, y)|^2 dy \right) dx \\ &\leq \int_{|x| < r} \left(\int_{0 < y < \sqrt{r^2 - |x|^2}} |\nabla w(x, y)|^2 dy \right) dx = \int_{B_r^+} |\nabla w|^2. \end{aligned}$$

(2) follows from the Cauchy-Schwarz inequality

$$f(t)^2 = \left(\int_0^t f'(y) dy \right)^2 \leq t \int_0^t |f'(y)|^2 dy. \quad (8.2)$$

Applying the Fubini's theorem as above proves the inequality. $\square$

Proposition 8.1.4. ([STV2, Lemma B.5]) Let $a \in (-1, 1)$ and $r > 0$. For $w \in C_c^1(\bar{B}_r^+ \setminus \mathbb{R}^n)$, we have following inequalities:

- (1) $C_a \int_{B_r^+} y^{a-2} w^2 \leq \int_{B_r^+} y^a |\nabla w|^2.$
- (2) $C_a \int_{(\partial B_r)^+} y^{a-1} w^2 \leq \int_{B_r^+} y^a |\nabla w|^2.$
- (3) $C_a \int_{(\partial B_r)^+} y^a w^2 \leq r \int_{B_r^+} y^a |\nabla w|^2.$

Here the constants C_a depend only on a . They may blow up as $|a| \rightarrow 1$.

Proof. Plugging in $f = y^{a/2} g$ in the inequality (8.1), we have

$$\begin{aligned} \frac{1}{4} \int_0^t y^{a-2} g(y)^2 dy &\leq \int_0^t \frac{a^2}{4} y^{a-2} g(y)^2 + y^a g'(y)^2 + a y^{a-1} g(y) g'(y) dy \\ &\leq \left(\frac{a^2 + 1}{8} \right) \int_0^t y^{a-2} g(y)^2 + C(a) \int_0^t y^a g'(y)^2 dy. \end{aligned}$$

by Young's inequality. As $|a| < 1$, this gives us

$$\int_0^t y^{a-2} g(y)^2 dy \leq \frac{8C(a)}{1-a^2} \int_0^t y^a g'(y)^2 dy. \quad (8.3)$$

which proves the inequality (1) by applying the Fubini's theorem.

For $f = y^{a/2}g$ in the inequality (8.2), we have

$$t^a g(t)^2 \leq t \int_0^t \frac{a^2}{4} y^{a-2} g(y)^2 + y^a g'(y)^2 + a y^{a-1} g(y) g'(y) dy.$$

By applying the inequality (8.3) to the right-hand side, we have

$$t^{a-1} g(y)^2 \leq C(a) \int_0^1 y^a g'(y)^2$$

from which the inequality (2) is claimed.

For (3), we apply the Poincaré-type inequality for $v = y^{a/2}w \in C_c^1(\bar{B}_r^+ \setminus \mathbb{R}^n)$. From the Poincaré-type inequality,

$$C \int_{(\partial B_r)^+} v^2 \leq r \int_{B_r^+} |\nabla v|^2,$$

we have

$$C \int_{(\partial B_r)^+} y^a w^2 \leq r \int_{B_r^+} \frac{a^2}{4} y^{a-2} w^2 + y^a |\nabla w|^2 + a y^{a-1} w \partial_y w.$$

Applying the inequality (1) on the right-hand side gives us the inequality (3). $\square$

8.2 Liouville-type theorem

In this section, we present the proof for the Liouville-type theorem for weak solutions of the following degenerate PDE.

$$\Delta_a u = 0 \text{ on } \mathbb{R}_+^{n+1} \quad (8.4)$$

Definition 8.2.1. Let $a \in (-1, 1)$. $w \in \tilde{H}^{1,a}(\mathbb{R}_+^{n+1})$ is a weak solution of the (8.4) if

$$\int_{\mathbb{R}_+^{n+1}} y^a \nabla w \cdot \nabla \phi = 0$$

for every $\phi \in C_c(\mathbb{R}_+^{n+1})$.

Observe that $\tilde{H}^{1,a}(\mathbb{R}_+^{n+1})$ is the natural functional space in which weak solutions of the degenerate elliptic PDE (8.4) reside. As the technique used to prove Theorem 7.2.1 is crucial for our later application, we include the proof.

