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Doctoral Dissertations, 2011-2026 Mathematics

2025

Spacelike singularities in General Relativity and the BKL proposal

Li, Warren Hua-Lun

Princeton, NJ : Princeton University

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Spacelike singularities in General Relativity and the BKL proposal

Warren Hua-Lun Li

A Dissertation

Presented to the Faculty

of Princeton University

in Candidacy for the Degree

of Doctor of Philosophy

Recommended for Acceptance

by the Department of

Mathematics

Adviser: Mihalis Dafermos

May 2025

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Abstract

In this dissertation, we present several results related to spacelike singularities arising in Einstein's theory of General Relativity, and their connection to the heuristics of Belinski, Khalatnikov and Lifshitz (BKL) in the physics literature. BKL proposed that spacelike singularities arising from the Einstein equations resemble an anisotropic Kasner metric, but where the rates and directions of compression and stretching depend on space and time. Further, they describe the time dependence as a cascade of Kasner-like regimes connected by *BKL bounces* where the rates and directions transition rapidly from one regime to another – often termed BKL's *chaotic and oscillatory approach to singularity.*

Our thesis has three main parts, each of which provides support of BKL's proposal within various contexts. In the first part, we show that a large class of spherically symmetric solutions to the Einstein–(Maxwell)–scalar field system possess a Kasner–like singularity in the above sense. Though the scalar field and symmetry assumptions suppress the above oscillations, remnants of their effects remain upon introducing the Maxwell field. This is explored further in joint work with M. Van de Moortel, where we study homogeneous black hole interiors with charged scalar hair.

In the second part, we characterize a wide class of symmetric, but *spatially inhomogeneous* spacetimes, both in vacuum and within the Einstein–Maxwell–scalar field model, which feature BKL bounces near their singularity. These are the first *inhomogeneous* examples of such bounces, and although our current work allows for only one bounce rather than a chaotic cascade, our method – combining nonlinear ODE analysis with energy estimates – is robust and we outline several potential further applications.

The final part analyzes the *linearized Einstein–scalar field system* near exact Kasner spacetimes and exhibits a *scattering isomorphism* between suitable Cauchy initial data and the asymptotic quantities (e.g. Kasner-like rates and directions) at the singularity. The scattering map reveals interesting gains and losses of derivatives, which depend sensitively on the anisotropy of the Kasner background.

Acknowledgements

First of all, I wish to express my deepest gratitude to my parents and my brother for their unwavering love and support. Without them, I would never have started my journey as a scientist and mathematician. Even from across the ocean, they remain my most enthusiastic cheerleaders.

I will be forever grateful to my PhD advisor, Mihalis Dafermos, for his mentorship and, more importantly, his friendship and camaraderie during my years at Princeton. With his unrivaled passion for General Relativity, Mihalis has shaped my perspective on the field as well as my perspective on the role of the mathematician. He is also full of charming anecdotes and I've hugely enjoyed our many discussions with topics ranging from physics to philosophy to Greek culture, and of course how to write effective exposition, and I'm very lucky to have him as my advisor.

Next, I wish to thank Igor Rodnianski for many enlightening discussions and comments on my work. I would like to thank Alex Ionescu, as well as Igor and Mihalis again, for agreeing to serve on my dissertation committee. I am also very grateful to Maxime Van de Moortel for being an exceedingly enthusiastic collaborator and mentor, and also serving as a reader for my dissertation. And, of course, I wish to thank Jill Leclair for making my time at graduate school a relatively hurdle-free experience, and for all her help navigating through bureaucracy, particularly with my time away in the first and last years of my PhD.

I am very fortunate to have been in Princeton at a time when mathematical relativity was booming, and to have overlapped with so many exceptional mathematical peers and mentors. I would like to thank Serban Cicortas for being willing to share so much time in front of a blackboard, and for being a collaborator in all but name. I'd also like to thank Grigorios Fournodavlos for sharing many insights about Big Bang singularities, as well as being an excellent host in my multiple visits to Crete. I would also like to acknowledge Christoph Kehle, Giorgos Moschidis, Bjoern Bringmann, John Anderson, Gabriele Benomio, Ryan Unger, István Kádár, Hamed Masaood, Allen Fang, Jaydeep Singh, Stefan Tudose, Tuomas Tuukkanen, Onyx Gautam, and Igor Medvedev for their friendship and words of wisdom. I also spent part of my 5th year in Cambridge, and would like to thank Rita Teixeira da Costa, Claude Warnick, Spyros Alexakis, Malcolm Perry, Grigalius Taujanskas, David Bick, Mónica Tapia del Moral, Matt Smith and Lili Fehértói-Nagy for their wonderful company during my visit.

Next, I wish to thank Jonathan Luk, Hans Ringström and Martin Taylor, particularly for their invaluable assistance in my job search. I would also like to acknowledge Sung-Jin Oh, Jared Speck, Yakov Shlapentokh-Rothman, Jan Sbierski, Jacques Smulevici, Arick Shao, Elena Giorgi, Shiwu Yang, Jérémie Szeftel, Xinliang An and Mahir Hadžić for enjoyable discussions and allowing me opportunities to give talks all across the world.

During my time at graduate school, I was extremely fortunate to be part of a wonderful community in Fine Hall and beyond, and for that I wish to acknowledge Giorgos Kotsovolis, Ian Fleschler,

Doug Stryker, Katy Woo, Anna Skorobogatova, Marcus Greenwood, Cindy Wang, Hezekiah Grayer II, Vikram Giri, Paul Minter, Thomas Massoni, Lorenzo Sarnataro, Noah Kravitz, Trajan Hammonds, Maciej Wlodek, Mayank Pandey, Anshul Adve and many others in the Fine Hall community. I would also like to acknowledge the Princeton University Badminton Club, particularly Riri Jiang, Yuewen Wu and Dave Singh, for giving me a place to blow off steam.

The final word, of course, belongs to Yue. Thank you for your love and kindness, and for putting up with a long distance mathmo all this time.

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Chapter 1

Introduction

A significant milestone in the understanding of Einstein's theory of General Relativity is the prediction of *singularities*, which arise, for instance, in the study of the *interior of black holes* or at the *Big Bang*. The *Einstein equations* of General Relativity [56] detail how the *curvature* of a spacetime $(\mathcal{M}^4, \mathbf{g})$ is coupled to *matter fields*:

$$\mathbf{Ric}_{\mu\nu}[\mathbf{g}] - \frac{1}{2}\mathbf{R}[\mathbf{g}]\mathbf{g}_{\mu\nu} = 2\,\mathbf{T}_{\mu\nu}. \tag{Ein}$$

Here, $\mathbf{g}$ is a Lorentzian metric on a $(1+3)$-dimensional manifold $\mathcal{M}^4$, while $\mathbf{Ric}[\mathbf{g}]$ and $\mathbf{R}[\mathbf{g}]$ represent the *Ricci curvature* and *scalar curvature* associated to $\mathbf{g}$, and $\mathbf{T}$ is a symmetric $(0,2)$-tensor on $\mathcal{M}$ representing the *energy-momenta* of matter; examples of $\mathbf{T}$ for various matter fields will be given in Chapter 2. In the case that $\mathbf{T} \equiv 0$, we call (Ein) the *Einstein vacuum equations*.

In mathematics, one studies solutions of the geometric equation (Ein) by making a suitable choice of coordinates on $\mathcal{M}$, thereby transforming (Ein) into a system of nonlinear PDEs. Many such choices, including the *wave coordinates* considered in the foundational result of Choquet-Bruhat [59], reveal the *hyperbolic* character of the Einstein equations. Therefore, (Ein) may be treated as a locally well-posed system of evolution equations, with initial data given on a spacelike *Cauchy hypersurface*. One is then interested in the global properties of dynamical spacetimes arising from such initial data.

Celebrated *singularity theorems* of Penrose [113] and Hawking [77] show that under several physically reasonable conditions, "singularities" are a robust dynamical prediction of the Einstein equations (Ein). Two examples of spacetimes exhibiting singularities are the (interior regions of the) spherically symmetric *Schwarzschild black holes* [134], and the *Kasner solutions* [89] often encountered in mathematical cosmology. One application of the Penrose–Hawking theorems is that the "singularity" of these spacetimes persists upon applying small perturbations.

However, one shortcoming of the Penrose–Hawking theorems is that the proof, while remarkable in its elegant use of Lorentzian geometry, reveals little about the *nature* of the predicted singularity.

In fact, one should remark that what the theorems predict is not singularity formation in a mathematical sense – suggesting *blow-up* of some quantity – but instead *(causal) geodesic incompleteness*. Indeed, there are spacetimes, including the *Kerr black holes* and the *Taub–NUT spacetimes*, which satisfy the conditions of the Penrose–Hawking theorems yet exhibit no coordinate-invariant blow-up; the geodesic incompleteness is instead due to *Cauchy horizons* across which these spacetimes may be smoothly extended.

Even restricting to a class of spacetimes whose singularity exhibits, say, curvature blow-up and such that the singularity may be realised as a *spacelike* boundary of $\mathcal{M}$ – thereby ignoring phenomena such as *weak null singularities* [107] – the question of how the spacetime metric $\mathbf{g}$ generically behaves near the singularity remains puzzling to mathematicians and physicists alike.

One proposal towards the resolution of this problem comes from a series of articles written by physicists Belinski, Khalatnikov and Lifshitz [95, 16, 17], often shortened to BKL. To describe their proposal, it will be convenient to write the explicit form of the (anisotropic) Kasner metric,

$$\mathbf{g}_{Kas} = -dt^2 + t^{2p_1}(dx^1)^2 + t^{2p_2}(dx^2)^2 + t^{2p_3}(dx^3)^2, \tag{Kas}$$

where $p_1, p_2, p_3 \in \mathbb{R}$ that determine the rates at which lengths contract or expand in the x^1, x^2 and x^3 directions. For $\mathbf{g}_{Kas}$ solving the Einstein vacuum equations, exactly one of the p_i is negative and the remaining two are positive, meaning that as one approaches the singularity at $t = 0$, one direction is infinitely streched while the other two are infinitely compressed. This phenomenon is colloquially known as *spaghettification*.

One notes, however, that $\mathbf{g}_{Kas}$ is highly symmetric and does not necessarily, at first glance, say anything about "generic" singularities. The remarkable proposal of BKL is that in a neighbourhood of a generic spacelike singularity, the metric $\mathbf{g}$ highly resembles that of (Kas) but where both the *rates* p_i and the *directions* dx^i of contraction and expansion are allowed to depend on space and time. We call this BKL's *Kasner-like ansatz*, see already (1.2.2).

Furthermore, they postulate that, sufficiently close to the singularity, the spatial x-dependence is dominated by the temporal t-dependence, and therefore that at any spatial point on the singularity, the dynamical properties of the metric $\mathbf{g}$, as well as any matter fields, are well-approximated by the near-singularity dynamics of a system of finite-dimensional, autonomous, ODEs. Under this assumption, often called the *asymptotically velocity term dominated* assumption, or AVTD for short, the dynamics of generic singularities thus reduce to understanding the dynamics of generic orbits of this ODE system.

We now distinguish two separate cases. The first case, often denoted the *subcritical regime*, corresponds to the ODE system having *stable*[1] *fixed points*. Such regimes arise when studying problems in symmetry, or when particular choices of "stiff" matter fields are present [14], or in studying the Einstein equations in high dimension [50]. In this case, one expects generic ODE

[1]Stability is understood as stable in the direction $t \to 0$, i.e. towards the singularity.

orbits to converge to one of these fixed points; as a result, the singularity consists of spatial points x each of which are associated to one of these stable fixed points.

On the other hand, in the second case, which we call the *oscillatory regime*, every fixed point of the ODE system is *unstable*. Here, BKL [16, 17] suggest that generic ODE orbits resemble an infinite cascade of *heteroclinic orbits*, each of which connects one unstable fixed point to another. As a result, near any point on the singularity, the spacetime resembles an infinite sequence of *Kasner epochs* – corresponding to the unstable fixed points – which are connected by so-called *BKL bounces* – corresponding to the heteroclinic orbit. Furthermore, the ODE system exhibits features reminiscent of chaos, and the proposal here is often called BKL's *chaotic and oscillatory approach to singularity*. Interest in this regime stems from the fact that the $(1 + 3)$-dimensional Einstein vacuum equations correspond to this regime. Further, this does not change upon coupling the Einstein equations to many physically reasonable matter fields e.g. perfect fluids, electromagnetism, particle-like matter, etc.

While there exists a plethora of work regarding the BKL proposal in the physics literature, see for instance the book of Belinski and Henneaux [13], mathematical results in this direction are relatively few. While there are examples, particularly in various symmetry classes, of rigorously constructed solutions to (Ein) with (curvature) singularities, their connection to BKL's work is rather underexplored.

Question 1. Are there large classes of solutions to the Einstein equations (Ein) possessing a (curvature) singularity Kasner-like in the sense of BKL? In particular, near the singularity can one quantitatively identify the rates and directions, and their dependence on t and x?

This question is (intentionally) vaguely phrased so as to leave phrases such as "large classes", "Kasner-like" and "quantitatively identify" somewhat open to interpretation. Question 1 can be asked in both the subcritical regime (where, close enough to the singularity, the t-dependence may be ignored), and the oscillatory regime (where one expects both t and x-dependence).

We answer Question 1 for the spherically symmetric Einstein–(Maxwell)–scalar field system in Theorem 1.3.1, where both the symmetry class and the scalar matter will help to enforce *subcriticality*. This model has longstanding use in the study of (spherically symmetric) gravitational collapse, see for instance [34, 99]. In [34] and references within, Christodoulou proves the Weak and Strong Cosmic Censorship Conjectures[2] in the simpler case of the Einstein–scalar field system, in particular Christodoulou proves that generic spacetimes with a black hole terminate in a spacelike singularity with curvature blow-up. One application of Theorem 1.3.1 is to show that this singularity in indeed Kasner-like.

Another showcase of Kasner-like behavior in black hole interiors appears in Theorem 1.4.1, which

[2]See [114] for an introduction to these conjectures; the Strong Cosmic Censorship conjecture will also be expanded upon in Section 1.1.

arose from joint work with M. Van de Moortel and concerns the interior of certain so-called "hairy black holes", which are spatially homogeneous solutions of the Einstein–(Maxwell)–charged scalar field system. These hairy black hole interiors exhibit a plethora of interesting nonlinear behavior, including growing and oscillatory behavior for the scalar field as one approaches the singularity, and settling to a Kasner-like regime even closer to the singularity. While, as before, the scalar matter enforces subcriticality, remnants of the oscillatory behavior – namely the BKL bounces – are present in the findings of Theorem 1.4.1.

Question 1 appears to be much more challenging in the oscillatory regime. In this setting, at present definitive answers are only available in *spatially homogeneous* spacetimes, where homogeneity means there is no x-dependence, and one can immediately reduce the dynamics to ODEs, as we do in the case of the Einstein–(Maxwell)–charged scalar field system with $\mathbb{R} \times \mathbb{S}^2$ spatial topology in Theorem 1.4.1. See also [82] for an example of another such ODE system for the Einstein vacuum equations with more general spatial topologies. Outside this spatially homogeneous case where the ODEs exactly govern the dynamics, progress towards Question 1 necessitates a proof of the AVTD assumption, so that terms involving x-derivatives may be considered error terms in the Kasner-like description. This encourages us to pose the following question.

Question 2. Can one prove that *generic* spacelike singularities arising in solutions to the Einstein equations (Ein) exhibit AVTD behaviour? Can one identify quantities that satisfy a system of autonomous, nonlinear ODEs, up to error terms whose contribution is negligible?

As we elaborate upon in Section 1.2, Question 2 is easier to answer in the subcritical regime, where subcriticality ensures no complicated nonlinear behaviour in a neighborhood of the singularity, see for instance [63]. To make progress in the oscillatory regime, one must consider Question 2 even in the presence of nonlinear, and possibly chaotic, effects. That is, Question 2 is asked in conjunction with the following.

Question 3. Can one describe the dynamical behaviour of the associated system of ODEs? For instance, can one identify stable fixed points, unstable fixed points, and heteroclinic orbits? What is the behaviour of generic orbits?

At first glance, Question 3 is a pure ODE problem whose analysis involves the theory of finite dimensional dynamical systems and is decoupled from the rest of the problem. Indeed, much of the literature focuses solely on analysis of the ODEs of [82], see [151, 120] and the recent [11]. We also refer the reader again to Theorem 1.4.1, where an explicit mechanism for BKL bounces, i.e. suitable heteroclinic orbits, is obtained for the spherically symmetric Einstein–Maxwell–scalar field model. However, as ultimatately the ODE is only an approximation and one must introduce error terms, we see it is the case that Question 3 and Question 2 must be understood in parallel.

It is clear that to resolve Question 1 for *generic* solutions of (Ein), resolutions of Question 2 and Question 3 will be key. As mentioned, this has been largely understood in the subcritical regime, at

least *close enough* to the singularity, while Question 2 remains challenging in the oscillatory regime. One model problem which morally interpolates between these regimes is to understand spacetimes which ultimately settle to the subcritical regime, but further from the singularity exhibit nonlinear behaviour that is well-described by ODEs, which must in turn be analysed to resolve Question 3.

This is the approach taken in our results regarding "BKL bounces outside homogeneity", see Section 1.5 and Theorems 1.5.2, 1.5.3, 1.5.5 and 1.5.6. In these results, we address various models solving (Ein) in symmetry classes, namely the Gowdy symmetric Einstein vacuum equations and surface symmetric Einstein–Maxwell–scalar field equations (where the ODE behavior was derived from that of the aforementioned Theorem 1.4.1). In these results, we construct a large, open, set of initial data such that the solutions possess a spacelike singularity which is "subcritical" but that in an *intermediate regime* away from the singularity the solutions exhibit nonlinear ODE-type behaviour reminiscent of BKL bounces.

Another problem, which we address in the final part of the dissertation, is a *scattering problem*, which is a step towards justifying BKL's claim of their Kasner-like ansatz being "generic". In [95, 16], BKL claim genericity via a "degrees of freedom" counting argument. In other words, they apply a combinatorial argument that the number of functional degrees of freedom in their ansatz matches the number of functional degrees of freedom in prescribing initial data for the Einstein equations (Ein). An interesting mathematical problem is to translate this function counting argument into an isomorphism, or scattering map, between suitable topological spaces. In other words, we ask the following.

Question 4. What is the precise mathematical nature of the (nonlinear) map between Cauchy initial data for the Einstein equations (Ein) and asymptotic information at the singularity?

One interprets the phrase "precise mathematical nature", for instance, as a diffeomorphism between suitably defined Banach spaces or Fréchet spaces – such as Sobolev spaces $W^{k,p}$ or the space C^∞ – or, to allow for greater flexibility, Banach manifolds or Fréchet manifolds. Ideally, if one can use Banach rather than Fréchet, Question 4 is answered such that there is no loss of information in going from initial data to asymptotic information, and vice versa. For instance, for Banach spaces based on Sobolev spaces one can hope to capture information about sharp and precise differences in regularity between initial data and asymptotic information.

The resolution of Question 4 is more straightforward in the subcritical regime, where "asymptotic information" is expected to be encoded by the (stable) fixed point of the ODE to which the spacetime converges in a neighbourhood of each spatial point $x = (x^1, x^2, x^3)$. In such cases, there is much recent progress in understanding formation of Kasner-like singularities from smooth initial data [63, 21, 69] as well as reconstructing a true solution of (Ein) given smooth asymptotic information [61, 127, 64]. Together, these works describe a $C^\infty \to C^\infty$ scattering map between an open set of initial data and an open set of asymptotic information at the singularity.

In the final part of this dissertation, we address the problem of upgrading the scattering map from C^∞ to C^∞ to a scattering map between Sobolev spaces / manifolds of finite regularity. In Theorem 7.3.1, we address the scattering problem for the *linearised Einstein equations* around a fixed Kasner spacetime in the subcritical regime. This builds upon a model problem of understanding (finite regularity) scattering for the *linear wave equation* in a fixed Kasner spacetime, see Theorem 1.6.2. These linear problems simplify the problem in that the scattering map becomes a linear isomorphism between Banach spaces, though we hope that the methods introduced may be applied in the nonlinear problem, which we hope to return to in future work.

In the remainder of the introduction, we provide further background on spacelike singularities in General Relativity and, in particular, the heuristics of BKL, and then briefly discuss the main results of this dissertation and their relationship to the BKL proposal and Questions 1 to 4 above.

1.1 Singularities in General Relativity

We recall several well-known results about the evolution problem in General Relativity and, in particular, the formation of singularities. Here, and in the rest of Chapter 1, we consider spacetimes $(\mathcal{M}^4, \mathbf{g})$ solving the Einstein equations (Ein) with matter fields $\mathbf{\Phi}$ and energy-momentum tensor $\mathbf{T} = \mathbf{T}[\mathbf{\Phi}]$ depending on $\mathbf{\Phi}$ and $\mathbf{g}$.

From [78, Chapter 7], initial data for the Einstein equations is given by a quadruple (Σ, g, k, Φ), where Σ^3 is a smooth 3-manifold, g is a Riemannian metric on Σ, k is a symmetric $(0,2)$-tensor on Σ, and Φ represents initial data for the matter fields $\mathbf{\Phi}$, which evolve according to their own equations of motion depending on the choice of $\mathbf{\Phi}$. The quadruple is also required to satisfy *constraint equations*:

$$ {}^{(3)}R[g] + |k|_g^2 - (\mathrm{tr}_g\, k)^2 = 4\rho[\Phi], \tag{1.1.1} $$

$$ d(\mathrm{tr}_g\, k) - \mathrm{div}_g\, k = 2j[\Phi], \tag{1.1.2} $$

where ${}^{(3)}R[g]$ represents the scalar curvature of g, and the scalar $\rho[\Phi]$ and 1-form $j[\Phi]$ can be derived from the form of $\mathbf{T}[\mathbf{\Phi}]$.

Starting from the seminal work of [59, 30] for the Einstein vacuum equations, and later generalised to many choices of matter field $\mathbf{\Phi}$, it is known that any such quadruple (Σ, g, k, Φ) satisfying (1.1.1)–(1.1.2) gives rise to a *Maximal Globally Hyperbolic Development* (or MGHD, see [78] for a definition) given by a triple $(\mathcal{M}, \mathbf{g}, \mathbf{\Phi})$ solving (Ein), such that Σ is a spacelike Cauchy hypersurface of $(\mathcal{M}, \mathbf{g})$, g and k represent the induced metric and second fundamental form of the embedding $\Sigma \hookrightarrow \mathcal{M}$, and Φ is a suitable restriction of $\mathbf{\Phi}$ onto Σ.

Armed with this initial value formulation, we now recall the Penrose–Hawking singularity theorems. We choose to state the Hawking theorem, abridged from [78, Chapter 8].

Theorem 1.1.1 (Singularity theorem)**.** *Suppose* (Σ, g, k, Φ) *is an initial data set for the Einstein equations* (Ein)*, such that*

(i) The weak energy condition is satisfied i.e. $\mathbf{T}(X,X) \geq 0$ *for all timelike vectors* X*.*
(ii) There exists $\delta > 0$ *such that* $\mathrm{tr}_g\, k \leq -\delta$ *uniformly on* Σ*.*

Then the MGHD $(\mathcal{M}, \mathbf{g}, \mathbf{\Phi})$ *is (future) causally geodesically incomplete.*

We reiterate that the singularity theorems never show "singularity" in the sense of coordinate invariant blow-up, but instead (causal) geodesic incompleteness. We recall to the reader the example of the Taub-NUT spacetime [78, Chapter 5], to which Theorem 1.1.1 applies but which features no coordinate invariant blow-up. Alternatively, one can consider the interior of the Kerr family and Reissner-Nordström family of black holes. In all such cases, the prediced geodesic incompleteness is instead due to the presence of *Cauchy horizons*, across which $(\mathcal{M}, \mathbf{g})$ has non-unique and non-globally hyperbolic smooth extensions.

There are, of course, known solutions to the Einstein equations where genuine singularities are present. The celebrated Schwarzschild family of spacetimes are vacuum solutions to (Ein) depending on a single parameter $M > 0$. They each contain a black hole interior region, described by the metric $\mathbf{g}_{Schw}$ on $\mathcal{M}_{Schw,int} = (0, 2M) \times \mathbb{R} \times \mathbb{S}^2$:

$$\mathbf{g}_{Schw} = -\left(\frac{2M}{r} - 1\right)^{-1} dr^2 + \left(\frac{2M}{r} - 1\right) dt^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{1.1.3}$$

Treating r as a past-directed timelike coordinate, the interior region is bounded to the future by a *spacelike singularity* at $r = 0$. The singular nature of this boundary is reflected in the fact that the *Kretschmann scalar* can be computed to be

$$\mathfrak{K} \coloneqq |\mathbf{Riem}[\mathbf{g}_{Schw}]|^2_{\mathbf{g}} = \frac{48M^2}{r^6}$$

and thus blows up towards the singularity at $r = 0$. We call this a *curvature singularity*. In fact, there does not even exist a C^0 extension of (1.1.3) across the $r = 0$ boundary [133].

As alluded to previously, the main protagonist of this discussion will be the Kasner spacetimes (Kas), defined on the manifold $\mathcal{M}_{Kas} = (0, +\infty) \times \mathbb{T}^3$. For $\mathbf{g}_{Kas}$ to be a solution to the Einstein vacuum equations, the real numbers p_i must satisfy the following *Kasner relations*.

$$\sum_{i=1}^{3} p_i = 1, \qquad \sum_{i=1}^{3} p_i^2 = 1. \tag{1.1.4}$$

That is, the p_i are constrained to lie on a circle in parameter space, often called the *Kasner circle*.

Like Schwarzschild, the Kasner spacetimes possess a singular boundary at $t = 0$, and the singular nature of the boundary is manifested by the following blow-up of the Kretschmann scalar as $t \to 0^+$:

$$\mathfrak{K} = \frac{4}{t^4} \cdot \left(\sum_{i=1}^{3} p_i^2 (p_i - 1)^2 + \sum_{i<j} p_i^2 p_j^2\right).$$

We note that the choice $(p_1, p_2, p_3) = (1, 0, 0)$, and permutations thereof, gives rise to exceptional Kasner solutions that are flat and thus locally isometric to Minkowski space. The $t = 0$ boundary of such flat Kasner spacetimes are Cauchy horizons, and for the most part we exclude these solutions from the subsequent discussion.

We also consider Kasner spacetimes which solve not the vacuum equations, but instead the Einstein–*scalar field* system, where the matter $\mathbf{\Phi}$ consists of a scalar function $\phi : \mathcal{M} \to \mathbb{R}$ solving the *wave equation* on $(\mathcal{M}, \mathbf{g})$:

$$\Box_g \phi = \mathbf{g}^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0, \tag{Scal}$$

and whose associated energy-momentum tensor is given by:

$$\mathbf{T}_{\mu\nu}[\phi] = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \mathbf{g}_{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi. \tag{1.1.5}$$

Then a *generalized Kasner spacetime* $(\mathcal{M}_{Kas}, \mathbf{g}_{Kas}, \phi_{Kas} = p_\phi \log t)$ solves the Einstein–scalar field system (Ein) and (Scal) if the *generalized Kasner relations* hold:

$$\sum_{i=1}^{3} p_i = 1, \qquad \sum_{i=1}^{3} p_i^2 + 2p_\phi^2 = 1. \tag{1.1.6}$$

Aside from the exceptional flat case above, these generalized Kasner spacetimes also exhibit curvature blow-up as $t \to 0$.

Before moving onto to the heuristic work of BKL in Section 1.2, we take this opportunity to introduce the Strong Cosmic Censorship conjecture of General Relativity. Vaguely speaking, this conjecture states that in physically reasonable contexts, "generic" solutions to the Einstein equations (Ein) for which the Penrose–Hawking singularity theorems apply indeed exhibit singularity formation in the stronger sense of blow-up. That is, although there are spacetimes such as Taub-NUT and the Kerr family possessing smooth Cauchy horizons, these will be non-generic. We present two versions of this conjecture, one of which applies in the astrophysical setting (e.g. whose singularities are expected to reside in black holes) and the other of which applies in the cosmological setting, which, in our paper, means spacetimes $(\mathcal{M}, \mathbf{g})$ with a closed spatial topology.

Conjecture 1.1.2 (Strong Cosmic Censorship – astrophysical version)**.** *Let $(\mathcal{M}, \mathbf{g}, \mathbf{\Phi})$ be a generic solution to the Einstein equations* (Ein) *with reasonable matter $\mathbf{\Phi}$ which arises as the MGHD of smooth, one-ended and asymptotically flat, initial data (Σ, g, k, Ψ). Then $(\mathcal{M}, \mathbf{g})$ is inextendible as a Lorentzian manifold in a suitable regularity class.*

Conjecture 1.1.3 (Strong Cosmic Censorship – cosmological version)**.** *Let $(\mathcal{M}, \mathbf{g}, \mathbf{\Phi})$ be a generic solution to the Einstein equations* (Ein) *with reasonable matter $\mathbf{\Phi}$ which arises as the MGHD of initial data (Σ, g, k, Φ) with Σ closed. Then $(\mathcal{M}, \mathbf{g})$ is inextendible as a Lorentzian manifold in a suitable regularity class.*

The precise mathematical formulation of these conjectures depends on the choice of definition for the underlined terms. These are all subtle issues, and we mention for one thing that Conjecture 1.1.2 is *false* if one chooses the regularity class to be C^0, see [43], but conjecturally *true* in the C^2 class. The spacetimes of [43] conjecturally contain a *weak null singularity* [107], meaning a C^0 Cauchy horizon at which the Christoffel symbols of $\mathbf{g}$ blow up. For most of this dissertation, we will consider scenarios where the singularity is *spacelike*, and largely ignore other singular behaviour such as weak null singularities and naked singularities.

1.2 The BKL proposal

We now elaborate upon the Kasner-like ansatz for the near-singularity solutions of Einstein's equations, proposed by Khalatnikov and Lifshitz in [95] and developed in the celebrated work of BKL [16, 17]. Starting with the Einstein vacuum equations, [95] considers the following well-known ADM decomposition of spacetime:

$$\mathcal{M} = (0, T) \times \Sigma, \qquad \mathbf{g} = -dt^2 + g(t). \tag{1.2.1}$$

Here $g(t)$ is a time-dependent smooth Riemannian metric on the 3-manifold Σ.

Influenced by the Kasner metric (Kas), [95] proposes the following ansatz for $\mathbf{g}$ in the neighborhood of a singularity at $\{t = 0\}$. For $(t, x) \in \mathcal{M}$, they write

$$\mathbf{g}(t, x) = -dt^2 + \sum_{I=1}^{3} t^{2p_I(x)} \omega^I(x) \otimes \omega^I(x) + \cdots . \tag{1.2.2}$$

In (1.2.2), $\omega^I(x)$ are smooth 1-forms on Σ spanning the cotangent space $T^*\Sigma$, while the exponents $p_I : \Sigma \to \mathbb{R}$ are smooth functions on satisfying the two *Kasner relations*, now functional identities:

$$\sum_{I=1}^{3} p_I(x) = 1, \qquad \sum_{I=1}^{3} p_I^2(x) = 1. \tag{1.2.3}$$

Note the time coordinate t is chosen such that the singularity is *synchronized* at $t = 0$.

Assume for ease of exposition that $p_1(x) < p_2(x) < p_3(x)$. We define $e_I(x)$ to be the vector fields dual to $\omega^I(x)$, and use the following notational convention: for any (p, q)-tensor field T defined on Σ we use lower case indices $T_{i_1 \cdots i_p}{}^{j_1 \cdots j_q}$ to denote the tensor field evaluated using the coordinate vector fields $\frac{\partial}{\partial x^i}$ and their respective coordinate 1-forms dx^i, while upper case indices $T_{I_1 \cdots I_p}{}^{J_1 \cdots J_q}$ denote the tensor field evaluated using the frame fields $e_I(x), \omega^I(x)$. Indices are raised and lowered with respect to the Riemannian metric $g(t)$.

In the gauge (1.2.1), the evolutionary components of the Einstein equations are

$$\frac{\partial}{\partial t} g_{ij} = -2k_{ij}, \tag{1.2.4}$$

$$\frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial t} \left(\sqrt{\det(g)} k_i{}^j \right) = {}^{(3)}\mathrm{Ric}[g]_i{}^j - \mathbf{T}_i{}^j + \delta_i{}^j \left(\mathbf{T}_0{}^0 + \mathbf{T}_k{}^k \right). \tag{1.2.5}$$

The metric on the right hand side of (1.2.2), without the error term denoted by $\cdots$, solves (1.2.4) and (1.2.5), so long as the right hand side of (1.2.5) is absent. It thus seems that a necessary condition for (1.2.2) to be self-consistent in vacuum (with $\mathbf{T} = 0$) is that the expression ${}^{(3)}\mathrm{Ric}[g]$ is suitably small. In [95], the authors argue that the required smallness is that, for some $\epsilon > 0$,

$$|{}^{(3)}\mathrm{Ric}[g]_{IJ}| = O(t^{p_I+p_J-2+\epsilon}). \tag{1.2.6}$$

On the other hand, assuming AVTD behaviour (more on this later), a computation of spatial Ricci curvature with respect to the approximate orthonormal frame $t^{-p_I(x)}e_I(x)$ allows one to obtain the following asymptotics for ${}^{(3)}\mathrm{Ric}[g]_{IJ}$, where λ^K_{IJ} are the structure coefficients of the frame, $d\omega^K = \lambda^K_{IJ}\,\omega^I \wedge \omega^J$, $\lambda^K_{IJ} = -\lambda^K_{JI}$.

$$\left|t^{-p_I-p_J+2}\,.\,{}^{(3)}\mathrm{Ric}[g]_{IJ}\right| = (\lambda^1_{23})^2 \cdot t^{2(1+p_1-p_2-p_3)} \cdot \delta_{IJ} + O(t^{\epsilon}) \tag{1.2.7}$$

Assuming (1.2.3) and $p_1 < p_2 < p_3$, it follows that $2(1 + p_1 - p_2 - p_3) = 4p_1 < 0$, and thus (1.2.6) can only hold if the structure coefficient $\lambda^1_{23}(x)$ vanishes identically on Σ. Together with a function counting argument, this provides an obstruction to (1.2.2) being a sufficiently general ansatz[3]. That is, outside of symmetry classes where λ^1_{23} may vanish by design, one expects p_I and ω_I to depend on t as well as x.

In [16], BKL then give heuristics suggesting what happens if (1.2.6) fails. Again assuming AVTD behaviour, which we recall means that ∂_x-derivatives are subdominant in comparison to ∂_t-derivatives. This has the effect that dynamics in the future light cones emanating from distinct points $(0, p) \in \{0\} \times \mathbf{\Sigma}$ on the singularity are decoupled. With this assumption they then compute that in such a light cone, the ansatz (1.2.2) is valid for $t \gg t_B$ for τ_B some critical time, and for $t \ll t_B$ the metric transitions to a form that again resembles (1.2.2) but with Kasner exponents $p_I(x)$ replaced by new exponents $\acute{p}_I(x)$.

Miraculously, in their computation they provide an explicit relationship between the exponents pre- and post-transition, often known as the *Kasner map*.

$$\acute{p}_1 = -\frac{p_1}{1+2p_1}, \quad \acute{p}_2 = \frac{p_2+2p_1}{1+2p_1}, \quad \acute{p}_3 = \frac{p_3+2p_1}{1+2p_1}. \tag{1.2.8}$$

Between these regimes, the spacetime undergoes a nonlinear transition from one regime to another, often denoted in the literature as a *BKL* or *Kasner bounce*. Typically these nonlinear transitions, or bounces, then cascade indefinitely, giving rise to BKL's *chaotic and oscillatory approach to singularity*.

The story changes slightly if, instead of considering the Einstein vacuum equations, one considers the Einstein–scalar field (Ein) together with (Scal). In [14], Belinski–Khalatnikov study near-singularity behaviour for this system, where together with the same Kasner-like ansatz (1.2.2) for

[3] In the original paper of Khalatnikov and Lifshitz [95], this observation together with a naïve function counting argument led them to believe that singularities are in fact non-generic! However in the later paper [16] this observation was instead understood as an instability; this is the modern viewpoint.

g they suggest the following ansatz for the scalar field ϕ:

$$\phi(t,x) = p_\phi(x)\log t + q(x) + \cdots . \tag{1.2.9}$$

In order for (1.2.2) and (1.2.9) to solve the Einstein-scalar field equations asymptotically, the Kasner relations (1.2.3) are modified to

$$\sum_{I=1}^{3} p_I(x) = 1, \qquad \sum_{I=1}^{3} p_I(x)^2 + 2p_\phi(x)^2 = 1. \tag{1.2.10}$$

The key observation made in [14] is that the modified Kasner relations (1.2.10) admit $p_i(x)$ with $0 < p_1(x) < p_2(x) < p_3(x)$. In such *subcritical* regimes, $2(1+p_1-p_2-p_3) = 4p_1$ is now positive, and the previously problematic term (1.2.7) can be treated as an error. Since one may also show that the matter terms $\mathbf{T}_i{}^j$ and $\mathbf{T}_0{}^0$ on the right hand side of (1.2.5) combine to produce a similarly negligible error. As a result, in such subcritical regimes, (1.2.2) and (1.2.9) actually combine to produce a self-consistent leading order ansatz with the correct number of functional degrees of freedom.

From this argument, we learn that a scalar field, often called a "stiff" matter field, allows one to access a subcritical regime near singularity where each spatial point x on the singularity corresponds exactly to a generalized Kasner solution with Kasner relations (1.1.6). In connection with the discussion preceding Section 1.1, we comment that these generalized Kasner solutions correspond exactly to stable fixed points of an ODE system. See for instance [120], with the caveat that in place of a scalar field [120] considers another "stiff" matter field, namely a perfect fluid with equation of state $p = \rho$.

Function counting and higher-dimensional gravity

Let us flesh out the function counting argument of [95, 16, 14] in the case of the Einstein–scalar field system above, and moreover, generalize the argument to $1+D$-dimensions, where the Einstein equation (Ein) and the wave equation (Scal), as well as the BKL ansatz (1.2.2) and (1.2.9), together with the Kasner relations (1.2.10), have natural generalizations.

Initial data sets for the Einstein–scalar field system are $(\Sigma, g, k, \phi_0, \psi_0)$, where ϕ_0 and ψ_0 will correspond to ϕ and its normal derivative restricted to Σ. Taking the constraint equations (1.2.4)–(1.2.5) as well as the choice of Σ into account, the number of functional degrees of freedom for such initial data quintuples is:

$$\underbrace{\frac{D(D+1)}{2}}_{\text{from } g} + \underbrace{\frac{D(D+1)}{2}}_{\text{from } k} + \underbrace{2}_{\text{from } \phi_0,\psi_0} - \underbrace{(D+1)}_{\text{due to constraints}} - \underbrace{1}_{\text{freedom for } \Sigma} = D^2.$$

On the other hand, the degrees of freedom for BKL's Kasner-like ansatz (1.2.2) and (1.2.9) is computed as follows, taking into account that $\omega^I(x), p_I(x), p_\phi(x), q(x)$ also satisfy D momentum

constraints that we have previously ignored:

$$\underbrace{D+2}_{\text{from } p_I,p_\phi,q} + \underbrace{D^2}_{\text{from components of } \omega^I} - \underbrace{2}_{\text{due to Kasner relations}} - \underbrace{D}_{\text{due to constraints}} = D^2.$$

Therefore, so long as there are no additional obstructions, the number of functional degrees of freedom for initial data and for the Kasner-like ansatz matches. This was BKL's original justification that, in the subcritical regime, where (1.2.6) provides no additional obstruction, their ansatz covers a "generic" class of spacetimes.

We further comment a little on the meaning of subcriticality in higher dimension. Following the work of Demaret–Henneaux–Spindel [50] the obstruction that comes from the analogue of (1.2.6) is dealt with if we impose the subcriticality condition

$$\min_{\substack{1\leq i,j,k\leq D\\ j\neq k}}(1+p_i-p_j-p_k)>0. \tag{1.2.11}$$

As mentioned previously, this is impossible in vacuum for $D=3$, and this extends to $3\leq D\leq 9$. Thus here, one needs the scalar field matter – or some other "stiff" matter – to access the subcritical regime. However, in high dimension $D\geq 10$, [50] observes there are also vacuum Kasner spacetimes satisfying (1.2.11). This is also exploited in the mathematical work of [63, 69].

A comment on electromagnetism

In Chapter 3 and Chapter 6, we will also introduce another matter field, namely Maxwell's electromagnetic field. This is described by a 2-form $\mathbf{F}\in\Omega^2(\mathcal{M})$, evolves according to the (source-free) Maxwell equations,

$$\nabla^\nu \mathbf{F}_{\mu\nu}=0, \qquad \nabla_{[\lambda}\mathbf{F}_{\mu\nu]}=0, \tag{Max}$$

and contributes to the right hand side of Einstein's equation (Ein) via

$$\mathbf{T}_{\mu\nu}[\mathbf{F}]=\mathbf{F}_{\mu\rho}\mathbf{F}_\nu{}^\rho-\frac{1}{4}\mathbf{g}_{\mu\nu}\,\mathbf{F}^{\rho\sigma}\,\mathbf{F}_{\rho\sigma}. \tag{1.2.12}$$

Belinski and Khalatnikov [15] also considered the effect of the electromagnetic field $\mathbf{F}$ – and in fact more general Yang–Mills fields – near the singularity. Unlike the scalar field, $\mathbf{F}$ is tensorial and anisotropic, and [15] show that it actually mimics, in some sense, the effects of gravitational perturbations. We base this discussion off [13, Chapter 4], where the authors suggest the following asymptotics for $\mathbf{F}$ alongside the Kasner-like ansatz (1.2.2):

$$E^I(x,t)=\frac{1}{t}\Phi^I_E(x)+\cdots, \qquad B^I(x,t)=\frac{1}{t}\Phi^I_B(x)+\cdots, \tag{1.2.13}$$

where $E_I\coloneqq\mathbf{F}(\partial_t,e_I)$ and $B_I\coloneqq *\mathbf{F}(\partial_t,e_I)$ is the usual $1+3$ electric-magnetic decomposition of $\mathbf{F}$, evaluated with respect to the spatial frame $\{e_I\}$.

However, even (1.2.13) turns out to be inconsistent outside the subcritical regime where $\min\{p_I(x)\} > 0$. Assuming $p_1(x) < p_2(x) < p_3(x)$, the $e_1(x)$ projections of Maxwell's equations (Max) in the BKL ansatz yield:

$$\frac{\partial}{\partial t}(tE^1(x)) \approx -(\lambda^1_{23})^2 t^{2p_1-1} \cdot tB^1(x), \quad \frac{\partial}{\partial t}(tB^1(x)) \approx +(\lambda^1_{23})^2 t^{2p_1-1} \cdot tE^1(x). \tag{1.2.14}$$

Hence if $p_1 < 0$ the right-hand side would appear to be non-integrable.

Nonetheless, the system (1.2.14) does suggest that the expression

$$\omega^2(t,x) \coloneqq t^2[E^1(x)^2 + B^1(x)^2] \tag{1.2.15}$$

is (approximately) constant in time, and, abusing notation slightly, we set $\omega^2(x) = \omega^2(t,x)$ in the following discussion. As well as ${}^{(3)}\mathrm{Ric}[g]$, we must also estimate the matter term on the right hand side of (1.2.5). Writing $\mathbf{T}_{\mu\nu}[\mathbf{F}]$ in terms of E^i and B^i, one finds the following expression:

$$-2\mathbf{T}_{ij} + g_{ij}(\mathbf{T}_0{}^0 + \mathbf{T}_k{}^k) = 2(E_iE_j + B_iB_j) - g_{ij}(E_kE^k + B_kB^k),$$

which in light of (1.2.15), and upon treating the contributions of E_2, B_2, E_3, B_3 as errors, yields

$$\left| t^{-p_I-p_J+2} \cdot [-2\mathbf{T}_{IJ} + g_{IJ}(\mathbf{T}_0{}^0 + \mathbf{T}_k{}^k)] \right| = \omega^2 \cdot t^{2p_1}\delta_{IJ} + O(t^\epsilon). \tag{1.2.16}$$

The term (1.2.16) with $\omega \neq 0$ sources a similar instability to that of (1.2.7) with $\lambda^1_{23} \neq 0$. Even more remarkably, the analysis of [15] suggests that as long as either ω or λ^1_{23} fails to vanish, we obtain a Kasner bounce with the transition map (1.2.8). This concludes our brief explanation of why the electromagnetic field acts as a proxy for spatial curvature in the BKL setting.

1.3 Spacelike singularities in spherical symmetry

Here, we introduce the concepts of Chapter 3 which study the Kasner-like description of spherically symmetric solutions of the Einstein equations (Ein), coupled to the scalar field ϕ via (Scal) and (1.1.5), and sometimes also an electromagnetic field $\mathbf{F}$ via (Max) and (1.2.12). Note that, by Birkhoff's theorem [22], spherically symmetric solutions to the Einstein <u>vacuum</u> equations are non-dynamical and are (locally) isometric to some region of the Schwarzschild solution with mass $M \in \mathbb{R}$, where $M = 0$ corresponds to Minkowski space, thus matter is introduced in order to study nontrivial dynamical features.

In studying the coupled Einstein–scalar field system in spherical symmetry, the obstruction to Kasner-like asymptotics, introduced in [95] and Section 1.2, will be suppressed in two different ways. Firstly, we observe that spherical symmetry causes the spatial curvature term sourcing the instability (1.2.6) to vanish by assumption. Secondly, the presence of scalar field matter allows for *subcritical* regimes where the spatial curvature no longer provides an instability.

For these reasons, the formation and asymptotics of general spacelike singularities is much better understood in this setting. In Chapter 3, we prove that spacelike singularities in the spherically symmetric Einstein-scalar field system can be described precisely, in the sense that in the neighborhood of a point p on the singular boundary, we can describe the leading order blow-up rates of all geometric quantities (including the Kretschmann scalar and the Hawking mass defined in (2.2.2)) and matter quantities such as ϕ and its derivatives, in terms of the area-radius r, defined in (2.1.1).

In doing so, we provide a precise correspondence between such spacelike singularities in the spherically symmetric Einstein–field model and the heuristics of [14] – in particular we associate to each point p on the singular boundary a triplet of Kasner exponents. This can be viewed as a rigorous version of the correspondence previously found in the physics literature [26] regarding these spherically symmetric singularities.

Our work builds upon that of An–Zhang [4], where the authors provide upper bound estimates for various geometric and matter quantities. Our improvement is that such upper bounds are upgraded to precise leading-order asymptotics, thus allowing us to make the correspondence to Kasner-like singularities. Another novel aspect of our work is that, in order to explore phenomenological distinctions between subcritical and oscillatory regimes, we introduce an electromagnetic field. As discussed in the end of Section 1.2, this causes a similar effect to that of gravitational perturbations outside symmetry, and it is thus of great interest to extend ideas of [4, 26] to the Einstein–Maxwell–scalar field.

Another reason for this is the prevalence of the spherically symmetric Einstein–Maxwell–scalar field in the study of gravitational collapse[4]. In a series of remarkable works [31, 32, 35], Christodoulou studied this system (without $\mathbf{F}$) as a toy model for understanding the Weak and Strong Cosmic Censorship conjectures (recall Conjecture 1.1.2). In these works, it is established that the maximal globally hyperbolic future development of *generic* one-ended initial data possesses a future-complete null infinity $\mathcal{I}^+$, and that the spacetime either disperses as in Figure 1.1, or possesses a black hole region bounded to the future by a spacelike boundary, at which the *area-radius* r tends to 0 and the Kretschmann scalar blows up, see Figure 1.2.

In the latter case, [31, 32] applies monotonicity arguments to establish the spacelike character of the singular $\{r = 0\}$ boundary, as well as the blow-up of the Hawking mass and Kretschmann scalar. In order to make a comparison to the BKL's Kasner-like ansatz, however, we shall require a more precise description of how such geometric quantities and the scalar field ϕ blow up.

Progress to this effect is achieved in [4, 3], where the authors derive *upper bounds* for ϕ and $r^2\partial\phi$. Subsequently, these upper bounds are used to show inverse polynomial upper bounds for

[4]We mention already the caveat that for such spherically symmetric spacetimes with nontrivial Maxwell field, there cannot be a regular center of symmetry Γ, and one is restricted to studying *two-ended black holes*. In order to study astrophysical one-ended collapse, one should instead study the spherically symmetric Einstein–Maxwell–*charged* scalar field model, see the start of Chapter 2.

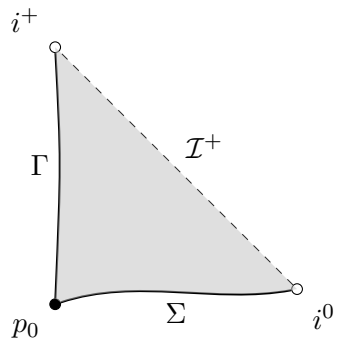


Figure 1.1: Penrose diagram representation of a dispersive solution to the Einstein-scalar field equations. This is a future causal geodesically complete spacetime with a complete future null infinity $\mathcal{I}^+$.

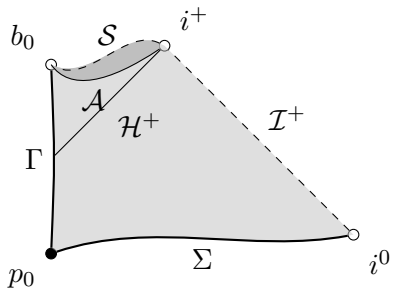


Figure 1.2: Penrose diagram representation of a gravitational collapse solution to the Einstein-scalar field equations. The solution possesses a complete future null infinity $\mathcal{I}^+$ as well as a black hole region bounded to the past by the *event horizon* $\mathcal{H}^+$. The black hole interior contains an *apparent horizon* $\mathcal{A}$, whose future is a *trapped region* $\mathcal{T}$ culminating at a spacelike boundary $\mathcal{S} = \{r = 0\}$ connecting i^+ with the first singularity at the center, denoted b_0.

other metric quantities and the Hawking mass m. Note that [4] produces such estimates in a local neighborhood of a small portion of the spacelike singularity, while [3] uses decay rates along the event horizon $\mathcal{H}^+$, originating from the heuristics of Price [116] and rigorously proved in [47], to prove improved bounds near future spacelike infinity i^+, showing that the singularity "approaches Schwarzschild" near i^+. See also [5] for related results for the Einstein–scalar field system with a positive cosmological constant $\Lambda > 0$.

In Chapter 3, we build upon the upper bounds [4], showing that in a regular double null gauge with null coordinates (u, v) (see Section 2.2), the quantities $r^2\partial_u\phi$ and $r^2\partial_v\phi$ can be extended as Hölder continuous functions all the way to the singular boundary $\{r = 0\}$. These asymptotics will in turn allow us to deduce exact rates for the polynomial blow-up of quantities such as the *null lapse* $\Omega^2(u, v)$, Hawking mass, and the Kretschmann scalar. For example, we identify a quantity $\Psi(u, v)$, bounded and Hölder continuous up to the $\{r = 0\}$, such that

$$\Omega^2(u, v) \cdot r^{1-\Psi^2(u,v)}(u, v) = C(u) + O(r). \tag{1.3.1}$$

Here, $C(u)$ should be thought of as a function 'at the $r = 0$ singularity'. We prove a similar, but *conditional*, result, for the Einstein–Maxwell–scalar field model in spherical symmetry.

This latter model has also received much attention in the mathematical community, see [41, 109, 110, 99]. These results, however, are largely concerned with the existence and properties of the Cauchy horizon in the interior of black holes. Indeed, the groundbreaking result of [109, 110] proves the C^2 formulation of the strong cosmic censorship conjecture in the context of the spherically symmetric Einstein-Maxwell-scalar field model. On the other hand, the objective of this present article is to study the behaviour of the scalar field and geometry when the singularity is no longer *null*, as in the case of the Cauchy horizon, but instead *spacelike*.

In what remains of this section, we present a rough version of our main theorems. Our theorems will apply on regions we call *strongly singular spacetimes*, which will be the maximal development of characteristic initial data in the trapped region, which terminate at an $\{r = 0\}$ singularity. See Figure 1.3, and Section 2.2 for details.

In the presence of a non-trivial electromagnetic field $\mathbf{F}$, we must make the following two additional assumptions on the strongly singular spacetime, where $R_0 > 0$ and $\alpha \in (0, 1)$ are fixed constants:

$$\inf_{(u,v)\in\mathcal{D}} \min\{-r\partial_v r(u,v), -r\partial_u r(u,v)\} \geq R_0, \tag{QTS}$$

$$\inf_{(u,v)\in\mathcal{D}} \min\left\{\left|r\frac{\partial_v\phi}{-\partial_v r}\right|(u,v), \left|r\frac{\partial_u\phi}{-\partial_u r}\right|(u,v)\right\} \geq \sqrt{1+\alpha}. \tag{SKE}$$

(QTS) and (SKE) are acronyms for *quantitatively trapped spheres* and *subcritical Kasner exponents* respectively. Note that (QTS), which holds by assumption when $\mathbf{F} = 0$, provides a mechanism to reach an $r = 0$ singularity in the first place, while the necessity of (SKE) comes from the discussion of Section 1.2.

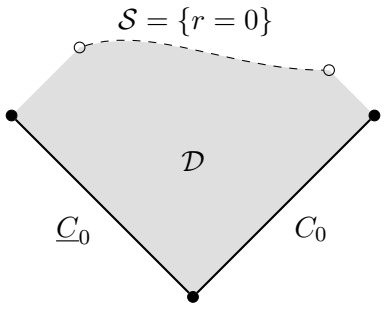


Figure 1.3: A strongly singular spacetime to which Theorem 1.3.1 applies. The region $\mathcal{D}$ is foliated by trapped 2-spheres, and possesses a future boundary $\mathcal{S}$ at which r extends continuously to 0. Examples of such strongly singular spacetimes exist both in the case $\mathbf{F} = 0$, and in the case that $\mathbf{F} \neq 0$, where one must additionally assume (QTS) and (SKE).

Theorem 1.3.1 below is a rough version of our main Theorems 3.1.1 and 3.1.2 in Chapter 3

Theorem 1.3.1 (Asymptotics – rough version)**.** *Consider solutions to the spherically symmetric Einstein-Maxwell-scalar field system* (Ein), (Scal), (Max) *on strongly singular spacetimes as in Figure 1.3. If either of the following holds:*

(A) The electromagnetic field vanishes i.e. $\mathbf{F}_{\mu\nu} = 0$*, or*
(B) The electromagnetic field is non-trivial, but we assume a priori that (QTS) *and* (SKE) *hold,*

then the dynamical variables $r(u,v)$*,* $\Omega^2(u,v)$ *and* $\phi(u,v)$ *obey the following asymptotics:*

(I) Asymptotics for r: $r^2(u,v)$ *is a* $C^{1,\beta}$ *function up to and including the singular boundary* $\mathcal{S}$*. Its first derivatives* $-r\partial_u r(u,v), -r\partial_v r(u,v)$ *are strictly positive* $C^{0,\beta}$ *functions up to* $\mathcal{S} = \{(u,v) : r(u,v) = 0\}$*, which is a* $C^{1,\beta}$ *curve in the* (u,v)*–plane.*

(II) Asymptotics for ϕ: *There exist functions* $\Psi(u,v)$ *and* $\Xi(u,v)$*, bounded and Hölder continuous up to and including the singular boundary* $\mathcal{S}$*, such that*[5]

$$\phi(u,v) = \Psi(u,v) \cdot \log\left(\frac{r_0}{r(u,v)}\right) + \Xi(u,v).$$

Moreover, in case (B) one always has $|\Psi| \geq 1$*.*

(III) Asymptotics for Ω^2: *There exists a function* $\mathfrak{M}(u,v)$*, with* $\log \mathfrak{M}$ *bounded and Hölder continuous up to and including the singular boundary* $\mathcal{S}$*, such that*

$$4\Omega^{-2}\partial_u r \partial_v r(u,v) = \mathfrak{M} \cdot \left(\frac{r}{r_0}\right)^{-(\Psi^2+1)}.$$

(IV) Estimates localized at a TIP[6]: *Let* $p \in \mathcal{S}$*. By (I)–(III),* $-r\partial_u r$*,* $-r\partial_v r$*,* Ψ*,* Ξ *and* $\mathfrak{M}$ *may all be defined at* p*, and we denote their values at* p *as* C_u*,* C_v*,* Ψ_∞*,* Ξ_∞ *and* $\mathfrak{M}_\infty$ *respectively.*

For such p*, consider its causal past* $J^-(p) = \{(u,v) \in \mathcal{D} : u \leq u_p, v \leq v_p\}$*. Then for* $(u,v) \in J^-(p)$*, one has, for some* $\alpha > 0$*:*

$$-r\partial_u r(u,v) = C_u + O(r^\alpha), \quad -r\partial_v r(u,v) = C_v + O(r^\alpha),$$

$$r^2\partial_u \phi(u,v) = C_u \cdot \Psi_\infty + O(r^\alpha), \quad r^2\partial_v \phi(u,v) = C_v \cdot \Psi_\infty + O(r^\alpha),$$

$$\phi(u,v) = \Psi_\infty \cdot \log\left(\frac{r_0}{r}\right) + \Xi_\infty + O(r^\alpha \log r^{-1}),$$

$$\Omega^2(u,v) = \frac{4C_u C_v}{r_0^2} \cdot \mathfrak{M}_\infty^{-1} \cdot \left(\frac{r}{r_0}\right)^{\Psi_\infty^2 - 1} + O(r^{\Psi_\infty^2 - 1 + \alpha} \log r^{-1}).$$

[5] r_0 here can be chosen to be any fixed constant

[6] TIP stands for *terminal indecomposable past*, see Chapter 6 of [78]. Strictly speaking, our use of the term TIP is an abuse of terminology as every $p \in \mathcal{S}$ actually represents a 2-sphere of TIPs.

(V) *Precise blow-up estimates:* *For $p \in \mathcal{S}$ and $J^-(p)$ as in (IV), the Hawking mass m, the Kretschmann scalar $\mathfrak{K} = \mathbf{Riem}[\mathbf{g}]_{\alpha\beta\gamma\delta}\mathbf{Riem}[\mathbf{g}]^{\alpha\beta\gamma\delta}$ and the scalar field blow up at the following rates:*

$$2m(u,v) = \mathfrak{M}_\infty r_0 \cdot \left(\frac{r_0}{r}\right)^{\Psi_\infty^2} + O(r^{-\Psi^2+\alpha}\log r^{-1}),$$

$$\mathfrak{K} = \mathfrak{M}_\infty^2 \cdot \left(\frac{r_0}{r}\right)^{2(\Psi_\infty^2+3)} \cdot \frac{4(3-2\Psi_\infty^2+2\Psi_\infty^4)}{r_0^4} + O(r^{-2\Psi_\infty^2-6+\alpha}\log r^{-1}),$$

$$\nabla_\alpha\phi\nabla^\alpha\phi = -\frac{\mathfrak{M}_\infty\Psi_\infty^2}{r_0^2} \cdot \left(\frac{r_0}{r}\right)^{\Psi_\infty^2+3} + O(r^{-\Psi_\infty^2-3+\alpha}\log r^{-1}).$$

We refer to Theorem 3.1.1 and Theorem 3.1.2 for the precise versions of Theorem 1.3.1 when $\mathbf{F} \equiv 0$ and $\mathbf{F} \neq 0$ respectively. We conclude with several brief comments about Theorem 1.3.1.

1. In light of (III), one may introduce the following (past-directed) time function:

$$\tau = \mathfrak{M}^{-1/2} \cdot \frac{2r_0}{3+\Psi^2} \cdot \left(\frac{r}{r_0}\right)^{\frac{3+\Psi^2}{3}}. \tag{1.3.2}$$

   In particular, τ is chosen such that $g(\nabla\tau,\nabla\tau) = -1 + O(r^\alpha \log r^{-1})$, and will become our candidate time function for an asymptotically Gaussian or asymptotically CMC description of our spacetime. See Corollary 3.1.3 for further details.

2. We further associate the following three p-dependent Kasner exponents to each $p \in \mathcal{S}$:

$$p_1(p) = \frac{\Psi_\infty^2(p)-1}{\Psi_\infty^2(p)+3}, \qquad p_2(p) = p_3(p) = \frac{2}{\Psi_\infty^2(p)+3}. \tag{1.3.3}$$

   Note that in case (B), the condition $|\Psi| \geq 1$ translates to $\min p_i(p) \geq 0$. This is expected, in the presence of a nontrivial electromagnetic field, due to the discussion at the end of Section 1.2.

3. Theorem 1.3.1 gives the leading order term of an asymptotic expansion for ϕ, Ω^2, and other related quantities in terms of r. It would be interesting to compute the next order terms in such expansions, as in the heuristic work [26], though we do not pursue that here.

1.4 A case study: Black holes with charged scalar hair

We now introduce the object of study in Chapter 4, namely a class of spherically symmetric and in fact *spatially homogeneous* solutions to the Einstein–Maxwell–*charged* scalar field system (to be introduced at the start of Chapter 2), which are based on the evolution of initial data posed on

two bifurcate null hypersurfaces $\mathcal{H}_L$ and $\mathcal{H}_R$ emulating the event horizons of a charged Reissner-Nordström black hole, the difference being that these two horizons carry non-decaying (in fact, constant) *charged scalar hair* of size ϵ. An intriguing consequence is that with "stiff" matter model, the resulting black hole interior will not have a Cauchy horizon but instead terminates at a spacelike singularity, which will have a precise Kasner-like structure.

While our result in Theorem 1.4.1 will hold[7] for any value of the *cosmological constant* $\Lambda \in \mathbb{R}$ introduced into the Einstein equation (see (Ein*) at the start of Chapter 2), it is perhaps most physically relevant in the $\Lambda < 0$ case, where a large class of asymptotically Anti-de-Sitter (AdS) *hairy black hole exteriors* has recently been constructed [155, 156] and their interior region corresponds to the spacetime we construct in our Theorem 1.4.1 (see Section 4.2.5). Note that though we only expect to be able to construct such an exterior region in the asymptotically AdS case, we nonetheless name our spacetimes *hairy black holes* for all choices of $\Lambda \in \mathbb{R}$.

Furthermore, for $\Lambda = 0$, our construction from Theorem 1.4.1 has bearings on the interior of (non-hairy) *asymptotically flat black holes* as well, as we explain in Section 4.2.1. To summarize: the domain of dependence property allows to consider the black hole interior region independently from the black hole exterior (see Figure 4.4); thus the repercussions of Theorem 1.4.1 extend significantly beyond asymptotically AdS hairy black holes.

To give a rough (but detailed) version of our findings regarding these hairy black holes, we first recall the interior region of the Reissner–Nordström-(dS/AdS) black hole, with metric $\mathbf{g}_{RN}$:

$$\mathbf{g}_{RN} = -\left(1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2}$$

with parameters $(M, \mathbf{e}, \Lambda)$. The hairy black hole constructed in the following Theorem 1.4.1 has initial data (1.4.1), (1.4.2) that are $O(\epsilon^2)$-perturbations of $\mathbf{g}_{RN}$, with scalar hair of initial size ϵ.

Theorem 1.4.1. *Fix the following characteristic initial data on bifurcate event horizons $\mathcal{H}_L \cup \mathcal{H}_R$:*

$$\phi \equiv \epsilon, \tag{1.4.1}$$

$$\mathbf{g} = \mathbf{g}_{RN} + O(\epsilon^2), \tag{1.4.2}$$

where $\mathbf{g}_{RN}$ is a sub-extremal Reissner–Nordström-(dS/AdS) metric with $\mathbf{e} \neq 0$.

Define $(\mathcal{M}, \mathbf{g}, \phi, \mathbf{F}, A_\mu)$ to be the maximal globally hyperbolic future development of this data with respect to the Einstein–Maxwell–charged scalar field system. Then $\mathcal{M} = \{(t, s, \omega) : t \in \mathbb{R}, s \in (-\infty, s_\infty, \omega \in \mathbb{S}^2\}$ and $\mathbf{g}$ is a spatially homogeneous, spherically symmetric metric, with $\mathbf{g}$, $\mathbf{F}$, and ϕ written as

$$\mathbf{g} = \Omega^2(s)(-ds^2 + dt^2) + r^2(s) d\sigma_{\mathbb{S}^2}, \quad \mathbf{F} = \frac{Q(s)}{r^2(s)} \Omega^2(s) ds \wedge dt, \quad \phi = \phi(s), \tag{1.4.3}$$

[7] We also note the existence of exponentially growing modes for charged scalar fields on Reissner–Nordström-de Sitter and Kerr–Newman–de Sitter obtained in [20], which might suggest compatibility with the existence of hairy black hole solutions in the $\Lambda > 0$ setting.

where $r(s)$ is a decreasing function of $s \in (-\infty, s_\infty)$ with $\lim_{s\to s_\infty} r(s) = 0$ with $s_\infty < +\infty$.

*Let $\eta > 0$ be sufficiently small. Then there exists $\epsilon_0(M, \mathbf{e}, \Lambda, m^2, q_0, \eta) > 0$ and a set $E_\eta \subset (-\epsilon_0, \epsilon_0)\setminus\{0\}$, satisfying $\frac{|(-\delta,\delta)\setminus E_\eta|}{2\delta} = O(\eta)$ for all $0 < \delta \leq \epsilon_0$, such that for all $\epsilon \in E_\eta$, the spacetime $(\mathcal{M}, g)$ terminates at a **spacelike singularity** $\mathcal{S} = \{r = 0\}$, asymptotically described by a Kasner metric of positive exponents $(p_1, p_2, p_3) \in (0, 1)^3$. The spacetime $(\mathcal{M}, \mathbf{g})$ may be partitioned into several regions, as illustrated by the Penrose diagram of Figure 1.4, and has the following features:*

1. *Almost formation of a Cauchy horizon. In the early regions, $(\mathbf{g}, \mathbf{F}, \phi)$ are uniformly close to those of the Reissner–Nordström-(dS/AdS) background, and the scalar field is approximated by a linearly-oscillating profile:*

$$\phi(s) = \epsilon \cdot \mathrm{Re}\left(2B(M, \mathbf{e}, \Lambda, m^2, q_0)e^{i\omega_{RN}(q_0,M,\mathbf{e},\Lambda)s}\right) + O(\epsilon^2), \tag{1.4.4}$$

   where $B(M, \mathbf{e}, \Lambda, m^2, q_0) \in \mathbb{C} \setminus \{0\}$ is a fixed scattering parameter, and the frequency is given by $\omega_{RN}(M, \mathbf{e}, \Lambda, q_0) = |q_0\mathbf{e}| \cdot (\frac{1}{r_-} - \frac{1}{r_+}) > 0$. Here $r_\pm(M, \mathbf{e}, \Lambda) > 0$ are respectively the radii of the event and Cauchy horizons of the background Reissner–Nordström-(dS/AdS) metric.

2. *Collapsed oscillations. The scalar field ϕ experiences **growing** Bessel-like oscillations while r shrinks from r_- to 0. For some $s_{\mathcal{O}} = O(\epsilon^{-2})$ representing the future boundary of this collapsed oscillations region, there exists $C(M, \mathbf{e}, \Lambda, m^2, q_0) \neq 0$ and another frequency $\omega_0(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$, such that if one defines*

$$\alpha(\epsilon) = C \cdot \sin\big(\omega_0 \cdot \epsilon^{-2} + O(\log(\epsilon^{-1}))\big) + O(\epsilon^2 \log(\epsilon^{-1})) \quad \text{as } |\epsilon| \to 0, \tag{1.4.5}$$

   so that, at $s = s_{\mathcal{O}}$, one has

$$-\frac{r\dot{\phi}}{\dot{r}}(s) = \alpha(\epsilon) \text{ and } r(s) \approx \epsilon. \tag{1.4.6}$$

   Note that this quantity is of order 1 despite the initial data for ϕ being $O(\epsilon)$.

3. *Formation of the first Kasner regime. A Kasner regime starts developing with Kasner exponents*

$$p_1 = P(\alpha) = \frac{\alpha^2 - 1}{3 + \alpha^2}, \quad p_2 = p_3 = \frac{2}{3 + \alpha^2}, \tag{1.4.7}$$

   where p_1 is the Kasner exponent in the t-direction and $p_2 = p_3$ are the exponents in the $\mathbb{S}^2$ directions in (1.4.3).

4. *The final Kasner regime. For any fixed $\sigma \in (0, 1)$, we introduce the following disjoint subsets of E_η.*

   - *(Non-bounce case) $\epsilon \in E'^{Nbo}_{\eta,\sigma} \subset E_\eta$ if $|\alpha(\epsilon)| \geq 1 + \sigma$ (i.e. $p_1 > 0$). When $\epsilon \in E'^{Nbo}_{\eta,\sigma}$, the first Kasner regime is also the final Kasner regime and continues all the way to $\mathcal{S} = \{r = 0\}$ in a monotonic fashion.*

- *(Kasner bounce case)* $\epsilon \in E^{'\,bo}_{\eta,\sigma} \subset E_\eta$ *if* $\eta \leq |\alpha(\epsilon)| \leq 1-\sigma$. *We have* $|E^{'\,bo}_{\eta,\sigma}| > 0$ *(in particular* $E^{'\,bo}_{\eta,\sigma} \neq \emptyset$*). When* $\epsilon \in E^{'\,bo}_{\eta,\sigma}$*, the above Kasner regime eventually transitions towards a (different) final Kasner regime with positive Kasner exponents*

$$p_1 = P\left(\frac{1}{\alpha}\right) = \frac{1-\alpha^2}{1+3\alpha^2}, \quad p_2 = p_3 = \frac{2\alpha^2}{1+3\alpha^2}. \tag{1.4.8}$$

The final Kasner regime then continues all the way to $\mathcal{S} = \{r = 0\}$ *in a monotonic fashion.*

5. *Charge retention of the Kasner singularity. The charge* $Q(s)$ *admits a limit* $Q_\infty \neq 0$ *as* $r \to 0$. *More precisely*

$$Q_\infty = (1 - \nu(M, \mathbf{e}, \Lambda)) \cdot \mathbf{e} + O(\epsilon) \text{ with } \nu(M, \mathbf{e}, \Lambda) \in \left(0, \frac{1}{2}\right), \tag{1.4.9}$$

$$\text{and } \nu(M, \mathbf{e}, \Lambda) \begin{cases} = \frac{1}{4} & \text{if } \Lambda = 0, \\ \in (0, \frac{1}{4}) & \text{if } \Lambda > 0, \\ \in (\frac{1}{4}, \frac{1}{2}) & \text{if } \Lambda < 0. \end{cases} \tag{1.4.10}$$

Remark. While Theorem 1.4.1 is in the setting of spherically symmetric spatially homogeneous spacetimes (also known as Kantowski–Sachs cosmologies), it also applies to spatially homogeneous spacetimes with other surface symmetry (see Section 2.1), for instance plane symmetry. While the Einstein–Maxwell–charged scalar field system takes a slightly different form in the plane symmetric setting, our analysis still carries through, and the conclusions of Theorem 1.4.1 hold mutatis mutandis.

Remark. The constants $\eta, \sigma > 0$, presumed small throughout this article, are present to ensure that $\alpha(\epsilon)$ is bounded away from $\{0, 1\}$. Note that we do not obtain a statement for every sufficiently small ϵ, but only for a set E_η, which features an $O(\eta)$ loss in the sense given above. However, the methods used to prove Theorem 1.4.1 allow for $|1-\alpha|\epsilon^{-0.01} > 1$ or $|\alpha|\epsilon^{-0.01} > 1$, with only minor adjustments. Understanding what happens when say $|1-\alpha| \ll \epsilon^2$ or $|\alpha| \ll \epsilon^2$, is an interesting open problem (see Section 4.2.4 and 4.2.5).

We will now discuss the relations between Theorem 1.4.1 and the study of both the BKL proposal and astrophysical gravitational collapse. One of the main features is the existence of a *Kasner bounce*: this in fact was the first example of a spacetime, dynamically formed from physically interesting initial data, in which **an unstable Kasner regime forms dynamically, and then disappears under the Kasner bounce**.

Differences and connections with non-hairy black holes The hairy black holes of Theorem 1.4.1 and the analogous ones from the case of uncharged scalar hair from [148] have a spacelike singularity $\mathcal{S} = \{r = 0\}$ and no Cauchy horizon and, in that, differ globally from the non-hairy black

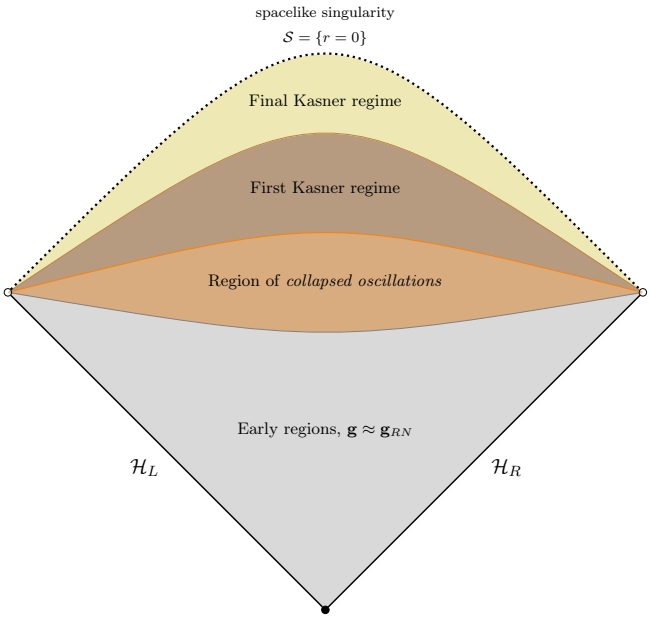


Figure 1.4: Penrose diagram of the hairy black hole interior from Theorem 1.4.1. If $|\alpha| \geq 1+\sigma > 1$, then the first Kasner regime matches the final Kasner regime and continues to $\{r = 0\}$. If $0 < \eta \leq |\alpha| \leq 1 - \sigma < 1$, then there is a Kasner bounce between the first and final Kasner regimes. A more detailed breakdown is given in Figure 4.1.

holes arising in astrophysical gravitational collapse. However, the domain of dependence property (in two-ended black holes) shows that the spacelike singularity $\mathcal{S}$ in non-hairy black holes can arise from initial data that is locally similar to the data of Theorem 1.4.1, see Figure 4.4. Therefore, our new theorem on hairy black holes may also dictate the qualitative behavior of the spacelike singularity $\mathcal{S}$ inside the such *non-hairy black holes*. Lastly, our proof opens the door to studying spacelike singularity beyond the spatially homogeneous case in subsequent works.

Comparison with hairy black hole interiors for other matter models The Kasner exponents' dependency on $\sin\left(\omega_0 \cdot \epsilon^{-2} + O(\log(\epsilon^{-1}))\right)$ found in Theorem 1.4.1 gives rise to fluctuations near $\epsilon = 0$: we term this phenomenon the *fluctuating collapse*. The fluctuating collapse and Kasner bounces from Theorem 1.4.1 contrast with the *violent nonlinear collapse* of the *uncharged* hairy black holes found in [148]. Recall from [148] that in the uncharged case, *there is no Kasner bounce*, and the final Kasner exponents and curvature are of the form

$$(p_1, p_2, p_3) = (1 - O(\epsilon^2),\ O(\epsilon^2),\ O(\epsilon^2)),\ K(r) \approx r^{-O(\epsilon^{-2})}.$$

The name "violent collapse" in the uncharged case comes from the $O(\epsilon^{-2})$ power in the blow-up rate of the curvature $K(r)$, which becomes *more singular* as $\epsilon \to 0$. This is an example of a singular limit, since $\epsilon = 0$ corresponds to Reissner–Nordström, while we can make sense of the $\epsilon \to 0$ (weak)

limit in the appropriate region using

$$\lim_{\epsilon\to 0}(p_1,p_2,p_3)(\epsilon) = (1,0,0), \text{ which corresponds to a subset of Minkowski.}$$

In contrast, in the charged case studied in our Theorem 1.4.1, there is *not even a weak limit* as $\epsilon \to 0$, since $(p_1,p_2,p_3)(\epsilon)$ depends on $\sin\big(\omega_0 \cdot \epsilon^{-2} + O(\log(\epsilon^{-1}))\big)$ as discussed above.

Lastly, we mention that previous numerics in the physics literature also predicted the presence of Kasner bounces in the interior of Einstein–Yang-Mills hairy black holes [55, 25, 150]. It is a very interesting open problem to compare these Kasner bounces with the one obtained in the setting of Theorem 1.4.1.

Collapsed oscillations and charge retention One new phenomenon driving the dynamics of Theorem 1.4.1 are what we term *collapsed oscillations*. Even at the linear level, a (spatially homogeneous) charged scalar field has (infinite) linear oscillations of the form (1.4.4) near the Cauchy horizon of the Reissner–Nordström interior spacetime [92]. In the nonlinear setting, such linear oscillations appear in some "early regions" in the dynamics. Subsequently, these oscillations start interacting nonlinearly with the collapse process (as r gets closer to 0), leading to a Bessel-function type behavior in terms of $z = \frac{r}{\epsilon}$. Because $\frac{r}{\epsilon}$ is a *decreasing* function of time, this Bessel behavior leads to *growth of the scalar field* which transitions from amplitude ϵ to amplitude $\alpha(\epsilon) = O(1)$. We will elaborate on collapsed oscillations and their mechanism in Section 4.2.3.

Another interesting question is whether spacelike singularities can retain charge/angular momentum. This issue is puzzling, as the only explicit black hole solution with a spacelike singularity is the Schwarzschild interior, which is uncharged and non-rotating. In point 5 of Theorem 1.4.1, we exhibit a *mechanism of discharge* of the black hole, passing from $\mathbf{e}$ at the event horizon $\mathcal{H} = \mathcal{H}_L \cup \mathcal{H}_R$ to $(1-\delta)\mathbf{e}$ at the spacelike singularity $\mathcal{S}$, where $\delta \in (0, \frac{1}{2})$. Note that the *discharge is not complete* and the spacelike singularity retains a non-zero final charge $(1-\delta)\mathbf{e}$. It is remarkable that, for $\Lambda = 0$, the discharge ratio $\delta = \frac{1}{4}$ is independent of the black hole parameters. To the best of our knowledge, the spacetime of Theorem 1.4.1 is the first example of a spacelike singularity retaining[8] charge. The main mechanism of discharge and charge retention occurs at the same time as the collapsed oscillations.

Kasner bounces In Theorem 1.4.1, we show that if

$$|\alpha(\epsilon)| = C \cdot |\sin\big(\omega_0 \epsilon^{-2} + O(\log(\epsilon^{-1}))\big)| + O(\epsilon^2 \log(\epsilon^{-1}))$$

has $\alpha(\epsilon) < 1$, then there will be a Kasner bounce. For $(M, \mathbf{e}, \Lambda, m^2, q_0)$ such that $C(M, \mathbf{e}, \Lambda, m^2, q_0) > 1$, the bounce condition $C \cdot |\sin\big(\omega_0 \epsilon^{-2} + O(\log(\epsilon^{-1}))\big)| < 1$ holds for all $\epsilon \in E^{bo}_\eta \cap (0, \epsilon_0)$, where E^{bo}_η has non-zero measure: in fact $|E^{bo}_\eta \cap (0,\epsilon_0)| \approx \frac{\epsilon_0}{C}$ if $C \gg 1$. On

[8] The hairy black holes with uncharged matter from [148] also had non-zero charge, but there $\mathbf{F}$ is not dynamical.

the other hand, if $C(M, \mathbf{e}, \Lambda, m^2, q_0) < 1$, then the bounce condition always holds. These Kasner bounces have the following features:

1. For some range of proper time $\{e^{-b^2(\alpha)\cdot\epsilon^{-2}} \ll \tau < \epsilon^{q^2(\alpha)}\}$ with $q(\alpha) > 0,\ b(\alpha) > 0$, the metric is uniformly close to a $(p_1, \frac{1-p_1}{2}, \frac{1-p_1}{2})$ Kasner-(like) metric, where $p_1(\epsilon) \approx \frac{\alpha^2-1}{3+\alpha^2} < 0$.

2. In the smallest values of proper time from the singularity $\{0 < \tau \ll e^{-b^2(\alpha)\cdot\epsilon^{-2}}\}$, the metric is uniformly close to, and indeed converges as $\tau \to 0$ towards, a $(\acute{p_1}, \frac{1-\acute{p_1}}{2}, \frac{1-\acute{p_1}}{2})$ Kasner-(like) metric, where $\acute{p_1}(\epsilon) \approx \frac{1-\alpha^2}{1+3\alpha^2} > 0$.

Remark. When writing (imprecisely) $e^{-b^2(\alpha)\cdot\epsilon^{-2}} \ll \tau$, we mean $\epsilon^{-N_1}\cdot e^{-b^2(\alpha)\cdot\epsilon^{-2}} < \tau$ for some $N_1 > 0$, and similarly $\tau \ll e^{-b^2(\alpha)\cdot\epsilon^{-2}}$ means $\tau < \epsilon^{N_2}\cdot e^{-b^2(\alpha)\cdot\epsilon^{-2}}$ for $N_2 > 0$, so that the transition region in between the two Kasner regimes has size $O(\log(\epsilon^{-1}))$ in terms of $\log(\tau^{-1})$, while $\log(\tau^{-1}) \approx \epsilon^{-2}$ at the times where the bounce is occurring. We interpret this to mean that the bounce occurs very quickly in terms of proper time τ.

Remark. The dichotomy of whether $|\alpha(\epsilon)| < 1$ or $|\alpha(\epsilon) > 1$ and these Kasner bounces is exactly consistent with Theorem 1.3.1 in Section 1.3. Indeed, the quantity $|\alpha(\epsilon)|$ appearing in point 3 of Theorem 1.4.1 is exactly the quantities appearing in the $\max\{\cdots\}$ in the "subcriticality condition" (SKE). That is, if $|\alpha(\epsilon)| > 1$ then one can apply Theorem 1.3.1 – which should generalize to the charged scalar field model considered – and yields a spacelike singularity with Kasner exponents of the type (1.4.7), exactly as in (1.3.3).

On the other hand, if $|\alpha(\epsilon)| < 1$ then the aforementioned charged retention means Theorem 1.3.1 does not apply; instead there is an instability due to Kasner bounces. This is the same BKL bounce that will be explored outside of the inhomogeneous case in Theorem 1.5.5 and Theorem 1.5.6 of Section 1.5. (In fact the homogeneous Kasner bounces studied in Theorem 1.4.1 was a key ingredient and motivation for these latter results.)

Holographic superconductors in the AdS-CFT correspondence The numerical study [75] already predicted the scenario of Theorem 1.4.1 and the presence of Kasner bounces in this context. Beyond the setting of Theorem 1.4.1 (which features zero, or one Kasner bounce depending on ϵ), [75] also discusses the possibility of having *two* (or more) Kasner bounces. The original motivation of [75] is related to a body of work on the physical significance of the hairy black hole (including its asymptotically AdS exterior region) of Theorem 1.4.1, claimed to be the model for a *holographic superconductor* in the context of the AdS-CFT theory.

The proof of Theorem 1.4.1 will be given in Chapter 4. More precisely, after some setup in Chapter 2, particularly Section 2.4, we state and prove the precise version of our results for these hairy black hole interiors, namely Theorem 4.1.1 and Theorem 4.1.2. Before moving to the proof in Chapter 2, we also briefly elaborate on the discussion points above.

1.5 BKL bounces outside homogeneity

Here, we give further background to Chapters 5 and 6 of this dissertation, concerning *BKL bounces outside homogeneity*. While the Kasner-like asymptotics of Chapter 3 only concern the subcritical regime and subcritical phenomena, in these chapters we explore a model problem which begins to provide an insight into the more nonlinear phenomena predicted by BKL, particularly the *BKL bounces*. That is, we begin to explore to Question 2 and Question 3 in problems where nonlinear BKL bounces are present.

As signalled by Question 2 and Question 3, the key ingredients to understanding BKL's proposal of [16] mathematically is (i) to show that, close to the singularity, ∂_x-derivatives are negligible in comparison to ∂_t-derivatives and the Einstein equations may be treated as ODEs and (ii) to show that these ODEs exhibit behaviour represented by a cascade of BKL bounces.

Our results of Chapter 5 and Chapter 6, which concern the Gowdy symmetric Einstein vacuum equations and the surface symmetric Einstein–Maxwell–scalar field equations respectively, will not quite exhibit an infinite cascade of bounces, since the symmetry reductions mean that our spacetimes ultimately settle to a subcritical regime. However, at an *intermediate time scale* preceding the singularity, one can observe one, or potentially some other finite number of, bounces.

To elaborate on this, we begin to introduce the symmetry classes we consider. A $\mathbb{T}^3$ *Gowdy symmetric* spacetime has topology $\mathbb{R}\times\mathbb{T}^3$ and has two spacelike Killing vector fields, whose orbits together span copies of $\mathbb{T}^2$ and whose associated *twist constants* vanich. See the original paper of Gowdy [67], or [36, 124]. The Einstein vacuum equations in Gowdy symmetry will be given in Section 2.6, but we highlight that the two main dynamical equations are:

$$(t\partial_t)^2P-(t\partial_\theta)^2P=e^{2P}(t\partial_tQ)^2-e^{2P}(t\partial_\theta Q)^2, \tag{1.5.1}$$

$$(t\partial_t)^2Q-(t\partial_\theta)^2Q=-2(t\partial_tP)(t\partial_tQ)+2(t\partial_\theta P)(t\partial_\theta Q), \tag{1.5.2}$$

where $P,Q:(0,+\infty)\times\mathbb{S}^1\to\mathbb{R}$ depend on a time variable t and spatial variable θ.

Solutions of the system (1.5.1)–(1.5.2) arising from regular initial data always persist remain regular in the entire interval $t\in(0,+\infty)$. As a result, $\mathbb{T}^3$ Gowdy symmetric spacetimes are a popular model for studying the Strong Cosmic Censorship Conjecture 1.1.3, with the strategy often to show geodesic completeness in the $t\to+\infty$ direction[9] and singularity formation in the $t\to 0$ direction. For the latter part – which is more relevant to our thesis – the essential step is to understand asymptotics for P as $t\to 0$. We state a remarkable result of Ringström, which is a key ingredient in his proof of Strong Cosmic Censorship in [123].

Theorem 1.5.1. *For solutions to the system* (1.5.1)–(1.5.2) *arising from initial data which is open and dense in a suitable topology, there exists a finite set* $S\subset\mathbb{S}^1$ *and a function* $V_\infty(\theta)$ *which is*

[9]Note that geodesic completeness is not necessary to show future inextendibility, see [44].

regular and obeys $0 < V(\theta) < 1$ *for all* $\theta \in \mathbb{S}^1 \setminus S$ *such that*

$$-t\partial_t P(t,\theta) \to V_\infty(\theta) \qquad \text{as } t \to 0.$$

Elements of the finite set S are referred to as *spikes* and we defer further discussion to Chapter 5. One highly nontrivial aspect is that for such generic solutions, one has the restriction $0 < V(\theta) < 1$, which notably differs from the results in *polarized Gowdy* – where $Q \equiv 0$ – due to Chrusciel–Isenberg–Moncrief [37].

However, understanding Strong Cosmic Censorship in full generality and the issue of spikes is not our main focus. Our starting point is instead the following two interrelated questions.

> Can one characterize a wide class of initial Cauchy data at $t = t_0 > 0$, including data which is not at first glance compatible with Ringström's asymptotics (e.g. data such that $0 < -t\partial_t P(t_0,\theta) < 1$ does not hold everywhere), such that the corresponding spacetime reaches a $t = 0$ singularity for which one has the asymptotics $-t\partial_t P(t,\theta) \to V_\infty(\theta)$ with quantitative estimates depending on initial data[10]?
>
> Moreover, for all such data can one fully understand the *intermediate dynamics* of the spacetime between the initial data at $t = t_0$ and the asymptotics at $t = 0$?

In the results of Chapter 5, we prove the existence of such a class of initial data (Theorem 1.5.2) together with a full quantitative understanding of the intermediate dynamics (Theorem 1.5.3), thus answering the above questions in the affirmative. Moreover this class is open in the C^∞ topology on initial data, thus our main results will represent a qualitative stability result in the sense that in the regime considered, certain aspects such as curvature blow-up, are stable to perturbation. We also uncover the interesting Corollary 5.1.3, interpreted as an *instability result* for non-polarized perturbations of polarized Gowdy spacetimes. See Section 5.1 for details.

We roughly state our two main theorems. The first, Theorem 1.5.2 is a *stability theorem*, and characterizes a subset in the moduli space of initial data at $t = t_0$ such that the corresponding solution is well controlled in both L^2 and L^∞ for $0 < t \leq t_0$. The second, Theorem 1.5.3, referred to as the *bounce theorem*, characterizes the (nonlinear) dynamics arising from initial data as above.

To precisely define this subset of initial data, we introduce two real-valued parameters $0 < \gamma < 1$ and $\zeta > 0$. Then the initial data is chosen to satisfy three conditions: (1) *Weak subcriticality*: The data at $t = t_0$ satisfies certain L^∞ bounds depending on γ, (2) *Energy boundedness*: For some $N = N(\gamma) \in \mathbb{N}$, an Nth order L^2 norm of initial data is bounded by ζ, and (3) *Closeness to singularity*: The time t_0 at which initial data is prescribed satisfies $0 < t_0 \leq t_*$ for some t_* depending on γ and ζ. We state our results roughly as follows.

[10] This differs from the approach in [121, 123], where the strategy is more akin to showing that the solutions not achieving the appropriate asymptotics are *non-generic* in some sense; in particular there is little use of the relationship between the asymptotic information and initial data at some $t_0 > 0$.

Theorem 1.5.2 (Stability, rough version)**.** *For $0 < \gamma < 1$ and $\zeta > 0$, consider initial data obeying the three conditions above. Then the corresponding solution (P, Q) to the Gowdy symmetric system* (1.5.1)–(1.5.2) *is such that for all $t \in (0, t_0)$,*

$$-t\partial_t P(t,\theta) \geq \gamma, \quad (1 + t\partial_t P(t,\theta))^2 + (e^P t\partial_\theta Q(t,\theta))^2 \geq (1-\gamma)^2.$$

Furthermore, there exist functions $V, q : \mathbb{S}^1 \to \mathbb{R}$, with V bounded and q continuous, such that $-t\partial_t P(t,\theta) \to V(\theta)$ pointwise and $Q(t,\theta) \to q(\theta)$ uniformly as $t \to 0$. If one further assumes $\gamma > \frac{1}{3}$, then we have that q is C^1 and the convergence $Q(t,\theta) \to q(\theta)$ holds in the C^1 topology.

Our second main theorem concerns the dynamical behaviour of certain quantities along causal curves. To fix notation, let $\gamma : I \to (0, +\infty) \times \mathbb{S}^1$ be any *causal curve* parameterised by its t-coordinate. Then define:

$$\mathcal{P}_\gamma(t) \coloneqq -t\partial_t P(\gamma(t)), \qquad \mathcal{Q}_\gamma(t) \coloneqq e^P t\partial_\theta Q(\gamma(t)).$$

It is key that our theorem holds uniformly in the choice of causal curve γ, meaning the ODEs for $\mathcal{P}_\gamma$ and $\mathcal{Q}_\gamma$ being independent for different choices of γ with distinct endpoints, at least up to error terms which reflect *AVTD behaviour*. This reflects explicitly that our result is *spatially inhomogeneous*.

Theorem 1.5.3 (Bounces, rough version)**.** *Let (P, Q) be a solution to the Gowdy symmetric system* (1.5.1)–(1.5.2) *arising from initial data as in Theorem 1.5.2. Then for $\gamma(t)$ a timelike curve as above, the quantities $\mathcal{P}_\gamma(t)$ and $\mathcal{Q}_\gamma(t)$ obey the ODEs:*

$$t\frac{d}{dt}\mathcal{P}_\gamma = \mathcal{Q}_\gamma^2 + \mathcal{E}_\mathcal{P}, \qquad t\frac{d}{dt}\mathcal{Q}_\gamma = (1-\mathcal{P})\mathcal{Q}_\gamma + \mathcal{E}_\mathcal{Q},$$

where the error terms vanish quickly as $t \to 0$.

Furthermore, $\mathcal{Q}_\gamma(t)$ converges to 0 as $t \to 0$, while there exists some $\mathcal{P}_{\gamma,\infty}$ such that $\mathcal{P}_\gamma(t) \to \mathcal{P}_{\gamma,\infty}$ as $t \to 0$. In fact, for V as in Theorem 1.5.2, $\mathcal{P}_{\gamma,\infty} = V(\theta_0)$ where $\theta_0 = \lim_{t\to 0}\theta(t)$ is the θ-coordinate of the past endpoint of γ. Finally, we estimate $\mathcal{P}_{\gamma,\infty}$ depending on the value of $\partial_\theta q(\theta_0) = \lim_{t\to 0}\partial_\theta Q(\gamma(t))$:

(i) If either the limit $\partial_\theta q(\theta_0)$ does not exist or $\partial_\theta q(\theta_0)$ exists and is nonzero, then necessarily $V(\theta_0) = \mathcal{P}_{\gamma,\infty} \leq 1$ and moreover in the case that $\mathcal{Q}_\gamma(t_0)$ is small,

$$\mathcal{P}_{\gamma,\infty} \approx \min\{\mathcal{P}_\gamma(t_0), 2 - \mathcal{P}_\gamma(t_0)\} + O(\mathcal{Q}_\gamma(t_0)).$$

(ii) Let $\gamma > \frac{1}{2}$. By Theorem 1.5.2, $\partial_\theta q(\theta_0)$ exists. If θ_0 is such that $\partial_\theta q(\theta_0) = 0$, then

$$\mathcal{P}_{\gamma,\infty} \approx \mathcal{P}_\gamma(t_0) + O(\mathcal{Q}_\gamma(t_0)).$$

We briefly the three conditions required in our results. Our first condition, weak subcriticality, is chosen such that we include spacetimes where the nonlinear bounces above can occur, but such that these bounces are not too wild enough to make quantities such as $\mathcal{P}_\gamma$ and $\mathcal{Q}_\gamma$ uncontrollable. The second condition of energy boundedness is natural as our proof relies on L^2 energy estimates at top order. We will, however, allow the initial energy bound, $\zeta > 0$, to be large, meaning we allow large spatial variation. In this case, our final condition, closeness to singularity, is required to prevent this spatial variation from playing a major role in the dynamics. Note that t_* and ζ play a similar role to that of ζ_1 and ζ_0 in [69, Theorem 12]; as in [69], one may take $t_* = 1$ so long as ζ is taken sufficiently small, but the result is equally, if not more, interesting in the case where ζ is large.

We further make a connection between the bounce result Theorem 1.5.3 and the bounces studied by BKL. To make this precise, one must change coordinates from the t variable used in (1.5.1)–(1.5.2) and the "proper time" foliation used in (1.2.2). It turns out that after performing this change of coordinates, one can make a formal correspondence between the asymptotic quantity $V(\theta)$ and the Kasner exponents, namely (5.1.6). Furthermore, the transition from $V(\theta) = \mathcal{P}_\gamma(t_0)$ to $\acute{V}(\theta) = 2 - \mathcal{P}_\gamma(t_0)$, as suggested in Theorem 1.5.3(i) when $\mathcal{P}_\gamma(t_0) > 1$, then formally corresponds exactly to the Kasner map (1.2.8). This concludes our initial discussion of our results in Gowdy symmetry.

We also briefly discuss our related result for the surface symmetric Einstein–Maxwell–scalar field system, to be proved in Chapter 6. This model is, in some sense, a mild generalization of the spherically symmetric model in Section 1.3, with the surfaces of symmetry now being general 2-surfaces of constant curvature. We once again have to introduce matter to break the rigidity from Birkhoff's theorem, and we choose a scalar field and electromagnetic field for reasons explained in Section 1.2.

For the results of Chapter 6 we do not, however, use the double null gauge as in Chapter 3, and instead use another gauge for surface symmetric gauge initially due to Rein [117]. See Section 2.5 for details. We focus on two of the equations coming from the Einstein–Maxwell–scalar field system in this gauge, namely

$$r\partial_r\mu = \frac{1}{2}\left((r\partial_r\phi)^2 + r^2e^{2(\mu-\lambda)}(\partial_x\phi)^2 + 1 - \frac{Q^2}{r^2}e^{2\mu}\right), \tag{1.5.3}$$

$$(r\partial_r)^2\phi = r^2e^{2(\mu-\lambda)}\partial_x^2\phi + r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x\phi - \frac{Q^2}{r^2}e^{2\mu}\, r\partial_r\phi, \tag{1.5.4}$$

where ϕ, μ, λ depend on a time variable r and spatial variable x, while $Q \in \mathbb{R}$ is a constant, which vanishes if and only if $\mathbf{F} = 0$. For a starting point of our discussion, in analogy to Ringström's result in Gowdy, we shall roughly translate / generalize aspects of our Kasner-like description in Theorem 1.3.1 to surface symmetric spacetimes in this alternative gauge, as follows.

Theorem 1.5.4. *For solutions to the Einstein–Maxwell scalar field system in surface symmetry in*

a region $(r, x) \in (0, R) \times \mathbb{S}^1$ *which moreover either has* $Q = 0$ *or satisfies a version of the conditions* (QTS) *and* (SKE) *in this alternative gauge, then there exists* $\Psi_\infty : \mathbb{S}^1 \to \mathbb{R}$ *such that*

$$r\partial_r\phi(r, x) \to \Psi_\infty(x) \qquad \text{as } r \to 0.$$

Moreover, when $Q \neq 0$ *one always has* $|\Psi| \geq 1$.

Unlike for Ringström's result in Gowdy symmetric spacetimes, this is conditional both on having a solution which reaches $r = 0$ and the two conditions (QTS) and (SKE). This is necessary, due to there being examples e.g. the Reissner-Nordström family, where $r = 0$ is not reached. Another difference from the Gowdy case is that there are no *spike points* where Ψ_∞ is discontinuous; we note, however, that such spikes reappear if we instead consider the Einstein–Maxwell–*charged* scalar field system. (See [99] for a description of this model.)

But in any case, the above theorem suggests we can ask similar questions in the Gowdy case. That is, one asks the following.

> Can one characterise a wide class of initial Cauchy data at $r = r_0$, including data which are not at first glance compatible with the above theorem (e.g. data violating the (QTS) condition in the sense that $Q \neq 0$ but $0 < r\partial_r\phi(r_0, x) < 1$), so that the corresponding solutions of the Einstein–Maxwell–scalar field system feature exist all the way to $\{r = 0\}$, and for which one still has the asymptotics $r\partial_r\phi(r, x) \to \Psi_\infty(x)$ with quantitative estimates?
>
> Further, for all data in this class can one fully understand the *intermediate dynamics* of the spacetime between the data at $r = r_0$ and the singularity at $r = 0$?

In Chapter 6, we answer these questions in a similar manner to the results in Gowdy symmetry. That, is, we prove a *stability / global existence theorem* in Theorem 1.5.5 and a *bounce theorem* in Theorem 1.5.6. As in the Gowdy case, we introduce two parameters, now $\eta > 1$ and $\zeta > 0$, from which we similarly introduce three conditions: (1) *Weak subcriticality*: Data satisfies L^∞ boundes depending on η, (2) *Energy boundedness*: For $N = N(\eta) \in \mathbb{N}$, an Nth order L^2 norm of data is bounded by ζ, and (3) *Closeness to singularity*: The time r_0 at which initial data is prescribed obeys $0 < r_0 \leq r_* = r_*(\eta, \zeta)$. Then we have the following.

Theorem 1.5.5 (Global existence, rough version)**.** *For some* $\eta > 2$ *and* $\zeta > 0$*, consider initial data for the surface symmetric Einstein–Maxwell–scalar field system at* $r_0 > 0$ *satisfying the three conditions (1)-(3). Then the corresponding solution is such that the maximal interval of existence extends all the way to* $r = 0$*, with quantitative* L^∞ *estimates.*

Our second main theorem concerns the dynamical behaviour of particular quantities along causal curves. To fix notation, let $\gamma : I \to \mathcal{Q} = I \times \mathbb{S}^1$ be any *causal curve* parameterised by its r-coordinate, and define:

$$\mathcal{P}_\gamma(r) \coloneqq r\partial_r\phi(\gamma(r)), \qquad \mathcal{Q}_\gamma(r) \coloneqq \frac{Q^2}{r^2}e^{2\mu}(\gamma(r)).$$

Then the corresponding bounce theorem is as follows. (See Section 6.1 for the interpretation regarding why this corresponds to the bounces predicted by BKL.)

Theorem 1.5.6 (Bounces, rough version)**.** *Consider a solution to the surface symmetric Einstein–Maxwell–scalar field system arising from initial data as in Theorem 1.5.5. Then for $\gamma(r)$ any causal curve as above, the quantities $\mathcal{P}_\gamma(r)$ and $\mathcal{Q}_\gamma(r)$ obey the ODEs:*

$$r\frac{d}{dr}\mathcal{P}_\gamma = -\mathcal{P}_\gamma\mathcal{Q}_\gamma + \mathcal{E}_\mathcal{P}, \qquad r\frac{d}{dr}\mathcal{Q}_\gamma = \mathcal{Q}_\gamma(\mathcal{P}_\gamma^2 - 1 - \mathcal{Q}_\gamma - \mathcal{E}_\mathcal{Q}),$$

where the error terms vanish quickly as $r \to 0$. Furthermore:

(i) If $Q = 0$, then $\mathcal{P}_\gamma(r)$ converges to some $\mathcal{P}_{\gamma,\infty}$ as $r \to 0$, satisfying $\mathcal{P}_{\gamma,\infty} \approx \mathcal{P}_\gamma(r_0)$.

(ii) If $Q \neq 0$, then $\mathcal{Q}_\gamma(r)$ converges to 0 as $r \to 0$, while $\mathcal{P}_\gamma(r)$ converges to some $\mathcal{P}_{\gamma,\infty}$ as $r \to 0$, necessarily satisfying $\mathcal{P}_{\gamma,\infty} \geq 1$ and in the particular case where $\mathcal{Q}_\gamma(r_0)$ is small,

$$\mathcal{P}_{\gamma,\infty} \approx \max\{\mathcal{P}_\gamma(r_0), \mathcal{P}_\gamma(r_0)^{-1}\} + O(\mathcal{Q}_\gamma(r_0)).$$

See Section 6.1 for a more precise version of these theorems. This completes our general overview of the BKL bounce results.

1.6 Scattering towards spacelike singularities

In the latter part of the dissertation, meaning from Chapter 7 onwards, we somewhat segue into a different topic, namely a *scattering problem* for spacelike singularities, which addresses Question 4 posed at the start of the introduction. While Chapter 7 provides further geometric setup and background, the main objectives to be achieved in Chapter 8 and Chapter 9 are to determine a *scattering theory* for two model linear hyperbolic problems in exact Kasner spacetimes (Kas), relating Cauchy initial data at $t = 1$ and suitable asymptotic data living at the $t = 0$ singularity. In these linear problems, this relation will take the form of a *Hilbert space isomorphism*.

The first of these model problems will be the linear scalar wave equation on a $(1+D)$-dimensional Kasner background, which we write in the form of a first-order hyperbolic system as follows:

$$t\partial_t\phi = \psi, \tag{1.6.1}$$

$$t\partial_t\psi = \sum_{i=1}^{D} t^{2-2p_i}\partial_{x^i}^2\phi. \tag{1.6.2}$$

Smooth initial data, say at $t = 1$, gives rise to a unique smooth solution in the entire spacetime $(t, x) \in (0, +\infty) \times \mathbb{T}^D$. However, in light of the singularity, one expects blow up as $t \to 0$, even for smooth Cauchy data. In fact, one obtains [1, 115, 125] the following asymptotics for ϕ as $t \to 0$:

Theorem 1.6.1 (Asymptotics for the wave equation in non-degenerate Kasner spacetimes)**.** *Let $\phi : \mathcal{M}_{Kas} \to \mathbb{R}$ be a smooth solution to* (1.6.1)–(1.6.2) *on a non-degenerate*[11] *Kasner spacetime. Then there exist smooth functions ψ_∞ and φ_∞ on $\mathbb{T}^D$ such that as $t \to 0$,*

$$\phi(t,x) = \psi_\infty(x)\log t + \varphi_\infty(x) + o(1), \quad \psi(t,x) = \psi_\infty + o(1). \tag{1.6.3}$$

We comment already that Theorem 1.6.1 suggests AVTD behaviour in the sense that (1.6.3) is exactly what one expects if the x-derivative term on the right hand side of (1.6.2) is ignored. Further, the asymptotics (1.6.3) suggest a notion of asymptotic data at the $t = 0$ singularity for the problem under consideration, namely the functions $\psi_\infty, \varphi_\infty : \mathbb{T}^D \to \mathbb{R}$. The full scattering problem for the wave equation may thus be formulated as follows:

(i) (*Existence of the scattering operator*): For what class of Cauchy data (ϕ_C, ψ_C) does there exist corresponding scattering data $(\psi_\infty, \varphi_\infty)$ such that the solution ϕ to the wave equation attains the asymptotics (1.6.3) in a suitable sense? What can be said about the map $\mathcal{S}_\downarrow : (\phi_C, \psi_C) \mapsto (\psi_\infty, \varphi_\infty)$?

(ii) (*Asymptotic completeness*): If instead we are given $(\psi_\infty, \phi_\infty)$, when is it possible to find Cauchy data (ϕ_C, ψ_C) such that the solution ϕ to the wave equation attaining the Cauchy data obeys the asymptotics (1.6.3) with the given ψ_∞ and ϕ_∞? What about the map $\mathcal{S}_\uparrow = (\mathcal{S}_\downarrow)^{-1} : (\psi_\infty, \phi_\infty) \mapsto (\phi_C, \psi_C)$.

(iii) (*Scattering isomorphism*): Can we find Hilbert spaces $\mathcal{H}_C$ and $\mathcal{H}_\infty$ such that the scattering operator $\mathcal{S}_\downarrow$ can be extended to a Hilbert space isomorphism $\mathcal{S}_\downarrow : \mathcal{H}_C \to \mathcal{H}_\infty$ with inverse $\mathcal{S}_\uparrow$?

The second model system addressed in this article is the linearized Einstein–scalar field system. This is given by the Fréchet derivative of the Einstein–scalar field system (Ein)–(Scal) with respect to the metric $\mathbf{g} = \mathbf{g}_{Kas} + \varepsilon h$ and scalar field $\phi = \phi_{Kas} + \varepsilon\mathfrak{\phi}$. This formal linearization yields the following two equations.

$$\left(\mathbf{g}_{Kas}^{-1}\right)^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta h_{\mu\nu} + \mathbf{D}_\mu\mathbf{D}_\nu\left(\operatorname{tr} h\right) - 2\mathbf{D}_{(\mu}\left(\operatorname{div} h\right)_{\nu)} + 2\,\mathbf{Riem}^{\alpha\ \ \beta}_{\ \mu\ \ \nu}\,h_{\alpha\beta} = 4\mathbf{D}_{(\mu}\mathfrak{\phi}\mathbf{D}_{\nu)}\phi_{Kas}, \tag{1.6.4}$$

$$\Box_{\mathbf{g}_{Kas}}\mathfrak{\phi} = h^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta\phi_{Kas} + \left(\operatorname{div} h\right)^\alpha\mathbf{D}_\alpha\phi_{Kas} - \frac{1}{2}\mathbf{D}^\alpha(\operatorname{tr} h)\,\mathbf{D}_\alpha\phi_{Kas}. \tag{1.6.5}$$

Here, any raising and lowering of indices is via the background metric $\mathbf{g}_{Kas}$, and the symbol $\mathbf{D}$ is the Levi-Civita connection of $\mathbf{g}_{Kas}$.

To have a well-posed initial value problem, the system (1.6.4)–(1.6.5) must be imbued with a choice of (linearized) gauge, most often inherited from a suitable gauge for the nonlinear Einstein–scalar field system. In this article we to adopt a linearised $(1 + D)$-dimensional ADM-type gauge with Constant Mean Curvature (CMC), where the spacetime is foliated using a time function t

[11] Non-degenerate means that the Kasner exponents are $(1, 0, \ldots, 0)$ or a permutation thereof.

whose level sets $\Sigma_t = \{t\} \times \mathbb{T}^D$ have constant mean curvature equal to $\operatorname{tr} k = -t^{-1}$, see Section 7.1 for details. In this gauge, the key evolutionary variables of the linearized problem will be $\mathfrak{\eta}_i{}^j$, $\mathfrak{\kappa}_i{}^j$, $\mathfrak{\phi}$, $\mathfrak{\psi}$, which represent linearized versions of the first fundamental form, the second fundamental form, the scalar field, and the normal derivative of the scalar field.

The point is that – under additional constraints that we defer to Chapter 7 – these evolutionary variables also satisfy a first-order (elliptic-)hyperbolic system, and given that the background Kasner spacetime is *subcritical* as in (1.2.11), can be shown to obey asymptotics of a similar sort to (1.6.3), with asymptotic quantities $(\mathfrak{\eta}_\infty)_i{}^j$, $(\mathfrak{\kappa}_\infty)_i{}^j$, $\mathfrak{\phi}_\infty$ and $\mathfrak{\psi}_\infty$. In analogy with the wave equation, we can ask similar questions about the scattering theory; note that there are additional subtleties due to the constraint equations and choice of spatial gauge, see Chapter 7.

In light of these additional complexities, within this section we only state our scattering theorem for our first model problem, concerning the *linear scalar wave equation*. Our theorem will involve a symbol (or pseudodifferential operator) $\mathcal{T}_*$, which we define as follows. See Appendix D for a review of Fourier decompositions and symbol calculus on $\mathbb{T}^D$.

Definition 1.6.1. Firstly, for $\lambda \in \mathbb{Z}^d \setminus \{0\}$, let $t_{\lambda *} > 0$ be the unique solution of $\tau_\lambda^2(t_{\lambda *}) \coloneqq \sum_{i=1}^D t_{\lambda *}^{2-2p_i} \lambda_i^2 = 1$, and let $t_{0*} = 1$. Then set $\mathcal{T}_*$ be the symbol associated (in the sense of Definition D.0.3) to the function $T_* : \mathbb{Z}^D \to \mathbb{R}$ which has $T_*(\lambda) = t_{\lambda *}$ for $\lambda \in \mathbb{Z}^D$. For any exponent $\alpha \in \mathbb{R}$, we also define $\mathcal{T}_*^\alpha$ and $\log(\mathcal{T}_*)$ following the standard symbol calculus.

Remark. In the language of [80, Chapter XVIII], we show in Lemma D.0.1 that $\mathcal{T}_*^\alpha$ lies in the symbol classes $S^{\max\{0,\alpha(1-p)^{-1}\}}$, where $p = \max_i p_i$, while $\log(\mathcal{T}_*)$ lies in S^ε for any $\varepsilon > 0$. In particular, for suitably chosen $s_1, s_2 \in \mathbb{R}$ the operators $\mathcal{T}_*^\alpha$ and $\log(\mathcal{T}_*)$ are bounded operators from H^{s_1} to H^{s_2}.

Having set up our first-order system (1.6.1)–(1.6.2), we now state Theorem 1.6.2, our scattering result for the scalar wave equation in non-degenerate Kasner spacetimes. The proof of Theorem 1.6.2 will be in Chapter 8.

Theorem 1.6.2 (Scattering for the wave equation in non-degenerate Kasner spacetimes)**.** *Let $(\mathcal{M}_{Kas}, \mathbf{g}_{Kas})$ be a non-degenerate Kasner spacetime. We consider solutions $\phi : \mathcal{M}_{Kas} \to \mathbb{R}$ to the wave equation* (1.6.1)–(1.6.2) *on $(\mathcal{M}_{Kas}, \mathbf{g}_{Kas})$. Then we have the following scattering results:*

(i) (Existence of the scattering operator) Let $s > \frac{D}{2} + 1$. Given Cauchy data $(\phi_C, \psi_C) \in H^{s+1} \times H^s$, there exists a unique (classical) solution $(\phi(t), \psi(t))$ to the system (1.6.1)–(1.6.2) *in $\mathcal{M}$ attaining the Cauchy data $\phi(1, \cdot) = \phi_C(\cdot)$ and $\partial_t \phi(1, \cdot) = \psi_C(\cdot)$. Let $\varphi(t)$ be defined by:*

$$\varphi(t) = \phi(t) - \psi(t) \cdot \log t. \tag{1.6.6}$$

Then there exist unique functions $\psi_\infty(x)$ and $\varphi_\infty(x)$ in H^s such that

$$\psi(t, x) \to \psi_\infty(x) \text{ in } H^s, \qquad \varphi(t, x) \to \varphi_\infty(x) \text{ in } H^s \quad \text{strongly as } t \to 0. \tag{1.6.7}$$

Furthermore, one has that

$$\psi_\infty \in H^{s+\frac{1}{2}}, \qquad \varphi_\infty + \log(\mathcal{T}_*)\psi_\infty \in H^{s+\frac{1}{2}}. \tag{1.6.8}$$

(ii) (Asymptotic completeness) Let $s > \frac{D}{2}+1$, and let ψ_∞ and φ_∞ be functions in $\mathbb{T}^D$ with Sobolev regularity as in (1.6.8)*. Then there exists a unique (classical) solution $(\phi(t), \psi(t))$ to the system* (1.6.1)–(1.6.2) *in $\mathcal{M}_{Kas}$, with regularity*

$$(\phi, \psi) \in C^0((0, +\infty), H^{s+1} \times H^s) \cap C^1((0, +\infty), H^s \times H^{s-1}),$$

such that if $\varphi(t)$ is defined as in (1.6.6)*, then the asymptotics* (1.6.7) *hold, strongly as $t \to 0$.*

(iii) (Scattering isomorphism) For $s \in \mathbb{R}$, define the following Hilbert spaces together with their norms:

$$\mathcal{H}^s_C = \{(\phi, \psi) : \phi \in H^{s+1}, \psi \in H^s\}, \quad \| (\phi, \psi) \|^2_{\mathcal{H}^s_C} \coloneqq \|\phi\|^2_{H^{s+1}} + \|\psi\|^2_{H^s}, \tag{1.6.9}$$

$$\mathcal{H}^s_\infty = \{(\psi, \varphi) : \psi \in H^{s+\frac{1}{2}}, \varphi + \log(\mathcal{T}_*)\psi \in H^{s+\frac{1}{2}}\},$$
$$\| (\psi, \varphi) \|^2_{\mathcal{H}^s_\infty} \coloneqq \|\psi\|^2_{H^{s+\frac{1}{2}}} + \|\varphi + \log(\mathcal{T}_*)\psi\|^2_{H^{s+\frac{1}{2}}}. \tag{1.6.10}$$

Then the map $\mathcal{S}_\downarrow$, which is defined to map (ϕ_C, ψ_C) from (i) to $(\psi_\infty, \varphi_\infty)$ in (1.6.7)*, may be extended to a Hilbert space isomorphism $\mathcal{S}_\downarrow : \mathcal{H}^s_C \to \mathcal{H}^s_\infty$, whose inverse $\mathcal{S}_\uparrow = \mathcal{S}_\downarrow^{-1}$ is exactly given by (a suitable extension of) the map taking $(\psi_\infty, \varphi_\infty)$ in (ii) to $(\phi(1, \cdot), \psi(1, \cdot))$.*

A remarkable feature of Theorem 1.6.2 is the gain or loss of half a derivative from Cauchy data to scattering data. In particular, the renormalized time derivative $\psi = t\partial_t\phi$ lies in H^s at any fixed time $t > 0$, but its limit ψ_∞ at $t = 0$ has its regularity increased to $H^{s+\frac{1}{2}}$. This as a sign that spatial derivatives not only become negligible compared to temporal derivatives near $t = 0$, but actually become better behaved as one approaches the singular boundary at $t = 0$.

Note than an almost-sharp correspondence between Cauchy data and asymptotic data was already found by Ringström in [125, Propositions 8.1 and 8.9], however in this work the derivative gain is from H^s to $H^{s+\frac{1}{2}-\varepsilon}$ rather than the sharp $H^{s+\frac{1}{2}}$. In order to achieve this sharp result, it is essential that we modify our Hilbert spaces using the symbol $\log(\mathcal{T}_*)$, a feature not present in [125]. This final modification was partly inspired by a scattering result for the wave equation in higher-dimensional de Sitter spacetimes [38].

One could also entertain scattering between Cauchy data at $t = 1$ and a suitable notion of asymptotic data at $t = +\infty$. The issue is that although one may obtain upper bounds for $\phi(t)$ and $\psi(t)$ as $t \to +\infty$, it is difficult to upgrade these to asymptotics since both $\phi(t)$ and $\psi(t)$ oscillate at late times. There are partial results in this direction, see [115] and [125, Chapter 6].

Another reason we do not study $t \to +\infty$ scattering is that the Kasner spacetimes are expected to be unstable in the $t \to +\infty$ direction as solutions to the Einstein–scalar field system, see [98]. One could conjecture that generic perturbations lead to either isotropic expansion or recollapse into a singularity, see for instance [6, 58]. This is also related to the "cosmic no-hair conjecture" in the presence of a positive cosmological constant, see [10, 24].

The analogous scattering theorem for the system of linearized gravity and the variables $(\mathfrak{h}_i{}^j, \mathfrak{k}_i{}^j, \mathfrak{f}, \mathfrak{p})$ is stated in Chapter 7 as Theorem 7.3.1. We mention two things: firstly, here we must impose the subcriticality condition (1.2.11), which we refer to in the stronger form:

$$\min_{\substack{1\leq i,j,k\leq D\\ j\neq k}}(1 + p_i - p_j - p_k) \geq \delta > 0. \tag{1.6.11}$$

We shall mention already that a new phenomenon arises upon introduction of the $(1,1)$-tensors $\mathfrak{h}_i{}^j$ and $\mathfrak{k}_i{}^j$. In that case, Theorem 7.3.1 involves also the symbol $\mathcal{T}_*^{-p_i+p_j}$, thus the anisotropy of the Kasner spacetime also affects the derivative gain and loss incurred in the scattering process, particularly in *off-diagonal* components of the tensors $\mathfrak{k}_i{}^j$ and $\mathfrak{r}_i{}^j$.

In fact, as one varies across the whole range of Kasner exponents satisfying the subcriticality condition (1.2.11), the symbol $\mathcal{T}_*^{-p_i+p_j}$ can be of arbitrarily high order. Thus the degeneration of top order energy estimates – without modification by such a symbol – as observed in the nonlinear setting of say [63] or [61], is a true phenomenon, and one expects that in any nonlinear scattering result one must allow for a large loss of derivatives in either direction.

1.7 Outline of the dissertation

Chapter 2: In this chapter, we set up various formulations of the Einstein vacuum equations, Einstein–Maxwell–(real) scalar field system and Einstein–Maxwell–charged scalar field system in symmetry classes that reduce the systems to $(1+1)$-dimensional PDEs, in preparation for Chapters 3 to 6. After a brief introduction to these different matter models and their physical relevance, in Section 2.1, we define the *surface symmetric equations* and discuss various properties including rigidity of the electromagnetic field in surface symmetry. In Section 2.2, we discuss the particular case of the Eisntein–Maxwell–(charged) scalar field system in spherical symmetry, using *double null* coordinates (u, v) to describe the metric. In Section 2.3, we specialize further to the Einstein–Maxwell–real scalar field system in spherical symmetry, and set up initial data in trapped regions in preparation for the results of Chapter 3. In Section 2.4, we discuss equations describing the spherically symmetric Einstein–Maxwell–*charged* scalar field equations for the homogeneous hairy black hole interiors studied in Chapter 4, where the evolution equations reduce to ODEs. In Section 2.5, we then reformulate the *surface symmetric* Einstein–Maxwell–real scalar field equations in another gauge we call *Rein gauge*, where the area-radius r is treated as a time coordinate. We also

briefly discuss well-posedness and continuation criteria for this gauge, in preparation for Chapter 6. In Section 2.6, we introduce the final symmetry class discussed in the dissertation, namely the T^2 Gowdy symmetry class for which we study inhomogeneous BKL bounces in Chapter 5.

Chapter 3: In this chapter, we study the Kasner-like description of singularities arising in the spherically symmetric Einstein–Maxwell–(real) scalar field model introduced in Section 1.3 and Theorem 1.3.1. In Section 3.1, we firstly state the detailed version of our results in Theorem 3.1.1 and Theorem 3.1.2 and provide a sketch of the proof, before giving further context and insight into applications in Chapter 3.2. The next three sections are devoted to the proofs of Theorem 3.1.1 and Theorem 3.1.2; in Section 3.3 we provide upper bounds for ϕ and Ω^2, in Section 3.4 we upgrade these to precise asymptotics, which are then used to conclude the proof in Section 3.5. In Section 3.6 we then provide several examples of spacetimes to which one may apply the results of Theorems 3.1.1 and 3.1.2.

Chapter 4: In this chapter, we present the results regarding the interior of homogeneous hairy black hole solutions, introduced in Section 1.4 and Theorem 1.4.1. In Section 4.1, we first introduce the various regions in our black hole interior, and state the precise version of our results in Theorem 4.1.1 and Theorem 4.1.2. This is followed by a more elaborate explanation of the physical and mathematical context regarding these constructed spacetimes in Section 4.2. The remainder of Chapter 4 is dedicated to the proof of Theorem 4.1.1 and Theorem 4.1.2: In Section 4.3 we analyse the "early regions" of the black hole interior, where the spacetimes remains close to the original Reissner-Nordström spacetime. In Section 4.4, we then study the collapsed oscillation region $\mathcal{O}$, where the scalar field ϕ exhibits oscillatory, Bessel-like behavior. In Section 4.5, we study the proto-Kasner region, where $\mathbf{g}$ starts to exhibit Kasner-like behavior. In Section 4.6, we do some preliminary analysis to study the sets of values of ϵ for which we can provide more detailed analysis of the Kasner-like regions. In Section 4.7 and Section 4.8, we then complete the proof by studying the precise Kasner-like behavior near the singularity, depending on the value of ϵ.

Chapter 5: In this chapter, we study BKL bounces outside homogeneity for Gowdy symmetric vacuum spacetimes (see Section 2.6 for a definition of Gowdy symmetry), introduced in Section 1.5 and Theorems 1.5.2 and 1.5.3. In Section 5.1, we state the precise version of our results in Theorem 5.1.1 and Theorem 5.1.2, and give a brief overview before proceeding to the proof. In Section 5.2 we introduce the bootstrap assumptions and energy quantities used in the proof, and also provide several lemmas regarding Sobolev spaces to be used in the proof. In Section 5.3, we carry out the energy estimates needed in the proof, and use these to derive ODEs with errors in Section 5.4. In Section 5.5, we now use these ODEs to prove certain L^∞ estimates for the variables in Gowdy symmetry. Finally, in Section 5.6 we conclude the proof of Theorem 5.1.1 and in Section 5.7 we

conclude the proof of Theorem 5.1.2.

Chapter 6: In this chapter, we study similar BKL bounces outside homogeneity for surface symmetric solutions to the Einstein–Maxwell–(real) scalar fields equations also introduced in Section 1.5, see Theorems 1.5.5 and 1.5.6. This chapter largely follows the structure of Chapter 5; in Section 6.1 we state the precise versions of our results in Theorem 6.1.1 and Theorem 6.1.2. In Section 6.2, we introduce the bootstrap assumptions and energy quantities used in the proof, and in Section 6.3 and Section 6.4 we carry out the energy estimates and the L^∞ / ODE estimates respectively. Finally, in Section 6.5 we conclude the proof of Theorem 6.1.1 and in Section 6.6 we complete the proof of Theorem 6.1.2.

Chapter 7: In this chapter, we begin to consider the full Einstein–scalar field equations in $(1+3)$-dimensions, without symmetry assumptions, with the aim of understanding the scattering problem towards spacelike singularities introduced in Section 1.6. In Section 7.1, we consider the Einstein–scalar field equations in the $(1+3)$-, or more generally $(1+D)$-dimensionaon, ADM decomposition of spacetime, and also discuss how to consider the Kasner spacetimes (Kas) within this context. In Section 7.2, we detail the linearization of the Einstein–scalar field equations around the Kasner solutions in the context of the ADM-decomposition, and introduce various notation to be used in Chapter 9. In Section 7.3, we then state the precise version of our scattering theorem for the linearized Einstein–scalar field system around Kasner, namely 7.3.1, which will be proven in Chapter 9. In Section 7.4, we then further contextualize our results, and consider how one could apply or generalize our results to scattering theory for the *nonlinear* Einstein–scalar field system.

Chapter 8: In this chapter, we study the scattering theory for the linear wave equation in a fixed Kasner background introduced in Section 1.6. That is, we prove Theorem 1.6.2. We start with an overview of the proof in Section 8.1, and provide some notation and preliminary lemmas regarding the frequency space (Fourier) decomposition in Section 8.2. The bulk of the proof of Theorem 1.6.2 is a combination of a "high-frequency" energy estimate given in Section 8.3 and a "low-frequency" energy estimate given in Section 8.4. These are then used to complete the proof in Section 8.5.

Chapter 9: In this chapter, we complete the dissertation by studying the scattering theory for the linearized Einstein–scalar field system around Kasner, which was introduced in Chapter 7. That is, we prove Theorem 7.3.1. In Section 9.1, we start with an overview of the new ideas introduced in the proof (in comparison with the wave case). In Section 9.2, we repurpose the linearized Einstein–scalar field system of Chapter 7 into a system of ODEs at each frequency $\lambda \in \mathbb{Z}^D$. These are then used to carry out a high-frequency energy estimate in Section 9.3 and a low-frequency energy estimate in Section 9.4. We complete this chapter by using these energy estimates (together ith

suitable gauge transformations) to complete the proof of Theorem 7.3.1.

Chapter 2

The Einstein equations in symmetry

Within this chapter we will introduce basic definitions and properties regarding the Einstein vacuum equations and various Einstein matter systems in symmetry, notably the Einstein–Maxwell–scalar field system and subsystems thereof, where we allow the scalar field to be *massive* and *charged*. We in particular consider the Einstein vacuum equations and Einstein–matter systems in the *surface symmetry*, T^2 *symmetry* and *Gowdy symmetry* classes, the purpose being to reduce the $(1+3)$-dimensional Einstein equations to a $(1+1)$-dimensional system.

Recall that the Einstein equations of General Relativity describe the dynamical coupling of a spacetime $(\mathcal{M}, \mathbf{g})$ to matter fields $\mathbf{\Phi}$ via the following geometric equation, where the Einstein tensor $\mathbf{G}_{\mu\nu}$ represents *curvature* while $\mathbf{T}_{\mu\nu}$ represents the *energy-momentum content* of matter. Note that we also introduce a cosmological constant $\Lambda \in \mathbb{R}$.

$$\mathbf{G}_{\mu\nu}[\mathbf{g}] + \Lambda \mathbf{g}_{\mu\nu} = 2\,\mathbf{T}_{\mu\nu}[\mathbf{\Phi}], \tag{Ein*}$$

$$\text{where } \mathbf{G}_{\mu\nu}[\mathbf{g}] \coloneqq \mathbf{Ric}_{\mu\nu}[\mathbf{g}] - \frac{1}{2}\mathbf{R}[\mathbf{g}]\mathbf{g}_{\mu\nu}.$$

While (Ein) is already non-trivial in the *vacuum* (meaning $\mathbf{T} = 0$), it is often informative to introduce additional matter fields, including perfect fluids, scalar and vector fields, electromagnetic waves, particle-type fields (obeying equations from kinetic theory), and so on, both for the purpose of studying the dynamics of such matter fields in curved spacetimes, as well as understanding their backreaction on $(\mathcal{M}, \mathbf{g})$ themselves via the right hand side of (Ein*).

We start with an electromagnetic field, represented by a 2-form $\mathbf{F} \in \Omega^2(\mathcal{M})$. This has energy-momentum content

$$\mathbf{T}_{\mu\nu}[\mathbf{F}] = \mathbf{F}_{\mu\rho}\mathbf{F}_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu}\mathbf{F}^{\rho\sigma}\mathbf{F}_{\rho\sigma}, \tag{2.0.1}$$

while the electromagnetic field $\mathbf{F}$, often referred to as the *Maxwell field*, evolves according to *Maxwell's equations*; we now also include a source term J_ν on the right hand side:

$$\nabla^\nu \mathbf{F}_{\mu\nu} = J_\nu, \qquad \nabla_{[\lambda}\mathbf{F}_{\mu\nu]} = 0. \tag{2.0.2}$$

The other type of matter we consider is a *scalar field* ϕ, which is a scalar-valued function on $\mathcal{M}$. We distinguish two cases. Firstly, there is the simple case of a real scalar field $\phi : \mathcal{M} \to \mathbb{R}$, with energy-momentum content

$$\mathbf{T}_{\mu\nu}[\phi] = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2}\mathbf{g}_{\mu\nu}\nabla^\rho \phi \nabla_\rho \phi. \tag{2.0.3}$$

The function ϕ then evolves according to the scalar wave equation on $(\mathcal{M}, \mathbf{g})$, namely

$$\Box_{\mathbf{g}}\phi := \mathbf{g}^{\mu\nu}\nabla_\mu \nabla_\nu \phi = 0. \tag{2.0.4}$$

Definition 2.0.1. We say that a triple $(\mathcal{M}, \mathbf{g}, \phi)$ solves the *Einstein–scalar field* system if they solve the equations (Ein*) and (2.0.4), with $\mathbf{T}[\mathbf{\Phi}]$ given by $\mathbf{T}[\phi]$ as in (2.0.3).

We say that a quadruple $(\mathcal{M}, \mathbf{g}, \mathbf{F}, \phi)$ solves the *Einstein–Maxwell–scalar field* system if they solve the equations (Ein*), (2.0.2) and (2.0.4), where $\mathbf{T}[\mathbf{\Phi}]$ is given by $\mathbf{T}[\mathbf{F}] + \mathbf{T}[\phi]$ as in (2.0.1) and (2.0.3), and the source term J_ν in (2.0.2) vanishes.

Remark. The *characteristics* of both the Maxwell equations (2.0.2) and the scalar wave equation (2.0.4) are exactly null vectors of $(\mathcal{M}, \mathbf{g})$, meaning that both electromagnetic waves and (massless) scalar waves propagate at the same speed as gravitational waves, i.e. at the speed of light. As a result, these matter fields are often used as a proxy for gravitational waves in vacuum.

We now introduce the second, and more complicated, type of scalar wave equation. This *charged* and possibly *massive* scalar wave will be complex valued $\phi : \mathcal{M} \to \mathbb{C}$ and *nonlinearly coupled* to the electromagnetic field $\mathbf{F}$ in the following sense: from the second equation of (2.0.2) it follows that $\mathbf{F}$ can (locally) be written as $\mathbf{F} = dA$ for some 1-form A_μ. This allows one to define a *gauge derivative* $\mathbf{D}_\mu = \nabla_\mu + iq_0 A_\mu$, where $q_0 \in \mathbb{R}$ and $\mathbf{D}$ acts on complex-valued tensor fields. Then the complex scalar ϕ has energy-momenta

$$\mathbf{T}_{\mu\nu}[\phi] = \mathrm{Re}\left(\mathbf{D}_\mu \phi \overline{\mathbf{D}_\nu \phi}\right) - \frac{1}{2}\mathbf{g}_{\mu\nu}\left(\mathbf{D}^\rho \phi \overline{\mathbf{D}_\rho \phi} + m^2|\phi|^2\right), \tag{2.0.5}$$

for some $m^2 \in \mathbb{R}$ – representing the mass of the scalar field – and ϕ obeys the gauge covariant wave equation with respect to $(\mathcal{M}, \mathbf{g}, A)$:

$$\mathbf{g}^{\mu\nu}\mathbf{D}_\mu \mathbf{D}_\nu \phi = m^2 \phi. \tag{2.0.6}$$

We comment that the tuple (A, ϕ) described above is subject to the following *gauge invariance*; for any real-valued $f : \mathcal{M} \to \mathbb{R}$ one can transform $A_\mu \mapsto A_\mu + \partial_\mu f$, $\phi \mapsto \phi e^{-iq_0 f}\phi$. This is related to the fact ϕ may be viewed as a section of a complex line bundle whose connection has curvature $\mathbf{F}$, see [99] for details.

Definition 2.0.2. We say that a quintuple $(\mathcal{M}, \mathbf{g}, \mathbf{F}, A, \phi)$ with $\mathbf{F} = dA$ solves the *Einstein–Maxwell–charged (and massive) scalar field system* if they solve (Ein*), (2.0.2), (2.0.6), with $q_0 \in \mathbb{R}$

and $m^2 \in \mathbb{R}$ ($m^2 = 0$ corresponds to massless), and $\mathbf{T}[\mathbf{\Phi}] = \mathbf{T}[\mathbf{F}] + \mathbf{T}[\phi]$ as in (2.0.1) and (2.0.5), and on the right hand side of (2.0.2),

$$J_\mu = -iq_0 \operatorname{Im}\left(\phi \overline{\mathbf{D}_\mu \phi}\right), \tag{2.0.7}$$

where we view two quintuples as equivalent if (A, ϕ) may be related by the gauge invariance above.

Remark. The Einstein–Maxwell–charged and massive scalar field system may be derived as the Euler–Lagrange equations of the following Einstein–Hilbert action, with matter:

$$S = \int_{\mathcal{M}} \left(R[\mathbf{g}] - 2\left(\mathbf{g}^{ab}\mathbf{g}^{cd}\mathbf{F}_{ac}\mathbf{F}_{bd} + \mathbf{g}^{ab}\mathbf{D}_a\phi\overline{\mathbf{D}_b\phi} + m^2|\phi|^2 \right) + \Lambda \right) \sqrt{-\det \mathbf{g}}\, d^4x.$$

In what follows, we will generally take $\Lambda = 0$ in the Einstein equation (Ein*), with the exception of Section 2.4, where we describe the equations used to study spatially homogeneous, spherically symmetric hairy black hole interiors in Chapter 4. For the remainder of Chapter 2, we will specialize the Einstein vacuum equations, the Einstein–(Maxwell)–scalar field equations and the Einstein–Maxwell–charged scalar field equations to various symmetry classes, and introduce various coordinate gauges such that these systems may be written as PDEs which are amenable to analysis.

2.1 Surface symmetric spacetimes

A spacetime $(\mathcal{M}^4, \mathbf{g})$ is said to be *surface symmetric* if it possesses two linearly independent spacelike Killing vector fields, which together generate a group of isometries whose orbits are each rescaled copies of a surface Σ^2 of constant sectional curvature, so that $\mathcal{M}^4 \cong \mathcal{Q}^2 \times \Sigma^2$ with $\mathcal{Q}$ being the quotient manifold, and the metric $\mathbf{g}$ has the following warped product form.

$$\mathbf{g} = \mathbf{g}_{\mathcal{Q}} + r^2 d\sigma_{\Sigma^2}. \tag{2.1.1}$$

Here $\mathbf{g}_{\mathcal{Q}}$ is a $(1+1)$-dimensional Lorentzian metric on the quotient manifold $\mathcal{Q}$, the function $r : \mathcal{Q} \to \mathbb{R}$ is positive on the interior of $\mathcal{Q}$ and is referred to as the *area-radius function*[1]. Finally, the quantity $d\sigma_{\Sigma^2}$ denotes the standard metric on the surface Σ^2, rescaled to have constant sectional curvature $\kappa \in \{1, 0, -1\}$. That is, in some local coordinate chart,

$$d\sigma_{\Sigma^2} = d\theta^2 + \sin_\kappa^2 \theta d\varphi^2 = \begin{cases} d\theta^2 + \sin^2\theta\, d\varphi^2 & \text{if } \Sigma \text{ has universal cover } \mathbb{S}^2, \\ d\theta^2 + d\varphi^2 & \text{if } \Sigma \text{ has universal cover } \mathbb{R}^2, \\ d\theta^2 + \sinh^2\theta\, d\varphi^2 & \text{if } \Sigma \text{ has universal cover } \mathbb{H}^2. \end{cases}$$

In the *spherically symmetric* case, where $\Sigma^2 = \mathbb{S}^2$, we also allow for the isometry group to have single-point orbits, which lie on the *centre* Γ of spherical symmetry. The centre Γ is often considered

[1]Indeed, when Σ^2 is a closed manifold, this is a characteristic length scale associated to each copy of Σ^2 in $(\mathcal{M}, \mathbf{g})$.

as a regular boundary of $\mathcal{Q}$, and the area-radius r is continuously extendible onto $\Gamma \subset \partial\mathcal{Q}$ with $r|_\Gamma = 0$.

Let us now consider a surface symmetric spacetime solving the Einstein equations (Ein). Since the left hand side of (Ein) must respect the warped product structure of (2.1.1), it follows that the energy-momentum tensor $\mathbf{T} = \mathbf{T}[\mathbf{\Phi}]$ must also obey a symmetry assumption. That is, we are always able to decompose $\mathbf{T}$ as

$$\mathbf{T} = T_{\mathcal{Q}} + \frac{\mathbf{S}}{2}\, d\sigma_{\Sigma^2}, \tag{2.1.2}$$

where $T_{\mathcal{Q}}$, often denoted as simply T or $T_{\alpha\beta}$ in the sequel, is a symmetric $(0,2)$-tensor on $\mathcal{Q}$ and $\mathbf{S} : \mathcal{Q} \to \mathbb{R}$ is a scalar representing a trace of $\mathbf{T}$ in the Σ^2 directions.

The simplest possibility is the case that $\mathbf{T}$ vanishes i.e. the Einstein vacuum equations. Unfortunately, at least in the spherically symmetric case, it is known due to Birkhoff [22] that any spherically symmetric solution to the Einstein vacuum equations is locally isometric to a member of the Schwarzschild family of black holes with mass parameter $M \in \mathbb{R}$, with $M = 0$ thought of as Minkowski space. A similar result is true for more general constant curvature surfaces Σ^2, and any value of the cosmological constant $\Lambda \in \mathbb{R}$. As a result, in surface symmetry one typically introduces additional matter fields to study nontrivial dynamics.

Electromagnetism in surface symmetry

Let us first consider coupling an electromagnetic field $\mathbf{F} \in \Omega^2(\mathcal{M})$ to the Einstein equations in surface symmetry. In light of the form of the energy momentum tensor (1.2.12) and the decomposition (2.1.2), it follows that such an electromagnetic field $\mathbf{F}$ must take the form:

$$\mathbf{F} = \frac{Q}{r^2}\boldsymbol{\omega}_{\mathcal{Q}} + B\,\boldsymbol{\omega}_{\Sigma^2}, \tag{2.1.3}$$

where $\boldsymbol{\omega}_{\Sigma^2} = \sin_\kappa \theta\, d\theta \wedge d\varphi$ is the natural volume form on Σ^2 and $\boldsymbol{\omega}_{\mathcal{Q}}$ is the volume form associated to $\mathbf{g}_{\mathcal{Q}}$, and $Q, B : \mathcal{Q} \to \mathbb{R}$ are functions we call the *electric charge* and *magnetic charge* respectively.

Another remarkable simplification of surface symmetry is that for $\mathbf{F}$ of the form (2.1.3) solving the *source free* Maxwell equations (Max), it follows that Q and B are constant.

Lemma 2.1.1. *Let $\mathbf{F}$ be a surface symmetric solution to the Maxwell equations (2.0.2) wiht $J_\mu = 0$, and $\mathbf{g}$ and $\mathbf{F}$ given by (2.1.1) and (2.1.3). Then $Q, B : \mathcal{Q} \to \mathbb{R}$ are constant.*

Proof. Consider the first of Maxwell's equations, $d\mathbf{F} = 0$. Since $dr, dQ, dB \in \Omega^1(\mathcal{Q}) \subset \Omega^1(\mathcal{M})$, and the 2-forms $\boldsymbol{\omega}_{\mathcal{Q}}$ and $\boldsymbol{\omega}_{\Sigma^2}$ are closed,

$$d\mathbf{F} = dB \wedge \boldsymbol{\omega}_{\Sigma^2} = 0 \quad \text{ implies } dB = 0.$$

The second of Maxwell's equations is equivalent to $d * \mathbf{F} = 0$, where $*$ is the Hodge star associated to $\mathbf{g}$. Note that – modulo choice of signs – one may compute that

$$*\mathbf{F} = -\frac{B}{r^2}\boldsymbol{\omega}_{\mathcal{Q}} + Q\,\boldsymbol{\omega}_{\Sigma^2}$$

Therefore, the same argument shows $dQ = 0$, as required. □

Lemma 2.1.1 shows that in surface symmetry, even the coupled Einstein–Maxwell equations display no nontrivial dynamics (indeed, there is an extension of Birkhoff's theorem in electrovacuum). In fact, Lemma 2.1.1 remains true so long as the Maxwell equations are source-free, and also applies for the Einstein–Maxwell–(real) scalar field system.

This is no longer true for the Einstein–Maxwell–*charged* scalar field system, since here, the scalar field $\phi : \mathcal{M} \to \mathbb{C}$ is complex valued and is itself coupled back to the electromagnetic field via the source term J_ν in (2.0.7). This allows one, for instance, to study one-ended gravitational collapse with an electromagnetic field, though for the most part we only analyse the Einstein–Maxwell–(neutral) scalar field system within the present dissertation, with the exception of Section 2.4 and Chapter 4.

In the sequel, following the physical heuristic that there are no "magnetic monopoles", we shall always assume $B = 0$, and $\mathbf{F}$ thereby takes the following form, with $Q \in \mathbb{R}$ constant for te Einstein–Maxwell–(real) scalar field model.

$$\mathbf{F} = \frac{Q}{r^2}\boldsymbol{\omega}_{\mathcal{Q}}. \tag{2.1.4}$$

In the Einstein–Maxwell–(charged) scalar field model, $Q : \mathcal{Q} \to \mathbb{R}$ is allowed to vary.

2.2 The Einstein–Maxwell–scalar field system in double null gauge

To study the Einstein–Maxwell–(possibly charged and massive) scalar field system in the surface symmetric setting, it remains to pick a gauge for the quotient metric $\mathbf{g}$. One convenient choice is the *double null gauge*, where coordinates (u, v) are transversal null coordinates chosen so that the warped product (2.1.1) takes the form

$$\mathbf{g} = -\Omega^2(u, v)\, du\, dv + r^2(u, v)\, d\sigma_{\Sigma^2}. \tag{2.2.1}$$

That is, the geometry is entirely described by two functions of (u, v), the area-radius $r(u, v) > 0$ and the *null lapse* $\Omega^2(u, v)$. For the rest of Section 2.2 and the whole of Chapter 3 and Chapter 4, we only consider the case $\Sigma^2 = \mathbb{S}^2$.

There is gauge freedom regarding the choice of the null coordinates u and v; given smooth and monotonically increasing $U = U(u)$ and $V = V(v)$ we may alternatively write

$$\mathbf{g} = -\tilde{\Omega}^2(U, V)\, dU dV + r^2(U, V)\, d\sigma_{\mathbb{S}^2},$$

so long as $\tilde{\Omega}^2(U(u), V(v)) \cdot \frac{dU}{du}\frac{dV}{dv} = \Omega^2(u, v)$. It will often be useful to consider quantities that are *gauge-invariant*, such as the area-radius $r(u, v)$ or the *Hawking mass* $m(u, v)$, given by

$$1 - \frac{2m}{r} \coloneqq \mathbf{g}(\nabla r, \nabla r) = -4\Omega^{-2}\partial_u r\, \partial_v r. \tag{2.2.2}$$

Now considering the full Einstein–Maxwell–scalar field system, the scalar field $\phi = \phi(u, v)$ now depends only on (u, v), while, in light of (2.1.4), the Maxwell field $\mathbf{F}$ can now be written as

$$\mathbf{F} = \frac{Q}{2r^2} \cdot \Omega^2 \, du \wedge dv, \tag{2.2.3}$$

where for our purposes $Q = Q(u, v)$ is the *electric charge* of our spacetime, and is a constant in the case of the uncharged scalar field.

In this choice of gauge, we use the Einstein–Maxwell–(possibly charged and massive) scalar field to write the following equations for the dynamical quantities $r(u, v)$, $\Omega^2(u, v)$, and $\phi(u, v)$. The $uu-$ and $vv-$components of (Ein) give the following *Raychaudhuri equations*, which can be interpreted as constraint equations for Ω^2 along null hypersurfaces of constant u or v.

$$\partial_u(\Omega^{-2}\partial_u r) = -\Omega^{-2} r |\mathbf{D}_u \phi|^2 \tag{2.2.4}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -\Omega^{-2} r |\mathbf{D}_v \phi|^2, \tag{2.2.5}$$

The remaining components of (Ein) imply the following propagation equations for r^2 and Ω^2:

$$\partial_u \partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r} + \frac{\Omega^2 Q^2}{4r^3} + \frac{\Omega^2 r(m^2|\phi|^2 + \Lambda)}{4}, \tag{2.2.6}$$

$$\partial_u \partial_v \log\big(\Omega^2\big) = \frac{\Omega^2}{2r^2} + \frac{2\partial_u r \partial_v r}{r^2} - \frac{\Omega^2 Q^2}{r^4} - 2\,\mathrm{Re}(\mathbf{D}_u \phi \overline{\mathbf{D}_v \phi}), \tag{2.2.7}$$

In the charged scalar field case, the Maxwell equations (2.0.2) with (2.0.7) become:

$$\partial_u Q = -q_0 r^2 \,\mathrm{Im}(\phi \overline{\mathbf{D}_u \phi}), \tag{2.2.8}$$

$$\partial_v Q = +q_0 r^2 \,\mathrm{Im}(\phi \overline{\mathbf{D}_v \phi}), \tag{2.2.9}$$

Finally, the wave equation (2.0.6) becomes

$$\mathbf{D}_u \mathbf{D}_v \phi = -\frac{\partial_u r \cdot \mathbf{D}_v \phi}{r} - \frac{\partial_v r \cdot \mathbf{D}_u \phi}{r} + \frac{i q_0 Q \Omega^2}{4r^2}\phi - \frac{m^2 \Omega^2}{4}\phi, \tag{2.2.10}$$

where the gauge field $A_\mu = A_\mu(u, v)$ will satisfy

$$\partial_u A_v - \partial_v A_u = \frac{Q\Omega^2}{2r^2}. \tag{2.2.11}$$

Following this, it is also convenient to make the following definitions of the *Hawking mass*[2] m and the *renormalized mass* ϖ by

$$m \coloneqq \frac{r}{2}(1 - g_{\mathcal{Q}}(\nabla r, \nabla r)) = \frac{r}{2}(1 - 4\Omega^{-2}\partial_u r \partial_v r). \tag{2.2.12}$$

$$\varpi = \rho + \frac{Q^2}{2r} - \frac{\Lambda r^3}{6}. \tag{2.2.13}$$

[2]We admit there is a conflict in notation in that m simultaneously represents Hawking mass and the mass parameter in the scalar field equation (2.0.6). In the sequel, it should be clear from context which of these is meant.

From (2.2.4)–(2.2.10) one can derive the following transport equations for the renormalized mass:

$$\partial_u \varpi = -2r^2(\Omega^{-2}\partial_v r)^{-1}|D_u\phi|^2 + \frac{m^2}{2}r^2|\phi|^2\partial_u r - q_0 Q r\,\mathrm{Im}(\phi\overline{D_u\phi}), \tag{2.2.14}$$

$$\partial_v \varpi = -2r^2(\Omega^{-2}\partial_u r)^{-1}|D_v\phi|^2 + \frac{m^2}{2}r^2|\phi|^2\partial_v r + q_0 Q r\,\mathrm{Im}(\phi\overline{D_v\phi}). \tag{2.2.15}$$

The equations (2.2.4)–(2.2.11) may be viewed as a system of wave and transport equations for $(r^2, \Omega^2, \phi, A, Q)$. It is straightforward to derive the following local existence result for *characteristic initial data* away from any center of symmetry Γ.

Proposition 2.2.1. *Let $k \geq 2$. Let $\Omega^2 > 0, r^2 > 0, \phi, Q, A_u, A_v$ be C^k functions defined on $(u,v) \in [u_0,u_1]\times\{v_0\}\cup\{u_0\}\times[v_0,v_1]$ such that the equations* (2.2.4)*,* (2.2.5)*,* (2.2.8) *and* (2.2.9) *hold on $[u_0,u_1]\times\{v_0\}\cup\{u_0\}\times[v_0,v_1]$, with appropriate C^k compatibility conditions at (u_0,v_0). Then there exists some $\delta > 0$, and C^k functions $\Omega^2 > 0, r^2 > 0, \phi, Q, A_u, A_v$ on $(u,v) \in [u_0,u_1]\times[v_0,v_0+\delta]\cup[u_0,u_0+\delta]\times[v_0,v_1]$, unique up to gauge transformations $A_\mu \mapsto A_\mu + \partial_\mu f$ and $\phi \mapsto \phi e^{-iq_0 f}$, satisfying the system* (2.2.4)–(2.2.11)*.*

Proof (sketch). For local existence, it is convenient to fix a gauge for A_μ by choosing f to make $A_v \equiv 0$ in the whole spacetime. Then upon differentiating (2.2.11) in u, one can write (2.2.6), (2.2.7), (2.2.10), (2.2.11) as a system of wave equations of the form

$$\partial_u\partial_v\Phi = \mathcal{N}(\Phi, \partial_u\Phi, \partial_v\Phi),$$

where Φ represents the tuple $\Phi = (r^2, \log\Omega^2, \phi, A_u, r^{-1})$ and $\mathcal{N}$ is a smooth function of its arguments.

For such a system, local existence in a region $(u,v) \in [u_0,u_0+\delta]\times[v_0,v_0+\delta]$ is straightforward, see for instance [45, Proposition 2], and one can extend to the region $(u,v) \in [u_0,u_1]\times[v_0,v_0+\delta]\cup[u_0,u_0+\delta]\times[v_0,v_1]$ using appropriate *null structure* for $\mathcal{N}$, similarly to [106]. The remaining equations (2.2.4), (2.2.5), (2.2.8) and (2.2.9) are viewed as *constraint* equations, which are propagated by the system of wave equations above, so long as they are satisfied on initial data. Uniqueness (up to gauge) is also a straightforward consequence of the local theory. □

Remark. In fact, by [99], the Einstein–Maxwell–charged scalar field system satisfies a *generalized extension principle* that gives one further control over the maximal domain of existence; in particular one expects *first singularities* $p \in \mathcal{Q}$ away from the center Γ to have $r \to 0$ near p.

2.3 Einstein–Maxwell–scalar field initial data in trapped regions

Specializing the Einstein–Maxwell–scalar field system (2.2.4)–(2.2.11) to the particular case where ϕ is real and $q_0, m^2, \Lambda = 0$ – so that in particular A_u, A_v are no longer necessary – we yield

$$\partial_u(-\Omega^{-2}\partial_u r) = \Omega^{-2} r(\partial_u\phi)^2, \tag{2.3.1}$$

$$\partial_v(-\Omega^{-2}\partial_v r) = \Omega^{-2} r(\partial_v\phi)^2. \tag{2.3.2}$$

$$\partial_u\partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u r\partial_v r}{r} + \frac{\Omega^2}{4r^3}Q^2, \tag{2.3.3}$$

$$\partial_u\partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2} + \frac{2\partial_u r\partial_v r}{r^2} - \frac{\Omega^2}{r^4}Q^2 - 2\partial_u\phi\partial_v\phi. \tag{2.3.4}$$

$$\partial_u\partial_v\phi = -\frac{\partial_u r\cdot\partial_v\phi}{r} - \frac{\partial_v r\cdot\partial_u\phi}{r}, \tag{2.3.5}$$

We shall later find it advantageous to rewrite (2.3.3) and (2.3.5) in the following forms:

$$\partial_u(-r\partial_v r) = \frac{\Omega^2}{4}\left(1 - \frac{Q^2}{r^2}\right), \tag{2.3.6}$$

$$\partial_v(-r\partial_u r) = \frac{\Omega^2}{4}\left(1 - \frac{Q^2}{r^2}\right), \tag{2.3.7}$$

$$\partial_u(r\partial_v\phi) = -\partial_u r\cdot\partial_v\phi, \tag{2.3.8}$$

$$\partial_v(r\partial_u\phi) = -\partial_v r\cdot\partial_u\phi. \tag{2.3.9}$$

Throughout Chapter 3 in particular, we call the system (2.3.1)–(2.3.9) the Einstein-Maxwell scalar field system in spherical symmetry (or EMSFSS for short). In the special case that $Q \equiv 0$, we instead denote the system simply as the Einstein-scalar field in spherical symmetry (or ESFSS for short).

There, we pose characteristic initial data for (r, Ω^2, ϕ) for the EMSFSS system in a *trapped region*. That is, as in Proposition 2.2.1, initial data is prescribed on two null hypersurfaces $C_0 = \{u_0\}\times[v_0, v_1]$ and $\underline{C}_0 = [u_0, u_1]\times\{v_1\}$, and this data yields a *maximal future hyperbolic development in a domain* as follows (see Figure 2.1):

$$\mathcal{D} \coloneqq \{(u,v) \in [u_0,u_1]\times[v_0,v_1] \subset \mathbb{R}^2 : r(u,v), \Omega^2(u,v), \phi(u,v) \text{ bounded and regular}\}.$$

Our analysis will be confined to the trapped region. As in [32], we take this to mean the region $\mathcal{T} \subset \mathcal{Q}$ with $\mathcal{T} \coloneqq \{(u,v) : \partial_u r(u,v) < 0, \partial_v r(u,v) < 0\}$. Due to the monotonicity of $\Omega^{-2}\partial_u r$ and $\Omega^{-2}\partial_v r$ from the Raychaudhuri equations (2.3.1) and (2.3.2), it suffices to have initial data in the trapped region, since

$$C_0 \cup \underline{C}_0 \subset \mathcal{T} \implies \mathcal{D} \subset \mathcal{T}.$$

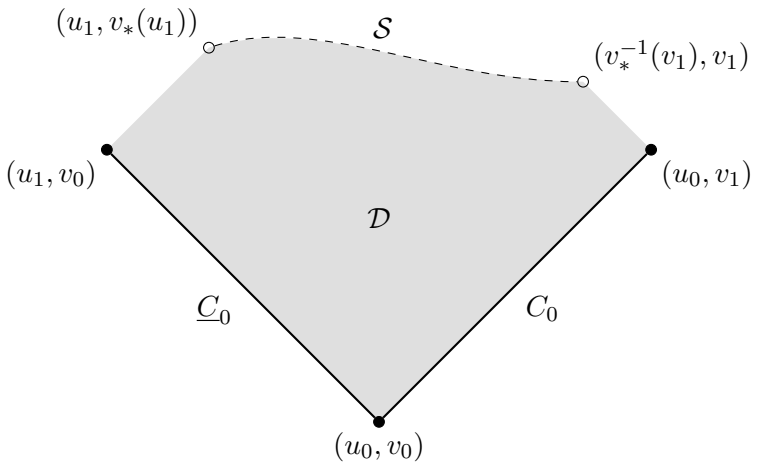


Figure 2.1: The characteristic initial value problem considered in Chapter 3. Regular initial data is prescribed at $C_0 \cup \underline{C}_0$ such that $C_0 \cup \underline{C}_0$ is trapped, yielding a maximal future hyperbolic development $\mathcal{D}$ with a singular boundary $\mathcal{S}$ at which $r = 0$.

Assuming $C_0 \cup \underline{C}_0 \subset \mathcal{T}$, we may fix the remaining gauge-freedom in (u, v) by prescribing

$$\partial_v r = -1 \text{ on } C_0, \qquad \partial_u r = -1 \text{ on } \underline{C}_0. \tag{$uv-$gauge}$$

Imposing ($uv-$gauge), the free data on $C_0 \cup \underline{C}_0$ will then be $\Omega^2(u_0, v_0) \in \mathbb{R}^+$, $r(u_0, v_0) \in \mathbb{R}^+$, and the characteristic data for the scalar field, namely $\partial_v \phi$ on C_0 and $\partial_u \phi$ on $\underline{C}_0$ as well as $\phi(u_0, v_0)$. For our purposes, it will mostly suffice to take $\partial_v \phi|_{C_0}$ and $\partial_u \phi|_{\underline{C}_0}$ to be C^1.

It is often advantageous to consider quantities which are *gauge-independent* with regard to the choice of u and v. We may, for instance, consider the Hawking mass m, defined in (2.2.2), in place of Ω^2. We also make frequent use of the following gauge-independent vector fields:

$$L = \frac{1}{-r\partial_v r}\partial_v, \qquad \underline{L} = \frac{1}{-r\partial_u r}\partial_u, \tag{2.3.10}$$

which are normalized such that $g(L, \underline{L}) = -2(2mr - r^2)^{-1}$. Observe that in ($uv-$gauge),

$$rL\phi = \partial_v \phi \text{ on } C_0, \qquad r\underline{L}\phi = \partial_u \phi \text{ on } \underline{C}_0.$$

2.3.1 Strongly singular maximal future developments

We are interested in solutions of the characteristic initial value problem containing singularities, or more specifically spacelike singularities. By [31, 32, 99], in particular the *generalized extension principle* of [99], such singularities in the ESFSS or EMSFSS system, at least those away from the center of symmetry Γ, are characterized by the area-radius $r(u, v)$ tending to 0.

For this purpose, we focus on maximal globally hyperbolic future developments $\mathcal{D}$ possessing a non-empty singular boundary $\mathcal{S}$ at which $r(u, v) = 0$, which may be parameterized as $(u, v_*(u))$.

In the ESFSS system (where $Q \equiv 0$), there is additional monotonicity in the system that simplifies the issue of finding $\mathcal{S}$. Note that (2.3.6) and (2.3.7) give:

$$\partial_u(-r\partial_v r) = \partial_v(-r\partial_u r) = \frac{\Omega^2}{4} > 0.$$

Hence $-r\partial_u r$ and $-r\partial_v r$ are bounded below by their minimum value on $C_0 \cup \underline{C}_0$. Thus $r^2(u,v)$ is monotonically decreasing at a rate which is bounded below, in both the u and v directions, must therefore reach 0 so long as the intervals $[u_0, u_1]$ and $[v_0, v_1]$ are large enough. We formalize this observation in the following lemma.

Lemma 2.3.1. *Consider the ESFSS system with characteristic initial data prescribed on $C_0 \cup \underline{C}_0$ as in Section 2.3. Then there exist constants $R_0, R_1 > 0$, depending only on data, such that for $(u,v) \in \mathcal{D}$,*

$$R_0 \leq -r\partial_u r(u,v) \leq R_1, \qquad R_0 \leq -r\partial_v r(u,v) \leq R_1. \tag{2.3.11}$$

For the EMSFSS system with $Q \neq 0$, the upper bound of (2.3.11) remains true.

If (2.3.11) holds and one supposes, moreover, that there exists $v_*(u_1) < v_1$ such that $(u_1, v) \in \mathcal{D}$ for $v_0 \leq v < v_*(u_1)$, but $r(u_0, v) \to 0$ as $v \to v_*(u_1)^-$. Then the region $\mathcal{D}$ has a singular boundary $\mathcal{S}$, parameterised by a continuous curve $v = v_*(u)$ for $v_*^{-1}(v_1) \leq u \leq u_1$, such that $r(u,v)$ extends continuously to 0 on $\mathcal{S}$. The boundary $\mathcal{S}$ is spacelike in the sense that $v_*(u)$ is strictly decreasing.

Proof. When $Q \equiv 0$, the monotonicity of (2.3.6) and (2.3.7) mentioned above immediately implies the lower bound in (2.3.11) for $R_0 = \min\{\min_{(u_0,v)\in C_0} -r\partial_v r(u_0,v), \min_{(u,v_0)\in \underline{C}_0} -r\partial_u r(u,v_0)\} > 0$. Regardless of the value of Q, the upper bound is found by integrating (2.3.6), (2.3.7) and then applying (2.3.1), (2.3.2). For instance,

$$\begin{aligned}
-r\partial_v r(u,v) &\leq -r\partial_v r(u_0,v) + \int_{u_0}^{u} \frac{\Omega^2}{4}(\tilde{u},v)\, d\tilde{u}, \\
&= -r\partial_v r(u_0,v) + \int_{r(u,v)}^{r(u_0,v)} \frac{\Omega^2}{-4\partial_u r}(\tilde{u},v)\, dr(\tilde{u},v), \\
&\leq -r\partial_v r(u_0,v) + \int_{r(u,v)}^{r(u_0,v)} \frac{\Omega^2}{-4\partial_u r}(u_0,v)\, dr(\tilde{u},v), \\
&\leq \max_{(u_0,\tilde{v})\in C_0} (-r\partial_v r(u_0,\tilde{v})) + r(u_0,v_0) \cdot \max_{(u_0,\tilde{v})\in C_0} \frac{\Omega^2}{-4\partial_u r}(u_0,\tilde{v}).
\end{aligned}$$

The inequality on the third line follows from the Raychaudhuri equation (2.3.1), which implies that $\frac{\Omega^2}{-\partial_u r}$ is decreasing in u. A similar computation for $-r\partial_u r(u,v)$ then implies the upper bound in (2.3.11).

Assuming the existence of $v_*(u_1)$ as in the lemma, the parameterization of the singular boundary $\mathcal{S}$ as $(u, v_*(u))$ follows straightforwardly from (2.3.11). $\square$

Remark. By the monotonicity $\partial_u\partial_v r^2 < 0$ in the case $Q \equiv 0$, a sufficient condition on data to guarantee the existence of such a $v_*(u_1) < v_1$ is the following inequality:

$$r(u_0, v_0)^2 \geq r(u_1, v_0)^2 + r(u_0, v_1)^2. \tag{2.3.12}$$

However this is not necessary, and need not restrict ourselves to scenarios where (2.3.12) holds.

We now consider the EMSFSS system with $Q \neq 0$. In this case, there is no monotonicity akin to $\partial_u\partial_v r^2 < 0$, and in order to proceed we instead make the *a priori assumption*

$$\inf_{(u,v)\in\mathcal{D}} \min\{-r\partial_v r(u,v), -r\partial_u r(u,v)\} \geq R_0. \tag{QTS}$$

By Lemma 2.3.1, the full inequality (2.3.11) holds so long as (QTS) holds.In the case $Q \neq 0$, we also need to make a second a priori assumption: for some $\alpha \in (0,1)$, we assume

$$\inf_{(u,v)\in\mathcal{D}} \min\left\{|r^2 L\phi(u,v)|^2, |r^2 \underline{L}\phi(u,v)|^2\right\} \geq 1+\alpha. \tag{SKE}$$

Here the null vector fields L and $\underline{L}$ are defined in (2.3.10). In Chapter 3, we often refer to the following definition.

Definition 2.3.1. A *strongly singular* future development $\mathcal{D}$ of (trapped) characteristic initial data on $C_0 \cup \underline{C}_0$ is a maximal future development $\mathcal{D}$, as described in Section 2.3, with the two additional properties:

(i) There exists some continuous, strictly decreasing function $v_*(u)$, defined on $v_*^{-1}(v_1) \leq u \leq u_1$ for some $v_*^{-1}(v_1) \in (u_0, u_1)$, such that $\mathcal{S} = \{(u, v_*(u)) : v_*^{-1}(v_1) \leq u \leq u_1\}$ is a future boundary of $\mathcal{D}$ such that the area-radius $r(u,v)$ extends continuously to 0 at $\mathcal{S}$.

(ii) Either the Maxwell charge Q is identically zero, or the strongly singular development must also satisfy the two conditions (QTS) and (SKE).

See Theorem 1.3.1 to recall the (rough version of the) results proven for such strongly singular spacetimes.

2.4 Equations for homogeneous hairy black hole interiors

In Chapter 4, we will be interested in charged hairy perturbations of sub-extremal Reissner-Nordström interiors. To define this, given some parameters $M > 0$, $\mathbf{e}, \Lambda \in \mathbb{R}$, consider the polynomial

$$P_{M,\mathbf{e},\Lambda}(X) = X^2 - 2MX + \mathbf{e}^2 - \tfrac{1}{3}\Lambda X^4. \tag{2.4.1}$$

Then the set of subextremal-parameters $(M, \mathbf{e}, \Lambda)$ is $\mathcal{P}_{se} = \mathcal{P}_{se}^{\Lambda<0} \cup \mathcal{P}_{se}^{\Lambda=0} \cup \mathcal{P}_{se}^{\Lambda>0}$, where $\mathcal{P}_{se}^{\Lambda<0}$ is such that $\Lambda < 0$ and the polynomial $P_{M,\mathbf{e},\Lambda}(X)$ has two distinct positive real roots $r_- < r_+$, $\mathcal{P}_{se}^{\Lambda=0}$

is such that $\Lambda = 0$ and the polynomial $P_{M,\mathbf{e},\Lambda}(X)$ has two distinct positive real roots $r_- < r_+$, while $\mathcal{P}_{se}^{\Lambda>0}$ is such that $\Lambda > 0$ and $P_{M,\mathbf{e},\Lambda}(X)$ has three distinct positive real roots $r_- < r_+ < r_c$.

The Reissner-Nordström(-dS/AdS) spacetime is a spherically symmetric solution to the Einstein–Maxwell system, and can be written in standard (t, r) coordinates as

$$g_{RN} = -\left(1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2}. \tag{2.4.2}$$

In particular, the Reissner-Nordström(-dS/AdS) interior metric is given by (2.4.2), restricted to the coordinate range $r_- < r < r_+$, $t \in \mathbb{R}$. Note that, in the interior, t is a spacelike coordinate while r is a timelike coordinate.

The Maxwell field is given by setting $Q \equiv \mathbf{e}$ in (2.2.3), and $\Omega^2 = \Omega^2_{RN}$ will be defined shortly. A choice gauge field A which will be consistent with the remainder of Section 2.4 and Chapter 4 is

$$A = -\left(\frac{\mathbf{e}}{r_+} - \frac{\mathbf{e}}{r}\right) dt. \tag{2.4.3}$$

To recast the metric (2.4.2) into the double null form (2.2.1), we define

$$\frac{dr}{dr^*} \coloneqq \frac{\Omega^2_{RN}}{4}, \quad \Omega^2_{RN} \coloneqq -4\left(1 - \frac{2M}{r} + \frac{\mathbf{e}^2}{r^2} - \frac{\Lambda r^2}{3}\right), \tag{2.4.4}$$

$$u \coloneqq \frac{r^* - t}{2}, \quad v \coloneqq \frac{r^* + t}{2}. \tag{2.4.5}$$

In this (u, v) coordinate system, the metric can now be written as

$$g_{RN} = -\Omega^2_{RN} du dv + r^2 d\sigma_{\mathbb{S}^2}. \tag{2.4.6}$$

In the sequel, we denote the Reissner-Nordström area-radius function by r_{RN}.

We define the surface gravity of the event horizon $2K_+$, and the surface gravity of the Cauchy horizon $2K_-$ by the following functions of black hole parameters:

$$2K_\pm = 2K(r = r_\pm) = \frac{2}{r_\pm^2}\left(M - \frac{\mathbf{e}^2}{r_\pm} - \frac{\Lambda r_\pm^3}{3}\right). \tag{2.4.7}$$

It is then a well-known fact that the null lapse Ω^2_{RN} obeys the following asymptotics, where $\alpha_\pm > 0$ are fixed constants depending on the black hole parameters:

$$\log\left(\frac{\Omega^2_{RN}}{\alpha_+ e^{2K_+(M,\mathbf{e},\Lambda)\cdot r^*}}\right) = O(e^{2K_+(M,\mathbf{e},\Lambda)\cdot r^*}) \text{ as } r^* = u + v \to -\infty, \tag{2.4.8}$$

$$\log\left(\frac{\Omega^2_{RN}}{\alpha_- e^{2K_-(M,\mathbf{e},\Lambda)\cdot r^*}}\right) = O(e^{2K_-(M,\mathbf{e},\Lambda)\cdot r^*}) \text{ as } r^* = u + v \to +\infty. \tag{2.4.9}$$

We note that, for subextremal black hole parameters, one has $2K_+ > 0$ while $2K_- < 0$.

Introducing the following "regular coordinates",

$$U = e^{2K_+ u}, \quad V = e^{2K_+ v}, \tag{2.4.10}$$

it is well-known that in the coordinate system (U, v), the metric g_{RN} can be smoothly extended beyond $U = 0$, and the right event horizon $\mathcal{H}_R = \{(U, v) : U = 0, v \in [-\infty, +\infty)\}$ is realised as a smooth null hypersurface. A similar construction can be made for the coordinates (u, V), with $\mathcal{H}_L = \{(u, V) : u \in [-\infty, +\infty), V = 0\}$. Using the coordinate system (U, V), the metric g_{RN} is defined for $0 \leq U, V < +\infty$, and can be smoothly beyond both $\mathcal{H}_R$ and $\mathcal{H}_L$, including the bifurcation sphere $\mathcal{H}_R \cap \mathcal{H}_L = \{U = 0, V = 0\}$.

2.4.1 Black hole interiors with charged scalar hair

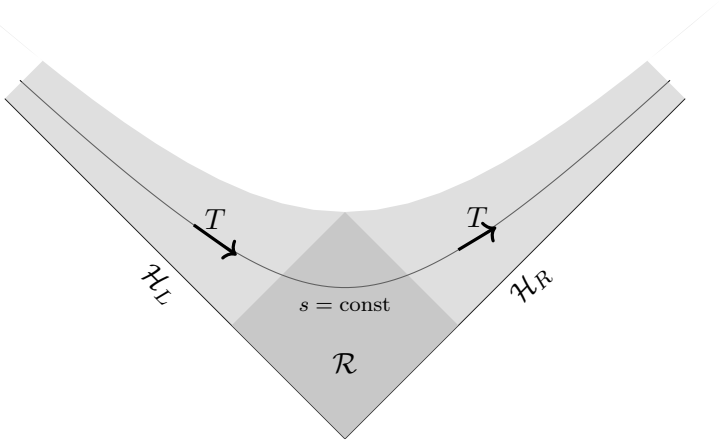


Figure 2.2: The local solution to the characteristic initial value problem for a spatially homogeneous black hole interior with charged scalar hair

Consider the characteristic initial value problem with initial data given on the two affine complete null hypersurfaces $\mathcal{H}_L = \{(U, V) : U \geq 0, V = 0\} = \{(u, v) : u \in [-\infty, +\infty), v = -\infty\}$ and $\mathcal{H}_R = \{(U, V) : U = 0, V \geq 0\} = \{(u, v) : u = -\infty, v \in [-\infty, +\infty)\}$, intersecting at the bifurcation sphere $(U, V) = (0, 0)$.

We normalize the regular coordinates (U, V), which are related to the usual interior coordinates (u, v) via (2.4.10), using the following gauge choice:

$$\Omega_R^2(U, v)|_{\mathcal{H}_R} = \frac{1}{2K_+} e^{-2K_+(M,\mathbf{e},\Lambda)\cdot u}\, \Omega^2(u, v)|_{\mathcal{H}_R} = \frac{\alpha_+}{2K_+} e^{2K_+(M,\mathbf{e},\Lambda)\cdot v}, \tag{2.4.11}$$

$$\Omega_L^2(u, V)|_{\mathcal{H}_L} = \frac{1}{2K_+} e^{-2K_+(M,\mathbf{e},\Lambda)\cdot v}\, \Omega^2(u, v)|_{\mathcal{H}_L} = \frac{\alpha_+}{2K_+} e^{2K_+(M,\mathbf{e},\Lambda)\cdot u}, \tag{2.4.12}$$

These choices are made such that, with respect to the *generalized Kruskal-Szekeres* coordinate system (U, V), Ω^2 is normalized as follows on $\mathcal{H} = \mathcal{H}_R \cup \mathcal{H}_L$:

$$\Omega_{reg}^2(U, V)|_{\mathcal{H}} = \frac{\alpha_+}{(2K_+)^2}. \tag{Ω-data}$$

In the context of this article, we pose the following characteristic initial data (recall (2.2.13) for the definition of ϖ):

$$r|_{\mathcal{H}} = r_+(M, \mathbf{e}, \Lambda), \tag{r-data}$$

$$\varpi|_{\mathcal{H}} = M > 0, \tag{ϖ-data}$$

$$Q|_{\mathcal{H}} = \mathbf{e} \in \mathbb{R} \setminus \{0\}, \tag{Q-data}$$

$$\phi|_{\mathcal{H}} = \epsilon \in \mathbb{R} \setminus \{0\}, \tag{ϕ-data}$$

as well as a gauge for the Maxwell field such that the components A_V, A_U vanish on $\mathcal{H}_R, \mathcal{H}_L$ respectively. In particular $D_V\phi$ and $D_U\phi$ will vanish on their respective horizon pieces. As a result, this data is compatible with the null constraints (2.2.4), (2.2.5).

In the following proposition, we show that the data (Ω-data), (r-data), (Q-data), (ϕ-data) uniquely specifies a solution to the Einstein-Maxwell-charged scalar field system, which is moreover spatially homogeneous. Of course, in order to have uniqueness we must impose a gauge for the Maxwell field A. In our setting, for the purpose of reducing the Einstein-Maxwell-charged scalar field system to a system of ODEs (see already Section 2.4.2 and (2.4.20)), it will be convenient to choose

$$UA_U + VA_V = 0. \tag{A-gauge}$$

We now identify the spacetime describing the hairy black hole interior spacetimes studied in this article, which we firstly describe in the regular coordinate system (U, V).

Proposition 2.4.1. *Consider characteristic initial data (Ω-data), (r-data), (Q-data), (ϕ-data) to the Einstein-Maxwell-charged scalar field system (2.2.4)–(2.2.11), with u, v replaced by U, V respectively. Then imposing also (A-gauge), there exists a unique maximal future development[3] of the system $(\Omega^2, r, \phi, Q, A_U, A_V)$ that is smooth up to and including the horizon $\mathcal{H} = \mathcal{H}_L \cup \mathcal{H}_R$.*

Furthermore, the domain of definition of this maximal development is given by $\{(U, V) : 0 \leq UV < D_{max}\}$ for some $D_{max}(M, \mathbf{e}, \Lambda, m^2, q_0, \epsilon) \in (0, +\infty]$, and letting T be the vector field (see Figure 2.2)

$$T \coloneqq 2K_+ \left(-U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V} \right), \tag{2.4.13}$$

one finds that T is a Killing vector field, satisfying $Tr = T\Omega^2 = TQ = T\phi = T(UA_U) = 0$. In particular, the spacetime is foliated by Cauchy hypersurfaces $UV = const$, which are each spatially homogeneous with isometry group $\mathbb{R} \times SO(3)$.

Proof. We appeal to a local existence result, for instance Proposition 2.2.1, with the caveat that our gauge choice (A-gauge) is not well-suited to such local well-posedness statements. Following Proposition 2.2.1 we firstly instead find a solution with some other choice of gauge field $A^{(0)}_\mu$, in a

[3]Maximality is meant in the sense that there is no larger causal region of the (U, V)-plane where one may smoothly extend the solution.

region $(U,V) \in [0,\delta] \times [0,\delta]$ with δ sufficiently small (see the darker shaded region of Figure 2.2), attaining the prescribed data, with A replaced by $A^{(0)}$ in the equations. In particular, Maxwell gauge-independent quantities such as Ω^2_{reg}, r, Q are already uniquely determined.

We seek a gauge transformation that relates any such $A^{(0)}$ to an A satisfying (A-gauge). The correct gauge transformation is $A = A^{(0)} - dh$, with h given by

$$h(U,V) \coloneqq \int_0^1 UA^{(0)}(Ux,Vx)\,dx. \tag{2.4.14}$$

The reason is that this choice of h implies that

$$\begin{aligned} U\frac{\partial h}{\partial U} + V\frac{\partial h}{\partial V} &= \int_0^1 \left(UA^{(0)}(Ux,Vx) + U^2 x \frac{\partial A^{(0)}}{\partial U}(Ux,Vx) + UVx\frac{\partial A^{(0)}}{\partial V}(Ux,Vx) \right) dx, \\ &= \int_0^1 \frac{d}{dx}(UxA^{(0)}(Ux,Vx))\,dx = UA^{(0)}(U,V), \end{aligned}$$

so that

$$UA_U + VA_V = UA^{(0)} - U\frac{\partial h}{\partial U} - V\frac{\partial h}{\partial V} = 0,$$

as required. Furthermore, the h chosen in (2.4.14) is the unique such gauge transformation that is regular at $(0,0)$: If $\tilde{h}$ was another such function, then the difference $g = h - \tilde{h}$ would satisfy

$$U\frac{\partial g}{\partial U} + V\frac{\partial g}{\partial V} = 0,$$

whose general solution is of the form $g(U,V) = G(U/V)$. So regularity at $(0,0)$ implies that G, and thus g, is constant, and $dh = d\tilde{h}$ after all.

Hence we have constructed a unique regular solution in the characteristic rectangle $[0,\delta] \times [0,\delta]$. We next show that the vector field T defined in (2.4.13) annihilates all the relevant quantities. For this purpose, we argue geometrically as follows: let $a > 0$ be any positive real number, and consider the double null coordinate transformation $U \mapsto U' = aU, V \mapsto V' = a^{-1}V$. Then in the (U',V') coordinate system, we note that it still holds that on $\mathcal{H} = \{(U',V') : U' = 0 \text{ or } V' = 0\}$, we have

$$\begin{gathered} \Omega'^2_{reg}(U',V') = \frac{\alpha_+}{(2K_+)^2}, \\ r = r_+, \\ Q = \mathbf{e}, \\ \phi = \epsilon, \\ U'A_{U'} + V'A_{V'} = 0. \end{gathered}$$

Hence by the aforementioned existence and uniqueness result, we have a unique solution in the characteristic rectangle $(U',V') \in [0,\delta] \times [0,\delta]$. Furthermore, this agrees with the solution in the original (U,V) coordinates, so that, for $f \in \{\Omega^2, r, Q, \phi, UA_U\}$, we have

$$f(U',V') = f(aU, a^{-1}V) = f(U,V). \tag{2.4.15}$$

Allowing a to vary across all positive reals, it is clear that we have a solution in the whole of $\{(U,V) \in \mathbb{R}^2_{\geq 0} : 0 \leq UV \leq \delta^2\}$, i.e. the lighter shaded region in Figure 2.2 that arises from sweeping out the darker shaded region for different choices of $a > 0$. Furthermore, it is immediate from (2.4.15) that T annihilates all such quantities f.

The extension to the region $\{0 \leq UV < D_{max}\}$ is then straightforward using standard extension principle arguments for nonlinear waves and again appealing to this geometric trick. □

Remark. By the generalized extension principle of [99], if the quantity D_{max} of Proposition 2.4.1 is finite, then one must have $r \to 0$ as $UV \to D_{max}$. However, Proposition 2.4.1 is qualitative in nature and says little about quantitative properties of the interior, or if and how any spacelike singularity is formed.