Theorem 8.2.2. [STV2, Theorem 3.1] Let $a \in (-1, 1)$. Suppose $w \in \tilde{H}^{1,a}(\mathbb{R}_+^{n+1})$ is a weak solution to the PDE $\Delta_a w = 0$ such that w is bounded. Then $w \equiv 0$.

Proof. Let $H(r) = \frac{1}{r^{n+a}} \int_{(\partial B_r)^+} y^a w^2$ and $E(r) = \frac{1}{r^{n+a-1}} \int_{B_r^+} y^a |\nabla w|^2$. Note that $H(r)$ and $E(r)$ are scale invariant in the sense that for any $\lambda > 0$, we have

$$\begin{aligned} \frac{1}{r^{n+a}} \int_{(\partial B_r)^+} y^a w(z)^2 dS(z) &= \frac{1}{(\lambda r)^{n+a}} \int_{(\partial B_{\lambda r})^+} (\lambda y)^a w(\lambda z)^2 dS(\lambda z) \\ \frac{1}{r^{n+a-1}} \int_{B_r^+} y^a |\nabla w|^2(z) dz &= \frac{1}{(\lambda r)^{n+a-1}} \int_{B_{\lambda r}^+} (\lambda y)^a |\nabla w|^2(\lambda z) d(\lambda z). \end{aligned}$$

Applying integral by parts formula, we have

$$\begin{aligned}
H'(r) &= \frac{1}{r^{n+a}} \int_{(\partial B_r)^+} y^a \partial_r(w^2) \\
&= \frac{1}{r^{n+a}} \int_{(B_r)^+} \operatorname{div}(y^a \nabla(w^2)) \\
&= \frac{2}{r^{n+a}} \int_{(B_r)^+} y^a |\nabla w|^2 + y^a w_s (\Delta w + \frac{a}{y} \partial_y w) \\
&= \frac{2}{r} E(r).
\end{aligned}$$

Then by the weighted trace inequality Proposition 8.1.4-(3), $H'(r) = \frac{2}{r} E(r) \geq \frac{C_a}{r} H(r)$ for some positive $C_a > 0$. This differential inequality is equivalent to $(r^{-C_a} H(r))' \geq 0$, and combining with the fact that w is bounded,

$$r^{n+a+C_a} \int_{(\partial B_1)^+} y^a w^2 \leq \int_{(\partial B_r)^+} y^a w^2 \leq C \int_{(\partial B_r)^+} y^a = C r^{n+a}.$$

Taking the limit $r \rightarrow \infty$, we immediatly see that $w \equiv 0$.

□

8.3 Scattering theory on CCE manifolds and adapted metrics

In this section, we introduce adapted metrics using the scattering theory on CCE manifolds. For more general statements, readers are referred to [GZ] and [MM].

Let $s = \frac{n}{2} + \gamma$.

First, we recall that for a CCE manifold (X, M, g_+) with conformal infinity $(M, [\hat{g}])$, there exists a unique defining function x such that

$$g_+ = x^{-2}(dx^2 + g_x), \text{ with } g_0 = \hat{g}$$

in a collar neighbourhood of M . x is called the geodesic defining function.

For $s \in \mathbb{C}$ such that $s(n-s) \notin \sigma_{pp}(-\Delta_+)$ and $f \in C^\infty(M)$, there exists a solution $\mathcal{P}(s)f$ to the Poisson equation

$$(-\Delta_+ - s(n-s))(\mathcal{P}(s)f) = 0 \text{ in } X$$

exists in the form

$$\begin{cases} \mathcal{P}(s)f = x^{n-s}F + x^sG & \text{if } s \notin n/2 + \mathbb{N}/2 \\ \mathcal{P}(s)f = x^{n/2-l/2}F + Gx^{n/2+l/2} \log x & \text{if } s = n/2 + l/2, l \in \mathbb{N} \end{cases}$$

where $F, G \in C^\infty(\overline{X})$ and $F|_M = f$. The operator $\mathcal{P}(s)$ is referred to as the Poisson operator and is known to be meromorphic in $\{\operatorname{Re} s > n/2\}$ with poles only for s such that $s(n-s) \in \sigma_{pp}(-\Delta_+)$. The scattering operator, denoted by $S(s)$, is defined by $S(s)f = G|_M$. The fractional GJMS operator is defined by

$$P(\gamma) := 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} S\left(\frac{n}{2} + \gamma\right)$$

where Γ is the Gamma function. Lastly, we note that F has an asymptotic expansion $F = f + f_{(2)}x^2 + f_{(4)}x^4 + \dots$ near M .

We now define a family of compactifications parametrized by a real parameter $s := \frac{n}{2} + \gamma$ for a given CCE manifold (X, M, g_+) .

Definition 8.3.1. For $s > \frac{n}{2}$, $s \neq n$, and $s(n-s) \notin \sigma_{pp}(-\Delta_+)$, let $v_s := \mathcal{P}(s)1$. It is known that $v_s > 0$ by [CC, Lemma 6.1]. Denote $\rho_s = v_s^{\frac{1}{n-s}}$. The compactified metric $g_s^* := \rho_s^2 g_+$ is called the adapted metric.

For $s = n$, let $v := -\frac{d}{ds}|_{s=n} \mathcal{P}(s)1$. From [FG, Theorem 4.1], v is a solution to the PDE $-\Delta_+ v = n$. In addition, $v - \log x$ is in $C^3(\overline{X})$ and vanishes on M . The metric $g^* := e^{2v} g_+$ is called the Fefferman-Graham compactified metric.

8.4 Conformal infinity with positive scalar curvature

In this section, we gather some geometric properties of CCE manifolds when the compactified metric on the boundary $(M, \hat{g})$ has positive scalar curvature. Firstly, it is known that $-\Delta_+$ has no pure point spectrum, and hence, the adapted metric is well-defined for $s > \frac{n}{2}$.

Theorem 8.4.1. *[L, Theorem A] Let (X, M, g_+) be a CCE manifold with conformal infinity $(M, [\hat{g}])$. If the Yamabe constant on the boundary is non-negative, then $-\Delta_+$ has no L^2 eigenfunctions.*

For $0 < \gamma < 1$, the positivity of the scalar curvature on the conformal infinity implies the following.

Theorem 8.4.2. *[GQ, Theorem 1.2] Let $0 < \gamma < 1$. $P(\gamma)1$ is positive provided that the scalar curvature $\hat{R}$ of M is positive.*

As $P(\gamma) := 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} S(\frac{n}{2} + \gamma)$ where $\Gamma(-\gamma)$ is negative and $\Gamma(\gamma)$ is positive, we have the asymptotic formula $v_s = x^{n-s}(1 + h_s x^{2\gamma} + \dots)$ for some negative function h_s defined on the boundary. This fact will be used later in the proof of Theorem 7.3.1.

For $1 \leq \gamma \leq \frac{n}{2} + 1$, the positivity of $\hat{R}$ implies the positivity of scalar curvature R_s^* .

Proposition 8.4.3. *[CC, Proposition 6.4] Let (X, M, g_+) be a CCE manifold and $\hat{g}$ be the compactified metric on the boundary M . Suppose the scalar curvature $\hat{R}$ of $(M^n, \hat{g})$ is positive. Then, $R_s^* > 0$ for $1 \leq \gamma \leq \frac{n}{2} + 1$.*

Finally, for $\gamma = \frac{1}{2}$, R_s^* is identically zero. Instead, the positivity of $\hat{R}$ implies the following.

Lemma 8.4.4. *[WZ, Lemma 5.4] If $\hat{R} > 0$, then for $\gamma = \frac{1}{2}$, $\left(\frac{1 - |\nabla_s^* \rho_s|^2}{\rho_s}\right) > 0$ on $\bar{X}$, and $R_s \equiv 0$.*

Chapter 9

Proofs of main results

9.1 Outline of the proofs

To prove Theorem 7.3.1, we claim that $1 - |\nabla_s^* \rho_s|^2 > 0$ in $\overline{X}$ for $0 < \gamma < 1$. This is motivated by Lemma 8.4.4. We compute the PDE for $\left(\frac{1 - |\nabla_s^* \rho_s|^2}{\rho_s^{2\gamma}}\right)$ and apply the continuity method to this quantity as in the proof of Proposition 8.4.3. Once we justify the assumption $|\nabla_s^* \rho_s| \leq 1$ for the non-compact setting, it is easy to see that $|\nabla_s^* \rho_s| \leq 1$ implies that $0 \leq w_s < 1$, and $0 \leq \partial_y w_s$. Then, Theorem 7.3.2 is a direction application of Theorem 7.2.1.