Following Proposition 2.4.1, we would also like to understand the transversal derivatives of r and ϕ along $\mathcal{H}$, for which we shall need the full system of equations (2.2.4)–(2.2.11). For r, we see that using (2.2.6) on $\mathcal{H}_R$,

$$\partial_V \partial_U r = \frac{\alpha_+}{4(2K_+)^2}\left(-\frac{1}{r_+} + \frac{\mathbf{e}^2}{r_+^3} + r_+\Lambda + r_+ m^2\epsilon^2\right). \tag{2.4.16}$$

Since $r_+ = r_+(M, \mathbf{e}, \Lambda)$ satisfies the equation

$$P_{M,\mathbf{e},\Lambda}(r_+) = r_+^2 - 2Mr_+ + \mathbf{e}^2 - \tfrac{1}{3}\Lambda r_+^4 = 0,$$

it is readily checked that the expression in the parentheses in (2.4.16) is equal to $-2K_+ + r_+ m^2\epsilon^2$. We then integrate (2.4.16), noting that $\partial_U r = 0$ at the bifurcation sphere $(U,V) = (0,0)$, to find

$$\partial_U r|_{\mathcal{H}_R} = \frac{\alpha_+ V}{8K_+}\left(-1 + \frac{r_+ m^2\epsilon^2}{2K_+}\right).$$

Returning to (u,v) coordinates, and performing a similar procedure on $\mathcal{H}_L$, we deduce

$$\lim_{u\to-\infty} \frac{-4\partial_u r}{\Omega^2}(u,v) = \lim_{v\to-\infty} \frac{-4\partial_v r}{\Omega^2}(u,v) = 1 - \frac{r_+ m^2\epsilon^2}{2K_+}. \tag{2.4.17}$$

We remark here that due to the presence of the charged scalar field mass, there is already an $O(\epsilon^2)$ deviation from the corresponding Reissner-Nordström quantity. A similar procedure applied to (2.2.10) will yield

$$\frac{2K_+}{\alpha_+ V} \cdot \mathbf{D}_U\phi(0,V) = \lim_{u\to-\infty} \frac{\mathbf{D}_u\phi}{\Omega^2}(u,v) = \beta_+\epsilon, \tag{2.4.18}$$

$$\frac{2K_+}{\alpha_+ U} \cdot \mathbf{D}_V\phi(U,0) = \lim_{v\to-\infty} \frac{\mathbf{D}_v\phi}{\Omega^2}(u,v) = \beta_+\epsilon, \tag{2.4.19}$$

where $\beta_+ = \beta_+(M, \mathbf{e}, \Lambda, m)$ is some fixed constant we do not explicitly determine.

2.4.2 System of ODEs for spatially homogeneous solutions

Define $s = u+v$, $t = v-u$, where the null coordinates (u, v) are fixed by the gauge choices (2.4.11), (2.4.12), (2.4.10). Since $\partial_t = \frac{1}{2}(\partial_v - \partial_u) = \frac{1}{2}T$, Proposition 2.4.1 proves that the maximal future development arising from the characteristic data of Section 2.4.1 obeys $\partial_t r = 0, \partial_t \Omega^2 = 0, \partial_t Q = 0, \partial_t \phi = 0$. So we may consider these just as functions of s.

Of course, this is only true after imposing (A-gauge). In the (u, v) coordinate system,

$$A = A_U dU + A_V dV = 2K_+(UA_U du + VA_V dv) =: \tilde{A}(du - dv) = -\tilde{A}dt, \tag{2.4.20}$$

where, due to Proposition 2.4.1, $\tilde{A} = UA_U(s)$ is a real-valued function of s, with $\lim_{s\to-\infty} \tilde{A}(s) = 0$. We next show that this choice of gauge will in fact constrain the scalar field ϕ to be real.

Lemma 2.4.2. *With (2.4.20) and the initial data of Section 2.4.1, $\phi = \phi(s)$ is everywhere real.*

Proof. Consider the transport equations (2.2.8), (2.2.9) for the quantity Q. Since $\partial_t Q = \frac{1}{2}(\partial_v - \partial_u)Q = 0$,

$$q_0 r^2 \operatorname{Im}(\phi\overline{\mathbf{D}_s\phi}) = \tfrac{1}{2} q_0 r^2 \operatorname{Im}(\phi\overline{(\mathbf{D}_u\phi + \mathbf{D}_v\phi)}) = 0.$$

Hence by (2.4.20) we must have $\operatorname{Im}(\phi\overline{\mathbf{D}_s\phi}) = \operatorname{Im}(\phi\overline{\partial_s\phi}) = 0$.

Next, we decompose ϕ into its modulus and argument; $\phi = \Phi e^{i\theta}$. Then

$$\operatorname{Im}(\phi\overline{\partial_s\phi}) = -\Phi^2\partial_s\theta = 0,$$

so that the phase θ is constant whenever ϕ is nonzero. But as ϕ is smooth in the variable s, it does not change phase when it reaches 0, hence ϕ is real everywhere. □

Using the identities $\partial_u = \partial_s - \partial_t, \partial_v = \partial_s + \partial_t$, we have that, for $f \in \{r(s), \Omega(s), \phi(s), Q(s), \tilde{A}(s)\}$, one has

$$\partial_u f = \partial_v f = \frac{df}{ds} =: \dot{f}.$$

We now proceed to rewrite the Einstein-Maxwell-Klein-Gordon system (2.2.4)–(2.2.11) as a system of ODEs.

Firstly, the Raychaudhuri equation becomes

$$\frac{d}{ds}(-\Omega^{-2}\dot{r}) = \Omega^{-2} r(|\dot{\phi}|^2 + |\tilde{A}|^2 q_0^2 |\phi|^2). \tag{2.4.21}$$

Defining the quantity $\kappa = -\frac{1}{4}\Omega^2\dot{r}^{-1}$, which is exactly 1 in Reissner-Nordström(-dS/AdS), we may rewrite this as

$$\frac{d}{ds}\kappa^{-1} = 4\Omega^{-2} r(|\dot{\phi}|^2 + |\tilde{A}|^2 q_0^2 |\phi|^2). \tag{2.4.22}$$

We also at times appeal to (2.4.21) in the form:

$$\frac{d}{ds}(-\dot{r}) - \frac{d}{ds}\log\big(\Omega^2\big)\cdot(-\dot{r}) = r(|\dot{\phi}|^2 + |\tilde{A}|^2 q_0^2 |\phi|^2). \tag{2.4.23}$$

The wave equation for r is now written as

$$\ddot{r} = -\frac{\Omega^2}{4r} - \frac{\dot{r}^2}{r} + \frac{\Omega^2}{4r^3}Q^2 + \frac{\Omega^2 r}{4}(m^2|\phi|^2 + \Lambda), \tag{2.4.24}$$

which may be conveniently rewritten as

$$\frac{d}{ds}(-r\dot{r}) = \frac{\Omega^2}{4} - \frac{\Omega^2 Q^2}{4r^2} - \frac{\Omega^2 r^2}{4}(m^2|\phi|^2 + \Lambda). \tag{2.4.25}$$

The wave equation (2.2.7) for the null lapse Ω^2 becomes

$$\frac{d^2}{ds^2}\log\big(\Omega^2\big) = \frac{\Omega^2}{2r^2} + 2\frac{\dot{r}^2}{r^2} - \frac{\Omega^2}{r^4}Q^2 - 2\dot{\phi}^2 + 2|\tilde{A}|^2 q_0^2|\phi|^2, \tag{2.4.26}$$

or alternatively

$$\frac{d^2}{ds^2}\log\big(r\Omega^2\big) = \frac{\Omega^2}{4r^2} - \frac{3}{4}\frac{\Omega^2 Q^2}{r^4} - 2|\dot{\phi}|^2 + 2|\tilde{A}|^2 q_0^2|\phi|^2 + \frac{\Omega^2 m^2}{4}|\phi|^2. \tag{2.4.27}$$

For the Maxwell field Q and the gauge field $\tilde{A}$, the equations (2.2.8), (2.2.9), (2.2.11) become

$$\dot{Q} = \tilde{A} q_0^2 r^2 |\phi|^2, \tag{2.4.28}$$

$$\dot{\tilde{A}} = -\frac{Q\Omega^2}{4r^2}. \tag{2.4.29}$$

Finally, the wave equation (2.2.10) for the scalar field may be written as the second order ODE

$$\ddot{\phi} = -\frac{2\dot{r}\dot{\phi}}{r} - q_0^2|\tilde{A}|^2\phi - \frac{m^2\Omega^2}{4}\phi, \tag{2.4.30}$$

which we often use in the form

$$\frac{d}{ds}(r^2\dot{\phi}) = -r^2 q_0^2|\tilde{A}|^2\phi - \frac{m^2\Omega^2 r^2}{4}\phi. \tag{2.4.31}$$

We also reformulate the initial data of Section 2.4.1 so as to satisfy the ODE system (2.4.21)–(2.4.31). Data is posed at the limit $s \to -\infty$, and is given by

$$\lim_{s\to-\infty} r(s) = r_+, \ \lim_{s\to-\infty} Q(s) = \mathbf{e}, \ \lim_{s\to-\infty}\phi(s) = \epsilon, \tag{2.4.32}$$

$$\lim_{s\to-\infty} \Omega^2(s)\cdot e^{-2K_+ s} = \alpha_+, \tag{2.4.33}$$

$$\lim_{s\to-\infty} -4\Omega^{-2}(s)\dot{r}(s) = 1 - \frac{r_+ m^2\epsilon^2}{2K_+}, \tag{2.4.34}$$

$$\lim_{s\to-\infty}\frac{d}{ds}\log\big(\Omega^2\big)(s) = 2K_+, \tag{2.4.35}$$

$$\lim_{s\to-\infty}\Omega^{-2}(s)\tilde{A}(s) = -\frac{\mathbf{e}}{8K_+ r_+^2}, \tag{2.4.36}$$

$$\lim_{s\to-\infty}\Omega^{-2}(s)\dot{\phi}(s) = \beta_+\epsilon. \tag{2.4.37}$$

This concludes the set-up for the analytical problem considered in this section and Chapter 4.

2.5 Equations in Rein gauge

In contrast to the results of Chapter 3 for spherically symmetric spacetimes, in later parts of the dissertation, namely Chapter 6 we shall turn our attention to more general surface symmetric spacetimes of Section 2.1, now written in a different choice of gauge, originally due to Rein [117] in the study of the Einstein–Vlasov system.

This gauge choice will make sense only in regions of the spacetime $(\mathcal{M}, \mathbf{g})$ such that the surfaces Σ^2 of symmetry are *trapped*, meaning the area-radius r may be treated as a *timelike coordinate*. In such trapped regions, we write the spacetime metric $\mathbf{f}$ of (2.1.1) in the following diagonal form:

$$\mathbf{g} = -e^{2\mu} dr^2 + e^{2\lambda} dx^2 + r^2 d\sigma_{\Sigma^2}. \tag{2.5.1}$$

Here, level sets of the coordinate x are orthogonal to leaves of a constant r-foliation, while μ, λ are real-valued functions of r and x.

While the gauge (2.5.1) has previously been considered for the surface symmetric Einstein–Vlasov–scalar field system in [139, 140, 141], to the author's knowledge it has not previously been used for other matter systems, including Maxwell; thus we choose to include a more thorough description of the gauge in the cyrrent section. We later record a local well-posedness result Proposition 2.5.2 and a continuation criterion in Lemma 2.5.3, differing slightly from that given in [141]. Given the decomposition (2.1.2) for the energy momentum tensor $\mathbf{T} = \mathbf{T}[\mathbf{\Phi}]$, one expresses the Einstein equations (Ein) with respect to the gauge (2.5.1) as follows, where $\operatorname{tr}_{\mathcal{Q}} T = e^{-2\mu} T_{rr} + e^{-2\lambda} T_x x$.

$$r\partial_r \lambda = r^2 T_{rr} - \frac{1}{2}\left(1 + \kappa e^{2\mu}\right), \tag{2.5.2}$$

$$r\partial_r \mu = r^2 e^{2(\mu-\lambda)} T_{xx} + \frac{1}{2}\left(1 + \kappa e^{2\mu}\right), \tag{2.5.3}$$

$$\partial_x \mu = r T_{xr}, \tag{2.5.4}$$

$$\begin{aligned}(r\partial_r)^2 \mu - r^2 e^{2(\mu-\lambda)} \partial_x^2 \mu =& (r\partial_r \lambda)(r^2 \operatorname{tr}_{\mathcal{Q}} T + \kappa) e^{2\mu} + r\partial_r \left[\left(r^2 \operatorname{tr}_{\mathcal{Q}} T + \kappa\right) e^{2\mu}\right] \\ &+ r^2 e^{2(\mu-\lambda)} \partial_x \mu \, \partial_x (\mu - \lambda) + r^2 e^{2(\mu-\lambda)} \left[T_{xx} - \frac{1}{2} \operatorname{tr}_{\mathcal{Q}} T e^{2\lambda} - \frac{\mathbf{S}}{2} e^{2\lambda}\right].\end{aligned} \tag{2.5.5}$$

We refer the reader to [117, Equations (1.3)–(1.6)] for the specific case where the matter field $\mathbf{\Phi}$ is Vlasov matter (though that our r and x are instead t and r in [117]). Compared to [117], we have manipulated the equations somewhat so that (a) our canonical choice of time derivative is $r\partial_r$ and (b) equation (2.5.5) has wave-like structure.

The claimed wave-like structure is made clear by the observation that with respect to the coordinates of (2.5.1), the wave equation (Scal) for ϕ is given by:

$$(r\partial_r)^2 \phi - r^2 e^{2(\mu-\lambda)} \partial_x^2 \phi = e^{2\mu} (r\partial_r \phi)\left(r^2 \operatorname{tr}_{\mathcal{Q}} T + \kappa\right) + r^2 e^{2(\mu-\lambda)} \partial_x \phi \, \partial_x (\mu - \lambda). \tag{2.5.6}$$

Thus (2.5.5) has similar structural features to (2.5.6), albeit with a more complicated right hand side.

In the remainder of this section, and subsequently Chapter 6, we assume that the quotient manifold $\mathcal{Q}$ is diffeomorphic to $I \times \mathbb{S}^1$, where $I \subset (0, +\infty)$ is an interval and we periodise the x-variable to lie in $\mathbb{S}^1$, exactly as in [140]. This is purely for convenience; upon localising to suitable causal subdomains one could also consider settings where x lies in an interval rather than $\mathbb{S}^1$.

2.5.1 The Einstein–Maxwell–scalar field system in Rein gauge

We specialize the above to the Einstein–Maxwell–scalar field system $(\mathcal{M}, \mathbf{g}, \phi, \mathbf{F})$. In light of (2.1.4) and the metric gauge choice (2.5.1), the electromagnetic tensor $\mathbf{F}$ is represented by the following 2-form, where $Q \in \mathbb{R}$ is the constant *electric charge*.

$$\mathbf{F} = \frac{2Q}{r^2} e^{\mu+\lambda} dr \wedge dx. \tag{2.5.7}$$

Proposition 2.5.1 (Surface symmetric Einstein–Maxwell–scalar field system)**.** *Let $(\mathcal{M}, \mathbf{g}, \phi, \mathbf{F})$ be a surface symmetric solution to the Einstein–Maxwell–scalar field system* (Ein)*,* (Scal) *and* (Max)*, with metric $\mathbf{g}$ given by* (2.5.1) *and Maxwell tensor $\mathbf{F}$ given by* (2.5.7)*. Then the dynamical quantities (λ, μ, ϕ) satisfy the following equations for $(r, x) \in \mathcal{Q}$.*

$$r\partial_r \lambda = \frac{1}{2}\left((r\partial_r \phi)^2 + r^2 e^{2(\mu-\lambda)} (\partial_x \phi)^2 - 1 - \kappa e^{2\mu} + \frac{Q^2}{r^2} e^{2\mu} \right), \tag{2.5.8}$$

$$r\partial_r \mu = \frac{1}{2}\left((r\partial_r \phi)^2 + r^2 e^{2(\mu-\lambda)} (\partial_x \phi)^2 + 1 + \kappa e^{2\mu} - \frac{Q^2}{r^2} e^{2\mu} \right), \tag{2.5.9}$$

$$\partial_x \mu = r\partial_r \phi \, \partial_x \phi, \tag{2.5.10}$$

$$\begin{aligned} (r\partial_r)^2 \mu - r^2 e^{2(\mu-\lambda)} \partial_x^2 \mu =& (r\partial_r \lambda) \left(-\frac{Q^2}{r^2} + \kappa \right) e^{2\mu} + r\partial_r \left[\left(-\frac{Q^2}{r^2} + \kappa \right) e^{2\mu} \right] \\ &+ r^2 e^{2(\mu-\lambda)} \partial_x \mu \, \partial_x (\mu - \lambda) + 2r^2 e^{2(\mu-\lambda)} (\partial_x \phi)^2 - \frac{Q^2}{r^2} e^{2\mu}, \end{aligned} \tag{2.5.11}$$

$$(r\partial_r)^2 \phi - r^2 e^{2(\mu-\lambda)} \partial_x^2 \phi = (r\partial_r \phi) \left(-\frac{Q^2}{r^2} + \kappa \right) e^{2\mu} + r^2 e^{2(\mu-\lambda)} \partial_x \phi \, \partial_x (\mu - \lambda). \tag{2.5.12}$$

Furthermore $\mu - \lambda$ obeys the transport equation

$$r\partial_r (\mu - \lambda) = 1 + \left(-\frac{Q^2}{r^2} + \kappa \right) e^{2\mu}. \tag{2.5.13}$$

Proof. The equations (2.5.8)–(2.5.12) are derived directly from (2.5.2)–(2.5.6), upon insertion of the following expressions for the decomposition of the energy momentum tensor $\mathbf{T}_{\mu\nu} = \mathbf{T}_{\mu\nu}[\phi] + \mathbf{T}_{\mu\nu}[\mathbf{F}]$

with respect to the coordinates of (2.5.1):

$$T_{rr}[\phi] = e^{2(\mu-\lambda)}T_{xx}[\phi] = \frac{1}{2}\left((\partial_r\phi)^2 + e^{2(\mu-\lambda)}(\partial_x\phi)^2\right),\ \mathbf{S}[\phi] = e^{-2\mu}(r\partial_r\phi)^2 - r^2e^{-2\lambda}(\partial_x\phi)^2,$$

$$T_{rr}[\mathbf{F}] = -e^{2(\mu-\lambda)}T_{xx}[\mathbf{F}] = \frac{1}{2}\frac{Q^2}{r^2}e^{2\mu},\ \mathbf{S}[\mathbf{F}] = \frac{4Q^2}{r^2}.$$

We leave the details to the reader. □

2.5.2 The initial value problem

We briefly outline local existence for the surface symmetric Einstein–Maxwell–scalar field system as given in Section 2.5.1. See also statements similar to the following Proposition 2.5.2 in [117] for the surface symmetric Einstein–Vlasov system and in [141] for the surface symmetric Einstein–Vlasov–scalar field system.

We treat the equations in a slightly different manner from [117], as to emphasize the wave-like nature of (2.5.9). As a result, though our system is reduced to a 2-dimensional problem and one can apply the method of characteristics as in [117], we clarify that local existence also holds in energy spaces $H^{s+1} \times H^s$, for $s > \frac{1}{2}$.

Proposition 2.5.2 (Local well-posedness)**.** *Let $Q \in \mathbb{R}$ be a constant and $\kappa \in \{-1, 0, +1\}$. Consider solutions (ϕ, μ, λ) to the surface symmetric Einstein–Maxwell–scalar field system* (2.5.8)–(2.5.12) *for $(r, x) \in I \times \mathbb{S}^1$, where $I \subset (0, +\infty)$ is an interval.*

Consider data as follows: let $r_0 > 0$ and let $\phi_D, \mu_D, \lambda_D, \dot{\phi}_D, \dot{\mu}_D, \dot{\lambda}_D : \mathbb{S}^1 \to \mathbb{R}$ be functions with regularity to be specified shortly, satisfying the following three constraint equations:

$$\dot{\mu}_D = \frac{1}{2}\left(\dot{\phi}_D^2 + r_0^2e^{2(\mu_D-\lambda_D)}(\partial_x\phi_D)^2 + 1 + \kappa e_D^{2\mu} - \frac{Q^2}{r_0^2}e^{2\mu_D}\right),$$

$$\dot{\lambda}_D = \frac{1}{2}\left(\dot{\phi}_D^2 + r_0^2e^{2(\mu_D-\lambda_D)}(\partial_x\phi_D)^2 - 1 - \kappa e_D^{2\mu} + \frac{Q^2}{r_0^2}e^{2\mu_D}\right),$$

$$\partial_x\mu_D = \dot{\phi}_D\,\partial_x\phi_D.$$

Let the Banach space X be either $C^{k+1} \times C^k$ for integers $k \geq 0$, or $H^{s+1} \times H^s$ for real numbers $s > \frac{1}{2}$. Then given $(f_D, \dot{f}_D) \in X$ for $f \in \{\phi, \mu, \lambda\}$, there exists a maximal interval I containing r_0 and a unique solution (ϕ, μ, λ) of the surface symmetric Einstein–Maxwell–scalar field system (2.5.8)–(2.5.12) *with*

$$(f, r\partial_r f) \in C(I, X) \text{ and } (f, r\partial_r f)|_{r=r_0} = (f_D, \dot{f}_D) \text{ for } f \in \{\phi, \mu, \lambda\}. \tag{2.5.14}$$

Proof. (Sketch) We prove local-wellposedness for this system by identifying several of the equations in (2.5.8)–(2.5.12) as evolution equations for which one may apply standard local well-posedness, and the remaining equations as constraint equations which are true at $r = r_0$ and are then propagated.

The equations we identify as evolution equations are (2.5.11), (2.5.12) and (2.5.13). Note that (2.5.11) and (2.5.12) are quasilinear wave equations with respect to the metric $\mathbf{g}$ in (2.5.1), while (2.5.13) is a transport equation whose transport vector field $r\partial_r$ is timelike with respect to $\mathbf{g}$. Thus:

- If $X = C^{k+1} \times C^k$, then one finds (ϕ, μ, λ) solving (2.5.11)–(2.5.13) and achieving (2.5.14) by applying the method of characteristics, with respect to the characteristic vector fields $L = \partial_r + e^{\mu-\lambda}\partial_x, \underline{L} = \partial_r - e^{\mu-\lambda}\partial_x$ for (2.5.11)–(2.5.12) and the timelike vector field $\partial_r = \frac{1}{2}(L + \underline{L})$ for (2.5.13).

- If $X = H^{s+1} \times H^s$ for $s > \frac{1}{2}$, then the existence and uniqueness of (ϕ, μ, λ) solving (2.5.11)–(2.5.13) and achieving (2.5.14) follows from the local well-posedness for systems of quasilinear wave equations, see for instance Hughes–Kato–Marsden [83]. This is easily extended to the wave–transport system (2.5.11)–(2.5.13), particularly since the right hand side of the transport equation (2.5.13) depends only on μ and not on any of its derivatives.

It remains to show that the remaining equations (2.5.8)–(2.5.10) are satisfied. In fact, since we already have (2.5.13) it suffices to show that (2.5.9)–(2.5.10) are satisfied. We do this via *constraint propagation*. The key computation is that from (2.5.8)–(2.5.10), if one defines:

$$\mathcal{C}_1 = r\partial_r\mu - \frac{1}{2}\left((r\partial_r\phi)^2 + r^2e^{2(\mu-\lambda)}(\partial_x\phi)^2 + 1 + \kappa e^{2\mu} - \frac{Q^2}{r^2}e^{2\mu}\right), \quad \mathcal{C}_2 = \partial_x\mu - r\partial_r\phi\,\partial_x\phi,$$

then one may derive the equations

$$r\partial_r\mathcal{C}_1 = re^{\mu-\lambda}\partial_x\mathcal{C}_2 + 2\left(-\frac{Q^2}{r^2} + \kappa\right)e^{2\mu}\,\mathcal{C}_1,$$
$$r\partial_r\mathcal{C}_2 = re^{\mu-\lambda}\partial_x\mathcal{C}_1 + 2\left(1 + \left(-\frac{Q^2}{r^2} + \kappa\right)e^{2\mu}\right)\mathcal{C}_2.$$

So $(\mathcal{C}_1, \mathcal{C}_2)$ obey a homogeneous first order linear hyperbolic system with coefficients determined by the previously found solution to (2.5.8)–(2.5.10). Therefore, by considering an energy of the form $\mathcal{E}_{\mathcal{C}}(r) = \frac{1}{2}\int_{\mathbb{S}^1}(\mathcal{C}_1^2(r,x) + \mathcal{C}_2^2(r,x))\,dx$, integration by parts yields that

$$\left|r\frac{d}{dr}\mathcal{E}_{\mathcal{C}}(r)\right| \leq C \cdot \mathcal{E}_{\mathcal{C}}(r)$$

for some C depending on the solution (ϕ, μ, λ). Since the initial data constraints imply $\mathcal{E}_{\mathcal{C}}(r_0) = 0$, we have that $\mathcal{E}_{\mathcal{C}}(r) = 0$ for all $r \in I$. Therefore $\mathcal{C}_1, \mathcal{C}_2 \equiv 0$ and the remaining equations (2.5.8)–(2.5.10) hold as required. □

Along with the local well-posedness result of Proposition 2.5.2, we also include the following continuation criteria, showing that if the past endpoint of I is not 0, then the quantity μ must blow up.

Lemma 2.5.3 (Continuation criterion)**.** *Let (ϕ, μ, λ) be a solution of the surface symmetric Einstein–Maxwell–scalar field system (2.5.8)–(2.5.12), arising from initial data as in Proposition 2.5.2. Then if $r_p = \inf I \in [0, r_0)$ is the past endpoint of I, then either $r_p = 0$, or one has*

$$\sup_{(r,x)\in(r_p,r_0)\times\mathbb{S}^1} \mu(r,x) = +\infty. \tag{2.5.15}$$

Proof. Suppose for contradiction that $r_p > 0$ and $\mu(r,x) \leq A < +\infty$ for all $(r,x) \in (r_p, r_0] \times \mathbb{S}^1$. By standard continuation criteria for the quasilinear wave–transport system (2.5.11)-(2.5.13) it will suffice to show that $\partial_x f$ and $\partial_r f$ remain uniformly bounded for $(r,x) \in (r_p, r_0]$ for each of $f = \phi, \mu, \lambda$.

The first observation is that if $\mu(r,x) \leq A$, then $e^{2\mu} \leq e^{2A}$ and $\frac{Q^2}{r^2} e^{2\mu} \leq Q^2 r_p^{-2} e^{2A}$. So the coefficients appearing in the wave equations (2.5.12) and (2.5.11) are uniformly bounded. We apply this first to the wave equation (2.5.12) for ϕ. This equation can be shown to be equivalent to:

$$(\partial_r \pm e^{\mu-\lambda}\partial_x)(\partial_r\phi \mp e^{\mu-\lambda}\partial_x\phi) = \frac{1}{r}\left(-\frac{Q^2}{r^2} + \kappa\right) e^{2\mu}(\partial_r\phi \pm e^{\mu-\lambda}\partial_x\phi).$$

Therefore, if one defines

$$A(\tilde{r}) = \max\left\{ \sup_{x\in\mathbb{S}^1} |\partial_r\phi(\tilde{r},x) + e^{\mu-\lambda}\partial_x\phi(\tilde{r},x)|, \sup_{x\in\mathbb{S}^1} |\partial_r\phi(\tilde{r},x) - e^{\mu-\lambda}\partial_x\phi(\tilde{r},x)| \right\},$$

integrating the above integral curves of $\partial_r \pm e^{\mu-\lambda}\partial_x$ yields the integral estimate

$$A(r) \leq A(r_0) + r_p^{-1}(Q^2 r_p^{-2} + 1)e^{2A} \int_r^{r_0} A(\tilde{r})\, d\tilde{r}.$$

Thus applying Grönwall's inequality, there is a uniform bound $A(r) \leq A_\phi$ for all $r \in (r_p, r_0]$. By the definition of $A(r)$, we therefore have $|\partial_r\phi(r,x)| \leq A_\phi$ and $|e^{\mu-\lambda}\partial_x\phi(r,x)| \leq A_\phi$.

We now play the same game with (2.5.11). There are more terms arising in this equation, and for instance that the $e^{2(\mu-\lambda)}(\partial_x\phi)^2$ in (2.5.11) needs to be controlled by the previous estimate $|e^{\mu-\lambda}\partial_x\phi| \leq A_\phi$. Omitting the details, one finds A_μ so that $|\partial_r\mu(r,x)| \leq A_\mu$ and $|e^{\mu-\lambda}\partial_x\mu(r,x)| \leq A_\mu$ for $r \in (r_p, r_0]$.

It remains to estimate derivatives of λ, or equivalently derivatives of $\mu - \lambda$. The expression $\partial_r(\mu-\lambda)$ may be estimated immediately using (2.5.13). We also commute (2.5.13) with ∂_x, yielding:

$$r\partial_r\partial_x(\mu-\lambda) = \left(-\frac{Q^2}{r^2} + \kappa\right) e^{2\mu}(2\partial_x\mu) = 2\left(-\frac{Q^2}{r^2} + \kappa\right) e^{\mu+\lambda} \cdot e^{\mu-\lambda}\partial_x\mu.$$

Having estimated $|e^{\mu-\lambda}\partial_x\mu| \leq A_\mu$, it suffices to find an upper bound for $e^{\mu+\lambda}$. One does this by using (2.5.8)–(2.5.9) to show that $r\partial_r(\mu+\lambda) \geq 0$, and therefore $e^{\mu+\lambda}$ decreases as r decreases towards r_p. So $r\partial_r\partial_x(\mu-\lambda)$ is uniformly bounded, and thus one finds A_λ such that $|\partial_x(\mu-\lambda)| \leq A_\lambda$.

The final step is that the previous μ and ϕ estimates yielded $|e^{\mu-\lambda}\partial_x\phi| \leq A_\phi$ and $|e^{\mu-\lambda}\partial_x\mu| \leq A_\mu$. To conclude, one must remove the $e^{\mu-\lambda}$ prefactors, which is easy since we have an estimate for $|\partial_r(\mu-\lambda)|$ and therefore for $|\mu-\lambda|$ itself. So have uniform bounds for $|\partial_x f|$ and $|\partial_r f|$ for each of $f = \phi, \mu, \lambda$. By standard continuation criteria for wave–transport systems, this allows us to extend the solution beyond $r = r_p$, yielding the required contradiction. □

2.6 Gowdy symmetry

We complete Chapter 2 with a discussion of the other symmetry class for which we present results, namely $\mathbb{T}^2$-Gowdy symmetry. This will be used in the results of Chapter 5 where we study BKL bounces for the Einstein vacuum[4] equations in Gowdy symmetry.

A spacetime $(\mathcal{M}, \mathbf{g})$ is said to be *Gowdy symmetric* if it contains two spacelike Killing vector fields X and Y, which together generate an isometry group G_{Gowdy} whose orbits are topologically diffeomorphic to a two surface Σ^2, where the *twist constant* $C_K = \boldsymbol{\varepsilon}_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma Y^\delta$ is set to vanish identically, where $\boldsymbol{\varepsilon}$ is the volume form associated to $\mathbf{g}$. See [67, 36] for further references. We note that unlike the case of the surface symmetric spacetimes, it not required that each copy of Σ^2 is isometric to a rescaled version of Σ^2 with a standard metric.

We shall consider the particular case of the $\mathbb{T}^3$ Gowdy symmetric spacetimes, where X and Y can be chosen to be coordinate vector fields on $\Sigma^2 \cong \mathbb{T}^2$. Here, a Gowdy symmetric spacetime metric $\mathbf{g}$ can be written in the relatively simple form (see [124]):

$$\mathbf{g} = -t^{-\frac{1}{2}} e^{\frac{\lambda}{2}} (-dt^2 + d\theta^2) + t[e^P (d\sigma + Q d\delta)^2 + e^{-P} d\delta^2], \tag{2.6.1}$$

where (t, θ) are coordinates on the quotient manifold $\mathcal{Q} = \mathcal{M}/G_{Gowdy}$ and (σ, δ) are coordinates on each copy of $\mathbb{T}^2$. In this form, $X = \partial_\sigma$ and $Y = \partial_\delta$, meaning that the functions λ, P, Q are functions of (t, θ) only. Note that, since P and Q are allowed to be non-constant, each copy of $\mathbb{T}^2$ is a stretched / sheared version of the standard metric on $\mathbb{T}^2$.

Since the area of each copy of $\mathbb{T}^2$ has area proportional to $t > 0$, the gauge (2.6.1) is often referred to as an *areal gauge* for a $\mathbb{T}^3$ Gowdy symmetric spacetime. In any case, in this gauge we consider $t \in (0, +\infty)$ to be a time coordinate, while $\theta \in \mathbb{S}^1$, is our natural spatial coordinate. With respect to (2.6.1), the Einstein vacuum equations are written as:

$$(t\partial_t)^2 P - (t\partial_\theta)^2 P = e^{2P}(t\partial_t Q)^2 - e^{2P}(t\partial_\theta Q)^2, \tag{2.6.2}$$

$$(t\partial_t)^2 Q - (t\partial_\theta)^2 Q = -2(t\partial_t P)(t\partial_t Q) + 2(t\partial_\theta P)(t\partial_\theta Q) \tag{2.6.3}$$

together with two equations for λ.

$$t\partial_t \lambda = (t\partial_t P)^2 + (t\partial_\theta P)^2 + e^{2P}(t\partial_t Q)^2 + e^{2P}(t\partial_\theta Q)^2, \tag{2.6.4}$$

$$\partial_\theta \lambda = 2(t\partial_t P\, \partial_\theta P + e^{2P} t\partial_t Q\, \partial_\theta Q). \tag{2.6.5}$$

Note that the former equations (2.6.2) and (2.6.3) decouple from the latter two equations for λ, and in studying the evolutionary problem one usually solves for P and Q using (2.6.2) and (2.6.3), then afterwards one integrates (2.6.4) to solve for λ. The final equation (2.6.5) is a constraint that

[4] To the author's knowledge, the literature is scarce for Einstein equations with matter in Gowdy symmetric settings. We would like to return to the Einstein–(Maxwell)–scalar field equations in Gowdy in future work.

is propagated by the remaining equations, as long as it is satisfied at some initial time. Due to this, we typically study only the equations (2.6.2) and (2.6.3).

A remarkable feature of $\mathbb{T}^3$ Gowdy symmetric spacetimes is that the equations (2.6.2) and (2.6.3) have a hidden geometric structure. It turns out these two equations are exactly a system of *wave maps* (see [137, Chapter 6] for a general definition of wave maps) between a domain manifold $\mathbb{R}_{>0} \times \mathbb{T}^2$ with metric

$$g_0 = -dt^2 + d\theta^2 + t^2 d\chi^2,$$

and a target manifold $\mathbb{R}^2$ with metric

$$g_R = dP^2 + e^{2P} dQ^2.$$

More precisely, the equations (2.6.2)–(2.6.3) describe wave maps from $(\mathbb{R}_{>0} \times \mathbb{T}^2, g_0)$ to $(\mathbb{R}^2, g_R)$ which are independent of χ. We comment that $(\mathbb{R}^2, g_R)$ is isometric to hyperbolic space. Indeed, the change of variables $x = Q$, $y = e^{-P}$ puts g_R into the familiar upper half plane model $g_R = y^{-2}(dx^2 + dy^2)$. See [121, 123] and references therein for further details regarding the wave map structure. What is important for us is that, for such wave maps, it is straightforward to show that the energy functional:

$$\mathcal{E}(t) \coloneqq \frac{1}{2} \int_{\mathbb{S}^1} \left[(\partial_t P)^2(t,\theta) + (\partial_\theta P)^2(t,\theta) + e^{2P}(\partial_t Q)^2(t,\theta) + e^{2P}(\partial_\theta Q)^2(t,\theta) \right] d\theta \tag{2.6.6}$$

obeys the derivative identity:

$$\frac{d}{dt}\mathcal{E}(t) = -\frac{1}{t} \cdot \frac{1}{2} \int_{\mathbb{S}^1} [(\partial_t P)^2(t,\theta) + e^{2P}(\partial_t Q)^2(t,\theta)] \, d\theta,$$

so that $-\frac{1}{t}\mathcal{E}(t) \leq \frac{d}{dt}\mathcal{E}(t) \leq 0$ and $\mathcal{E}(t)$ thereby remains bounded within any compact interval in time, $t \in [T_0, T_1] \subset (0, +\infty)$.

Since the key evolutionary equations (2.6.2)–(2.6.3) is simply a (semilinear) system of wave maps, the initial value problem here is standard (see [124]), for initial data given by functions $P_D, \dot{P}_D, Q_D, \dot{Q}_D$ which will be presribed at $t = t_0 > 0$:

$$(P_D, \dot{P}_D, Q_D, \dot{Q}_D) = (P, t\partial_t P, Q, t\partial_t Q)|_{t=t_0}.$$

A consequence of the energy boundedness above is that in fact, for suitably regular initial data, the corresponding solution of the wave map system (2.6.2)–(2.6.3) always exists and remains regular in the entire interval $t \in (0, +\infty)$. That is, we have the following.

Proposition 2.6.1. *Let $k \geq 0$ be an integer, and suppose (P, Q) be is a (maximally extended) solution of the Gowdy wave maps system (2.6.2)–(2.6.3), arising from initial data $P_D, \dot{P}_D, Q_D, \dot{Q}_D$ at $t = t_0 > 1$, with $P_D, Q_D \in C^{k+1}(\mathbb{S}^1)$ and $\dot{P}_D, \dot{Q}_D \in C^k(\mathbb{S}^1)$. Then (P, Q) exist and have regularity C^{k+1} inside the entire domain $(t, \theta) \in (0, +\infty) \times \mathbb{S}^1$.*

In light of Proposition 2.6.1, the more interesting problem is instead to study the *asymptotic behaviour* of (P, Q) in the limits $t \to 0$ and $t \to \infty$. In particular, for the study of the Strong Cosmic Censorship Conjecture 1.1.3, a common strategy is to use such asymptotics to show geodesic completeness in the $t \to \infty$ direction and singularity formation, or more precisely curvature blowup, in the $t \to 0$ direction. We elaborate more in Section 5.1.

We conclude this section, and Chapter 2, with a definition of a subclass of the $\mathbb{T}^3$ Gowdy spacetimes, called the *polarized Gowdy spacetimes*.

Definition 2.6.1. A *polarized Gowdy spacetime* is a $\mathbb{T}^3$ Gowdy symmetric solution to the Einstein vacuum equation with $Q \equiv 0$ everywhere.

For polarized Gowdy spacetimes, the wave map system (2.6.2)–(2.6.3) then reduces to a single linear hyperbolic PDE:

$$(t\partial_t)^2 P - (t\partial_\theta)^2 P = 0. \tag{2.6.7}$$

This will be the starting point of our discussion in Chapter 5.

Chapter 3

Singularities in spherically symmetric spacetimes with scalar matter

In Chapter 3, we state and prove our main results regarding the *Kasner-like description of spacelike singularities in spacetimes with scalar matter*, based on my article [103]. Throughout Chapter 3, we shall refer to the spherically symmetric Einstein–(Maxwell)–scalar field equations in double null gauge (2.3.1)–(2.3.9), which we commonly shorten to the E(M)SFSS system.

3.1 Statement of main results and overview

3.1.1 The Hölder regularity norms

Before diving into the theorem statements, we first recall what it means for a function to be $C^{k,\beta}$, and define our notion of logarithmically modified Hölder regularity, $C^{k,\beta,\log}$.

Definition 3.1.1. For $\mathcal{D}$ bounded in $\mathbb{R}^2$ and $\beta \in (0,1)$, we define the $C^{0,\beta}$ norm of a function $f : \mathcal{D} \to \mathbb{R}$ to be[1]:

$$\|f\|_{C^{0,\beta}} \coloneqq \sup_{(u,v)\in\mathcal{D}} |f(u,v)| + \sup_{\substack{\{u\}\times[\tilde{v}_0,\tilde{v}_1]\\ \subset\mathcal{D}}} \frac{|f(u,\tilde{v}_1)-f(u,\tilde{v}_0)|}{|\tilde{v}_1-\tilde{v}_0|^\beta} + \sup_{\substack{[\tilde{u}_0,\tilde{u}_1]\times\{v\}\\ \subset\mathcal{D}}} \frac{|f(\tilde{u}_1,v)-f(\tilde{u}_0,v)|}{|\tilde{u}_1-\tilde{u}_0|^\beta}. \quad (3.1.1)$$

If $\|f\|_{C^{0,\beta}} < +\infty$ we say that f is *Hölder continuous* in $\mathcal{D}$, and can be defined up to and including the boundary ∂D (in a unique Hölder continuous fashion).

Similarly, define the logarithmically modified $C^{0,\beta,\log}$ norm of $f : \mathcal{D} \to \mathbb{R}$ to be

$$\begin{aligned}\|f\|_{C^{0,\beta,\log}} \coloneqq& \sup_{(u,v)\in\mathcal{D}} |f(u,v)| \\ &+ \sup_{\substack{\{u\}\times[\tilde{v}_0,\tilde{v}_1]\\ \subset\mathcal{D}}} \frac{|f(u,\tilde{v}_1)-f(u,\tilde{v}_0)|}{|\tilde{v}_1-\tilde{v}_0|^\beta(1+|\log(\tilde{v}_1-\tilde{v}_0)|)} + \sup_{\substack{[\tilde{u}_0,\tilde{u}_1]\times\{v\}\\ \subset\mathcal{D}}} \frac{|f(\tilde{u}_1,v)-f(\tilde{u}_0,v)|}{|\tilde{u}_1-\tilde{u}_0|^\beta(1+|\log(\tilde{u}_1-\tilde{u}_0)|)}. \quad (3.1.2)\end{aligned}$$

[1] As $\mathcal{D}$ is bounded in $\mathbb{R}^2$, it is straightforward to see that this is equivalent to the usual $C^{0,\beta}$ norm.

If $\|f\|_{C^{0,\beta,\log}} < +\infty$ we say that f is *log-Hölder continuous*.

Finally, we define f to be $C^{k,\beta}$ (respectively $C^{k,\beta,\log}$) for some $k \in \mathbb{N}$ if f is C^k and its kth derivatives $\partial_u^{k_1}\partial_v^{k_2} f$ with $k_1 + k_2 = k$ are each $C^{0,\beta}$ (respectively $C^{0,\beta,\log}$).

Remark. Since $1+\log x^{-1} \lesssim x^{-\gamma}$ for x small, it holds that for $0 < \beta' < \beta < 1$, there exists $C(\beta, \beta', \mathcal{D})$ such that $\|f\|_{C^{0,\beta'}} \leq C\|f\|_{C^{0,\beta,\log}}$. Hence log-Hölder continuity implies Hölder continuity for any smaller exponent.

3.1.2 Theorems 3.1.1 and 3.1.2: Asymptotics at singularity in double null coordinates

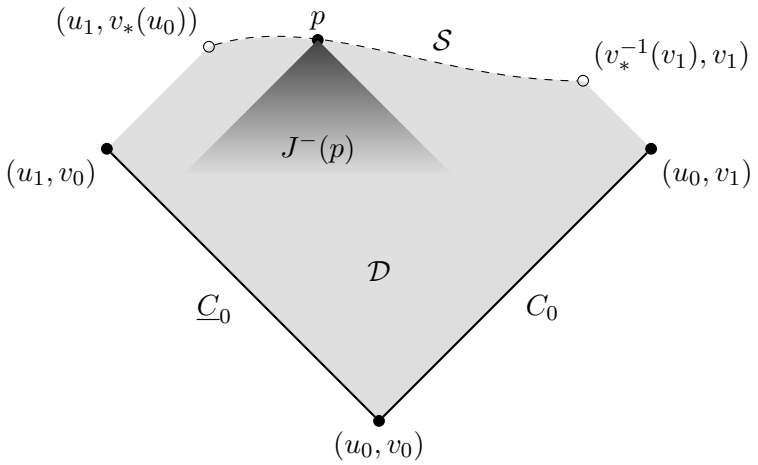


Figure 3.1: Theorems 3.1.1 and 3.1.2 give precise asymptotics for the dynamical quantities $r(u,v)$, $\Omega^2(u,v)$ and $\phi(u,v)$ inside the strongly singular maximal future development $\mathcal{D}$. Localizing to the TIP $J^-(p)$, one may use the values of Ψ, Ξ and $\mathfrak{M}$ (defined in Theorems 3.1.1 and 3.1.2) at the singular point $p \in \mathcal{S}$ to describe the asymptotics in $J^-(p)$.

We now state Theorems 3.1.1 and 3.1.2, which are the precise versions of Theorem 1.3.1 for the ESFSS and the EMSFSS system respectively. We encourage the reader to recall the notion of a strongly singular spacetime, and the gauge condition ($uv-$gauge), from Section 2.2, and often refer to Figure 3.1 for the Penrose diagram representing both results.

Theorem 3.1.1. *Consider characteristic initial data for the ESFSS system on the bifurcate null hypersurface $C_0 \cup \underline{C}_0$, obeying the gauge condition ($uv-$gauge). Suppose that the maximal development $\mathcal{D}$ is strongly singular, as described in Section 2.3.1. Then:*

(I) Asymptotics for r: The squared area-radius $r^2(u,v)$ is a $C^{1,1/2}$ function in $\mathcal{D}$. There exist constants $R_0, R_1 > 0$, such that for $(u,v) \in \mathcal{D}$,

$$R_0 \leq -r\partial_u r(u,v) \leq R_1, \qquad R_0 \leq -r\partial_v r(u,v) \leq R_1. \tag{3.1.3}$$

Consequently, we identify the $r = 0$ singularity as a $C^{1,1/2}$ curve, denoted $\mathcal{S}$, which we parameterize as $\mathcal{S} = \{(u, v_*(u)) : u_0 \leq u \leq v_*^{-1}(v_1)\}$. We define $\bar{\mathcal{D}} = \mathcal{D} \cup \mathcal{S}$, and by $C_u(p), C_v(p)$ the continuous limits of $-r\partial_u r, -r\partial_v r$ respectively at $p \in \mathcal{S}$.

(II) *Asymptotics for ϕ: There exists a $C^{0,1/2}$ function $\Psi(u, v)$ and a $C^{0,1/2,\log}$ function $\Xi(u, v)$ such that for $(u, v) \in \mathcal{D}$,*

$$\phi(u, v) = \Psi(u, v) \cdot \log\left(\frac{r_0}{r}\right) + \Xi(u, v). \tag{3.1.4}$$

We may extend Ψ and Ξ continuously to the singular boundary $\mathcal{S}$, and denote by $\Psi_\infty(p)$ and $\Xi_\infty(p)$ their values at $p \in \mathcal{S}$.

(III) *Asymptotics for Ω^2: There exists a $C^{0,1/2,\log}$ function $\mathfrak{M}(u, v)$, continuously extendible as a positive function on $\bar{\mathcal{D}}$, such that we have the following asymptotics for the gauge-invariant quantity $4\Omega^{-2}\partial_u r \partial_v r$:*

$$4\Omega^{-2}\partial_u r \partial_v r(u, v) = \mathfrak{M}(u, v) \cdot \left(\frac{r}{r_0}\right)^{-(\Psi^2+1)}. \tag{3.1.5}$$

We denote by $\mathfrak{M}_\infty(p)$ the value of $\mathfrak{M}$ at $p \in \mathcal{S}$.

(IV) *Estimates localized at a TIP: Let $p = (u_p, v_*(u_p)) \in \mathcal{S}$, and consider its causal past $J^-(p) = \{(u, v) \in \mathcal{D} : u \leq u_p, v \leq v_*(u_p)\}$. Then there exists a constant $D > 0$ such that for $(u, v) \in J^-(p)$:*

$$|-r\partial_u r(u, v) - C_u(p)| \leq D \cdot r, \tag{3.1.6}$$

$$|-r\partial_v r(u, v) - C_v(p)| \leq D \cdot r, \tag{3.1.7}$$

$$|r^2\partial_u \phi(u, v) - C_u(p) \cdot \Psi_\infty(p)| \leq D \cdot r, \tag{3.1.8}$$

$$|r^2\partial_v \phi(u, v) - C_v(p) \cdot \Psi_\infty(p)| \leq D \cdot r, \tag{3.1.9}$$

$$\left|\phi(u, v) - \Psi_\infty(p) \cdot \log\left(\frac{r_0}{r}\right) - \Xi_\infty(p)\right| \leq D \cdot r \log r^{-1}, \tag{3.1.10}$$

$$\left|\Omega^2 - \frac{4C_u(p)C_v(p)}{r_0^2} \cdot \mathfrak{M}_\infty^{-1}(p) \cdot \left(\frac{r}{r_0}\right)^{\Psi_\infty(p)^2-1}\right| \leq D \cdot r^{\Psi_\infty(p)^2} \log r^{-1}. \tag{3.1.11}$$

(V) *Precise blow-up estimates: For $p \in \mathcal{S}$ and $J^-(p)$ as in (IV), we have the following blow-up rates for the Hawking mass and the Kretschmann scalar:*

$$\left|2m(u, v) - \mathfrak{M}_\infty(p) r_0 \cdot \left(\frac{r_0}{r}\right)^{\Psi_\infty^2}\right| \leq D \cdot r^{-\Psi_\infty(p)^2+1} \log r^{-1}, \tag{3.1.12}$$

$$\left| \mathrm{Riem}_{\alpha\beta\gamma\delta}\mathrm{Riem}^{\alpha\beta\gamma\delta} - \mathfrak{M}_\infty(p)^2 \cdot \left(\frac{r_0}{r}\right)^{2(\Psi_\infty(p)^2+3)} \cdot \frac{4(3-2\Psi_\infty(p)^2+2\Psi_\infty(p)^4)}{r_0^4} \right| \leq D \cdot r^{-2\Psi_\infty(p)^2-5} \log r^{-1}. \quad (3.1.13)$$

For the scalar field, we have the following gauge-invariant blow-up rate:

$$\left| \nabla_\alpha\phi\nabla^\alpha\phi + \frac{\mathfrak{M}_\infty(p)\Psi_\infty(p)^2}{r_0^2} \cdot \left(\frac{r_0}{r}\right)^{\Psi_\infty(p)^2+3} \right| \leq D \cdot r^{-\Psi_\infty(p)^2-2} \log r^{-1}. \quad (3.1.14)$$

Theorem 3.1.2. *Consider characteristic initial data for the ESFSS system on the bifurcate null hypersurface $C_0 \cup \underline{C}_0$, obeying the gauge condition (uv−gauge). Suppose a priori that the maximal development $\mathcal{D}$ produced by this data terminates at an $r = 0$ spacelike singularity, and moreover that (QTS) and (SKE) hold in $\mathcal{D}$. Then for $\beta = \alpha/2$,*

(I) Asymptotics for r: The squared area-radius $r^2(u,v)$ is a $C^{1,\beta}$ function in $\mathcal{D}$. Building upon (QTS), we find an additional upper bound: for $(u,v) \in \mathcal{D}$, there exists some $R_1 > R_0$ such that:

$$R_0 \leq -r\partial_u r(u,v) \leq R_1, \qquad R_0 \leq -r\partial_v r(u,v) \leq R_1. \quad (3.1.15)$$

Consequently, we can identify the $r = 0$ singularity as a $C^{1,\beta}$ curve, denoted $\mathcal{S}$, which we parameterize by $\mathcal{S} = \{(u, v_*(u)) : u_0 \leq u \leq v_*^{-1}(v_1)\}$. We then define $\bar{\mathcal{D}}$, C_u and C_v as in (I) of Theorem 3.1.1.

(II) Asymptotics for ϕ: There exists a $C^{0,\beta}$ function $\Psi(u,v)$ and a $C^{0,\beta,\log}$ function $\Xi(u,v)$ such that for $(u,v) \in \mathcal{D}$,

$$\phi(u,v) = \Psi(u,v) \cdot \log\left(\frac{r_0}{r}\right) + \Xi(u,v). \quad (3.1.16)$$

We may extend Ψ and Ξ continuously to the singular boundary $\mathcal{S}$, and denote by $\Psi_\infty(p)$ and $\Xi_\infty(p)$ their values at $p \in \mathcal{S}$.

(III) Asymptotics for Ω^2: There exists a $C^{0,\beta,\log}$ function $\mathfrak{M}(u,v)$, continuously extendible as a positive function on $\bar{\mathcal{D}}$, such that we have the following asymptotics for the gauge-invariant quantity $4\Omega^{-2}\partial_u r\partial_v r$:

$$4\Omega^{-2}\partial_u r\partial_v r(u,v) = \mathfrak{M}(u,v) \cdot \left(\frac{r}{r_0}\right)^{-(\Psi^2+1)}. \quad (3.1.17)$$

We denote by $\mathfrak{M}_\infty(p)$ the value of $\mathfrak{M}$ at $p \in \mathcal{S}$.

(IV) Estimates localized at a TIP: Let $p = (u_p, v_*(u_p)) \in \mathcal{S}$, and consider its causal past $J^-(p) = \{(u,v) \in \mathcal{D} : u \leq u_p, v \leq v_*(u_p)\}$. Then there exists a constant $D > 0$ such that for

$(u, v) \in J^-(p)$*:*

$$| - r\partial_u r(u, v) - C_u(p)| \leq D \cdot r^\alpha, \tag{3.1.18}$$

$$| - r\partial_v r(u, v) - C_v(p)| \leq D \cdot r^\alpha, \tag{3.1.19}$$

$$|r^2\partial_u \phi(u, v) - C_u(p) \cdot \Psi_\infty(p)| \leq D \cdot r^\alpha, \tag{3.1.20}$$

$$|r^2\partial_v \phi(u, v) - C_v(p) \cdot \Psi_\infty(p)| \leq D \cdot r^\alpha, \tag{3.1.21}$$

$$\left|\phi(u, v) - \Psi_\infty(p) \cdot \log\left(\frac{r_0}{r}\right) - \Xi_\infty(p)\right| \leq D \cdot r^\alpha \log r^{-1}, \tag{3.1.22}$$

$$\left|\Omega^2 - \frac{4C_u(p)C_v(p)}{r_0^2} \cdot \mathfrak{M}_\infty^{-1}(p) \cdot \left(\frac{r}{r_0}\right)^{\Psi_\infty(p)^2-1}\right| \leq D \cdot r^{\Psi_\infty(p)^2-1+\alpha} \log r^{-1}. \tag{3.1.23}$$

(V) *Precise blow-up estimates: For $p \in \mathcal{S}$ and $J^-(p)$ as in (IV), we have the following blow-up rates for the Hawking mass and the Kretschmann scalar:*

$$\left|2m(u, v) - \frac{\mathfrak{M}_\infty(p)r_0}{2} \cdot \left(\frac{r_0}{r}\right)^{\Psi_\infty^2}\right| \leq D \cdot r^{-\Psi_\infty(p)^2+\alpha} \log r^{-1}, \tag{3.1.24}$$

$$\left|\mathrm{Riem}_{\alpha\beta\gamma\delta}\mathrm{Riem}^{\alpha\beta\gamma\delta} - \mathfrak{M}_\infty(p)^2 \cdot \left(\frac{r_0}{r}\right)^{2(\Psi_\infty(p)^2+3)} \cdot \frac{4(3 - 2\Psi_\infty(p)^2 + 2\Psi_\infty(p)^4)}{r_0^4}\right| \leq D \cdot r^{-2\Psi_\infty(p)^2-6+\alpha} \log r^{-1}. \tag{3.1.25}$$

We treat also the matter fields. For the scalar field ϕ, we have the following gauge-invariant blow-up rate:

$$\left|\nabla_\alpha \phi \nabla^\alpha \phi + \frac{\mathfrak{M}_\infty(p)\Psi_\infty(p)^2}{r_0^2} \cdot \left(\frac{r_0}{r}\right)^{\Psi_\infty(p)^2+3}\right| \leq D \cdot r^{-\Psi_\infty(p)^2-3+\alpha} \log r^{-1}, \tag{3.1.26}$$

while the electromagnetic 2-form F satisfies:

$$F_{\mu\nu}F^{\mu\nu} = -\frac{2Q^2}{r^4}. \tag{3.1.27}$$

3.1.3 Corollary 3.1.3: A BKL-like expansion

We now state Corollary 3.1.3, which interprets the strongly singular spacetimes of Theorem 3.1.1 and Theorem 3.1.2 in the context of the BKL ansatz. In particular, we define a foliation using the level sets of a time function τ, given by (3.1.28), and show that this foliation obeys several BKL-like properties. We refer the reader to [126] to justify how Parts 1 to 5 indeed detail a correspondence between our foliation and the BKL picture.

Note that throughout Corollary 3.1.3, we refer to error terms of the form $O(r^\alpha \log r^{-1})$. To see that these are genuine lower order terms, note that via (3.1.28), one has $O(r^\alpha \log r^{-1}) = O(\tau^{\frac{2\alpha}{\Psi^2+3}} \log \tau^{-1})$, and thereby converge to 0 as $\tau \to 0$.

Corollary 3.1.3. *Let $(\mathcal{M}, g)$ be a smooth, spherically symmetric, strongly singular spacetime solving the EMSFSS system, and described by either Theorem 3.1.1 or Theorem 3.1.2. Let Ψ, Ξ and $\mathfrak{M}$ be described by (II) and (III) in these theorems. Now define the function $\tau : \mathcal{D} \to \mathbb{R}$ by*

$$\tau := \frac{2r_0}{\Psi^2+3} \cdot \mathfrak{M}^{-1/2} \cdot \left(\frac{r}{r_0}\right)^{\frac{\Psi^2+3}{2}}. \tag{3.1.28}$$

We foliate the spacetime $(\mathcal{M}, g)$ by hypersurfaces of constant τ, denoted S_τ. Then there exists some τ_0 such that:

1. *The spacetime $\mathcal{M} \cap \{\tau \leq \tau_0\}$ is globally hyperbolic and foliated by the spacelike hypersurfaces S_τ. Furthermore, the time-function τ is asymptotically normalized in the sense that*

$$|g(\nabla\tau, \nabla\tau) + 1| \leq D\, r^\alpha \log r^{-1}. \tag{3.1.29}$$

2. *The foliation is also asymptotically CMC (constant mean curvature) in the sense that the mean curvature k with respect to the foliation obeys*

$$|\tau k - 1| \leq D\, r^\alpha \log r^{-1}. \tag{3.1.30}$$

3. *Define an orthonormal frame as follows: let $e_0 = n^{-1}\nabla\tau$ with $n = (-g(\nabla\tau, \nabla\tau))^{1/2}$ the ADM lapse function, let $e_2 = \frac{1}{r}\partial_\theta$ and $e_3 = \frac{1}{r\sin\theta}\partial_\phi$ be tangent to the 2-spheres of symmetry, and finally choose e_1 to complete the orthonormal basis with direction chosen such that $g(e_1, \partial_v) < 0$. Then the frame is approximately parallel transported in the following sense:*

$$\nabla_{e_0} e_i = c_i e_0, \qquad \text{where } c_i \text{ obeys } |\tau c_i| \leq D\, r^\alpha \log r^{-1}. \tag{3.1.31}$$

4. *Defining k_{ij} to be the second fundamental form of the embedding $S_\tau \hookrightarrow (\mathcal{M}, g)$, let $\mathcal{K}_i{}^j$ denote the respective expansion normalized Weingarten map, defined by $\mathcal{K}_i{}^j = k^{-1} \cdot k_i{}^j = (\operatorname{tr} k)^{-1} k_i{}^j$. Then the eigenvectors of $\mathcal{K}_i{}^j$ are exactly the frame vectors e_1, e_2, e_3, and their respective eigenvalues p_1, p_2 and p_3 satisfy*

$$p_1 = \frac{\Psi^2-1}{\Psi^2+3} + O(r^\alpha \log r^{-1}), \quad p_2 = p_3 = \frac{2}{\Psi^2+3} + O(r^\alpha \log r^{-1}). \tag{3.1.32}$$

5. *Finally, the scalar field has the following logarithmic behavior:*

$$\phi = \frac{2\Psi}{\Psi^2+3} \cdot \log \tau^{-1} + O(1), \qquad -\tau e_0 \phi = \frac{2\Psi}{\Psi^2+3} + O(r^\alpha \log r^{-1}). \tag{3.1.33}$$

3.1.4 Sketch of the proof

We briefly describe several of the key steps in the proof of Theorem 3.1.1 and Theorem 3.1.1. For simplicity, we focus on part (IV), i.e. estimates localized at single TIP $p \in \mathcal{S}$ and its causal past $J^-(p)$. The essential claims will be that for any $(u, v) \in J^-(p)$, there exists some $\alpha > 0$ such that:

(I) There exist positive constants $C_u, C_v > 0$ such that in $J^-(p)$, one has

$$-r\partial_u r = C_u + O(r^\alpha), \qquad -r\partial_v r = C_v + O(r^\alpha).$$

(II) There exist constants $\Psi_\infty, \Xi_\infty \in \mathbb{R}$ such that in $J^-(p)$, one has

$$r^2\partial_u\phi = C_u \cdot \Psi_\infty + O(r^\alpha), \qquad r^2\partial_v\phi = C_v \cdot \Psi_\infty + O(r^\alpha),$$

$$\phi = \Psi_\infty \log r^{-1} + \Xi_\infty + O(r^\alpha).$$

(III) There exists some $C_\Omega > 0$ such that in $J^-(p)$, one has

$$\Omega^2 = C_\Omega \cdot r^{\Psi_\infty^2 - 1} + O(r^{\Psi_\infty^2 - 1 + \alpha}).$$

In the full proof, each of these must be shown simultaneously. For the purpose of illustrating the ideas, we sketch a proof of each of (I)–(III) assuming the others. We start with (III), which is proven in Section 3.5.1, particularly Proposition 3.5.1. The idea is to rewrite the Raychaudhuri equation (see already (2.3.2)) as:

$$\partial_v \log\left(\frac{\Omega^2}{-\partial_v r}\right) = \frac{\partial_v r}{r} \cdot \left(\frac{r^2\partial_v\phi}{-r\partial_v r}\right)^2.$$

Assuming (I) and (II), the rightmost expression can be written as $\Psi_\infty^2 + O(r^\alpha)$. Therefore, subtracting $\Psi_\infty^2 \cdot \partial_v(\log r) = \Psi_\infty^2 \cdot \frac{\partial_v r}{r}$ from both sides, one finds

$$\partial_v \log\left(\frac{\Omega^2}{-\partial_v r} \cdot r^{-\Psi_\infty^2}\right) = \partial_v r \cdot O(r^{\alpha-1}).$$

The right hand side is now integrable towards $r = 0$, hence $\frac{\Omega^2}{-\partial_v r} \cdot r^{-\Psi_\infty^2}$ tends to some positive constant as $r \to 0$. Combining again with (I), and performing similar integrations in the u-direction, we indeed find (III).

Next, we move onto (I), see already (3.5.13) in Section 3.5.2 (note though that in the course of our actual proof we largely circumvent the need for (I) by using instead the gauge-invariant derivatives (2.3.10), introduced in Section 2.2). For this, we use the following wave-like evolution equation for $r^2(u, v)$:

$$\partial_u(-r\partial_v r) = \partial_v(-r\partial_u r) = \frac{\Omega^2}{4}\left(1 - \frac{Q^2}{r^2}\right).$$

But by (III), we may bound Ω^2 by a power of r, giving

$$\partial_u(-r\partial_v r) = \begin{cases} O(r^{\Psi_\infty^2 - 1}) & \text{if } Q \equiv 0, \\ O(r^{\Psi_\infty^2 - 3}) & \text{if } Q \not\equiv 0. \end{cases}$$

To integrate this, we change variables from u to r. This is where we use (QTS), which tells us that $du = O(r)dr$. Furthermore, when $Q \neq 0$, we require $\Psi_\infty^2 > 1$ in order for the integral of $O(r^{\Psi^2 - 3})$

in u not to diverge as $r \to 0$ – justifying why we need the assumption (SKE). If either $Q \equiv 0$, or $Q \neq 0$ but (QTS) and (SKE) hold, then we get (I) as claimed. Note the $O(r^\alpha)$ error term in (I) is not optimal, and that the true first-order correction will be $O(r^\gamma)$ with γ depending on Ψ_∞^2. However, for our goal of deriving the leading order asymptotics, our $O(r^\alpha)$ errors are sufficient.

Finally, we derive (II), giving asymptotics for the main dynamical quantity $\phi(u, v)$. This is the main objective of Section 3.4, particularly Proposition 3.4.4. There, we use the wave equation in the following form:

$$\partial_u(r^2\partial_v\phi) = -r\partial_v r \cdot \partial_u\phi + r\partial_u r \cdot \partial_v\phi = \tilde{X}\phi,$$

where $\tilde{X}$ is defined as $\tilde{X} = -r\partial_v r\partial_u + r\partial_u r\partial_v$. The procedure will be to first estimate $|\tilde{X}\phi|$ and then integrate this equation towards $r = 0$.

The fact $\tilde{X}\phi$ appears is crucial. If the vector field[2] $\tilde{X}$ were to be replaced by, say, the null vector field ∂_v, then anticipating (I) the modified right-hand side would obey $|\partial_v\phi| = O(r^{-2})$. However, r^{-2} is <u>not</u> integrable in u towards $r = 0$, recalling that $du = O(r)dr$ but one still sees the logarithmic divergence $\int r^{-2}du = \int O(r^{-1})dr = O(\log r^{-1})$. Fortunately, $\tilde{X}$ will be a "better" derivative than ∂_v; since $\tilde{X}r = 0$, $\tilde{X}$ is a "spatial" derivative tangent to the spacelike hypersurfaces of constant r, and is expected to behave better than "temporal derivatives" in line with the AVTD expectation of BKL.

Indeed, we shall obtain $|\tilde{X}\phi| \lesssim r^{-2+\alpha}$, the proof of which is the main focus of Section 3.4.2, particularly Proposition 3.4.3. Using this, we now integrate the above equation and find that $r^2\partial_v\phi = C_v \cdot \Psi_\infty + O(r^\alpha)$, for some $\Psi_\infty \in \mathbb{R}$. The remaining claims of (II) are then straightforward – noting that we use $|\tilde{X}\phi| \lesssim r^{-2+\alpha}$ again to show that the Ψ_∞ arising in the leading order terms of $r^2\partial_v\phi$ and $r^2\partial_u\phi$ coincide. This concludes our sketch of the proof of our main theorems.

3.2 Applications and future directions

Before moving onto the proof of Theorem 3.1.1, Theorem 3.1.2 and Corollary 3.1.3 in Sections 3.3 to 3.5, we first elaborate on a few further questions and applications stemming from our results.

3.2.1 Stability and instability of spacelike singularities

Our main results provides leading order asymptotics for all geometric and matter field quantities near a spacelike singularity $\mathcal{S} = \{r = 0\}$ <u>provided we already know it exists</u>. Returning to our initial discussion regarding the generality of such singularities, it is prudent to ask to what extent our results model singularities arising from a general class of initial data.

[2]The vector field $\tilde{X}$ is also a rescaled version of the Kodama vector field [97], most often used to find energy estimates in the exterior region, where r is instead a spacelike variable and $\tilde{X}$ is timelike.

For instance, one may consider stability properties of the strongly singular spacetimes described in Theorems 3.1.1 and 3.1.2. Even for perturbations in spherical symmetry, this is a subtle problem, since the whereabouts of the eventual singular boundary is only determined a posteriori, and one must track the location of the new singularity $\hat{\mathcal{S}}$ arising from perturbed data, in any choice of coordinate gauge.

In particular, to make quantitative comparisons between the perturbed spacetime and the background spacetime, we require a further gauge transformation $(u, v) \mapsto (\hat{u}(u), \hat{v}(v))$ such that the position of $\hat{\mathcal{S}}$ in the $(\hat{u}, \hat{v})$-plane coincides with that of $\mathcal{S}$ in the (u, v)-plane. We state the following theorem, which we do not prove in the current dissertation but we instead comment that it should follow from the methods used in the proof of Theorem 6.1.0 in Appendix B[3].

Theorem I (Stability – rough version)**.** *Let (r, Ω^2, ϕ, Q) be a spherically symmetric and strongly singular solution to the Einstein-Maxwell-scalar field system* (Ein)*,* (Scal)*,* (Max)*, as described in Theorems 3.1.1 and 3.1.2, with characteristic initial data $(r_i, \Omega_i^2, \phi_i, Q_i)$ given on $C_0 \cup \underline{C}_0$. The domain of existence $\mathcal{D}$ and spacelike singularity $\mathcal{S}$ are given by the following subsets in the (u, v)-plane:*

$$\mathcal{D} = \{(u, v) \in [u_0, u_1] \times [v_0, v_1] \subset \mathbb{R}^2 : v < v_*(u) \text{ when } v_*^{-1}(v_1) \leq u \leq u_1\},$$

$$\mathcal{S} = \{(u, v_*(u)) : v_*^{-1}(v_1) \leq u \leq u_1\}.$$

By Theorems 3.1.1 and 3.1.2, $v_*(u)$ is a $C^{1,\beta}$ curve in the (u, v)-plane.

Consider perturbed characteristic initial data on $C_0 \cup \underline{C}_0$ given by $(\hat{r}_i, \hat{\Omega}_i^2, \hat{\phi}_i, \hat{Q}_i)$, with the perturbation being of size ϵ in a suitable norm, and suppose further that either

(A) $Q_i \equiv \hat{Q}_i \equiv 0$, or

(B) the subcriticality assumption (SKE) holds[4] for the background solution (r, Ω^2, ϕ, Q).

Then for ϵ sufficiently small, the maximal future development $(\hat{r}, \hat{\Omega}^2, \hat{\phi}, \hat{Q})$ of the perturbed data, defined in a new domain of development $\hat{\mathcal{D}}$, is also strongly singular with spacelike singularity $\hat{\mathcal{S}}$ and satisfies the assumptions of Theorems 3.1.1 and 3.1.2.

Furthermore, after a gauge transformation $u \mapsto \hat{u}(u), v \mapsto \hat{v}(v)$, with $\|\hat{u}(u) - u\|_{C^{1,\beta}}, \|\hat{v}(v) - v\|_{C^{1,\beta}} \lesssim \epsilon$, the locations of $\hat{\mathcal{D}}$ and $\hat{\mathcal{S}}$ in the $(\hat{u}, \hat{v})$-plane coincide with that of $\mathcal{D}$ and $\mathcal{S}$ in the (u, v)-plane. Abusing notation somewhat, for $(u, v) \in \mathcal{D}$ as above we define $(f - \hat{f})(u, v)$ to be $f(u, v) - \hat{f}(\hat{u}^{-1}(u), \hat{v}^{-1}(v))$. If $\hat{\Psi}$, $\hat{\Xi}$ and $\hat{\mathfrak{M}}$ are the quantities arising in (I)–(III) of Theorems 3.1.1 and 3.1.2 corresponding to the perturbed spacetime, one finally has the quantitative stability estimate

$$\|r^2 - \hat{r}^2\|_{C^{1,\beta}} + \|\Psi - \hat{\Psi}\|_{C^0} + \|\Xi - \hat{\Xi}\|_{C^0} + \|\mathfrak{M} - \hat{\mathfrak{M}}\|_{C^0} \lesssim \epsilon.$$

[3]The gauge transformation $u \to \hat{u}(u), v \to \hat{v}(u)$ could be described, for instance, by concatenating two gauge transformations between double null gauge and the Rein gauge used in Theorem 6.1.0

[4]By a Cauchy stability argument, it is only necessary for (SKE) to hold near $\mathcal{S}$, which translates to $\inf_{p \in \mathcal{S}} \Psi_\infty^2(p) > 1 + \alpha$.

This stability theorem is consistent with the expectations of BKL, since either of (A) or (B) suggests that the right hand side of equation (1.2.5) in Section 1.2 is suitably small. It is more interesting to consider perturbations that violate both (A) and (B). Concretely, this means taking a strongly singular solution with $Q \equiv 0$ and having $\Psi^2 < 1$ somewhere near $\mathcal{S}$, but then perturbing with $\hat{Q} \not\equiv 0$.

The subtle nature of this problem is illustrated by the following scenario (see Figure 3.2): take the unperturbed spacetime to be a strongly singular portion $\mathcal{D}$ of the Schwarzschild interior, then choose the perturbation to be such that $Q \equiv \epsilon \neq 0$ but ϕ still vanishes. This perturbed spacetime is therefore a portion of the Reissner-Nordström interior (see Chapter 5.5 of [78] for the explicit Reissner-Nordström metric). However, the Reissner-Nordström interior contains no singularities; it instead possesses a bifurcate null Cauchy horizon. Hence the perturbed maximal development $\hat{\mathcal{D}}$ arising from compact data contains no singularities, and is smoothly extendible in all directions, no matter the smallness of ϵ.

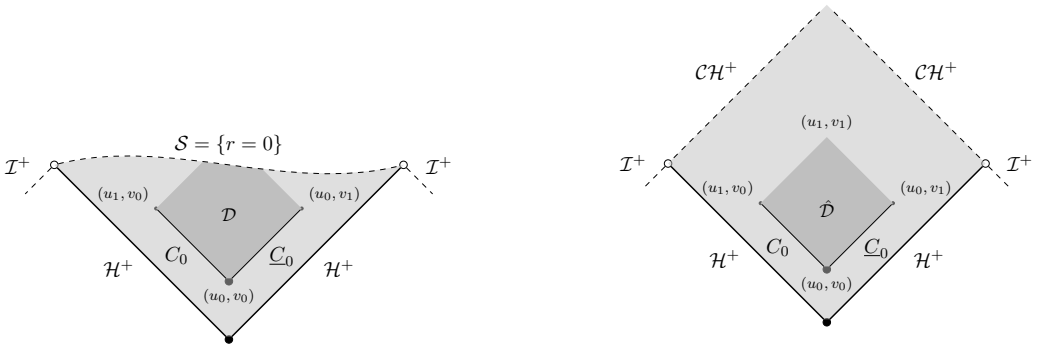


Figure 3.2: Penrose diagrams representing the Schwarzschild interior (left) and the Reissner-Nordström interior (right). The darker shaded region of the Schwarzschild interior represents a strongly singular spacetime in the sense of Theorems 3.1.1 and 3.1.2, whose singular boundary $\mathcal{S}$ is destroyed upon perturbations adding a non-trivial charge $Q \neq 0$. This effect is due to the presence of a (global) Cauchy horizon $\mathcal{CH}^+$ in the Reissner-Nordström interior.

This example illustrates that perturbations violating (A) and (B) can in fact destroy the very presence of the spacelike singularity, both locally and globally. In fact, using the methods of this paper, it is possible to qualitatively characterize the instability in our context. We argue that a charged perturbation of a strongly singular spacetime violating (SKE) will either fail to be singular, or contain a Kasner bounce such that the new singularity instead obeys the subcriticality assumption (SKE), so long as the double-null gauge remains reasonable in the sense that (QTS) still holds. The proof of the following corollary is omitted, but follows straightforwardly from the methods of Chapter 3.

Corollary II (Instability)**.** *Let (r, Ω^2, ϕ, Q) be a spherically symmetric and strongly singular solution to the spherically symmetric Einstein-Maxwell scalar field equations* (Ein)*,* (Scal)*,* (Max)*, such that*

$Q \equiv 0$. *Suppose furthermore that (SKE) is violated near the singular boundary* $\mathcal{S}$ *– in fact impose the stronger assumption that* $\sup_{p\in\mathcal{S}} \Psi^2_\infty(p) < 1$.

Given that the above spacetime arises from characteristic initial data $(r_i, \Omega_i^2, \phi_i, Q_i = 0)$ *on* $C_0 \cup \underline{C}_0$*, consider now perturbed initial data* $(\hat{r}_i, \hat{\Omega}_i^2, \hat{\phi}_i, \hat{Q}_i)$ *with* $\hat{Q}_i \neq 0$*. Upon solving (Ein), (Scal), (Max), the maximal future development of the perturbed data, given by functions* $(\hat{r}, \hat{\Omega}^2, \hat{\phi}, \hat{Q})$ *defined on a new domain of existence* $\hat{\mathcal{D}}$*, is such that one of the following holds:*

1. *(Disappearance of singularity.) The domain of existence is the entire causal rectangle* $\hat{\mathcal{D}} = [u_0, u_1] \times [v_0, v_1]$*, so that* $\inf_{\hat{\mathcal{D}}} r(u, v) > 0$ *and each of* $(\hat{r}, \hat{\Omega}^2, \hat{\phi}, \hat{Q})$ *are bounded and regular in* $\hat{\mathcal{D}}$*. The singular boundary* $\mathcal{S}$ *disappears.*

2. *(Gauge degeneration.) The perturbed spacetime remains singular in the sense that* $\inf_{\hat{\mathcal{D}}} r(u, v) = 0$*, but the quantitative trapped sphere condition (QTS) now fails to hold, i.e.*

$$\inf_{(u,v)\in\hat{\mathcal{D}}} \min\{-r\partial_u r(u, v), -r\partial_v r(u, v)\} = 0.$$

3. *(Bounce to subcritical regime.) The perturbed spacetime remains strongly singular and contains a new* $\hat{r} = 0$ *singular boundary* $\hat{\mathcal{S}}$*. The quantitative trapped sphere condition (QTS) still holds, however the new function* $\hat{\Psi}$ *quantitatively departs from that of the unperturbed spacetime, in the sense that*

$$\liminf_{r_*\to 0} \sup_{r(p)\leq r_*} \hat{\Psi}^2(p) \geq 1.$$

   Furthermore, if $\hat{\Psi}$ *can be shown to be continuously extendible to the boundary, and* $\hat{\Psi}_\infty$ *is its restriction to* $\mathcal{S}$*, then* $\inf_{p\in\mathcal{S}} \hat{\Psi}_\infty(p) > 1$*. In particular, (SKE) holds in the a sufficiently small neighborhood of* $\hat{\mathcal{S}}$ *in the perturbed spacetime.*

In each case of these three cases, some quantity will differ in a large sense from its value in the unperturbed spacetime, no matter how small $\epsilon \neq 0$ *may be.*

We now discuss circumstances in which each of the possibilities 1–3 may hold. An example of possibility 1 is the Schwarzschild to Reissner-Nordström perturbation described earlier. However, one could argue that this scenario is fine-tuned in the sense that ϕ is chosen to vanish identically in both the background spacetime and the perturbed spacetime, so the scalar field provides no access to subcriticality. It may be reasonable to expect that ϕ plays a significant role in the instability mechanism, and hence possibility 1 is conjecturally non-generic.

We turn next to possibility 2 in Corollary II. This is somewhat undesirable, as we cannot use our methods to describe the Kasner exponents at any point $p \in \hat{\mathcal{S}}$ at which (QTS) fails. Note for instance that the failure of (QTS) allows for portions of the $\hat{r} = 0$ singular boundary which are *null* rather than spacelike. Unfortunately, it is difficult to rule out the failure of (QTS), and one expects

to be able to find constructions where possibility 2 occurs. Nevertheless, one could hope that such instances are again non-generic, particularly in the context of Corollary II where the size ϵ of the perturbation is sufficiently small.

This leaves possibility 3, which suggests (at least in the case where $\hat{\Psi}$ remains bounded and continuously extendible to $\mathcal{S}$) the presence of Kasner bounces as described in Section 1.2. In the context of the Einstein-Maxwell scalar field system, this is perhaps the most interesting of the possibilities in Corollary II (and is expected to be the generic outcome when ϵ is small).

We do mention, for instance, that the results outlined in Section 1.4, namely Theorem 1.4.1, provides examples exhibiting a Kasner bounce which are spherically symmetric, spatially homogeneous solutions to the Einstein-Maxwell-*charged* scalar field system, referred to as the interior of hairy black holes. These spacetimes contain an $r = 0$ singular boundary $\mathcal{S}$, and the near-singularity region of the spacetimes features, in some cases, a "Kasner inversion" where a quantity morally equivalent to Ψ "inverts" from $\Psi \approx \alpha \in (0, 1)$ (in a non-subcritical Kasner-like regime $\mathcal{K}_1$), to $\Psi \approx \alpha^{-1} > 1$ in the final subcritical Kasner-like regime $\mathcal{K}_2$ containing the singularity $\mathcal{S}$. See already Figure 4.2.

The spatial homogeneity assumed in this construction (which is investigated in detail in Chapter 4 reduces the dynamics to a finite-dimensional ODE system. The inversion mechanism is then described via deriving a one-dimensional dynamical system for Ψ:

$$\frac{d\Psi}{dR} \approx -\Psi(\Psi - \alpha)(\Psi - \alpha^{-1}).$$

Here $R := \log r^{-1}$ is such that $R \to +\infty$ as $r \to 0$, and the inversion from $\Psi \approx \alpha$ to $\Psi \approx \alpha^{-1}$ is exactly the progression of Ψ away from the unstable fixed point α and towards the stable fixed point α^{-1}. We note, finally, that via the correspondence (1.3.3) between Ψ and the Kasner exponents p_1, p_2, p_3, the inversion $\alpha \mapsto \alpha^{-1}$ corresponds exactly to the Kasner map (1.2.8).

More precise and quantitative control of the instability possibility 3, particularly in the case that neither the background nor the perturbation are spatially homogeneous – even if one stays in spherical symmetry – remains interesting and open. One major difficulty here is the presence of spatial derivatives, forbidding one from staying in the realm of ODEs. We also refer the reader to Chapter 9 of this dissertation (see also Section 1.5, where we find a large class of spacetimes solving the Einstein–Maxwell–scalar field system – albeit written in the *Rein gauge* of Section 2.5 – exhibiting such bounces outside spatial homogeneity; furthermore, a subclass of these can be checked to dynamically verify the conditions (QTS) and (SKE) upon transforming back to double null gauge.

3.2.2 Applications to spherically symmetric collapse

As discussed, a familiar setting under which the system (Ein), (Scal), (Max) has been studied is that of one- or two-ended spherically gravitational collapse. We discuss first the $\mathbf{F} \equiv 0$ case, where

Christodoulou [31, 32, 35] shows that the generic Penrose diagram representing collapse of one-ended asymptotically flat data to a black hole is that of Figure 1.2.

Theorems 3.1.1 and 3.1.2 shows that for such spacetimes, we can associate to any trapped 2-sphere $(u, v) \in \mathcal{T} : \{(u, v) : \partial_u r(u, v) < 0, \partial_v r(u, v) < 0\}$, the functions Ψ, Ξ and $\mathfrak{M}$, which each extend in a continuous manner to functions Ψ_∞, Ξ_∞ and $\mathfrak{M}_\infty$ defined along $\mathcal{S}$. We focus on Ψ_∞, which determines the Kasner exponents associated to any $p \in \mathcal{S}$, via (1.3.3). There are two regions of the Penrose diagram which are of particular interest – the region near spacelike infinity i^+, and the region near the first singularity at the center b_0; each of these is depicted in Figure 3.3.

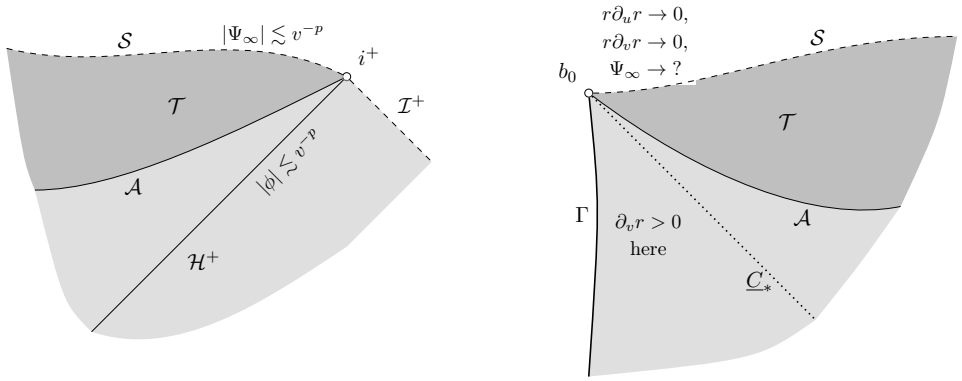


Figure 3.3: Zooming in on the portions of the singular boundary $\mathcal{S}$ near i^+ (left) and b_0 (right), indicating the relevance of Theorems 3.1.1 and 3.1.2 in these regions

For the purpose of studying the near-i^+ region, [3] uses Price's law decay of the scalar field ϕ along the event horizon $\mathcal{H}^+$, namely that for an outgoing Eddington-Finkelstein type coordinate v and some $p > 1$,

$$|\phi|_{\mathcal{H}^+}(v)| \lesssim v^{-p}. \tag{3.2.1}$$

Decay of the form (3.2.1) was first proven in the spherically symmetric setting in [47] – in fact [47] also allows for $\mathbf{F}_{\mu\nu} \neq 0$ so long as the black hole is sub-extremal in the limit.

Returning to $\mathbf{F}_{\mu\nu} = 0$, [3] propagates the decay (3.2.1) towards $\mathcal{S}$, showing that for the same coordinate v, one has $|\Psi_\infty(v)| \lesssim v^{-p}$ for large v. Via Theorems 3.1.1 and 3.1.2 it is thus reasonable to say that the singularity is "almost-Schwarzschildian" near i^+. Note, in particular, that (SKE) is necessarily violated in this region, and we expect instability should we turn on either electromagnetic or non-symmetric perturbations.

We turn now to the region near b_0. Though Theorems 3.1.1 and 3.1.2 attributes a value for Ψ_∞, and hence the Kasner exponents, for all $p \in \mathcal{S} \setminus \{b_0\}$, it is highly unclear what happens at the point b_0 itself. For b_0, one cannot apply the analysis presented in this article, since b_0 is not preceded by a trapped region – in fact $J^-(b_0) \cap \mathcal{T} = \emptyset$. Hence new techniques are needed to understand, for instance, if $\Psi_\infty(p)$ has a limit as $p \to b_0$. This may be related to the subtle issue of naked singularities, see also [33, 34, 35].

Moving on, we introduce a non-trivial Maxwell field $\mathbf{F}_{\mu\nu} \neq 0$. The Einstein-Maxwell-scalar field model considered in this article forces us to consider only two-ended asymptotically flat black hole spacetimes. For this model, [40, 41] and later [109, 110] show that the possible Penrose diagrams are as in Figure 3.4, and the black hole contains a non-empty Cauchy horizon $\mathcal{CH}^+$ across which the metric is C^0-extendible.

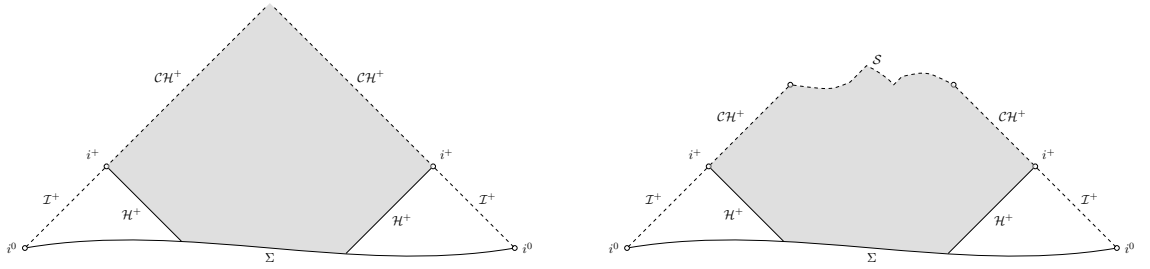


Figure 3.4: Penrose diagrams representing two-ended spherically symmetric gravitational collapse spacetime. The shaded regions correspond to the black hole interiors ($\mathcal{A}$ and $\mathcal{T}$ are no longer drawn), which always possess a non-empty Cauchy-horizon $\mathcal{CH}^+$ and in some cases an $r = 0$ spacelike singularity $\mathcal{S}$. If $\mathcal{S} \neq \emptyset$ as in the rightmost picture, it is unknown whether $\mathcal{S}$ could contain null pieces.

Since the Cauchy horizon is null and $r|_{\mathcal{CH}^+} \neq 0$, Theorems 3.1.1 and 3.1.2 does not apply there[5]. In fact, there are spacetimes e.g. perturbations of Reissner-Nordström [40] for which the Penrose diagram is as in the left of Figure 3.4. Nevertheless, for large data an $r = 0$ singularity $\mathcal{S}$ could arise, as in the right of Figure 3.4.

In such cases, Theorems 3.1.1 and 3.1.2 *could* apply, though we must be careful as (QTS) may not necessarily hold. In particular it is not clear a priori whether $\mathcal{S}$ could contain null segments. It is worth mentioning, however, that one could refer to the charged examples of Section 3.6.1 and apply a straightforward gluing argument[6] to guarantee that at least part of $\mathcal{S}$ obeys the assumptions of Theorems 3.1.1 and 3.1.2.

Finally, we mention the spherically symmetric Einstein-Maxwell-*charged* scalar field model. This matter model allows for one-ended gravitational collapse, while retaining the possibility of a non-empty Cauchy horizon. Therefore all of the issues mentioned here could arise. We refer the reader to [99, 143, 145, 92, 147] for further discussions of the black hole interior in this model.

3.2.3 The scattering problem for spacelike singularities

As in [9, 60, 61], one could take the approach of starting with asymptotic data on the singularity, and ask whether one can construct a singular solution to Einstein's equation (Ein) realizing such

[5]Even so, $\mathcal{CH}^+$ is generically singular in a C^2 sense, see [41, 109]. This is historically significant in showing that singularities in General Relativity are not always spacelike, in contrast to the original predictions of BKL.

[6]See [91] for a more detailed discussion of gluing for Einstein's equation in spherical symmetry.

asymptotics. In light of Theorems 3.1.1 and 3.1.2, the appropriate asymptotic data should be the functions $\Psi_\infty(p)$, $\Xi_\infty(p)$ and $\mathfrak{M}_\infty(p)$.