For Theorem 7.3.3, we first note that we assume $w_s \in \tilde{H}^{1,0}(\mathbb{R}_+^{n+1})$ rather than $\tilde{H}^{1,\mu}(\mathbb{R}_+^{n+1})$. Although $\tilde{H}^{1,\mu}(\mathbb{R}_+^{n+1})$ seems to be a more natural functional space for weak solutions, the space $\tilde{H}^{1,0}(\mathbb{R}_+^{n+1})$ is still sufficient to control the boundary behavior of w_s . Also note that by Proposition 8.1.2, $\tilde{H}^{1,\mu}(\mathbb{R}_+^{n+1}) \subset \tilde{H}^{1,0}(\mathbb{R}_+^{n+1})$ when $\gamma \geq \frac{1}{2}$.

The quantity $\frac{1}{r^{n+a}} \int_{\partial B_r} y^a w_s^2$ used in the proof of Theorem 9.1 seems to be not appropriate as the boundedness of w_s does not ensure it is finite when $\gamma \geq 1$. This is the reason that Theorem 7.2.2 assumes $|w(z)| \leq C y^{2-a}$ near $\mathbb{R}^n$. The idea is to use the quantity $\frac{1}{r^n} \int_{\partial B_r} w_s^2$ instead and show that it satisfies the differential inequality

inequality $H'(r) \geq \frac{C}{r}H(r)$ by utilizing non-negativity of the scalar curvature R_s^* .

Theorem 7.3.4 follows from the same strategy of Theorem 7.3.3 except for that we need an additional assumption that w is bounded near $\mathbb{R}^n$ as the non-negativity of the scalar curvature does not imply w is bounded in this case.

9.2 Proof for Theorem 7.3.1

First we recall some conformal transformation rules for curvature quantities.

Lemma 9.2.1. *Let $\frac{n}{2} < s < n$ and $f_s := 1 - |\nabla_s^* \rho_s|^2$. Denote scalar curvature, traceless Ricci curvature, and Ricci curvature of g_s^* by R_s^*, E_s^*, Ric_s^* respectively. Then we have following identities:*

- (1) $\Delta_s^* \rho_s = -s(\frac{f_s}{\rho_s})$.
- (2) $R_s^* = (s - \frac{n+1}{2})\frac{f_s}{\rho_s^2}$.
- (3) $E_s^* = -(n-1)\rho_s^{-1}(\nabla_s^{*2} \rho_s + \frac{s}{n+1}\frac{f_s}{\rho_s}g_s^*)$.
- (4) $Ric_s^* = -(n-1)\rho_s^{-1}(\nabla_s^{*2} \rho_s + \frac{s}{n+1}\frac{f_s}{\rho_s}g_s^*) + (\frac{s-\frac{n+1}{2}}{n+1})\frac{f_s}{\rho_s^2}g_s^*$.
- (5) $Ric_s^*(\nabla_s^* \rho_s, \nabla_s^* \rho_s) = -(n-1)\frac{\nabla_s^{*2} \rho_s(\nabla_s^* \rho_s, \nabla_s^* \rho_s)}{\rho_s} + [\frac{s-\frac{n+1}{2}}{n+1} - \frac{(n-1)s}{n+1}]\frac{f_s}{\rho_s^2}|\nabla_s^* \rho_s|^2$.

Proof. See (2.2), (2.3), (4.3), (4.4), and (4.5) of [WZ]. □

Theorem 9.2.2. *Let $(X^{n+1}, \partial X = M, g_+)$ be a CCE manifold and $\hat{g}$ be the compactified metric on the boundary M . Suppose the scalar curvature $\hat{R}$ of $(M^n, \hat{g})$ is positive. Let $0 < \gamma < 1$. Then $|\nabla_s^* \rho_s|^2 < 1$ on $\bar{X}$, or $\text{sign}(R_s^*) = \text{sign}(\gamma - \frac{1}{2})$ where ∇_s^*, R_s^* are the covariant derivative and the scalar curvature of the adapted metric g_s^**

Proof. By straightforward computation utilizing Lemma 9.2.1, it is easy to see that

$$\Delta_s^* f_s = -2|\nabla_s^{*2} \rho_s|^2 + (2s - n + 1)\frac{\nabla_s^* \rho_s}{\rho} \cdot \nabla_s^* f_s - 2(2s - n)f_s \frac{|\nabla_s^* \rho_s|^2}{\rho_s^2}. \quad (9.1)$$

We compute the asymptotic expansion of f_s . Let x be the geodesic defining function, denote $g := x^2 g_+$ and ∇ be the covariant derivative with respect to g . By

Theorem 8.4.2,

$$v_s = x^{n-s}(1 + h_s x^{2\gamma} + \dots)$$

for some negative function h_s on M . By direct computation,

$$\rho_s = x(1 + \frac{h_s}{n-s} x^{2\gamma} + \dots)$$

from which we have

$$\begin{aligned} f_s &= 1 - |\nabla_s^* \rho_s|^2 = 1 - \left(\frac{x}{\rho_s}\right)^2 |\nabla \rho_s|^2 \\ &= 1 - \left(1 + \frac{h_s}{n-s} x^{2\gamma}\right)^{-2} \left(1 + (1+2\gamma) \frac{h_s}{n-s} x^{2\gamma}\right)^2 \\ &= -4\gamma \frac{h_s}{n-s} x^{2\gamma} + \dots \end{aligned}$$

Since $h_s < 0$, $\frac{f_s}{\rho_s^{2\gamma}}$ is strictly positive on the boundary M . This is the reason we chose the exponent 2γ at the denominator of the quantity $\frac{f_s}{\rho_s^{2\gamma}}$.

Next we compute the equation for $\frac{f_s}{\rho_s^{2\gamma}}$. Using the identity (9.1), it is easy to check

$$\Delta_s^* \left(\frac{f_s}{\rho_s^{2\gamma}} \right) = -\frac{2|\nabla_s^* \rho_s|^2}{\rho_s^{2\gamma}} + (1-2\gamma) \nabla_s^* \left(\frac{f_s}{\rho_s^{2\gamma}} \right) \cdot \left(\frac{\nabla_s^* \rho_s}{\rho_s} \right) + 2\gamma s \frac{f_s^2}{\rho_s^{2\gamma+2}}. \quad (9.2)$$

Now we apply the continuity method to prove $\frac{f_s}{\rho_s^{2\gamma}}$ is positive. For $s = \frac{n}{2} + \frac{1}{2}$, or $\gamma = \frac{1}{2}$, it is already known by Lemma 8.4.4. Now we apply the continuity method to prove the assertion. Since v_s is a continuous family, openness follows. For closedness, suppose $f_s \geq 0$ and $f_s = 0$ at some point. $f_s = 0$ should be achieved at an interior point as f_s is strictly positive on the boundary due to the asymptotic expansion. We apply Serrin's strong maximum principle to the equation (9.2) to derive a contradiction. Hence $f_s > 0$ in $\bar{X}$ and the sign of R_s^* follows from the identity $R_s^* = (\gamma - \frac{1}{2}) \frac{f_s}{\rho_s^2}$. $\square$

9.3 Proofs for Liouville-type theorems

We begin with the following simple implication of the condition $1 - |\nabla_s^* \rho_s|^2 \geq 0$.

Lemma 9.3.1. *Let $\frac{n}{2} < s < n$ and v_s be a positive solution of the PDE*

$$-\Delta_+ v_s - s(n-s)v_s = 0$$

on $\mathbb{R}_+^{n+1}$ such that $1 - |\nabla_s^* \rho_s|^2 \geq 0$. Let $w_s := 1 - \frac{v_s}{y^{n-s}}$. Assume further that $w_s \equiv 0$ on $\mathbb{R}^n$. Then, $0 \leq \partial_y w_s$, and $0 \leq w_s < 1$ on $\mathbb{R}_+^{n+1}$.