Unfortunately, there is one major shortcoming of Theorems 3.1.1 and 3.1.2 – namely that to have any hope of solving (Ein) backwards, one needs to obey the following *asymptotic momentum constraint* at $\mathcal{S}$:

$$\tilde{\nabla} \log \mathfrak{M}_\infty(p) = 2\Psi_\infty(p) \cdot \tilde{\nabla}\, \Xi_\infty(p), \tag{3.2.2}$$

where $\tilde{\nabla}$ is a suitable spatial derivative on $\mathcal{S}$. See Section 5 of [26] for heuristics deriving (3.2.2) from the usual momentum constraint. However, Theorems 3.1.1 and 3.1.2 gives only Hölder regularity for $\log \mathfrak{M}_\infty$ and Ξ_∞, hence (3.2.2) cannot even be interpreted meaningfully. Finding a suitable gauge, and notion of spatial derivative, such that (3.2.2) can be verified remains an open problem in the present setting.

Nonetheless, one could begin with smooth quantities $\Psi_\infty(p)$, $\Xi_\infty(p)$ and $\log \mathfrak{M}_\infty(p)$, satisfying (3.2.2). Then one expects, using methods from [9, 61], that one could reconstruct the strongly singular spacetime solving (Ein) and realizing Theorems 3.1.1 and 3.1.2 with Ψ_∞, Ξ_∞ and $\mathfrak{M}_\infty$ as given. We hope to return to this problem in the future. However, due to the above regularity issues, we are still far from a satisfactory scattering theory linking asymptotic data at the singularity to data on any spacelike or bifurcate null hypersurface.

3.3 Upper bounds

We now begin the proof of our main Theorems 3.1.1 and 3.1.2. In this section, we recover the upper bounds of [4], and generalize their results to the EMSFSS model with $Q \neq 0$. Throughout Section 3.3, D will represent a constant depending on the initial data at $C_0 \cup \underline{C}_0$, as well as the constants $R_0, \alpha > 0$ in (QTS), (SKE) in the case $Q \neq 0$.

3.3.1 Initial bounds on Ω^2 using Raychaudhuri

We begin with an upper bound for Ω^2 that will be crucial for integrability purposes. Note that at this point, we must already assume (QTS) and (SKE) when $Q \neq 0$. One may compare this to Section 3 of [4].

Lemma 3.3.1. *For $(u, v) \in \mathcal{D}$, one has the following upper bounds for Ω^2 and related quantities:*

(A) For the ESFSS system, one has

$$\max\left\{\frac{\Omega^2}{-\partial_u r}(u,v), \frac{\Omega^2}{-\partial_v r}(u,v)\right\} \leq D \tag{3.3.1}$$

as well as

$$\Omega^2(u,v) \leq \frac{D}{r(u,v)}. \tag{3.3.2}$$

(B) Now consider the EMSFSS system with $Q \neq 0$. Assuming (SKE), one has the stronger bound

$$\max\left\{\frac{\Omega^2}{-\partial_u r}(u,v), \frac{\Omega^2}{-\partial_v r}(u,v)\right\} \leq D\, r(u,v)^{1+\alpha}. \tag{3.3.3}$$

Assuming also (QTS), one further finds, with $r_0 = r(u_0, v_0)$:

$$\Omega^2(u,v) \leq D\, r(u,v)^{\alpha} \leq D\, r_0^{\alpha}. \tag{3.3.4}$$

Proof. (3.3.1) follows trivially from the Raychaudhuri equations (2.3.1), (2.3.2). Indeed, these imply that $-\Omega^{-2}\partial_u r$ and $-\Omega^{-2}\partial_v r$ are nondecreasing in u and v respectively, so the quantities on the left hand side of (3.3.1) are bounded by their respective maxima on $C_0 \cup \underline{C}_0$. To get (3.3.2), simply combine (3.3.1) with the lower bounds on $-r\partial_u r, -r\partial_v r$ from Lemma 2.3.1.

Moving on to the case (B) with $Q \neq 0$, we recast the Raychaudhuri equations as transport equations for the following logarithmic quantities:

$$\partial_u \log\left(\frac{\Omega^2}{-\partial_u r}\right) = \frac{\partial_u r}{r} \cdot \left(\frac{r\partial_u \phi}{\partial_u r}\right)^2 = \frac{\partial_u r}{r} \cdot (r^2 \underline{L}\phi)^2. \tag{3.3.5}$$

$$\partial_v \log\left(\frac{\Omega^2}{-\partial_v r}\right) = \frac{\partial_v r}{r} \cdot \left(\frac{r\partial_v \phi}{\partial_v r}\right)^2 = \frac{\partial_v r}{r} \cdot (r^2 L\phi)^2. \tag{3.3.6}$$

Now assume the lower bound (SKE). Since $\partial_u r, \partial_v r < 0$, (SKE) applied to (3.3.5) and (3.3.6) yields

$$\partial_u \log\left(\frac{\Omega^2}{-\partial_u r}\right) \leq \partial_u \log r \cdot (1+\alpha),$$

$$\partial_v \log\left(\frac{\Omega^2}{-\partial_v r}\right) \leq \partial_v \log r \cdot (1+\alpha).$$

From this, the expression $\frac{\Omega^2}{-\partial_u r} \cdot r^{-(1+\alpha)}$ is also monotonically decreasing in u. This observation, alongside the analogous observation with v in place of u, asserts that

$$\max\left\{\frac{\Omega^2}{-\partial_u r}(u,v) \cdot r(u,v)^{-(1+\alpha)}, \frac{\Omega^2}{-\partial_v r}(u,v) \cdot r(u,v)^{-(1+\alpha)}\right\} \leq D.$$

This is exactly (3.3.3), and the final estimate (3.3.4) follows from combining (3.3.3) and the lower bound (QTS) on $-r\partial_u r$ and $-r\partial_v r$. □

3.3.2 Upper bounds for $r^2 L\phi$ and $r^2 \underline{L}\phi$

We now apply the upper bounds on Ω^2 found in Section 3.3.1 to bound appropriately weighted derivatives of the scalar field ϕ. We use a similar Grönwall argument to that appearing in Section 6 of [4], though we instead opt to estimate the gauge-independent quantities $L\phi$ and $\underline{L}\phi$.

As in [4], we foliate our spacetime $\mathcal{D}$ using spacelike constant-r hypersurfaces. Define, for $\tilde{r} \leq r_0 = r(u_0, v_0)$, the hypersurface

$$\Sigma_{\tilde{r}} := \{(u,v) \in \mathcal{D} : r(u,v) = \tilde{r}\}.$$

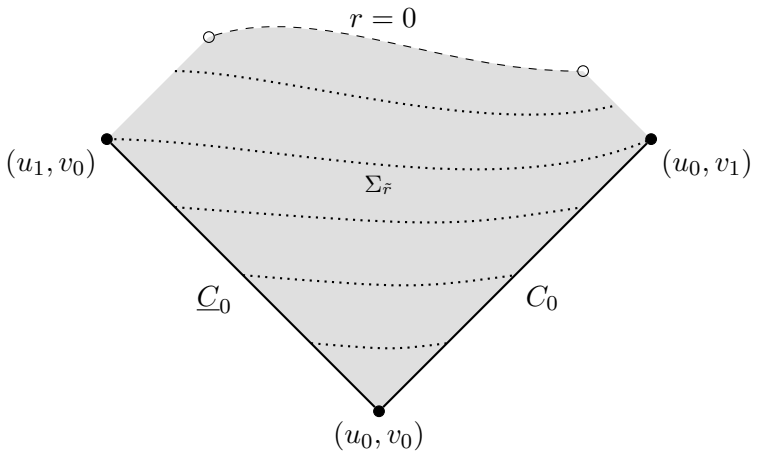


Figure 3.5: Foliation of $\mathcal{D}$ by constant-r hypersurfaces Σ_r, used in a Grönwall argument

We next use the EMSFSS system to determine null transport equations for the weighted quantities $rL\phi$ and $r\underline{L}\phi$. From the equations (2.3.8) and (2.3.6), one finds the following propagation equations:

$$\partial_u(rL\phi) = \frac{\partial_u\phi}{r} - \frac{\Omega^2}{-4r\partial_v r}\left(1 - \frac{Q^2}{r^2}\right)\cdot rL\phi, \tag{3.3.7}$$

$$\partial_v(r\underline{L}\phi) = \frac{\partial_v\phi}{r} - \frac{\Omega^2}{-4r\partial_u r}\left(1 - \frac{Q^2}{r^2}\right)\cdot r\underline{L}\phi. \tag{3.3.8}$$

The following proposition uses these to produce the required upper bounds for $r^2L\phi$ and $r^2\underline{L}\phi$.

Proposition 3.3.2. *Let $(u, v) \in \mathcal{D}$ for either (A) the ESFSS system with general initial data, or (B) the EMSFSS system supplemented with (QTS) and(SKE), one has*

$$|r^2L\phi(u, v)| \leq D, \qquad |r^2\underline{L}\phi(u, v)| \leq D. \tag{3.3.9}$$

Proof. Define the function $A(r)$ to be

$$A(\tilde{r}) = \sup_{\tilde{r}\leq\hat{r}\leq r_0} \max\left\{\max_{(u,v)\in\Sigma_{\hat{r}}} |rL\phi(u, v)|, \max_{(u,v)\in\Sigma_{\hat{r}}} |r\underline{L}\phi(u, v)|\right\}.$$

The objective is to show that $A(r) \leq D/r$ for some constant D depending only on data, R_0 and α.

We integrate (3.3.7) in the interval $[u_0, u]$; one finds after a change of variables $\tilde{u} \mapsto r(\tilde{u}, v)$ that

$$\begin{aligned} rL\phi(u, v) &= \int_{u_0}^{u}\left[\frac{\partial_u\phi}{r}(\tilde{u}, v) - \frac{\Omega^2}{-4r\partial_v r}\left(1 - \frac{Q^2}{r^2}\right)\cdot rL\phi(\tilde{u}, v)\right] d\tilde{u} + rL\phi(u_0, v), \\ &= \int_{r(u_0,v)}^{r(u,v)}\left[-\frac{1}{r}\cdot r\underline{L}\phi - \frac{\Omega^2}{-4r\partial_u r\partial_v r}\left(1 - \frac{Q^2}{r^2}\right)\cdot rL\phi\right] dr(\tilde{u}, v) + rL\phi(u_0, v). \end{aligned}$$

One deduces from this the integral inequality

$$|rL\phi(u,v)| \leq \int_{r(u,v)}^{r_0} \left[\frac{1}{\hat{r}} + \sup_{(\tilde{u},\tilde{v})\in\Sigma_{\hat{r}}} \frac{\Omega^2}{-4r\partial_u r\partial_v r}\cdot\left|1-\frac{Q^2}{r^2}\right|(\tilde{u},\tilde{v})\right]\cdot A(\hat{r})\,d\hat{r} + \sup_{\tilde{v}\in[v_0,v_1]}|rL\phi(u_0,\tilde{v})|.$$

A similar integral inequality can be found for $|r\underline{L}\phi(u,v)|$. Taking the maximum over $(u,v)\in\Sigma_{\tilde{r}}$, we find

$$A(\tilde{r}) \leq \int_{\tilde{r}}^{r_0} \left[\frac{1}{\hat{r}} + \sup_{(\tilde{u},\tilde{v})\in\Sigma_{\hat{r}}} \frac{\Omega^2}{-4r\partial_u r\partial_v r}\cdot\left|1-\frac{Q^2}{r^2}\right|(\tilde{u},\tilde{v})\right]\cdot A(\hat{r})\,d\hat{r} + \sup_{C_0\cup\underline{C}_0}\max\{|rL\phi|,|r\underline{L}\phi|\}. \tag{3.3.10}$$

The last term is controlled by data, and we now apply Grönwall's inequality. The $1/\hat{r}$ appearing in this inequality is responsible for the $1/\tilde{r}$ growth of $A(\tilde{r})$, while the term involving Ω^2 will be controlled by Lemma 3.3.1:

$$\frac{\Omega^2}{-4r\partial_u r\partial_v r}\cdot\left|1-\frac{Q^2}{r^2}\right| \lesssim \begin{cases} 1 & \text{if } Q=0, \\ r^{\alpha-1} & \text{if } Q\neq 0 \text{ but (QTS) holds.}\end{cases} \tag{3.3.11}$$

Since $\alpha>0$, this expression is integrable in r as $r\to 0$, and there exists some $\tilde{D}$, such that

$$\int_{\tilde{r}}^{r_0}\left[\frac{1}{\hat{r}} + \sup_{(\tilde{u},\tilde{v})\in\Sigma_{\hat{r}}} \frac{\Omega^2}{-4r\partial_u r\partial_v r}\cdot\left|1-\frac{Q^2}{r^2}\right|(\tilde{u},\tilde{v})\right]d\hat{r} \leq \log\left(\frac{r_0}{\tilde{r}}\right)+\tilde{D}. \tag{3.3.12}$$

Hence applying Grönwall to (3.3.10) and using this estimate (3.3.12), we find that

$$A(\tilde{r}) \leq e^{\tilde{D}}\cdot\sup_{C_0\cup\underline{C}_0}\max\{|rL\phi|,|r\underline{L}\phi|\}\cdot\frac{r_0}{\tilde{r}}. \tag{3.3.13}$$

Multiplying both sides by $\tilde{r}$, we have a uniform bound on $|r^2L\phi(u,v)|$ and $|r^2\underline{L}\phi(u,v)|$ as required. □

Remark. Proposition 3.3.2 provides upper bounds for the scalar field ϕ appearing in the ESFSS or EMSFSS systems. However, since only the wave equation (2.3.5) is used in the Grönwall argument, it follows from the proof that a similar estimate would holds if ϕ were any spherically symmetric scalar function evolving via to the scalar wave equation in the *fixed background* $(\mathcal{M},g)$.

Note that the r^{-2} growth of $L\phi,\underline{L}\phi$, and the associated logarithmic growth of ϕ itself, is ubiquitous in the study of scalar waves in backgrounds involving a Kasner-like spacelike singularity, see for instance [62, 1].

It will be useful in the sequel to control not only solutions of the scalar wave equation but also solutions of *inhomogeneous* wave equations in the background $(\mathcal{M},g)$, where the inhomogeneity has a prescribed blow-up rate. This will be the aim of the following Section 3.3.3.

3.3.3 Upper bound estimates for general inhomogeneous wave equations

We prove a result, analogous to Proposition 3.3.2, producing sharp upper bounds for scalar quantities ψ satisfying a (possibly inhomogeneous) wave equation in the region $\mathcal{D}$. This will be useful when commuting the wave equation with various vector fields, as will be necessary in Section 3.3.4 and Section 3.4.

Proposition 3.3.3. *Let ψ be a function on $\mathcal{D} \subset \mathbb{R}^2$ obeying the following inhomogeneous wave equation:*

$$\partial_u \partial_v \psi = -\frac{\partial_u r \cdot \partial_v \psi}{r} - \frac{\partial_v r \cdot \partial_u \psi}{r} + F. \tag{3.3.14}$$

Here F is a function of $(u, v) \in \mathcal{D}$, such that for some $\gamma \in \mathbb{R}$ and $F_0 > 0$ fixed, F satisfies:

$$\left| \frac{F(u,v)}{\partial_u r \, \partial_v r} \right| \leq F_0 \, r^{\gamma}(u, v). \tag{3.3.15}$$

Then there exists a constant C, depending only upon the initial data for (Ω^2, r, ϕ, Q) at $C_0 \cup \underline{C}_0$, the exponent γ, and the quantities R_0 and α in (QTS) and (SKE) when $Q \neq 0$, such that we have the following estimates for the first derivatives $L\psi$ and $\underline{L}\psi$:

$$|r^2 L\psi(u,v)| + |r^2 \underline{L}\psi(u,v)| \leq C \sup_{C_0 \cup \underline{C}_0} \max\{|rL\psi|, |r\underline{L}\psi|\} + CF_0 \cdot \begin{cases} 1 & \text{if } \gamma > -2, \\ \log\left(\frac{r_0}{r(u,v)}\right) & \text{if } \gamma = -2, \\ r(u,v)^{\gamma+2} & \text{if } \gamma < -2. \end{cases} \tag{3.3.16}$$

Furthermore, the quantity ψ itself obeys the following estimate:

$$|\psi(u,v)| \leq |\psi(u_0, v_0)| + C \sup_{C_0 \cup \underline{C}_0} \max\{|rL\psi|, |r\underline{L}\psi|\} \log\left(\frac{r_0}{r(u,v)}\right) + CF_0 \cdot \begin{cases} \log\left(\frac{r_0}{r(u,v)}\right) & \text{if } \gamma > -2, \\ \log^2\left(\frac{r_0}{r(u,v)}\right) & \text{if } \gamma = -2, \\ r(u,v)^{\gamma+2} & \text{if } \gamma < -2. \end{cases} \tag{3.3.17}$$

Proof. The proof follows that of Proposition 3.3.2. We first rewrite (3.3.14) as propagation equations for the quantities $rL\psi$ and $r\underline{L}\psi$. One finds

$$\partial_u(rL\psi) = +\frac{\partial_u \psi}{r} - \frac{\Omega^2}{-4\partial_v r}\left(1 - \frac{Q^2}{r^2}\right) \cdot rL\psi + \frac{F}{-\partial_v r}, \tag{3.3.18}$$

$$\partial_v(r\underline{L}\psi) = +\frac{\partial_v \psi}{r} - \frac{\Omega^2}{-4\partial_u r}\left(1 - \frac{Q^2}{r^2}\right) \cdot r\underline{L}\psi + \frac{F}{-\partial_u r}. \tag{3.3.19}$$

One now integrates (3.3.18) and (3.3.19) from C_0 and $\underline{C}_0$ respectively. Integrating (3.3.18) and changing variables to $r(\tilde{u}, v)$ as before yields:

$$rL\psi(u,v) = \int_{r(u_0,v)}^{r(u,v)} \left[-\frac{1}{r} \cdot r\underline{L}\psi - \frac{\Omega^2}{-4r\partial_u r \partial_v r} \left(1 - \frac{Q^2}{r^2}\right) \cdot rL\psi \right] dr(\tilde{u}, v) + rL\psi(u_0, v) + \int_{r(u_0,v)}^{r(u,v)} \frac{F}{-\partial_u r \partial_v r}\, dr(\tilde{u}, v). \quad (3.3.20)$$

We wish to find an integral inequality similar to (3.3.10). To simplify notation, we define

$$\beta(\tilde{r}) = \frac{1}{\tilde{r}} + \sup_{(\tilde{u},\tilde{v})\in\Sigma_{\tilde{r}}} \frac{\Omega^2}{-4r\partial_u r\partial_v r} \cdot \left|1 - \frac{Q^2}{r^2}\right| (\tilde{u}, \tilde{v}). \quad (3.3.21)$$

Due to (3.3.11), there exist constants $D_0 < D_1$ depending only on the data (r, Ω^2, ϕ, Q) at $C_0 \cup \underline{C}_0$ as before, such that we have the following bounds:

$$\frac{D_0}{\tilde{r}} \leq \beta(r) \leq \frac{D_1}{\tilde{r}}, \quad (3.3.22)$$

$$\frac{D_0 r_0}{\tilde{r}} \leq \gamma(\tilde{r}) \coloneqq \exp\left(\int_{\tilde{r}}^{r_0} \beta(\hat{r})\, d\hat{r} \right) \leq \frac{D_1 r_0}{\tilde{r}}. \quad (3.3.23)$$

We also use (3.3.15) to bound the rightmost term appearing in (3.3.20). To avoid having to distinguish cases further based on whether $\gamma \geq -1$, we write

$$\int_{r(u_0,v)}^{r(u,v)} \frac{F}{-\partial_u r \partial_v r}\, dr(\tilde{u}, v) \leq C F_0\, r(u,v)^{\min\{\gamma+1, -1/2\}}. \quad (3.3.24)$$

We can now write down the inequality analogous to (3.3.10) in Proposition 3.3.2. Define

$$A_\psi(\tilde{r}) = \sup_{\tilde{r}\leq\hat{r}\leq r_0} \max \left\{ \max_{(u,v)\in\Sigma_{\hat{r}}} |rL\psi(u,v)|, \max_{(u,v)\in\Sigma_{\hat{r}}} |r\underline{L}\psi(u,v)| \right\}.$$

Then using (3.3.20) and the corresponding equation for $r\underline{L}\psi$, as well as (3.3.24), one finds

$$A_\psi(\tilde{r}) \leq \int_{\tilde{r}}^{r_0} \beta(\hat{r}) A_\psi(\hat{r})\, d\hat{r} + P_0 + C F_0\, \tilde{r}^{\min\{\gamma+1,-1/2\}}. \quad (3.3.25)$$

Here $P_0 = \sup_{C_0\cup\underline{C}_0} \max\{|rL\psi|, |r\underline{L}\psi|\}$ is the term arising from the initial data for ψ.

Since the usual integral Grönwall inequality applied to (3.3.25) would produce a weaker estimate than the required (3.3.16), we instead use (3.3.25) to produce a differential inequality for the quantity

$$B_\psi(\tilde{r}) = \int_{\tilde{r}}^{r_0} \beta(\hat{r}) A_\psi(\hat{r})\, d\hat{r}.$$

From (3.3.25), we find the following inequality for $B_\psi(\tilde{r})$:

$$\frac{d}{d\tilde{r}} B_\psi(\tilde{r}) = -\beta(\tilde{r}) A_\psi(\tilde{r}) \geq -\beta(\tilde{r}) B_\psi(\tilde{r}) - P_0 - C F_0\, \beta(\tilde{r})\, \tilde{r}^{\min\{\gamma+1,-1/2\}}.$$

Defining $\gamma(\tilde{r})$ as in (3.3.23), one thus finds

$$\frac{d}{d\tilde{r}}(\gamma^{-1}(\tilde{r})B_\psi(\tilde{r})) \geq -P_0\gamma^{-1}(\tilde{r}) - CF_0\gamma^{-1}(\tilde{r})\beta(\tilde{r})\,\tilde{r}^{\min\{\gamma+1,-1/2\}}. \tag{3.3.26}$$

Integrating backwards in $\tilde{r}$, and noting from (3.3.22) and (3.3.23) that $\gamma^{-1}(\tilde{r}) \sim \tilde{r}$ and $\gamma^{-1}(\tilde{r})\beta(\tilde{r}) \sim 1$, one therefore finds that (where cases are distinguished based on the integrability of the right hand side of (3.3.26)):

$$\tilde{r}B_\psi(\tilde{r}) \leq CP_0\tilde{r} + CF_0 \cdot \begin{cases} 1 & \text{if } \gamma > -2, \\ \log\left(\frac{r_0}{\tilde{r}}\right) & \text{if } \gamma = -2, \\ \tilde{r}^{\gamma+2} & \text{if } \gamma < -2. \end{cases} \tag{3.3.27}$$

Inserting (3.3.27) in the equation (3.3.25), we find that $\tilde{r}B_\psi(\tilde{r})$ in the left hand side of (3.3.27) can be replaced by $\tilde{r}A_\psi(\tilde{r})$, at the cost of increasing the constant C. This estimate is then exactly the desired (3.3.16).

To find the final estimate (3.3.17) for ψ itself, note that the vector fields rL and $r\underline{L}$ are morally r-derivatives, taken in null directions. To make this concrete, given $G(r)$ any real function of r and $\zeta : \mathcal{D} \subset \mathbb{R}^2$ a C^1 function satisfying $|rL\zeta(u,v)|, |r\underline{L}\zeta(u,v)| \leq G(r(u,v))$, then

$$\begin{aligned} |\zeta(u,v)| &\leq |\zeta(u_0,v_0)| + |\zeta(u_0,v) - \zeta(u_0,v_0)| + |\zeta(u,v) - \zeta(u_0,v)|, \\ &\leq |\zeta(u_0,v_0)| + \int_{v_0}^{v} |\partial_v\zeta(u_0,\tilde{v})|\,d\tilde{v} + \int_{u_0}^{u} |\partial_u\zeta(\tilde{u},v)|\,d\tilde{u}, \\ &= |\zeta(u_0,v_0)| + \int_{r_0}^{r(u_0,v)} |rL\zeta(u_0,\tilde{v})|\,dr(u_0,\tilde{v}) + \int_{r(u_0,v)}^{r(u,v)} |r\underline{L}\zeta(\tilde{u},v)|\,dr(\tilde{u},v), \\ &\leq |\zeta(u_0,v_0)| + \int_{r(u,v)}^{r_0} G(\tilde{r})\,d\tilde{r}. \end{aligned} \tag{3.3.28}$$

Using (3.3.16) and applying (3.3.28) with $\zeta = \psi$, we deduce (3.3.17) as claimed. □

3.3.4 Higher order estimates

As in Section 7 of [4], it will be important in later sections to control higher derivatives of the quantities (r, Ω^2, ϕ). We start with the geometric quantities r and Ω^2:

Proposition 3.3.4. *We have the following estimates for derivatives of $\Omega^2(u,v)$:*

(A) For the ESFSS system, there exists a constant D such that

$$|\partial_u\Omega^2| \leq \frac{D}{r^3(u,v)}, \qquad |\partial_v\Omega^2| \leq \frac{D}{r^3(u,v)}. \tag{3.3.29}$$

(B) For the EMSFSS system supplemented with QTS) and (SKE), we have instead that:

$$|\partial_u\Omega^2| \leq \frac{D}{r^{4-\alpha}(u,v)}, \qquad |\partial_v\Omega^2| \leq \frac{D}{r^{4-\alpha}(u,v)}. \tag{3.3.30}$$

Moving onto estimates for derivatives of $r^2(u,v)$, we have

(A) For the ESFSS system, with D as above, we have

$$|\partial_u(-r\partial_v r)| = |\partial_v(-r\partial_u r)| \leq \frac{D}{r(u,v)}, \tag{3.3.31}$$

$$|\partial_u(-r\partial_u r)| \leq \frac{D}{r(u,v)}, \qquad |\partial_v(-r\partial_v r)| \leq \frac{D}{r(u,v)}. \tag{3.3.32}$$

(B) For the EMSFSS system, with D as above, we have

$$|\partial_u(-r\partial_v r)| = |\partial_v(-r\partial_u r)| \leq \frac{D}{r^{2-\alpha}(u,v)}, \tag{3.3.33}$$

$$|\partial_u(-r\partial_u r)| \leq \frac{D}{r^{2-\alpha}(u,v)}, \qquad |\partial_v(-r\partial_v r)| \leq \frac{D}{r^{2-\alpha}(u,v)}. \tag{3.3.34}$$

(C) In both cases, one also has

$$|\partial_u\partial_u r| \leq \frac{D}{r^3(u,v)}, \qquad |\partial_u\partial_v r| \leq \frac{D}{r^3(u,v)}, \qquad |\partial_v\partial_v r| \leq \frac{D}{r^3(u,v)}. \tag{3.3.35}$$

Remark. Regarding (3.3.29)–(3.3.35), due to (2.3.11) it is straightforward to derive similar estimates except with the derivatives ∂_u, ∂_v replaced by the gauge-invariant derivatives $\underline{L}, L$ respectively.

Proof. Consider the evolution equation (2.3.4) for $\log\Omega^2$. Using Lemma 3.3.1 and Proposition 3.3.2, we find from this equation the following estimate:

$$|\partial_u\partial_v \log\Omega^2| \lesssim \frac{1}{r^4}. \tag{3.3.36}$$

We now integrate (3.3.36) in the u direction; since $-r\partial_u r \geq R_0 \gtrsim 1$ we have

$$\begin{aligned} |\partial_v \log\Omega^2|(u,v) &\lesssim |\partial_v \log\Omega^2|(u_0,v) + \int_{u_0}^{u} \frac{1}{r^4}(\tilde{u},v)\, d\tilde{u}, \\ &\lesssim |\partial_v \log\Omega^2|(u_0,v) + \int_{r(u,v)}^{r(u_0,v)} \frac{1}{r^3}(\tilde{u},v)\, dr(\tilde{u},v) \lesssim \frac{1}{r^2(u,v)}, \end{aligned}$$

noting that the first term on the right hand side arises from data and can be ignored.

Writing $|\partial_v\Omega^2| = \Omega^2 \cdot |\partial_v \log\Omega^2|$ and using Lemma 3.3.1 to bound Ω^2, one deduces the $\partial_v\Omega^2$ estimates in (3.3.29) and (3.3.30). The $\partial_u\Omega^2$ estimates follow analogously.

We turn to the second derivative estimates for $r^2(u,v)$. (3.3.31) and (3.3.33) follow easily from the evolution equations (2.3.6) and (2.3.7), where Lemma 3.3.1 is used to bound the Ω^2 arising in these equations. On the other hand, to deal with $\partial_u(-r\partial_u r)$ for instance, we differentiate (2.3.7) in u to find:

$$\partial_v\partial_u(-r\partial_u r) = \frac{\partial_u\Omega^2}{4}\left(1 - \frac{Q^2}{r^2}\right) + \frac{\Omega^2}{4}\frac{Q^2\partial_u r}{r^3}. \tag{3.3.37}$$

Using (3.3.29) or (3.3.30) alongside Lemma 3.3.1, one therefore finds:

$$|\partial_v \partial_u(-r\partial_u r)| \leq \begin{cases} D\, r(u,v)^{-3} & \text{for ESFSS,} \\ D\, r(u,v)^{-4+\alpha} & \text{for EMSFSS, given (QTS), (SKE).} \end{cases} \tag{3.3.38}$$

Integrating (3.3.38) and changing the integration variable from v to r in a familiar fashion, one gets

$$|\partial_u(-r\partial_u r)| \leq \begin{cases} D\, r(u,v)^{-1} & \text{for ESFSS,} \\ D\, r(u,v)^{-2+\alpha} & \text{for EMSFSS, given (QTS), (SKE).} \end{cases} \tag{3.3.39}$$

An analogous estimate holds for $\partial_v(-r\partial_v r)$, yielding exactly (3.3.32) and (3.3.34). The final estimate (3.3.35) is straightforward from combining all previous estimates, e.g.

$$|\partial_u \partial_u r| = \frac{1}{r}\left|(\partial_u r)^2 + \partial_u(-r\partial_u r)\right| \lesssim \frac{1}{r^3}.$$

This completes the proof of Proposition 3.3.4. □

We now use Propositions 3.3.4 and 3.3.3 to derive estimates for second order derivatives of $\phi(u,v)$.

Proposition 3.3.5. *For either the ESFSS system or EMSFSS supplemented with (QTS), (SKE), there exists a constant D such that the second order derivatives of ϕ satisfy*

$$|\partial_u \partial_u \phi| \leq \frac{D}{r^4(u,v)}, \qquad |\partial_u \partial_v \phi| \leq \frac{D}{r^4(u,v)}, \qquad |\partial_v \partial_v \phi| \leq \frac{D}{r^4(u,v)}. \tag{3.3.40}$$

As in Proposition 3.3.4, we may find a similar estimate to (3.3.35) with ∂_u, ∂_v replaced by $\underline{L}, L$ respectively.

Proof. Commute the wave equation (2.3.5) for ϕ with the coordinate derivative ∂_u. One finds:

$$\partial_u \partial_v(\partial_u \phi) = -\frac{\partial_u r \cdot \partial_v(\partial_u \phi)}{r} - \frac{\partial_v r \cdot \partial_u(\partial_u \phi)}{r} + F_u. \tag{3.3.41}$$

Here the inhomogeneity F is given by:

$$F_u = \left(\frac{(\partial_u r)^2}{r^2} - \frac{\partial_u \partial_u r}{r}\right)\partial_v \phi + \left(\frac{\partial_u r \partial_v r}{r^2} - \frac{\partial_u \partial_v r}{r}\right)\partial_u \phi. \tag{3.3.42}$$

Using Proposition 3.3.4, particularly (3.3.35), and Proposition 3.3.2, one estimates

$$\left|\frac{F_u}{\partial_u r \partial_v r}\right| \lesssim \frac{1}{r^4}.$$

In light of (3.3.41), one now applies Proposition 3.3.3 with $\psi = \partial_u \phi$ and $\gamma = -4$ to find that

$$|r^2 L \partial_u \phi| + |r^2 \underline{L} \partial_u \phi| \lesssim \frac{1}{r^2}.$$

Since L and $\underline{L}$ are uniformly equivalent to ∂_v and ∂_u respectively, one therefore deduces the $\partial_u \partial_u \phi$ and $\partial_u \partial_v \phi$ estimate in (3.3.40). The $\partial_v \partial_v \phi$ estimate follows by commuting with ∂_v instead of ∂_u. □

3.4 Precise asymptotics for the scalar field

The aim of the present section will be to prove the parts of Theorems 3.1.1 and 3.1.2 concerning the scalar field ϕ. The key step will be to show that the weighted derivatives $r^2 L\phi$, $r^2 \underline{L}\phi$ can be extended in a $C^{0,\beta}$ fashion to the $r = 0$ boundary.

As discussed in Section 3.1, the key ingredient will be the following: treating instead $r^2\partial_v\phi = -r\partial_v r \cdot r^2 L\phi$ in this preliminary discussion, one has from (2.3.8) the equation

$$\partial_u(r^2\partial_v\phi) = -r\partial_v r \cdot \partial_u\phi + r\partial_u r \cdot \partial_v\phi.$$

The results of Section 3.3, particularly Proposition 3.3.2, imply that the right-hand side is $O(r^{-2})$, which is borderline non-integrable in the variable u as $r \to 0$.

Thankfully, the operator $\tilde{X} = -r\partial_v r \cdot \partial_v + r\partial_u r \cdot \partial_u$ turns out to be a better derivative than the standard coordinate derivatives ∂_u, ∂_v. Since $\tilde{X}r = 0$, the vector field $\tilde{X}$ is adapted to the constant-r spacelike hypersurfaces $\Sigma_{\tilde{r}}$, see Figure 3.5. Hence one expects that such a "spatial" derivative behaves better.

Indeed, we shall see from Proposition 3.4.3 that $\tilde{X}\phi$ is integrable up to the $r = 0$ singularity, allowing us to extend $r^2\partial_u\phi$ and $r^2\partial_v\phi$ to the singularity in a continuous manner. As in Section 3.3, the argument is streamlined upon replacing ∂_u and ∂_v by their gauge-independent counterparts $\underline{L}$ and L. In particular, instead of treating $\tilde{X}$ as above we will study the spatial vector field $X = L - \underline{L}$.

3.4.1 The vector field $X = L - \underline{L}$

Define the vector field X by

$$X = L - \underline{L} = \frac{1}{-r\partial_v r}\partial_v - \frac{1}{-r\partial_u r}\partial_u. \tag{3.4.1}$$

From (3.4.1), it immediately follows that $Xr = 0$, hence X is tangent to the constant-r hypersurfaces $\Sigma_{\tilde{r}}$. In particular, X is spacelike, and we can further compute

$$g(X, X) = -2g(L, \underline{L}) = \frac{\Omega^2}{r^2\partial_u r\,\partial_v r} > 0. \tag{3.4.2}$$

We wish to show that $X\phi$ obeys better estimates than those of $L\phi$, $\underline{L}\phi$ found in Proposition 3.3.2. For this purpose, we commute the wave equation (2.3.5) with X, and apply Proposition 3.3.3. We therefore first produce some preliminary estimates regarding the various commutator terms that arise.

Lemma 3.4.1. *For X as defined in (3.4.1), decompose the commutator vector field $[\partial_u, X]$ with respect to the coordinate basis ∂_u, ∂_v in the usual way:*

$$[\partial_u, X] = [\partial_u, X]^u\partial_u + [\partial_u, X]^v\partial_v,$$

and similarly for $[\partial_v, X]$ *and* $[L, X] = -[L, \underline{L}] = [\underline{L}, X]$. *Then there exists a constant* $D > 0$, *depending on the data at* $C_0 \cup \underline{C}_0$, *such that*

$$|[\partial_v, X]^u| + |[\partial_u, X]^v| + |[\partial_v, X]^u| + |[\partial_v, X]^v| + |[L, X]^u| + |[L, X]^v| \\ \leq \begin{cases} D\, r(u,v)^{-1} & \text{for ESFSS,} \\ D\, r(u,v)^{-2+\alpha} & \text{for EMSFSS, given (QTS), (SKE).} \end{cases} \tag{3.4.3}$$

Alternatively, one may decompose the commutator vector fields with respect to the basis $L, \underline{L}$:

$$[\partial_u, X] = [\partial_u, X]^{\underline{L}} \underline{L} + [\partial_v, X]^L L,$$

and similarly for $[\partial_v, X]$ *and* $[L, X]$. *Then (3.4.3) holds with the* u, v *indices replaced by* $\underline{L}, L$ *indices.*

Proof. This will follow from the higher derivative estimates of Section 3.3.4. Using (3.4.1), we have

$$[\partial_u, X] = \partial_u \left(\frac{1}{-r\partial_v r} \right) \partial_v - \partial_u \left(\frac{1}{-r\partial_u r} \right) \partial_u,$$

$$= -\frac{\partial_u(-r\partial_v r)}{(-r\partial_v r)^2} \partial_v + \frac{\partial_u(-r\partial_u r)}{(-r\partial_u r)^2} \partial_u.$$

Hence we may read off $[\partial_u, X]^v$ and $[\partial_v, X]^u$, and bound these using Proposition 3.3.4 and the lower bounds for $-r\partial_u r$ and $-r\partial_v r$. The commutators $[\partial_v, X]$ and $[L, X]$ are dealt with similarly.

For the final statement of the lemma, note that changing basis from ∂_u, ∂_v to $\underline{L}$, L simply involves multiplying and dividing by $-r\partial_u r$ and $-r\partial_v r$. Since these are upper and lower bounded (see Section 2.3.1 and (QTS)), the estimate (3.4.3) still holds when computing components of the various commutator vector fields with respect to the L, $\underline{L}$ basis. □

We also need to consider commutation with the second order wave operator $\partial_u \partial_v$:

Lemma 3.4.2. *We write the second order operator* $[\partial_u \partial_v, X]$ *in the form*

$$[\partial_u \partial_v, X] = \mathcal{X}_{uu} \partial_u \partial_u + \mathcal{X}_{uv} \partial_u \partial_v + \mathcal{X}_{vv} \partial_v \partial_v + \mathcal{X}_u \partial_u + \mathcal{X}_v \partial_v, \tag{3.4.4}$$

where $\mathcal{X}_{uu}, \mathcal{X}_{uv}, \mathcal{X}_{vv}, \mathcal{X}_u, \mathcal{X}_v$ *are each scalar functions of* $(u, v) \in \mathcal{D}$. *Then we have the following bounds:*

$$|\mathcal{X}_{uu}| + |\mathcal{X}_{uv}| + |\mathcal{X}_{vv}| \leq \begin{cases} D\, r(u,v)^{-1} & \text{for ESFSS,} \\ D\, r(u,v)^{-2+\alpha} & \text{for EMSFSS, given (QTS), (SKE),} \end{cases} \tag{3.4.5}$$

$$|\mathcal{X}_u| + |\mathcal{X}_v| \leq \begin{cases} D\, r(u,v)^{-3} & \text{for ESFSS,} \\ D\, r(u,v)^{-4+\alpha} & \text{for EMSFSS, given (QTS), (SKE).} \end{cases} \tag{3.4.6}$$

Proof. Once again, we may write explicit formulae for each of $\mathcal{X}_{uu}, \mathcal{X}_{uv}, \mathcal{X}_{vv}, \mathcal{X}_u, \mathcal{X}_v$, then appeal to (3.3.4). Using $[\partial_u\partial_v, X] = \partial_u \circ [\partial_v, X] + [\partial_u, X] \circ \partial_v$, we find that:

$$\mathcal{X}_{uu} = [\partial_v, X]^u = \partial_v\left(\frac{-1}{-r\partial_u r}\right),$$

$$\mathcal{X}_{uv} = [\partial_v, X]^v + [\partial_u, X]^u = \partial_v\left(\frac{1}{-r\partial_v r}\right) + \partial_u\left(\frac{-1}{-r\partial_u r}\right),$$

$$\mathcal{X}_{vv} = [\partial_u, X]^v = \partial_u\left(\frac{1}{-r\partial_v r}\right),$$

$$\mathcal{X}_u = \partial_u([\partial_v, X]^u) = \partial_u\partial_v\left(\frac{-1}{-r\partial_u r}\right),$$

$$\mathcal{X}_v = \partial_u([\partial_v, X]^v) = \partial_u\partial_v\left(\frac{1}{-r\partial_v r}\right).$$

In particular, (3.4.5) follows immediately from (3.4.3), while for (3.4.6), we first use the equations (2.3.6) and (2.3.7). For instance, we rewrite $\mathcal{X}_u$ using (2.3.7) as

$$\mathcal{X}_u = \partial_u\left(\frac{1}{(-r\partial_u r)^2}\cdot\frac{\Omega^2}{4}\left(1-\frac{Q^2}{r^2}\right)\right).$$

The ∂_u derivative could hit any of $(-r\partial_u r)^{-1}$, Ω^2 or r^{-1}. However, we see from Lemma 3.3.1 and Proposition 3.3.4 that in any of these cases, the result remains bounded by the right hand side of (3.4.6). The estimate for $\mathcal{X}_v$ is analogous, concluding the proof of Lemma 3.4.2. □

3.4.2 Estimates on $X\phi$

In this section we prove Proposition 3.4.3, producing upper bounds on $X\phi$ and its derivatives. One achieves this by commuting (2.3.5) with the vector field X, then appealing to Proposition 3.3.3. For convenience, we record (2.3.5) again here:

$$\partial_u\partial_v\phi = -\frac{\partial_u r\cdot\partial_v\phi}{r} - \frac{\partial_v r\cdot\partial_u\phi}{r}. \tag{3.4.7}$$

Proposition 3.4.3. *For the ESFSS system or the EMSFSS system supplemented with the additional assumptions (QTS), (SKE), we have the following upper bounds for $X\phi$ and its weighted derivatives:*

$$\begin{aligned}&|X\phi(u,v)| + |r^2 LX\phi(u,v)| + |r^2\underline{L}X\phi(u,v)|\\ &\qquad\leq \begin{cases} D\, r^{-1}(u,v) & \text{for ESFSS,}\\ D\, r^{-2+\alpha}(u,v) & \text{for EMSFSS, given (QTS), (SKE).}\end{cases}\end{aligned} \tag{3.4.8}$$

Proof. In light of Proposition 3.3.3, we seek an equation of the form (3.3.14), with $\psi = X\phi$. We apply X to equation (3.4.7); since $Xr = 0$ we find

$$\partial_u \partial_v X\phi = -\frac{\partial_u r \cdot \partial_v X\phi}{r} - \frac{\partial_v r \cdot \partial_u X\phi}{r} + F_X, \tag{3.4.9}$$

where the inhomogeneous term F_X is given exactly by:

$$F_X = [\partial_u \partial_v, X]\phi + \frac{\partial_u r}{r} \cdot [\partial_v, X]\phi + \frac{\partial_v r}{r} \cdot [\partial_u, X]\phi - \frac{[\partial_u, X]r}{r} \cdot \partial_v \phi - \frac{[\partial_v, X]r}{r} \cdot \partial_u \phi. \tag{3.4.10}$$

It simply remains to estimate this inhomogeneous term F_X using the results of Section 3.4.1. We introduce some schematic notation; we write $\mathcal{X}_{**}$ to represent some element of $\{[\partial_u, X]^u, [\partial_u, X]^v, [\partial_v, X]^u, [\partial_v, X]^v\}$ or any linear combination thereof. Note that $\mathcal{X}_{uu}, \mathcal{X}_{uv}, \mathcal{X}_{vv}$ are included in $\mathcal{X}_{**}$, see the proof of Lemma 3.4.2. Likewise, denote $\mathcal{X}_*$ to represent one of $\mathcal{X}_u$ or $\mathcal{X}_v$ as in Lemma 3.4.2. Finally, denote ∂_* to be one of the two null coordinate derivatives ∂_u, ∂_v.

Using this notation and Lemma 3.4.2, F_X can be written in the schematic form:

$$F_X = \mathcal{X}_* \cdot \partial_* \phi + \mathcal{X}_{**} \cdot \partial_* \partial_* \phi + \mathcal{X}_{**} \cdot \frac{\partial_* r}{r} \cdot \partial_* \phi. \tag{3.4.11}$$

Now, from Lemmas 3.4.1 and 3.4.2, we may deduce that

$$|r^2 \mathcal{X}_*| + |\mathcal{X}_{**}| \lesssim \begin{cases} r^{-1}(u,v) & \text{for ESFSS,} \\ r^{-2+\alpha}(u,v) & \text{for EMSFSS, given (QTS), (SKE).} \end{cases}$$

We also need to control $\partial_* r$, $\partial_* \phi$ and $\partial_* \partial_* \phi$. The upper bounds for $-r\partial_* r$ give that $-\partial_* r \lesssim r^{-1}$, while derivatives of ϕ are controlled using Proposition 3.3.2 for $\partial_* \phi$ and Proposition 3.3.5 for $\partial_* \partial_* \phi$. These yield

$$|\partial_* \phi| \lesssim r^{-2}, \qquad |\partial_* \partial_* \phi| \lesssim r^{-4}.$$

Applying all of these bounds to (3.4.11), we find that

$$|F_X| \lesssim \begin{cases} r^{-5}(u,v) & \text{for ESFSS,} \\ r^{-6+\alpha}(u,v) & \text{for EMSFSS, given (QTS), (SKE).} \end{cases}$$

Using again lower bounds for $-r\partial_* r$, we therefore have that

$$\left| \frac{F_X}{\partial_u r \, \partial_v r} \right| \leq F_{X,0} \cdot r^{\gamma}(u,v). \tag{3.4.12}$$

Here $F_{X,0}$ is some constant depending only on data, while $\gamma = -3$ and $\gamma = -4 + \alpha$ in the context of the ESFSS and the EMSFSS system respectively. We are now in a position to apply Proposition 3.3.3 to (3.4.9).

Since we have $\gamma < -2$ in both cases, the final term on the right hand sides of (3.3.16) and (3.3.17) arising from the inhomogeneity dominate any terms arising from the data for $\psi = X\phi$ at $C_0 \cup \underline{C}_0$. Therefore the estimates (3.3.16) and (3.3.17) applied to $\psi = X\phi$ yield (3.4.8), as required. $\square$

3.4.3 Asymptotics for the scalar field at the singularity

Armed with Proposition 3.4.3, we now prove the asymptotics for ϕ given in Theorems 3.1.1 and 3.1.2, including the appropriate Hölder continuity for the quantities Ψ and Ξ.

Proposition 3.4.4. *Let (r, Ω^2, ϕ, Q) be a strongly singular solution to the EMSFSS system with an $r = 0$ spacelike singularity as in Section 2.3.1, where we assume (QTS), (SKE) to hold if $Q \neq 0$. Define the functions $\Psi : \mathcal{D} \to \mathbb{R}$ and $\Xi : \mathcal{D} \to \mathbb{R}$ to be:*

$$\Psi(u, v) \coloneqq r^2 \cdot \frac{(L + \underline{L})\phi(u, v)}{2}, \tag{3.4.13}$$

$$\Xi(u, v) \coloneqq \phi(u, v) - \Psi(u, v) \cdot \log\left(\frac{r_0}{r(u, v)}\right). \tag{3.4.14}$$

Then Ψ is $C^{0,\beta}$ in $\mathcal{D}$, with Hölder exponent $\beta = 1/2$ for the ESFSS system and $\beta = \alpha/2$ for the EMSFSS system with (QTS), (SKE). Ξ, on the other hand, is $C^{0,\beta,\log}$ (see (3.1.2) for a definition of log-Hölder continuity).

In fact, $\|\Psi\|_{C^{0,\beta}}$ and $\|\Xi\|_{C^{0,\beta,\log}}$ are bounded by a constant D depending on the data:

$$\|\Psi\|_{C^{0,\beta}} + \|\Xi\|_{C^{0,\beta,\log}} \leq D. \tag{3.4.15}$$

Proof. We study the derivatives of the quantities Ψ and Ξ. Using the wave equation (3.4.7), we produce some null transport equations for $r^2 L\phi$ and $r^2 \underline{L}\phi$. Using (2.3.8) as well as (2.3.6), one finds

$$\underline{L}(r^2 L\phi) = -X\phi + \frac{\Omega^2}{4r^2 \partial_u r \partial_v r}\left(\frac{Q^2}{r^2} - 1\right) \cdot r^2 L\phi. \tag{3.4.16}$$

One finds an analogous equation for $r^2 \underline{L}\phi$:

$$L(r^2 \underline{L}\phi) = X\phi + \frac{\Omega^2}{4r^2 \partial_u r \partial_v r}\left(\frac{Q^2}{r^2} - 1\right) \cdot r^2 \underline{L}\phi. \tag{3.4.17}$$

As Ψ contains both L and $\underline{L}$ derivatives, it is useful to also include the following null transport equations for $r^2 X\phi$. Using that $Lr^2 = \underline{L}r^2 = -2$, we have:

$$\underline{L}(r^2 X\phi) = r^2 \underline{L} X\phi - 2X\phi, \tag{3.4.18}$$

$$L(r^2 X\phi) = r^2 L X\phi - 2X\phi. \tag{3.4.19}$$

By Proposition 3.4.3 and Lemma 3.3.1, all the terms on the right hand sides of (3.4.16), (3.4.17), (3.4.18), (3.4.19) are $O(r^{-1})$ when $Q = 0$ and $O(r^{-2+\alpha})$ when $Q \neq 0$. This implies that these equations can be integrated up to $r = 0$. These equations, along with the definition (3.4.13) for Ψ, yield

$$\underline{L}\Psi = -\frac{1}{2} r^2 \underline{L} X\phi + \frac{\Omega^2}{4r^2 \partial_u r \, \partial_v r}\left(\frac{Q^2}{r^2} - 1\right) \cdot r^2 \underline{L}\phi. \tag{3.4.20}$$

$$L\Psi = \frac{1}{2} r^2 L X\phi + \frac{\Omega^2}{4r^2 \partial_u r \, \partial_v r}\left(\frac{Q^2}{r^2} - 1\right) \cdot r^2 L\phi. \tag{3.4.21}$$

Applying Proposition 3.4.3, there exists a constant D, depending on the data, R_0 and α, such that

$$|\underline{L}\Psi| + |L\Psi| \leq D\, r(u,v)^{-2+2\beta}. \tag{3.4.22}$$

Here β is as defined in the statement of Proposition 3.4.4. We now use this to prove the desired $C^{0,\beta}$ regularity: we first control the Hölder seminorm in the u-direction:

$$\begin{aligned}
|\Psi(\tilde{u}_1, v) - \Psi(\tilde{u}_0, v)| &= \left| \int_{\tilde{u}_0}^{\tilde{u}_1} \partial_u \Psi(\tilde{u}, v)\, d\tilde{u} \right|, \\
&\leq \int_{\tilde{u}_0}^{\tilde{u}_1} |r\partial_u r \cdot \underline{L}\Psi(\tilde{u}, v)|\, d\tilde{u}, \\
&\leq D \int_{r(\tilde{u}_1, v)}^{r(\tilde{u}_0, v)} \tilde{r}^{-1+2\beta}\, d\tilde{r} = \frac{D}{2\beta} \cdot (r(\tilde{u}_0, v)^{2\beta} - r(\tilde{u}_1, v)^{2\beta}).
\end{aligned}$$

We next use the inequality $x^\beta - y^\beta \leq (x-y)^\beta$ for $x > y > 0$ to get

$$|\Psi(\tilde{u}_1, v) - \Psi(\tilde{u}_0, v)| \leq \frac{D}{2\beta} \cdot |r^2(\tilde{u}_0, v) - r^2(\tilde{u}_1, v)|^\beta. \tag{3.4.23}$$

Since the Hölder seminorm is defined with respect to u and not r, we need to again use the upper bounded for $-r\partial_u r$; we have $|r^2(\tilde{u}_0, v) - r^2(\tilde{u}_1, v)| \leq |\tilde{u}_1 - \tilde{u}_0| \cdot \sup_{\tilde{u}_0 \leq \tilde{u} \leq \tilde{u}_1}(-2r\partial_u r(\tilde{u}, v))$. Since $-r\partial_u r \leq R_1$ is uniformly bounded by Lemma 2.3.1, we deduce from (3.4.23) that

$$\sup_{\substack{[\tilde{u}_0, \tilde{u}_1] \times \{v\} \\ \subset \mathcal{D}}} \frac{|\Psi(\tilde{u}_1, v) - \Psi(\tilde{u}_0, v)|}{|\tilde{u}_1 - \tilde{u}_0|^\beta} \leq D. \tag{3.4.24}$$

The β-Hölder seminorm in the v-direction is similar, while the boundedness of $|\Psi|$ follows immediately from the seminorm bounds and the fact that $\mathcal{D}$ is a bounded domain. This completes the estimates for Ψ.

We now move onto the quantity Ξ defined in (3.4.14), describing the $O(1)$ term in the asymptotic expansion of ϕ. From (3.4.14), (3.4.13), and $L \log r = \underline{L} \log r = -\frac{1}{r^2}$, one finds

$$\underline{L}\Xi = \underline{L}\phi - \frac{\Psi}{r^2} - \underline{L}\Psi \cdot \log\left(\frac{r_0}{r}\right) = -\frac{1}{2} X\phi - \underline{L}\Psi \cdot \log\left(\frac{r_0}{r}\right), \tag{3.4.25}$$

$$L\Xi = L\phi - \frac{\Psi}{r^2} - L\Psi \cdot \log\left(\frac{r_0}{r}\right) = +\frac{1}{2} X\phi - L\Psi \cdot \log\left(\frac{r_0}{r}\right). \tag{3.4.26}$$

From Proposition 3.4.3 that $|X\phi| \lesssim r^{-2+2\beta}$, while the terms involving $L\Psi$ and $\underline{L}\Psi$ on the right hand side are bounded using (3.4.22). Unfortunately, the presence of the $\log(r_0/r)$ term on the right hand side produces some loss on the exponent, and we therefore only have

$$|L\Xi| + |\underline{L}\Xi| \leq \tilde{D}\, r^{-2+2\beta} \log r^{-1}. \tag{3.4.27}$$

Given (3.4.27), the remainder of the proof follows the same strategy as for Ψ, with appropriate modifications due to the $\log r^{-1}$ term. This concludes the proof of the proposition. □

Remark. The important steps of this proof are (3.4.22) and (3.4.27). Once these estimates are found, there are other ways to characterise the regularity of Ψ and Ξ. For instance, one can instead find uniform Sobolev regularity on null hypersurfaces of the form:

$$\Psi \in L^\infty_u W^{1,p_\beta^-}_v \cap L^\infty_v W^{1,p_\beta^-}_u, \quad \Xi \in L^\infty_u W^{1,p_\beta^-}_v \cap L^\infty_v W^{1,p_\beta^-}_u, \quad p_\beta = \frac{1}{1-\beta}.$$

3.5 Proofs of the main theorems

In this section, we conclude the proofs of Theorems 3.1.1 and 3.1.2, as well as the proof of Corollary 3.1.3.

3.5.1 Asymptotics for Ω^2

We firstly provide asymptotics for the null lapse $\Omega^2(u,v)$, and related geometric quantities.

Proposition 3.5.1. *Consider a strongly singular solution to the EMSFSS system as in Section 2.3.1. Recalling the function $\Psi : \mathcal{D} \to \mathbb{R}$ from* (3.4.13) *in Proposition 3.4.4, there exists $D > 0$ depending on data, R_0 and α, such that*

$$\left| \log \left(\Omega^2 \cdot \left(\frac{r_0}{r} \right)^{-\Psi^2+1} \right) \right| \leq D. \tag{3.5.1}$$

In fact we find the following precise asymptotics for the gauge-invariant quantity $\mathbf{g}(\nabla r, \nabla r) = 4\Omega^{-2}\partial_u r\, \partial_v r$. There exists a positive function $\mathfrak{M} : \mathcal{D} \to \mathbb{R}$, $C^{0,\beta,\log}$ up to the $\{r = 0\}$ boundary, with

$$4\Omega^{-2}\partial_u r\, \partial_v r =: \mathfrak{M} \cdot \left(\frac{r}{r_0} \right)^{-(\Psi^2+1)}, \tag{3.5.2}$$

Here the Hölder exponent β is as in Proposition 3.4.4, and one finds the following bounds on $\mathfrak{M}$:

$$\| \log \mathfrak{M} \|_{C^{0,\beta,\log}} + \| \mathfrak{M} \|_{C^{0,\beta,\log}} \leq D. \tag{3.5.3}$$

Proof. We use the Raychaudhuri equation (2.3.1), which we record again for convenience:

$$\partial_u(-\Omega^{-2}\partial_u r) = \Omega^{-2} r |\partial_u \phi|^2,$$

We rewrite this equation as a null transport equation for the following logarithmic quantity involving Ω^2;

$$\partial_u \log \left(\frac{\Omega^2}{-4\partial_u r} \right) = \frac{r}{\partial_u r} \cdot |\partial_u \phi|^2.$$

Converting to the gauge-invariant derivative $\underline{L}$, one finds:

$$\underline{L} \log \left(\frac{\Omega^2}{-4\partial_u r} \right) = -\frac{1}{r^2} \cdot |r^2 \underline{L}\phi|^2. \tag{3.5.4}$$

Of course, there is also the analogous equation:

$$L\log\left(\frac{\Omega^2}{-4\partial_v r}\right) = -\frac{1}{r^2}\cdot|r^2 L\phi|^2. \tag{3.5.5}$$

By Proposition 3.3.2, the right hand sides of (3.5.4) and (3.5.5) are $O(r^{-2})$, and thus borderline non-integrable as $r \to 0$. We solve this issue by subtracting from (3.5.4) and (3.5.5) the leading order term, which is exactly $\Psi^2 \cdot r^{-2}$. Since $L\log r = \underline{L}\log r = -\frac{1}{r^2}$, we therefore find:

$$\underline{L}\log\left(\frac{\Omega^2}{-4\partial_u r}\right) - \underline{L}\left(\Psi^2\log\left(\frac{r}{r_0}\right)\right) = -\frac{1}{r^2}\cdot(|r^2\underline{L}\phi|^2 - \Psi^2) - 2\Psi(\underline{L}\Psi)\log\left(\frac{r}{r_0}\right)$$

Taking the difference of two squares on the right hand side, and combining the terms on the left, we get:

$$\underline{L}\log\left(\frac{\Omega^2}{-4\partial_u r}\cdot\left(\frac{r}{r_0}\right)^{-\Psi^2}\right) = \frac{1}{2}(X\phi)(r^2\underline{L}\phi + \Psi) - 2\Psi(\underline{L}\Psi)\log\left(\frac{r}{r_0}\right). \tag{3.5.6}$$

We now follow a similar method to the proof of Proposition 3.4.4. By (3.4.8) and (3.4.22), the right hand side of (3.5.6) is bounded by $r^{-2+2\beta}\log r^{-1}$:

$$\left|\underline{L}\log\left(\frac{\Omega^2}{-4\partial_u r}\cdot\left(\frac{r}{r_0}\right)^{-\Psi^2}\right)\right| \leq D\cdot r^{-2+2\beta}\log\left(\frac{r_0}{r}\right), \tag{3.5.7}$$

Hence integrating in the $\underline{L}$ direction, where as usual this involves multiplying by r then integrating in r, we observe that the right-hand side is integrable towards $r = 0$, and

$$\left|\log\left(\frac{\Omega^2}{-4\partial_u r}\cdot\left(\frac{r}{r_0}\right)^{-\Psi^2}\right)\right| \lesssim 1.$$

We finally exponentiate this expression, and use that $-r\partial_u r \lesssim 1$ to get the sharp upper bound (3.5.1) for Ω^2 as a polynomial power of r.

To find the sharp asymptotics for $\mathfrak{M}$ as defined in (3.5.2), the additional ingredient is to use (2.3.6) to get:

$$\underline{L}\log(-r\partial_v r) = \frac{\Omega^2}{4\partial_u r\partial_v r}\cdot\left(1 - \frac{Q^2}{r^2}\right). \tag{3.5.8}$$

Combining (3.5.8) with (3.5.6), we therefore find that

$$\underline{L}\log\mathfrak{M} = \frac{\Omega^2}{4\partial_u r\partial_v r}\cdot\left(1 - \frac{Q^2}{r^2}\right) - \frac{1}{2}(X\phi)(r^2\underline{L}\phi + \Psi) + 2\Psi(\underline{L}\Psi)\log\left(\frac{r}{r_0}\right). \tag{3.5.9}$$

Similarly, in the v-direction we have

$$L\log\mathfrak{M} = \frac{\Omega^2}{4\partial_u r\partial_v r}\cdot\left(1 - \frac{Q^2}{r^2}\right) + \frac{1}{2}(X\phi)(r^2 L\phi + \Psi) + 2\Psi(L\Psi)\log\left(\frac{r}{r_0}\right). \tag{3.5.10}$$

Now we may proceed in the same manner we used to derived Hölder continuity for Ψ and Ξ in Proposition 3.4.4. Using Proposition 3.4.3, Proposition 3.4.4, and Lemma 3.3.1, one has

$$|L\log\mathfrak{M}| + |\underline{L}\log\mathfrak{M}| \leq D\cdot r^{-2+2\beta}\log\left(\frac{r_0}{r}\right). \tag{3.5.11}$$

The remainder of the proof then follows exactly as in the proof of Proposition 3.4.4. □

Remark. Taking the difference of (3.5.10) and (3.5.9) yields the equation:

$$X \log \mathfrak{M} = 2(X\phi)\Psi + 2\Psi(X\Psi) \log\left(\frac{r}{r_0}\right) = 2\Psi \cdot X\Xi. \tag{3.5.12}$$

This can be viewed as a momentum constraint relating the quantities $\log \mathfrak{M}$, Ψ and Ξ on any constant-r hypersurfaces $\Sigma_{\tilde{r}}$.

Since $\mathfrak{M}$, Ψ and Ξ extend continuously to the $\{r = 0\}$ singularity, one would also like to view (3.5.12) as an asymptotic momentum constraint at the singularity, see (3.2.2). However, in order for this to be true, we need $\log \mathfrak{M}$ and Ξ to be regular enough to make sense of the spatial derivative $\tilde{\nabla}$ along the singularity, which is unfortunately not achieved by Theorems 3.1.1 and 3.1.2.To prove an identity of the form (3.2.2), it seems we need to at least be able to upgrade the Hölder regularity of Ψ_∞, Ξ_∞ and $\mathfrak{M}_\infty$ to Lipschitz regularity.

3.5.2 Proof of Theorems 3.1.1 and 3.1.2

We finally combine the results of Section 3.3, Section 3.4 and Proposition 3.5.1 above to complete the proofs of Theorems 3.1.1 and 3.1.2. We shall prove both theorems simultaneously using the now familiar notation

$$\beta = \begin{cases} 1/2 & \text{if } Q = 0, \\ \alpha/2 & \text{if } Q \neq 0 \text{ and (QTS), (SKE) hold.} \end{cases}$$

With brevity in mind, when we refer to equations in Theorems 3.1.1 and 3.1.2 we generally only mention those of the latter, e.g. equation (3.1.18) will refer to (3.1.6) also.

Proof of (I). The estimate (3.1.3) holds in the ESFSS case by Lemma 2.3.1, while the analogous (3.1.15) in the EMSFSS model follows, assuming (QTS), by the same lemma. To prove that $r^2(u, v)$ is $C^{1,\beta}$, we use Proposition 3.3.4, specifically estimates (3.3.31), (3.3.32), (3.3.33) and (3.3.34).

These estimates, combined with (3.1.3), give that

$$|L(-r\partial_u r)| + |\underline{L}(-r\partial_u r)| + |L(-r\partial_v r)| + |\underline{L}(-r\partial_v r)| \lesssim \frac{1}{r^{-2+2\beta}}. \tag{3.5.13}$$

From here, a familiar argument (e.g. from the proof of Proposition 3.4.4) yields that the expressions $-r\partial_u r(u, v)$ and $-r\partial_v r(u, v)$ are $C^{0,\beta}$ in $\bar{\mathcal{D}} = \mathcal{D} \cup \mathcal{S}$. These expressions are exact multiples of the first derivatives of $r^2(u, v)$, hence r^2 is $C^{1,\beta}$ in $\mathcal{D}$ with respect to the double-null coordinates (u, v).

Furthermore, by (3.1.3) the gradient of $r^2(u, v)$ is non-vanishing in $\bar{\mathcal{D}}$, so the level sets of r^2, including the limiting level set $\mathcal{S} = \{r(u, v) = 0\}$, are $C^{1,\beta}$ curves. It is then clear that one may parameterize using the $C^{1,\beta}$ function $v_*(u)$ as described in the theorem.

Proof of (II). This follows immediately from Proposition 3.4.4.

Proof of (III). This is exactly equation (3.5.2) of Proposition 3.5.1, and the following estimate (3.5.3).

Proof of (IV). We pick now a point $p \in \mathcal{S}$, and consider its causal past $J^-(p)$. We shall prove the various estimates of (IV) by integrating "backwards" from $p \in \mathcal{S}$, given that we already know that the limiting values $C_u(p) > 0$, $C_v(p) > 0$, $\Psi_\infty(p)$, $\Xi_\infty(p)$ and $\mathfrak{M}_\infty(p) > 0$ exist.

We illustrate this by giving a complete proof of (3.1.18) in Theorem 3.1.2. Let $p = (u_p, v_*(u_p))$, and suppose $(u, v) \in J^-(p)$. Then one has:

$$\begin{aligned} |-r\partial_u r(u,v) - C_u(p)| &\leq |-r\partial_u r(u,v) + r\partial_u r(u_p,v)| + |-r\partial_u r(u_p,v) + r\partial_u r(u_p, v_*(u_p))|, \\ &\leq \int_u^{u_p} |\partial_u(-r\partial_u r)(\tilde{u},v)|\, d\tilde{u} + \int_v^{v_*(u_p)} |\partial_v(-r\partial_u r)(u_p,\tilde{v})|\, d\tilde{v}, \\ &= \int_{r(u_p,v)}^{r(u,v)} |r\underline{L}(-r\partial_u r)|\, dr + \int_0^{r(u_p,v)} |rL(-r\partial_u r)|\, dr \lesssim r^{2\beta}(u,v). \end{aligned}$$

Note that the final inequality followed by using (3.5.13) above, and evaluating the integral $\int_0^r r^{-1+2\beta}\, dr = \frac{1}{2\beta} r^{2\beta}$. This proves (3.1.18), and (3.1.19) is similar.

For the scalar field estimates, note that a similar procedure with Ψ or Ξ in place of $-r\partial_u r$, gives

$$|\Psi(u,v) - \Psi_\infty(p)| \lesssim r^{2\beta}, \tag{3.5.14}$$

$$|\Xi(u,v) - \Xi_\infty(p)| \lesssim r^{2\beta} \log r^{-1}. \tag{3.5.15}$$

Here we used (3.4.22) and (3.4.27) to estimate the appropriate null derivatives. The definition of Ξ in (3.4.14) then yields (3.1.22).

To translate the estimates on gauge-invariant quantities such as Ψ to estimates on the coordinate derivatives $\partial_u\phi$ and $\partial_v\phi$, we use the definition of Ψ in (3.4.13), as well as the definition of the vector fields L, $\underline{L}$ and X. For instance, we write

$$r^2\partial_u\phi = (-r\partial_u r) \cdot \left(\Psi - \tfrac{1}{2} r^2 X\phi\right).$$

Using (3.1.18) and (3.5.14) above, as well as (3.4.8) to estimate the remaining term $r^2 X\phi$, one straightforwardly deduces (3.1.20). (3.1.21) follows analogously.

Finally, (3.1.23) follows by integrating (3.5.11) backwards from p, exactly as before, to find that

$$|\log \mathfrak{M}(u,v) - \log \mathfrak{M}_\infty(p)| \lesssim r^{2\beta} \log r^{-1}.$$

Hence by the definition (3.5.2) of $\mathfrak{M}$, we find

$$\left| \frac{\Omega^2}{4\partial_u r \partial_v r}(u,v) - \mathfrak{M}_\infty^{-1} \cdot \left(\frac{r}{r_0}\right)^{\Psi_\infty(p)^2+1} \right| \lesssim r^{\Psi_\infty(p)^2+1+2\beta} \log r^{-1}.$$

(3.1.23) then follows upon multiplying by $4\partial_u r \partial_v r$, and using (3.1.18), (3.1.19).

Proof of (V). The estimate (3.1.24) for the Hawking mass $m(u,v)$ is immediate from (3.1.23), (3.1.18), (3.1.19), in light of the following expression for $m(u,v)$:

$$m(u,v) = \frac{r}{2} \cdot (1 + 4\Omega^{-2}\partial_u r \partial_v r) = \frac{r}{2} + \frac{2}{r\Omega^2}(-r\partial_u r)(-r\partial_v r).$$

To estimate the Kretschmann scalar, we list all the non-vanishing components of the Riemann tensor, whose components are evaluated with respect to the following null frame:

$$L = \frac{1}{-r\partial_v r}\partial_v, \quad \underline{L} = \frac{1}{-r\partial_u r}\partial_u, \quad e_1 = r^{-1}\partial_\theta, \quad e_2 = (r\sin\theta)^{-1}\partial_\varphi.$$

For the EMSFSS system, one computes that, modulo the symmetries of the Riemann tensor: $\mathbf{Riem}(V_1,V_2,V_3,V_4) = \mathbf{Riem}(V_3,V_4,V_1,V_2) = -\mathbf{Riem}(V_2,V_1,V_3,V_4)$, the only non-vanishing components are (where $A,B \in \{1,2\}$):

$$\mathbf{Riem}(L,e_A,L,e_B) = \frac{1}{r^4}|r^2 L\phi|^2\,\delta_{AB},$$

$$\mathbf{Riem}(\underline{L},e_A,\underline{L},e_B) = \frac{1}{r^4}|r^2 \underline{L}\phi|^2\,\delta_{AB},$$

$$\mathbf{Riem}(L,e_A,\underline{L},e_B) = \frac{1}{r^4}\left[1 + \frac{\Omega^2}{4\partial_u r\partial_v r}\left(1 - \frac{Q^2}{r^2}\right)\right]\delta_{AB},$$

$$\mathbf{Riem}(e_1,e_2,e_1,e_2) = \frac{1}{r^2}\left(1 + 4\Omega^{-2}\partial_u r\partial_v r\right),$$

$$\mathbf{Riem}(L,\underline{L},L,\underline{L}) = -\frac{\Omega^2}{4\partial_u r\partial_v r}\cdot\frac{1}{r^6}\left[1 - (r^2 L\phi)(r^2\underline{L}\phi) + \frac{\Omega^2}{4\partial_u r\partial_v r}\left(1 - \frac{2Q^2}{r^2}\right)\right].$$

Using these, alongside the estimates of (IV), one can deduce the estimate (3.1.25), describing the leading order polynomial blow-up of the Kretschmann scalar.

Finally, (3.1.26) can be deduced using (3.1.20), (3.1.21) and (3.1.23), while (3.1.27), which is an exact equality, can be computed using the form (2.2.3) of the electromagnetic 2-form $\mathbf{F}$. This completes the proof of Theorem 3.1.1 and Theorem 3.1.2. □

3.5.3 Several higher order estimates

We move now to Corollary 3.1.3. In order to make precise statements about the foliation using the time function τ, defined in (3.1.28), such as statements regarding the second fundamental form of the embedding $S_\tau \hookrightarrow \mathcal{M}$, we must first understand the second-order derivatives of τ.

As the quantity Ψ appears in (3.1.28), this means that we must now understand third-order derivatives of ϕ. Hence we now assume the initial data to be smooth (in fact C^3 is sufficient), and use the following:

Proposition 3.5.2. *Consider a strongly singular solution to the EMSFSS system as described in Theorems 3.1.1 and 3.1.2. Assume moreover that (r^2, Ω^2, ϕ) are smooth in $\mathcal{D}$. Then there exists D,*

depending only on data, such that

$$|LLX\phi| + |L\underline{L}X\phi| + |\underline{L}LX\phi| + |\underline{L}\underline{L}X\phi| \leq D\, r^{-6+\alpha}. \tag{3.5.16}$$

Furthermore, the quantities Ψ and $\mathfrak{M}$ from Theorems 3.1.1 and 3.1.2 obey:

$$|LL\Psi| + |L\underline{L}\Psi| + |\underline{L}L\Psi| + |\underline{L}\underline{L}\Psi| \leq D\, r^{-4+\alpha}, \tag{3.5.17}$$

$$|LL\log\mathfrak{M}| + |L\underline{L}\log\mathfrak{M}| + |\underline{L}L\log\mathfrak{M}| + |\underline{L}\underline{L}\log\mathfrak{M}| \leq D\, r^{-4+\alpha}\log r^{-1}. \tag{3.5.18}$$

Proof. We start with (3.5.16). We first rewrite (3.4.9) in terms of L and $\underline{L}$; one gets

$$\underline{L}LX\phi = \frac{1}{r^2}LX\phi + \frac{1}{r^2}\underline{L}X\phi - \frac{\Omega^2}{4r^2\partial_u r\partial_v r}\cdot\left(1-\frac{Q^2}{r^2}\right)LX\phi + \frac{F_X}{(-r\partial_u r)(-r\partial_v r)}. \tag{3.5.19}$$

We use this directly to get the $|\underline{L}L\phi|$ estimate in (3.5.16). We estimate the first three terms on the right hand side of (3.5.19) using (3.4.22), as well as (3.5.2) to deal with the Ω^2 term. The F_X term is then estimated using (3.4.12). Combining all these allows us to conclude that $|\underline{L}LX\phi| \lesssim r^{-6+\alpha}$, and $|L\underline{L}X\phi| \lesssim r^{-6+\alpha}$ follows similarly.

To get the $LLX\phi$ estimate, we commute the equation (3.5.19) with the vector field L. We find:

$$\underline{L}LLX\phi = K_1(u,v)\cdot LLX\phi + K_2(u,v), \tag{3.5.20}$$

where $K_1, K_2 : \mathcal{D} \to \mathbb{R}$ are given by the following expressions:

$$K_1 = \frac{1}{r^2} + [\underline{L}, L]^L - \frac{\Omega^2}{4r^2\partial_u r\partial_v r}\cdot\left(1-\frac{Q^2}{r^2}\right). \tag{3.5.21}$$

$$\begin{aligned} K_2 = [\underline{L}, L]^{\underline{L}}\cdot\underline{L}LX\phi + \frac{2}{r^4}LX\phi + \frac{2}{r^4}\underline{L}X\phi + \frac{1}{r^2}L\underline{L}X\phi \\ - L\left[\frac{\Omega^2}{4r^2\partial_u r\partial_v r}\cdot\left(1-\frac{Q^2}{r^2}\right)\right]\cdot LX\phi + L\left[\frac{F_X}{(-r\partial_u r)(-r\partial_v r)}\right]. \end{aligned} \tag{3.5.22}$$

By Lemma 3.4.1 and Proposition 3.5.1, we have that $K_1 = \frac{1}{r^2} + O(r^{-2+\alpha})$. Furthermore, combining all the results of Sections 3.4 and 3.5, we can also deduce that $|K_2| \lesssim r^{-8+\alpha}$. We mention here that to estimate the final term appearing in (3.5.22), we go back to the expression (3.4.11) for F_X. Taking an L-derivative of F_X therefore entails taking a further derivative of the commutator terms appearing in (3.4.11). But we have explicit expressions for the commutator terms in Lemmas 3.4.1 and 3.4.2, and one crucial ingredient will be to estimate the third-order derivatives $\partial_*^3(r^2)$, where ∂_* is either of ∂_u or ∂_v.

This would require pushing the results of Section 3.3.4 to one derivative further, which is straightforward, and we refer the reader to Proposition 7.4 and Proposition 7.5 of [4]. The strategy in that article applies also to the $Q \neq 0$ case supplemented with (QTS), (SKE), and the eventual estimate

is that $\partial_*^3(r^2) \lesssim r^{-4+\alpha}$. The remainder of the argument deducing the estimate $|K_2| \lesssim r^{-8+\alpha}$ is left to the reader. The upshot is that (3.5.20) can now be written as:

$$\underline{L}LLX\phi = \left(\frac{1}{r^2} + O(r^{-2+\alpha})\right) \cdot LLX\phi + O(r^{-8+\alpha}).$$

We then integrate this expression using a familiar Grönwall argument to deduce $|LLX\phi| \lesssim r^{-6+\alpha}$. The estimate for $|\underline{L}\underline{L}X\phi|$ is similar, completing our proof of (3.5.16).

To go from (3.5.16) to (3.5.17), we recall for instance (3.4.21), and apply L, to get

$$LL\Psi = \frac{1}{2}r^2 LLX\phi - LX\phi + L\left[\frac{\Omega^2}{4r^2\partial_u r\partial_v r} \cdot \left(\frac{Q^2}{r^2} - 1\right) \cdot r^2 L\phi\right].$$

This, combined with (3.5.16) and the estimates of Theorems 3.1.1 and 3.1.2, yields $|LL\Psi| \lesssim r^{-4+\alpha}$. The remaining estimates of (3.5.17) follow similarly. The final estimate (3.5.18) then follows upon applying the vector fields L and $\underline{L}$ to (3.5.9) and (3.5.10). We leave the details to the reader. □

Remark. It is apparent from the proof of Proposition 3.5.2 that similar estimates hold for higher derivatives, so long as the data has the required regularity to begin with. A general rule of thumb is that taking an L or an $\underline{L}$ derivative (or equivalently ∂_u or ∂_v) costs a factor of r^{-2}. Note, however, that we do not claim anything about improving such derivative estimates when differentiating in spatial directions.

We now apply Proposition 3.4.4, Proposition 3.5.1, and the above Proposition 3.5.2 to analyze the first and second derivatives of the function τ, defined in (3.1.28). Recall for convenience that

$$\tau := \frac{2r_0}{\Psi^2+3} \cdot \mathfrak{M}^{-1/2} \cdot \left(\frac{r}{r_0}\right)^{\frac{\Psi^2+3}{2}}.$$

Corollary 3.5.3. *The first and second derivatives of τ obey the following estimates: for $\tilde{L}_1, \tilde{L}_2 \in \{L, \underline{L}\}$ one has*

$$\left|\tilde{L}_1\tau + \mathfrak{M}^{-1/2}r_0^{-1} \cdot \left(\frac{r}{r_0}\right)^{\frac{\Psi^2-1}{2}}\right| \leq D\, r^{\frac{\Psi^2-1}{2}} \cdot r^{\alpha} \log r^{-1}, \tag{3.5.23}$$

$$\left|\tilde{L}_2(|\log \tilde{L}_1\tau|) + \frac{\Psi^2-1}{2} \cdot \frac{1}{r^2}\right| \leq D\, r^{-2+\alpha} \log r^{-1}. \tag{3.5.24}$$

Proof. We compute $\tilde{L}_1\tau$ explicitly. One finds

$$\tilde{L}_1\tau = -\mathfrak{M}^{-1/2}r_0^{-1} \cdot \left(\frac{r}{r_0}\right)^{\frac{\Psi^2-1}{2}} \left[1 + \frac{1}{\Psi^2+3} \cdot \left(\frac{r}{r_0}\right)^2 \cdot \mathfrak{E}_{\tilde{L}_1}\right], \tag{3.5.25}$$

where the error term $\mathfrak{E}_{\tilde{L}_1}$ is given by:

$$\mathfrak{E}_{\tilde{L}_1} = \frac{4\Psi}{\Psi^2+3} \cdot \tilde{L}_1\Psi + 2\Psi \cdot \tilde{L}_1\Psi \cdot \log\left(\frac{r_0}{r}\right) + \tilde{L}_1 \log \mathfrak{M}. \tag{3.5.26}$$

By (3.4.20), (3.4.21), (3.5.9) and (3.5.10), we find that $|\mathfrak{E}_{\tilde{L}_1}| \lesssim r^{-2+\alpha}\log r^{-1}$. Inserting this into (3.5.25), it is straightforward to deduce (3.5.23). Furthermore, from Proposition 3.5.2 and (3.5.26), we find also $|\tilde{L}_2\mathfrak{E}_{\tilde{L}_1}| \lesssim r^{-4+\alpha}$. Then (3.5.24) follows by taking a logarithm of (3.5.25), and then applying $\tilde{L}_2$. □

3.5.4 Proof of Corollary 3.1.3

We conclude this section with the proof of Corollary 3.1.3, indicating how our strongly singular spacetimes fit into the BKL picture. To show Part 1, we first write

$$\nabla\tau = \frac{1}{\mathbf{g}(L,\underline{L})}\left[L\tau\cdot\underline{L} + \underline{L}\tau\cdot L\right]. \tag{3.5.27}$$

Therefore, using (2.3.10) to determine $\mathbf{g}(L,\underline{L})$, we expand $\mathbf{g}(\nabla\tau,\nabla\tau)$ to deduce

$$\mathbf{g}(\nabla\tau,\nabla\tau) = \frac{2(L\tau)(\underline{L}\tau)}{\mathbf{g}(L,\underline{L})} = -4\Omega^{-2}r^2\partial_u r\partial_v r(L\tau)(\underline{L}\tau) = r^2\cdot\mathfrak{M}\left(\frac{r_0}{r}\right)^{\Psi^2+1}\cdot(L\tau)(\underline{L}\tau).$$

Then (3.1.29) follows immediately from Corollary 3.5.3, particularly (3.5.23). So τ is asymptotically normalized.

We now write down the frame e_0, e_1, e_2, e_3 from Part 3 in terms of L, $\underline{L}$ and τ:

$$e_0 = -\gamma\left(\frac{L}{L\tau}+\frac{\underline{L}}{\underline{L}\tau}\right),\quad e_1 = \gamma\left(\frac{L}{L\tau}-\frac{\underline{L}}{\underline{L}\tau}\right),\quad e_2 = r^{-1}\partial_\theta,\quad e_3 = r^{-1}\sin^{-1}\theta\,\partial_\phi,$$

where γ is chosen such that e_0 and e_1 are unit vectors. That is, $2\gamma^2\mathbf{g}(L,\underline{L}) = -(L\tau)(\underline{L}\tau)$. In order to evaluate the second fundamental form, we must evaluate connection coefficients with respect to this frame. Standard observations allow us to write

$$\nabla_{\frac{L}{L\tau}}\left(\frac{L}{L\tau}\right) = \lambda_{LL}\frac{L}{L\tau},\qquad \nabla_{\frac{\underline{L}}{\underline{L}\tau}}\left(\frac{L}{L\tau}\right) = \lambda_{\underline{L}L}\frac{L}{L\tau},$$

$$\nabla_{\frac{L}{L\tau}}\left(\frac{\underline{L}}{\underline{L}\tau}\right) = \lambda_{L\underline{L}}\frac{\underline{L}}{\underline{L}\tau},\qquad \nabla_{\frac{\underline{L}}{\underline{L}\tau}}\left(\frac{\underline{L}}{\underline{L}\tau}\right) = \lambda_{\underline{L}\underline{L}}\frac{\underline{L}}{\underline{L}\tau}.$$

We wish to understand the leading order behavior of these coefficients $\lambda_{LL}, \lambda_{\underline{L}L}, \lambda_{L\underline{L}}, \lambda_{\underline{L}\underline{L}}$. In fact an explicit computation using the EMSFSS system (2.3.1)–(2.3.5) will yield, for $\tilde{L}_1, \tilde{L}_2 \in \{L, \underline{L}\}$:

$$\lambda_{\tilde{L}_1\tilde{L}_2} = \frac{1}{\tilde{L}_1\tau}\cdot\begin{cases}\left[-\tilde{L}_1\log\left(|\tilde{L}_2\tau|\right)+\frac{1}{r^2}(1-|r^2\tilde{L}_1\phi|^2)\right] & \text{if } \tilde{L}_1 = \tilde{L}_2,\\ \left[-\tilde{L}_1\log\left(|\tilde{L}_2\tau|\right)\right] & \text{if } \tilde{L}_1 \neq \tilde{L}_2.\end{cases}$$

Applying Corollary 3.5.3, particularly the estimate (3.5.24), and Theorems 3.1.1 and 3.1.2, one finds

$$\lambda_{\tilde{L}_1\tilde{L}_2} = \frac{1}{\tilde{L}_1\tau}\cdot\begin{cases}\left[-\frac{1}{r^2}\frac{\Psi^2-1}{2}+O(r^{-2+\alpha}\log r^{-1})\right] & \text{if } \tilde{L}_1 = \tilde{L}_2,\\ \left[+\frac{1}{r^2}\frac{\Psi^2-1}{2}+O(r^{-2+\alpha}\log r^{-1})\right] & \text{if } \tilde{L}_1 \neq \tilde{L}_2.\end{cases} \tag{3.5.28}$$

We now use (3.5.28) to find the connection coefficients of $(\mathcal{M}, g)$ with respect to the frame e_0, e_1, e_2, e_3. We begin with the components of the second fundamental form, $k_{ij} = \mathbf{g}(\nabla_{e_i} e_0, e_j)$. It follows from spherical symmetry that k_{ij} is diagonal with respect to our choice of frame. To evaluate k_{11}, use

$$\begin{aligned}\mathbf{g}(\nabla_{e_1} e_0, e_1) &= -\gamma^3 g\left(\nabla_{\frac{L}{L\tau}-\frac{\underline{L}}{\underline{L}\tau}}\left(\frac{L}{L\tau}+\frac{\underline{L}}{\underline{L}\tau}\right), \frac{L}{L\tau}-\frac{\underline{L}}{\underline{L}\tau}\right)\\ &= -\gamma^3\cdot(-\lambda_{LL}+\lambda_{L\underline{L}}+\lambda_{\underline{L}L}-\lambda_{\underline{L}\underline{L}})\cdot g\left(\frac{L}{L\tau}, \frac{\underline{L}}{\underline{L}\tau}\right)\\ &= \frac{\gamma}{2}\cdot(-\lambda_{LL}+\lambda_{L\underline{L}}+\lambda_{\underline{L}L}-\lambda_{\underline{L}\underline{L}}).\end{aligned}$$

Combining this with (3.5.28), the estimate (3.5.24), and Theorems 3.1.1 and 3.1.1, we find

$$\begin{aligned}\mathbf{g}(\nabla_{e_1} e_0, e_1) &= -(-2\mathbf{g}(L,\underline{L}))^{-1/2}\cdot\left(\frac{\Psi^2-1}{r^2}+O(r^{-2+\alpha}\log r^{-1})\right)\\ &= -\mathfrak{M}^{1/2}\cdot\left(\frac{r}{r_0}\right)^{-\frac{\Psi^2+3}{2}} r_0^{-1}\cdot\left(\frac{\Psi^2-1}{2}+O(r^{\alpha}\log r^{-1})\right).\end{aligned} \tag{3.5.29}$$

For the e_2 and e_3 components of the second fundamental form, one also explicitly computes

$$\mathbf{g}(\nabla_{e_2} e_0, e_2) = \frac{\gamma}{r^2}\left(\frac{1}{L\tau}+\frac{1}{\underline{L}\tau}\right) = -\mathfrak{M}^{1/2}\cdot\left(\frac{r}{r_0}\right)^{-\frac{\Psi^2+3}{2}} r_0^{-1}\cdot\left(1+O(r^{\alpha}\log r^{-1})\right), \tag{3.5.30}$$

$$\mathbf{g}(\nabla_{e_3} e_0, e_3) = \mathbf{g}(\nabla_{e_2} e_0, e_2) = -\mathfrak{M}^{1/2}\cdot\left(\frac{r}{r_0}\right)^{-\frac{\Psi^2+3}{2}} r_0^{-1}\cdot\left(1+O(r^{\alpha}\log r^{-1})\right). \tag{3.5.31}$$

From the expressions (3.5.29), (3.5.30) and (3.5.31), Parts 2 and 4 of Corollary 3.1.3 are immediate. For Part 3, note that a similar computation to (3.5.29) yields

$$\mathbf{g}(\nabla_{e_0} e_1, e_0) = \frac{\gamma}{2}\cdot(\lambda_{LL}+\lambda_{\underline{L}L}-\lambda_{L\underline{L}}+\lambda_{\underline{L}\underline{L}}). \tag{3.5.32}$$

However, now substituting (3.5.28) will lead to a cancellation at the leading order $O(r^{-(\Psi^2+3)/2})$, and we therefore find that $\mathbf{g}(\nabla_{e_0} e_1, e_0) = \tau^{-1}\cdot O(r^{\alpha}\log r^{-1})$. Furthermore, we easily compute $\mathbf{g}(\nabla_{e_0} e_2, e_0) = \mathbf{g}(\nabla_{e_0} e_3, e_0) = 0$. Hence Part 3 also follows.

Finally, we study the scalar field. Rewriting the expression (3.1.16) in terms of τ, one gets

$$\phi = \frac{2\Psi}{\Psi^2+3}\left(\log\tau^{-1}-\frac{1}{2}\log\mathfrak{M}+\log\frac{2r_0}{\Psi^2+3}\right)+\Xi. \tag{3.5.33}$$

Therefore the first expression in (3.1.33) is immediate, while the second is straightforward after combining (3.5.33) with Proposition 3.4.4, Proposition 3.5.1 and Proposition 3.5.2 as per usual. We leave the details to the reader. This concludes our proof of Corollary 3.1.3. □

3.6 Several important examples

In this final section, we mention several explicit examples of spherically symmetric spacetimes possessing an $r = 0$ spacelike singularity, mostly taken from [26]. Each of these are solutions to the EMSFSS system (i.e. (Ein) coupled to (Scal) and (Max), in fact these examples are such that $F_{\mu\nu} \equiv 0$ aside from those of Section 3.6.1). We then consider each of these spacetimes in the context of Theorem 3.1.1 and Theorem 3.1.2.