Proof. Note that $1 - |\nabla_s^* \rho_s|^2 \geq 0 \iff |\nabla_s^* v_s^{1/(n-s)}| \leq 1 \iff \frac{y}{v_s^{1/(n-s)}} |\nabla v_s^{1/(n-s)}| \leq 1$ where ∇ is the covariant derivative with respect to the Euclidean metric. In particular,

$$\frac{v_s^{1/(n-s)}}{y} \geq |\nabla v_s^{1/(n-s)}| \geq \partial_y v_s^{1/(n-s)} = v_s^{1/(n-s)-1} \left(\frac{1}{n-s} \right) \partial_y v_s$$

from which we obtain $(n-s)\frac{v_s}{y} \geq \partial_y v_s$ as $s < n$.

Then, we have $\partial_y w_s = \partial_y \left(1 - \frac{v_s}{y^{n-s}} \right) = \frac{1}{y^{n-s}} ((n-s)\frac{v_s}{y} - \partial_y v_s) \geq 0$. $w_s < 1$ follows from $0 < v_s$. As $w_s \equiv 0$ on $\mathbb{R}^n$ and w_s is non-decreasing in the y -direction, w_s is non-negative. $\square$

The integral identity below is the key identity for our proofs

Lemma 9.3.2. *Let $\frac{n}{2} < s < n$ and v_s be a positive solution of the PDE*

$$-\Delta_+ v_s - s(n-s)v_s = 0$$

on $\mathbb{R}_+^{n+1}$ such that $1 - |\nabla_s^* \rho_s|^2 \geq 0$. Let $w_s := 1 - \frac{v_s}{y^{n-s}}$. Assume $w_s \in \tilde{H}^{1,0}(\mathbb{R}_+^{n+1})$. Then, for almost every $\rho > 0$, we have

$$\int_{(\partial B_r)^+} \partial_r (w_s^2) = 2 \int_{(B_r)^+} (|\nabla w_s|^2 - \left(\frac{a}{y} \right) w_s \partial_y w_s).$$

Proof. Let $a = 1 - 2\gamma$. By direct computation, the PDE for w_s with respect to the Euclidean metric is

$$\Delta_a w_s = 0.$$

Since this PDE is linear in the interior $\mathbb{R}_+^{n+1}$, w_s is smooth on $\mathbb{R}_+^{n+1}$. Integral by parts formula gives us

$$\begin{aligned} \int_{(\partial B_r)^+ \cap \{y > \epsilon\}} \partial_r(w_s^2) &= \int_{B_r^+ \cap \{y > \epsilon\}} \operatorname{div}(\nabla w_s^2) + \int_{B_r^+ \cap \{y = \epsilon\}} \partial_y w_s^2 \\ &= 2 \int_{B_r^+ \cap \{y > \epsilon\}} (|\nabla w_s|^2 + w_s \Delta w_s) + \int_{B_r^+ \cap \{y = \epsilon\}} \partial_y w_s^2 \\ &= 2 \int_{B_r^+ \cap \{y > \epsilon\}} (|\nabla w_s|^2 - \left(\frac{a}{y}\right) w_s \partial_y w_s) + \int_{B_r^+ \cap \{y = \epsilon\}} \partial_y w_s^2 \end{aligned}$$

for small $\epsilon > 0$. As $w_s \in \tilde{H}^{1,0}(\mathbb{R}_+^{n+1})$, $\int_{(\partial B_r)^+} |\partial_r(w_s^2)| < \infty$ for almost every r by Fubini's theorem. For such r , the left-hand side and the first term of the right-hand side converges to $\int_{(\partial B_r)^+} \partial_r(w_s^2)$ and $2 \int_{(B_r)^+} (|\nabla w_s|^2 - \left(\frac{a}{y}\right) w_s \partial_y w_s)$ respectively as $\epsilon \rightarrow 0$ by the Lebesgue dominated convergence theorem. Let $p(\epsilon) := \int_{B_r^+ \cap \{y = \epsilon\}} |\partial_y w_s^2|$. It suffices to prove that there exists a sequence $\epsilon_k \rightarrow 0$ such that $p(\epsilon_k) \rightarrow 0$. We observe that

$$\begin{aligned} \int_0^r \frac{p(\epsilon)}{\epsilon} d\epsilon &= \int_{B_r^+} y^{-1} |\partial_y w_s^2| \\ &\leq 4 \int_{B_r^+} y^{-2} w_s^2 + |\nabla w_s|^2 \leq C \int_{B_r^+} |\nabla w_s|^2 < \infty \end{aligned}$$

by (1) of Proposition 8.1.4. If there exists $\delta, \epsilon' > 0$ such that $p(\epsilon) > \delta$ for every $\epsilon < \epsilon'$, then

$$\int_0^r \frac{p(\epsilon)}{\epsilon} d\epsilon \geq \int_0^{\epsilon'} \frac{\delta}{\epsilon} d\epsilon = \infty$$

which is a contradiction. Hence, there exists a sequence $\epsilon_k \rightarrow 0$ such that $p(\epsilon_k) \rightarrow 0$ and the proof is complete. $\square$

Now we are ready to prove our desired Liouville-type theorems.

Proof of Theorem 7.3.2. By direct computation, the PDE for w_s with respect to the Euclidean metric is

$$\Delta_a w_s = 0.$$

Observe that $0 \leq w_s < 1$ by Lemma 9.3.1. As $w_s \in \tilde{H}^{1,a}(\mathbb{R}_+^{n+1})$, and $0 \leq w_s \leq 1$ we may apply Proposition 9.1 to conclude $w_s \equiv 0$, or $v_s = y^{n-s}$. $\square$

Proof of Theorem 7.3.3. Firstly, $0 \leq \partial_y w_s$, and $0 \leq w_s \leq 1$ by Lemma 9.3.1. Let $H(r) = \frac{1}{r^n} \int_{(\partial B_r)^+} w_s^2$ and $E(r) = \frac{1}{r^{n-1}} \int_{B_r^+} |\nabla w_s|^2$. Note that $H(r)$ and $E(r)$ are scale invariant. Applying Lemma 9.3.2, we obtain the following inequality for almost every $r > 0$.

$$\begin{aligned} H'(r) &= \frac{1}{r^n} \int_{(\partial B_r)^+} \partial_r(w_s^2) \\ &= \frac{2}{r^{n-1}} \int_{(B_r)^+} (|\nabla w_s|^2 - \left(\frac{a}{y}\right) w_s \partial_y w_s) \\ &\geq \frac{2}{r^{n-1}} \int_{(B_r)^+} |\nabla w_s|^2 = \frac{2}{r} E(r) \end{aligned}$$

where we use $w_s, \partial_y w_s \geq 0$ at the last inequality. Then by (3) of Proposition 8.1.4, $H'(r) \geq \frac{C_0}{r} H(r)$ for a.e. r . Integrating this differential inequality, and using the fact w_s is bounded,

$$r^{n+C_0} \int_{(\partial B_1)^+} w_s^2 \leq \int_{(\partial B_r)^+} w_s^2 \leq \int_{(\partial B_r)^+} 1 = Cr^n$$

Taking the limit $r \rightarrow \infty$, we immediatly see that $w_s \equiv 0$ $\square$

Below is for $\gamma = \frac{n}{2}$, or the Fefferman-Graham compactification.

Proof of Theorem 7.3.4. w satisfies followings:

$$\begin{cases} \Delta w = (n-1) \frac{\partial_y w}{y} \\ \Delta w + \left(\frac{n-1}{2}\right) |\nabla w|^2 \leq 0 \\ w = 0 \text{ on } \mathbb{R}^n. \end{cases}$$

The non-negativity of scalar curvature implies that

$$0 \geq \Delta w + \left(\frac{n-1}{2}\right)|\nabla w|^2 \geq (n-1)\frac{\partial_y w}{y} + \left(\frac{n-1}{2}\right)|\partial_y w|^2 = (n-1)\partial_y w\left(\frac{1}{y} + \frac{\partial_y w}{2}\right).$$

Thus, we have $0 \geq \partial_y w \geq -\frac{1}{2y}$ and since w is bounded near $\mathbb{R}^n$, it is easy to see that $-C(\log(y+1)+1) \leq w \leq 0$ for some constant $C > 0$.