To the best of our knowledge, every such explicit solution is such that the quantity $\Psi_\infty(p)$ arising in Theorem 3.1.1 will be constant along $\mathcal{S}$. However, as illustrated in Section 3.6.1, we show every value of $\Psi_\infty \in \mathbb{R}$ can be achieved. Further, in Sections 3.6.2 and 3.6.3, we show that the remaining asymptotic quantities $\Xi_\infty(p)$ and $\mathcal{M}_\infty(p)$ need not be constant, though in all cases the asymptotic momentum constraint (3.2.2) is satisfied.

3.6.1 Spatially homogeneous cosmologies

A spherically symmetric spacetime that is also spatially homogeneous[7], known in the literature as a Kantowski-Sachs cosmology [88], has a metric that we write in the following form:

$$\mathbf{g} = -h(s)ds^2 + h(s)^{-1}dx^2 + r^2(s)d\sigma_{\mathbb{S}^2}. \tag{3.6.1}$$

The associated massless scalar field ϕ and the electromagnetic field $\mathbf{F}$ are given by:

$$\phi = \phi(s), \qquad \mathbf{F} = \frac{Q}{r^2(s)} ds \wedge dx. \tag{3.6.2}$$

There are explicit families of solutions to the system (Ein), (Scal), (Max) of the above form, known in the $\mathbf{F} \equiv 0$ case as the *Just spacetimes* [87, 105, 154]. When $Q = 0$, the functions $r^2(s)$, $h(s)$ and $\phi(s)$ are given by:

$$r^2(s) = s(1-s)h(s), \quad h(s) = \left(\frac{s}{1-s}\right)^b, \quad \phi(s) = \pm\frac{\sqrt{1-b^2}}{2}\log\left(\frac{1-s}{s}\right). \tag{3.6.3}$$

Here $b \in [-1, 1]$ is a freely chosen parameter, while s lies in the open interval $s \in (0, 1)$. Note that the particular choice $b = 1$ is exactly the Schwarzschild interior metric (1.1.3), with $2M = 1$.

If, on the other hand, there is a non-trivial Maxwell field, then $Q(s) \equiv Q \neq 0$ is constant, and the remaining functions $r^2(s)$, $h(s)$ and $\phi(s)$ are given by

$$\begin{gathered} r^2(s) = s(1-s)h(s), \quad h(s) = \frac{Q^2}{b^2}\left[\left(\frac{s}{1-s}\right)^{b/2} + \left(\frac{s}{1-s}\right)^{-b/2}\right]^2, \\ \phi(s) = \pm\frac{\sqrt{1-b^2}}{2}\log\left(\frac{1-s}{s}\right). \end{gathered} \tag{3.6.4}$$

[7]The metric (3.6.1) has the additional Killing field $\mathbf{K} = \frac{\partial}{\partial x}$ – note that in this setting any such $\mathbf{K}$ is also hypersurface orthogonal.

As $h(s)$ is undefined at $b = 0$, we now constrain b to lie in $(0, 1]$. It turns out that $b = 1$ now corresponds to a Reissner-Nordström interior spacetime.

We now write these spacetimes in a double null coordinate gauge (2.2.1). Choosing u and v such that $2du = -h(s)ds + dx$, $2dv = -h(s)ds - dx$, one writes

$$r^2(u, v) = r^2(s), \quad \Omega^2(u, v) = \frac{4}{h(s)}, \quad \text{where } s = s(u, v) \text{ is defined by } \int_0^s h(\tilde{s})\, d\tilde{s} = -u - v.$$

Note that s is a past-directed timelike coordinate, and the boundary at $s = 0$ is now located at the line $u + v = 0$ in the (u, v)-plane. Since $r(u, v) = 0$ here, we interpret this as the spacelike singularity $\mathcal{S}$ in Theorems 3.1.1 and 3.1.2. These theorems will be applicable only in the trapped region where $\frac{dr}{ds} > 0$, which includes an s-neighborhood of 0.

By spatial homogeneity, the functions $\Psi(u, v)$, $\Xi(u, v)$ and $\mathfrak{M}(u, v)$ arising in these Theorems are now solely functions of s. We compute $\Psi(s)$ to be:

$$\Psi(s) = \begin{cases} \dfrac{\sqrt{1-b^2}}{1-2s-b}, & \text{in the } Q = 0 \text{ case (3.6.3)}, \\ \dfrac{\sqrt{1-b^2}}{1-2s+b\tanh\left(\frac{b}{2}\log\left(\frac{s}{1-s}\right)\right)}, & \text{in the } Q \neq 0 \text{ case (3.6.4)}. \end{cases} \tag{3.6.5}$$

In both cases, Ψ_∞ is given by $\lim_{s\to 0} \Psi(s) = \sqrt{\frac{1+b}{1-b}}$. Therefore, as b varies across $[-1, 1]$, Ψ_∞ is allowed to take all real values in the $Q = 0$ case. However, in the $Q \neq 0$ case we must obey $\Psi_\infty > 1$, as only $b \in (0, 1]$ is admissible. This is not surprising, as $\Psi_\infty > 1$ is equivalent to the subcriticality condition (SKE) holding in a neighborhood of $\mathcal{S} = \{r = 0\}$.

Alternatively, we can view the metrics corresponding to (3.6.3) and (3.6.4) as being singular at $s = 1$ instead, with the caveat that the "singularity" will disappear when $b = \pm 1$. The $s = 1$ singularity is a past boundary rather than a future boundary. We compute $\Psi(s)$ to be (3.6.5) as before, but evaluating (3.6.5) as $s \to 1$ we have a different value for Ψ_∞ depending on whether $Q = 0$ or $Q \neq 0$. Penrose diagrams for these spacetimes are given in Figure 3.6.

As mentioned in the introduction, the existence of the $Q \neq 0$ example (3.6.4) is significant not only in showing that the non-emptiness of the set of spacetimes to which Theorem 3.1.2 applies, but also, via a straightforward gluing of some compact part of the right diagram in Figure 3.6 to a two-ended asymptotically flat exterior, shows that there exist spacetimes with the right Penrose diagram in Figure 3.4, and where Theorem 3.1.2 applies to at least part of the singular boundary $\mathcal{S}$.

3.6.2 The FLRW spacetimes

The most famous cosmological solutions to Einstein's equation (Ein) are the spatially homogeneous and isotropic *Friedmann-Lemaître-Robertson-Walker* (FLRW) spacetimes. These metrics are of the

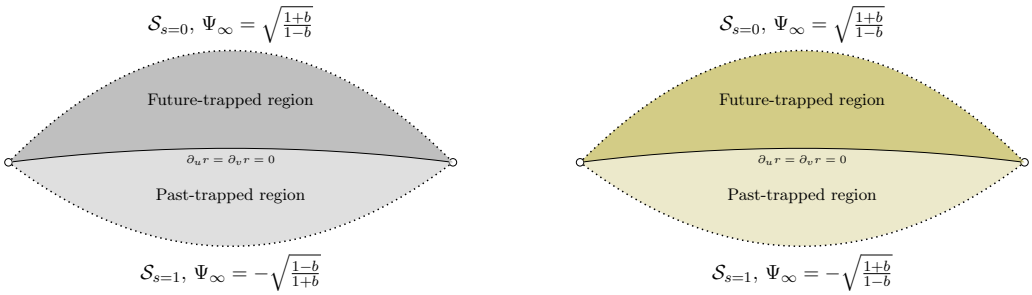


Figure 3.6: Penrose diagrams representing the Kantowski-Sachs cosmologies (3.6.1). Ignoring the borderline cases $b = \pm 1$, the spacetimes are bounded both in the future and in the past by a spacelike singularity. When $Q = 0$ (left), the value of Ψ_∞ at either singularity takes all real values as b varies in $[-1, 1]$. When $Q \neq 0$ (right), we must have $b \in (0, 1]$, and hence the value of Ψ_∞ at both boundaries obeys $|\Psi_\infty|^2 > 1$.

form:

$$\mathbf{g}_{FLRW} = -dt^2 + a^2(t)\, d\sigma_{\Sigma_0}, \tag{3.6.6}$$

where Σ_0 is a Riemannian manifold of constant curvature (here we take $\Sigma_0 = \mathbb{R}^3$ or $\Sigma_0 = \mathbb{H}^3$), and $d\sigma_{\Sigma_0}$ is the standard metric on Σ_0. The proper time variable t takes values in $(0, +\infty)$, and $a(t) > 0$ is such that $\lim_{t\downarrow 0} a(t) = 0$.

We study FLRW metrics solving the Einstein-scalar field system (with $\mathbf{F} = 0$). Focusing on $\Sigma_0 = \mathbb{R}^3$ or $\mathbb{H}^3$, we write down the following explicit forms of the metric and scalar field in *conformal coordinates*:

- If $\Sigma_0 = \mathbb{R}^3$, then we have the following for $T > 0$:

$$\mathbf{g}_{FLRW_E} = T(-dT^2 + dX^2 + dY^2 + dZ^2), \qquad \phi = \pm\frac{\sqrt{3}}{2}\log T. \tag{3.6.7}$$

- If $\Sigma_0 = \mathbb{H}^3$, then we denote $R^2 = X^2 + Y^2 + Z^2$, and the metric and scalar field can be written in the following form, valid for $T^2 - R^2 > 1$:

$$\begin{aligned} \mathbf{g}_{FLRW_H} &= \left(1 - \frac{1}{(T^2 - R^2)^2}\right)(-dT^2 + dX^2 + dY^2 + dZ^2), \\ \phi &= \pm\frac{\sqrt{3}}{2}\log\left(\frac{T^2 - R^2 - 1}{T^2 - R^2 + 1}\right). \end{aligned} \tag{3.6.8}$$

In both cases, we express the metric in double-null coordinates using $u = \frac{-T-R}{2}, v = \frac{-T+R}{2}$. Then the lapse Ω^2, the area-radius r and the scalar field ϕ can be written as follows.

- In the Euclidean case $\Sigma_0 = \mathbb{R}^3$, we have for $(u, v) \in \mathcal{Q} = \{(u, v) : v \geq u,\, u + v < 0\}$:

$$\Omega^2(u, v) = 4(-u - v), \quad r^2(u, v) = (-u - v)(v - u)^2, \quad \phi(u, v) = \pm\frac{\sqrt{3}}{2}\log(-u - v). \tag{3.6.9}$$

- In the hyperbolic case $\Sigma_0 = \mathbb{H}^3$, we have for $(u,v) \in \mathcal{Q} = \{(u,v) : v \geq u,\, 4uv > 1,\, u < 0\}$:

$$\Omega^2(u,v) = 4\left(1 - \frac{1}{16u^2v^2}\right), \quad r^2(u,v) = \left(1 - \frac{1}{16u^2v^2}\right)(v-u)^2,$$
$$\phi(u,v) = \pm\frac{\sqrt{3}}{2}\log\left(\frac{4uv-1}{4uv+1}\right). \tag{3.6.10}$$

It is straightforward to check that (3.6.9) and (3.6.2) solve the system (2.3.1), (2.3.2), (2.3.3), (2.3.4), (2.3.5) with $Q \equiv 0$. In both cases there are two parts of the boundary $\partial\mathcal{Q}$ where $r = 0$: the regular center Γ corresponding to the fixed points of the $SO(3)$ isometry group, and the singular boundary $\mathcal{S}$ located at $u + v = 0$ and at $uv = 4$ in (3.6.9) and (3.6.2) respectively.

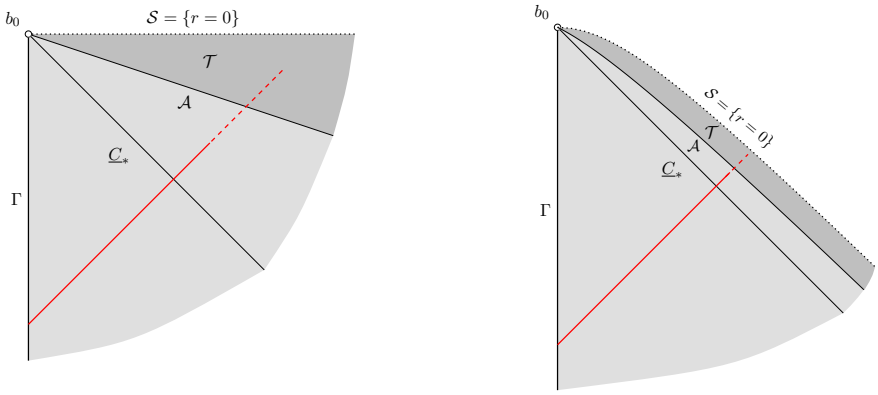


Figure 3.7: (Non-compactified) Penrose diagrams representing the FLRW spacetimes (3.6.7) (left) and (3.6.8) (right). Both spacetimes feature a spacelike singularity $\mathcal{S}$ at which $\Psi_\infty = \mp\sqrt{3}$. The singular boundary $\mathcal{S}$ is preceded by a trapped region $\mathcal{T}$ and an apparent horizon $\mathcal{A}$. In both cases, we can consider initial data given on a regular null cone emanating from Γ which coincides exactly with the induced FLRW data on the thick red line above. Extending this initial cone to be asymptotically flat, the maximal development of this data will have a singular boundary, a portion of which exactly coincides with that of FLRW.

In both cases, $\partial_u r < 0$ everywhere, and the singular boundary $\mathcal{S}$ is preceded by a trapped region $\mathcal{T} = \{(u,v) : \partial_v r < 0\}$, depicted in Figure 3.7. Therefore, we may apply Theorem 3.1.1 to subregions which are strongly singular, and associate quantities Ψ, Ξ and $\mathfrak{M}$ to all $(u,v) \in \mathcal{T} \cup \mathcal{S}$. In particular, we compute the quantities $r^2 L\phi$ and $r^2 \underline{L}\phi$ appearing in (3.4.13) to be

$$r^2 L\phi = \mp\sqrt{3}\left[1 + \frac{2(u+v)}{-3u-v}\right],\ r^2\underline{L}\phi = \mp\sqrt{3}\left[1 - \frac{2(u+v)}{u+3v}\right] \text{ for (3.6.9)},$$
$$r^2 L\phi = \mp\sqrt{3}\left[1 + \frac{(4uv-1)(4v^2+1)}{1-16uv^3}\right],\ r^2\underline{L}\phi = \mp\sqrt{3}\left[1 - \frac{(4uv-1)(4u^2+1)}{16u^3v-1}\right] \text{ for (3.6.2)}.$$

In both cases, it is apparent that we have $\Psi_\infty = \mp\sqrt{3}$. Using the correspondence (1.3.3), the Kasner exponents at $\mathcal{S}$ are therefore always $p_1 = p_2 = p_3 = \frac{1}{3}$. This is in accordance with the isotropy

assumption i.e. the assumption that all spatial directions should be equivalent. Furthermore, these exponents agree with the fact that $a(t) \sim t^{1/3}$ in (3.6.6) for both of these spacetimes.

We compute also the values of Ξ_∞ and $\mathfrak{M}_\infty$ for these spacetimes. Setting $r_0 = 1$ in the expressions (3.1.4) and (3.1.5), one finds:

$$\Xi_\infty = \mp\sqrt{3}\log(v-u) = \mp\sqrt{3}\log R, \quad \mathfrak{M}_\infty = \frac{1}{4}(v-u)^6 = \frac{R^6}{4}, \quad \text{for (3.6.9)},$$

$$\Xi_\infty = \mp\sqrt{3}\log\left(\frac{v-2}{2}\right) = \mp\sqrt{3}\log\frac{R}{2}, \quad \mathfrak{M}_\infty = 4(v-u)^6 = 4R^6, \quad \text{for (3.6.2)}.$$

Note, in particular, that these obey the asymptotic momentum constraint (3.2.2), with $\tilde{\nabla} = \frac{d}{dR}$.

Our analysis is valid for the whole spacelike singularity $\mathcal{S}$, excluding the first singularity at the center, $b_0 = \Gamma \cap \mathcal{S}$. As noted in Section 3.1, the point b_0 is excluded from our analysis as it is not preceded by trapped 2-spheres. We note, however, that in the specific context of the FLRW spacetimes, upon reverting to the metric (3.6.6) the point b_0 need not be distinguished from other points on the spacelike singularity, and still has Kasner exponents $p_1 = p_2 = p_3 = 1/3$.

Finally, one notes that the FLRW metrics (3.6.7) and (3.6.8) may be considered within the context of gravitational collapse. To see this, observe that if we consider a null cone emanating from $b \in \Gamma$, continuing it beyond $J^-(b_0)$ but not into the trapped region $\mathcal{T}$, and instead extending the cone to be asymptotically flat (see the red lines in Figure 3.7), the Penrose diagram resulting from the maximal future development of such data would be that of Figure 1.2.

Moreover, the metric in a neighborhood of b_0 would be exactly FLRW, and have subcritical Kasner exponents of $p_1 = p_2 = p_3 = 1/3$. This contruction is particularly interesting in light of the fact that FLRW singularities are known to be stable outside of symmetry in some settings [130, 131, 136, 57].

3.6.3 Scale-invariant collapsing spacetimes

Our final explicit example of a spherically symmetric spacetime possessing a spacelike singularity are the scale-invariant solutions [153, 32]. These are spherically symmetric solutions to (Ein), (Scal) possessing a conformal Killing vector field $\mathbf{S}$, known as the *scaling vector field*, which obeys $\mathcal{L}_{\mathbf{S}} g = 2g$.

In a suitable double-null gauge, where $\mathbf{S}$ can be written as $\mathbf{S} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, the metric has the following form, with $\Omega^2 \equiv 1$:

$$g = -dudv + r^2(u,v)d\sigma_{\mathbb{S}^2},$$

where the area-radius $r(u,v)$ and scalar field $\phi(u,v)$ are as follows: for $v < 0$,

$$r = \tfrac{1}{2}(v-u), \quad \phi \equiv 0,$$

while, for $v \geq 0$, one has for some $p \in \mathbb{R}$:

$$r^2 = \tfrac{1}{4}[(1-p)v - u][(1+p)v - u], \quad \phi = -\frac{1}{2}\log\left[\frac{(1-p)v-u}{(1+p)v-u}\right]. \tag{3.6.11}$$

These solutions feature a loss of regularity across the ingoing null hypersurface $\{v = 0\}$; along any outgoing null cone $C_0 = \{(u, v) : v \geq u\}$ with $u < 0$, the quantity $\partial_v\phi$ will have a jump discontinuity at $v = 0$. Nonetheless, these solutions remain relevant in light of the fact that the Einstein-scalar field equations in spherical symmetry remain well-posed for $\partial_v\phi$ merely of bounded variation [32]. In any case, for this article we are only concerned with the interior region $v > 0$, where all quantities are smooth.

This interior spacetime, whose area-radius r and scalar field ϕ are given by (3.6.11), is singular if the parameter p is such that $|p| > 1$. Assuming without loss of generality that $p > 1$, the singular boundary lies at $\mathcal{S} = \{(u, v) : u + (p-1)v = 0, v > 0\}$. This boundary $\mathcal{S}$ is preceded by a trapped region $\mathcal{T}$, bounded to the past by an apparent horizon at $\mathcal{A} = \{(u, v) : u + (p^2-1)v = 0, v > 0\}$.

Applying Theorem 3.1.1, we compute the quantity $\Psi(u, v)$ for $(u, v) \in \mathcal{T}$ to be:

$$\Psi(u, v) = \frac{1}{2}\left(\frac{pu}{(1-p^2)v - u} + \frac{pv}{v-u}\right). \tag{3.6.12}$$

By considering (3.6.12) in the limit $u + (p-1)v \uparrow 0$, one finds that $\Psi_\infty(u, v) = 1$ for all $(u, v) \in \mathcal{S}$. Setting $r_0 = 1$ in (3.1.4) and (3.1.5), we also compute Ξ_∞ and $\mathfrak{M}_\infty$ to be:

$$\Xi_\infty = \log\left[\frac{(1+p)v-u}{2}\right] = \log pv, \quad \mathfrak{M}_\infty = \tfrac{1}{4}(v-u)[(p^2-1)v + u] = \tfrac{1}{4}p^2(p-1)v^2.$$

Here we used that $(p-1)v + u = 0$ on $\mathcal{S}$. In particular, it is straightforward to check that the asymptotic momentum constraint (3.2.2) holds with spatial derivative given by $\tilde{\nabla} = \frac{\partial}{\partial v}$.

One should mention that the Penrose diagram corresponding to these spacetimes coincides with the left picture in Figure 3.7, though there are key differences in that the region to the past of $\underline{C}_*$ is now exactly Minkowski and that the value of Ψ_∞ on $\mathcal{S}$ is now identically 1 rather than $\sqrt{3}$.

This concludes our discussion of the scale-invariant spacetimes. Note that one could also apply this analysis to the k-self-similar spacetimes[8] constructed by Christodoulou in [33]. However, since these spacetimes are not fully explicit, and instead constructed via orbits of a 2-dimensional dynamical system, we choose not to discuss them any further here. We instead refer the reader to [39] for a detailed discussion in the specific case $k^2 = \frac{1}{3}$, and we note that it is expected that the methods of [39] will generalize to all relevant values of k.

[8]Note that the purpose of [33] was to obtain self-similar spacetimes containing a naked singularity. Nonetheless, there are also k-self-similar spacetimes in [33] which contain a trapped region and spacelike singularity $\mathcal{S}$ in the interior, and we can apply Theorem 3.1.1 to these spacetimes.

Chapter 4

A case study: Black holes with charged scalar hair

In Chapter 4, we state and prove our main results regarding a class of *spatially homogeneous black hole interiors with charged scalar hair*, based on the article [104] written in collaboration with Maxime Van de Moortel. See Section 1.4 for a brief introduction to the setup. The analysis of this Chapter will use the formulation of the spherically metric Einstein–Maxwell–*charged and massive* scalar field system introduced in Section 2.1, or in particular the ODEs corresponding to our spatially homogeneous spacetimes given in Section 2.3, namely (2.4.21)–(2.4.31), with data given in the $s \to -\infty$ limit and given by (2.4.32)–(2.4.37).

4.1 Statement of main results and overview

4.1.1 Definition of the spacetime sub-regions

We now give a precise definition of the various regions appearing inside our homogeneous black hole interiors, represented in Figure 4.1. Using the gauge choice (Ω-data), the following regions are part of the spacetime described in Proposition 2.4.1. The following regions, which each correspond to solutions of the ODE system (2.4.21)–(2.4.31) with s being inside a non-empty interval $I \subset \mathbb{R}$, are such that, for ϵ chosen sufficiently small, their union covers the entire spacetime of Theorem 4.1.1.

- The red-shift region is $\mathcal{R} \coloneqq \{-\infty < s \leq -\Delta_{\mathcal{R}}\}$ for some $\Delta_{\mathcal{R}} \gg 1$: Here, we make strong use of the positive surface gravity of the event horizon (red-shift effect, e.g. see [40, 48] and subsequent works).
- The no-shift region is $\mathcal{N} \coloneqq \{-\Delta_{\mathcal{R}} \leq s \leq S\}$ for some $S \gg 1$: Here, we use a Cauchy stability argument and Grönwall's inequality to show that quantities are still ϵ^2-close to their Reissner-Nordström values.

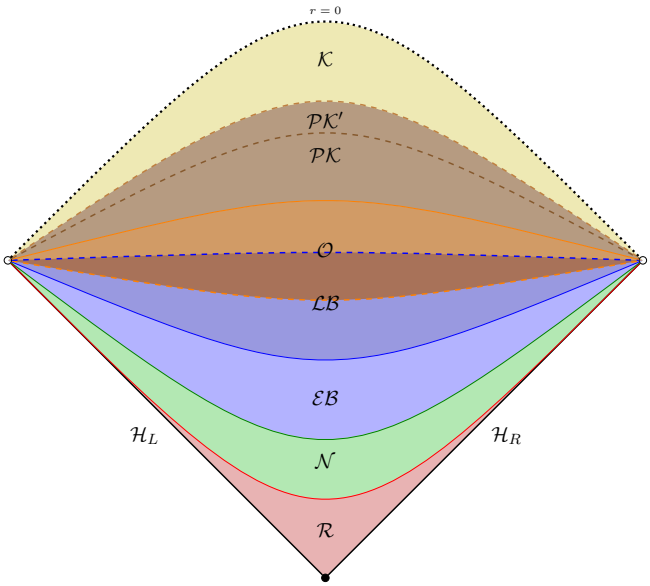


Figure 4.1: A more detailed version of Figure 1.4, partitioning the hairy black hole interior into the different regions $\mathcal{R}, \mathcal{N}, \mathcal{EB}, \mathcal{LB}, \mathcal{O}, \mathcal{PK}, \mathcal{K}$..

- The early blue-shift region is $\mathcal{EB} := \{S \leq s \leq s_{lin}(\epsilon) := |2K_-|^{-1} \log\left(\nu\epsilon^{-1}\right)\}$ for $\nu > 0$: Here, we begin to exploit the *blue-shift* effect of the Cauchy horizon of Reissner-Nordström, which is due to negative surface gravity $2K_- < 0$.

- The late blue-shift region is $\mathcal{LB} := \{s_{lin}(\epsilon) \leq s \leq \Delta_{\mathcal{B}}\epsilon^{-1}\}$ for some some $\Delta_{\mathcal{B}} > 0$: Here, the spacetime geometry begins to depart from that of Reissner-Nordström, and we provide the key ingredients to help us with the analysis of subsequent regions. In particular, this region starts to see a growth of the Hawking mass (a relic of mass inflation, see [112, 145] and the introduction of [148]).

- The oscillation region is $\mathcal{O} := \{s_O(\epsilon) := 50 s_{lin}(\epsilon) \leq s \leq s_{PK}(\epsilon)\}$ where $r(s_{PK}) = 2|B|\mathfrak{W}r_-\epsilon$ for $\mathfrak{W} = \mathfrak{W}(M, \mathbf{e}, \Lambda, q_0) > 0$ defined in (4.3.51): Here, the Bessel-type behavior kicks in, leading to the collapsed oscillations discussed in Section 4.2.3.

- The proto-Kasner region is $\mathcal{PK} := \{s_{PK}(\epsilon) \leq s \leq s_i(\epsilon)\}$ where $r(s_i) = e^{-\delta_0\epsilon^{-2}}$ for $\delta_0 = \delta_0(M, \mathbf{e}, \Lambda, q_0) > 0$ defined in (4.5.1): Here, the Bessel-type behavior continues but exhibits logarithmic growth rather than oscillations.

  We also define the sub-region $\mathcal{PK}' := \{s_{K_1}(\epsilon) \leq s \leq s_i(\epsilon)\} \subset \mathcal{PK}$ where $r(s_{K_1}) = 2|B|\mathfrak{W}r_-\epsilon^2$. In this sub-region, we will establish that Kasner-type behavior starts to take place.

- The Kasner region is $\mathcal{K} := \{s_i(\epsilon) \leq s < s_\infty(\epsilon)\}$, where $\lim_{s\to s_\infty} r(s) = 0$. In this region, we prove that the metric is described by either one Kasner regime, or two Kasner regimes

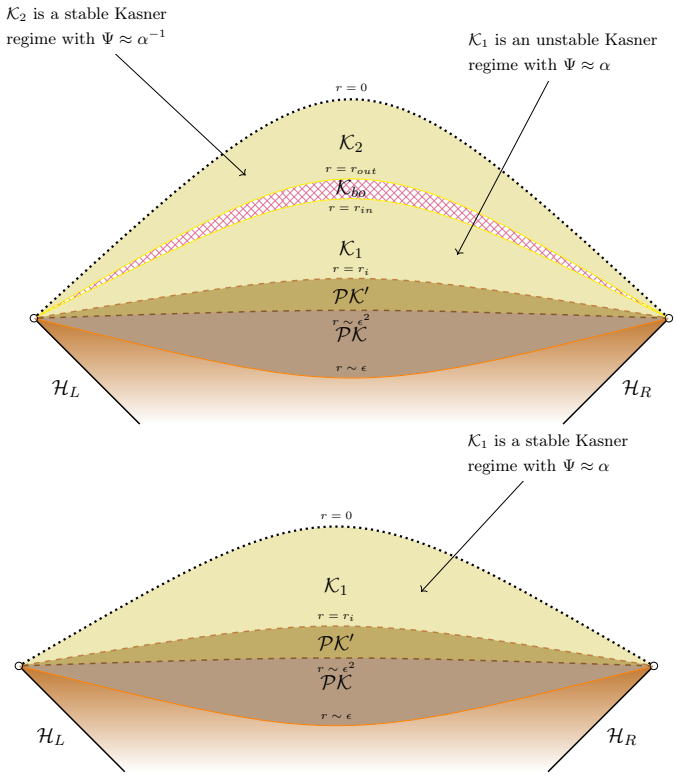


Figure 4.2: A zoom on $\mathcal{PK} \cup \mathcal{K}$ in Figure 4.1, with the top picture representing the bounce case $(\mathcal{K}_2, \mathcal{K}_{bo} \neq \emptyset)$ and the bottom picture the no bounce case $(\mathcal{K}_2, \mathcal{K}_{bo} = \emptyset)$. Note the inclusion $\mathcal{PK}' \subset \mathcal{PK}$.

connected by a bounce. In $\mathcal{K}$, the scalar field is governed by a first-order ODE.

We also introduce the additional sub-regions of $\mathcal{PK} \cup \mathcal{K}$ depicted in Figure 4.2, where Ψ is defined by $\Psi = -\frac{r\dot{\phi}}{\dot{r}}$ and α is defined in (1.4.5).

- The first Kasner region $\mathcal{K}_1 \subset \mathcal{PK}' \cup \mathcal{K}$ overlaps with $\mathcal{PK}$ and $\mathcal{K}$. In this region, we will show the metric is in a Kasner regime. Anticipating Theorem 4.1.2, we will find that $\mathcal{K}_1 = \mathcal{PK}' \cup \mathcal{K}$ in the no Kasner bounce case (4.1.22) (i.e. the first Kasner regime is the final Kasner regime, with all Kasner exponents being positive) and $\mathcal{K}_1 = \{s_{K_1} \leq s \leq s_{in}\} \neq \mathcal{PK}' \cup \mathcal{K}$ in the Kasner bounce case (4.1.21) (with one negative Kasner exponent, which is thus expected to be unstable). Here $s_{in} := \min\{s \in \mathcal{K} : |\Psi(s)| = |\alpha| + \epsilon^2\}$.

- The Kasner bounce region $\mathcal{K}_{bo} \subset \mathcal{K}$. We have $\mathcal{K}_{bo} = \emptyset$ in the no Kasner bounce case (4.1.22), and $\mathcal{K}_{bo} = \{s_{in} \leq s \leq s_{out}\}$ in the Kasner bounce case (4.1.21), where $s_{out} := \min\{s \in \mathcal{K} : |\Psi(s)| = |\alpha|^{-1} - \epsilon^2\}$. We have weaker control of the metric in $\mathcal{K}_{bo}$, but we show that it is very short in terms of proper time.

- The second Kasner region $\mathcal{K}_2 \subset \mathcal{K}$. We will have $\mathcal{K}_2 = \emptyset$ in the no Kasner bounce case (4.1.22), and $\mathcal{K}_2 = \{s_{out} \leq s < s_\infty\}$ in the Kasner bounce case (4.1.21), where we exhibit a second Kasner regime with positive Kasner exponents.

4.1.2 First statement: formation of a spacelike singularity

We first start with our main result, which covers part of Theorem 1.4.1 (namely the formation of the spacelike singularity). We reiterate that Theorem 4.1.1 contains the statements 1, 2, 5 of our rough Theorem 1.4.1; more precisely statement 1 of Theorem 1.4.1 corresponds to statement i of Theorem 4.1.1, while statements 2 and 5 of Theorem 1.4.1 are covered by statement ii of Theorem 4.1.1 as well as the estimate (4.1.15).

The statements iii and iv of Theorem 4.1.1 will also lay the groundwork towards proving the more specific Kasner asymptotics claimed in statements 3 and 4 of Theorem 1.4.1; the precise nature of these asymptotics will be covered in Theorem 4.1.2, upon further restricting ϵ.

In the following theorem and the rest of the paper, for $A, B \geq 0$ we will use the notation $A \lesssim B$ to denote that there exists $C(M, \mathbf{e}, \Lambda, q_0, m^2, \eta) > 0$ such that $A \leq CB$.

Theorem 4.1.1. *Let $(M, \mathbf{e}, \Lambda) \in \mathcal{P}_{se}$ with $\mathbf{e} \neq 0$, $q_0 \neq 0$, $m^2 \in \mathbb{R}$ be subextremal Reissner-Nordström(-dS/AdS) parameters (see Section 2.4). Then, for $\eta > 0$ chosen sufficiently small, there exists $\epsilon_0(M, \mathbf{e}, \Lambda, q_0, m^2, \eta) > 0$ and a subset $E_\eta \subset (-\epsilon_0, \epsilon_0) \setminus \{0\}$ satisfying $\frac{|(-\delta,\delta)\setminus E_\eta|}{2\delta} = O(\eta)$ for any $0 < \delta \leq \epsilon_0$, such that the following holds: For all $\epsilon \in E_\eta$, the maximal future development $\mathcal{M}$ for the Einstein–Maxwell–charged (and possibly massive) scalar field system of the characteristic*

data from Section 2.4 (i.e. (Ω-data), (r-data), (Q-data), *and* (ϕ-data)*) terminates at a spacelike singularity $\mathcal{S}$ on which r extends continuously to* 0*, and the Penrose diagram is given by Figure 4.1.*

More precisely, there exists a foliation of $\mathcal{M}$ by spacelike hypersurfaces Σ_s, with $s \in (-\infty, s_\infty(\epsilon))$, where s is defined in (Ω-data)*, and $s_\infty = \frac{|K_-|}{4|B|^2\omega_{RN}^2}\epsilon^{-2} + O(\log(\epsilon^{-1}))$, where $B(M, \mathbf{e}, \Lambda, m^2, q_0) \neq 0$ is defined in Proposition 4.1.3, and $2K_-(M, \mathbf{e}, \Lambda) < 0$, $\omega_{RN}(M, \mathbf{e}, \Lambda, q_0) > 0$ are defined in Section 2.4. The subsequent spacetime dynamics are described as follows:*

i. (Almost formation of a Cauchy horizon). In the late blue-shift region $\mathcal{LB} \coloneqq \{s_{lin}(\epsilon) \leq s \leq \Delta_{\mathcal{B}}\epsilon^{-1}\}$, we have the following stability estimates with respect to the Reissner–Nordström(-dS/AdS) metric:

$$\epsilon^{-1}|\phi(s) - 2\epsilon\,\mathrm{Re}(Be^{i\omega_{RN}s})| + \epsilon^{-1}\left|\frac{d}{ds}[\phi(s) - 2\epsilon\,\mathrm{Re}(Be^{i\omega_{RN}s})]\right| + |r(s) - r_-| + |Q(s) - \mathbf{e}| + \left|\frac{d}{ds}\log\big(\Omega^2\big)(s) - 2K_-\right| \lesssim \epsilon^2 s \lesssim \epsilon. \quad (4.1.1)$$

For $s \in \mathcal{LB}$, we find also the following estimate for $-r\dot{r}(s)$:

$$\left|-r\dot{r}(s) - \frac{4|B|^2\omega_{RN}^2\epsilon^2 r_-^2}{2|K_-|}\right| \lesssim e^{2K_- s} + \epsilon^4 s. \quad (4.1.2)$$

ii. (Collapsed oscillations and loss of charge). In the oscillation region $\mathcal{O} \coloneqq \{s_O(\epsilon) \leq s \leq s_{PK}(\epsilon)\}$, we have the following Bessel-type oscillations for the scalar field: for some $\xi_0 = \frac{|K_-|}{4|B|^2\omega_{RN}} + O(\epsilon^2\log(\epsilon^{-1}))$,

$$\left|\phi(s) - \left(C_J(\epsilon)J_0\left(\frac{\xi_0 r^2(s)}{r_-^2\epsilon^2}\right) + C_Y(\epsilon)Y_0\left(\frac{\xi_0 r^2(s)}{r_-^2\epsilon^2}\right)\right)\right| \lesssim \epsilon^2\log(\epsilon^{-1}), \quad (4.1.3)$$

$$\left|\frac{d}{ds}\left(\phi(s) - \left(C_J(\epsilon)J_0\left(\frac{\xi_0 r^2(s)}{r_-^2\epsilon^2}\right) + C_Y(\epsilon)Y_0\left(\frac{\xi_0 r^2(s)}{r_-^2\epsilon^2}\right)\right)\right)\right| \lesssim \epsilon^2\log(\epsilon^{-1}), \quad (4.1.4)$$

where the constants $C_J(\epsilon)$, $C_Y(\epsilon)$ are highly oscillatory in ϵ; namely for $\mathfrak{W}(M, \mathbf{e}, q_0, \Lambda) > 0$ given by

$$\mathfrak{W}(M, \mathbf{e}, \Lambda, q_0) = \sqrt{\frac{\omega_{RN}}{2|K_-|}} = \sqrt{\frac{|q_0\mathbf{e}|}{|\frac{\mathbf{e}^2}{r_-^2} - \frac{\Lambda}{3}r_+(r_+ + 2r_-)|}} > 0, \quad (4.1.5)$$

one finds that

$$\left|C_J(\epsilon) - \frac{\sqrt{\pi}}{2}\mathfrak{W}^{-1}\cos(\Theta(\epsilon))\right| + \left|C_Y(\epsilon) - \frac{\sqrt{\pi}}{2}\mathfrak{W}^{-1}\sin(\Theta(\epsilon))\right| \lesssim \epsilon^2\log(\epsilon^{-1}), \quad (4.1.6)$$

$$\left|\Theta(\epsilon) - \frac{\epsilon^{-2}}{8|B|^2\mathfrak{W}}\right| \lesssim \log(\epsilon^{-1}). \quad (4.1.7)$$

For $s \in \mathcal{O}$, one has (note this improves upon (4.1.2) *in $\mathcal{LB} \cap \mathcal{O}$):*

$$\left|-r\dot{r}(s) - \frac{4|B|^2\omega_{RN}^2\epsilon^2 r_-^2}{2|K_-|}\right| \lesssim \epsilon^4\log(\epsilon^{-1}). \quad (4.1.8)$$

Moreover, the charge Q transitions from $\mathbf{e}$ to $Q_\infty(M,\mathbf{e},\Lambda) := \frac{3}{4}\mathbf{e}+\Lambda\frac{r_-^2 r_+(2r_-+r_+)}{12\mathbf{e}}$ (up to $O(\epsilon^{2-})$ errors), and

$$\left|Q(s)-\mathbf{e}+(\mathbf{e}-Q_\infty)\left(1-\frac{r^2(s)}{r_-^2}\right)\right| \lesssim \epsilon^2\log\big(\epsilon^{-1}\big) \text{ for all } s\in\mathcal{O}, \tag{4.1.9}$$

$$\frac{Q_\infty(M,\mathbf{e},0)}{\mathbf{e}} = \frac{3}{4}, \tag{4.1.10}$$

$$\left\{\frac{Q_\infty(M,\mathbf{e},\Lambda)}{\mathbf{e}}:\ (M,\mathbf{e},\Lambda)\in\mathcal{P}_{se},\ \Lambda<0\right\} = \left(\frac{1}{2},\frac{3}{4}\right), \tag{4.1.11}$$

$$\left\{\frac{Q_\infty(M,\mathbf{e},\Lambda)}{\mathbf{e}}:\ (M,\mathbf{e},\Lambda)\in\mathcal{P}_{se},\ \Lambda>0\right\} = \left(\frac{3}{4},1\right). \tag{4.1.12}$$

iii. (Logarithmic Bessel-type divergence). In the proto-Kasner region $\mathcal{PK} := \{s_{PK}(\epsilon)\le s\le s_i(\epsilon)\}$, the Bessel-type behavior persists with modified coefficients, namely there exist

$$C_{YK}(\epsilon) = C_Y(\epsilon)+O(\epsilon^2\log\big(\epsilon^{-1}\big)),$$
$$C_{JK}(\epsilon) = C_J(\epsilon)+O(\epsilon^2\log\big(\epsilon^{-1}\big)), \xi_K = \xi_0+O(\epsilon^2\log\big(\epsilon^{-1}\big)),$$

such that (4.1.3), (4.1.4) remain true for all $s\in\mathcal{PK}$, after replacing $(C_Y(\epsilon),C_J(\epsilon),\xi_0)$ by $(C_{YK}(\epsilon),C_{JK}(\epsilon),\xi_K)$. Consequently, we have the following logaritheoremic growth: for all $s\in\mathcal{PK}$,

$$\left|\phi(s)+\frac{2}{\pi}C_{YK}(\epsilon)\log\left(\frac{r_-^2\epsilon^2}{\xi_K r^2(s)}\right)-C_{YK}(\epsilon)-2\pi^{-1}(\gamma-\log 2)C_{JK}(\epsilon)\right| \lesssim \epsilon^2\log\big(\epsilon^{-1}\big), \tag{4.1.13}$$

$$\left|\frac{d}{ds}\left(\phi(s)+\frac{2}{\pi}C_{YK}\log\left(\frac{r_-^2\epsilon^2}{\xi_K r^2(s)}\right)\right)\right| \lesssim \epsilon^2\log\big(\epsilon^{-1}\big). \tag{4.1.14}$$

Moreover, (4.1.8) still holds for $s\in\mathcal{PK}$, while the charge Q remains close to its value at s_{PK}; for all $s\in\mathcal{PK}$:

$$|Q(s)-Q_\infty| \lesssim \epsilon^2\log\big(\epsilon^{-1}\big). \tag{4.1.15}$$

Finally, in the sub-region $\mathcal{PK}' = \{s_{K_1}(\epsilon)\le s\le s_i(\epsilon)\}\subset\mathcal{PK}$, the quantity $\Psi(s)$ defined by

$$\Psi(s) := -r(s)\frac{d\phi}{dr}(s) = \frac{r(s)}{-\frac{dr}{ds}(s)}\frac{d\phi}{ds}(s)$$

obeys the estimates:

$$\left|\Psi(s)+\frac{2}{\sqrt{\pi}}\mathfrak{W}^{-1}\sin(\Theta(\epsilon))\right| \lesssim \epsilon^2\log\big(\epsilon^{-1}\big). \tag{4.1.16}$$

$$|\Psi(s)-\Psi(s_i)| \lesssim r^2(s)\log\big(\epsilon^{-1}\big) \lesssim \epsilon^4\log\big(\epsilon^{-1}\big). \tag{4.1.17}$$

iv. In the Kasner region $\mathcal{K} := \{s_i \leq s < s_\infty(\epsilon)\}$, $-r\dot{r}(s)$ *obeys the lower bound*

$$-r\dot{r}(s) \geq \frac{4|B|^2\omega_{RN}^2\epsilon^2 r_-^2}{2|K_-|} \cdot \frac{1}{2}\eta^2. \tag{4.1.18}$$

The quantity Ψ *obeys the following ODE: introducing* $R := \log\left(\frac{r_-}{r}\right)$,

$$\frac{d\Psi}{dR} = -\Psi(\Psi-\alpha)(\Psi-\alpha^{-1}) + \mathcal{F}, \tag{4.1.19}$$

$$|\alpha - \Psi(s_i)| \lesssim e^{-\delta_0\epsilon^{-2}},\ \mathcal{F}(R) \lesssim e^{-\delta_0\epsilon^{-2}} r(R). \tag{4.1.20}$$

Finally, the charge retention remains in the sense that (4.1.15) *is also valid for all* $s \in \mathcal{K}$.

4.1.3 Second statement: Kasner asymptotics in the $\mathcal{PK}$ and $\mathcal{K}$ regions

We now enter into the details of the regions $\mathcal{PK}' \cup \mathcal{K}$, in which the Kasner-like behavior manifests. The following theorem requires a (slightly) stronger assumption on ϵ than Theorem 4.1.1.

Theorem 4.1.2. *Let* $(M, \mathbf{e}, \Lambda) \in \mathcal{P}_{se}$ *with* $\mathbf{e} \neq 0$, $q_0 \neq 0$, $m^2 \in \mathbb{R}$. *Then, for any sufficiently small* $\eta, \sigma > 0$, *there exists* $\epsilon_0(M, \mathbf{e},\ \Lambda, q_0, m^2, \eta, \sigma) > 0$ *such that there exists a subset* $E'_{\eta,\sigma} \subset E_\eta$ *satisfying* $\frac{|(-\delta,\delta)\setminus E'_{\eta,\sigma}|}{2\delta} = O(\eta+\sigma)$ *for any* $0 < \delta \leq \epsilon_0$, *and such that* $E'_{\eta,\sigma}$ *may be expressed as a disjoint union* $E'_{\eta,\sigma} = E'^{\,bo}_{\eta,\sigma} \cup E'^{\,Nbo}_{\eta,\sigma}$, *where*

$$\epsilon \in E'^{\,bo}_{\eta,\sigma} \text{ if } \eta < |\Psi(s_i)| = \frac{2}{\sqrt{\pi}}\mathfrak{W}^{-1}|\sin(\Theta(\epsilon))| + O(\epsilon^2\log(\epsilon^{-1})) \leq 1-\sigma < 1. \tag{4.1.21}$$

$$\epsilon \in E'^{\,Nbo}_{\eta,\sigma} \text{ if } |\Psi(s_i)| = \frac{2}{\sqrt{\pi}}\mathfrak{W}^{-1}|\sin(\Theta(\epsilon))| + O(\epsilon^2\log(\epsilon^{-1})) \geq 1+\sigma > 1. \tag{4.1.22}$$

We call these respectively the Kasner bounce case and the no bounce inversion case.

Moreover, we have the following two possibilities.

- *If* $\mathfrak{W}(M, \mathbf{e}, \Lambda, q_0) \geq \frac{2}{\sqrt{\pi}}$, *then* $E'^{\,Nbo}_{\eta,\sigma} = \emptyset$, *and therefore* $\frac{|(-\epsilon,\epsilon)\setminus E'^{\,bo}_{\eta,\sigma}|}{2\epsilon} = O(\eta+\sigma)$, *for* η, σ *small.*
- *If* $\mathfrak{W}(M, \mathbf{e}, \Lambda, q_0) < \frac{2}{\sqrt{\pi}}$, *then* $|E'^{\,Nbo}_{\eta,\sigma}|, |E'^{\,bo}_{\eta,\sigma}| > 0$, *and for* η, σ *small:*

$$\frac{|E'^{\,Nbo}_{\eta,\sigma} \cap (-\epsilon,\epsilon)|}{2\epsilon} = \frac{2}{\pi}\arcsin\left(1 - \frac{\sqrt{\pi}}{2}\mathfrak{W}\right) + O(\sigma),$$
$$\frac{|E'^{\,bo}_{\eta,\sigma} \cap (-\epsilon,\epsilon)|}{2\epsilon} = 1 - \frac{2}{\pi}\arcsin\left(1 - \frac{\sqrt{\pi}}{2}\mathfrak{W}\right) + O(\eta+\sigma).$$

Then, for all $\epsilon \in E'_{\eta,\sigma}$, *we have the following Kasner-like behavior, where, in what follows,* $b_- = \frac{|B|\omega_{RN}}{|K_-|}$.

1. *In the first Kasner region* $\mathcal{K}_1$, *we have the following, recalling* α *from* (4.1.20).

(a) *In the no Kasner bounce case* (4.1.22)*, we have* $|\alpha| > 1$*,* $\mathcal{K}_1 = \mathcal{PK}' \cup \mathcal{K}$*, and for all* $s \in \mathcal{K}_1$*,*

$$|\Psi(s) - \alpha| \lesssim \epsilon^2 \cdot \left(\frac{r(s)}{r(s_{PK})}\right)^{\beta}, \tag{4.1.23}$$

where we define $\beta := \min\{\frac{1}{2}, \alpha^2 - 1\} > 0$*. Moreover, the metric takes the following Kasner-like form*

$$g = -d\tau^2 + \mathcal{X}_1(1 + \mathfrak{E}_{X,1}(\tau))\, \tau^{\frac{2(\alpha^2-1)}{\alpha^2+3}}\, dt^2 + \mathcal{R}_1(1 + \mathfrak{E}_{R,1}(\tau))\, r_-^2 \tau^{\frac{4}{\alpha^2+3}}\, d\sigma_{\mathbb{S}^2}, \tag{4.1.24}$$

$$\left|\log \mathcal{X}_1 + \frac{\alpha^2+1}{\alpha^2+3}\frac{4|K_-|^2}{|B|^2}\,\epsilon^{-2}\right| + \left|\log \mathcal{R}_1 - \frac{1}{\alpha^2+3}\frac{4|K_-|^2}{|B|^2}\,\epsilon^{-2}\right| \lesssim \log\big(\epsilon^{-1}\big), \tag{4.1.25}$$

$$|\mathfrak{E}_{X,1}(\tau)| + |\mathfrak{E}_{R,1}(\tau)| \lesssim \epsilon^2 \cdot \left(\frac{\tau}{\tau(s_{K_1})}\right)^{\frac{2\beta}{\alpha^2+3}}, \tag{4.1.26}$$

where τ *is the proper time*[1]*, and we call* $(p_1, p_2, p_3) = (\frac{\alpha^2-1}{\alpha^2+3}, \frac{2}{\alpha^2+3}, \frac{2}{\alpha^2+3}) \in (0,1)^3$ *the Kasner exponents.*

(b) *In the Kasner bounce case* (4.1.21)*, we have* $|\alpha| < 1$*,* $\mathcal{K}_1 = \{s_{K_1} \le s \le s_{in}\} \subset \mathcal{PK}' \cup \mathcal{K}$*, and for all* $s \in \mathcal{K}_1$*,*

$$|\Psi(s) - \alpha| \lesssim \epsilon^2, \tag{4.1.27}$$

where $s_{in} = \frac{b_-^{-2}}{4|K_-|} + O(\log(\epsilon^{-1}))$ *is such that*

$$r(s_{in}) = r_- \cdot \exp\left(-\frac{\epsilon^{-2}}{2b_-^2 \cdot (1-\alpha^2)} + O(\log(\epsilon^{-1}))\right). \tag{4.1.28}$$

Moreover, the metric takes the following Kasner-like form, where $\tau_0 > 0$ *is a constant.*

$$g = -d\tau^2 + \mathcal{X}_1 \cdot (1 + \mathfrak{E}_{X,1}(\tau))\, (\tau - \tau_0)^{\frac{2(\alpha^2-1)}{\alpha^2+3}}\, dt^2 + \mathcal{R}_1 \cdot (1 + \mathfrak{E}_{R,1}(\tau))\, r_-^2 (\tau - \tau_0)^{\frac{4}{\alpha^2+3}}\, d\sigma_{\mathbb{S}^2}. \tag{4.1.29}$$

We call $(p_1, p_2, p_3) = (\frac{\alpha^2-1}{\alpha^2+3}, \frac{2}{\alpha^2+3}, \frac{2}{\alpha^2+3}) \in (-\frac{1}{3}, 0) \times (\frac{1}{2}, \frac{2}{3})^2$ *the first Kasner exponents. Moreover,* $\mathcal{X}_1$ *and* $\mathcal{R}_1$ *obey* (4.1.25)*, and for all* $\tau(s_{K_1}) \le \tau \le \tau(s_{in})$*:*

$$|\mathfrak{E}_{X,1}(\tau)| + |\mathfrak{E}_{R,1}(\tau)| \lesssim \epsilon^2. \tag{4.1.30}$$

2. *In the Kasner bounce region* $\mathcal{K}_{bo}$ *(only in the Kasner bounce case* (4.1.21)*), we have for all* $s \in \mathcal{K}_{bo}$*,*

$$r(s) = r_- \cdot \exp\left(-\frac{\epsilon^{-2}}{2b_-^2 \cdot (1-\alpha^2)} + O(\log(\epsilon^{-1}))\right). \tag{4.1.31}$$

Moreover, in terms of proper time, we have

$$0 < \tau(s_{in}) - \tau(s_{out}) \lesssim \exp\left(-\frac{\epsilon^{-2}}{b_-^2 \cdot (1-\alpha^2)} + O(\log(\epsilon^{-1}))\right). \tag{4.1.32}$$

[1]More precisely, τ a is past-directed timelike variable, orthogonal to the hypersurfaces Σ_s, and normalized such that $g(d\tau, d\tau) = -1$ and $\tau = 0$ at the spacelike singularity $\{r = 0\}$.

3. *In the second Kasner region $\mathcal{K}_2$ (only in the Kasner bounce case (4.1.21)), we define $\beta := \min\{\frac{1}{2}, \alpha^{-2}-1\} > 0$ (since $|\alpha|<1$). We have for all $s \in \mathcal{K}_1$:*

$$|\Psi(s) - \alpha^{-1}| \lesssim \epsilon^2 \cdot \left(\frac{r(s)}{r(s_{out})}\right)^{\beta}. \tag{4.1.33}$$

Moreover, the metric takes the following Kasner-like form, for all $0 < \tau \leq \tau(s_{out})$

$$g = -d\tau^2 + \mathcal{X}_2 \cdot (1+\mathfrak{E}_{X,2}(\tau))\, \tau^{\frac{2(1-\alpha^2)}{1+3\alpha^2}}\, dt^2 + \mathcal{R}_2 \cdot (1+\mathfrak{E}_{R,2}(\tau))\, r_-^2 \tau^{\frac{4\alpha^2}{1+3\alpha^2}}\, d\sigma_{\mathbb{S}^2}. \tag{4.1.34}$$

$$\left|\log \mathcal{X}_2 + \frac{1+\alpha^{-2}}{1+3\alpha^2}\frac{4|K_-|^2}{|B|^2}\,\epsilon^{-2}\right| + \left|\log \mathcal{R}_2 - \frac{1}{1+3\alpha^2}\frac{4|K_-|^2}{|B|^2}\,\epsilon^{-2}\right| \lesssim \log\big(\epsilon^{-1}\big), \tag{4.1.35}$$

$$|\mathfrak{E}_{X,2}(\tau)| + |\mathfrak{E}_{R,2}(\tau)| \lesssim \epsilon^2 \cdot \left(\frac{\tau}{\tau(s_{out})}\right)^{\frac{2\beta}{\alpha^{-2}+3}}. \tag{4.1.36}$$

We call $(p_1,p_2,p_3) = (\frac{1-\alpha^2}{1+3\alpha^2}, \frac{2\alpha^2}{1+3\alpha^2}, \frac{2\alpha^2}{1+3\alpha^2}) \in (0,1)^3$ the second Kasner exponents.

Finally, in terms of proper time, we have (recalling that τ is normalized so that $\lim_{s\to s_\infty}\tau(s) = 0$):

$$\exp\left(-\frac{1+\alpha^{-2}}{2b_-^2\cdot(1-\alpha^2)}\,\epsilon^{-2} + O(\log(\epsilon^{-1}))\right) \lesssim \tau(s_{out}) \lesssim \exp\left(-\frac{1+\alpha^{-2}}{2b_-^2\cdot(1-\alpha^2)}\,\epsilon^{-2} + O(\log(\epsilon^{-1}))\right). \tag{4.1.37}$$

Without loss of generality, we will choose $\epsilon > 0$ in all the subsequent sections.

4.1.4 Linear scattering in the Reissner-Nordström interior

In the proof of Theorem 4.1.1, we often need to make comparisons to various quantities in exact Reissner-Nordström(-dS/AdS). While this is straightforward for (r, Ω^2, Q), the scalar field ϕ vanishes in Reissner-Nordström, and we instead compare ϕ to solutions to the linear (*charged*) covariant Klein-Gordon equation in the Reissner-Nordström interior:

$$\mathbf{g}_{RN}^{\mu\nu}\mathbf{D}_\mu\mathbf{D}_\nu\phi = m^2\phi. \tag{4.1.38}$$

Here $\mathbf{g}_{RN}$ and $\mathbf{D}$ are the metric and covariant derivative in Reissner-Nordström(-dS/AdS), with A as specified in Section 2.4 and $m^2 \in \mathbb{R}$ and $q_0 \neq 0$ are fixed.

Since we are interested in the spatially homogeneous problem, we consider only solutions $\phi = \phi_{\mathcal{L}}$ to (4.1.38) that satisfy $T\phi_{\mathcal{L}} = 0$, $S\phi_{\mathcal{L}} = 0$, where $T = \partial_t$ is the Killing vector field of Reissner-Nordström(-dS/AdS) associated to stationarity and S is any vector field on the sphere.

Then denoting $\psi(s) = \phi_{\mathcal{L}}(t=0, r^* = s)$, where r^*, t are as defined in Section 2.4, it can be checked that ψ satisfies the following second order ODE in s (see, for instance, the equation (2.4.30)):

$$\ddot{\psi} = \frac{\Omega_{RN}^2(s)}{4}\frac{\dot{\psi}}{2r_{RN}(s)} - \frac{\Omega_{RN}^2(s)}{4}m^2\psi - q_0^2\left(\frac{\mathbf{e}}{r_+} - \frac{\mathbf{e}}{r_{RN}(s)}\right)^2\psi. \tag{4.1.39}$$

We define the quantities

$$\tilde{A}_{RN,\infty} = \frac{\mathbf{e}}{r_+} - \frac{\mathbf{e}}{r_-} \neq 0, \qquad \omega_{RN} \coloneqq |q_0 \tilde{A}_{RN,\infty}| > 0. \tag{4.1.40}$$

Then, following [94], we define four functions solving (4.1.39): $\psi_{\mathcal{H},1}, \psi_{\mathcal{H},2}, \psi_{\mathcal{CH},1}$ and $\psi_{\mathcal{CH},2}$. These satisfy the following asymptotics towards the event horizon $\mathcal{H} = \{s = -\infty\}$ and the Cauchy horizon $\mathcal{CH} = \{s = +\infty\}$

$$\psi_{\mathcal{H},1}(s) = 1 + o(1) \qquad \text{as } s \to -\infty, \tag{4.1.41}$$

$$\psi_{\mathcal{H},2}(s) = s + o(1) \qquad \text{as } s \to -\infty, \tag{4.1.42}$$

$$\psi_{\mathcal{CH},1}(s) = e^{i\omega_{RN}s} + o(1) \qquad \text{as } s \to +\infty, \tag{4.1.43}$$

$$\psi_{\mathcal{CH},2}(s) = \overline{\psi_{\mathcal{CH},1}}(s) = e^{-i\omega_{RN}s} + o(1) \qquad \text{as } s \to +\infty. \tag{4.1.44}$$

The results of [94] then imply the following, where the coefficient $B = B(M, q_0, \mathbf{e}, \Lambda, m^2)$ appearing below is exactly that of Theorem 4.1.1).

Proposition 4.1.3. *Recalling the definition of $2K_-(M, \mathbf{e}, \Lambda) < 0$ from Section 2.4 and $\alpha_-(M, \mathbf{e}, \Lambda) > 0$ from (2.4.9), there exists some constant $C > 0$ such that*

$$|\psi_{\mathcal{CH},1}(s) - e^{i\omega_{RN}s}| + \left|\frac{d\psi_{\mathcal{CH},1}}{ds}(s) - i\omega_{RN}e^{i\omega_{RN}s}\right| \leq C\Omega^2_{RN}(s) \leq 2C\alpha_- e^{2K_-s}, \tag{4.1.45}$$

$$|\psi_{\mathcal{CH},2}(s) - e^{-i\omega_{RN}s}| + \left|\frac{d\psi_{\mathcal{CH},2}}{ds}(s) + i\omega_{RN}e^{-i\omega_{RN}s}\right| \leq C\Omega^2_{RN}(s) \leq 2C\alpha_- e^{2K_-s}. \tag{4.1.46}$$

Furthermore, there exists a scattering coefficient $B = B(M, q_0, \mathbf{e}, \Lambda) \in \mathbb{C} \setminus \{0\}$ such that

$$\psi_{\mathcal{H},1}(s) = B\psi_{\mathcal{CH},1}(s) + \overline{B}\psi_{\mathcal{CH},2}(s) = 2\,\mathrm{Re}(B\psi_{\mathcal{CH},1}(s)). \tag{4.1.47}$$

Corollary 4.1.4. *Let $\phi_{\mathcal{L}}$ be the solution to (4.1.38) with constant data $\phi = \epsilon$ on the event horizon $\mathcal{H} = \{s = -\infty\}$. Then there exists $C(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ and $\tilde{S}(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ such that, for $s \geq \tilde{S}$, one has*

$$\left|\phi_{\mathcal{L}}(s) - B\epsilon e^{i\omega_{RN}s} - \overline{B}\epsilon e^{-i\omega_{RN}s}\right| \leq C\epsilon e^{2K_-s}. \tag{4.1.48}$$

$$\left|\dot{\phi}_{\mathcal{L}}(s) - i\omega_{RN}B\epsilon e^{i\omega_{RN}s} + i\omega_{RN}\overline{B}\epsilon e^{-i\omega_{RN}s}\right| \leq C\epsilon e^{2K_-s}. \tag{4.1.49}$$

4.2 Further context on the constructed hairy black holes

Before proceeding to the proof of Theorem 4.1.1 and Theorem 4.1.2, we elaborate slightly on the physical context of the hairy black holes constructed in our result, and their relevance to astrophysical gravitational collapse (in the $\Lambda = 0$ case), to other hairy black holes, to cosmology, and to so-called holographic superconductors (in the $\Lambda < 0$ case via the famous AdS-CFT correspodence). We also outline the key new phenomena together with ideas in their proof, particularly in the collapsed oscillations region $\mathcal{O}$ and the Kasner-like region $\mathcal{K}$.

4.2.1 Differences and connections with non-hairy black holes

The spacetimes constructed in Theorem 1.4.1 are spatially homogeneous and we interpret them as the interior region of so-called "hairy black holes". The main distinctive feature of "hairy black holes" is the presence of so-called *scalar hair*, meaning that the scalar field ϕ *does not decay on the event horizon* $\mathcal{H}^+$ and tends to a non-zero constant instead. (In the case of the hairy black holes of Theorem 1.4.1, ϕ is identically equal to this constant on $\mathcal{H} = \mathcal{H}_L \cup \mathcal{H}_R$, see (1.4.1)).

We have already explained that it is not possible to obtain an asymptotically flat spherically symmetric hairy black hole exterior solution of the Einstein–Maxwell–charged scalar field system with $\Lambda = 0$; however, as we will discuss in Section 4.2.5, the construction of a such hairy black hole exterior is possible for asymptotically-AdS data, with $\Lambda < 0$ in (Ein*). Such a construction will necessarily involve a static exterior region, while the corresponding interior is spatially homogeneous. This is due to the fact that for a t-independent black hole, t is a timelike coordinate in the exterior which, by definition, becomes a spacelike coordinate in the black hole interior.

Nonetheless, one may also wish to consider asymptotically flat (where $\Lambda = 0$) spacetimes which are relevant to the study of astrophysical black holes. In this context, one anticipates that solutions of the Einstein–Maxwell–charged scalar field system with regular Cauchy data *decay* towards a Reissner–Nordström exterior solution, in particular ϕ tends to 0 on $\mathcal{H}^+$ (in spherical symmetry, see [46, 110] for the Einstein–Maxwell–real scalar field system with $m^2 = q_0 = 0$, and also [146] for small $|q_0\mathbf{e}|$ on a fixed Reissner–Nordström exterior). The resulting black holes thus feature ϕ decaying on $\mathcal{H}^+$ at the following rate: for all $v > 1$,

$$|\phi|_{|\mathcal{H}^+}(v) + |D_v\phi|_{|\mathcal{H}^+}(v) \lesssim v^{-s}, \tag{4.2.1}$$

where v is a standard Eddington–Finkenstein type advanced-time coordinate and $s > \frac{1}{2}$. We will call such black holes "non-hairy" in the sequel, to mark the contrast with the black holes from Theorem 1.4.1.

The interior of the non-hairy black hole solving the Einstein–Maxwell–charged scalar field system in spherical symmetry was studied in [94, 142, 144, 147, 145] and their Penrose diagram was completely characterized (modulo issues related to locally naked singularities, see the second paragraph below). In this section, we will briefly explain these results and provide contrast with our hairy black hole interiors from Theorem 1.4.1. We also comment that one can use our hairy black hole interiors as a tool to retrieve information on non-hairy black holes.

Local structure of the non-hairy black hole interior near infinity i^+ We discuss the terminal boundary of the black hole interior. In the case of the hairy black holes of Theorem 1.4.1, it is entirely spacelike $\mathcal{S} = \{r = 0\}$. In contrast, for (spherically symmetric) non-hairy black holes solutions of, the terminal boundary is *not* entirely spacelike and admits a null component near i^+

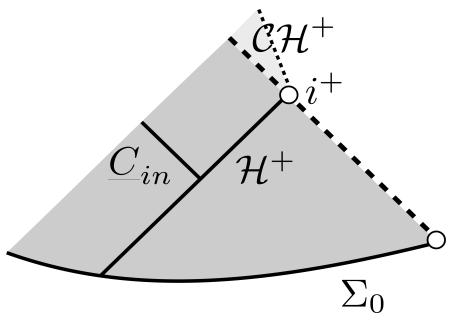


Figure 4.3: Penrose diagram of the spacetime corresponding to Theorem 4.2.1

– called the Cauchy horizon $\mathcal{CH}^+ \neq \emptyset$. This fact constitutes the most important difference between hairy and non-hairy black holes.

Theorem 4.2.1 ([142])**.** *Consider regular spherically symmetric characteristic data on $\mathcal{H}^+ \cup \underline{C}_{in}$, where $\mathcal{H}^+ := [1, +\infty)_v \times \mathbb{S}^2$, converging to a sub-extremal Reissner–Nordström exterior as* (4.2.1) *. Then, restricting $\underline{C}_{in}$ to be sufficiently short, the future domain of dependence of $\mathcal{H}^+ \cup \underline{C}_{in}$ is bounded by a Cauchy horizon $\mathcal{CH}^+$, namely a null boundary emanating from i^+ and foliated by spheres of strictly positive area-radius r, as depicted in Figure 4.3.*

Remark. We note that the presence of a Cauchy horizon $\mathcal{CH}^+ \neq \emptyset$ in the interior of dynamical black holes is not specific to spherical symmetry: for instance, it has been obtained for perturbations of Kerr for the Einstein vacuum equations (without symmetry) in [43]. Whether Theorem 4.2.1 can be generalized to the Einstein–Maxwell–charged scalar field equations without spherical symmetry is an interesting open problem, in view of the particularly slow decay of the form (4.2.1) one has to assume in the presence of matter (see [93, 108, 144] for an extended discussion of slow decay).

Global structure of non-hairy black hole interiors Theorem 4.2.1 only gives information on a local region located near i^+ (see Figure 4.3). The *global nature of the terminal boundary*, as it turns out (see Theorem 4.2.2), depends on the topology of the initial data; we distinguish two important cases:

a. The two-ended case (topology of time-slices: $\mathbb{R} \times \mathbb{S}^2$). The maximally-extended Schwarzschild, Reissner-Nordström, and Kerr spacetimes, and our hairy black holes from Theorem 1.4.1 possess the two-ended topology.

b. The one-ended case (topology of time-slices: $\mathbb{R}^3$) is the topology suitable to studying the gravitational collapse of a star into a black hole [31, 99, 108, 147] (referred to as "gravitational collapse" for short).

In the one-ended case b, the following is known about the terminal boundary.

Theorem 4.2.2 (Black hole interior in gravitational collapse, [147, 146, 145])**.** *We consider a one-ended black hole interior, under the assumptions of Theorem 4.2.1 and additional inverse-polynomial lower bounds on ϕ consistent with* (4.2.1)*. Then, assuming the absence of locally naked singularities emanating from the center Γ, there is a (non-empty) spacelike singularity $\mathcal{S} = \{r = 0\}$ and the Penrose diagram is given by the left-most of Figure 1.2.*

Remark. A locally naked singularity is an (outgoing) null boundary $\mathcal{CH}_\Gamma$ emanating from the center Γ. Assuming their absence in Theorem 4.2.2 is unavoidable, since examples of locally naked singularities have been constructed [33] for the Einstein–Maxwell–charged scalar field system. However, at least within solutions where $\mathbf{F} \equiv 0$, such locally naked singularities are non-generic within spherical symmetry [32, 35] and one may conjecture the same statement in the more general situation where $\mathbf{F} \neq 0$. See [99, 147] for an extended discussion of this delicate issue.

For two-ended black holes (irrelevant, however, to gravitational collapse), Theorem 4.2.2 is false because small perturbations of Reissner–Nordstrom obeying (4.2.1) feature no spacelike singularity [42]. However, it is conjectured [42, 49, 99] that, even in the two-ended case, large perturbations would yield a spacelike singularity $\mathcal{S} = \{r = 0\} \neq \emptyset$.

Remark. Note that for a two-ended black hole, the causal past of any compact subset of $\mathcal{S}$ intersects the event horizon $\mathcal{H}^+$ on a set with compact closure. This observation will be important in the discussion of the next paragraph, see Figure 4.4.

Connections between hairy and non-hairy black holes and related open problems We come back to our original motivation: understanding spacelike singularities inside black holes. Our goal is to construct a large class of (asymptotically flat) black holes with *both* a spacelike singularity $\mathcal{S} = \{r = 0\}$ and a null Cauchy horizon $\mathcal{CH}^+$, with precise quantitative information on (at least part of) $\mathcal{S} = \{r = 0\}$.

It is an open problem to provide a quantitative description of the spacelike singularity inside any charged[2] or rotating black hole (note indeed that Theorem 4.2.2 does not describe $\mathcal{S}$ quantitatively). In the two-ended case, however, we propose a construction of a black hole featuring both a null Cauchy horizon $\mathcal{CH}^+ \neq \emptyset$, and a spacelike singularity $\mathcal{S}$ which is partially described by the terminal boundary in Theorem 1.4.1.

Parameterize the two event horizons by $\mathcal{H}_R = \{(-\infty, v),\ v \in \mathbb{R}\}$, $\mathcal{H}_L = \{(u, -\infty),\ u \in \mathbb{R}\}$.

i. Fix $\phi_{|\mathcal{H}_R^+}(v) \equiv \epsilon$ for $v \leq A$, and $\phi_{|\mathcal{H}_L^+}(v) \equiv \epsilon$ for $u \leq A$, for some large A depending on ϵ. We evolve (non-uniquely) this characteristic data on $(\mathcal{H}_R \cap \{v \leq A\}) \cup (\mathcal{H}_L \cap \{u \leq A\})$ towards the past in the following fashion:

[2]Note, however, that it is possible to describe asymptotically Schwarzschild one-ended uncharged spherically symmetric black holes [3].

- Attach a small past-directed ingoing (respectively outgoing) cone $\underline{C}_R^+$ (respectively C_L^+) to the future-endpoint of $\mathcal{H}_R \cap \{v \leq A\}$) (respectively $(\mathcal{H}_L \cap \{u \leq A\})$).
- By local well-posedness, the past domain of influence of $\mathcal{H}_R \cap \{v \leq A\}) \cup \underline{C}_R^+$ (respectively $\mathcal{H}_L \cap \{y \leq A\} \cup C_L^+$) is the causal rectangle with past ingoing boundary $\underline{C}_R^-$ (respectively C_L^-).
- Evolving the initial data on $\underline{C}_R^- \cup C_L^-$ towards the past and appealing to local well-posedness again provides another smaller causal rectangle. This rectangle combines with the two previously-constructed rectangles into the (future) domain of influence of some $C_{out} \cup \underline{C}_{in}$.

We obtain a (non-unique) solution to the Einstein–Maxwell–charged scalar field system up to the bifurcate null cones $C_{out} \cup \underline{C}_{in}$ (see Figure 4.4).

ii. Extend C_{out} (respectively $\underline{C}_{in}$) into an outgoing (respectively ingoing) cone which is asymptotically flat using a standard truncation argument (see [33, 128]). The bifurcate null cones $\tilde{C}_{out} \cup \tilde{\underline{C}}_{in}$ thus obtained intersect (what should be thought of as) future null infinity $\mathcal{I}^+ = \mathcal{I}_L^+ \cup \mathcal{I}_R^+$, and $\phi_{|\tilde{C}_{out} \cup \tilde{\underline{C}}_{in}}$ decays towards $\mathcal{I}^+$.

iii. Solve *forward* for the above characteristic data on $\tilde{C}_{out} \cup \tilde{\underline{C}}_{in}$. By the domain of dependence property, the following spacetime region (consisting of the dark grey, orange, brown and yellow regions in Figure 4.4)

$$\{(u,v) : u \leq A,\ v \leq A\} \cap \mathcal{D}^+((\mathcal{H}_R \cap \{v \leq A\}) \cup (\mathcal{H}_L \cap \{u \leq A\})),$$

where $\mathcal{D}^+$ denotes the domain of dependence,

is isometric to a subset of the hairy black hole g_ϵ of Theorem 1.4.1 that contains a large portion of the spacelike singularity $\mathcal{S}$ (in green in Figure 4.4).

iv. Using that $\phi_{|\tilde{C}_{out} \cup \tilde{\underline{C}}_{in}}$ decays towards $\mathcal{I}^+$, prove that the decay condition (4.2.1) is satisfied on the new event horizons $\mathcal{H}_R^*$ and $\mathcal{H}_L^*$ (note that the event horizons $\mathcal{H}_R^*$, $\mathcal{H}_L^*$ for the newly constructed black hole do not coincide with the original event horizons $\mathcal{H}_R$, $\mathcal{H}_L$ of the hairy black hole). As a consequence of Theorem 4.2.1, obtain the existence of Cauchy horizons $\mathcal{CH}_L, \mathcal{CH}_R \neq \emptyset$.

We want to point out that the only unknown step is the proof of (4.2.1) (step iv) which relies on establishing polynomial decay on the event horizon. We note, furthermore, that for small charge $|q_0\mathbf{e}| \ll 1$ and $m^2 = 0$, step iv should follow from (a slight generalization) of [146].

Remark. Note that A must be chosen to be large enough (depending on ϵ) for the construction to feature any part of the spacelike singularity constructed in Theorem 1.4.1. As a consequence, the weighted norm $\|(1 + |v|^s)\phi\|_{L^\infty}$ (where $s > \frac{1}{2}$) is large, thus we are not in the small-data regime

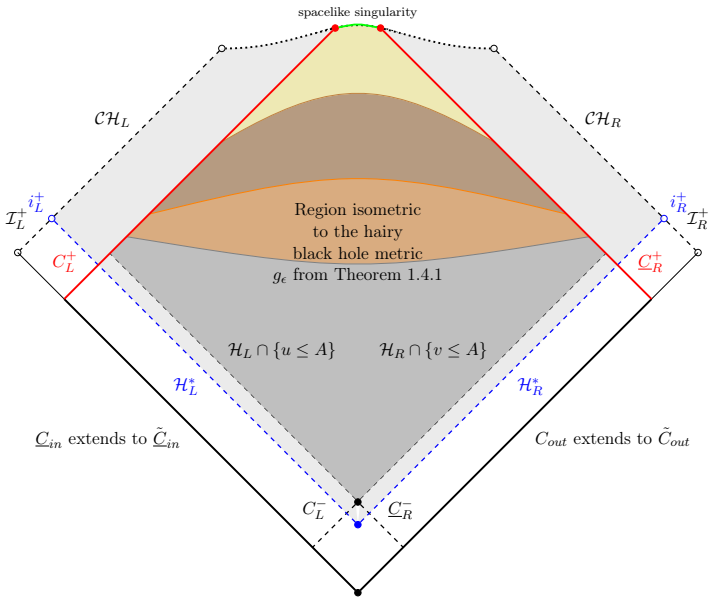


Figure 4.4: The proposed construction of a two-ended black hole with a spacelike singularity $\mathcal{S}$ via gluing argument. The union of the dark grey, orange, brown and yellow regions (including the green part of the spacelike singularity) is isometric to a subset of the hairy black hole of Figure 1.4. $\mathcal{I}_L^+$ and $\mathcal{I}_R^+$ are the components of null infinity $\mathcal{I}^+$.

considered in [42] (this is consistent with the non-empty spacelike singularity our construction, recalling the earlier discussion on the global structure of two-ended black holes).

Running steps i-iv successfully provides a (two-ended) black hole with a Cauchy horizon $\mathcal{CH}^+ \neq \emptyset$ and a spacelike singularity $\mathcal{S} \neq \emptyset$ partly given by $\mathcal{S}$ in the hairy black hole of Theorem 1.4.1. We formalize the above strategy into the following open problem.

Open problem a. Construct a (one or two)-ended black hole with a Cauchy horizon $\mathcal{CH}^+ \neq \emptyset$ and a spacelike singularity $\mathcal{S} \neq \emptyset$, which coincides with the hairy black hole singularity $\mathcal{S}$ from Theorem 1.4.1 away from $\mathcal{CH}^+ \cap \mathcal{S}$.

Our road-map towards a resolution of Open Problem a shows that the fluctuations and Kasner bounces of Theorem 1.4.1 **can play a role in the interior of asymptotically flat, non-hairy black holes**. We find it striking that, even when restricted to spherical symmetry, the spacelike singularity inside a black hole can obey such intricate dynamics. It is an interesting open problem to study the stability of these dynamics outside of any symmetry assumption and thus gauge their relevance to generic black hole solutions.

To understand a larger class of black holes with both a Cauchy horizon $\mathcal{CH}^+ \neq \emptyset$ and a spacelike singularity $\mathcal{S} \neq \emptyset$, it is of interest to perturb the hairy black hole of Theorem 1.4.1 within spherical symmetry but relaxing spatial homogeneity. Subsequently following steps i-iv, where g_ϵ is replaced by a perturbed spacetime, will yield even more general insights than Open Problem a into spherically

symmetric spacelike inside black holes, which we formalize in the following open problem (note Open Problem b is the charged ($q_0 \neq 0$) version of Open Problem v in [148]).

Open problem b. Consider (two-ended) initial data on $\mathcal{H}^+$ such that, instead of (1.4.1):

$$|\phi_{|\mathcal{H}^+}(v) - \epsilon| \leq |\epsilon|^N \cdot e^{-C_0 v} \tag{4.2.2}$$

for $\epsilon \in E_\eta$, as defined in Theorem 1.4.1, with $N > 0$ and $C_0 > 0$ sufficiently large constants. Prove (or disprove) that the terminal boundary is spacelike, and provide (reasonably) precise quantitative estimates.

Then, construct a (one or two)-ended black hole with a Cauchy horizon $\mathcal{CH}^+ \neq \emptyset$ and a spacelike singularity $\mathcal{S} \neq \emptyset$, which coincides with the above perturbed hairy black hole singularity $\mathcal{S}$ away from $\mathcal{CH}^+ \cap \mathcal{S}$.

We finally want to emphasize that our quantitative methods give hope to transpose some results of Theorem 1.4.1 to towards Open Problem b. We hope to return to this these very interesting questions in future work.

4.2.2 Comparison with hairy black hole interiors for other matter models

The charged hairy black holes with uncharged matter from [148] An alternative to studying the Einstein–Maxwell–charged scalar field system with $q_0 \neq 0$ is to study the uncharged scalar field case $q_0 = 0$ where the Maxwell field $\mathbf{F} \neq 0$ does not interact with ϕ. This was first done numerically in [74] and then rigorously by the second author [148] and qualified as "violent nonlinear collapse". It is remarkable that the behavior in the $q_0 = 0$ case differs drastically from what we found in the $q_0 \neq 0$ case in Theorem 1.4.1, as the following result shows.

Theorem 4.2.3 ([148])**.** *We make the same assumptions as Theorem 1.4.1, except that now $q_0 = 0$, hence $\mathbf{F} = \frac{\mathbf{e}}{r^2(s)}\Omega^2 ds \wedge dt$, where $\mathbf{e} \neq 0$. Then, for almost all sub-extremal parameters $(M, \mathbf{e}, \Lambda, m^2)$, there exists $\epsilon_0(M, \mathbf{e}, \Lambda, m^2) > 0$ such that, for all $0 < |\epsilon| < \epsilon_0$, the spacetime $(\mathcal{M}, \mathbf{g})$ ends at a spacelike singularity $\mathcal{S} = \{r = 0\}$ asymptotically described by a Kasner metric with exponents $(p_1, p_2, p_3) = (1, 0, 0) + O(\epsilon^2) \in (0, 1)^3$ and given by Figure 4.5. Moreover, the Kretschmann scalar $\mathcal{K} = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ blows up at a rate $r^{-C\cdot\epsilon^{-2}+O(\epsilon^{-1})}$ on $\mathcal{S} = \{r = 0\}$ for $C(M, \mathbf{e}, \Lambda, m^2) > 0$.*

We point out the following similarities and differences between Theorem 4.2.3 and Theorem 1.4.1.

1. In both cases, the terminal boundary is a spacelike singularity $\mathcal{S} = \{r = 0\}$ approximately described by a Kasner-like metric with positive Kasner exponents (compare Figure 1.4 and Figure 4.5).

2. In both cases, the early regions are similar and governed by the almost formation of a Cauchy horizon.

3. In both cases, the Maxwell charge Q is uniformly bounded away from 0: in Theorem 4.2.3, this is trivial ($Q = \mathbf{e} \neq 0$ is constant), in Theorem 1.4.1, this is item 5, a surprising property that we call "charge retention".

4. Even at the linear level (i.e. (2.0.6) on a fixed Reissner–Nordström interior), (1.4.4) is not true if $q_0 = 0$: the scalar field does not oscillate, it grows instead like $\phi \approx \epsilon \cdot s$, where $\mathcal{CH}^+ = \{s = \infty\}$ (except possibly for an exceptional set of $(M, \mathbf{e}, \Lambda, m^2)$ of 0-Lebesgue measure that leads to the absence of growth, see [90, 148], which is why one restricts to *almost* every parameter in Theorem 4.2.3).

5. The oscillating profile (1.4.4) turns it a Bessel-type growing oscillation as $r \to 0$, after which $|\phi|$ becomes $O(1)$ around $r \approx \epsilon$ (collapsed oscillations). There is no such mechanism in Theorem 4.2.3.

6. The final Kasner exponent $p_1 \in (0, 1)$ in Theorem 1.4.1 is related to $|\sin\big(\omega_0 \epsilon^{-2} + O(\log\big(\epsilon^{-1}\big))\big)|$, but we restrict ϵ so that p_1 is bounded away from 0 and 1, so there is no overlap (by our assumptions) with any of the Kasner exponents obtained in Theorem 4.2.3 where $|p_1 - 1| \lesssim \epsilon^2$.

7. As a consequence, the collapse in Theorem 1.4.1 is *not violent* but instead rapidly fluctuating in ϵ: one can easily see that $\mathcal{K}$ blows up at a rate r^{-q}, where q depends on $\sin\big(\omega_0 \cdot \epsilon^{-2} + O(\log\big(\epsilon^{-1}\big))\big)$.

8. There is no Kasner bounce in the $q_0 = 0$ setting: in fact, in Theorem 4.2.3 one proves that the final exponent p_1 lies in $(0, 1)$, so there is no mechanism triggering a Kasner bounce. In contrast, in Theorem 1.4.1, in the regimes where $|\sin\big(\omega_0 \cdot \epsilon^{-2} + O(\log\big(\epsilon^{-1}\big))\big)|$ is too small, a Kasner regime with $\acute{p_1} < 0$ forms, which is unstable, and ultimately disappears under Kasner bounce, giving rise to a second Kasner regime with $p_1 \in (0, 1)$.

We also remark that Theorem 1.4.1 restricts ϵ to a subset of $(-\epsilon_0, \epsilon_0) \setminus \{0\}$, which is the complement of a set of small measure, while Theorem 4.2.3 does not have this restriction. The restriction is to ensure that the final p_1 is bounded away from $\{0, 1\}$ in Theorem 1.4.1, as we explained above.

Finally, we note that Theorem 4.2.3 should lead to a resolution of Open Problem a in the case of an uncharged, massive scalar field (i.e. with $q_0 = 0$, $m^2 \neq 0$), upon the proof of (4.2.1) for $m^2 \neq 0$ (i.e. step iv in the last paragraph of Section 4.2.1).

Other hairy black holes The study of spatially homogeneous hairy black holes has been abundant both in the mathematics and physics literature: we first mention the important examples of Einstein–Yang–Mills hairy black holes [23, 25, 28, 55, 132, 135]. For the Einstein–Yang–Mills black holes, the above work suggest the presence of a spacelike singularity preceded by (potentially many)

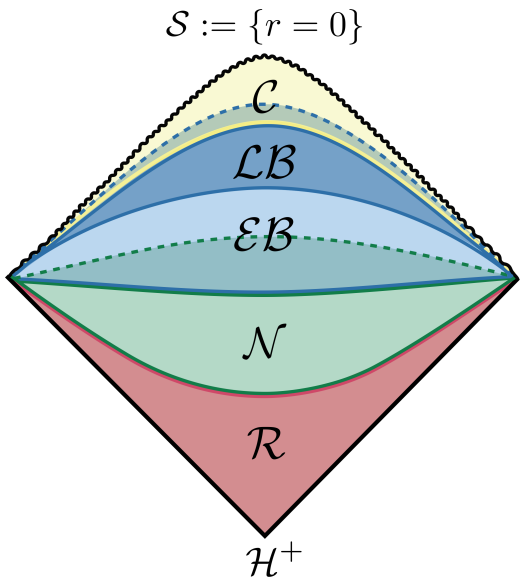


Figure 4.5: The Penrose diagram of the hairy black hole interiors constructed in Theorem 4.2.3.

Kasner bounces, although the qualitative behavior is different than what we obtained for charged scalar fields in Theorem 1.4.1. Finally, we mention the existence of rotating hairy black holes with massive Klein–Gordon fields [29, 51, 79]; The study of the interior of these rotating hairy black holes remains an open problem. We refer the reader to the introductions of [29, 148] for an extended discussion of various hairy black holes.

4.2.3 The collapsed oscillations resulting from the charge of the scalar field

The collapsed oscillations occur in a region $\mathcal{O} = \{\epsilon \lessapprox r \lessapprox r_-\}$ (see Figure 1.4). The key point is that, schematically, ϕ will be shown to obey the following Bessel equation of order 0 in $\mathcal{O}$, with respect to a new variable which is the renormalized square of the area-radius $z := \frac{r^2}{\epsilon^2}$:

$$\frac{d}{dz}\left(z\frac{d\phi}{dz}\right) + \xi_0^2 z\phi = error. \tag{4.2.3}$$

Here $\xi_0 \neq 0$ is a constant proportional[3] to q_0. To simplify the discussion here, we normalize $\xi_0 = 1$. Since $1 \lessapprox z \lessapprox \epsilon^{-2}$, we need to understand the behavior for large z: it is given by damped oscillations of the form

$$Y_0(z) \sim \sqrt{\frac{2}{\pi z}}\cos\Big(z - \frac{\pi}{4}\Big) \text{ or } J_0(z) \sim \sqrt{\frac{2}{\pi z}}\sin\Big(z - \frac{\pi}{4}\Big) \text{ as } z \to +\infty. \tag{4.2.4}$$

Note, however, that z is a past-directed timelike variable, so the damping is "backwards-in-time". Thus $|Y_0|(z)$, $|J_0|(z) \lesssim \epsilon$ on the past boundary $z \sim \epsilon^{-2}$ of $\mathcal{O}$, but $|Y_0|(z)$, $|J_0|(z) \lesssim 1$ on the

[3]One sees, as predicted by Theorem 4.2.3 (see [148]), that in the $q_0 = 0$ case, we have $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) \approx 0$, hence $r\frac{d\phi}{dr} \approx constant = \epsilon^{-1}$, which is why in the $q_0 = 0$ case, the behavior is violent, and not fluctuating as in the $q_0 \neq 0$ case, see also the discussion in Section 4.2.2.

future boundary $z \approx 1$ of $\mathcal{O}$; modulo the oscillations, this means that *the scalar field amplitude has experienced growth of size* ϵ^{-1} in $\mathcal{O}$.

Remark. Note that, as long as r is bounded away from 0, (4.2.3) is consistent with (linear) oscillations giving rise to (1.4.4): it is only as r gets close to 0 that these oscillations provide growth, hence the name "collapsed oscillations". We will show, however, that as soon as $r \ll \epsilon$, ϕ no longer oscillates, see Section 4.2.4.

The algebraic relations connecting (4.2.4) to the ODE initial conditions $\phi(r \approx r_-)$ will ultimately show that ϕ has the following schematic form at the exit of the collapsed oscillations region $\mathcal{O}$:

$$\phi(r) \approx C_J \cos\big(\omega_0 \cdot \epsilon^{-2} + O(\log(\epsilon^{-1}))\big) J_0\left(\xi_0 \frac{r^2}{\epsilon^2}\right) + C_Y \sin\big(\omega_0 \cdot \epsilon^{-2} + O(\log(\epsilon^{-1}))\big) Y_0\left(\xi_0 \frac{r^2}{\epsilon^2}\right). \quad (4.2.5)$$

Contrary to appearances, (4.2.5) is not symmetric in J_0 and Y_0: when $r \ll \epsilon$, the function $Y_0\left(\xi_0 \frac{r^2}{\epsilon^2}\right)$ dominates $J_0\left(\xi_0 \frac{r^2}{\epsilon^2}\right)$, since the Bessel functions $J_0(z)$ and $Y_0(z)$ obey the asymptotics

$$J_0(z) = O(1) \text{ and } Y_0(z) \sim \log\big(z^{-1}\big) \text{ as } z \to 0. \quad (4.2.6)$$

Hence, for $e^{-\delta_0 \epsilon^{-2}} \ll r \ll \epsilon$ (the lower bound will be explained in Section 4.2.4), we schematically show

$$\phi(r) \approx C_Y \sin\big(\omega_0 \cdot \epsilon^{-2} + O(\log(\epsilon^{-1}))\big) \log\left(\xi_0^{-1} \frac{\epsilon^2}{r^2}\right) \approx C_Y \sin\big(\omega_0 \cdot \epsilon^{-2} + O(\log(\epsilon^{-1}))\big) \log\Big(\frac{\epsilon}{r}\Big). \quad (4.2.7)$$

Remark. Most other quantities, such as the charge Q, already determine their final values at $r = 0$ inside $\mathcal{O}$ (up to $O(\epsilon)$-errors). Therefore, the charge retention mechanism from Theorem 1.4.1 results from an explicit computation in $\mathcal{O}$, see Lemma 4.4.7.

For more details on Bessel equations and functions, we refer the reader to our Appendix A.

4.2.4 The Kasner bounce mechanism within our spacetimes

We will now explain the schematic mechanism behind the Kasner bounce phenomenon as obtained in our Theorem 1.4.1, which is the second main novelty compared to the $q_0 = 0$ case of [148]. We will show that the Kasner bounce, *in the regimes where it occurs* (namely, for $\epsilon \in E'^{\,bo}_{\eta,\sigma}$ of positive measure), is located in a region of the following form; for constants $D > 0$, $N > 0$:

$$\mathcal{K}_{bo} \subset \{e^{-D \cdot \epsilon^{-2}} \epsilon^N \lesssim r \lesssim e^{-D \cdot \epsilon^{-2}} \epsilon^{-N}\}. \quad (4.2.8)$$

We define the key quantity Ψ, which is a dimensionless derivative of ϕ: for $r_-(M, \mathbf{e}, \Lambda) > 0$ and $\delta_0(M, \mathbf{e}, \Lambda) > 0$ to be defined later, let

$$\Psi := -r\frac{d\phi}{dr}, \text{ and define } \Psi_i := \Psi\big|_{r=e^{-\delta_0 \cdot \epsilon^{-2}} r_-}. \tag{4.2.9}$$

The condition for the presence of a bounce will end up being

$$\eta \leq |\Psi_i| \leq 1 - \sigma, \text{ for some } \eta, \sigma > 0 \text{ independent of } \epsilon. \tag{bo}$$

The reason for assuming $\eta \leq |\Psi_i|$ is that, based on numerics (see Section 4.2.5), there could be multiple Kasner bounces when$|\Psi_i|$ is close to 0, and the dynamics would be even more complicated. If $|\Psi_i|$ is too close to 1, though we are still able to produce a spacelike singularity (see already Theorem 4.1.1), we do not claim further quantitative estimates, as some Kasner exponents degenerate towards 0 in this case

As a consequence of (4.2.7), we find that, for some $C(M, \mathbf{e}, \Lambda, m^2, q_0) \neq 0$,

$$|\Psi_i| \approx |C| \cdot |\sin\big(\omega_0 \cdot \epsilon^{-2} + O(\log\big(\epsilon^{-1}\big))\big)|. \tag{4.2.10}$$

Combining (bo) and (4.2.10) explains heuristically why the presence of a bounce depends on ϵ and why any small neighborhood of 0 of the form $(-\delta, \delta)$ still contains infinitely many spacetimes featuring a bounce.

Our non-bounce condition in Theorem 1.4.1 is not the complement of (bo), it is instead

$$|\Psi_i| \geq 1 + \sigma, \text{ for some } \sigma > 0 \text{ independent of } \epsilon., \tag{no-bo}$$

where, as above, we arrange that $|\Psi_i|$ is not too close to 1 to avoid the Kasner exponents degenerating.

We now explain why it is that if (bo) is satisfied, there is a bounce, whereas if (no-bo) is satisfied then there is no bounce. Since the wave equation is second-order, Ψ should satisfy a first-order ODE. Though our system is highly nonlinear, this ODE surprisingly turns out to be presentable in a simple form, written schematically[4] as follows:

$$\frac{d\Psi}{dR} = -\Psi(\Psi - \Psi_i)(\Psi - \Psi_i^{-1}) + error, \text{ where } R := \log\Big(\frac{r_-}{r}\Big), \tag{4.2.11}$$

where the error terms are of size $O(e^{-\delta_0 \epsilon^{-2}})$ for some $\delta_0 > 0$ and, thus, have no impact on the qualitative behavior of the ODE. The dynamics of Ψ relies on the linearized stability of (4.2.11) near $\Psi = \Psi_i$ of the schematic form

$$\frac{d(\delta\Psi)}{dR} = -(\Psi_i^2 - 1) \cdot \delta\Psi + error.$$

[4]Note that the discussion of the ODE (4.2.11) and its solutions was already present in [75] at the heuristic and numerical level.

If $|\Psi_i| > 1$, then $\Psi = \Psi_i$ is a stable fixed point as $R \to +\infty$ (corresponding to $r \to 0$): this is what happens if (no-bo) is true and then, there is no bounce and $\Psi \approx \Psi_i$ up to $r = 0$. In contrast, if $|\Psi_i| < 1$, then $\Psi = \Psi_i$ is an *unstable* fixed point, but $\Psi = \Psi_i^{-1}$ is a stable fixed point. If (bo) is true, then we find that Ψ gets inverted from Ψ_i to Ψ_i^{-1} in the region (4.2.8), over which the change in R is $\Delta R = O(\log(\epsilon^{-1}))$.

Once the behavior of Ψ has been quantified, one can immediately retrieve the Kasner exponents from Theorem 1.4.1 using similar techniques as in [148]. The most important (but algebraically trivial) feature of these relations is that $p_1 = P(\alpha) > 0$ if and only if $|\alpha| > 1$, see the formula (1.4.7), (1.4.8) (where $\alpha \approx \Psi_i$): thus, the final Kasner regime always has $p_1 > 0$ in Theorem 1.4.1. Hence, the final Kasner regime indeed lies in the subcritical regime of exponents as explained earlier in Section 4.2.4.

4.2.5 Holographic superconductors in the AdS-CFT correspondence

In this section, we will discuss two separate questions, which end up being connected by a vast literature.

- In what sense is the spacetime of Theorem 1.4.1 the interior of a hairy black hole, i.e. can we construct a corresponding "hairy black hole exterior"?
- What do numerics tell us about the hairy black hole interior of Theorem 1.4.1?

Note that Theorem 1.4.1 is valid for any value of the cosmological constant $\Lambda \in \mathbb{R}$. However, in the $\Lambda = 0$ case, there are no non-trivial static, spherically symmetric solutions of the Einstein–Maxwell–charged scalar field system, see [12] (although [12] assumes $q_0 = 0$).

On the other hand, in the $\Lambda < 0$ case, recent works of Zheng [155, 156] construct asymptotically Anti-de-Sitter hairy black hole exteriors bifurcating off of (a large subset of) sub-extremal Reissner–Nordström–AdS spacetimes. The interior of these black holes corresponds to the spacetimes considered in Theorem 1.4.1 (in the $q_0 \neq 0$ case) and Theorem 4.2.3 (in the $q_0 = 0$ case), thereby resolving Open Problem iv in [148].

Anti-de-Sitter asymptotics impose that, for a negative cosmological constant $\Lambda < 0$ in (Ein*):

$$g = \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + o(r^{-1})\right)[-dt^2 + ds^2] + r^2(s)d\sigma_{\mathbb{S}^2}, \tag{4.2.12}$$

which gives the following asymptotics on ϕ: for $m^2 < 0$, there exists constants $\phi_{(0)}$, $\phi_{(1)}$ such that

$$\begin{gathered}\phi(r) = \phi_{(0)} \cdot u_D(r) + \phi_{(1)} \cdot u_N(r), \\ \text{where } u_D(r) \sim r^{-\frac{3}{2}+\sqrt{\frac{9}{4}-m^2}} \text{ and } u_N(r) \sim r^{-\frac{3}{2}-\sqrt{\frac{9}{4}-m^2}} \text{ as } r \to +\infty.\end{gathered} \tag{4.2.13}$$

Here $\phi_{(0)} = 0$ corresponds to Dirichlet-type boundary conditions, while $\phi_{(1)} = 0$ is Neumann-type. Both boundary conditions are allowed in [155, 156]. We note that the Dirichlet case is the $\Lambda < 0$

analogue of asymptotic flatness in the $\Lambda = 0$ case and thus truly corresponds to what should be called an asymptotically-AdS "hairy black hole".

Previous works in physics already argued for the existence of such stationary asymptotically-AdS black holes [70, 73, 72] as part of the broader field of AdS/CMT (for Condensed Matter Theory) aiming at improving the understanding of condensed matter on flat spacetime by embedding it in a higher-dimensional AdS spacetime (see e.g. [76] or the related discussion in [148]). In the charged scalar field case ($q_0 \neq 0$), such black holes represent holographic analogues of superconductors due to a spontaneous breaking of the $\mathbb{U}(1)$-symmetry that the charged scalar field is associated with. The presence of black hole hair (meaning a non-trivial stationary black hole) reflects the phenomenon of condensation, with Dirichlet boundary condition $\phi_{(0)} \neq 0$ corresponding to a so-called stimulated emission, and $\phi_{(0)} = 0$ to spontaneous emission, in analogy with real-life superconductors (see e.g. [81] for more details).

In summary, the first question is answered in the affirmative: the regions considered in Theorem 1.4.1 and Theorem 4.2.3 indeed correspond to the interior of an asymptotically-AdS black hole (with Dirichlet or Neumann boundary conditions) recently rigorously constructed in [155, 156], whose existence had already been speculated in physics works [70, 72] in view of their importance as holographic models of superconductors.

We turn to the second question, regarding previous numerical works in the interior of asymptotically-AdS hairy black holes. Prior to discussing the $q_0 \neq 0$ case as studied in Theorem 1.4.1, let us mention that the hairy black hole interior with $\mathbf{F} \equiv 0$ (uncharged black hole) was studied in [65], and with $\mathbf{F} \neq 0$, $q_0 = 0$ (uncharged matter) in [74], see [148] for an extended discussion. We also mention the interesting follow-up works [27, 52, 53, 68, 71] for different matter models.

Turning to the case with $q_0 \neq 0$, we start by remarking that the collapsed oscillations and Kasner bounces from Theorem 1.4.1 were previously anticipated[5] numerically in [75]. This scenario was verified numerically and is entirely consistent with our findings from Theorem 1.4.1. The motivation to investigate the interior of holographic superconductors in [65, 74, 75] is two-fold: at first, it originates as a rudimentary model for quantum effects in the interior of black holes, following the AdS/CFT paradigm. In this context, the authors of [75] promote a *holographic version of Strong Cosmic Censorship* roughly asserting that generic *stationary*[6] (asymptotically-AdS) black holes (corresponding to thermal states) possess an internal singularity. Secondly, another motivation originates in the paradigm of AdS/CMT to understand real-life superconductors using holographic

[5] While [75] is in the setting of planar symmetry which obeys slightly different equations, it happens to make no difference to the fluctuating collapse and Kasner bounce phenomena and the conclusion of Theorem 1.4.1 also hold in this setting, as we already noted in Remark 1.4.

[6] In contrast, the usual notion of Strong Cosmic Censorship is not tied to stationary solutions and formulates genericity considerations inside a moduli space of initial data leading to a large class of dynamical spacetimes, see, for instance, the discussion in [149].

models (as already formulated in [70, 72, 73] and pursued in [65, 74, 75]). In this context, an analogy was established between the oscillations found inside holographic superconductors and the (AC) Josephson effect inside a "real-life" superconductor, see [85, 86], ultimately causing the Kasner bounce in some regime. The physical interpretation of these Kasner bounces inside holographic superconductors is still being debated [75].

Finally, note that, by the formula (1.4.7), when $\alpha \approx \Psi_i$ gets close to 0, then the Kasner exponents $p_2 = p_3$ get close to 0: when this happens, [75] numerically observes a *second Kasner bounce* and have also mentioned the possibility of arbitrarily many Kasner bounces. These very interesting aspects are not covered by Theorem 1.4.1, since its assumptions *specifically require to choose* ϵ so that $\alpha(\epsilon)$ is bounded away from 0 (at the cost, of course, of reducing the measure of the set of eligible ϵ by an arbitrarily small amount $\eta > 0$). It would be of great interest to prove rigorous results confirming the numerical breakthroughs of [75].

Open problem c. Generalize the conclusion of Theorem 1.4.1 for a larger set of $\epsilon \in (-\epsilon_0, \epsilon_0) \setminus \{0\}$ such that $|\Psi_i| \lesssim \epsilon^N$ are allowed for a suitably chosen $N > 0$, where Ψ_i defined in (4.2.9) is schematically of the form (4.2.10). In the $|\Psi_i| \lesssim \epsilon^N$ situation, control the occurrence of two (or more) Kasner bounces.

4.2.6 Outline of the proof in Chapter 4

The remainder of Chapter 4 will be devoted to the proof of Theorem 4.1.1 and Theorem 4.1.2, and will follow the various regions $\mathcal{R}$, $\mathcal{N}$, $\mathcal{EB}$, $\mathcal{LB}$, $\mathcal{O}$, $\mathcal{PK}$ and $\mathcal{K}$ depicted on Figure 4.1. Recall that the system of ODEs and initial data was already given in Section 2.4.2.

- In Section 4.3, we prove estimates in the red-shift region $\mathcal{R}$, the no-shift region $\mathcal{N}$, the early blue-shift region $\mathcal{EB}$, and the late blue-shift region $\mathcal{LB}$. These estimates are similar to the ones appearing in [148] and feature the almost formation of a Cauchy horizon.

- In Section 4.4, we prove estimates in the oscillation region $\mathcal{O}$. This section corresponds to the collapsed oscillations discussed in Section 4.2.3.

- In Section 4.5, we prove estimates in the proto-Kasner region $\mathcal{PK}$. In this section, we demonstrate the onset of a Kasner geometry transitioning from the collapsed oscillations in $\mathcal{O}$ to the Kasner behavior in $\mathcal{K}$.

- In Section 4.6, we linearize the system of Einstein equations to control precisely the phase $\Theta(\epsilon) = \omega_0 \epsilon^{-2} + O(\log(\epsilon^{-1}))$ appearing in (4.2.10); this step is essential in constructing the set E_η of acceptable ϵ in Theorem 1.4.1.

- In Section 4.7, we prove estimates in the Kasner region $\mathcal{K}$. In particular, we prove the Kasner bounce phenomenon discussed in Section 4.2.4.

- In Section 4.8, we conclude the proof of (the precise version of) Theorem 1.4.1 by providing precise geometric estimates characteristic of the Kasner behavior in a sub-region of $\mathcal{PK} \cup \mathcal{K}$.

4.3 Almost formation of a Cauchy horizon

In this section, we provide estimates up to a region $s \sim \epsilon^{-1}$. The analysis will be perturbative in nature, and we always bear in mind the comparison to the linear charged scalar field problem in Reissner-Nordström, see Section 4.1.4. The analysis will largely follow [148], with minor modifications due to the dynamical nature of the charge Q and the charge of the scalar field.

As in [148], the estimates up to $s \sim \epsilon^{-1}$ will be divided into the four regions $\mathcal{R}$, $\mathcal{N}$, $\mathcal{EB}$, and $\mathcal{LB}$ (see Figure 4.1). Where differences from [148] are minor, we aim to be relatively brief, and focus on the new techniques required to deal with the charge and the scalar field.

4.3.1 Estimates up to the no-shift region

We begin with some notation. As the arguments of this section are perturbative, it will help to introduce differences between the quantities $(r, \Omega^2, \phi, Q, \tilde{A})$ and their Reissner-Nordström(-dS/AdS) values. Therefore define

$$\delta r(s) = r(s) - r_{RN}(s), \qquad \delta\Omega^2(s) = \Omega^2(s) - \Omega^2_{RN}(s),$$

$$\delta\phi(s) = \phi(s) - \phi_{\mathcal{L}}(s),$$

$$\delta Q(s) = Q(s) - \mathbf{e}, \qquad \delta\tilde{A}(s) = \tilde{A}(s) + \left(\frac{\mathbf{e}}{r_+} - \frac{\mathbf{e}}{r_{RN}(s)}\right),$$

where we used (2.4.3) to provide the Maxwell gauge field $\tilde{A}_{RN}$ to which we compare, and $\phi_{\mathcal{L}}$ is the solution to the linear charged scalar field scattering problem in Reissner-Nordström, see Corollary 4.1.4.

We also use the quantity $\delta \log \Omega^2(s) = \log \Omega^2(s) - \log \Omega^2_{RN}(s)$. We commonly use that for $\delta \log \Omega^2(s) \lesssim 1$,

$$\delta\Omega^2(s) \lesssim \Omega^2(s)\, \delta \log \Omega^2(s) \lesssim \delta\Omega^2(s). \tag{4.3.1}$$

We now proceed to the estimates in the red-shift region $\mathcal{R}$.

Proposition 4.3.1. *There exist $D_R(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ and $\Delta_{\mathcal{R}}(M, \mathbf{e}, \Lambda) \gg 1$ such that, in the red-shift region $\mathcal{R} = \{-\infty < s < -\Delta_R\}$, the following estimates hold:*

$$|\phi| + |\Omega^{-2}\dot{\phi}| \leq D_R \epsilon, \tag{4.3.2}$$

$$|\kappa^{-1} - 1| \leq D_R \epsilon^2, \tag{4.3.3}$$

$$|\varpi - M| \leq D_R \epsilon^2, \tag{4.3.4}$$

$$\left|\frac{d\log(\Omega^2)}{ds} - 2K(s)\right| + |\log(\alpha_+^{-1}\Omega^2) - 2K_+ s| \leq D_R\Omega^2 \ll 1, \tag{4.3.5}$$

$$|\delta\log(\Omega^2)| + |\delta r| + |\delta\dot{r}| + |\delta\tilde{A}|\Omega^{-2} + |\delta Q| \leq D_R\epsilon^2, \tag{4.3.6}$$

$$|\delta\phi| + |\delta\dot{\phi}| \leq D_R\epsilon^3. \tag{4.3.7}$$

Proof. We provide a short sketch. For more details, see Proposition 4.5 in [142] or Lemma 4.1 in [148]. We make the following bootstrap assumptions:

$$\left|\frac{d}{ds}\log(\Omega^2) - 2K_+\right| \leq K_+, \tag{RS1}$$

$$|\phi| \leq 4\epsilon, \tag{RS2}$$

$$|Q - \mathbf{e}| \leq \frac{\mathbf{e}}{2}, \tag{RS3}$$

$$|r - r_+| \leq \frac{r_+}{2}. \tag{RS4}$$

These will hold in neighborhood of $s = -\infty$ due to the asymptotic data (2.4.32) and (2.4.35). The bootstrap assumption (RS1) will give the important estimate

$$\int_{-\infty}^{s} \Omega^2(s')\,ds' \lesssim \int_{-\infty}^{s} \frac{d}{ds}\log(\Omega^2)\cdot\Omega^2(s')\,ds' = \int_{-\infty}^{s}\frac{d}{ds}\Omega^2(s') = \Omega^2(s). \tag{4.3.8}$$

The evolution equation (2.4.29) for $\tilde{A}$ will immediately yield that $|\tilde{A}| \lesssim \Omega^2$. Therefore, turning to the equation (2.4.31), we use (4.3.8) to see that

$$|r^2\dot{\phi}| \lesssim \int_{-\infty}^{s}\Omega^2(s')\epsilon\,ds \lesssim \Omega^2(s)\epsilon,$$

which, after another round of integration, gives $|\phi| \leq \epsilon + C\Omega^2(s)\epsilon$, for some positive constant C.

Since the right hand side of the equation (2.4.22) is now bounded by $C\Omega^2\epsilon^2$, the estimate (4.3.3) is straightforward, and from this point (4.3.4) and (4.3.5) are also immediate. In particular, the remaining bootstraps are all improved, so long as $\Delta_{\mathcal{R}}$ is chosen large enough that $\Omega^2(s) < C^{-1}$ and ϵ is sufficiently small.

In order to get the difference estimates (4.3.6), (4.3.7), we make yet another bootstrap assumption:

$$|\delta r| + |\delta\dot{r}| + |\delta\log(\Omega^2)| + |\delta\tilde{A}| + \epsilon^{-1}|\delta\phi| + \epsilon^{-1}|\delta\dot{\phi}| \leq \epsilon^2. \tag{RS5}$$

The claim is that we can then use the ODEs for differences (found by subtracting from the relevant ODE in Section 2.4.2 the analogous ODE in Reissner-Nordström) to then improve the RHS of (RS5) by $C\Omega^2\epsilon^2$, hence improving the bootstrap for $\Delta_{\mathcal{R}} > 0$ sufficiently large.

We demonstrate this for the $\delta\phi$ estimate as an illustration. Taking the differences of (2.4.30), we have

$$|\delta\ddot{\phi}| \lesssim |\delta\dot{r}||\dot{\phi}| + |\dot{r}||\delta\dot{\phi}| + |\dot{r}\dot{\phi}||\delta r| + |\delta\tilde{A}||\tilde{A}||\phi| + (\tilde{A}^2 + \Omega^2)|\delta\phi| + |\delta\Omega^2||\phi|.$$

Using (4.3.2)–(4.3.5) as well as (RS5) to estimate the RHS by appropriate powers of ϵ and Ω^2 ((4.3.1) is also useful here), we see that

$$|\delta\ddot{\phi}| \lesssim \Omega^2\epsilon^3,$$

so that, integrating this from $s = -\infty$ once and then twice, using (4.3.8) we indeed get

$$|\delta\dot{\phi}| + |\delta\phi| \leq C\Omega^2\epsilon^3$$

as claimed. A similar procedure can be executed for the remaining equations of Section 2.4.2. This improves (RS5) and completes the proof of this proposition. □

In the sequel, we fix $\Delta_{\mathcal{R}}$ such that Proposition 4.3.1 applies. Next, we use a Grönwall argument to provide estimates in the no-shift region $\mathcal{N}$.

Proposition 4.3.2. *Take $S > 0$ to be any fixed real number. Then there exists some $C(M, \mathbf{e}, \Lambda, m^2, q_0) > 1$ and $D_N(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ such that, in the region $\mathcal{N} = \{-\Delta_{\mathcal{R}} \leq s \leq S\}$, the following estimates hold for ϵ sufficiently small:*

$$|\phi| + |\dot{\phi}| \leq D_N C^s \epsilon, \tag{4.3.9}$$

$$|\kappa^{-1} - 1| \leq D_N C^s \epsilon^2, \tag{4.3.10}$$

$$|Q - \mathbf{e}| \leq D_N C^s \epsilon^2, \tag{4.3.11}$$

$$|\varpi - M| \leq D_N C^s \epsilon^2, \tag{4.3.12}$$

$$|\delta r| + |\delta\dot{r}| + |\delta\log(\Omega^2)| + \left|\frac{d}{ds}\delta\log(\Omega^2)\right| + |\delta\tilde{A}| \leq D_N C^s \epsilon^2, \tag{4.3.13}$$

$$|\delta\phi| + |\delta\dot{\phi}| \leq D_N C^s \epsilon^3. \tag{4.3.14}$$

In the sequel, we only apply these estimates at $s = S$, so that allowing D_N to depend also on S, the terms involving C^s can be absorbed into D_N – indeed we define $D'_N = C^S D_N$ for convenience.

Proof. Here, we begin with some bootstrap assumptions on the geometry and the Maxwell field which clearly hold in a neighborhood of $s = -\Delta_{\mathcal{R}}$:

$$|Q| \leq 2|\mathbf{e}|, \tag{NS1}$$

$$|\tilde{A}| \leq \frac{2|\mathbf{e}|}{r_+}, \tag{NS2}$$

$$|\kappa^{-1} - 1| \leq \tfrac{1}{2}, \tag{NS3}$$

$$r \geq \tfrac{r_-}{2}, \tag{NS4}$$

$$\Omega^2 \leq 2\Omega^2_{\max}(M, \mathbf{e}, \Lambda) \coloneqq 2\sup_{s\in\mathbb{R}} \Omega^2_{RN}(s). \tag{NS5}$$

Using these bootstraps and (2.4.30), it is straightforward to see that

$$\left|\frac{d}{ds}(\phi,\dot{\phi})\right| \leq C_1(M,\mathbf{e},\Lambda,q_0,m^2)|(\phi,\dot{\phi})|,$$

which immediately yields (4.3.9). Using the evolution equation (2.4.28) for Q then gives (4.3.11), improving (NS1) – in fact we have $|Q-\mathbf{e}| \lesssim C_1^{2s}\epsilon^2$.

Next, we consider the following tuple of geometric and gauge quantities:

$$\mathbf{X} = (\delta r, \delta\dot{r}, \delta\log\Omega^2, \tfrac{d}{ds}\delta\log\Omega^2, \delta\tilde{A}).$$

Taking the differences from Reissner-Nordström for the equations (2.4.24), (2.4.26), (2.4.29), along with the bootstrap assumptions, for some $C_2(M,\mathbf{e},\Lambda,m^2,q_0)>0$, we get

$$\left|\frac{d\mathbf{X}}{ds}\right| \leq C_2(|\mathbf{X}| + C_1^{2s}\epsilon^2). \tag{4.3.15}$$

Thus, using Grönwall, we deduce (4.3.13) for some C with appropriate dependence on C_1, C_2. For ϵ small, this improves (NS2), (NS4), (NS5).

Next, for $s \geq -\Delta_{\mathcal{R}}$, we know $\Omega^{-2} \leq C_3 e^{-2K_- s}$ for some $C_3(M,\mathbf{e},\Lambda,m^2,q_0)>0$. Hence, (2.4.22),

$$|\kappa^{-1}(s) - \kappa^{-1}(-\Delta_{\mathcal{R}})| \lesssim C_3 e^{-2K_- s} C_1^{2s}\epsilon^2. \tag{4.3.16}$$

So, for C chosen sufficiently large, we get (4.3.10) and improve (NS3). The final estimate (4.3.14) is straightforward. □

Corollary 4.3.3. *Let $S' > S$ be another arbitrarily chosen real number. Then for $S \leq s \leq S'$ and for ϵ sufficiently small, there exists some $D_-(M,\mathbf{e},\Lambda,m^2,q_0,S,S')>0$ such that:*

$$|r(s) - r_-| \leq D_-(\epsilon^2 + e^{2K_- s}), \tag{4.3.17}$$

$$\left|\tilde{A} + \frac{\mathbf{e}}{r_+} - \frac{\mathbf{e}}{r_-}\right| \leq D_-(\epsilon^2 + e^{2K_- s}), \tag{4.3.18}$$

$$D_-^{-1} e^{2K_- s} \leq \Omega^2 \leq D_- e^{2K_- s}, \tag{4.3.19}$$

$$|\phi - B\epsilon e^{i\omega_{RN} s} - \overline{B}\epsilon e^{-i\omega_{RN} s}| + |\dot{\phi} - i\omega_{RN} B\epsilon e^{i\omega_{RN} s} + i\omega_{RN}\overline{B}\epsilon e^{-i\omega_{RN} s}| \leq D_-(\epsilon^3 + \epsilon e^{2K_- s}), \tag{4.3.20}$$

$$\left|\frac{d\log(\Omega^2)}{ds} - 2K_-\right| \leq D_-(\epsilon^2 + e^{2K_- s}) \leq \frac{|K_-|}{100}. \tag{4.3.21}$$

Proof. Note the previous proposition holds in the same way for S replaced by S'. Therefore, the corollary follows from the difference estimates of that proposition, Corollary 4.1.4, and some further basic computations on Reissner-Nordström such as

1. $r_{RN}(s) - r_- \lesssim e^{2K_- s}$ for $s \geq S \gg 1$.
2. $\Omega^2_{RN} \sim e^{2K_- s}$ for $s \geq S \gg 1$.

The final part of (4.3.21) clearly follows by taking S large and ϵ small. In the sequel, we will take advantage of the fact that S and S' can be taken to be as large as required. □

4.3.2 Estimates on the early blue-shift region

The early blue-shift $\mathcal{EB}$ is defined by $\mathcal{EB} = \{S \leq s \leq s_{lin} = (|2K_-|)^{-1}\log(\nu\epsilon^{-2})\}$, where $\nu(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ is chosen such that (4.3.28) holds. In the blue-shift regions, we begin to exploit the fact that Ω^2 is exponentially decaying in s.

Proposition 4.3.4. *Take the quantity S in Proposition 4.3.2 to be sufficiently large, and $\nu(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ to be sufficiently small. Then there exists some $D_E(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ such that the following hold:*

$$|\delta\log(\Omega^2)| + s\left|\frac{d}{ds}\delta\log(\Omega^2)\right| \leq D_E\epsilon^2 s^2, \tag{4.3.22}$$

$$|\phi| + |\dot{\phi}| \leq D_E|B|\epsilon, \tag{4.3.23}$$

$$|\delta r|s^{-1} + |\delta\dot{r}| + |\delta\tilde{A}| \leq D_E\epsilon^2, \tag{4.3.24}$$

$$\left|\kappa^{-1} - 1\right| \leq \tfrac{1}{100}, \tag{4.3.25}$$

$$|\varpi - M| + |Q - \mathbf{e}| \leq D_E\epsilon^2 s, \tag{4.3.26}$$

$$|\delta\phi| + |\delta\dot{\phi}| \leq D_E\epsilon^3 s. \tag{4.3.27}$$

Proof. Recall the quantities D'_N from Proposition 4.3.2 and $B \in \mathbb{C}$ from Proposition 4.1.3. We bootstrap the following estimates:

$$|\delta\log\Omega^2| \leq 4D'_N\epsilon^2 s^2, \tag{EB1}$$

$$|\phi| + \omega_{RN}^{-1}|\dot{\phi}| \leq 20|B|\epsilon, \tag{EB2}$$

$$r \geq \tfrac{r_-}{2}, \tag{EB3}$$

$$|\tilde{A}| \leq \tfrac{2\mathbf{e}}{r_+}. \tag{EB4}$$

To start with, note that, by (EB1) and the Reissner-Nordström asymptotics, we have that, for some $C > 0$,

$$C^{-1}e^{2K_-s} \leq \Omega^2 \leq Ce^{2K_-s}.$$

Note also that $|\delta Q| \lesssim \epsilon^2 s$ is immediate from (4.3.11) and the bootstraps.

Next, we use (EB2) along with the Raychaudhuri equation (2.4.22) to get the estimate on κ:

$$\left|\kappa^{-1} - 1\right| \leq D'_N\epsilon^2 + C|B|^2\epsilon^2 e^{2|K_-|s} \leq \frac{1}{100}, \tag{4.3.28}$$

where ν is chosen such that the second inequality holds. One can rewrite the left hand side of this expression as

$$\left|\kappa^{-1} - 1\right| = 4\Omega_{RN}^{-2}|\delta\dot{r} + \dot{r}(e^{-\delta\log\Omega^2} - 1)|.$$

Hence, we can use (4.3.28) and (EB1) to produce a preliminary estimate on $\delta\dot{r}$:

$$|\delta\dot{r}| \lesssim \Omega^2|\delta\log\Omega^2| + e^{2K_-s}\epsilon^2 + |B|^2\epsilon^2 \lesssim e^{2K_-s}(D'_N + D'_N s^2)\epsilon^2 + |B|^2\epsilon^2. \tag{4.3.29}$$

Integrating this up[7] from $s = S$, we get $|\delta r| \lesssim \epsilon^2 s$, or to be more specific we have the following, where C depends on the black hole parameters, m^2 and q_0 but not the choice of S (unlike the quantity $D'_N = C^S D_N$ which does have exponential dependence on S):

$$|\delta r| \leq C\epsilon^2(|B|^2 s + D'_N + D'_N S^2 e^{2K_- S}) \leq C\epsilon^2(|B|^2 s + D'_N). \tag{4.3.30}$$

Here, we use that $S^2 e^{2K_- S}$ is uniformly bounded for $S > 0$.

We wish to improve the bootstrap (EB1). So we use the equation (2.4.26) and take differences from Reissner-Nordström, leading to following inequality

$$\begin{aligned}\left|\frac{d^2}{ds^2}\delta \log \Omega^2\right| &\lesssim \Omega^2(|\delta \log \Omega^2| + |\delta r| + |\delta \dot{r}| + |\delta Q|) + |\phi|^2 + |\dot{\phi}|^2, \\ &\lesssim e^{2K_- s}(D'_N s^2 + |B|^2 s + D'_N)\epsilon^2 + |B|^2\epsilon^2.\end{aligned}$$

Integrating this expression up from $s = S$ twice, we get

$$\begin{aligned}\left|\frac{d}{ds}\delta \log \Omega^2\right| &\lesssim D'_N\epsilon^2 + |B|^2\epsilon^2 s, \\ |\delta \log \Omega^2| &\lesssim D'_N\epsilon^2 s + |B|^2\epsilon^2 s^2.\end{aligned}$$

So, for S sufficiently large, we improve the bootstrap (EB1) – note that $D'_N = C^S D_N$ grows exponentially in S so that $D'_N \gg |B|^2$ for S large.

Then taking differences between our ODEs and the Reissner-Nordström ODEs will lead to the estimates (4.3.24) and (4.3.26), thus improving upon the bootstraps (EB3) and (EB4).

We now turn to the estimates on the charged scalar field. We shall use the equation (2.4.31) and consider the quantity

$$H = r^4\dot{\phi}^2 + r^4 q_0^2|\tilde{A}|^2\phi^2. \tag{4.3.31}$$

As we already control r and $|\tilde{A}|$ from below in this region, producing an upper bound for H would give us desired estimates on $|\phi|, |\dot{\phi}|$. Using (2.4.31) as well as (2.4.29), we compute

$$\frac{dH}{ds} = -\frac{m^2\Omega^2 r^4}{2}\phi\dot{\phi} - \frac{r^2 q_0^2 \tilde{A} Q \Omega^2}{2}\phi^2 + 4r^3\dot{r}q_0^2|\tilde{A}|^2\phi^2.$$

The last term is negative since $\dot{r} < 0$, so we can apply Cauchy-Schwarz (again recalling the lower bounds on r and $|\tilde{A}|$) to see that

$$\frac{dH}{ds} \leq C_H\Omega^2 H,$$

for some $C_H = C_H(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$. Therefore, by Grönwall, so long as we pick S large enough such that

$$\int_S^{s_{lin}} C_H\Omega^2\, ds \leq CC_H \int_S^\infty e^{2K_- s}\, ds < \log 2,$$

[7]We use here a straightforward calculus lemma: given $\alpha > 0$ and $N \in \mathbb{N}$, then for $s_0 \in \mathbb{R}$ large, $\int_{s_0}^\infty s^N e^{-\alpha s}\, ds \lesssim_{\alpha,N} s_0^N e^{-\alpha s_0}$.

we deduce that $H(s) \leq 2H(S) \leq 16 r_-^4 \omega_{RN}^2 |B|^2 \epsilon^2$, where the latter estimate for $H(S)$ is computed using (4.3.20). This improves (EB2).

Finally, for the difference estimate on the scalar field we use the modified quantity $\tilde{H}$, given by

$$\tilde{H} = r^4 |\delta\dot{\phi}|^2 + r^4 q_0^2 |\tilde{A}|^2 |\delta\phi|^2. \tag{4.3.32}$$

To find the analogous differential inequality here, we first need to find the difference version of (2.4.30):

$$\delta\ddot{\phi} = -\frac{2\dot{r}}{r}\delta\dot{\phi} - q_0^2|\tilde{A}|^2\delta\phi - \frac{m^2\Omega^2}{4}\delta\phi + \mathcal{E}_{EB}, \tag{4.3.33}$$

where the error[8] term $\mathcal{E}_{EB}$ is estimated by $|\mathcal{E}_{EB}| \lesssim \epsilon^3(1 - s\dot{r} + s^2\Omega_{RN}^2)$.

Using a similar procedure as before, one can use this equation to get the differential inequality:

$$\frac{d\tilde{H}}{ds} \lesssim \Omega^2\tilde{H} + \epsilon^3(1 - s\dot{r} + s^2\Omega_{RN}^2)\sqrt{\tilde{H}}. \tag{4.3.34}$$

Rewriting this expression as a differential inequality in terms of $\sqrt{\tilde{H}}$ and using the usual Grönwall inequality with a (nondecreasing) inhomogeneous term, we get

$$\sqrt{\tilde{H}}(s) \lesssim \sqrt{\tilde{H}}(S) + \int_S^s \epsilon^3(1 - s\dot{r} + s^2\Omega_{RN}^2)\, ds \lesssim \epsilon^3 s$$

as required. This concludes the proof of (4.3.27) and the proposition. □

4.3.3 Estimates on the late blue-shift region

To close this section, we give the estimates on the late blue-shift region. Here, we leave the regime where Cauchy stability holds, in particular showing that $\dot{r}$ remains at size $O(\epsilon^2)$ rather than continuing to decay exponentially.

Proposition 4.3.5. *Choose $\Delta_{\mathcal{B}} > 0$ sufficiently small, and define $b_- = \frac{2|B|\omega_{RN}}{2|K_-|}$. Then, in the region $\mathcal{LB} = \{s_{lin} \leq s \leq \Delta_{\mathcal{B}}\epsilon^{-1}\}$, there exists some $D_L(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ such that the following estimates hold, recalling the quantity α_- from (2.4.9):*

$$|r(s) - r_-| \leq D_L\epsilon^2 s, \tag{4.3.35}$$

$$\left| -\dot{r} - \frac{4|B|^2\omega_{RN}^2\epsilon^2 r(s)}{2|K_-|} \right| \leq D_L(\Omega^2 + \epsilon^4 s), \tag{4.3.36}$$

$$\left| \frac{d}{ds}\log\big(\Omega^2\big) - 2K_- \right| \leq D_L\epsilon^2 s, \tag{4.3.37}$$

$$\left| \frac{d}{ds}\log\Big(r(s)^{-b_-^{-2}\epsilon^{-2}} \cdot \Omega^2(s)\Big) \right| \leq D_L(\epsilon^{-2}\Omega^2 + \epsilon^2 s), \tag{4.3.38}$$

[8] In this expression for the error term, our estimates in the region $\mathcal{EB}$ show that both $-s\dot{r}$ and $s^2\Omega_{RN}^2$ are $O(1)$. Nonetheless, we choose to write it in this form both for clarity and because we shall use the same expression again in the late blue-shift region, where both of these are still $O(1)$ but for different reasons.

$$\left|\log\left(\frac{\Omega^2}{\alpha_- e^{2K_- s}}\right)\right| \leq D_L\epsilon^2 s^2, \tag{4.3.39}$$

$$|\phi| + |\dot{\phi}| \leq D_L\epsilon, \tag{4.3.40}$$

$$|Q-\mathbf{e}|s^{-1} + |\delta\tilde{A}| \leq D_L\epsilon^2, \tag{4.3.41}$$

$$|\phi - B\epsilon e^{i\omega_{RN}s} - \overline{B}\epsilon e^{-i\omega_{RN}s}| + |\dot{\phi} - i\omega_{RN}B\epsilon e^{i\omega_{RN}s} + i\omega_{RN}\overline{B}\epsilon e^{-i\omega_{RN}s}| \leq D_L\epsilon^3 s. \tag{4.3.42}$$

Proof. We use the following bootstrap assumptions, where the constant $B_1(M,\mathbf{e},\Lambda,m^2,q_0) > 0$ will be described later in the proof:

$$\left|\frac{d}{ds}\log\Omega^2 - 2K_-\right| \leq |K_-|, \tag{LB1}$$

$$|\tilde{A}| \leq \tfrac{2|\mathbf{e}|}{r_+}, \tag{LB2}$$

$$-\dot{r} \leq \frac{8|B|^2\omega_{RN}^2 r_-}{|2K_-|}\epsilon^2, \tag{LB3}$$

$$|Q| \leq 2|\mathbf{e}|, \tag{LB4}$$

$$|\delta\log\Omega^2| \leq B_1\epsilon^2 s^2. \tag{LB5}$$

From (LB3), we have the trivial estimate $|\delta\dot{r}| \lesssim \epsilon^2$, and thus $|\delta r| \lesssim \epsilon^2 s$.

Due to (LB1), we have that in $\mathcal{LB}$,

$$\int_{s_{lin}}^{s} \Omega^2(s')\,ds' \leq |K_-|^{-1}\Omega^2(s_{lin}) \lesssim \epsilon^2,$$

So given all the bootstrap assumptions, we may proceed exactly as in the proof of Proposition 4.3.4 to recover the estimates (4.3.40) and (4.3.42). Note that the error $\mathcal{E}_{LB}$ in (4.3.33) is replaced by $\mathcal{E}_{LB}$ which still satisfies the same estimate

$$|\mathcal{E}_{LB}| \lesssim \epsilon^3(1 - s\dot{r} + s^2\Omega_{RN}^2) \lesssim \epsilon^3,$$

so the proof follows in an identical fashion.

We also get the estimate (4.3.41) just as in Proposition 4.3.4, improving the two bootstraps (LB2) and (LB4).

Using the difference version of (2.4.27), we get the inequality

$$\left|\frac{d^2}{ds^2}\delta\log\big(r\Omega^2\big)\right| \lesssim B_1\Omega^2\epsilon^2 s^2 + \epsilon^2. \tag{4.3.43}$$

Integrating once, we deduce that

$$\left|\frac{d}{ds}\delta\log\big(r\Omega^2\big)\right| \lesssim B_1\epsilon^2 s_{lin}^2(\epsilon)\cdot\epsilon^2 + \epsilon^2 s \lesssim \epsilon^2 s. \tag{4.3.44}$$

Therefore, in light of the previous estimates on $\delta r, \delta\dot{r}$, for ϵ sufficiently small we have

$$\left|\frac{d}{ds}\log\big(r\Omega^2\big)(s) - 2K_-\right| + \left|\frac{d}{ds}\log\big(\Omega^2\big)(s) - 2K_-\right| \leq C\epsilon^2 s \leq \frac{|K_-|}{10}, \tag{4.3.45}$$

where the final inequality follows for $\Delta_{\mathcal{B}}$ taken sufficiently small.

This improves the bootstrap (LB1), and moreover, integrating (4.3.44) once again will yield

$$|\delta \log(r\Omega^2)| \leq C(\epsilon^2 s_{lin}^2(\epsilon) + B_1 \epsilon^4 s_{lin}^2(\epsilon) + \epsilon^2 s^2), \tag{4.3.46}$$

which improves (LB5) after once again accounting for δr, so long as B_1 is chosen sufficiently large (i.e. larger than $4C$ for the C appearing in (4.3.46)).

It remains to improve upon the bootstrap (LB3). For this, as in the uncharged scalar field model of [148], we use the Raychaudhuri equation in the convenient form (2.4.23). We begin by estimating the expression involving the scalar field; using (4.3.42) and (4.3.41), we have

$$|\dot{\phi}|^2 + |\tilde{A}|^2 q_0^2 |\phi|^2 = |B\omega_{RN}\epsilon e^{i\omega_{RN}s} - \overline{B}\omega_{RN}\epsilon e^{-i\omega_{RN}s}|^2 + \omega_{RN}^2 |B\epsilon e^{i\omega_{RN}s} + \overline{B}\epsilon e^{-i\omega_{RN}s}|^2 + \mathcal{E}_B.$$

Expanding out the trigonometric expressions on the right, we can see this can be rewritten as

$$|\dot{\phi}|^2 + |\tilde{A}|^2 q_0^2 |\phi|^2 = 4|B|^2 \omega_{RN}^2 \epsilon^2 + \mathcal{E}_B, \tag{4.3.47}$$

where the error term is bounded by $|\mathcal{E}_B| \lesssim \epsilon^4 s$.

Therefore, using also (4.3.35) and (4.3.44), one finds that (2.4.23) may be written in the form

$$\frac{d}{ds}(-\dot{r}) - 2K_-(-\dot{r}) = 4|B|^2 \omega_{RN}^2 r_- \epsilon^2 + \mathcal{E}_B(s), \tag{4.3.48}$$

with the error $\mathcal{E}_B(s)$ once again satisfying $|\mathcal{E}_B(s)| \lesssim \epsilon^4 s$.

We now use a classical integrating factor of $e^{-2K_- s}$ to integrate (4.3.48) between s_{lin} and s, yielding

$$-\dot{r}(s) = -e^{2K_-(s-s_{lin})}\dot{r}(s_{lin}) + \int_{s_{lin}}^{s} 4|B|^2 \omega_{RN}^2 r_- \epsilon^2 e^{2K_-(s-s')} + \mathcal{E}_B(s') e^{2K_-(s-s')}\, ds'. \tag{4.3.49}$$

Using $|\mathcal{E}_B(s')| \lesssim \epsilon^4 s'$ and computing these integrals, we find that

$$\left| -\dot{r}(s) - \frac{4|B|^2 \omega_{RN}^2 r_- \epsilon^2}{|2K_-|} \right| \lesssim e^{2K_-(s-s_{lin})} \left| -\dot{r}(s_{lin}) - \frac{4|B|^2 \omega_{RN}^2 r_- \epsilon^2}{|2K_-|} \right| + \epsilon^4 s. \tag{4.3.50}$$

From this and prior estimates, we yield (4.3.36), thus improving the remaining bootstrap (LB3).

The estimate (4.3.38) simply comes from combining (4.3.36) and (4.3.37). Finally, (4.3.39) follows from combining (2.4.9) with (LB5), plus using that $e^{2K_- s} \leq e^{2K_- s_{lin}} \lesssim \epsilon^2$ for $s \in \mathcal{LB}$. □

We conclude this section with a straightforward corollary concerning quantities evaluated at a specific point $s = s_O(\epsilon) \coloneqq 50 s_{lin}(\epsilon)$ that will be useful in the next region. For ease of notation, we first define the dimensionless parameter $\mathfrak{W}(M, \mathbf{e}, \Lambda, q_0) > 0$ as:

$$\mathfrak{W} \coloneqq \sqrt{\frac{\omega_{RN}(M, \mathbf{e}, \Lambda, q_0)}{|2K_-(M, \mathbf{e}, \Lambda)|}}. \tag{4.3.51}$$

Corollary 4.3.6. *Let $s_O(\epsilon) = 50 s_{lin}(\epsilon) \in \mathcal{LB}$. Then, upon defining the quantities*

$$r_0 = r(s_O), \quad \omega_0 = |q_0 \tilde{A}(s_O)|, \quad \xi_0 = \omega_0 \left(-\frac{d}{ds} \left(\frac{r^2}{r_-^2 \epsilon^2} \right) \right)^{-1} (s_O), \quad Q_0 = Q(s_O), \tag{4.3.52}$$

we have the following estimates:

$$|r_0 - r_-| + \log\big(\epsilon^{-1}\big)|\omega_0 - \omega_{RN}| + \left| \xi_0 - \frac{1}{8|B|^2 \mathfrak{W}^2} \right| + |Q_0 - \mathbf{e}| \leq D_L \epsilon^2 \log\big(\epsilon^{-1}\big). \tag{4.3.53}$$

In particular, the following upper and lower bounds hold for ξ_0.

$$\xi_0 \leq \frac{1}{4|B|^2 \mathfrak{W}^2}, \quad \xi_0^{-1} \leq 16 |B|^2 \mathfrak{W}^2. \tag{4.3.54}$$

Finally, one has

$$|\phi(s_O)| + |\dot{\phi}(s_O)| \leq D_L \epsilon. \tag{4.3.55}$$

4.4 The collapsed oscillations

From this point, the results begin to differ from that of [148]. We next study the region of 'collapsed oscillations' (see Section 1.4.6), a region where r eventually becomes $O(\epsilon)$ small. The dynamics in this section are largely driven by the charged scalar field, which remains oscillatory but also exhibits a slowly growing behavior, in that the amplitude grows from $O(\epsilon)$ at start of the regime to $O(1)$ at the end of the regime. Indeed, after a change of variables, the behavior of ϕ will be well approximated by a *Bessel function* of order 0. Schematically, in the region $\epsilon \lesssim r \ll r_-$, the scalar field behaves as

$$\phi(r) \approx \frac{C\epsilon}{r} \cdot \cos\left(\frac{r^2 - r_-^2}{\epsilon^2} + O(\log\big(\epsilon^{-1}\big)) \right). \tag{4.4.1}$$

We will, in fact, need to track the growth-oscillatory behavior more precisely using the explicit Bessel functions $J_0(x)$ and $Y_0(x)$, defined as the two linearly independent solutions of Bessel's equation of order 0:

$$\frac{d}{dz}\left(z \frac{df}{dz} \right) + zf = 0. \tag{4.4.2}$$

They exhibit the following asymptotic behavior (see also Facts A.0.1, A.0.2 and A.0.4 in Appendix A for more detailed asymptotics):

$$J_0(z) = \begin{cases} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right) + O(z^{-3/2}) & \text{as } z \to +\infty, \\ 1 + O(z) & \text{as } z \to 0, \end{cases} \tag{4.4.3}$$

$$Y_0(z) = \begin{cases} \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4}\right) + O(z^{-3/2}) & \text{as } z \to +\infty, \\ \frac{2}{\pi}\left(\ln\left(\frac{z}{2}\right) + \gamma\right) + O(z) & \text{as } z \to 0, \end{cases} \tag{4.4.4}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. We see, therefore, that the Bessel functions $J_0(z)$ and $Y_0(z)$ both oscillate and decay[9] at a slow inverse polynomial rate as $z \to \infty$, but exhibit very different behavior in the $z \to 0$ regime. We later show the schematic estimate (4.4.1) takes the following more precise form involving the renormalized variable $z = \frac{r^2}{\epsilon^2}$ and the frequency ξ_0 defined in Corollary 4.3.6:

$$\phi(r) \approx C_J(\epsilon) \cdot J_0\left(\frac{\xi_0}{r_-^2} \cdot \frac{r^2}{\epsilon^2}\right) + C_Y(\epsilon) \cdot Y_0\left(\frac{\xi_0}{r_-^2} \cdot \frac{r^2}{\epsilon^2}\right),$$

$$C_J(\epsilon) \approx \sqrt{2\pi\xi_0} \cdot |B| \cdot \cos\big(\xi_0 \cdot \epsilon^{-2}\big),\ C_Y(\epsilon) \approx \sqrt{2\pi\xi_0} \cdot |B| \cdot \sin\big(\xi_0 \cdot \epsilon^{-2}\big).$$

In light of (4.4.3)–(4.4.4), tracking the size of the coefficients $C_J(\epsilon)$ and $C_Y(\epsilon)$ precisely will be important in later sections.

The main objective of Section 4.4 is to prove these schematic estimates. In view of the logarithmic growth of $Y_0(z)$ for $z \ll 1$, the precise value of the coefficient $C_Y(\epsilon)$ will be crucial in the geometry of the later regions where $r \ll \epsilon$ (see Section 4.5).

Once the scalar field asymptotics are understood, we also account for the scalar field *backreaction* onto both the Maxwell field and the geometry. In this region, the backreaction on the spacetime geometry is minimal; however the backreaction on the Maxwell field has a rather curious effect. That is, within the oscillatory region, the charge $Q(s)$ will decrease in absolute value from (approximately) $\mathbf{e}$ to (approximately) Q_∞, where $Q_\infty(M, \mathbf{e}, \Lambda)$ depends on the black hole parameters and lies strictly between $\frac{\mathbf{e}}{2}$ and $\mathbf{e}$, see (4.4.13) and Lemma 4.4.7. Q_∞ being bounded away from 0 is causing the *charge retention* in Theorem 1.4.1, see Section 4.2.3.

We now state the main result of this section, where $\mathcal{O} = \{s \geq s_O(\epsilon) : r(s) \geq 2|B|\mathfrak{W}\epsilon r_-\}$ and we recall that $\mathfrak{W}$ was defined in (4.3.51).

Proposition 4.4.1. *For $s \in \mathcal{O}$, the lapse Ω^2 obeys the following upper bound, where we recall α_- from* (2.4.9).

$$\Omega^2(s) \leq \alpha_- \exp(K_- s). \tag{4.4.5}$$

There exists some $D_O(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ such that, for $s \in \mathcal{O}$, we have the following estimates:

$$|-r\dot{r}(s) - 4|B|^2\mathfrak{W}^2 r_-^2 \epsilon^2 \omega_{RN}| \leq D_O \epsilon^4 \log \epsilon^{-1}, \tag{4.4.6}$$

$$\left|\frac{d}{ds}\log\Omega^2 - 2K_-\right| \leq D_O\epsilon^2(\log\big(\epsilon^{-1}\big) + r^{-2}(s)). \tag{4.4.7}$$

$$|q_0\tilde{A}(s) - q_0\tilde{A}_{RN,\infty}| \leq D_O\epsilon^2, \tag{4.4.8}$$

$$\left|Q(s) - \mathbf{e} + \frac{1}{4}\frac{2K_-}{\tilde{A}_{RN,\infty}}(r_-^2 - r^2(s))\right| \leq D_O\epsilon^2\log\big(\epsilon^{-1}\big). \tag{4.4.9}$$

[9] As z is decreasing in s in our setting, and we are interested in behavior as z decreases, the decay of Bessel functions exhibits itself as inverse polynomial growth of the amplitude of the scalar field ϕ.

Finally, there exist coefficients $C_J(\epsilon)$ and $C_Y(\epsilon)$ determined via the formula (4.4.28) and satisfying (4.4.29) and (4.4.30) such that we have the following two estimates on the scalar field: for all $s \in \mathcal{O}$,

$$\left|\phi(s) - \left(C_J(\epsilon)J_0\left(\frac{\xi_0 r^2(s)}{r_-^2\epsilon^2}\right) + C_Y(\epsilon)Y_0\left(\frac{\xi_0 r^2(s)}{r_-^2\epsilon^2}\right)\right)\right| \leq D_O\epsilon^2\log\big(\epsilon^{-1}\big), \tag{4.4.10}$$

$$\left|\dot{\phi}(s) - \omega_0\left(C_J(\epsilon)J_1\left(\frac{\xi_0 r^2(s)}{r_-^2\epsilon^2}\right) + C_Y(\epsilon)Y_1\left(\frac{\xi_0 r^2(s)}{r_-^2\epsilon^2}\right)\right)\right| \leq D_O\epsilon^2\log\big(\epsilon^{-1}\big). \tag{4.4.11}$$

Here, ξ_0 and ω_0 are as defined in Corollary 4.3.6.

Later, it will be convenient to refer to (4.4.9), (4.4.10) and (4.4.11) evaluated at the right endpoint of $\mathcal{O}$.

Corollary 4.4.2. *Let $s_{PK}(\epsilon) > 0$ be such that $r(s_{PK}) = 2|B|\mathfrak{W}\epsilon r_-$. Then*

$$\big|s_{PK} - (8|B|^2\mathfrak{W}^2\omega_{RN})^{-1}\epsilon^{-2}\big| \leq D_O\log\big(\epsilon^{-1}\big). \tag{4.4.12}$$

Further, if one defines Q_∞ as

$$Q_\infty(M,\mathbf{e},\Lambda) \coloneqq \mathbf{e} - \frac{2K_-(M,\mathbf{e},\Lambda)\cdot r_-^2(M,\mathbf{e},\Lambda)}{4\tilde{A}_{RN,\infty}(M,\mathbf{e},\Lambda)}, \tag{4.4.13}$$

one may infer the following estimates at $s = s_{PK}$.

$$|Q(s_{PK}) - Q_\infty| \leq D_O\epsilon^2\log\big(\epsilon^{-1}\big), \tag{4.4.14}$$

$$\big|\phi(s_{PK}) - \big(C_J(\epsilon)J_0(4\xi_0|B|^2\mathfrak{W}^2) + C_Y(\epsilon)Y_0(4\xi_0|B|^2\mathfrak{W}^2)\big)\big| \leq D_O\epsilon^2\log\big(\epsilon^{-1}\big), \tag{4.4.15}$$

$$\Big|\dot{\phi}(s_{PK}) - \omega_0\big(C_J(\epsilon)J_1(4\xi_0|B|^2\mathfrak{W}^2) + C_Y(\epsilon)Y_1(4\xi_0|B|^2\mathfrak{W}^2)\big)\Big| \leq D_O\epsilon^2\log\big(\epsilon^{-1}\big). \tag{4.4.16}$$

This section will be organized into three parts. In Section 4.4.1, we introduce the main bootstrap assumptions for the region $\mathcal{O}$, and make several preliminary observations. In Section 4.4.2, we derive the main scalar field estimates, establishing the aforementioned Bessel type behavior. In Section 4.4.3, we use the results on the scalar field to improve the bootstrap assumptions, proving in particular the estimates (4.4.5), (4.4.7) and (4.4.9) regarding $\Omega^2(s)$ and $Q(s)$, then complete the proof of Proposition 4.4.1.

4.4.1 Bootstraps and preliminary estimates

Throughout the proof of Proposition 4.4.1, we make reference to the following three bootstrap assumptions:

$$\Omega^2 \leq \epsilon^{40}, \tag{O1}$$

$$|\phi| + |\dot{\phi}| \leq \epsilon^{-1}, \tag{O2}$$

$$|Q| \leq \epsilon^{-2}. \tag{O3}$$

By Proposition 4.3.5 and Corollary 4.3.6, these hold within a neighborhood of $s = s_O(\epsilon)$. It will be convenient to define the bootstrap region $\mathcal{O}_{boot}$ as the connected component of $\{s \in \mathcal{O} : s \leq \epsilon^{-4}, \text{(O1)–(O3) apply}\}$ such that $s_O \in \mathcal{O}_{boot}$. We now make some preliminary estimates using these bootstraps and Proposition 4.3.5. Morally, the following lemma will allow us to treat $-r\dot{r}$ and $\tilde{A}$ as constant inside $\mathcal{O}$, at least up to an extremely small error.

Lemma 4.4.3. *Any $s \in \mathcal{O}_{boot}$ satisfies $s \lesssim \epsilon^{-2}$. Furthermore, there exists some constant $D_O(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ such that, for $s \in \mathcal{O}_{boot}$,*

$$\left|\frac{d}{ds}(-r\dot{r})\right| + |r\dot{r}(s) - r\dot{r}(s_O)| \leq D_O\epsilon^{30}, \tag{4.4.17}$$

$$\left|-r\dot{r}(s) - 4|B|^2\mathfrak{W}^2\omega_{RN}r_-^2\epsilon^2\right| \leq D_O\epsilon^4\log\left(\epsilon^{-1}\right), \tag{4.4.18}$$

$$|q_0\tilde{A}(s) - q_0\tilde{A}(s_O)| \leq D_O\epsilon^{30}, \tag{4.4.19}$$

$$||q_0\tilde{A}|(s) - \omega_{RN}| \leq D_O\epsilon^2. \tag{4.4.20}$$

Proof. These estimates are immediate from (2.4.25) and (2.4.29). Indeed, letting f be either $-r\dot{r}$ or $\tilde{A}$, we see that

$$\left|\frac{df}{ds}\right| \leq \Omega^2 \cdot P_f(|\phi|, |Q|, |\dot{\phi}|, r^{-2})$$

for some polynomial P_f of degree less than 2. Then, using that (O1) provides a large power of ϵ, the remaining bootstraps plus the fact that $r(s) \gtrsim \epsilon$ give the estimates (4.4.17) and (4.4.19). The other two estimates then follow straightforwardly from Proposition 4.3.5, (4.4.17) and (4.4.19).

As $-r\dot{r} \gtrsim \epsilon^2$ and $r^2(s_O) = r_-^2 + O(\epsilon^2\log\left(\epsilon^{-1}\right))$, it is immediate that $s \lesssim \epsilon^{-2}$ for $s \in \mathcal{O}_{boot}$. □

We use Lemma 4.4.3 to rewrite several of the equations of Section 2.4.2 as simplified equations with constant coefficients, plus an error term. Recalling ξ_0 from (4.3.52), to simplify notation, we denote

$$x = x(s) \coloneqq \frac{r^2(s)}{r_-^2\epsilon^2}, \text{ so that } \xi_0 = \left|\left(-\frac{dx}{ds}\right)^{-1} \cdot q_0\tilde{A}\right|(s = s_O). \tag{4.4.21}$$

We then proceed to rewrite (2.4.31) and (2.4.28) in terms of the new variable x. Performing this change of variables, one gets the two equations:

$$\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) + \xi_0^2 x\phi = \mathcal{E}_\phi, \tag{4.4.22}$$

$$\frac{dQ}{dx} = \text{sgn}(\mathbf{e})q_0r_-^2\xi_0\epsilon^2 x|\phi|^2 + \mathcal{E}_Q. \tag{4.4.23}$$

We should like to estimate the error terms $\mathcal{E}_\phi$ and $\mathcal{E}_Q$. To do so, we apply Lemma 4.4.3 extensively to prove the following corollary.

Corollary 4.4.4. *Recall x and ξ_0 from (4.4.21). Then, for $s \in \mathcal{O}_{boot}$, one finds*

$$\left| \xi_0 - \left(-\frac{dx}{ds} \right)^{-1} \cdot |q_0 \tilde{A}|(s) \right| \leq D_O \epsilon^{25}. \tag{4.4.24}$$

Furthermore, in the equations (4.4.22) and (4.4.23), one has the following control on the error terms:

$$|\mathcal{E}_\phi| + |\mathcal{E}_Q| \leq D_O \epsilon^{20}.$$

Proof. For (4.4.24), we first use that, from (2.4.25),

$$\left| \frac{d^2 x}{ds^2} \right| = \frac{2}{r_-^2 \epsilon^2} \left| \frac{d}{ds}(-r\dot{r}) \right| = \frac{\Omega^2}{2 r_-^2 \epsilon^2} \left| 1 - \frac{Q^2}{r^2} - r^2 |\phi|^2 - r^2 \Lambda \right| \lesssim \epsilon^{25}, \tag{4.4.25}$$

where the final estimate comes from the bootstrap assumptions (O1)–(O3). Integrating this once between s_O and $s \in \mathcal{O}_{boot}$, and combining with (4.4.21), (4.4.24) follows.

For the error terms, making the substitution $s \mapsto x$ in (2.4.31) and (2.4.28) yields:

$$|\mathcal{E}_\phi| + |\mathcal{E}_Q| \lesssim \left(\Omega^2 + \left| \frac{d^2 x}{ds^2} \right| + \left| \xi_0 - \left(-\frac{dx}{ds} \right)^{-1} |q_0 \tilde{A}| \right| \right) \cdot P(\epsilon^{-1}, r^{-1}, |\phi|, |\dot{\phi}|),$$

for some polynomial P of degree less than five. Upon inserting (O1), (4.4.25) and (4.4.24), the corollary follows. □

4.4.2 Scalar field oscillations

The focus of this subsection will be understanding the behavior of the *charged* scalar field $\phi(s)$ via the equation (4.4.22). This equation is a Bessel type ODE that we would like to understand as x decreases from $\epsilon^{-2} + O(\log(\epsilon^{-1}))$ to $r^2(s_{PK})(r_-^2 \epsilon^2)^{-1} = 4|B|^2 \mathfrak{W}^2$.

Proposition 4.4.5. *For $s \in \mathcal{O}_{boot}$, there exists a constant $D_O > 0$, and coefficients $C_J(\epsilon)$ and $C_Y(\epsilon)$ such that*[10]

$$|\phi - C_J(\epsilon) J_0(\xi_0 x) - C_Y(\epsilon) Y_0(\xi_0 x)| \leq D_O \epsilon^{10}, \tag{4.4.26}$$

$$\left| \frac{d\phi}{dx} + \xi_0 C_J(\epsilon) J_1(\xi_0 x) + \xi_0 C_Y(\epsilon) Y_1(\xi_0 x) \right| \leq D_O \epsilon^{10}. \tag{4.4.27}$$

Defining $x_{\mathcal{B}} = x(s_O)$, the coefficients $C_J(\epsilon)$ and $C_Y(\epsilon)$ are determined by

$$\begin{bmatrix} C_J(\epsilon) \\ C_Y(\epsilon) \end{bmatrix} = \frac{\pi x_{\mathcal{B}}}{2} \begin{bmatrix} -\xi_0 Y_1(\xi_0 x_{\mathcal{B}}) & -Y_0(\xi_0 x_{\mathcal{B}}) \\ \xi_0 J_1(\xi_0 x_{\mathcal{B}}) & J_0(\xi_0 x_{\mathcal{B}}) \end{bmatrix} \begin{bmatrix} \phi(x_{\mathcal{B}}) \\ \frac{d\phi}{dx}(x_{\mathcal{B}}) \end{bmatrix}. \tag{4.4.28}$$

The coefficients obey the estimates

$$\left| C_J(\epsilon) - \frac{\sqrt{\pi}}{2} \mathfrak{W}^{-1} \cos(\Theta(\epsilon)) \right| \leq D_O \epsilon^2 \log(\epsilon^{-1}), \tag{4.4.29}$$

[10] For a definition of the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$, see Appendix A and Fact A.0.3.

$$\left|C_Y(\epsilon) - \frac{\sqrt{\pi}}{2}\mathfrak{W}^{-1}\sin(\Theta(\epsilon))\right| \leq D_O\epsilon^2\log\left(\epsilon^{-1}\right), \tag{4.4.30}$$

where the phase function $\Theta(\epsilon)$ *is given by*

$$\Theta(\epsilon) = |q_0\tilde{A}|(s)\cdot r^2(s)\cdot\left(-\frac{d}{ds}r^2(s)\right)^{-1} + \omega_{RN}s + \arg B - \frac{\pi}{4}\Bigg|_{s=s_O}. \tag{4.4.31}$$

Finally, one has the following upper bounds for $\phi(s)$ *and* $\dot{\phi}(s)$*:*

$$\max\{|\phi|, \omega_{RN}^{-1}|\dot{\phi}|\} \leq \frac{100|B|\epsilon r_-}{r}. \tag{4.4.32}$$

Proof. Ignoring the error term $\mathcal{E}_\phi$, the equation (4.4.22) is exactly Bessel's equation of order 0 after a rescaling $z = \xi_0 x$. We should like to separate the analysis of the linear behavior arising from Bessel's equation and analysis of the error term.

For this purpose, we recast the equation (4.4.22) as a first-order linear evolution equation in the variables $(\phi, \frac{d\phi}{dx})$, treating the error term as an inhomogeneity. We get the following (see also the discussion preceding Lemma A.0.6):

$$\frac{d}{dx}\begin{bmatrix}\phi(x)\\ \frac{d\phi}{dx}(x)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\xi_0^2 & -\frac{1}{x}\end{bmatrix}\begin{bmatrix}\phi(x)\\ \frac{d\phi}{dx}(x)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{x}\mathcal{E}_\phi\end{bmatrix}. \tag{4.4.33}$$

The analysis of this first-order system then follows from Lemma A.0.6, or more precisely Corollary A.0.7. Following these, we aim to write the solution using Duhamel's principle; let $\mathbf{S}(z_1; z_0)$ be the linear solution operator for the usual Bessel's equation defined in Lemma A.0.6, and $\mathbf{S}_{\xi_0}(x_1; x_0) \coloneqq \mathbf{Q}_{\xi_0}\circ\mathbf{S}(\xi_0 x_1; \xi_0 x_0)\circ\mathbf{Q}_{\xi_0}^{-1}$ as in Corollary A.0.7, where $\mathbf{Q}_\chi : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear stretching operator $\mathbf{Q}_\chi : (A, B) \mapsto (A, \chi B)$.

Then, following Corollary A.0.7, $\mathbf{S}_{\xi_0}(x_1; x_0)$ can be interpreted as the linear solution (semigroup) operator for the first-order system (4.4.33) without the inhomogeneous term $\mathcal{E}_\phi$. Defining $x_{\mathcal{B}}$ as in the statement of the proposition, we have from Corollary 4.3.6 that $|x_{\mathcal{B}} - \epsilon^{-2}| \lesssim \log\left(\epsilon^{-1}\right)$. Duhamel's principle (i.e. Corollary A.0.7) applied to first-order systems[11] gives that

$$\begin{bmatrix}\phi\\ \frac{d\phi}{dx}\end{bmatrix}(x) = \mathbf{S}_{\xi_0}(x; x_{\mathcal{B}})\begin{bmatrix}\phi\\ \frac{d\phi}{dx}\end{bmatrix}(x_{\mathcal{B}}) + \int_{x_{\mathcal{B}}}^{x}\mathbf{S}_{\xi_0}(x;\tilde{x})\begin{bmatrix}0\\ \frac{1}{\tilde{x}}\mathcal{E}_\phi(\tilde{x})\end{bmatrix}d\tilde{x}. \tag{4.4.34}$$

We define:

$$\begin{bmatrix}\phi_B\\ \frac{d\phi_B}{dx}\end{bmatrix}(x) \coloneqq \mathbf{S}_{\xi_0}(x; x_{\mathcal{B}})\begin{bmatrix}\phi\\ \frac{d\phi}{dx}\end{bmatrix}(x_{\mathcal{B}}), \qquad \begin{bmatrix}\phi_e\\ \frac{d\phi_e}{dx}\end{bmatrix}(x) \coloneqq \int_{x_{\mathcal{B}}}^{x}\mathbf{S}_{\xi_0}(x;\tilde{x})\begin{bmatrix}0\\ \frac{1}{\tilde{x}}\mathcal{E}_\phi(\tilde{x})\end{bmatrix}d\tilde{x}. \tag{4.4.35}$$

Thus $\phi_B(x)$ represents exactly the solution of the homogeneous linear Bessel ODE with data given at $x = x_{\mathcal{B}}$, while $\phi_e(x)$ is to be treated as an error term due to the small inhomogeneity. To treat $\phi_B(x)$ and $\phi_e(x)$, we shall use parts (1) and (2) of Corollary A.0.7 respectively.

[11] Note that in this section, we are always integrating backwards in x. Nonetheless, the standard theory of first order systems will still apply, with the usual convention that $\int_{x_0}^{x_1} = -\int_{x_1}^{x_0}$.

As $\phi_B(x)$ must be a solution to (a rescaled) Bessel's equation, the solution must be exactly (using Fact A.0.3 to justify the appearance of $J_1(\xi_0 x)$ and $Y_1(\xi_0 x)$ in the derivatives):

$$\phi_B(x) = C_J J_0(\xi_0 x) + C_Y Y_0(\xi_0 x), \tag{4.4.36}$$

$$\frac{d\phi_B}{dx}(x) = -\xi_0 C_J J_1(\xi_0 x) - \xi_0 C_Y Y_1(\xi_0 x). \tag{4.4.37}$$

In these equations, the coefficients $C_J = C_J(\epsilon)$ and $C_Y = C_Y(\epsilon)$ are chosen such that $\phi(x_{\mathcal{B}}) = \phi_B(x_{\mathcal{B}})$ and $\frac{d\phi}{dx}(x_{\mathcal{B}}) = \frac{d\phi_B}{dx}(x_{\mathcal{B}})$, i.e. so that $\phi_B(x)$ agrees with $\phi(x)$ to first order at $x = x_{\mathcal{B}}$. (It is immediate from (4.4.35) that $\phi_e(x)$ vanishes to first order at $x = x_{\mathcal{B}}$.) Part (1) of Corollary A.0.7 then implies (4.4.28).

In order to deal with $\phi_e(x)$, we recall part 2 of Corollary A.0.7. Using Corollary 4.4.2, in particular the upper and lower bounds for ξ_0 in (4.3.54), the operator norm of the rescaled solution operator satisfies

$$\|\mathbf{S}_{\xi_0}(x_1; x_0)\|_{l^2 \to l^2} \leq \max\left\{\frac{1}{4|B|^2\mathfrak{W}^2}, 16|B|^2\mathfrak{W}^2\right\} \cdot \max\left\{1, \frac{x_0}{x_1}\right\}.$$

Integrating backwards in the variable x, i.e. $x < \bar{x}$ in (4.4.34), we can see that upon defining the l^∞ norm on $\mathbb{R}^2$ as $\|(x,y)\|_{l^\infty} = \max\{|x|, |y|\}$,

$$\left\| \mathbf{S}_{\xi_0}(x, \tilde{x}) \begin{bmatrix} 0 \\ \frac{1}{\tilde{x}}\mathcal{E}_\phi(\tilde{x}) \end{bmatrix} \right\|_{l^\infty} \lesssim \max\left\{\frac{1}{|B|^2\mathfrak{W}^2}, |B|^2\mathfrak{W}^2\right\} \cdot \frac{1}{x} \sup|\mathcal{E}_\phi| \lesssim \epsilon^{20},$$

where we apply Corollary 4.4.4 and $x \geq 4|B|^2\mathfrak{W}^2$ in the last step. Furthermore, the length of the interval of integration in (4.4.34) is bounded by $x_{\mathcal{B}} - x \leq x_{\mathcal{B}} \lesssim 2\epsilon^{-2}$, so we have

$$\left\| \begin{bmatrix} \phi_e(x) \\ \frac{d\phi_e}{dx}(x) \end{bmatrix} \right\|_{l^\infty} = \left\| \int_{x_{\mathcal{B}}}^{x} \mathbf{S}_{\xi_0}(x; \tilde{x}) \begin{bmatrix} 0 \\ \frac{1}{\tilde{x}}\mathcal{E}_\phi(\tilde{x}) \end{bmatrix} dx' \right\|_{l^\infty} \lesssim \epsilon^{10} \tag{4.4.38}$$

as required. Combining (4.4.36), (4.4.37) and (4.4.38) then gives both (4.4.26) and (4.4.27).

We now recover the precise form for the coefficients $C_J(\epsilon), C_Y(\epsilon)$. We carefully compute $C_J(\epsilon)$; the other coefficient $C_Y(\epsilon)$ shall follows in an analogous manner. By (4.4.28), we have the formula

$$C_J(\epsilon) = \frac{\pi x_{\mathcal{B}}}{2}\left[-\xi_0 Y_1(\xi_0 x_{\mathcal{B}})\phi(x_{\mathcal{B}}) - Y_0(\xi_0 x_{\mathcal{B}})\frac{d\phi}{dx}(x_{\mathcal{B}})\right]. \tag{4.4.39}$$

We therefore obtain a precise expression for $C_J(\epsilon)$ using the large-z asymptotics for the Bessel function $Y_\nu(z)$, and the scalar field estimates at the comparison point $s = s_O$ given by (4.3.42) in Proposition 4.3.5. By Fact A.0.4 and $x_{\mathcal{B}} = \epsilon^{-2} + O(\log(\epsilon^{-1}))$,

$$Y_0(\xi_0 x_{\mathcal{B}}) = \sqrt{\frac{2}{\pi\xi_0 x_{\mathcal{B}}}}\sin\left(\xi_0 x_{\mathcal{B}} - \frac{\pi}{4}\right) + O(\epsilon^3), \tag{4.4.40}$$

$$Y_1(\xi_0 x_{\mathcal{B}}) = -\sqrt{\frac{2}{\pi\xi_0 x_{\mathcal{B}}}}\cos\left(\xi_0 x_{\mathcal{B}} - \frac{\pi}{4}\right) + O(\epsilon^3). \tag{4.4.41}$$

On the other hand, writing B as $B = |B|(\cos\arg B + i\sin\arg B)$ in (4.3.42) gives for $\phi(x_{\mathcal{B}}) = \phi(s)|_{s=s_O}$:

$$\phi(x_{\mathcal{B}}) = 2|B|\epsilon\cos(\omega_{RN}s_O + \arg B)| + O(\epsilon^3\log(\epsilon^{-1})). \tag{4.4.42}$$

For $\frac{d\phi}{dx}$, we proceed in several steps, using in particular Lemma 4.4.3, (4.4.24), (4.4.21) and (4.3.53):

$$\begin{aligned}\frac{d\phi}{dx}(x_{\mathcal{B}}) &= \left(\frac{dx}{ds}\right)^{-1}\dot{\phi}\Big|_{s=s_O}\\ &= -2|B|\epsilon\,\omega_{RN}\left(\frac{dx}{ds}\right)^{-1}\sin(\omega_{RN}s + \arg B)\Big|_{s=s_O} + O(\epsilon^3\log(\epsilon^{-1}))\\ &= +2|B|\epsilon\,\omega_{RN}(8|B|^2\mathfrak{W}^2\omega_{RN})^{-1}\sin(\omega_{RN}s_O + \arg B)| + O(\epsilon^3\log(\epsilon^{-1}))\\ &= 2|B|\epsilon\,\xi_0\sin(\omega_{RN}s_O + \arg B)| + O(\epsilon^3\log(\epsilon^{-1})).\end{aligned} \tag{4.4.43}$$

Substituting all of (4.4.40), (4.4.41), (4.4.42), (4.4.43) into (4.4.39), one deduces that

$$C_J(\epsilon) = \frac{\pi x_{\mathcal{B}}}{2}\sqrt{\frac{2\xi_0}{\pi x_{\mathcal{B}}}}\cdot 2|B|\epsilon\cos\left(\xi_0 x_{\mathcal{B}} - \frac{\pi}{4} + \omega_{RN}s_O + \arg B\right) + O(\epsilon^2\log(\epsilon^{-1})). \tag{4.4.44}$$

To conclude, one uses Corollary 4.3.6 and $x_{\mathcal{B}} = \epsilon^{-2} + O(\log(\epsilon^{-1}))$ to yield, since $\sqrt{8\xi_0}|B| = \mathfrak{W} + O(\epsilon\log(\epsilon^{-1}))$:

$$C_J(\epsilon) = \frac{\sqrt{\pi}}{2}\mathfrak{W}^{-1}\cos(\Theta(\epsilon)) + O(\epsilon^2\log(\epsilon^{-1})), \tag{4.4.45}$$

where by the definition of ξ_0 in Corollary 4.3.6, $\Theta(\epsilon)$ is as in (4.4.31).

Finally, to obtain the upper bound (4.4.32), note first of all that $\xi_0 x \geq \xi_0 4|B|^2\mathfrak{W}^2 \geq 1/4$ by Corollary 4.3.6. Since one can check that, for $z \geq \frac{1}{4}$,

$$\max\{|J_0(z)|, |J_1(z)|, |Y_0(z)|, |Y_1(z)|\} \leq 10z^{-1/2},$$

one can use (4.4.26) and (4.4.27) along with (4.4.29) and (4.4.30) to deduce that

$$\max\left\{|\phi(x)|, \xi_0^{-1}\left|\frac{d\phi}{dx}\right|\right\} \leq \sqrt{\pi}\,\mathfrak{W}^{-1}\cdot 10(\xi_0 x)^{-1/2} + O(\epsilon^2\log(\epsilon^{-1})) \leq \frac{60|B|\epsilon r_-}{r}. \tag{4.4.46}$$

Then, one can use Lemma 4.4.3 to change the x-derivative to an s-derivative, with negligible error, and thus find (4.4.32). □

4.4.3 Precise estimates for Q and Ω^2

Note that while (O2) has already been improved by (4.4.26), it remains to estimate Q and Ω^2 in the region $\mathcal{O}_{boot}$, and close the remaining bootstraps (O3) and (O1). We shall show that Q changes to (a value close to) the quantity Q_∞, while Ω^2 remains exponentially decaying in s.

Proposition 4.4.6. *For $s \in \mathcal{O}_{boot}$, we have the following estimates for $Q(s)$ and $\Omega^2(s)$, where $D_O > 0$ may be larger than that of Lemma 4.4.3 and Proposition 4.4.5 if necessary.*

$$\left| Q(s) - \mathbf{e} + \frac{1}{4}\frac{2K_-}{\tilde{A}_{RN,\infty}}(r_-^2 - r^2(s)) \right| \leq D_O \epsilon^2 \log\big(\epsilon^{-1}\big), \tag{4.4.47}$$

$$\left| \frac{d}{ds}\log\Omega^2 - 2K_- \right| \leq D_O \epsilon^2 \left(\log\big(\epsilon^{-1}\big) + r^{-2}(s)\right). \tag{4.4.48}$$

Proof. We shall prove (4.4.47) using (4.4.23) and Proposition 4.4.5. By Corollary 4.3.6, particularly the estimate $|Q_0 - \mathbf{e}| \lesssim \epsilon^2 \log\big(\epsilon^{-1}\big)$ from (4.3.53), it remains to estimate the integral:

$$\mathrm{sgn}(\mathbf{e}) \int_{x_{\mathcal{B}}}^{x} q_0 r_-^2 \xi_0 \epsilon^2 \tilde{x} |\phi|^2(\tilde{x})\, d\tilde{x}, \tag{4.4.49}$$

where we recall for convenience that $x = (\frac{r}{r_-})^2 \epsilon^{-2}$.

Firstly, using (4.4.26), (4.4.29), (4.4.30), and the Bessel function asymptotics in (4.4.3), (4.4.4), one may find that

$$\left| \phi - \sqrt{\frac{1}{2\xi_0 x}} \mathfrak{W}^{-1} \cos(\xi_0 x - \pi/4 - \Theta(\epsilon)) \right| \lesssim \epsilon^2 \log\big(\epsilon^{-1}\big) x^{-1/2} + x^{-3/2}, \tag{4.4.50}$$

which we may use to deduce the estimate

$$\left| \xi_0 x \phi^2 - \frac{\mathfrak{W}^{-2}}{2} \cos^2(\xi_0 x - \pi/4 - \Theta(\epsilon)) \right| \lesssim \epsilon^2 \log\big(\epsilon^{-1}\big) + x^{-1}. \tag{4.4.51}$$

Therefore, recalling that $\mathfrak{W}^{-2} = \frac{|2K_-|}{\omega_{RN}} = \frac{|2K_-|}{q_0|\tilde{A}_{RN,\infty}|}$ by definition, we can integrate this to estimate (4.4.49) by

$$\begin{aligned} &\left| \int_{x_{\mathcal{B}}}^{x} q_0 r_-^2 \xi_0 \epsilon^2 \tilde{x} |\phi|^2(\tilde{x})\, d\tilde{x} - \frac{2|K_-| r_-^2}{2|\tilde{A}_{RN,\infty}|} \epsilon^2 \int_{x_{\mathcal{B}}}^{x} \cos^2(\xi_0 \tilde{x} - \pi/4 - \Theta(\epsilon))\, d\tilde{x} \right| \\ &\qquad \lesssim \epsilon^2 (\epsilon^2 \log\big(\epsilon^{-1}\big) x_{\mathcal{B}} + \log(x_{\mathcal{B}}/x)) \lesssim \epsilon^2 \log\big(\epsilon^{-1}\big). \end{aligned} \tag{4.4.52}$$

Then integrating $\cos^2(\xi_0 x - \pi/4 - \Theta(\epsilon)) = \frac{1}{2}(1 + \cos(2\xi_0 x - \pi/2 - 2\Theta(\epsilon))$ in the usual manner,

$$\left| \int_{x_{\mathcal{B}}}^{x} q_0 r_-^2 \xi_0 \epsilon^2 \tilde{x} |\phi|^2(\tilde{x})\, d\tilde{x} - \frac{1}{4}\frac{2|K_-|}{|\tilde{A}_{RN,\infty}|}(r^2(x) - r^2(x_{\mathcal{B}})) \right| \lesssim \epsilon^2 \log\big(\epsilon^{-1}\big). \tag{4.4.53}$$

By replacing $r^2(x_{\mathcal{B}})$ by r_-^2 (which carries an error of $O(\epsilon^2 \log\big(\epsilon^{-1}\big))$), we get the estimate (4.4.47) – to ensure we have the right sign in (4.4.47), recall that $\mathbf{e}$ and $\tilde{A}_{RN,\infty}$ have opposite signs, while $2K_-$ is negative.

We now move onto the estimates for Ω^2. For this, we consider (2.4.27), and integrate by parts using (2.4.30) as follows, where we use (O1) to control error terms involving Ω^2:

$$\begin{aligned}\left|\frac{d}{ds}\log\big(r\Omega^2\big)-2K_-\right|(s) &\lesssim \left|\frac{d}{ds}\delta\log\big(r\Omega^2\big)\right|(s_O)+\left|\int_{s_O}^{s}(-|\dot\phi|^2+q_0^2|\tilde A|^2|\phi|^2)\,ds\right|+\epsilon^{30}\\ &\lesssim \epsilon^2\log\big(\epsilon^{-1}\big)+|\phi\dot\phi(s)|+|\phi\dot\phi(s_O)|+\left|\int_{s_O}^{s}\phi(\ddot\phi+q_0^2|\tilde A|^2\phi)\,ds\right|\\ &\lesssim \epsilon^2\log\big(\epsilon^{-1}\big)+|\phi\dot\phi(s)|+\left|\int_{s_O}^{s}\frac{-\dot r\phi\dot\phi}{r}\,ds\right|. \end{aligned}\tag{4.4.54}$$

In the third line, we used Corollary 4.3.6 to absorb $|\phi\dot\phi(s_{\mathcal{O}})|$ into the $\epsilon^2\log\big(\epsilon^{-1}\big)$ error.

By (4.4.32) in Proposition 4.4.5, for $s\in\mathcal{O}_{boot}$, one has

$$|\phi\dot\phi(s)|\leq\frac{10^4|B|^2\omega_{RN}\epsilon^2r_-^2}{r^2}.$$

Inserting this estimate into all instances of $\phi\dot\phi$ in (4.4.54) and then evaluating the integral, one eventually arrives at the estimate (4.4.48), after applying (4.4.18) again. Indeed, one has

$$\left|\frac{d}{ds}\log\Omega^2-2K_-\right|\leq\left|\frac{d}{ds}\log\big(r\Omega^2\big)-2K_-\right|+\frac{-\dot r}{r}\lesssim\epsilon^2\left[\log\big(\epsilon^{-1}\big)+r^{-2}(s)\right].$$

This completes the proof of the proposition. □

Proof of Proposition 4.4.1 and Corollary 4.4.2. Suppose the bootstrap assumptions (O1), (O3), (O2) hold in $\mathcal{O}_{boot}\subset\mathcal{O}$. Then the conclusions of Lemma 4.4.3 and Propositions 4.4.5 and 4.4.6 hold. In particular, (4.4.32) and (4.4.47) show that (O2) and (O3) are indeed improved in the bootstrap region. Since, by Lemma 4.4.3, we know that $s\lesssim\epsilon^{-2}$ for all $s\in\mathcal{O}_{boot}$, in order to show $\mathcal{O}=\mathcal{O}_{boot}$ it remains only to improve (O1).

For this purpose, we simply need to integrate (4.4.48). By (4.4.18), we know that $-\dot r r\gtrsim\epsilon^2$, hence due to (4.3.39) and $s\geq s_O\gtrsim\log\big(\epsilon^{-1}\big)$, one has

$$\begin{aligned}|\log\Omega^2(s)-2K_-s-\log\alpha_-| &\lesssim \epsilon^2\log\big(\epsilon^{-1}\big)^2+\int_{s_O}^{s}\left(\epsilon^2\log\big(\epsilon^{-1}\big)-\frac{\dot r}{r}\right)\\ &\lesssim \epsilon^2\log\big(\epsilon^{-1}\big)s+\log\frac{r(s_O)}{r(s)}. \end{aligned}\tag{4.4.55}$$

In fact, the final term on the right hand side is also bounded by $\epsilon^2\log\big(\epsilon^{-1}\big)s$. To see this, note that if s is such that $r(s)\geq\frac{r_-}{2}$, then (4.4.18) implies that

$$\log\frac{r(s_O)}{r(s)}=\int_{s_O}^{s}\frac{-\dot r(\tilde s)}{r(\tilde s)}\,d\tilde s\lesssim\int_{s_O}^{s}\frac{\epsilon^2}{r(\tilde s)^2}\,d\tilde s\lesssim\epsilon^2 s.$$

On the other hand, if $r(s)\leq\frac{r_-}{2}$ then (4.4.18) implies that $s\gtrsim\epsilon^{-2}$, and thus $\log\Big(\frac{r(s_O)}{r(s)}\Big)\lesssim\log\big(\epsilon^{-1}\big)\lesssim\epsilon^2\log\big(\epsilon^{-1}\big)s$.

Hence, (4.4.55) implies that

$$\Omega^2 \leq \alpha_- e^{(2K_- + D_O \epsilon^2 \log(\epsilon^{-1}))s}, \tag{4.4.56}$$

which clearly implies (4.4.5) for ϵ small. Moreover, since $e^{K_- \cdot s_O(\epsilon)} \lesssim \epsilon^{50}$, we have improved the final bootstrap (O1). Therefore, $\mathcal{O}_{boot} = \mathcal{O}$ and its right endpoint is $s = s_{PK}$, where $r(s_{PK}) = 2|B|\mathfrak{W}\epsilon r_-$.

Integrating (4.4.18) in the region $\mathcal{O}$, one gets:

$$\left| \frac{1}{2} r^2(s_O) - \frac{1}{2} r^2(s_{PK}) - 4|B|^2 \mathfrak{W}^2 \omega_{RN} r_-^2 \epsilon^2 (s_{PK} - s_O) \right| \lesssim \epsilon^2 \log(\epsilon^{-1}),$$

which simplifies after applying the estimates of Proposition 4.3.5 to

$$\left| r_-^2 - 8|B|^2 \mathfrak{W}^2 \omega_{RN} r_-^2 \epsilon^2 s_{PK} \right| \lesssim \epsilon^2 \log(\epsilon^{-1}).$$

This yields the estimate (4.4.12) for s_{PK}. The remaining parts of both Proposition 4.4.1 and Corollary 4.4.2 are straightforward. □

Remarkably, (4.4.14) shows that the spacetime exhibits a nonzero, yet controlled *discharge* within the oscillatory region $\mathcal{O}$. We conclude this section with a (purely algebraic) lemma revealing that the final charge $Q_\infty(M, \mathbf{e}, \Lambda)$ lies strictly between $\mathbf{e}$ and $\mathbf{e}/2$.