Let $H(r) = \frac{1}{r^n} \int_{(\partial B_r)^+} w^2$ and $E(r) = \frac{1}{r^{n-1}} \int_{B_r^+} |\nabla w|^2$. Note that $H(r)$ and $E(r)$ are scale invariant. Applying integral by parts formula,

$$\begin{aligned} H'(r) &= \frac{1}{r^n} \int_{(\partial B_r)^+} \partial_r(w^2) \\ &= \frac{2}{r^n} \int_{(B_r)^+} |\nabla w|^2 + (n-1)w\left(\frac{1}{y}\partial_y w\right) \\ &\geq \frac{2}{r^n} \int_{(B_r)^+} |\nabla w|^2 = \frac{2}{r} E(r) \end{aligned}$$

where we used $w, \partial_y w \leq 0$ at the last inequality. Then by Sobolev trace inequality, $H'(r) \geq \frac{C_n}{r} H(r)$ for some $C_n > 0$. Integrating this differential inequality, and using the lower bound of w , we have

$$r^{n+C_n} \int_{(\partial B_1)^+} w^2 \leq \int_{(\partial B_r)^+} w^2 \leq C \int_{(\partial B_r)^+} (1 + \log(y+1))^2 \leq C_\epsilon \int_{(\partial B_r)^+} (y^\epsilon + 1)$$

for some $0 < \epsilon < C_n$ and $C_\epsilon > 0$. Taking the limit $r \rightarrow \infty$, this shows $w \equiv 0$.

□

Bibliography

- [CC] J. Case and S.-Y. A. Chang, On fractional GJMS operators, *Comm. Pure Appl. Math.* 69 (2016), 1017-1061.
- [CG] S.-Y. A. Chang and Y. Ge, Compactness of conformally compact Einstein manifolds in dimension 4, *Adv. Math.* 340 (2018), 588-652.
- [CGJQ] S.-Y. A. Chang, Y. Ge, S. Jin and J. Qing, On compactness conformally compact Einstein manifolds and uniqueness of Graham-Lee metrics, III, *arXiv:2107.03075*.
- [CGQ] S.-Y. A. Chang, Y. Ge, Compactness of conformally compact Einstein 4-manifolds II, *Adv. Math.* 373 (2020), 107325.
- [CS] L. Caffarelli, L. Silvestre. An extension problem related to the fractional Laplacian. *Communications in partial differential equations* 32 (2007), no. 8, 1245-1260.
- [FG] C. Fefferman, and R. Graham, Q-curvature and Poincaré metrics, *Math. Res. Lett.*, 9 (2002), no. 2-3, 139-151.
- [GQ] C. Guillarmou and J. Qing, Spectral characterization of Poincaré-Einstein manifolds with infinity of positive Yamabe type, *Int. Math. Res. Not.* (2010), 1720-1740.

- [GZ] C. R. Graham and M. Zworski, Scattering matrix in conformal geometry, *Invent. Math.* 152 (2003), 89-118.
- [L] J. Lee, The spectrum of an asymptotically hyperbolic Einstein manifold. *Comm. Anal. Geom.* 3 (1995), no. 1-2, 253–271.
- [MM] R. Mazzeo and R. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, *J. Funct. Anal.* 113 (1991), 25-45.
- [STV1] Y. Sire, S. Terracini, S. Vita, Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions. *Comm. Partial Differential Equations* 46-2 (2021), 310-361.
- [STV2] Y. Sire, S. Terracini, S. Vita, Liouville type theorems and regularity of solutions to degenerate or singular problems part II: odd solutions. *Math. Eng.* 3-1 (2021), 1-50.
- [WZ] F. Wang and H. Zhou, A note on the compactness of Poincarè-Einstein manifolds, *arXiv:2106.01704*, 2021.
- [Y] R. Yang, On higher order extensions for the fractional Laplacian, *arXiv:1302.4413*, preprint.