Lemma 4.4.7. *Define Q_∞ as in* (4.4.13)*. Then one has the following alternative form for Q_∞, where we recall $\mathcal{P}_{se} = \mathcal{P}_{se}^{\Lambda<0} \cup \mathcal{P}_{se}^{\Lambda=0} \cup \mathcal{P}_{se}^{\Lambda>0}$ from Section 2.4.*

$$Q_\infty = \frac{3}{4}\mathbf{e} + \frac{\Lambda r_-^2 r_+ (2r_- + r_+)}{12\mathbf{e}}. \tag{4.4.57}$$

From this, we make the following observations regarding Q_∞:

(i) If $\Lambda = 0$, then $Q_\infty = \frac{3}{4}\mathbf{e}$.

(ii) If $\Lambda < 0$, then Q_∞ lies strictly between $\frac{1}{2}\mathbf{e}$ and $\frac{3}{4}\mathbf{e}$. Furthermore, as $(M, \mathbf{e}, \Lambda)$ varies across the sub-extremal parameter space $\mathcal{P}_{se}^{\Lambda<0}$, $Q_\infty/\mathbf{e}$ achieves all values in $(\frac{1}{2}, \frac{3}{4})$.

(iii) If $\Lambda > 0$, then Q_∞ lies strictly between $\frac{3}{4}\mathbf{e}$ and $\mathbf{e}$. Furthermore, as $(M, \mathbf{e}, \Lambda)$ varies across the sub-extremal parameter space $\mathcal{P}_{se}^{\Lambda>0}$, $Q_\infty/\mathbf{e}$ achieves all values in $(\frac{3}{4}, 1)$.

In particular, in all cases, one has $|Q_\infty| > \frac{1}{2}|\mathbf{e}| > 0$.

Proof. To get (4.4.57), we need to find a clean expression for $2K_-$ in terms of the black hole parameters. For this purpose, recall the polynomial (2.4.1), and define the function $f(X)$ as

$$f(X) = X^{-2} P_{M,\mathbf{e},\Lambda}(X) = X^{-2}(r_+ - X)(r_- - X)\left(\frac{\mathbf{e}^2}{r_+ r_-} - \frac{\Lambda}{3}(r_+ + r_-)X - \frac{\Lambda}{3}X^2\right),$$

from which we may alternatively define the surface gravity $2K_-$ as:

$$2K_-(M, \mathbf{e}, \Lambda) = f'(X)|_{X=r_-} = r_-^{-2}(r_- - r_+)\left(\frac{\mathbf{e}^2}{r_+ r_-} - \frac{\Lambda}{3} r_-(r_+ + r_-) - \frac{\Lambda}{3} r_-^2\right).$$

Recalling once more that $\tilde{A}_{RN,\infty}(M,\mathbf{e},\Lambda) = \frac{\mathbf{e}}{r_+} - \frac{\mathbf{e}}{r_-} = \frac{\mathbf{e}(r_- - r_+)}{r_+ r_-}$, we therefore get (4.4.57).

Once we have (4.4.57), the case (i) is immediate. For the first statement of (iii), one may assume without loss of generality that $\mathbf{e} > 0$. Then, for $\Lambda > 0$, (4.4.57) gives $Q_\infty > \frac{3}{4}\mathbf{e}$, while (4.4.13) gives $Q_\infty < \mathbf{e}$, since $\tilde{A}_{RN,\infty} < 0$ and $2K_- < 0$. For the $\Lambda < 0$ case, we require one final observation, namely

$$0 = r_+^{-1} P_{M,\mathbf{e},\Lambda}(r_+) - r_-^{-1} P_{M,\mathbf{e},\Lambda}(r_-) = \mathbf{e}^2 \left(\frac{1}{r_+} - \frac{1}{r_-} \right) + (r_+ - r_-) - \frac{\Lambda}{3}(r_+^3 - r_-^3).$$

Dividing by $r_+ - r_-$ and multiplying by $r_+ r_-$, this gives

$$\tfrac{\Lambda}{3} r_+ r_- (r_+^2 + r_+ r_- + r_-^2) = -\mathbf{e}^2 + r_+ r_-.$$

In particular, in the case $\Lambda < 0$, we then have

$$0 < -\tfrac{\Lambda}{3} r_-^2 r_+ (2r_- + r_+) < -\tfrac{\Lambda}{3} r_+ r_- (r_+^2 + r_+ r_- + r_-^2) < \mathbf{e}^2.$$

Substituting this into (4.4.57), we get the first statement of (ii).

To prove the second statements of (ii), (iii), we first state without proof several facts about the sub-extremal parameter spaces. Without loss of generality, we fix $\mathbf{e}$ to be positive. Then letting $\mathcal{P}_{se}^+ = \mathcal{P}_{se} \cap \{\mathbf{e} > 0\}$ be the space of subextremal parameters with positive $\mathbf{e}$,

- $\mathcal{P}_{se}^+$ is an open, connected set in $\mathbb{R}^3$.
- Define the set $\mathcal{P}_{ex}^{\Lambda<0}$ to be the set of $(M,\mathbf{e},\Lambda) \in \mathbb{R}^+ \times \mathbb{R} \times (-\infty, 0)$ such that the polynomial $P_{M,\mathbf{e},\Lambda}(X)$ has a single repeated positive root $X = R$, and $\mathcal{P}_{ex}^{\Lambda>0}$ to be the set of $(M,\mathbf{e},\Lambda) \in \mathbb{R}^+ \times \mathbb{R} \times (0, +\infty)$ such that the polynomial $P_{M,\mathbf{e},\Lambda}(X)$ has two positive roots $X = R$ and $X = R_C$, with $X = R$ having multiplicity 2 and $R < R_C$. Then, $\mathcal{P}_{ex}^+ \coloneqq (\mathcal{P}_{ex}^{\Lambda<0} \cup \mathcal{P}_{ex}^{\Lambda>0}) \cap \{\mathbf{e} > 0\}$ is a subset of the boundary $\partial\mathcal{P}_{se}^+$.
- For $(M,\mathbf{e},\Lambda) \in \mathcal{P}_{se}^+$, one must have $-\infty < \Lambda M^2 < \frac{2}{9}$, i.e. we are constrained to have $\Lambda M^2 < \frac{2}{9}$ in order for there to exist a choice of $\mathbf{e} > 0$ such that the polynomial $P_{M,\mathbf{e},\Lambda}(X)$ has the correct number of distinct positive roots. Moreover, all these values are achieved when restricting to the extremal case i.e. $\{\Lambda M^2 : (M,\mathbf{e},\Lambda) \in \mathcal{P}_{ex}^+\} = (-\infty, \frac{2}{9})$.
- The functions $r_+(M,\mathbf{e},\Lambda)$ and $r_-(M,\mathbf{e},\Lambda)$ are continuous in $\mathcal{P}_{se}^+$. Furthermore, they may be continuously extended onto $\mathcal{P}_{ex}^+ \subset \partial\mathcal{P}_{se}^+$, upon defining $r_- = r_+ = R$ on $\mathcal{P}_{ex}^+$.

In light of these facts, Q_∞ is a continuous function of $(M,\mathbf{e},\Lambda)$ in $\mathcal{P}_{se}^+$, and may be continuously extended onto the space of extremal parameters $\mathcal{P}_{ex}^+$, where we have

$$Q_\infty = \frac{3}{4}\mathbf{e} + \frac{\Lambda R^4}{4\mathbf{e}}. \tag{4.4.58}$$

We would like to estimate ΛR^4. For extremal parameters, we must have $P_{M,\mathbf{e},\Lambda}(R) = \frac{d}{dX}P_{M,\mathbf{e},\Lambda}(R) = 0$, from which we can get the two identities[12]:

$$-2\Lambda R^3 + 3R - 3M = 0, \tag{4.4.59}$$

$$R^2 - 3MR + 2\mathbf{e}^2 = 0. \tag{4.4.60}$$

Using (4.4.59) and then (4.4.60) in (4.4.58), we then get

$$Q_\infty = \frac{3}{4}\mathbf{e}\left(1 + \frac{R(R-M)}{2\mathbf{e}^2}\right) = \frac{3}{4}\mathbf{e}\left(1 + \frac{R-M}{3M-R}\right) = \frac{3}{2}\mathbf{e}\cdot\frac{1}{3-R/M}. \tag{4.4.61}$$

It remains to work out the range of values that R/M can take. We rearrange (4.4.59) to get the equation:

$$1 - \frac{R}{M} + \frac{2}{3}\Lambda M^2\left(\frac{R}{M}\right)^3 = 0. \tag{4.4.62}$$

As ΛM^2 varies between $-\infty$ and $\frac{2}{9}$, we can see that the unique positive root of this cubic expression in R/M varies between 0 and $\frac{3}{2}$, not including the endpoints. (Note that if $\Lambda M^2 > \frac{2}{9}$, then in fact there are no positive roots! This is why ΛM^2 must be upper bounded.)

Therefore, as ΛM^2 and thus R/M vary within their allowed ranges, from (4.4.61) we have that Q_∞ is allowed to take every value between $\mathbf{e}/2$ and $\mathbf{e}$, not including the endpoints.

Of course, this computation was done for extremal parameters in $\mathcal{P}^+_{ex}$ rather than the subextremal parameters in $\mathcal{P}^+_{se}$. But $\mathcal{P}^+_{se}$ is connected, so it is straightforward to show that indeed $\{Q_\infty(M,\mathbf{e},\Lambda) : (M,\mathbf{e},\Lambda) \in \mathcal{P}^+_{se}\} = (\mathbf{e}/2, \mathbf{e})$ also. This concludes the proof of Lemma 4.4.7. □

4.5 The Proto-Kasner Region

4.5.1 Estimates beyond the oscillatory region – statement of Proposition 4.5.1

In the previous section, we considered the region $\mathcal{O} = \{s \geq s_O : r(s) \geq 2|B|\mathfrak{W}\epsilon r_-\} = \{s_O \leq s \leq s_{PK}\}$, where $s_O(\epsilon) = 50 s_{lin}$ and we $s_{PK} = O(\epsilon^{-2})$ was defined such that $r(s_{PK}) = 2|B|\mathfrak{W}\epsilon r_-$. The reason it was necessary to end this region at $r \sim \epsilon r_-$ was twofold:

1. Considering the Bessel function form of ϕ,

$$\phi = C_J(\epsilon)J_0\left(\frac{\xi_0 r^2}{r_-^2\epsilon^2}\right) + C_Y(\epsilon)Y_0\left(\frac{\xi_0 r^2}{r_-^2\epsilon^2}\right) + \text{error},$$

then the Bessel functions $J_0(z), Y_0(z)$ will change behavior from oscillatory (at large z) to convergent or logarithmically growing (at small z) once $r^2(s) \leq \xi_0^{-1} r_-^2\epsilon^2 \approx 8|B|^2\mathfrak{W}^2\epsilon^2 r_-^2$. (See (4.4.3), (4.4.4))

[12] These will arise from $\frac{d}{dX}P_{M,\mathbf{e},\Lambda}(R) = 0$ and $\frac{d}{dX}(X^{-4}P_{M,\mathbf{e},\Lambda})(R) = 0$ respectively.

2. We tackled many error terms using the smallness of $\Omega^2(s)$ to dominate polynomial powers of r^{-1} and $|\phi|$. At the start of the region $\mathcal{O}$, i.e. $s = s_O(\epsilon) = 50 s_{lin}(\epsilon)$, we only had that $\Omega^2 \lesssim \epsilon^{100}$, so in order for the errors to be controlled we required r^{-1} to be at worst an inverse power of ϵ.

Nevertheless, we will show that we may extend many of the important estimates beyond $s = s_{PK}$ to a region $\mathcal{PK} = \{s \geq s_{PK} : r(s) \geq e^{-\delta_0 \epsilon^{-2}} r_-\}$, which for reasons that will later become clear we denote by the *proto-Kasner* region. Here, the dimensionless constant $\delta_0(M, \mathbf{e}, \Lambda, q_0)$ is selected to be (recall that $b_- = \frac{2|B|\omega_{RN}}{|2K_-|} = 2|B|\mathfrak{W}^2$)

$$\delta_0 \coloneqq \frac{1}{2000} |B(M, \mathbf{e}, \Lambda)|^{-2} \mathfrak{W}^{-4}(M, \mathbf{e}, \Lambda, q_0) = \frac{1}{500} b_-^{-2}. \tag{4.5.1}$$

In the region $\mathcal{PK}$, we will overcome difficulty 1 by instead using new bootstrap assumptions that reflect the now monotonic behavior of ϕ and $\dot{\phi}$. The more fundamental difficulty 2 is dealt with by now using the fact that Ω^2 is now such that $-\log \Omega^2 \gtrsim -K_- s \gtrsim \epsilon^{-2}$ in regimes such as $\mathcal{PK}$ where s is of order ϵ^{-2}.

Proposition 4.5.1. *Choose $\delta_0(M, \mathbf{e}, \Lambda, q_0) > 0$ as in (4.5.1). Then, in the region $\mathcal{PK} = \{2|B|\mathfrak{W}\epsilon r_- \geq r(s) \geq e^{-\delta_0 \epsilon^{-2}} r_-\}$, there exists some $D_{PK}(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ such that that one has the following exponential behavior for the lapse $\Omega^2(s)$:*

$$\Omega^2(s) \leq D_{PK} \exp\big(-50 \delta_0 \epsilon^{-2}\big). \tag{4.5.2}$$

Recalling the quantity Q_∞ from (4.4.13), we have the following for all $s \in \mathcal{PK}$:

$$\big|-r\dot{r}(s) - 4|B|^2 \mathfrak{W}^2 \omega_{RN} r_-^2 \epsilon^2\big| \leq D_{PK} \epsilon^4 \log\big(\epsilon^{-1}\big), \tag{4.5.3}$$

$$\Big|q_0 \tilde{A}(s) - q_0 \tilde{A}_{RN,\infty}\Big| \leq D_{PK} \epsilon^2, \tag{4.5.4}$$

$$|Q(s) - Q_\infty| \leq D_{PK} \epsilon^2 \log\big(\epsilon^{-1}\big). \tag{4.5.5}$$

Using these, and recalling that $\omega_{RN} = |q_0 \tilde{A}_{RN,\infty}|$, we define the quantities ω_K and ξ_K by

$$\omega_K \coloneqq |q_0 \tilde{A}|(s_{PK}) = \omega_{RN} + O(\epsilon^2), \tag{4.5.6}$$

$$\xi_K \coloneqq \omega_K \left(-\frac{d}{ds} \frac{r^2}{r_-^2 \epsilon^2}\right)^{-1} (s_{PK}) = \frac{1}{8|B|^2 \mathfrak{W}^2} + O(\epsilon^2 \log\big(\epsilon^{-1}\big)). \tag{4.5.7}$$

For the scalar field ϕ, there will exist coefficients $C_{JK}(\epsilon)$ and $C_{YK}(\epsilon)$ obeying, for $C_J(\epsilon)$ and $C_Y(\epsilon)$ as in Proposition 4.4.5, the estimates $|C_{JK}(\epsilon) - C_J(\epsilon)| + |C_{YK}(\epsilon) - C_Y(\epsilon)| \leq D_{PK} \epsilon^2 \log\big(\epsilon^{-1}\big)$, such that for all $s \in \mathcal{PK}$:

$$\left|\phi(s) - \left(C_{JK}(\epsilon) J_0\left(\frac{\xi_K r^2(s)}{r_-^2 \epsilon^2}\right) + C_{YK}(\epsilon) Y_0\left(\frac{\xi_K r^2(s)}{r_-^2 \epsilon^2}\right)\right)\right| \leq D_{PK} \epsilon^2 \log\big(\epsilon^{-1}\big), \tag{4.5.8}$$

$$\left|\dot{\phi}(s) - \omega_K \left(C_{JK}(\epsilon) J_1 \left(\frac{\xi_K r^2(s)}{r_-^2 \epsilon^2} \right) + C_{YK}(\epsilon) Y_1 \left(\frac{\xi_K r^2(s)}{r_-^2 \epsilon^2} \right) \right) \right| \leq D_{PK} \epsilon^2 \log\big(\epsilon^{-1}\big). \tag{4.5.9}$$

Moreover, the following upper bounds for ϕ and $\dot{\phi}$ are satisfied for all $s \in \mathcal{PK}$:

$$|\phi(s)| \leq 100\, \mathfrak{W}^{-1} \left(1 + \log \left(\frac{8|B|^2 \mathfrak{W}^2 r_-^2 \epsilon^2}{r^2} \right) \right), \tag{4.5.10}$$

$$|\dot{\phi}| \leq 100\, \omega_{RN} \cdot \frac{8|B|^2 \mathfrak{W} r_-^2 \epsilon^2}{r^2}. \tag{4.5.11}$$

Remark. Both the statement and the proof of Proposition 4.5.1 will show large similarities to Proposition 4.4.1 in Section 4.4. One notes, however, that, in order to obtain the necessary error estimates, such as those of Lemma 4.5.2, the quantities ξ_0 and ω_0, which are defined as the values of certain quantities evaluated at $s = s_O$, to the quantities ξ_K and ω_K, which are the values of the same quantities but evaluated at $s = s_{PK}$ instead.

4.5.2 Proof of Proposition 4.5.1

We often make reference to the following four bootstrap assumptions in the region $\mathcal{PK}$.

$$\Omega^2 \leq \alpha_- \exp\big(-50\delta_0 \epsilon^{-2}\big), \tag{PK1}$$

$$|\phi| \leq 100\, \mathfrak{W}^{-1} \left(1 + \log \left(\frac{8|B|^2 \mathfrak{W}^2 r_-^2 \epsilon^2}{r^2} \right) \right), \tag{PK2}$$

$$|\dot{\phi}| \leq 100\, \omega_{RN} \cdot \frac{8|B|^2 \mathfrak{W} r_-^2 \epsilon^2}{r^2}, \tag{PK3}$$

$$|Q| \leq 2|\mathbf{e}|. \tag{PK4}$$

From Corollary 4.4.2 and Lemma 4.4.7, it follows that that the bootstraps (PK2), (PK3), (PK4) hold in a neighborhood of $s = s_{PK}$. On the other hand, (PK1) holds in a neighborhood of $s = s_{PK}$ due to (4.4.5) and (4.4.12): note that, for ϵ chosen sufficiently small, using (4.5.1) and (4.4.12), we have

$$K_- s_{PK} \leq -\frac{2|K_-|}{\omega_{RN}} \frac{1}{32|B|^2 \mathfrak{W}^2} \epsilon^{-2} \leq -\frac{1}{32|B|^2 \mathfrak{W}^4} \epsilon^{-2} < -51\delta_0 \epsilon^{-2}. \tag{4.5.12}$$

We now proceed in largely the same manner as in the region $\mathcal{O}$. Always assuming the bootstraps, we begin as in Section 4.4.1 with some preliminary estimates on $-r\dot{r}$ and $\tilde{A}$, then as in Section 4.4.2 we use these together with the equation (2.4.31) written in Bessel-like form to find precise asymptotics for the scalar field ϕ. Finally one concludes using these Bessel asymptotics to close estimates for Q and Ω^2 as in Section 4.4.3. We shall aim to be rather terse, as modifications from the analysis of the region $\mathcal{O}$ are generally minor.

Before proceeding, we briefly address how to deal with the difficulty 2 mentioned in Section 4.5.1. The idea is that the new bootstrap (PK1) now means the smallness of Ω^2 dominates by powers of

r^{-1}, since $r^{-1} \lesssim e^{-\delta_0 \epsilon^{-2}}$ in the region under consideration. For instance, consider the following expression arising in (2.4.25); in light of the bootstrap assumptions (PK1)–(PK4), one estimates

$$\frac{\Omega^2}{4}\left(\frac{Q^2}{r^2} + m^2 r^2 |\phi|^2\right) \lesssim e^{-48\delta_0\epsilon^{-2}}.$$

We now proceed with the proof in a series of lemmas similar to those of Section 4.4. As in that section, it is convenient to define $\mathcal{PK}_{boot}$ as the connected component of the set $\{s \in \mathcal{PK} : s \leq \epsilon^{-4}, \text{(PK1)–(PK4) apply}\}$ such that $s_{PK} \in \mathcal{PK}_{boot}$. Then we have the following.

Lemma 4.5.2. *For $s \in \mathcal{PK}_{boot}$, one finds*

$$\left|\frac{d}{ds}(-r\dot{r})(s)\right| + |-r\dot{r}(s) + r\dot{r}(s_{PK})| \leq D_{PK} e^{-40\delta_0\epsilon^{-2}}, \tag{4.5.13}$$

$$|q_0\tilde{A}(s) - q_0\tilde{A}(s_{PK})| \leq D_{PK} e^{-40\delta_0\epsilon^{-2}}. \tag{4.5.14}$$

Further, (4.5.3) and (4.5.4) hold. Moreover, letting $x(s) = \frac{r^2(s)}{r_-^2\epsilon^2}$, one finds that

$$\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) + \xi_K^2 x f = \mathcal{E}_\phi, \tag{4.5.15}$$

where the error term $\mathcal{E}_\phi$ is bounded by

$$|\mathcal{E}_\phi(s)| \leq D_{PK} e^{-30\delta_0\epsilon^{-2}}.$$

Proof. The proof is almost identical to that of Lemma 4.4.3 and 4.4.4, in light of the comments above. □

Lemma 4.5.3. *Recalling $x(s) = \frac{r^2(s)}{r_-^2\epsilon^2}$, let $x_{PK} = x(s_{PK})$, and define the coefficients $C_{JK}(\epsilon)$ and $C_{YK}(\epsilon)$ via*

$$\begin{bmatrix} C_{JK}(\epsilon) \\ C_{YK}(\epsilon) \end{bmatrix} = \begin{bmatrix} J_0(\xi_K x_{PK}) & Y_0(\xi_K x_{PK}) \\ -\xi_K J_1(\xi_K x_{PK}) & -\xi_K Y_1(\xi_K x_{PK}) \end{bmatrix}^{-1} \begin{bmatrix} \phi(x_{PK}) \\ \frac{d\phi}{dx}(x_{PK}) \end{bmatrix}. \tag{4.5.16}$$

Then one has that $|C_{JK}(\epsilon) - C_J(\epsilon)| + |C_{YK}(\epsilon) - C_Y(\epsilon)| \leq D_{PK}\epsilon^2 \log(\epsilon^{-1})$, and that, for $x = x(s)$ corresponding to $s \in \mathcal{PK}_{boot}$,

$$|\phi(x) - C_{JK}(\epsilon)J_0(\xi_K x) - C_{JK}(\epsilon)Y_0(\xi_K x)| \leq D_{PK} e^{-20\delta_0\epsilon^{-2}}, \tag{4.5.17}$$

$$\left|\frac{d\phi}{dx}(s) + \xi_K C_{JK}(\epsilon)J_1(\xi_K x) + \xi_K C_{YK}(\epsilon)Y_1(\xi_K x)\right| \leq D_{PK} e^{-20\delta_0\epsilon^{-2}}. \tag{4.5.18}$$

Proof. The idea is that in light of (4.5.15), we can use the equation (4.4.34) describing the solution for the scalar field using the solution operator to the linear Bessel equation, just as in Proposition 4.4.5. To get improved error estimates, however, we evolve the system from $x = x_{PK}$ as opposed to $x = x_{\mathcal{B}}$, hence the appearance of the updated Bessel coefficients $C_{JK}(\epsilon)$ and $C_{YK}(\epsilon)$ in (4.5.16).

We first show these are close to the original coefficients as claimed; from Proposition 4.4.5, we have

$$\begin{bmatrix} \phi(x_{PK}) \\ \frac{d\phi}{dx}(x_{PK}) \end{bmatrix} = \begin{bmatrix} J_0(\xi_0 x_{PK}) & Y_0(\xi_0 x_{PK}) \\ -\xi_0 J_1(\xi_0 x_{PK}) & -\xi_0 Y_1(\xi_0 x_{PK}) \end{bmatrix} \begin{bmatrix} C_J(\epsilon) \\ C_Y(\epsilon) \end{bmatrix} + O(\epsilon^{10}).$$

Moreover, using $x_{PK} = 4|B|^2\mathfrak{W}^2$, from (4.3.53) and Corollary 4.4.4, we know that $\xi_0 x_{PK}$ and $\xi_K x_{PK}$ lie within the interval $(\frac{1}{4}, 1)$ and that $|\xi_0 x_{PK} - \xi_K x_{PK}| \lesssim \epsilon^{25} \lesssim \epsilon^2 \log(\epsilon^{-1})$. Thus, since the derivatives of $J_\nu(z)$ and $Y_\nu(z)$ are bounded for $1/4 \leq z \leq 1$, and $|C_J(\epsilon)|, |C_Y(\epsilon)| \lesssim 1$, we can modify the above formula to

$$\begin{bmatrix} \phi(x_{PK}) \\ \frac{d\phi}{dx}(x_{PK}) \end{bmatrix} = \begin{bmatrix} J_0(\xi_K x_{PK}) & Y_0(\xi_K x_{PK}) \\ -\xi_K J_1(\xi_K x_{PK}) & -\xi_K Y_1(\xi_K x_{PK}) \end{bmatrix} \begin{bmatrix} C_J(\epsilon) \\ C_Y(\epsilon) \end{bmatrix} + O(\epsilon^2 \log(\epsilon^{-1})). \tag{4.5.19}$$

Finally, since by Lemma A.0.5 the inverse matrix

$$\begin{bmatrix} J_0(\xi_K x_{PK}) & Y_0(\xi_K x_{PK}) \\ -\xi_K J_1(\xi_K x_{PK}) & -\xi_K Y_1(\xi_K x_{PK}) \end{bmatrix}^{-1} = \frac{2}{\pi x_{PK}} \begin{bmatrix} -\xi_K Y_1(\xi_K x_{PK}) & -Y_0(\xi_K x_{PK}) \\ \xi_K J_1(\xi_K x_{PK}) & J_0(\xi_K x_{PK}) \end{bmatrix}$$

has uniformly bounded entries, combining (4.5.19) with (4.5.16) shows, as claimed, the relation

$$\begin{bmatrix} C_{JK}(\epsilon) \\ C_{YK}(\epsilon) \end{bmatrix} = \begin{bmatrix} C_J(\epsilon) \\ C_Y(\epsilon) \end{bmatrix} + O(\epsilon^2 \log(\epsilon^{-1})).$$

The remainder of the proof is as in Proposition 4.4.5. Adopting the same notation from the proof of this proposition, we record the analogue of (4.4.34) again here:

$$\begin{bmatrix} \phi \\ \frac{d\phi}{dx} \end{bmatrix}(x) = \mathbf{S}_{\xi_K}(x; x_{PK}) \begin{bmatrix} \phi \\ \frac{d\phi}{dx} \end{bmatrix}(x_{PK}) + \int_{x_{PK}}^{x} \mathbf{S}_{\xi_K}(x; \tilde{x}) \begin{bmatrix} 0 \\ \frac{1}{\tilde{x}} \mathcal{E}_\phi(\tilde{x}) \end{bmatrix} d\tilde{x}. \tag{4.5.20}$$

The first term on the right hand side will correspond to a solution of the linear Bessel equation after a rescaling $z = \xi_K x$, just as before. Using once again part (1) of Corollary A.0.7 we see this corresponds to the objects on the left hand sides of (4.5.17) and (4.5.18).

For the second term on the right hand side of (4.5.20), we once again appeal to part (2) of Corollary A.0.7. From this corollary, the appearance of the operator $\mathbf{S}_{\xi_K}(x; x_{PK})$ will contribute at worst an additional factor of $x^{-1} \leq \epsilon^2 e^{2\delta_0 \epsilon^{-2}}$ in our l^∞ estimates. Since the length of the integration interval $|x - x_{PK}| \leq x_{PK}$ is uniformly bounded, (4.5.17) and (4.5.18) follow straightforwardly. □

Completing the proof of Proposition 4.5.1. We are now in a position to close all the bootstraps and complete the proof. We first use Lemma 4.5.3 to improve (PK2) and (PK3). We shall only show the latter; the former follows in a similar fashion.

By Lemma 4.5.3 and the bounds on the coefficients in Proposition 4.4.5, within the region $\mathcal{PK}_{boot}$, one has

$$\left|\frac{d\phi}{dx}\right| \leq 2\sqrt{\pi}\, \mathfrak{W}^{-1}\, \xi_K \cdot (|J_1(\xi_K x)| + |Y_1(\xi_K x)|).$$

In light of Facts A.0.1 and A.0.2, one may check that $\left|\frac{d\phi}{dx}\right| \leq 4\sqrt{\pi}\,\mathfrak{W}^{-1}x^{-1}$. Hence, using $-\frac{dx}{ds} = 8|B|^2\mathfrak{W}^2\omega_{RN} + O(\epsilon^2\log(\epsilon^{-1}))$ from Lemma 4.5.2, we improve (PK3). As mentioned, improving (PK2) is similar.

We now move onto Q and Ω^2. For $Q(s)$, it remains to understand only how Q changes in the region $\mathcal{PK}$, in particular that it changes only by $O(\epsilon^2\log(\epsilon^{-1}))$. Looking at (2.4.28), we need to study

$$\left|\int_{s_{PK}}^{s} \tilde{A}q_0^2 r^2|\phi|^2(s')\,ds'\right|.$$

Now, we use (4.4.20) and (4.5.10) to estimate

$$\left|\int_{s_{PK}}^{s} \tilde{A}q_0^2 r^2|\phi|^2(s')\,ds'\right| \lesssim \int_{s_{PK}}^{s} \epsilon^2\cdot\frac{r^2}{r_-^2\epsilon^2}\left(1+\left|\log\left(\frac{\epsilon^2}{r^2}\right)\right|\right)^2(s')\,ds'. \tag{4.5.21}$$

Substituting $x = \frac{r^2}{r_-^2\epsilon^2}$ as usual, and noting that $-\frac{dx}{ds}\sim 1$, we therefore see that

$$\left|\int_{s_{PK}}^{s} \tilde{A}q_0^2 r^2|\phi|^2(s')\,ds'\right| \lesssim \int_0^{4|B|^2\mathfrak{W}^2} \epsilon^2 x(1+|\log(x)|)^2\,dx \lesssim \epsilon^2. \tag{4.5.22}$$

Combined with (4.4.14), this will yield (4.5.5), and hence improve the bootstrap (PK4).

Last of all, for the quantity Ω^2, we proceed using that the Raychaudhuri equation (2.4.21) implies $-\Omega^{-2}\dot{r}(s)$ is nonincreasing. Therefore, for $s\in\mathcal{PK}_{boot}$,

$$\Omega^2(s) \leq \frac{\Omega^2(s_{PK})}{-\dot{r}(s_{PK})}\cdot(-\dot{r}(s)) = \Omega^2(s_{PK})\cdot\frac{-r\dot{r}(s)}{-r\dot{r}(s_{PK})}\cdot\frac{r(s_{PK})}{r(s)}.$$

Therefore, one applies (4.5.3) and (4.4.5), alongside $r(s_{PK}) = 2|B|\mathfrak{W}r_-\epsilon$ and $r(s)\geq e^{-\delta_0\epsilon^{-2}}$ for $s\in\mathcal{PK}$, to find

$$\Omega^2(s) \leq (1+2D_O\epsilon^2\log(\epsilon^{-1}))\cdot\alpha_- e^{K_-s_{PK}}\cdot 2|B|\mathfrak{W}r_-\epsilon\cdot e^{\delta_0\epsilon^{-2}}. \tag{4.5.23}$$

By (4.5.12), this means that $\Omega^2(s)\leq\alpha_- e^{-50\delta_0\epsilon^{-2}}\cdot(1+2D_O\epsilon^2\log(\epsilon^{-1}))2|B|\mathfrak{W}r_-\epsilon$. Thus, for ϵ chosen sufficiently small, it is straightforward to bound the right hand side such that we improve the final bootstrap (PK1). As a result, $\mathcal{PK}_{boot} = \mathcal{PK}$, and the remaining assertions of Proposition 4.5.1 are straightforward. □

4.5.3 The onset of the Kasner-like geometry

Following the proof of Proposition 4.5.1, we mention a corollary of Proposition 4.5.1 that will be important in later sections, and in interpreting the subset $\mathcal{PK}' = \mathcal{K}_1\cap\mathcal{PK} = \{2|B|\mathfrak{W}\epsilon^2 r_- \geq r(s)\geq e^{-\delta_0\epsilon^{-2}}r_-\}$ as a genuine Kasner-like region. (Note that, for this step, we restrict to $r(s)\lesssim\epsilon^2 r_-$ rather than $r(s)\lesssim\epsilon r_-$ in order to ignore the $O(1)$ terms in the Bessel function asymptotics.)

Corollary 4.5.4. *Consider the region $s \in \mathcal{PK}' = \{2|B|\mathfrak{W}\epsilon^2 r_- \geq r(s) \geq e^{-\delta_0\epsilon^{-2}} r_-\}$. In this region, we have the following forms for $(\phi, \dot{\phi})$, where $c_1 = 1$ and $c_2 = 2\pi^{-1}(\gamma - \log 2)$ and $\gamma \approx 0.577$ is the Euler-Mascheroni constant.*

$$\left|\phi(s) + \frac{2}{\pi} C_{YK}(\epsilon) \log\left(\frac{r_-^2\epsilon^2}{r^2(s)\xi_K}\right) - c_1 C_{JK}(\epsilon) - c_2 C_{YK}(\epsilon)\right| \leq D_{PK}\epsilon^2 \log\big(\epsilon^{-1}\big), \tag{4.5.24}$$

$$\left|\dot{\phi}(s) + \frac{2}{\pi} C_{YK}(\epsilon) \frac{r_-^2 \omega_K \epsilon^2}{r^2(s)\xi_K}\right| \leq D_{PK}\epsilon^2 \log\big(\epsilon^{-1}\big). \tag{4.5.25}$$

Next, recall the function $\Theta(\epsilon)$ arising in (4.4.31). Then, defining $\Psi(s) := \frac{r^2\dot{\phi}}{-r\dot{r}}(s)$ [see already (4.6.1) *in the next section], one finds that*

$$\left|\Psi(s) + \frac{2}{\sqrt{\pi}} \mathfrak{W}^{-1} \sin(\Theta(\epsilon))\right| \leq D_{PK}\epsilon^2 \log\big(\epsilon^{-1}\big). \tag{4.5.26}$$

Furthermore, if we let $\Psi_i = \Psi(s_i)$, where $r(s_i) = e^{-\delta_0\epsilon^{-2}}$, then for $s \in \mathcal{PK}'$,

$$|\Psi(s) - \Psi_i| \leq D_{PK} r^2(s) \log\big(\epsilon^{-1}\big) \leq D_{PK}^2 \epsilon^4 \log\big(\epsilon^{-1}\big). \tag{4.5.27}$$

Finally, one obtains the following estimates for the lapse Ω^2:

$$\left|\frac{\frac{d}{ds}\log\Omega^2(s)}{\frac{d}{ds}\log r(s)} - \Psi_i^2 + 1\right| \leq D_{PK}\epsilon^4 \log\big(\epsilon^{-1}\big), \tag{4.5.28}$$

$$\left|\log\left[\Omega^2(s)\left(\frac{r(s)}{r_-}\right)^{1-\Psi_i^2}\right] + \frac{1}{2} b_-^{-2}\epsilon^{-2}\right| \leq D_{PK}\log\big(\epsilon^{-1}\big). \tag{4.5.29}$$

Proof. The equations (4.5.24) and (4.5.25) follow immediately from (4.5.8), (4.5.9) and the Bessel function asymptotics in Facts A.0.1 and A.0.2. Indeed, by restricting $r(s) \leq 2|B|\mathfrak{W}\epsilon^2 r_-$, we guarantee that

$$\frac{\xi_K r^2(s)}{r_-^2\epsilon^2} \leq \epsilon^2,$$

hence Facts A.0.1 and A.0.2 guarantee that

$$\left|J_0\left(\frac{\xi_K r^2(s)}{r_-^2\epsilon^2}\right) - 1\right| \lesssim \epsilon^4, \quad \left|J_1\left(\frac{\xi_K r^2(s)}{r_-^2\epsilon^2}\right)\right| \lesssim \epsilon^2,$$

$$\left|Y_0\left(\frac{\xi_K r^2(s)}{r_-^2\epsilon^2}\right) + \frac{2}{\pi}\log\frac{r_-^2\epsilon^2}{\xi_K r^2(s)} - \frac{2}{\pi}(\gamma - \log 2)\right| \lesssim \epsilon^4, \quad \left|Y_1\left(\frac{\xi_K r^2(s)}{r_-^2\epsilon^2}\right) + \frac{2}{\pi}\frac{r_-^2\epsilon^2}{\xi_K r^2(s)}\right| \lesssim \epsilon^2\log\big(\epsilon^{-1}\big).$$

Substituting these into (4.5.8) and (4.5.9), we get (4.5.24) and (4.5.25) with $c_1 = 1$ and $c_2 = 2\pi^{-1}(\gamma - \log 2)$.

Next, combining (4.5.25) with the estimates (4.5.6) and (4.5.7) it is straightforward to get

$$\left|r^2\dot{\phi}(s) + 16\pi^{-1}C_{YK}(\epsilon)|B|^2\mathfrak{W}^2\omega_{RN}r_-^2\epsilon^2\right| \lesssim \epsilon^4\log\big(\epsilon^{-1}\big). \tag{4.5.30}$$

Hence from (4.5.3) we find that

$$\big|\Psi(s) + 4\pi^{-1}C_{YK}(\epsilon)\big| = \left|\frac{r^2\dot{\phi}(s)}{-r\dot{r}(s)} + 4\pi^{-1}C_{YK}(\epsilon)\right| \lesssim \epsilon^2\log\big(\epsilon^{-1}\big), \tag{4.5.31}$$

thus (4.5.26) follows from $|C_{YK}(\epsilon) - C_Y(\epsilon)| \lesssim \epsilon^2\log\big(\epsilon^{-1}\big)$ and (4.4.30).

To get (4.5.27), note that (4.5.25) will also yield $|r^2\dot{\phi}(s) - r^2\dot{\phi}(s_i)| \lesssim \epsilon^2\log\big(\epsilon^{-1}\big)r^2(s)$, while we also know from (4.5.13) that $|-r\dot{r}(s)+r\dot{r}(s_i)| \lesssim e^{-40\delta_0\epsilon^{-2}} \lesssim \epsilon^2\log\big(\epsilon^{-1}\big)r^2(s)$ changes little in the region $\mathcal{PK}$. Since $-r\dot{r}(s) \sim \epsilon^2$ by (4.5.3), the computation

$$\left|\frac{r^2\dot{\phi}}{-r\dot{r}}(s) - \frac{r^2\dot{\phi}}{-r\dot{r}}(s_i)\right| \lesssim \frac{1}{-r\dot{r}(s)}\cdot\Big[|r^2\dot{\phi}(s) - r^2\dot{\phi}(s_i)| + |-r\dot{r}(s) + r\dot{r}(s_i)|\Big],$$

together with $r(s) \lesssim \epsilon^2$ for $s \in \mathcal{PK}'$, yields (4.5.27).

We now move on to the estimate (4.5.28) for the derivative of $\Omega^2(s)$. We proceed here using the Raychaudhuri equation in the form (2.4.23), which after multiplying by $r(s)(\dot{r}(s))^{-2}$ gives

$$\frac{r}{\dot{r}}\frac{d}{ds}\log\Omega^2 - \frac{r\ddot{r}}{\dot{r}^2} - \frac{r^2\dot{\phi}^2}{\dot{r}^2} = \frac{r^2|\tilde{A}|^2q_0^2|\phi|^2}{\dot{r}^2}.$$

Using (2.4.24) to substitute for $\ddot{r}$, and noticing that the rightmost term on the left hand side is exactly Ψ^2, one finds

$$\frac{r}{\dot{r}}\frac{d}{ds}\log\Omega^2 + 1 - \Psi^2 = (-r\dot{r})^{-2}\left[r^4\tilde{A}^2q_0^2|\phi|^2 + \frac{\Omega^2}{4}\left(Q^2 - r^2 + r^4\Lambda + r^4m^2|\phi|^2\right)\right].$$

From Proposition 4.5.1 and (4.5.24), the right hand side of this is bounded by a multiple of $\epsilon^{-4}(r^2\log\big(\frac{1}{r^2}\big))^2 \lesssim \epsilon^4\log\big(\epsilon^{-1}\big)$, so that if we also use (4.5.27), then we get (4.5.28) as required.

Finally, for (4.5.29), one would like to integrate (4.5.28). However, one needs to first estimate $\Omega^2(s_{K_1})$, where $s = s_{K_1}$ is such that $r(s_{K_1}) = 2|B|\mathfrak{W}\epsilon^2r_-$ (corresponding to the past boundary of $\mathcal{PK}'$). For this purpose, one finds exactly as in the proof of Proposition 4.4.1[13] that, for $s \in \mathcal{O}\cup(\mathcal{PK}\setminus\mathcal{PK}') = \{s_O \le s \le s_{K_1}\}$,

$$\left|\frac{d}{ds}\log\Omega^2(s) - 2K_-\right| \lesssim \epsilon^2[\log\big(\epsilon^{-1}\big) + r^{-2}(s)]. \tag{4.5.32}$$

Integrating this for $s \in \mathcal{O}\cup(\mathcal{PK}\setminus\mathcal{PK}')$, using that the length of the integration interval is of $O(\epsilon^{-2})$, one gets

$$\big|\log\Omega^2(s_{K_1}) - 2K_-s_{K_1}\big| \lesssim \log\big(\epsilon^{-1}\big) + |\log\Omega^2(s_O)| \lesssim \log\big(\epsilon^{-1}\big).$$

[13]We need to extend from $r(s) \gtrsim \epsilon$ to $r(s) \gtrsim \epsilon^2$, but the proof still applies as we only needed $|\log(r/r_-)| \lesssim \log\big(\epsilon^{-1}\big)$.

We also need to estimate the expression $2K_- s_{K_1}$. This follows from an identical computation to that of s_{PK} as in (4.4.12), and one finds that

$$2K_- s_{K_1} = -\frac{|2K_-|}{8|B|^2\mathfrak{W}^2\omega_{RN}\epsilon^2} + O(\log(\epsilon^{-1})) = -\frac{1}{2}b_-^2\epsilon^{-2} + O(\log(\epsilon^{-1})).$$

Since $\frac{r(s_{K_1})}{r_-} \sim \epsilon^2$, so that additional factors of $\log\left(\frac{r(s_{K_1})}{r_-}\right)$ may be added, we arrive at

$$\left|\log\left[\Omega^2(s_{K_1})\left(\frac{r(s_{K_1})}{r_-}\right)^{1-\Psi_i^2}\right] + \frac{1}{2}b_-^2\epsilon^{-2}\right| \lesssim \log(\epsilon^{-1}). \tag{4.5.33}$$

Finally, we rewrite (4.5.28) as

$$\left|\frac{d}{ds}\log\left[\Omega^2(s)\left(\frac{r(s)}{r_-}\right)^{1-\Psi_i^2}\right]\right| \lesssim \epsilon^2\log(\epsilon^{-1})\frac{-\dot{r}(s)}{r(s)}.$$

Integrating this for $s \in \mathcal{PK}'$, and using (4.5.33) to estimate the boundary term at $s = s_{K_1}$, we find (4.5.29). □

4.6 Construction of the sets E_η and $E'_{\eta,\sigma}$ for further quantitative estimates

So far, we have a description of the hairy black hole interior up to the region $\mathcal{PK}$, where $s \approx (8|B|^2\mathfrak{W}^2\omega_{RN}\epsilon^2)^{-1}$ and $r(s) \geq \exp(-\delta_0\epsilon^{-2})r_-$. These estimates hold for $0 < \epsilon \leq \epsilon_0$, where $\epsilon_0(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ is taken sufficiently small, and δ_0 is a fixed quantity determined by (4.5.1). In particular, up to this point, we have placed no restriction on the value of ϵ, other than its smallness.

However, within the present article, to proceed further we will need to restrict attention to a smaller subset of values of ϵ verifying a certain condition. We define the following important quantity:

$$\Psi := \frac{r^2\dot{\phi}}{-r\dot{r}} = -r\frac{d\phi}{dr}. \tag{4.6.1}$$

Then the precise condition on ϵ we shall use is that

$$|\Psi_i| := |\Psi(s_i)| \geq \eta > 0, \tag{†}$$

where s_i marks the end of the region $\mathcal{PK}$, i.e. $r(s_i) = e^{-\delta_0\epsilon^{-2}}r_-$, and $\eta > 0$ is an arbitrary small constant. Before using (†) to analyse the spacetime beyond $\mathcal{PK}$ in Section 4.7, we first quantitatively characterize the set of ϵ for which a condition such as (†) holds. In light of Corollary 4.5.4, we are required to study the quantity $\Theta(\epsilon)$ defined in (4.4.31).

4.6.1 Improved estimates on $\Theta(\epsilon)$

From Proposition 4.3.5, $\Theta(\epsilon)$ is identified to have the following dependence on ϵ:

$$\Theta(\epsilon) = \frac{1}{8|B|^2\mathfrak{W}^2\epsilon^2} + O(\log(\epsilon^{-1})). \tag{4.6.2}$$

However, the $O(\log(\epsilon^{-1}))$ error prevents us from having any quantitative control on quantities such as $\sin(\Theta(\epsilon))$, i.e. given only the above expression, for any fixed $\eta > 0$, if $|\cdot|$ denotes Lebesgue measure, the *limiting density*

$$\limsup_{\epsilon_0 \to 0} \epsilon_0^{-1} |\{\epsilon \in (0, \epsilon_0] : |\sin(\Theta(\epsilon))| > \eta\}|$$

could be arbitrarily small, or even vanish asymptotically.

To overcome this issue, we instead consider the quantity:

$$\frac{d}{d\epsilon}\Theta(\epsilon) = \left(\frac{\partial}{\partial\epsilon} + \frac{d\, s_O(\epsilon)}{d\epsilon}\frac{\partial}{\partial s}\right)\left(|q_0\tilde{A}|(s)\cdot r^2(s)\cdot\left(-\frac{d}{ds}r^2(s)\right)^{-1} + \omega_{RN}s\right), \tag{4.6.3}$$

where to make sense of the right hand side, we now interpret $f(s) \in \{r(s), \log\Omega^2(s), Q(s), \tilde{A}(s), \phi(s), \dot{\phi}(s)\}$ as (smooth) functions of ϵ as well as s.

Denoting the ϵ-derivatives by using a subscript, i.e. $f_\epsilon = \frac{\partial}{\partial\epsilon}f$, while still using $\dot{f} = \frac{\partial}{\partial s}f$ to denote s-derivatives, we shall take an ϵ-derivative of the system (2.4.21)–(2.4.31) to find a system of *linear* evolution equations for the quantities $f_\epsilon(s)$. For instance, the ϵ-derivative of (2.4.29) is

$$\dot{\tilde{A}}_\epsilon = -\frac{\Omega^2}{4r^2}\left(Q_\epsilon + Q(\log\Omega^2)_\epsilon - \frac{2Qr_\epsilon}{r}\right),$$

while the corresponding evolution equations for the other linearized quantities are more complicated and will not be written explicitly. We can also perform this linearization for the difference quantities

$$\delta\phi_\epsilon = \frac{\partial}{\partial\epsilon}\left(\phi(s) - \phi_{\mathcal{L}}(s)\right), \quad \delta\dot{\phi}_\epsilon = \frac{\partial}{\partial s}\delta\phi_\epsilon.$$

Along with the evolution equations, one must pose data for the quantities $f_\epsilon(s)$ in the $s \to -\infty$ limit. For this purpose, one should return to the $1+1$-dimensional formulation of the problem in the regular (U, V) coordinates as in Section 2.4.1, and take the appropriate ϵ-derivatives there. The correct asymptotic data is

$$\lim_{s\to-\infty} r_\epsilon(s) = 0, \ \lim_{s\to-\infty} Q_\epsilon(s) = 0, \ \lim_{s\to-\infty}\phi_\epsilon = 1, \tag{4.6.4}$$

$$\lim_{s\to-\infty}(\log\Omega^2)_\epsilon = \lim_{s\to-\infty}\frac{d}{ds}(\log\Omega^2)_\epsilon = 0, \tag{4.6.5}$$

$$\lim_{s\to-\infty}\Omega^{-2}\tilde{A}_\epsilon(s) = 0, \tag{4.6.6}$$

$$\lim_{s\to-\infty} 4\Omega^{-2}\dot{r}_\epsilon(s) = \frac{2r_+m^2\epsilon}{2K_+}, \tag{4.6.7}$$

$$\lim_{s\to-\infty}\Omega^{-2}\dot{\phi}_\epsilon = \beta_+. \tag{4.6.8}$$

The plan is now to use a similar procedure to Section 4.3 to find sufficiently strong estimates for the quantities $f_\epsilon \in \{r_\epsilon, (\log\Omega^2)_\epsilon, Q_\epsilon, \tilde{A}_\epsilon, \phi_\epsilon\}$ up to the late blue shift region $\mathcal{LB}$, where we have nontrivial overlap with the oscillatory region $\mathcal{O}$, and compute $\frac{d}{d\epsilon}\Theta(\epsilon)$. The most crucial estimate will be to determine that $\dot{r}_\epsilon$ is comparable to ϵ.

Proposition 4.6.1. *For $s \in \mathcal{EB} \cup \mathcal{LB} = \{S \leq s \leq \Delta_{\mathcal{B}}\epsilon^{-1}\}$, there exists some constant $D_{LE}(M, \mathbf{e}, \Lambda, m^2, q_0)$ such that:*

$$|\dot{r}_\epsilon| + s^{-1}|r_\epsilon| \leq D_{LE}\epsilon, \tag{4.6.9}$$

$$\left|\frac{d}{ds}(\log\Omega^2)_\epsilon\right| + s^{-1}|(\log\Omega^2)_\epsilon| \leq D_{LE}\epsilon s, \tag{4.6.10}$$

$$|\tilde{A}_\epsilon| + s^{-1}|Q_\epsilon| \leq D_{LE}\epsilon, \tag{4.6.11}$$

$$|\dot{\phi}_\epsilon| + |\phi_\epsilon| \leq D_{LE}, \tag{4.6.12}$$

$$|\delta\dot{\phi}_\epsilon| + |\delta\phi_\epsilon| \leq D_{LE}\epsilon^2 s. \tag{4.6.13}$$

Furthermore, we have the more precise estimate for $s_O = 50 s_{lin} \leq s \leq \Delta_{\mathcal{B}}\epsilon^{-1}$:

$$\left|-\dot{r}_\epsilon(s) - 8|B|^2\mathfrak{W}^2\omega_{RN}r_-\epsilon\right| \leq D_{LE}\epsilon^3 s. \tag{4.6.14}$$

Corollary 4.6.2. *Consider the expression (4.6.3). Then*

$$\left|\frac{d}{d\epsilon}\Theta(\epsilon) + \frac{1}{4|B|^2\mathfrak{W}^2\epsilon^3}\right| \leq D_{LE}\epsilon^{-1}\log\left(\epsilon^{-1}\right). \tag{4.6.15}$$

Proof of Corollary 4.6.2 given Proposition 4.6.1. First consider the $\frac{\partial}{\partial s}$ derivative in (4.6.3). One checks from (2.4.25), (2.4.29) and Proposition 4.3.5, that one has

$$\left|\frac{\partial}{\partial s}\left(|q_0\tilde{A}|(s)\cdot r^2(s)\cdot\left(-\frac{d}{ds}r^2(s)\right)^{-1} + \omega_{RN}s\right)\right| \lesssim \epsilon^{-4}\,\Omega^2\cdot(1+|\phi|^2) + |\omega_{RN} - |q_0\tilde{A}|| \lesssim \epsilon^2$$

where the final step follows from $\Omega^2 \lesssim \epsilon^{100}$ at $s = s_O$ and Proposition 4.3.5. Hence even with the $\frac{ds_O}{d\epsilon} \sim \epsilon^{-1}$ factor in front, this term contributes at worst $O(\epsilon)$ and can be ignored.

The main term $(4|B|^2\mathfrak{W}^2\epsilon^3)^{-1}$ on the left hand side of (4.6.15) comes from taking the ϵ-derivative of $\left(-\frac{d}{ds}r^2(s)\right)^{-1}$. Indeed, the expression that arises from this is

$$I = |q_0\tilde{A}|(s)\cdot r^2(s)\cdot\left(-\frac{d}{ds}r^2(s)\right)^{-2}\cdot 2(r_\epsilon(s)\dot{r}(s) + r(s)\dot{r}_\epsilon(s)).$$

Using Propositions 4.3.5 and 4.6.1, particularly (4.6.14), we can evaluate (note $\dot{r}(s)r_\epsilon(s) = O(\epsilon^3 s)$ is treated as part of the error):

$$\begin{aligned} I &= \frac{\omega_{RN}r_-^2}{(8|B|^2\mathfrak{W}^2r_-^2\omega_{RN}\epsilon^2)^2}\cdot 2(-8|B|^2\mathfrak{W}^2\omega_{RN}r_-^2\epsilon) + O(\epsilon^{-1}\log\left(\epsilon^{-1}\right)) \\ &= -\frac{1}{4|B|^2\mathfrak{W}^2\epsilon^3} + O(\epsilon^{-1}\log\left(\epsilon^{-1}\right)). \end{aligned}$$

Therefore, the ϵ-derivative on the left hand side of (4.6.15) can be evaluated using Proposition 4.6.1 as:

$$I + (|q_0\tilde{A}|_\epsilon(s)r^2(s) + 2|q_0\tilde{A}|(s)r(s)r_\epsilon(s)) \cdot \left(-\frac{d}{ds}r^2(s)\right)^{-1} = -\frac{1}{4|B|^2\mathfrak{W}^2\epsilon^3} + O(\epsilon^{-1}\log(\epsilon^{-1})),$$

completing the proof of the corollary. □

Proof of Proposition 4.6.1. The first step will be to find the upper bounds. For this purpose, we will have to write out the full linear system of ODEs, however, as mentioned previously, this would be cumbersome, and we therefore only include upper bounds for $|\dot{f}_\epsilon|$, often using that $r \sim 1, -\dot{r} \lesssim 1, |\phi| \lesssim \epsilon, |\dot{\phi}| \lesssim \epsilon, Q \sim 1$ and $\tilde{A} \lesssim 1$ within this region. Differentiating (2.4.24)–(2.4.30) in ϵ, the appropriate inequalities and equations are:

$$|\ddot{r_\epsilon}| \lesssim \Omega^2(|r_\epsilon| + |(\log\Omega^2)_\epsilon| + |Q_\epsilon|) + (-\dot{r})(|r_\epsilon| + |\dot{r}_\epsilon|) + \Omega^2|\phi\phi_\epsilon|, \tag{4.6.16}$$

$$\left|\frac{d^2}{ds^2}(\log\Omega^2)_\epsilon\right| \lesssim \Omega^2(|r_\epsilon| + |(\log\Omega^2)_\epsilon| + |Q_\epsilon|) + (-\dot{r})(|r_\epsilon| + |\dot{r}_\epsilon|) + |\tilde{A}||\phi\phi_\epsilon| + |\dot{\phi}\dot{\phi}_\epsilon| + |\tilde{A}_\epsilon||\phi|^2, \tag{4.6.17}$$

$$|\dot{Q}_\epsilon| \lesssim (|\tilde{A}_\epsilon| + |\tilde{A}r_\epsilon|)|\phi|^2 + |\tilde{A}\phi\phi_\epsilon|, \tag{4.6.18}$$

$$|\dot{\tilde{A}}_\epsilon| \lesssim \Omega^2(|r_\epsilon| + |(\log\Omega^2)_\epsilon| + |Q_\epsilon|), \tag{4.6.19}$$

$$\ddot{\phi}_\epsilon = -\frac{2\dot{r}\dot{\phi}_\epsilon}{r} - q_0^2\tilde{A}^2\phi_\epsilon - \frac{m^2\Omega^2}{4}\phi_\epsilon + J_\phi, \qquad |J_\phi| \lesssim |\ddot{r}_\epsilon - \dot{r}r_\epsilon||\dot{\phi}| + |\tilde{A}_\epsilon||\phi| + \Omega^2(\log\Omega^2)_\epsilon|\phi|. \tag{4.6.20}$$

Note that, for the final equation (4.6.20), it is necessary to exactly keep the terms corresponding to ϕ being a solution of the linear charged scalar wave equation. We now proceed through the regions $\mathcal{R}$, $\mathcal{N}$, $\mathcal{EB}$ and $\mathcal{LB}$ exactly as in Section 4.3.

Step 1: The redshift region $\mathcal{R} = \{-\infty < s \leq -\Delta_{\mathcal{R}}\}$

In this region, we consider the following bootstrap assumptions, which hold in a neighborhood of $s = -\infty$ by the asymptotic data above:

$$|r_\epsilon| + |\dot{r}_\epsilon| + |(\log\Omega^2)_\epsilon| + |Q_\epsilon| + |\Omega^{-2}\tilde{A}_\epsilon| \leq \epsilon, \tag{4.6.21}$$

$$|\phi_\epsilon| + |\dot{\phi}_\epsilon| \leq 2. \tag{4.6.22}$$

Note that by Proposition 4.3.1, we know already that $-\dot{r}, |\tilde{A}| \lesssim \Omega^2$ and $|\phi|, |\dot{\phi}| \lesssim \epsilon$, so given these bootstrap assumptions and the inequality (4.3.8) we may simply integrate up the equations (4.6.16)–(4.6.20) to get

$$|r_\epsilon| + |\dot{r}_\epsilon| + |(\log\Omega^2)_\epsilon| + \left|\frac{d}{ds}(\log\Omega^2)_\epsilon\right| + |Q_\epsilon| + |\Omega^{-2}\tilde{A}_\epsilon| \lesssim \epsilon\Omega^2, \tag{4.6.23}$$

$$|\phi_\epsilon - 1| + |\dot{\phi}_\epsilon| \lesssim \Omega^2. \tag{4.6.24}$$

(Recall that the asymptotic data for r_ϵ, $\dot{r}_\epsilon$, $\log\Omega^2_\epsilon$ and Q_ϵ is 0.) Hence with a choice of $\Delta_{\mathcal{R}}$ large enough, the bootstraps are easily improved.

Step 2: The no-shift region $\mathcal{N} = \{-\Delta_{\mathcal{R}} \leq s \leq S\}$

Since we are integrating only in a finite s-region, the no-shift region is easily dealt with using Grönwall. To do this, let $\mathbf{X}_\epsilon$ and $\mathbf{\Phi}_\epsilon$ denote the tuples:

$$\mathbf{X}_\epsilon = \left(r_\epsilon, \dot{r}_\epsilon, (\log\Omega^2)_\epsilon, \frac{d}{ds}(\log\Omega^2)_\epsilon, Q_\epsilon, \tilde{A}_\epsilon\right),\ \mathbf{\Phi}_\epsilon = (\phi_\epsilon, \dot{\phi}_\epsilon),$$

then in light of Proposition 4.3.2, the system (4.6.16)–(4.6.20) can be translated into

$$|\dot{\mathbf{X}}_\epsilon| \lesssim |\mathbf{X}_\epsilon| + \epsilon|\mathbf{\Phi}_\epsilon|,\ |\dot{\mathbf{\Phi}}_\epsilon| \lesssim |\mathbf{\Phi}_\epsilon| + \epsilon|\mathbf{X}_\epsilon|.$$

Hence a straightforward use of Grönwall in the bounded s-region $s \in [-\Delta_{\mathcal{R}}, S]$ yields

$$\sup_{s\in\mathcal{N}}|\mathbf{X}_\epsilon|(s) \lesssim |\mathbf{X}_\epsilon|(-\Delta_{\mathcal{R}}) + \epsilon\sup_{s\in\mathcal{N}}|\mathbf{\Phi}_\epsilon|(s) \lesssim |\mathbf{X}_\epsilon|(-\Delta_{\mathcal{R}}) + \epsilon|\mathbf{\Phi}_\epsilon|(-\Delta_{\mathcal{R}}) + \epsilon^2\sup_{s\in\mathcal{N}}|\mathbf{X}_\epsilon|(s)$$

So, for ϵ sufficiently small, we absorb the rightmost term into the left hand side, and repeat the argument for $\mathbf{\Phi}_\epsilon(s)$, to derive that

$$\sup_{s\in\mathcal{N}}|\mathbf{X}_\epsilon|(s) \lesssim |\mathbf{X}_\epsilon|(-\Delta_{\mathcal{R}}) + \epsilon|\mathbf{\Phi}_\epsilon|(-\Delta_{\mathcal{R}}),$$

$$\sup_{s\in\mathcal{N}}|\mathbf{\Phi}_\epsilon|(s) \lesssim |\mathbf{\Phi}_\epsilon|(-\Delta_{\mathcal{R}}) + \epsilon|\mathbf{X}_\epsilon|(-\Delta_{\mathcal{R}}).$$

Inserting our bounds from the red-shift region $\mathcal{R}$, there exists $D_{NE}(M, \mathbf{e}, \Lambda, m^2, q_0)$ such that, for all $s \in \mathcal{N}$,

$$|\mathbf{X}_\epsilon|(s) \leq D_{NE}\epsilon,\ |\mathbf{\Phi}_\epsilon|(s) \leq D_{NE}. \tag{4.6.25}$$

Step 3: Upper bounds in the blue shift regions $\mathcal{EB} \cup \mathcal{LB} = \{S \leq s \leq \Delta_{\mathcal{B}}\epsilon^{-1}\}$.

This step will be much simpler than the corresponding nonlinear estimates in Sections 4.3.2 and 4.3.3. We use the bootstrap assumptions:

$$|\dot{r}_\epsilon| \leq 10D_{NE}\epsilon, \tag{4.6.26}$$

$$|(\log\Omega^2)_\epsilon| \leq 10D_{NE}\epsilon s^3, \tag{4.6.27}$$

$$|Q_\epsilon| \leq 10D_{NE}\epsilon s^2. \tag{4.6.28}$$

Note that the first of these trivially implies $|r_\epsilon| \leq 10D_{NE}\epsilon s$. Then integration of (4.6.19) gives that $|\tilde{A}_\epsilon| \lesssim D_{NE}\epsilon$.

We next use these bootstraps to estimate ϕ_ϵ and $\dot{\phi}_\epsilon$. Note that the expression J_ϕ in (4.6.20) now obeys the estimate $|J_\phi| \lesssim D_{NE}\epsilon^2$. We now follow the proof of Proposition 4.3.4 and consider the quantity

$$H^{(\epsilon)} = r^4\dot{\phi}^2_\epsilon + r^4 q_0^2|\tilde{A}|^2\phi^2_\epsilon.$$

Completely analogously to before, one finds that

$$\dot{H}^{(\epsilon)} \lesssim \Omega^2 H^{(\epsilon)} + |J_\phi||\dot{\phi}| \lesssim \Omega^2 H^{(\epsilon)} + D_{NE}\epsilon^2\sqrt{H^{(\epsilon)}}.$$

So, as $\int_S^{\Delta_{\mathcal{B}}\epsilon^{-1}} \Omega^2(\tilde{s})\, d\tilde{s} \lesssim 1$, one can apply Grönwall to the quantity $\sqrt{H^{(\epsilon)}}$ to yield

$$\sqrt{H^{(\epsilon)}}(s) \lesssim \sqrt{H^{(\epsilon)}}(S) + D_{NE}\epsilon^2 s \lesssim |\phi_\epsilon|(S) + |\dot{\phi}_\epsilon|(S) + D_{NE}\epsilon^2 s \lesssim D_{NE}.$$

Since $H^{(\epsilon)} \sim \phi_\epsilon^2 + \dot{\phi}_\epsilon^2$, one gets the estimate (4.6.12).

The next step is to integrate (4.6.16). Note that, by Propositions 4.3.4 and 4.3.5, we have $-\dot{r} \lesssim \max\{\Omega^2, \epsilon^2\}$, $|\phi| \lesssim \epsilon$ and $\Omega^2 \lesssim e^{2K_- s}$, so

$$|\dot{r}_\epsilon| \leq D_{NE}\epsilon + CD_{NE}\int_S^{\Delta_{\mathcal{B}}\epsilon^{-1}} (e^{2K_- s'} s'^3\epsilon + \epsilon^3 s')\, ds' \leq 2D_{NE}\epsilon,$$

where C is a constant independent of ϵ and D_{NE}. For the second inequality above, note that evaluating the integral gives a quantity bounded by a multiple of $\epsilon(e^{2K_- S}S^3 + \Delta_{\mathcal{B}}^2)$, hence the inequality follows for S chosen sufficiently large and $\Delta_{\mathcal{B}}$ chosen sufficiently small. This improves (4.6.26) and in fact yields (4.6.9) after a further integration.

The estimates (4.6.10) and (4.6.11) then follow from integration of (4.6.17), (4.6.18) and (4.6.19), and this also improves the remaining two bootstrap assumptions.

Step 4: Precise bounds for the scalar field

We now move onto the estimate (4.6.13). Note that $\delta\phi_\epsilon = \phi_\epsilon - (\phi_{\mathcal{L}})_\epsilon$, but recall that since $\phi_{\mathcal{L}}$ is the exactly the solution to the linear charged wave equation in a Reissner-Nördstrom background having initial data $\lim_{s\to-\infty}\phi_{\mathcal{L}} = \epsilon$, it is clear that $(\phi_{\mathcal{L}})_\epsilon$ is exactly the solution to the linear charged wave equation in Reissner-Nördstrom with $\lim_{s\to-\infty}(\phi_{\mathcal{L}})_\epsilon(s) = 1$. Namely, we have

$$(\ddot{\phi}_{\mathcal{L}})_\epsilon = -\frac{2\dot{r}_{RN}(\dot{\phi}_{\mathcal{L}})_\epsilon}{r_{RN}} - q_0^2\tilde{A}_{RN}^2(\phi_{\mathcal{L}})_\epsilon - \frac{m^2\Omega_{RN}^2}{4}(\phi_{\mathcal{L}})_\epsilon.$$

Hence subtracting this equation from (4.6.20), and using both the estimates of Section 4.3 and earlier within the proof of this proposition, one finds

$$\delta\dot{\phi}_\epsilon = -\frac{2\dot{r}\delta\dot{\phi}_\epsilon}{r} - q_0^2\tilde{A}^2\delta\phi - \frac{m^2\Omega^2}{4}\delta\phi_\epsilon + J_\phi + \tilde{J}_\phi, \tag{4.6.29}$$

where J_ϕ is as in (4.6.20), $\tilde{J}_\phi$ arises from taking the differences of $r, \dot{r}, \tilde{A}, \Omega^2$ from their Reissner-Nordström quantities, and one gets the estimate

$$|J_\phi(s)| + |\tilde{J}_\phi(s)| \lesssim \begin{cases} \epsilon^2\Omega^2 & \text{for } s \in \mathcal{R}, \\ \epsilon^2 & \text{for } s \in \mathcal{N} \cup \mathcal{EB} \cup \mathcal{LB}. \end{cases}$$

We now proceed in the usual way using the quantity

$$\tilde{H}^{(\epsilon)} = r^4|\delta\dot{\phi}_\epsilon|^2 + r^4 q_0^2|\tilde{A}|^2|\delta\phi_\epsilon|^2.$$

Then by using the equation (4.6.29), one finds that for all $s \in \mathcal{R} \cup \mathcal{N} \cup \mathcal{EB} \cup \mathcal{LB}$, we get

$$\frac{d}{ds}\tilde{H}^{(\epsilon)} \lesssim \Omega^2 \tilde{H}^{(\epsilon)} + (|J_\phi| + |\tilde{J}_\phi|)\sqrt{\tilde{H}^{(\epsilon)}}.$$

Using now the fact that $\lim_{s\to-\infty} \tilde{H}^{(\epsilon)}(s) = 0$ and $\int_{-\infty}^{\Delta_{\mathcal{B}}\epsilon^{-1}} \Omega^2$ is uniformly bounded in ϵ, Grönwall applied to this differential inequality gives:

$$\sqrt{\tilde{H}^{(\epsilon)}} \lesssim \int_{-\infty}^{\Delta_{\mathcal{B}}\epsilon^{-1}} (|J_\phi(\tilde{s})| + |\tilde{J}_\phi(\tilde{s})|)\, d\tilde{s} \lesssim \epsilon^2 s. \tag{4.6.30}$$

This of course will yield the estimate (4.6.13). Note that by Corollary 4.1.4, one then has for $s \in \mathcal{LB}$,

$$\left|\phi_\epsilon - Be^{i\omega_{RN}s} - \overline{B}e^{-i\omega_{RN}s}\right| \lesssim \epsilon^2 s, \tag{4.6.31}$$

$$\left|\dot{\phi}_\epsilon - i\omega_{RN}Be^{i\omega_{RN}s} + i\omega_{RN}\overline{B}e^{-i\omega_{RN}s}\right| \lesssim \epsilon^2 s. \tag{4.6.32}$$

Step 5: The precise estimate for $\dot{r}_\epsilon$

We finally move to the estimate (4.6.14). We now use the differentiated version of the Raychaudhuri equation in the convenient form (2.4.23). We see that using (4.6.9), (4.6.10) and (4.6.11) we can find

$$\frac{d}{ds}(-\dot{r}_\epsilon) - \frac{d}{ds}\log\big(\Omega^2\big)\cdot(-\dot{r}_\epsilon) = 2r(\dot{\phi}\dot{\phi}_\epsilon + |\tilde{A}|^2 q_0^2 \phi\phi_\epsilon) + J_\Omega,$$

where the error J_Ω satisfies $|J_\Omega| \lesssim \epsilon^3 s$.

Now we use (4.6.31), (4.6.32), alongside the estimates (4.3.35), (4.3.37), (4.3.42), to find further that

$$\frac{d}{ds}(-\dot{r}_\epsilon) - 2K_-(-\dot{r}_\epsilon) = 8|B|^2\omega_{RN}^2 r_-\epsilon + J_\Omega + \tilde{J}_\Omega,$$

where $|\tilde{J}_\Omega| \lesssim \epsilon^3 s$. We now use a standard integrating factor to integrate between s_{lin} and $s \in \mathcal{LB}$, yielding

$$-\dot{r}_\epsilon(s) = e^{2K_-(s-s_{lin})}(-\dot{r}_\epsilon)(s_{lin}) + \int_{s_{lin}}^{s} e^{2K_-(s-s')}(8|B|^2\omega_{RN}^2 r_-\epsilon + J_\Omega(s') + \tilde{J}_\Omega(s'))\, ds'.$$

One may then simply compute the relevant integrals to find that

$$\left|-\dot{r}_\epsilon - \frac{8|B|^2\omega_{RN}^2 r_-\epsilon}{2K_-}\right| \lesssim \epsilon^{-1}\Omega^2 + \epsilon^3 s.$$

But for $s \geq s_O$, we know $\Omega^2 \lesssim \epsilon^{100}$, and (4.6.14) follows immediately. □

4.6.2 The measure of the set E_η

We now use Proposition 4.6.1, or more precisely Corollary 4.6.2, to control the measure of the set of values of ϵ such that condition (†) holds. Let $\epsilon_0(M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ be such that the results of Section 4.5 hold for $0 < |\epsilon| < \epsilon_0$. Then we have the following corollary:

Corollary 4.6.3. *Let $\eta > 0$ be a sufficiently small constant. We define E_η to be the set of ϵ such that the hairy black hole interior corresponding to $\phi = \epsilon$ on $\mathcal{H}$ obeys the condition (†) at $s = s_i$:*

$$E_\eta = \{\epsilon \in (0, \epsilon_0) : |\Psi_i| \geq \eta\}. \tag{4.6.33}$$

Then the set E_η is non-empty, has 0 as a limit point, and we have the following upper bound for the limiting density of values of ϵ violating (†): there exists some constant K such that

$$\limsup_{\tilde{\epsilon} \downarrow 0} \tilde{\epsilon}^{-1} |(0, \tilde{\epsilon}) \setminus E_\eta| \leq K \mathfrak{W} \eta. \tag{4.6.34}$$

Proof. We will estimate the measure of the set $F_{\eta,\tilde{\epsilon}} = (0, \tilde{\epsilon}) \setminus E_\eta = \{\epsilon \in (0, \tilde{\epsilon}) : |\Psi_i| < \eta\}$. We first use Corollary 4.5.4 to change variable from ϵ to $\tilde{\Theta} = \Theta(\epsilon)$. For ϵ_0 and η sufficiently small,

$$\begin{aligned} |F_{\eta,\tilde{\epsilon}}| &= \int_0^{\tilde{\epsilon}} \mathbb{1}_{\{|\Psi_i| < \eta\}} \, d\epsilon \\ &\leq \int_0^{\tilde{\epsilon}} \mathbb{1}_{\{|\sin(\Theta(\epsilon))| < \sqrt{\pi}\mathfrak{W}\eta\}} \, d\epsilon \\ &= \int_{\Theta(\tilde{\epsilon})}^{+\infty} \mathbb{1}_{\{|\sin\tilde{\Theta}| < \sqrt{\pi}\mathfrak{W}\eta\}} \left| \frac{d}{d\epsilon}\Theta(\epsilon)\Big|_{\epsilon=\Theta^{-1}(\tilde{\Theta})} \right|^{-1} d\tilde{\Theta}. \end{aligned}$$

We now apply Corollary 4.6.2 along with the previous estimate (4.6.2) for $\Theta(\epsilon)$. This will yield:

$$\frac{d}{d\epsilon}\Theta(\epsilon) = -4\sqrt{2}|B|\mathfrak{W}\,\Theta(\epsilon)^{3/2} + O(\Theta^{1/2}\log\Theta). \tag{4.6.35}$$

Combining with the above, we therefore see that

$$|F_{\eta,\tilde{\epsilon}}| \leq \int_{\Theta(\tilde{\epsilon})}^{+\infty} \mathbb{1}_{\{|\sin\tilde{\Theta}| < \sqrt{\pi}\mathfrak{W}\eta\}} \, \frac{1}{4|B|\mathfrak{W}} \, \tilde{\Theta}^{-3/2} \, d\tilde{\Theta}.$$

To evaluate this integral, note that if η is taken sufficiently small, then the set $\{|\sin\Theta| < \sqrt{\pi}\mathfrak{W}\eta\}$ is simply a union of intervals of width $2\sqrt{\pi}\mathfrak{W}\eta + O(\eta^2)$ and centered on the integer lattice $\pi\mathbb{Z}$. So the integral is akin to taking a discrete sum of the form $\sum_{n=\Theta(\tilde{\epsilon})}^{+\infty} \frac{\eta}{n^{3/2}}$, multiplied by appropriate weights. Keeping only the weights $|B|$ and $\mathfrak{W}$ which depend on the background parameters, we have

$$|F_{\eta,\tilde{\epsilon}}| \lesssim |B|^{-1}\,\Theta(\tilde{\epsilon})^{-1/2}\eta \lesssim \mathfrak{W}\eta\tilde{\epsilon}, \tag{4.6.36}$$

using (4.6.2) again in the last step. This yields (4.6.34), and the remainder of the corollary follows easily. □

Remark. Recalling Theorem 4.1.2, as well as studying the values of ϵ obeying (†), we also need to investigate the measure of the set of values of ϵ obeying the 'bounce' condition $\eta \leq |\Psi_i| \leq 1 - \sigma$ or the 'non-bounce' condition $|\Psi_i| \geq 1 + \sigma$. This can be done in the same way as the proof of Corollary 4.6.3.

4.7 Kasner regimes and bounces

Now, assuming the condition (†), i.e that ϵ lies in E_η as defined in Theorem 4.1.1, we will complete the proof of Theorem 4.1.1. In particular, we claim there exists some $\epsilon_0(\eta) > 0$ depending on η as well as the usual parameters $M, \mathbf{e}, \Lambda, m^2, q_0$, such that if $0 < \epsilon \leq \epsilon_0 = \epsilon_0(\eta)$ **and** (†) holds, then our corresponding hairy black hole interior contains a crushing spacelike singularity, with more quantitative Kasner-like asymptotics to follow.

Firstly, we briefly describe the expected dynamics between the end of the region $\mathcal{PK}$ and the eventual spacelike singularity at $s = s_\infty$. It turns out the intermediate dynamics will be highly sensitive to the value of $\Psi(s_i)$, often denoted as Ψ_i in the sequel, where $\Psi(s)$ is given by (4.6.1).

Following the discussion of the introduction, particularly Section 4.2.4, if $|\Psi_i| > 1$, then the region $\mathcal{PK}' \subset \mathcal{PK}$ (see Corollary 4.5.4) should already lie in a regime associated with positive Kasner exponents. For ϵ sufficiently small, we will show that many of the estimates of Proposition 4.5.1 will persist all the way to the spacelike singularity $\{s = s_\infty\}$, meaning a single (stable) Kasner-like regime.

On the other hand, if $\eta \leq |\Psi_i| < 1$, then initially, the spacetime lies in a regime associated with one negative and two positive Kasner exponents – known to be unstable in the cosmological setting. In this case, we shall observe the aforementioned *Kasner bounce* phenomenon. In simplified terms, between $s = s_i$ and $s = s_\infty$, the quantity Ψ will *invert* from its initial value $\Psi_i = \Psi(s_i)$ to a final value $\Psi_f \approx \Psi_i^{-1}$ satisfying $|\Psi_f| > 1$. The spacetime in turn evolves into a regime associated with positive Kasner exponents, which in turn persists up to the singularity.

The cause and nature of such Kasner bounces, as well as why this allows for spacelike singularity formation, will be the focus of this section. Denote by $\mathcal{K}$ the region $\{s \geq s_i\} = \{s : r(s) \leq e^{-\delta_0 \epsilon^{-2}} r_-\}$. The section will be organized as follows:

- In Section 4.7.1, we give further background into why we distinguish between $|\Psi| < 1$ and $|\Psi| > 1$, and hence the need to study Kasner bounces. We will also introduce new renormalized quantities and the equations that they obey.

- In Section 4.7.2, we state the main Proposition 4.7.1 regarding the quantity Ψ in the region $\mathcal{K}$. We then state the main bootstrap assumptions used in this proof, and prove some preliminary results used in the proof of Proposition 4.7.1.

- In Section 4.7.3, we provide the proof of Proposition 4.7.1. This will entail a detailed estimate for all the error terms involved in deriving an ODE of the form (4.2.11). Once completed, this proof shows that the spacetime will exist up to a spacelike singularity at some $s = s_\infty$ with $r(s_\infty) = 0$.

- In Section 4.7.4, we apply our results regarding Ψ in Proposition 4.7.1 to find quantitative

estimates on geometric quantities such as Ω^2. This will be crucial in showing quantitative closeness to Kasner spacetimes in Section 4.8.

4.7.1 Background on Kasner bounces

In this section, we briefly explore the important role played by the quantity Ψ, and the differing evolutionary dynamics that occur when $|\Psi_i| = |\Psi(s_i)|$ is greater than or less than 1. The two key equations are the evolution equation (2.4.25) for $-r\dot{r}$ and the Raychaudhuri equation (2.4.21). It will be useful to rewrite (2.4.21) as an evolution equation for $\log\big(\Omega^2/(-\dot{r})\big)$, and express the first term on the right hand side in terms of the quantity Ψ:

$$\frac{d}{ds}\log\left(\frac{\Omega^2}{-\dot{r}}\right) = \frac{\dot{r}}{r}\Psi^2 + \frac{r^2}{r\dot{r}}|\tilde{A}|^2 q_0^2|\phi|^2. \tag{4.7.1}$$

Supposing for now that the rightmost term is integrable and small, one sees that the value of Ψ^2 determines the leading order behavior of Ω^2 in r. Letting $\alpha^2 = \inf \Psi^2$ and $\beta^2 = \sup \Psi^2$ in our region of interest, integrating (4.7.1) gives:

$$C_{lower} r^{\beta^2} \leq \frac{\Omega^2}{-\dot{r}} \leq C_{upper} r^{\alpha^2}. \tag{4.7.2}$$

For this heuristic discussion, it is only important to note that C_{lower} and C_{upper} are independent or r, indeed the important feature will be the powers of r.

In light of (4.7.2), we turn to the evolution equation for $-r\dot{r}(s)$ written above. Inside a region such as $\mathcal{K}$ where r is small, one expects that the dominant term on the right hand side is the term $\frac{Q^2\Omega^2}{4r^2}$, and (4.7.2) will yield upper and lower bounds of the form:

$$-C_{lower} r^{\beta^2-2}\dot{r} \lesssim -\frac{d}{ds}(-r\dot{r}) \lesssim -C_{upper} r^{\alpha^2-2}\dot{r}. \tag{4.7.3}$$

Integrating this expression, and ignoring the degenerate case where α or β are equal to 1, we have

$$|r^{\beta^2-1}(s) - r^{\beta^2-1}(s_i)| \lesssim r\dot{r}(s) - r\dot{r}(s_i) \lesssim |r^{\alpha^2-1}(s) - r^{\alpha^2-1}(s_i)|. \tag{4.7.4}$$

We can therefore make two observations:

- If $\beta < 1$, then $|r(s)^{\beta^2-1} - r(s_i)^{\beta^2-1}|$ is unbounded as $r(s) \to 0$, which suggests that $-r\dot{r}$ is unbounded. In fact, since this suggests $r\dot{r}(s)$ will at some point become positive, we have somehow exited the trapped region. So something must have gone wrong, either in our a priori assumptions or in assuming that $\beta < 1$. In our context, we shall show the latter issue arises; there must be a 'bounce' which forces $\beta \geq 1$.

- If $\alpha > 1$, then $|r(s)^{\alpha^2-1} - r(s_i)^{\alpha^2-1}|$ is bounded by a multiple of $r(s_i)^{\alpha^2-1}$, which in our case has order of magnitude $O(e^{-(\alpha-1)\delta_0\epsilon^{-2}}) \ll -r\dot{r}(s_i)$. So $-r\dot{r}$ changes little from its value at

$s = s_i$. The insight is that if $\Psi(s_i) > 1$ initially, then there is a hope of closing estimates in a bootstrap argument, such that $-r\dot{r}$ only changes by this extremely small quantity, and thus allows for formation of a spacelike singularity.

4.7.2 The bootstraps, preliminary estimates and statement of Proposition 4.7.1

In this section, we state and initiate the proof of Proposition 4.7.1. As well as asserting the eventual formation of an $r = 0$ spacelike singularity, the key content of this proposition is that we can control the quantity $\Psi(s)$ via a certain nonlinear ODE – which, as we show in Section 4.7.4, in turn allows us to control the other geometric quantities including $\Omega^2(s)$.

Proposition 4.7.1. *Fix some $\eta \in \mathbb{R}$ with $0 < \eta < \min\{\frac{1}{2}\mathfrak{W}, \frac{1}{4}\}$. Then there exists some $\epsilon_0(\eta) > 0$ depending on η as well as the usual parameters $M, \mathbf{e}, \Lambda, m^2, q_0$, such that if both*

$$0 < \epsilon \leq \epsilon_0(\eta) \quad \textbf{\textit{and}} \quad |\Psi_i| \coloneqq |\Psi(s_i)| \geq \eta, \tag{$*$}$$

then the solution of the system (2.4.21)–(2.4.31) exists in the interval $s \in (-\infty, s_\infty)$, with $s_i < s_\infty < +\infty$ and $r(s) \to 0$ as $s \to s_\infty$. In fact, for $s \in \mathcal{K} = \{s \geq s_i : r(s) > 0\}$, one obtains the following bounds on $-r\dot{r}(s)$.

$$-r\dot{r}(s) \geq 2|B|^2\mathfrak{W}^2 r_-^2 \omega_{RN}\eta^2\epsilon^2, \quad -r\dot{r}(s) \leq 8|B|^2\mathfrak{W}^2 r_-^2\omega_{RN}\epsilon^2. \tag{4.7.5}$$

Regarding the quantity Ψ defined in (4.6.1), there exists some constant $D_K(\eta, M, \mathbf{e}, \Lambda, m^2, q_0) > 0$, a real number $\alpha = \alpha(\epsilon)$ and a function $\mathcal{F}(s)$ satisfying:

$$|\alpha - \Psi_i| \leq D_K \exp\big(-\delta_0\epsilon^{-2}\big), \quad |\mathcal{F}(s)| \leq D_K \exp\big(-\delta_0\epsilon^{-2}\big) r(s) = D_K r_- e^{-\delta_0\epsilon^{-2}-R},$$

where we define $R = \log(r_-/r(s))$, so that $\Psi = \Psi(R)$ satisfies the following ODE in the region $\mathcal{K}$:

$$\frac{d\Psi}{dR} = -\Psi\left(\Psi - \alpha\right)\left(\Psi - \frac{1}{\alpha}\right) + \mathcal{F}. \tag{4.7.6}$$

Finally, one has the following upper and lower bounds for $|\Psi(s)|$ for $s \in \mathcal{K}$:

$$\min\{|\Psi_i|, |\Psi_i^{-1}|\} - D_K e^{-\delta_0\epsilon^{-2}} \leq |\Psi(s)| \leq \max\{|\Psi_i|, |\Psi_i^{-1}|\} + D_K e^{-\delta_0\epsilon^{-2}}. \tag{4.7.7}$$

Remark. We use the same notation $D_K = D_K(\eta, M, \mathbf{e}, \Lambda, m^2, q_0) > 0$ throughout the various lemmas and propositions of Section 4.7. Note that it is possible to track the dependence of D_K, and any constants implied by the notation $\lesssim$, on η, and thereby strengthen the result from having a fixed η in $(*)$ to allowing $\eta = \eta(\epsilon)$ to decay as $\epsilon \to 0$, for instance $\eta = \epsilon^{0.01}$. However, for the purpose of simpler exposition, we do not pursue that here.

The proof of Proposition 4.7.1 will be broken into several lemmas in Sections 4.7.2 and 4.7.3. We first list the three main bootstrap assumptions of the region $\mathcal{K}$:

$$|\Psi| \leq 4\eta^{-1}, \tag{K1}$$

$$-r\dot{r}(s) \geq |B|^2\mathfrak{W}^2 r_-^2 \omega_{RN}\eta^2\epsilon^2, \tag{K2}$$

$$\frac{|Q_\infty|}{2} \leq |Q(s)| \leq 2|Q_\infty|. \tag{K3}$$

Here, $Q_\infty = Q_\infty(M, \mathbf{e}, \Lambda) \neq 0$ is defined in (4.4.13); by Lemma 4.4.7, Q_∞ lies strictly between $\mathbf{e}/2$ and $\mathbf{e}$. In light of Proposition 4.5.1 and Corollary 4.5.4, these bootstrap assumptions hold in a neighborhood of $s = s_i$.

In the remainder of Section 4.7.2, we state and prove two preliminary lemmas, the first of which provides estimates for the Maxwell quantities $\tilde{A}(s)$ and $Q(s)$, as well as the scalar field $\phi(s)$. The second lemma then produces a crucial lower bound for $|\Psi|$ as well as a useful preliminary upper bound on $\Omega^2(s)$.

Lemma 4.7.2. *Assuming the bootstraps (K1), (K2), (K3), we have the following preliminary estimates on ϕ as well as the Maxwell quantities Q and $\tilde{A}$:*

$$|\phi(s)| \leq 2\eta^{-1}\log\left(\frac{8|B|^2\mathfrak{W}^2 r_-^2\epsilon^2}{r^2(s)}\right), \tag{4.7.8}$$

$$|Q(s) - Q(s_i)| \leq D_K \exp\bigl(-\delta_0\epsilon^{-2}\bigr), \tag{4.7.9}$$

$$||q_0\tilde{A}|(s) - \omega_{RN}| \leq D_K\epsilon^2. \tag{4.7.10}$$

In particular, using (4.7.9), we immediately improve the bootstrap assumption (K3).

Lemma 4.7.3. *Assuming the bootstraps (K1), (K2), (K3), we have the following upper bound on $-r\dot{r}$ and estimate on $r^2\dot{\phi}$ which we use to get a corresponding lower bound on $|\Psi|$:*

$$-r\dot{r}(s) \leq 4|B|^2\mathfrak{W}^2\omega_{RN}r_-^2\epsilon^2 + D_K\epsilon^4\log\bigl(\epsilon^{-1}\bigr), \tag{4.7.11}$$

$$\left|\frac{r^2\dot{\phi}(s)}{r^2\dot{\phi}(s_i)} - 1\right| \leq D_K\exp\bigl(-\delta_0\epsilon^{-2}\bigr), \tag{4.7.12}$$

$$|\Psi| \geq \eta - D_K\exp\bigl(-\delta_0\epsilon^{-2}\bigr). \tag{4.7.13}$$

In particular, Ψ will never vanish, and thus never change sign, in the region $\mathcal{K}$. Moreover, we have the following initial upper bound for $\Omega^2(s)$:

$$\frac{\Omega^2}{-\dot{r}}(s) \leq \frac{\Omega^2}{-\dot{r}}(s_i) \leq \exp\bigl(-50\delta_0\epsilon^{-2}\bigr). \tag{4.7.14}$$

Proof of Lemma 4.7.2. To prove the preliminary bound on ϕ, recall that we may rewrite Ψ as $-r\frac{d\phi}{dr}$. So we use (K1) as follows:

$$|\phi(s) - \phi(s_i)| \leq \int_{r(s)}^{r(s_i)} \frac{|\Psi(r)|}{r}\, dr \leq 4\eta^{-1} \log\left(\frac{r(s_i)}{r(s)}\right) \leq 2\eta^{-1} \log\left(\frac{r^2(s_i)}{r^2(s)}\right). \tag{4.7.15}$$

But, from (4.5.24) in Corollary 4.5.4,

$$\begin{aligned} |\phi(s_i)| \leq \frac{2}{\pi} C_{YK}(\epsilon)\mathfrak{W}^{-1} \log\left(\frac{r_-^2\epsilon^2}{\xi_K r^2(s_i)}\right) + |c_1 C_{JK}(\epsilon)| + |c_2 C_{YK}(\epsilon)| \\ \leq 2\eta^{-1} \log\left(\frac{8|B|^2\mathfrak{W}^2 r_-^2\epsilon^2}{r^2(s_i)}\right). \end{aligned} \tag{4.7.16}$$

In the second step, we used $|C_{YK}(\epsilon)|, |C_{JK}(\epsilon)| \leq \sqrt{\pi}\mathfrak{W}$ from Propositions 4.4.5 and 4.5.1, as well as $\eta < \frac{1}{2}\mathfrak{W}$ and the definition (4.5.7). In particular, $r(s_i) = e^{-\delta_0\epsilon^{-2}}$ is small enough that the contribution of $|c_1 C_{JK}(\epsilon)| + |c_2 C_{YK}(\epsilon)|$ is negligible. Combining the two inequalities (4.7.15) and (4.7.16) will clearly yield (4.7.8).

For the gauge field $\tilde{A}(s)$, we use the following trick: For $s \in \mathcal{K}$, we have $r(s) \leq \exp\bigl(-\delta_0\epsilon^{-2}\bigr) r_- \leq \min\left\{\frac{|Q_\infty|}{8}, \frac{1}{2\sqrt{|\Lambda|}}\right\}$, and $|m^2 r^2 |\phi|^2| < 1$ via (4.7.8), so

$$\frac{\Omega^2}{4} - \frac{\Omega^2 Q^2}{4r^2} - \frac{\Omega^2 r^2}{4}(m^2|\phi|^2 + \Lambda) \leq -\frac{\Omega^2 Q^2}{8r^2}.$$

Thus, (2.4.25) tells us that $-r\dot{r}(s)$ is decreasing in $\mathcal{K}$, and one further has a bound on the integral

$$\int_{s_i}^{s} \frac{Q^2\Omega^2}{4r^2}(\tilde{s})\, d\tilde{s} \leq -2r\dot{r}(s_i). \tag{4.7.17}$$

Hence, using the lower bound of (K3) once more, the equation (2.4.29) yields

$$|q_0\tilde{A}(s) - q_0\tilde{A}(s_i)| \leq \int_{s_i}^{s} \frac{|Q|\Omega^2}{4r^2}(\tilde{s})\, d\tilde{s} \leq \frac{2}{|Q_\infty|}\int_{s_i}^{s} \frac{Q^2\Omega^2}{4r^2}(\tilde{s})\, d\tilde{s} \leq \frac{-4r\dot{r}(s_i)}{|Q_\infty|}. \tag{4.7.18}$$

Thus, (4.5.3) and (4.5.4) of Proposition 4.5.1 allows one to deduce (4.7.10).

To close the estimate on Q, we simply integrate (2.4.28):

$$|Q(s) - Q(s_i)| \leq \int_{s_i}^{s} q_0^2 |\tilde{A}| r^2 |\phi|^2(\tilde{s})\, d\tilde{s} \lesssim \int_{s_i}^{s} r^2(\tilde{s}) \log^2\left(\frac{8|B|^2\mathfrak{W}^2 r_-^2\epsilon^2}{r^2(\tilde{s})}\right) d\tilde{s}. \tag{4.7.19}$$

But in the region $\mathcal{K}$, where $r(s) \leq e^{-\delta_0\epsilon^{-2}} r_-$ and the interval of integration has length $|s - s_i| = O(\epsilon^{-2} e^{-2\delta_0\epsilon^{-2}})$ by (K2), it is straightforward to determine (4.7.9) and improve the bootstrap (K3). □

Proof of Lemma 4.7.3. The first estimate (4.7.11) is immediate by monotonicity. Indeed, as argued in the proof of Lemma 4.7.2, the quantity $-r\dot{r}(s)$ is decreasing in s. So the estimate (4.5.3) evaluated at $s = s_i$ yields (4.7.11).

For the lower bound on $|\Psi|$, we first produce the estimate (4.7.12) regarding $r^2\dot{\phi}$. Using the monotonicity of $-\Omega^{-2}\dot{r}(s)$ from the Raychaudhuri equation (2.4.21), and Proposition 4.5.1, we first find the preliminary estimate

$$\frac{\Omega^2}{-\dot{r}}(s) \leq \frac{\Omega^2}{-\dot{r}}(s_i) \leq \frac{\Omega^2 r(s_i)}{-r\dot{r}(s_i)} \lesssim \exp\big(-50\delta_0\epsilon^{-2}\big)\cdot\epsilon^{-2}\exp\big(-\delta_0\epsilon^{-2}\big) \lesssim \exp\big(-50\delta_0\epsilon^{-2}\big). \tag{4.7.20}$$

This is (4.7.14), and we use this together with (4.7.11) and the conclusions of Lemma 4.7.2 to yield from (2.4.31)

$$\left|\frac{d}{ds}(r^2\dot{\phi})(s)\right| \lesssim \exp\big(-\delta_0\epsilon^{-2}\big)\cdot r(s)\log\left(\frac{8|B|^2\mathfrak{W}^2 r_-^2\epsilon^2}{r^2(s)}\right) \lesssim \epsilon^2\exp\big(-\delta_0\epsilon^{-2}\big). \tag{4.7.21}$$

Integrating this up, and using once again that the interval of integration has length bounded by $O(\epsilon^{-2}e^{-2\delta_0\epsilon^{-2}})$, we therefore find that

$$|r^2\dot{\phi}(s) - r^2\dot{\phi}(s_i)| \lesssim \exp\big(-3\delta_0\epsilon^{-2}\big). \tag{4.7.22}$$

Finally, since $|\Psi(s_i)| \geq \eta$ and $-r\dot{r}(s_i)$ is bounded below using (4.4.18), we have $|r^2\dot{\phi}(s_i)| = -r\dot{r}(s_i)\cdot|\Psi(s_i)| \gtrsim \epsilon^2$. Combining this with (4.7.22) we obtain (4.7.12).

Therefore, for ϵ sufficiently small, we see that

$$|\Psi(s)| = \frac{-r\dot{r}(s_i)}{-r\dot{r}(s)}\cdot\frac{r^2\dot{\phi}(s)}{r^2\dot{\phi}(s_i)}\cdot|\Psi(s_i)| \geq \eta - D_K\exp\big(-\delta_0\epsilon^{-2}\big). \tag{4.7.23}$$

Here, we used (4.7.12) and that $-r\dot{r}(s)$ is decreasing for $s\in\mathcal{K}$, as well as the original assumption $(*)$. □

4.7.3 The dynamical system for Ψ and the proof of Proposition 4.7.1

In this section, we complete the proof of Proposition 4.7.1. The main step will be to find the ODE (4.7.6). For this purpose, we start with a lemma concerning the first and second derivatives of Ψ with respect to the timelike variable r.

Lemma 4.7.4. *Assume that the bootstraps (K1), (K2), (K3) hold. If $\mathcal{E}$ is defined such that*

$$\frac{d\Psi}{dr} = \Psi\frac{1}{-r\dot{r}}\frac{\Omega^2}{-4\dot{r}}\left(1-\frac{Q^2}{r^2}-m^2r^2|\phi|^2-r^2\Lambda\right)+\mathcal{E}, \tag{4.7.24}$$

then we have the following estimates in the region $s\in\mathcal{K}$:

$$|\mathcal{E}| \leq D_K\exp\big(-2\delta_0\epsilon^{-2}\big)\cdot r, \qquad \left|\frac{d\mathcal{E}}{dr}\right| \leq D_K\exp\big(-2\delta_0\epsilon^{-2}\big)\cdot r^{-1}. \tag{4.7.25}$$

For the second derivative, if the error term $\mathcal{F}_1$ is defined such that

$$\frac{d^2\Psi}{dr^2} - 2\Psi^{-1}\left(\frac{d\Psi}{dr}\right)^2 - \frac{\Psi^2-2}{r}\frac{d\Psi}{dr} = \mathcal{F}_1, \tag{4.7.26}$$

then for $s\in\mathcal{K}$, the expression $\mathcal{F}_1$ satisfies the following bound:

$$\mathcal{F}_1 \leq D_K\exp\big(-2\delta_0\epsilon^{-2}\big)r^{-1}. \tag{4.7.27}$$

Proof of Lemma 4.7.4. Assuming the bootstraps, the conclusions of Lemmas 4.7.2 and 4.7.3 will hold. In light of (2.4.25) and (2.4.31), differentiating Ψ in the variable r one yields:

$$\frac{d\Psi}{dr} = \frac{1}{\dot{r}}\frac{d\Psi}{ds} = \Psi \frac{1}{-r\dot{r}} \frac{\Omega^2}{-4\dot{r}} \left(1 - \frac{Q^2}{r^2} - m^2 r^2 |\phi|^2 - r^2\Lambda\right) + \mathcal{E}, \tag{4.7.28}$$

where the error $\mathcal{E}$ is given by

$$\mathcal{E} \coloneqq \frac{1}{-r\dot{r}} \frac{d}{dr}(r^2\dot{\phi}) = \frac{1}{-r\dot{r}} \left(\frac{q_0^2 |\tilde{A}|^2 r^3 \phi}{-r\dot{r}} + \frac{m^2\Omega^2 r^2 \phi}{-4\dot{r}} \right). \tag{4.7.29}$$

We seek the estimates (4.7.25). Using Lemmas 4.7.2 and 4.7.3 together with (K2), it is straightforward to get

$$|\mathcal{E}| \lesssim \epsilon^{-4} r^3 \log\left(\frac{\epsilon^2}{\xi^* r^2}\right) + \exp\bigl(-2\delta_0 \epsilon^{-2}\bigr)\epsilon^{-2} r^2 \log\left(\frac{\epsilon^2}{\xi^* r^2}\right) \lesssim \exp\bigl(-2\delta_0\epsilon^{-2}\bigr) r. \tag{4.7.30}$$

For the derivative estimate in (4.7.25), we simply differentiate the expression (4.7.29) term by term. This is rather cumbersome, and we simplify the exposition by merely considering the new multiplicative factors that arise when differentiating the various terms, noting that it is most crucial to keep track of the additional powers of r^{-1} that arise:

(i) Differentiating $(-r\dot{r})^{-1}$ in r will yield an extra multiplicative factor of

$$\frac{\Omega^2}{-4\dot{r}} \frac{1}{-r\dot{r}} \left(\frac{Q^2}{r^2} + m^2 r^2 |\phi|^2 + r^2\Lambda - 1 \right).$$

Since $Q \neq 0$, this will contribute a multiplicative factor of r^{-2}. Note that though one could fear that additional powers of ϵ^{-1} will appear (e.g. in the $-r\dot{r}$ appearing in the denominator), these will be negated by the smallness of $\frac{\Omega^2}{-\dot{r}}$, see (4.7.14).

(ii) Differentiating $\tilde{A}$ in r will introduce a similar factor of

$$\frac{Q\Omega^2}{-4r^2\dot{r}}$$

which we treat in exactly the same way as (i).

(iii) Differentiating ϕ in r simply yields

$$\frac{d\phi}{dr} = -\Psi r^{-1},$$

so that in light of bootstrap (K1) providing an upper bound on $|\Psi|$, differentiating this term contributes only one power of r^{-1}.

(iv) Differentiating the $\frac{\Omega^2}{4\dot{r}}$ in the second term of (4.7.29) introduces, by the Raychaudhuri equation (4.7.1), an extra multiplicative factor of

$$\frac{r}{\dot{r}^2}(|\dot{\phi}|^2 + q_0^2|\tilde{A}|^2\phi^2) = \frac{\Psi^2}{r} + \frac{r^3 q_0^2 |\tilde{A}|^2 \phi^2}{(-r\dot{r})^2} \leq \frac{\Psi^2}{r} + 1,$$

where we used the bootstrap (K2) and Lemma 4.7.2. So differentiating this term contributes at worst one power of r^{-1}.

(v) Finally, differentiating any powers of r that arise in (4.7.29) will trivially only lose one power of r.

From (i)–(v) we get the estimate (4.7.25).

Due to (4.7.25), (4.7.8), (4.7.14) and the bootstrap assumptions, (4.7.24) yields the upper bound

$$\left|\frac{d\Psi}{dr}\right| \leq D_K \exp\bigl(-50\delta_0\epsilon^{-2}\bigr)r^{-2} + D_K \exp\bigl(-2\delta_0\epsilon^{-2}\bigr)r \leq D_K \exp\bigl(-5\delta_0\epsilon^{-2}\bigr)r^{-2}. \tag{4.7.31}$$

We now treat the second derivative. We differentiate the expression (4.7.24) again in r, and find

$$\begin{aligned}\frac{d^2\Psi}{dr^2} = \Psi^{-1}\left(2\frac{d\Psi}{dr} - \mathcal{E}\right)\left(\frac{d\Psi}{dr} - \mathcal{E}\right) + \left(\frac{d\Psi}{dr} - \mathcal{E}\right)\left(\frac{\Psi^2}{r} + \frac{r}{\dot{r}^2}q_0^2|\tilde{A}|^2\phi^2\right) \\ + \left(\frac{d\Psi}{dr} - \mathcal{E}\right)\frac{d}{dr}\log\left|1 - \frac{Q^2}{r^2} - m^2r^2\phi^2 - r^2\Lambda\right| + \frac{d\mathcal{E}}{dr}.\end{aligned} \tag{4.7.32}$$

To explain the derivation of the equation (4.7.32), we take (4.7.24), subtract $\mathcal{E}$ from both sides, take a logarithm, and then finally differentiate, to write

$$\frac{d}{dr}\log\left|\frac{d\Psi}{dr} - \mathcal{E}\right| = \frac{d}{dr}\left(\log|\Psi| - \log(-r\dot{r}) + \log\left(\frac{\Omega^2}{-4\dot{r}}\right) + \log\left|1 - \frac{Q^2}{r^2} - m^2r^2\phi^2 - r^2\Lambda\right|\right),$$

from which (4.7.32) follows after using the usual evolution equations (2.4.25) and (4.7.1).

We now wish to recover the ODE (4.7.26) along with (4.7.27) i.e. we just need to show that all error terms present within (4.7.32) are bounded by $\exp\bigl(-\delta_0\epsilon^{-2}\bigr)r^{-1}$. This is mostly straightforward by (4.7.25) and (4.7.31), with the most complicated term being the one involving the log.

Firstly, using (4.7.31) and (4.7.25) as well as the lower bound (4.7.13) for $|\Psi|$, we see that

$$\left|\Psi^{-1}\left(2\frac{d\Psi}{dr} - \mathcal{E}\right)\left(\frac{d\Psi}{dr} - \mathcal{E}\right) - 2\Psi^{-1}\left(\frac{d\Psi}{dr}\right)^2\right| \lesssim \exp\bigl(-2\delta_0\epsilon^{-2}\bigr)r^{-1}. \tag{4.7.33}$$

Next, note that by (4.7.11), (4.7.8) and (4.7.10), the expression involving ϕ^2 can be bounded by

$$\frac{r}{\dot{r}^2}q_0^2|\tilde{A}|^2\phi^2 \lesssim \epsilon^{-4}r^3\log\bigl(r^{-1}\bigr) \leq \exp\bigl(-\delta_0\epsilon^{-2}\bigr)r^2 \leq \exp\bigl(-2\delta_0\epsilon^{-2}\bigr)r,$$

from which we may also deduce

$$\left|\left(\frac{d\Psi}{dr} - \mathcal{E}\right)\left(\frac{\Psi^2}{r} + \frac{r}{\dot{r}^2}q_0^2|\tilde{A}|^2\phi^2\right) - \frac{\Psi^2}{r}\frac{d\Psi}{dr}\right| \lesssim \exp\bigl(-2\delta_0\epsilon^{-2}\bigr)r^{-1}. \tag{4.7.34}$$

We now move onto the more tedious term involving $\frac{d}{dr}\log(\cdots)$. Computing the derivative using the system of equations (2.4.21)–(2.4.31) one eventually finds

$$\frac{d}{dr}\log\left|1 - \frac{Q^2}{r^2} - m^2r^2\phi^2 - r^2\Lambda\right| = -\frac{2}{r}\frac{1-f}{1-g},$$

where the expressions f and g are given by

$$f = \frac{r^3}{Q^2}\left(-\frac{Q\tilde{A}q_0^2\phi^2}{-\dot{r}} + m^2 r\phi^2 - m^2 r\phi\Psi + r\Lambda\right),$$

$$g = \frac{r^2}{Q^2}\cdot(1 - m^2r^2\phi^2 - r^2\Lambda).$$

Using the bootstraps along with Lemmas 4.7.2 and 4.7.3, it is straightforward to deduce that $|f| + |g| \lesssim r^2$. The conclusion of this computation is therefore that

$$\left|\frac{d}{dr}\log\left|1 - \frac{Q^2}{r^2} - m^2r^2\phi^2\right| + \frac{2}{r}\right| \lesssim r, \tag{4.7.35}$$

which along with (4.7.31) and (4.7.25) finally yields the required estimate

$$\left|\left(\frac{d\Psi}{dr} - \mathcal{E}\right)\frac{d}{dr}\log\left|1 - \frac{Q^2}{r^2} - m^2r^2\phi^2\right| + \frac{2}{r}\frac{d\Psi}{dr}\right| \lesssim \exp\bigl(-2\delta_0\epsilon^{-2}\bigr)r^{-1}. \tag{4.7.36}$$

The desired equation (4.7.26) is then found by combining the identity (4.7.32) with the estimates (4.7.33), (4.7.34), (4.7.36) and (4.7.25) to get the required error term $\mathcal{F}_1$ satisfying (4.7.27). □

We have now finished all the preparation for the proof of Proposition 4.7.1. Before turning to the actual proof below, we first give a brief sketch for the benefit of the reader. After changing variables to $R = \log(r_-/r(s))$, we may integrate up the second-order ODE (4.7.26) to derive a first-order ODE of a similar form to (4.7.6). More precisely, one finds an equation of the form

$$-r\frac{d\Psi}{dr} = \frac{d\Psi}{dR} = -\Psi(\Psi^2 - K\Psi + 1) + \text{error}, \tag{4.7.37}$$

where the expression K appearing here is a constant of integration. By evaluating (4.7.37) at $s = s_i$ we find that $K \approx \Psi_i + \Psi_i^{-1}$.

We treat (4.7.37) as a one-dimensional dynamical system for the unknown $\Psi = \Psi(R)$, and consider what happens as $r \to 0$, i.e. $R \to +\infty$. Assuming the error to be negligible, the dynamical system (4.7.37) will prohibit Ψ from growing too large – in particular allowing us to improve the bootstrap (K1). This, in turn, will allow us to improve the bootstrap (K2) due to the definition of Ψ and the upper bound on $r^2\dot{\phi}$ from (4.7.12). Recall also that bootstrap (K3) has already been improved in Lemma 4.7.2.

Having improved all the bootstraps, the lower bound (K2) allows us to continue the solution all the way to some $s = s_\infty$ with $r(s_\infty) = 0$. To obtain (4.7.6), we again integrate up the second-order ODE (4.7.26), but by determining the constant of integration K teleologically at $R = +\infty$ (i.e. $r \to 0$), we get the precise bound for the error term $\mathcal{F}$ as in Proposition 4.7.1.

Finally, some soft arguments using known upper and lower bounds for $|\Psi|$ will allow us to determine that $K \geq 2$, so that, writing $K = \alpha + \alpha^{-1}$, we get the required ODE (4.7.6). The remaining assertions are then straightforward. We now make this argument precise.

Proof of Proposition 4.7.1. Performing the change of variables $R = \log(r_-/r(s)) = \log r_- - \log r(s)$ on (4.7.26), one finds the ODE

$$\frac{d^2\Psi}{dR^2} - 2\Psi^{-1}\left(\frac{d\Psi}{dR}\right)^2 + (\Psi^2 - 1)\frac{d\Psi}{dR} = r^2\mathcal{F}_1. \tag{4.7.38}$$

Multiplying by the integrating factor Ψ^{-2}, we write the left hand side as a total derivative:

$$\frac{d}{dR}\left(\Psi^{-2}\frac{d\Psi}{dR} + \Psi + \frac{1}{\Psi}\right) = \Psi^{-2}r^2\mathcal{F}_1. \tag{4.7.39}$$

Due to (4.7.13) and (4.7.27), the right hand side of (4.7.39) can be bounded by $4\,D_K\eta^{-2}\exp\big(-2\delta_0\epsilon^{-2}\big)r \leq \exp\big(-\delta_0\epsilon^{-2}\big)e^{-R}$. So this error is integrable as $R \to +\infty$, and we proceed by integrating up (4.7.39).

For now, we can only integrate (4.7.39) in a finite bootstrap region; for $R_0 > R_i := \log(r_-/r(s_i)) = \delta_0\epsilon^{-2}$ lying in our bootstrap region, there exists a constant of integration K_{R_0} and an error term $\mathcal{F}_{R_0}(R)$, such that, for $R \in [R_i, R_0]$, the following holds:

$$\Psi^{-2}\frac{d\Psi}{dR} + \Psi - K_{R_0} + \frac{1}{\Psi} = \mathcal{F}_{R_0}(R). \tag{4.7.40}$$

The choice of K_{R_0} is made such that $\mathcal{F}_{R_0}(R_0) = 0$ and hence from the aforementioned bound on $\mathcal{F}_1$, one has $\mathcal{F}_{R_0}(R) \leq \exp\big(-\delta_0\epsilon^{-2}\big)e^{-R}$. We also rewrite the above in the form

$$\frac{d\Psi}{dR} = -\Psi(\Psi^2 - K_{R_0}\Psi + 1) + \Psi^2\mathcal{F}_{R_0}. \tag{4.7.41}$$

In order to proceed, we must estimate the constant of integration K_{R_0}. For this purpose, we evaluate (4.7.40) at $R = R_i$, and then apply Proposition 4.5.1 ((4.5.2) specifically). This proposition, together with (4.7.24), will give

$$\left|\Psi^{-2}\frac{d\Psi}{dR}(R_i)\right| \leq \exp\big(-2\delta_0\epsilon^{-2}\big),$$

so that, when evaluating (4.7.40) at $R = R_i$, one finds the estimate

$$\left|K_{R_0} - \Psi_i - \frac{1}{\Psi_i}\right| \leq 2\exp\big(-2\delta_0\epsilon^{-2}\big). \tag{4.7.42}$$

In particular, for ϵ chosen sufficiently small one can get the key upper bound $|K_{R_0}| \leq \frac{5}{3}\eta^{-1}$. Here we used the estimate (4.5.26) at $s = s_i$ in Corollary 4.5.4 and that we chose $\eta \leq \min\{\frac{1}{2}\mathfrak{W}, \frac{1}{4}\}$, as well as $\pi^{-\frac{1}{2}} < \frac{2}{3}$. We can use this to improve the bootstraps (K1) and (K2). We first improve the upper bound (K1) on $|\Psi|$; we prove $|\Psi| \leq \frac{5}{3}\eta^{-1}$. Without loss of generality, suppose that Ψ, and hence K_{R_0}, are positive for $s \in \mathcal{K}$, and suppose for contradiction that $\sup_{s\in\mathcal{K}}\Psi > \frac{5}{3}\eta^{-1}$.

So we may choose R_1 to be $R_1 = \inf\{R > R_i : \Psi(R) = \frac{5}{3}\eta^{-1}\}$. This trivially implies that $\frac{d\Psi}{dR}(R_1) \geq 0$. However, looking at (4.7.41) for any $R_0 \geq R_1$ we get

$$\frac{d\Psi}{dR}(R_1) = -\left(\tfrac{5}{3}\eta^{-1}\right)^2\left(\tfrac{5}{3}\eta^{-1} - K_{R_0}\right) - \tfrac{5}{3}\eta^{-1} + \Psi^2\mathcal{F}_{R_0}(R_1) < 0,$$

where the final step follows from $K_{R_0} \leq \frac{5}{3}\eta^{-1}$ and $\Psi^2|\mathcal{F}_{R_0}(R_1)| \leq \Psi^2 \exp\big(-2\delta_0\epsilon^{-2}\big) \leq \eta^{-1}$ for ϵ small. This is a contradiction, and thus ensures that $\sup_{s\in\mathcal{K}}|\Psi(s)| \leq \frac{5}{3}\eta^{-1}$. This improves the bootstrap (K1).

For the remaining bootstrap (K2), we combine the above with the estimate (4.7.12). To be precise, we have

$$-r\dot{r}(s) = -r\dot{r}(s_i)\cdot\frac{\Psi(s_i)}{\Psi(s)}\cdot\frac{r^2\dot{\phi}(s)}{r^2\dot{\phi}(s_i)} \geq \frac{3}{5}\eta^2\cdot(-r\dot{r}(s_i))\cdot\frac{r^2\dot{\phi}(s)}{r^2\dot{\phi}(s_i)}. \tag{4.7.43}$$

Thus, upon using Proposition 4.5.1 and (4.7.12), one finds (with D_K made larger if necessary) the lower bound

$$-r\dot{r}(s) \geq \frac{3}{5}\eta^2\cdot[4|B|^2\mathfrak{W}^2\omega_{RN}r_-^2\epsilon^2 - D_K\epsilon^4\log\big(\epsilon^{-1}\big)], \tag{4.7.44}$$

which clearly improves (K2) for ϵ sufficiently small. So the bootstrap argument is complete, and in light of (4.7.44), we conclude that the spacetime extends all the way to $r = 0$, i.e. $R = +\infty$.

To find the final one-dimensional dynamical system (4.7.6) we require a argument involving taking limits. Consider the identity (4.7.40); another consequence is that for $R_0 > R_1 \geq R_i$,

$$|K_{R_0} - K_{R_1}| = |\mathcal{F}_{R_0}(R_1)| \leq e^{-\delta_0\epsilon^{-2}}e^{-R_1}.$$

In particular, if $(R_n)_{n\in\mathbb{N}}$ is a sequence with $R_n \to +\infty$, then the sequence $(K_{R_n})_{n\in\mathbb{N}}$ is Cauchy, and the limit is independent of the sequence taken. So there exists some $K \in \mathbb{R}$ such that $K_R \to K$ as $R \to +\infty$, which by (4.7.42) also satisfies

$$|K - \Psi_i - \Psi_i^{-1}| \leq 2\exp\big(-2\delta_0\epsilon^{-2}\big). \tag{4.7.45}$$

Hence we may take the limit in equation (4.7.41) where we fix R and take $R_0 \to +\infty$. We find that there will exist some function $\mathcal{F}$ with $\Psi^2\mathcal{F}_{R_0}(R) \to \mathcal{F}(R)$ as $R_0 \to +\infty$, also satisfying $|\mathcal{F}| \leq D_K e^{-\delta_0\epsilon^{-2}}e^{-R}$, such that one has

$$\frac{d\Psi}{dR} = -\Psi(\Psi^2 - K\Psi + 1) + \mathcal{F}. \tag{4.7.46}$$

By construction, K has the same sign as $\Psi(s)$ in $\mathcal{K}$. We argue now that $|K| \geq 2$. Suppose otherwise that $|K| < 2$, which implies that $\Psi^2 - K\Psi + 1$ is bounded below by a positive constant β. Then (4.7.46) implies that

$$\frac{d|\Psi|}{dR} \leq -\beta|\Psi| + |\mathcal{F}|.$$

We then apply Grönwall's inequality, finding that

$$|\Psi(R)| \leq e^{-\beta(R-R^*)}|\Psi(R^*)| + \int_{R^*}^{R} e^{-\beta(R-\tilde{R})}|\mathcal{F}(\tilde{R})|\,d\tilde{R} \xrightarrow{R\to+\infty} 0.$$

But this is a contradiction to the lower bound (4.7.13)! Hence $|K| \geq 2$, and we may write $K = \alpha + \alpha^{-1}$ for some $\alpha \in \mathbb{R}$. By $|K| \leq \frac{5}{3}\eta^{-1}$, it is clear that $|\alpha| \leq \frac{5}{3}\eta^{-1}$ also. From (4.7.45), we can deduce that

$$|(\alpha - \Psi_i)(\alpha - \Psi_i^{-1})| \lesssim \exp\big(-2\delta_0\epsilon^{-2}\big)\eta^{-1} \lesssim \exp\big(-2\delta_0\epsilon^{-2}\big),$$

so, interchanging $\alpha \leftrightarrow \alpha^{-1}$ if necessary, we may choose α accordingly such that $|\alpha - \Psi_i| \lesssim \exp\big(-\delta_0 \epsilon^{-2}\big)$ as required. Therefore, we have arrived at the ODE (4.7.6).

For the final statement in Proposition 4.7.1, note that the upper bound follows from rather straightforward analysis of this ODE, while the lower bound arises from a straightforward modification from the proof of (4.7.13) in Lemma 4.7.3. □

4.7.4 Geometric features of the region $\mathcal{K}$ in the bounce case $|\Psi_i| < 1$

In this section, we make use of Proposition 4.7.1 to derive more quantitative information regarding the quantities $r(s)$ and $\Omega^2(s)$ in the region $s \in \mathcal{K}$, focusing for now on the interesting case where $|\Psi_i| < 1$ and there is a Kasner bounce. In particular, we will estimate the value of $r(s)$ where the bounce occurs, and bound $\Omega^2(s)$ in such a way that we can infer quantitative closeness to Kasner-like spacetimes before and after the bounce.

To make precise statements about the closeness of these regions to Kasner-like geometries, we will have to assume further that Ψ_i, and therefore the quantities α and α^{-1} of Proposition 4.7.1, are bounded strictly away from 1 in absolute value, where several important quantities will begin to degenerate[14].

So, for this section, we strengthen the condition $(*)$ to the following assumption on Ψ_i, any $0 < \sigma < 1/4$:

$$\eta \leq |\Psi_i| \leq 1 - \sigma. \tag{$**$}$$

In light of Corollary 4.6.3, or more precisely the remark following it, the assumption $(**)$ is not vacuous; for η sufficiently small and any choice of $\sigma \in \big(0, \frac{1}{4}\big)$, there are certainly arbitrarily small ϵ such that $(**)$ is satisfied, and in fact the measure of this set of ϵ is controlled, as claimed in Theorem 4.1.2.

Assuming $(**)$, we make precise the region of spacetime where the bounce occurs in the following lemma.

Lemma 4.7.5. *Given $n \in \mathbb{N}$ with $n \geq 2$, there exists $\epsilon_0(n, \eta, \sigma) > 0$ such that if $0 < |\epsilon| < \epsilon_0(n, \eta, \sigma)$ **and** the assumption $(**)$ holds, then for any $z > 0$ such that $z \in [|\alpha| + \epsilon^n, |\alpha|^{-1} - \epsilon^n]$, there exists a unique $s_z \in \mathcal{K}$ such that $|\Psi(s_z)| = z$. For this domain of z, the function $z \mapsto s_z$ is increasing, smooth and invertible, we may define the inversion interval $\mathcal{K}^n_{bo}$ to be $\{s_{in}(\epsilon) \leq s \leq s_{out}(\epsilon)\}$ with $|\Psi(s_{in})| = (|\alpha| + \epsilon^n)$ and $|\Psi(s_{out})| = (|\alpha|^{-1} - \epsilon^n)$.*

Moreover there exists a constant $D_I(M, \mathbf{e}, \Lambda, m^2, q_0, \eta, \sigma, n) > 0$ depending on η and σ as well as the usual parameters $M, \mathbf{e}, \Lambda, m^2, q_0$ such that we have the following for all $s \in \mathcal{K}^n_{bo}$:

$$\left| \epsilon^2 \cdot \log \frac{r_-}{r(s)} - \frac{b_-^{-2}}{2(1-\alpha^2)} \right| \leq D_I \epsilon^2 \log\big(\epsilon^{-1}\big). \tag{4.7.47}$$

[14] Assuming the Kasner correspondence of Section 4.2.4, having $|\alpha| = 1$ would imply a spacetime of Kasner exponents 0, 1/2 and 1/2, which already begins to display degenerate features in the BKL picture, see Section 4.2.4.

Remark. The idea is that Lemma 4.7.5 identifies precisely the region where the quantity Ψ transitions from having absolute value smaller than 1 to having absolute value greater than 1, and that this transition occurs entirely within a region where $\log\left(\frac{r_-}{r(s)}\right)$ is of size $\sim \epsilon^{-2}$, but the $\log\left(\frac{r_-}{r(s)}\right)$-difference within the region is only $O(\log(\epsilon^{-1}))$.

In what follows, we will, for the most part, take $n = 2$ and define $\mathcal{K}_{bo} := \mathcal{K}_{bo}^{n=2}$.

Proof. We shall prove Lemma 4.7.5 in three steps:

- First, we show using equations (4.7.1) and (2.4.25) that when $\log\left(\frac{r_-}{r(s)}\right) \leq \frac{1}{2}b_-^{-2}\epsilon^{-2}(1-\alpha^2)^{-1} - O(\log(\epsilon^{-1}))$, we still have $-r\dot{r}(s) \approx -r\dot{r}(s_i)$ and $\Psi(s) \approx \Psi_i \approx \alpha$.
- Next we show using the same equations, that conversely, once we proceed to $\log\left(\frac{r_-}{r(s)}\right) \geq \frac{1}{2}b_-^{-2}\epsilon^{-2}(1-\alpha^2)^{-1} + O(\log(\epsilon^{-1}))$, we must have that $\Psi(s) \geq |\alpha| + \epsilon^n$, hence identifying an $s = s_{in}(\epsilon)$ which which satisfies (4.7.47).
- Now applying Proposition 4.7.1, particularly the ODE (4.7.6), we deduce that $\frac{d\Psi}{dR}$ is positive and bounded strictly away from 0 for $|\Psi| \in [|\alpha|+\epsilon^n, |\alpha^{-1}|-\epsilon^n]$. Using this, we then show that one proceeds from $|\Psi| = |\alpha| + \epsilon^n$ to $|\Psi| = |\alpha|^{-1} - \epsilon^n$ in an R-interval of length $O(\log(\epsilon^{-1}))$, thus proving (4.7.47) for $s \in \mathcal{K}_{bo}$, as well as the remaining assertions of the lemma.

For ease of notation we suppose, without loss of generality, that $\Psi, \alpha > 0$ in this proof.

By (4.7.7), we know that, $\Psi \geq \alpha - \epsilon^2$ in $\mathcal{K}$. Therefore, one has from (4.7.1) that

$$\frac{d}{ds}\log\left[\frac{\Omega^2(s)}{-\dot{r}}\left(\frac{r(s)}{r_-}\right)^{-(\alpha-\epsilon^2)^2}\right] \leq \frac{d}{ds}\log\left(\frac{\Omega^2(s)}{-\dot{r}(s)}\right) - \frac{\dot{r}}{r}\Psi^2 \leq 0.$$

So $\frac{\Omega^2}{-\dot{r}}(s)\left(\frac{r(s)}{r_-}\right)^{-(\alpha-\epsilon^2)^2}$ is decreasing. But by Corollary 4.5.4, specifically (4.5.29), and $|\alpha - \Psi_i| \lesssim \epsilon^2\log(\epsilon^{-1})$, as well as the fact that $\frac{1}{|\dot{r}(s_i)|} \lesssim r(s_i)\cdot\epsilon^{-2} \lesssim \exp\left(2\log(\epsilon^{-1})\right)$, we have

$$\frac{\Omega^2}{-\dot{r}}(s)\left(\frac{r(s)}{r_-}\right)^{-(\alpha-\epsilon^2)^2} \leq \frac{\Omega^2}{-\dot{r}}(s_i)\left(\frac{r(s_i)}{r_-}\right)^{-(\alpha-\epsilon^2)^2} \leq \exp\left(-\frac{1}{2}b_-^{-2}\epsilon^{-2}\right)\cdot\exp\left(D_K\log(\epsilon^{-1})\right). \tag{4.7.48}$$

Furthermore we used that, because $\Psi_i^2 - (\alpha-\epsilon^2)^2 = O(\epsilon^2)$, one has $(\frac{r(s_i)}{r_-})^{\Psi_i^2-(\alpha-\epsilon^2)^2} \lesssim e^{O(\epsilon^{-2})\cdot O(\epsilon^2)} \lesssim 1$.

We now use this upper bound when integrating the equation (2.4.25). Due to Lemma 4.7.2, we

can use the following estimate for the integral of the right hand side of (2.4.25):

$$\begin{aligned}
|-r\dot{r}(s)+r\dot{r}(s_i)| &\leq \int_{s_i}^{s} \frac{Q_\infty^2\Omega^2(\tilde{s})}{2r^2(\tilde{s})}\,d\tilde{s} + \int_{s_i}^{s} \frac{\Omega^2(\tilde{s})r^2(\tilde{s})}{4}(|\Lambda|+m^2|\phi|^2)\,d\tilde{s} + \int_{s_i}^{s} \frac{\Omega^2(\tilde{s})}{4}\,d\tilde{s} \\
&\lesssim \int_{r(s)}^{r(s_i)} \frac{\Omega^2}{-\dot{r}}\frac{1}{r^2}\,dr \\
&\lesssim \exp\left(-\frac{1}{2}b_-^{-2}\epsilon^{-2}\right)\cdot\exp\big(D_K\log\big(\epsilon^{-1}\big)\big)\int_{r(s)}^{r(s_i)}\cdot\left(\frac{r}{r_-}\right)^{(\alpha-\epsilon^2)^2-2}dr \\
&\lesssim \left(\frac{r_-}{r(s)}\right)^{1-(\alpha-\epsilon^2)^2}\cdot\exp\left(-\frac{1}{2}b_-^{-2}\epsilon^{-2}\right)\cdot\exp\big(D_K\log\big(\epsilon^{-1}\big)\big).
\end{aligned}$$

Here, the last step follows as $1-(\alpha-\epsilon^2)^2 \geq \sigma > 0$ for sufficiently small ϵ depending on σ. If $\frac{r(s)}{r_-} \geq \exp\big((1-\alpha^2)^{-1}\big(-\frac{1}{2}b_-^{-2}\epsilon^{-2}+(D_K+6n)\log\big(\epsilon^{-1}\big)\big)\big)$, then one finds

$$\begin{aligned}
\left(\frac{r_-}{r(s)}\right)^{1-(\alpha-\epsilon^2)^2} &\leq \exp\left(-\frac{1-(\alpha-\epsilon^2)^2}{1-\alpha^2}\left(-\frac{1}{2}b_-^{-2}\epsilon^{-2}+(D_K+6n)\log\big(\epsilon^{-1}\big)\right)\right) \\
&\lesssim \epsilon^{6n}\cdot\exp\left(\frac{1}{2}b_-^{-2}\epsilon^{-2}\right)\cdot\exp\big(-D_K\log\big(\epsilon^{-1}\big)\big).
\end{aligned}$$

Putting this in the above one sees that $|-r\dot{r}(s)+r\dot{r}(s_i)| \lesssim \epsilon^{6n}$, or

$$\left|\frac{-r\dot{r}(s)}{-r\dot{r}(s_i)}-1\right| \lesssim \epsilon^{6n-2}.$$

But $-r\dot{r}(s)$ changing little from its initial value means that $\Psi(s)$ also changes little from its initial value; to see this use also (4.7.12), which combined with the above implies $|\Psi(s)-\Psi_i| \lesssim \epsilon^{6n-2}$. As $|\alpha-\Psi_i| \lesssim e^{-\delta_0\epsilon^{-2}}$, we thus know that for $\log r(s) \geq -\frac{1}{2}(1-\alpha^2)^{-1}b_-^{-2}\epsilon^{-2}+O(\log\big(\epsilon^{-1}\big))$ as specified, we have not yet entered the regime $\mathcal{K}_{bo}$.

On the other hand, we show that for $s_{in} = \sup\{s \in \mathcal{K} : \Psi(s) \leq \alpha+\epsilon^n\}$, we must have $\log r(s_{in}) \geq -\frac{1}{2}(1-\alpha^2)^{-1}b_-^{-2}\epsilon^{-2}-O(\log\big(\epsilon^{-1}\big))$. For this purpose, we use the Raychaudhuri equation (4.7.1) to see that, for $s_i \leq s \leq s_{in}$, one has

$$\frac{d}{ds}\log\left[\frac{\Omega^2(s)}{-\dot{r}}\left(\frac{r(s)}{r_-}\right)^{-(\alpha+\epsilon^n)^2}\right] \geq \frac{d}{ds}\log\left(\frac{\Omega^2(s)}{-\dot{r}(s)}\right)-\frac{\dot{r}}{r}\Psi^2 \geq -\frac{r^2}{-r\dot{r}}|\tilde{A}|^2q_0^2|\phi|^2.$$

Since $r(s) \leq e^{-\delta_0\epsilon^{-2}}r_-$ for $s \in \mathcal{K}$, one sees using Proposition 4.7.1 and Lemma 4.7.2 that the integral of the right hand side is bounded below by $-\log 2$, say. Therefore a similar application of Corollary 4.5.4 will yield that

$$\frac{\Omega^2}{-\dot{r}}\left(\frac{r(s)}{r_-}\right)^{-(\alpha+\epsilon^n)^2} \geq \frac{1}{2}\exp\left(-\frac{1}{2}b_-^{-2}\epsilon^{-2}\right)\cdot\exp\big(-D_K\log\big(\epsilon^{-1}\big)\big).$$

One now again integrates the equation (2.4.25), or more precisely (4.7.17), getting now the upper bound

$$\begin{aligned}|-r\dot{r}(s)+r\dot{r}(s_i)| &\geq \int_{s_i}^{s}\frac{Q^2\Omega^2(\tilde{s})}{8r^2(\tilde{s})}\,d\tilde{s} \gtrsim \int_{r(s)}^{r(s_i)}\frac{\Omega^2}{-\dot{r}}\frac{1}{r^2}\,dr\\ &\gtrsim \exp\left(-\frac{1}{2}b_-^{-2}\epsilon^{-2}\right)\cdot\exp\big(-D_K\log\big(\epsilon^{-1}\big)\big)\int_{r(s)}^{r(s_i)}\left(\frac{r}{r_-}\right)^{(\alpha+\epsilon^n)^2-2}dr\\ &\gtrsim \left(\frac{r_-}{r(s)}\right)^{1-(\alpha+\epsilon^n)^2}\cdot\exp\left(-\frac{1}{2}b_-^{-2}\epsilon^{-2}\right)\cdot\exp\big(-D_K\log\big(\epsilon^{-1}\big)\big).\end{aligned}$$

But by Proposition 4.7.1, we always have $\epsilon^2\lesssim -r\dot{r}\lesssim\epsilon^2$, so that

$$\left(\frac{r(s)}{r_-}\right)^{1-(\alpha+\epsilon^n)^2}\gtrsim\exp\left(-\frac{1}{2}b_-^{-2}\epsilon^{-2}\right)\cdot\exp\big((2-D_K)\log\big(\epsilon^{-1}\big)\big).$$

Hence we do indeed find that for $s_i\leq s\leq s_{in}$, we must have $\log(r(s)/r_-)\geq-\frac{1}{2}(1-\alpha^2)^{-1}b_-^{-2}\epsilon^{-2}-O(\log\big(\epsilon^{-1}\big))$ as claimed. This identifies $s=s_{in}$ obeying (4.7.47).

The remainder of this proof then proceeds entirely using the ODE (4.7.6), which for $R=\log\Big(\frac{r_-}{r(s)}\Big)$ we record again here as

$$\frac{d\Psi}{dR}=-\Psi(\Psi-\alpha)(\Psi-\alpha^{-1})+\mathcal{F},\quad |\mathcal{F}(R)|\leq D_Ke^{-\delta_0\epsilon^{-2}-R}.\tag{4.7.49}$$

We have identified $R_{in}=R(s_{in})=\frac{1}{2}(1-\alpha^2)^{-1}b_-^{-2}\epsilon^{-2}+O(\log\big(\epsilon^{-1}\big))$ such that $\Psi(R_{in})=\alpha+\epsilon^n$. We want to use that α is an unstable fixed point and α^{-1} a stable fixed point of this one-dimensional dynamical system.

Note that for $\Psi(s)\in[\alpha+\epsilon^n,\alpha^{-1}-\epsilon^n]$, one knows that (i) $\Psi\geq\eta$, and (ii) $\alpha^{-1}-1\geq1-\alpha\geq\sigma-O(e^{-\delta_0\epsilon^{-2}})$, so we absorb the error term $\mathcal{F}$ and quantify the stability and instability of the fixed points to find

$$\frac{d}{dR}(\Psi-\alpha)\geq\frac{\eta\sigma}{2}(\Psi-\alpha)\quad\text{ if }\Psi\in[\alpha+\epsilon^n,1],\tag{4.7.50}$$

$$\frac{d}{dR}(\alpha^{-1}-\Psi)\leq-\frac{\sigma}{2}(\alpha^{-1}-\Psi)\quad\text{ if }\Psi\in[1,\alpha^{-1}-\epsilon^n].\tag{4.7.51}$$

In particular, $\frac{d\Psi}{dR}>0$ as long as $\Psi\in[\alpha+\epsilon^n,\alpha^{-1}-\epsilon^n]$. From (4.7.50), one finds that $\Psi-\alpha$ proceeds from ϵ^n to $1-\alpha$ in $O(\log\big(\epsilon^{-1}\big))$ time in R, and from (4.7.51) that $\alpha^{-1}-\Psi$ proceeds from $\alpha^{-1}-1$ to ϵ^n also in $O(\log\big(\epsilon^{-1}\big))$ time in R. Therefore defining R_{out} to be the minimal R such that $\Psi(R_{out})=\alpha^{-1}-\epsilon^n$, one finds $R_{out}=R_{in}+O(\log\big(\epsilon^{-1}\big))=\frac{1}{2}(1-\alpha^2)^{-1}b_-^{-2}\epsilon^{-2}+O(\log\big(\epsilon^{-1}\big))$ also.

Finally, since $\frac{d\Psi}{dR}\geq\frac{1}{2}\epsilon^n\eta\sigma>0$ when $\Psi\in[\alpha+\epsilon^n,\alpha^{-1}-\epsilon^n]$, we have that, upon entering this region of Ψ, it is impossible to return, and the remaining claims of the lemma are immediate. □

4.8 Quantitative Kasner-like asymptotics

Let us recapitulate the various regions so far:

1. In Section 4.5, we worked in the Proto-Kasner region $\mathcal{PK} = \{s_{PK} \leq s \leq s_i\} = \{2|B|\mathfrak{W}\epsilon \geq \frac{r}{r_-} \geq e^{-\delta_0\epsilon^{-2}}\}$.
   We now also define the restricted region $\mathcal{PK}' = \{s_{K_1} \leq s \leq s_i\} = \{2|B|\mathfrak{W}\epsilon^2 \geq \frac{r}{r_-} \geq e^{-\delta_0\epsilon^{-2}}\}$.
2. In Section 4.7, we have worked in the Kasner region $\mathcal{K} = \{s_i \leq s < s_\infty\} = \{0 < \frac{r}{r_-} \leq e^{-\delta_0\epsilon^{-2}}\}$ and showed that $\lim_{s\to s_\infty} r(s) = 0$.

In what follows, we prove quantitative estimates on the "Kasner-like behavior" of the metric; for this, we will first have to restrict $\mathcal{PK}$ to its aforementioned subset $\mathcal{PK}'$ on which $r \gtrsim \epsilon^2$ (recall that $r(s_{K_1}) = 2|B|\mathfrak{W}r_-\epsilon^2$). While in the previous sections, the analysis was so far oblivious to the absence/presence of a Kasner bounce, we will now also be obliged to distinguish both cases and treat them differently.

As we will soon show, the "No Kasner bounce" condition will be that, for some $0 < \sigma < 1$,

$$|\Psi(s_i)| \geq 1 + \sigma, \tag{$***$}$$

whereas the "Kasner bounce" condition will be that, for some $0 < \sigma < 1$ and $\eta > 0$,

$$\eta \leq |\Psi(s_i)| \leq 1 - \sigma. \tag{$**$}$$

We note that even combining $(***)$ and $(**)$ does not cover the range of $\Psi(s_i)$ included in $(*)$. In particular, we do not say anything further in the case where $\Psi(s_i) = 1$. In each of these two cases, we will further sub-divide $\mathcal{PK} \cup \mathcal{K}$ as follows:

i. If the "No Kasner bounce" condition $(***)$ is satisfied, we define the first Kasner region by

$$\mathcal{K}_1 = \mathcal{PK}' \cup \mathcal{K}. \tag{4.8.1}$$

   Because there is no bounce in that case, we show indeed that the metric is close to a single Kasner spacetime on the whole of $\mathcal{K}_1 = \mathcal{PK}' \cup \mathcal{K}$ in Theorem 4.8.1.

ii. If the "Kasner bounce" condition $(**)$ is satisfied, we define the first Kasner region by

$$\mathcal{K}_1 = \{s_{K_1} \leq s \leq s_{in}\} \supset \mathcal{PK}', \tag{4.8.2}$$

   where $s_{in} \in \mathcal{K}$ will be defined shortly. We also define

$$\mathcal{K}_{bo} = \{s_{in} \leq s \leq s_{out}\} \tag{4.8.3}$$

   to be the Kasner bounce region, and

$$\mathcal{K}_2 = \{s_{out} \leq s < s_\infty\} \tag{4.8.4}$$

to be the second Kasner region, where s_{in}, $s_{out} \in \mathcal{K}$ given by Lemma 4.7.5 (applied to $n = 2$) are defined such that (4.7.47) is satisfied on $\mathcal{K}_{bo}$. Because of the Kasner bounce, we show that the metric is close to a first Kasner spacetime in $\mathcal{K}_1$, and close to a second, different Kasner spacetime in $\mathcal{K}_2$ in Theorem 4.8.2. By (4.7.47), the transition region $\mathcal{K}_{bo}$ will be shown to be small in an appropriate sense.

We now state the two main theorems of this section, Theorem 4.8.1 and 4.8.2.

Theorem 4.8.1. *Let $(r, \Omega^2, \phi, Q, \tilde{A})$ be a solution to the system (2.4.21)–(2.4.31), which, by Proposition 4.5.1, exists at least up to the value $s = s_i = \frac{b_-^{-2}\epsilon^{-2}}{4|K_-|} + O(\log(\epsilon^{-1}))$ at which $r(s) = e^{-\delta_0\epsilon^{-2}} r_-$ (see Proposition 4.3.5 for a definition of b_-). Suppose that for some given $0 < \sigma < 1$, $(***)$ is satisfied.*

Then there exists some $\epsilon_0(M, \mathbf{e}, \Lambda, m^2, q_0, \sigma) > 0$, such that, if $0 < |\epsilon| < \epsilon_0$, then there exists some $s_\infty > s_i$ such that the solution of (2.4.21)–(2.4.31) exists for $s \in (-\infty, s_\infty)$ with $\lim_{s\to s_\infty} r(s) = 0$.

Furthermore, we have the following Kasner-like asymptotics: denote s_{K_1} such that $r(s_{K_1}) = 2|B|\mathfrak{W}\epsilon^2 r_-$, then in the region $\mathcal{K}_1 = \{s_{K_1} \leq s < s_\infty\}$, one may write the metric in the following form, where α is as determined in Proposition 4.7.1:

$$g = -d\tau^2 + \mathcal{X}_1 \cdot (1 + \mathfrak{E}_{X,1}(\tau))\, \tau^{\frac{2(\alpha^2-1)}{\alpha^2+3}}\, dt^2 + \mathcal{R}_1 \cdot (1 + \mathfrak{E}_{R,1}(\tau))\, r_-^2 \tau^{\frac{4}{\alpha^2+3}}\, d\sigma_{\mathbb{S}^2}. \tag{4.8.5}$$

Here, $\mathcal{X}_1$ and $\mathcal{R}_1$ are constants, and $\mathfrak{E}_{X,1}(\tau)$ and $\mathfrak{E}_{R,1}(\tau)$ are small functions of τ satisfying the following bounds for $\beta = \min\{\frac{1}{2}, 1 - \alpha^2\}$

$$\left|\log \mathcal{X}_1 + \frac{\alpha^2+1}{\alpha^2+3} b_-^{-2}\epsilon^{-2}\right| + \left|\log \mathcal{R}_1 - \frac{1}{\alpha^2+3} b_-^{-2}\epsilon^{-2}\right| \leq C_K \log(\epsilon^{-1}), \tag{4.8.6}$$

$$|\mathfrak{E}_{X,1}(\tau)| + |\mathfrak{E}_{R,1}(\tau)| \leq C_K \epsilon^2 \cdot \left(\frac{\tau}{\tau(s_{K_1})}\right)^{\frac{2\beta}{\alpha^2+3}}. \tag{4.8.7}$$

Hence, the spacetime corresponds to a Kasner-like spacetime, with Kasner exponents $p_1 = \frac{\alpha^2-1}{\alpha^2+3}$, $p_2 = p_3 = \frac{2}{\alpha^2+3}$.

Theorem 4.8.2. *Let $(r, \Omega^2, \phi, Q, \tilde{A})$ be a solution to the system (2.4.21)–(2.4.31), which by Proposition 4.7.1 exists at least up to the value $s = s_i = \frac{b_-^{-2}\epsilon^{-2}}{4|K_-|} + O(\log(\epsilon^{-1}))$ at which $r(s) = e^{-\delta_0\epsilon^{-2}} r_-$. Suppose that for given $\eta > 0$ and $0 < \sigma < 1$, $(**)$ is satisfied.*

Then there exists some $\epsilon_0(M, \mathbf{e}, \Lambda, m^2, q_0, \eta, \sigma) > 0$, such that, if $0 < |\epsilon| < \epsilon_0$, then there exists some $s_\infty > s_i$ such that the solution of (2.4.21)–(2.4.31) exists for $s \in (-\infty, s_\infty)$ with $\lim_{s\to s_\infty} r(s) = 0$. We further single out two different regions with Kasner-like asymptotics, between which there is an intermediate region *where the Kasner bounce occurs. Letting s_{K_1} be such that $r(s_{K_1}) = 2|B|\mathfrak{W}\epsilon^2 r_-$, we define the following three regions:*

$$\mathcal{K}_1 = \{s_{K_1} \leq s \leq s_{in}\},\ \mathcal{K}_{bo} = \{s_{in} \leq s \leq s_{out}\},\ \mathcal{K}_2 = \{s_{out} \leq s \leq s_\infty\}.$$

We will describe Kasner-like asymptotics for the two regions $\mathcal{K}_1$ and $\mathcal{K}_2$.

In the region $\mathcal{K}_1$, one writes the metric in the following form, for α as in Proposition 4.7.1:

$$g = -d\tau^2 + \mathcal{X}_1 \cdot (1 + \mathfrak{E}_{X,1}(\tau)) (\tau - \tau_0)^{\frac{2(\alpha^2-1)}{\alpha^2+3}} dt^2 + \mathcal{R}_1 \cdot (1 + \mathfrak{E}_{R,1}(\tau)) r_-^2 (\tau - \tau_0)^{\frac{4}{\alpha^2+3}} d\sigma_{\mathbb{S}^2}. \quad (4.8.8)$$

Here $\tau_0 > 0$, $\mathcal{X}_1$ and $\mathcal{R}_1$ are constants, and $\mathfrak{E}_{X,1}(\tau)$ and $\mathfrak{E}_{R,1}(\tau)$ are functions of τ satisfying

$$\left| \log \mathcal{X}_1 + \frac{\alpha^2+1}{\alpha^2+3} b_-^{-2} \epsilon^{-2} \right| + \left| \log \mathcal{R}_1 - \frac{1}{\alpha^2+3} b_-^{-2} \epsilon^{-2} \right| \leq C_K \log\big(\epsilon^{-1}\big), \quad (4.8.9)$$

$$|\mathfrak{E}_{X,1}(\tau)| + |\mathfrak{E}_{R,1}(\tau)| \leq C_K \epsilon^2. \quad (4.8.10)$$

On the other hand, in the region $\mathcal{K}_2$, one instead has the following form for the metric

$$g = -d\tau^2 + \mathcal{X}_2 \cdot (1 + \mathfrak{E}_{X,2}(\tau)) \tau^{\frac{2(1-\alpha^2)}{1+3\alpha^2}} dt^2 + \mathcal{R}_2 \cdot (1 + \mathfrak{E}_{R,2}(\tau)) r_-^2 \tau^{\frac{4\alpha^2}{1+3\alpha^2}} d\sigma_{\mathbb{S}^2}. \quad (4.8.11)$$

The constants $\mathcal{X}_2$ and $\mathcal{R}_2$ are constants, and the functions $\mathfrak{E}_{X,2}(\tau)$ and $\mathfrak{E}_{R,2}(\tau)$ now satisfying the following bounds for $\beta = \min\{\frac{1}{2}, 1 - \alpha^2\}$

$$\left| \log \mathcal{X}_2 + \frac{1+\alpha^{-2}}{1+3\alpha^2} b_-^{-2} \epsilon^{-2} \right| + \left| \log \mathcal{R}_2 - \frac{1}{1+3\alpha^2} b_-^{-2} \epsilon^{-2} \right| \leq C_K \log\big(\epsilon^{-1}\big), \quad (4.8.12)$$

$$|\mathfrak{E}_{X,2}(\tau)| + |\mathfrak{E}_{R,2}(\tau)| \leq C_K \epsilon^2 \cdot \left(\frac{\tau}{\tau(s_{out})} \right)^{\frac{2\beta}{\alpha^{-2}+3}}. \quad (4.8.13)$$

One sees that the spacetime exhibits a Kasner bounce from the Kasner-like region $\mathcal{K}_1$ with Kasner exponents of $p_1 = \frac{\alpha^2-1}{\alpha^2+3}, p_2 = p_3 = \frac{2}{\alpha^2+3}$ to another Kasner-like region $\mathcal{K}_2$ with exponents of $p_1 = \frac{1-\alpha^2}{1+3\alpha^2}, p_2 = p_3 = \frac{2\alpha^2}{1+3\alpha^2}$. We further provide the following estimates regarding the proper time length of the regions $\mathcal{K}_2$ and $\mathcal{K}_{bo}$. For $\mathcal{K}_2$, the proper time τ from the singularity varies between 0 and $\tau(s_{out})$, obeying

$$\left| \log \tau(s_{out}) - \frac{1}{2} \frac{\alpha^{-2}+1}{1-\alpha^2} \cdot b_-^{-2} \epsilon^{-2} \right| \leq C_K \log\big(\epsilon^{-1}\big). \quad (4.8.14)$$

On the other hand, we have the following (non-sharp) upper bound for the size of $\mathcal{K}_{bo} = [s_{in}, s_{out}]$, where the proper time variable $\tau = \tau(s)$ satisfies $\frac{d\tau}{ds} = -\frac{\Omega(s)}{2}$.

$$0 \leq \tau(s_{in}) - \tau(s_{out}) \leq \exp\left(-\frac{b_-^{-2}\epsilon^{-2}}{1-\alpha^2} \right) \cdot \exp\big(C_K \log\big(\epsilon^{-1}\big)\big). \quad (4.8.15)$$

4.8.1 Asymptotics for Ψ near the $\{r = 0\}$ singularity

The first step in proving that the $\{r = 0\}$ singularity has Kasner-like asymptotics relies on showing that Ψ tends to the appropriate constant α or α^{-1} sufficiently quickly near the singularity. The aim will be to show that $\Psi = \alpha + O((\frac{r}{r_-})^\beta)$ or $\Psi = \alpha^{-1} + O((\frac{r}{r_-})^\beta)$ for some positive exponent $\beta > 0$.

After the usual change of coordinates $r = e^{-R} r_-$, this translates to showing that Ψ decays exponentially to α or α^{-1} in the variable R. For this purpose, we shall need to use the ODE (4.7.6) from Proposition 4.7.1. By standard dynamical systems theory, one expects that Ψ tends towards its stable fixed point at an exponential rate. We quantify this in the following lemma:

Lemma 4.8.3. *Fix constants $0 < \sigma, \eta < \frac{1}{2}$. For all $\alpha \in \mathbb{R}$ satisfying $1 + \sigma \leq |\alpha| \leq \eta^{-1}$, consider the following ODE for the function $\Psi = \Psi(R)$:*

$$\frac{d\Psi}{dR} = -\Psi(\Psi - \alpha)(\Psi - \alpha^{-1}) + \mathcal{F}(R), \qquad |\mathcal{F}(R)| \leq e^{-R}. \tag{4.8.16}$$

Define $\beta = \min\{\frac{1}{2}, \alpha^2 - 1\} \geq \sigma$. Then there exists some $\nu_0 = \nu_0(\sigma, \eta) > 0$ such that, for all $|\nu| < \nu_0$ and R_* satisfying both

$$|\Psi(R_*) - \alpha| \leq \nu^2 \quad \textbf{\textit{and}} \quad e^{-R_*} \leq \nu^2, \tag{4.8.17}$$

one finds that, for $R \geq R_*$, $\Psi(R)$ decays to α at the following exponential rate

$$|\Psi(R) - \alpha| \leq 8\nu^2 e^{-\beta(R - R_*)}. \tag{4.8.18}$$

The above lemma will be applied with $\nu \sim \epsilon^n$ or $\nu \sim e^{-\frac{\delta_0}{2}\epsilon^{-2}}$ in what follows.

Proof. We use a bootstrap argument along with Grönwall's inequality. We take the bootstrap assumption to be the desired estimate (4.8.18). Assuming this holds in some bootstrap region, we have

$$\Psi(\Psi - \alpha^{-1}) \geq \alpha^2 - 1 - |2\alpha - \alpha^{-1}| \cdot 8\nu^2 e^{-\beta(R - R_*)}.$$

Therefore, for R and $\tilde{R}$ in the bootstrap region, one computes that, for ν small enough,

$$\begin{aligned} -\int_{\tilde{R}}^{R} \Psi(R')(\Psi(R') - \alpha^{-1})\, dR' &\leq -(\alpha^2 - 1)(R - \tilde{R}) + \beta^{-1}|2\alpha - \alpha^{-1}| \cdot 8\nu^2 e^{-\beta(\tilde{R} - R_*)} \\ &\leq -\beta(R - \tilde{R}) + \log 2. \end{aligned}$$

Hence, after finding from (4.8.16) the differential inequality

$$\frac{d}{dR}|\Psi - \alpha| \leq -\Psi(\Psi - \alpha^{-1})|\Psi - \alpha| + e^{-R},$$

one uses Grönwall to deduce that

$$|\Psi(R) - \alpha| \leq 2\nu^2 e^{-\beta(R - R_*)} + \int_{R_*}^{R} 2e^{-\tilde{R}} e^{-\beta(R - \tilde{R})}\, d\tilde{R} \leq 2(\nu^2 + (1 - \beta)^{-1} e^{-R_*}) e^{-\beta(R - R_*)}.$$

So taking into account the second assumption of (4.8.17) and $\beta \leq 1/2$, we improve the bootstrap assumption (4.8.18). So this estimate is true for all $R \geq R_*$. □

4.8.2 First case: absence of Kasner bounce

In this subsection, assuming that (∗∗∗) holds, we prove the quantitative Kasner asymptotics in the region $\mathcal{K}_1 = \{s_{K_1} \leq s < s_\infty\}$. The essential ingredient is the following lemma:

Lemma 4.8.4. *Consider a solution $(r, \Omega^2, \phi, Q, \tilde{A})$ to the system (2.4.21)–(2.4.31) as in Proposition 4.7.1. Assuming also (***), one finds that there exists some $\mathcal{Y}_1 > 0$ satisfying, for $s_{K_1} \leq s < s_\infty$:*

$$\left| \log\left(\frac{\Omega^2}{-\dot{r}} \left(\frac{r}{r_-} \right)^{-\alpha^2} \right)(s) - \log \mathcal{Y}_1 \right| \leq D_1 \epsilon^2 \left(\frac{r(s)}{r_-} \right)^{\beta}, \tag{4.8.19}$$

with $\mathcal{Y}_1$ satisfying

$$\left| \log \mathcal{Y}_1 + \frac{1}{2} b_-^{-2} \epsilon^{-2} \right| \leq D_1 \cdot \log\big(\epsilon^{-1}\big). \tag{4.8.20}$$

Furthermore, one may find a constant $\mathcal{Z}_1 > 0$ such that

$$|-r\dot{r}(s) - \mathcal{Z}_1| \leq D_1 \epsilon^4 \left(\frac{r(s)}{r_-} \right)^{\beta}, \tag{4.8.21}$$

with $\mathcal{Z}_1$ satisfying

$$\big| \mathcal{Z}_1 - 4|B|^2 \mathfrak{W}^2 \omega_{RN} r_-^2 \epsilon^2 \big| \leq D_1 \cdot \epsilon^4 \log\big(\epsilon^{-1}\big). \tag{4.8.22}$$

Proof. We will use the Raychaudhuri equation (4.7.1). In light of Lemma 4.8.3, it is preferable to change variables once again, now to the R-coordinate:

$$\frac{d}{dR} \log\left(\frac{\Omega^2}{-\dot{r}} \right) = -\Psi^2 - \frac{r^4 q_0^2 \tilde{A}^2 \phi^2}{(-r\dot{r})^2}. \tag{4.8.23}$$

Proposition 4.7.1 and Lemma 4.7.2 tell us that the second term on the right hand side of this expression is $O(\epsilon^{-4} e^{-4R} R^2)$. Adding α^2 to both sides and using that Ψ is bounded, we have

$$\left| \frac{d}{dR} \log\left(\frac{\Omega^2}{-\dot{r}} \right) + \alpha^2 \right| \lesssim |\Psi - \alpha| + \epsilon^{-4} e^{-4R} R^2. \tag{4.8.24}$$

The crucial observation is that the right-hand side will be integrable (and with small integral) in the region $\mathcal{K}_1$. Indeed, we claim that if $R_{K_1} = -\log\left(\frac{r_{K_1}}{r_-}\right) = -\log\big(2|B|\mathfrak{W}\epsilon^2\big)$, then for $R_{K_1} \leq R < +\infty$, we have

$$\int_R^{+\infty} |\Psi(\tilde{R}) - \alpha| \, d\tilde{R} + \int_R^{+\infty} \epsilon^{-4} e^{-4\tilde{R}} \tilde{R}^2 \, d\tilde{R} \lesssim \epsilon^2 e^{-\beta R}. \tag{4.8.25}$$

The bound for the latter integral follows by straightforward calculus, using also $\beta < 1/2$ and $e^{-R} \leq e^{-R_{K_1}} \sim \epsilon^2$ to get the correct dependence on ϵ. For the former integral, we proceed in two steps; first we use Lemma 4.8.3 to deal with the region $\mathcal{K}$, then use Corollary 4.5.4 to handle the remaining part $\mathcal{PK} \cap \mathcal{K}_1$.

Note that, for the region $\mathcal{K}$, where we have access to the ODE (4.7.6), we have the bound $|\Psi(R_i) - \alpha| \lesssim e^{-\delta_0 \epsilon^{-2}}$ from Proposition 4.7.1. Hence, by Lemma 4.8.3 with $\nu^2 \sim e^{-\delta_0 \epsilon^{-2}}$, we know that, for $R \geq R_i$, we have $|\Psi - \alpha| \lesssim e^{-\delta_0 \epsilon^{-2}} e^{-\beta(R - R_i)}$. Thus, for $R \geq R_i$, one has

$$\int_R^{+\infty} |\Psi(\tilde{R}) - \alpha| \, d\tilde{R} \lesssim e^{-\delta_0 \epsilon^{-2}} e^{-\beta(R - R_i)} \lesssim e^{-(1-\beta)\delta_0 \epsilon^{-2}} e^{-\beta R}.$$

The last line here follows from the definition $R_i = \delta_0 \epsilon^{-2}$. The smallness of the expression $e^{-(1-\beta)\delta_0 \epsilon^{-2}}$ thus proves (4.8.25) for $R \geq R_i$.

For the remaining portion $R \in [R_{K_1}, R_i]$, we use Corollary 4.5.4. The point is that, in this region, we have $|\Psi - \alpha| \leq |\Psi - \Psi_i| + |\Psi_i - \alpha| \lesssim e^{-2R}\log\big(\epsilon^{-1}\big) + e^{-\delta_0\epsilon^{-2}}$. Therefore, one finds that

$$\int_R^{R_i} |\Psi(\tilde{R}) - \alpha|\, d\tilde{R} \lesssim e^{-2R}\log\big(\epsilon^{-1}\big) + e^{-\delta_0\epsilon^{-2}} R_i \lesssim \epsilon^2 e^{-\beta R}.$$

So we have proved the estimate (4.8.25). From (4.8.24), this shows that the expression

$$\log\left(\frac{\Omega^2}{-\dot{r}}\right) + \alpha^2 R = \log\left(\frac{\Omega^2}{-\dot{r}}\left(\frac{r}{r_-}\right)^{-\alpha^2}\right) \tag{4.8.26}$$

indeed has a finite limit $\log \mathcal{Y}_1$ as $R \to \infty$, and moreover satisfies the estimate (4.8.19).

We next estimate the constant $\log \mathcal{Y}_1$. To do this, we evaluate (4.8.19) at $r = r_{K_1}$ i.e. $s = s_{K_1}$ and use the estimate (4.5.29) from Corollary 4.5.4. As $|\log r(s_{K_1})|, |\log(-\dot{r}(s_{K_1}))| = O(\log\big(\epsilon^{-1}\big))$ here, we find that

$$\left|\log \mathcal{Y}_1 + \frac{1}{2} b_-^{-2}\epsilon^{-2}\right| \lesssim \log\big(\epsilon^{-1}\big). \tag{4.8.27}$$

Finally, we show the estimate (4.8.21) by considering the equation (2.4.25) after changing variables to r; Propositions 4.5.1 and 4.7.1 and then the now-known (4.8.19) tell us that for $s \in \mathcal{K}_1$, we have

$$\left|\frac{d}{dr}(-r\dot{r})\right| \lesssim \frac{\Omega^2}{-r^2\dot{r}} \lesssim \mathcal{Y}_1 \cdot \left(\frac{r}{r_-}\right)^{\alpha^2 - 2}.$$

Integrating this expression then yields (4.8.21) – of course it is essential here that $\alpha^2 - 1 \geq \frac{\sigma}{2} > 0$, and we have the smallness (4.8.27) for the expression $\mathcal{Y}_1$. □

4.8.3 Second case: presence of a Kasner bounce

Assuming instead that (∗∗) holds, Proposition 4.7.1 and Lemma 4.7.5 (for $n = 2$) show that the spacetime must exhibit a Kasner bounce. We proceed to show that if we define $\mathcal{K}_1$ and $\mathcal{K}_2$ as in (4.8.2), (4.8.4), there are Kasner-like asymptotics in both regimes. As in Section 4.8.2, we first find the precise asymptotics for the lapse Ω^2, as well as $-\dot{r}$.

The pre-bounce Kasner regime

We start to look at the region $\mathcal{K}_1$ (pre-bounce regime).

Lemma 4.8.5. *Consider a solution $(r, \Omega^2, \phi, Q, \tilde{A})$ to the system (2.4.21)–(2.4.25) as in Proposition 4.7.1. Assuming now the condition (∗∗) on the value of Ψ at $s = s_i$, there will exist some $\mathcal{Y}_1$ satisfying, for $s_{K_1} \leq s \leq s_{in}$,*

$$\left|\log\left(\frac{\Omega^2}{-\dot{r}}\left(\frac{r}{r_-}\right)^{-\alpha^2}\right)(s) - \log \mathcal{Y}_1\right| \leq D_1\epsilon^2, \tag{4.8.28}$$

with $\mathcal{Y}_1$ satisfying

$$\left|\log \mathcal{Y}_1 + \frac{1}{2} b_-^{-2}\epsilon^{-2}\right| \lesssim \log\bigl(\epsilon^{-1}\bigr). \tag{4.8.29}$$

Furthermore, one may find a constant $\mathcal{Z}_1 > 0$ such that

$$|-r\dot{r}(s) - \mathcal{Z}_1| \leq D_1\epsilon^4, \tag{4.8.30}$$

with $\mathcal{Z}_1$ satisfying

$$\bigl|\mathcal{Z}_1 - 4|B|^2\mathfrak{W}^2\omega_{RN}r_-^2\epsilon^2\bigr| \lesssim \epsilon^4\log\bigl(\epsilon^{-1}\bigr). \tag{4.8.31}$$

Proof. We follow a similar template to the proof of Lemma 4.8.4, though unlike in that case, we do not need $(\frac{r}{r_-})^\beta$ decay rates. In particular, we are able to integrate from $R = R_{K_1}$ rather than backwards from $R = +\infty$. Using the same Raychaudhuri equation (4.8.23), we need to prove the analogue of (4.8.25), i.e. for $R \geq R_{K_1}$,

$$\int_{R_{K_1}}^{R} |\Psi(\tilde{R}) - \alpha|\, d\tilde{R} + \int_{R_{K_1}}^{R} \epsilon^{-4}e^{-4\tilde{R}}\tilde{R}^2\, d\tilde{R} \lesssim \epsilon^2. \tag{4.8.32}$$

As in Lemma 4.8.4, the latter integral is straightforward, and we focus on the former. Following Corollary 4.5.4 and Proposition 4.7.1, we know that for $s \in \mathcal{PK}' = \mathcal{PK} \cap \mathcal{K}_1$, where $R \in [R_{K_1}, R_i]$, we have $|\Psi(R) - \alpha| \leq |\Psi(R) - \Psi_i| + |\Psi_i - \alpha| \lesssim e^{-2R}\log\bigl(\epsilon^{-1}\bigr) + e^{-\delta_0\epsilon^{-2}}$. So we know as before that

$$\int_{R_{K_1}}^{R_i} |\Psi(\tilde{R}) - \alpha|\, d\tilde{R} \lesssim \epsilon^2. \tag{4.8.33}$$

On the other hand, when integrating between R_i and R_{in}, Ψ is growing away from α exponentially as opposed to exponentially decaying, so we need a new tactic. The key observation is that following the definition of R_{in} from Lemma 4.7.5 for $n = 2$, we know a priori that $|\Psi(R) - \alpha| \leq \epsilon^2$ when $R \in [R_i, R_{in}]$. The same lemma also tells us that $|R_i - R_{K_1}| \lesssim \epsilon^{-2}$, so these two observations combined tell us that $\int_{R_i}^{R_{in}} |\Psi(\tilde{R}) - \alpha|\, d\tilde{R} \lesssim 1$.

This is not quite the claimed estimate (4.8.32). To improve the $O(1)$ bound to $O(\epsilon^2)$, we apply Lemma 4.7.5 with $n = 4$. Defining R'_{in} to instead be the unique $R > R_i$ with $|\Psi(R'_{in}) - \alpha| = \epsilon^4$, we find that, for $R \in [R'_{in}, R_{in}]$, we have $|\Psi(R) - \alpha| \leq e^{\frac{\eta\sigma}{2}(R - R_{in})}\epsilon^2$. This is because, within this region, (4.7.50) holds and may be "integrated backwards" from $R = R_{in}$. Furthermore, for $R \in [R_i, R'_{in}]$, we know a priori that $|\Psi(R) - \alpha| \leq \epsilon^4$. So

$$\int_{R_i}^{R_{in}} |\Psi(\tilde{R}) - \alpha|\, d\tilde{R} \leq \int_{R_i}^{R'_{in}} \epsilon^4\, d\tilde{R} + \int_{R'_{in}}^{R_{in}} e^{\frac{\eta\sigma}{2}(\tilde{R} - R_{in})}\epsilon^2\, d\tilde{R} \lesssim \epsilon^2. \tag{4.8.34}$$

Combining this with (4.8.33) yields the desired (4.8.32). The estimate (4.8.28) then follows exactly as in the proof of (4.8.19), as does the estimate (4.8.29) on $\mathcal{Y}_1$.

For the estimate (4.8.30), let us define $\mathcal{Z}_1$ as $-r\dot{r}(s_i)$. By Proposition 4.5.1, it is easy to integrate (2.4.25) and deduce (4.8.30) in the region $s \in [s_{K_1}, s_i]$. The difficulty lies in showing (4.8.30) in the region $s \in [s_i, s_{in}]$, not least because $-r\dot{r}$ is expected to change value during the inversion process.

What helps us here is that $-r\dot{r}$ is approximately inversely proportional to the quantity Ψ, given that $r^2\dot{\phi} = -r\dot{r}\cdot\Psi$ is approximately constant. To be quantitatively precise, we combine the estimate (4.7.12) with the trivial a priori observation that $\left|\frac{\Psi(s)}{\Psi_i} - 1\right| \lesssim \epsilon^2$ for $s \in [s_i, s_{in}]$ to get

$$\left|\frac{-r\dot{r}(s)}{-r\dot{r}(s_i)} - 1\right| = \left|\frac{\Psi(s_i)}{\Psi(s)}\cdot\frac{r^2\dot{\phi}(s)}{r^2\dot{\phi}(s_i)} - 1\right| \lesssim \epsilon^2. \tag{4.8.35}$$

Since $-r\dot{r}(s_i) \sim \epsilon^2$, the estimate (4.8.30) follows immediately. □

The post-bounce Kasner regime

Finally, we consider the post-bounce regime, i.e. the region $\mathcal{K}_2 = \{s_{out} \leq s < s_\infty\}$.

Lemma 4.8.6. *Consider a solution $(r, \Omega^2, \phi, Q, \tilde{A})$ to the system (2.4.21)–(2.4.25) as in Proposition 4.7.1. Assuming the condition (**) on the value of Ψ at $s = s_i$, then for $s_{out} \leq s \leq s_\infty$, there exists some constant $\mathcal{Y}_2$ such that*

$$\left|\log\left(\frac{\Omega^2}{-\dot{r}}\left(\frac{r}{r_-}\right)^{-\alpha^{-2}}\right)(s) - \log\mathcal{Y}_2\right| \leq D_2\epsilon^2\cdot\left(\frac{r(s)}{r(s_{out})}\right)^{\beta}, \tag{4.8.36}$$

with $\mathcal{Y}_2 > 0$ satisfying

$$\left|\log\mathcal{Y}_2 - \frac{1}{2}\alpha^{-2}b_-^{-2}\epsilon^{-2}\right| \leq D_2\cdot\log(\epsilon^{-1}). \tag{4.8.37}$$

Here $\beta = \min\{\alpha^{-2} - 1, \frac{1}{2}\}$. One may also find a constant $\mathcal{Z}_2 > 0$ with $|\mathcal{Z}_2| \sim \epsilon^2$ such that, in the same region,

$$|-r\dot{r}(s) - \mathcal{Z}_2| \leq D_2\epsilon^4\cdot\left(\frac{r(s)}{r(s_{out})}\right)^{\beta}. \tag{4.8.38}$$

Proof. We again begin the proof of (4.8.36) by using the Raychaudhuri equation (4.8.23). In this case, this equation alongside Proposition 4.7.1 will tell us that

$$\left|\frac{d}{dR}\log\left(\frac{\Omega^2}{-\dot{r}}\right) - \alpha^{-2}\right| \lesssim |\Psi(R) - \alpha^{-1}| + \epsilon^{-4}e^{-4R}R^2. \tag{4.8.39}$$

To estimate $|\Psi - \alpha^{-1}|$ in (4.8.39), we apply Lemma 4.8.3 with $R_* = R_{out}$, $\nu = \epsilon$ and α replaced by α^{-1} [note that $\alpha^{-1} > 1+\sigma > 0$ assuming (**)]. The lemma thus tells us that, for ϵ chosen sufficiently small, we have $|\Psi - \alpha^{-1}| \leq 8\epsilon^2 e^{-\beta(R-R_{out})}$ for all $R \geq R_{out}$. Hence, using that $R \geq R_{out} \gtrsim \epsilon^{-2}$ which in turn yields $\epsilon^{-4}e^{-4R}R^2 \lesssim e^{-\beta R}\epsilon^2$, the inequality (4.8.39) implies that

$$\left|\frac{d}{dR}\log\left(\frac{\Omega^2}{-\dot{r}}\left(\frac{r}{r_-}\right)^{\alpha^{-2}}\right)\right| \lesssim \epsilon^2 e^{-\beta(R-R_{out})}.$$

As the right hand side is integrable, $\frac{\Omega^2}{-\dot{r}}\left(\frac{r}{r_-}\right)^{\alpha^{-2}}$ has a well defined limit $\mathcal{Y}_2$ as $R \to +\infty$, and we obtain (4.8.36).

For (4.8.38), we will not estimate $-r\dot{r}(s)$ directly but instead use Ψ and $r^2\dot{\phi} = -r\dot{r}\cdot\Psi$. We already have $|\Psi - \alpha^{-1}| \lesssim \epsilon^2 e^{-\beta(R-R_{out})}$, while we integrate (2.4.31) backwards from $R = +\infty$ to find that

$$|r^2\dot{\phi}(s) - \lim_{\tilde{s}\to s_\infty} r^2\dot{\phi}(\tilde{s})| \lesssim \int_s^{s_\infty} r^2|\phi|(\tilde{s}) + \Omega^2 r^2|\phi|(\tilde{s})\, d\tilde{s} \lesssim \int_0^{r(s)} \frac{r^3|\phi|}{-r\dot{r}} + \frac{\Omega^2}{-\dot{r}} r^2|\phi|\, dr.$$

Using Proposition 4.7.1 and that $\frac{\Omega^2}{-\dot{r}}$ is monotonically decreasing in s and thus uniformly small (see e.g. (4.7.14)), it is straightforward to find that defining $\mathcal{P}_2 = \lim_{s\to s_\infty} r^2\dot{\phi}(s) \sim \epsilon^2$, one (easily, since the RHS is $O(r^{4-})$) has

$$|r^2\dot{\phi}(s) - \mathcal{P}_2| \lesssim \epsilon^4 e^{-\beta(R-R_{out})}. \tag{4.8.40}$$

Combining this with the aforementioned $|\Psi - \alpha^{-1}| \lesssim \epsilon^2 e^{-\beta(R-R_{out})}$, we deduce (4.8.38) for $\mathcal{Z}_2 = \alpha\mathcal{P}_2$.

Finally, we provide the estimate (4.8.37) for $\log\mathcal{Y}_2$. Note that (4.8.39) is valid in the whole region $\mathcal{K} = \{R_i \le R < +\infty\}$, so that, in particular, we may integrate in the interval $R \in [R_{in}, +\infty)$ to find

$$\left|\log\mathcal{Y}_2 - \log\left(\frac{\Omega^2}{-\dot{r}}\left(\frac{r}{r_-}\right)^{-\alpha^{-2}}\right)(R_{in})\right| \lesssim \epsilon^2 + \int_{R_{in}}^{R_{out}} |\Psi(R) - \alpha^{-1}|\, dR. \tag{4.8.41}$$

We now compare this to the estimate (4.8.28) evaluated at $R = R_{in}$. As we are changing the exponent from α^{-2} to α^2, we generate an extra term on the left hand side:

$$\left|\log\mathcal{Y}_2 - \log\mathcal{Y}_1 - (\alpha^{-2} - \alpha^2)R_{in}\right| \lesssim \epsilon^2 + \int_{R_{in}}^{R_{out}} |\Psi(R) - \alpha^{-1}|\, dR. \tag{4.8.42}$$

Now we appeal to Lemma 4.7.5. By this lemma, we know that $|R_{out} - R_{in}| \lesssim \log(\epsilon^{-1})$, so the integral on the RHS is $O(\log(\epsilon^{-1}))$. Using also (4.7.47) to estimate R_{in} and the estimate (4.8.27) for $\log\mathcal{Y}_1$, we find that

$$\left|\log\mathcal{Y}_2 - \frac{1}{2}\alpha^{-2}b_-^{-2}\epsilon^{-2}\right| \lesssim \log(\epsilon^{-1}). \tag{4.8.43}$$

This completes the proof of the lemma. □

4.8.4 Kasner-like asymptotics in synchronous coordinates in both cases

To complete the proofs of Theorems 4.8.1 and 4.8.2, we simply need to use the estimates of Lemmas 4.8.4, 4.8.5 and 4.8.6 to put the metric $\mathbf{g}$ in the Kasner-like form stated. The following lemma will justify the change of coordinates.

Lemma 4.8.7. *Let $(\mathcal{M}, \mathbf{g})$ be a spherically symmetric spacetime with metric (2.2.1). Defining the coordinates $s = u + v$, $t = v - u$, suppose further that $T = \partial_t = \frac{1}{2}(\partial_v - \partial_u)$ is a Killing field for the metric, i.e. $r(s)$ and $\Omega^2(s)$ are functions of s, and that we are in a trapped region with $\dot{r}(s) < 0$.*

Suppose that for some interval $J \subset \mathbb{R}$, there exist constants $\mathcal{Y}, \mathcal{Z} > 0$ and an exponent $\gamma \geq 0$, as well as sufficiently small "lower-order terms" $\mathfrak{E}(s)$, such that we have the following asymptotics for the expressions $\frac{\Omega^2}{-\dot{r}}$ and $-r\dot{r}$ when $s \in J$:

$$\frac{\Omega^2}{-\dot{r}}(s) = \mathcal{Y} \cdot \left(\frac{r(s)}{r_-}\right)^{\gamma} \cdot (1 + \mathfrak{E}(s)), \tag{4.8.44}$$

$$-r\dot{r}(s) = \mathcal{Z} \cdot (1 + \mathfrak{E}(s)). \tag{4.8.45}$$

We quantify the required smallness of $\mathfrak{E}$ in the following way: there exists $\epsilon_* > 0$ small and a non-increasing function $\bar{\mathfrak{E}}(s)$ such that $|\mathfrak{E}(s)| \leq \bar{\mathfrak{E}}(s) \leq \epsilon_*$.

Upon defining (up to translation) the past-directed proper time coordinate τ such that

$$\frac{d\tau}{ds} = -\frac{\Omega(s)}{2}, \tag{4.8.46}$$

then there exist constants $\mathcal{X}$ and $\mathcal{R}$ depending on $\mathcal{Y}, \mathcal{Z}, \gamma, r_-$, and $\tau_0 \in \mathbb{R}$, such that one has the following asymptotics for Ω^2 and r^2 with respect to τ:

$$\Omega^2(\tau) = 4\mathcal{X} \cdot (\tau - \tau_0)^{\frac{2(\gamma-1)}{\gamma+3}} \cdot (1 + \mathfrak{E}_X(\tau)), \tag{4.8.47}$$

$$r^2(\tau) = \mathcal{R} \cdot r_-^2 \cdot (\tau - \tau_0)^{\frac{4}{\gamma+3}} \cdot (1 + \mathfrak{E}_R(\tau)). \tag{4.8.48}$$

Here, there exists some $C = C(\gamma) > 0$ such that $|\mathfrak{E}_X(\tau)|, |\mathfrak{E}_R(\tau)| \leq C\bar{\mathfrak{E}}(s(\tau))$.

In particular, the metric $\mathbf{g}$ can be written in the form

$$\mathbf{g} = -d\tau^2 + \mathcal{X} \cdot (\tau - \tau_0)^{\frac{2(\gamma-1)}{\gamma+3}} \cdot (1 + \mathfrak{E}_X(\tau))\, dt^2 + \mathcal{R} \cdot (\tau - \tau_0)^{\frac{4}{\gamma+3}} \cdot (1 + \mathfrak{E}_R(\tau)) \cdot r_-^2\, d\sigma_{\mathbb{S}^2}. \tag{4.8.49}$$

Finally, one can find the following relationships between $\mathcal{X}, \mathcal{R}$ and $\mathcal{Y}$:

$$\left|\log \mathcal{X} - \frac{2(\gamma+1)}{\gamma+3} \log \mathcal{Y}\right| \lesssim 1 + |\log \mathcal{Z}|, \tag{4.8.50}$$

$$\left|\log \mathcal{R} + \frac{2}{\gamma+3} \log \mathcal{Y}\right| \lesssim 1 + |\log \mathcal{Z}|. \tag{4.8.51}$$

Proof. In this proof we schematically write all lower-order terms as $\mathfrak{E}$, leaving the precise claims on these errors to the reader. The crucial estimate is to find an expression for τ in terms of r. Using (4.8.46), (4.8.44) and (4.8.45), we find

$$\frac{d\tau}{dr} = \frac{\Omega}{-2\dot{r}} = \frac{\mathcal{Y}^{1/2} r_-^{1/2}}{2\mathcal{Z}^{1/2}} \cdot \left(\frac{r}{r_-}\right)^{\frac{\gamma+1}{2}} \cdot (1 + \mathfrak{E}). \tag{4.8.52}$$

Although $\inf_{s \in J} r(s)$ is not necessarily 0, we may artificially extend the interval J to an interval $\tilde{J}$, and the functions $r(s), \tau(s)$ on $\tilde{J}$ so that $\inf_{s \in \tilde{J}} r(s) = 0$ and (4.8.52) holds for all $s \in \tilde{J}$. After doing so, we may normalize τ_0 so that $r(\tau_0) = 0$, and find that, for $s \in \tilde{J}$,

$$\tau - \tau_0 = \frac{\mathcal{Y}^{1/2} r_-^{3/2}}{(\gamma+3)\mathcal{Z}^{1/2}} \cdot \left(\frac{r}{r_-}\right)^{\frac{\gamma+3}{2}} \cdot (1 + \mathfrak{E}). \tag{4.8.53}$$

The remaining assertions are immediate upon combining (4.8.53) with the assumptions (4.8.44) and (4.8.45). □

Conclusion of the proof of Theorem 4.8.1 and 4.8.2. Theorem 4.8.1 follows immediately from combining Lemmas 4.8.4 and 4.8.7, taking $\tau_0 = 0$ (note indeed that, in the No Kasner bounce case $(***)$, (4.8.52) is true up to $\{r = 0\}$, so $\tau_0 = 0$). Here $\gamma = \alpha^2$, and in order to obtain the final error estimate (4.8.7), we also need (4.8.52) to change variables from r to τ.

For the proof of Theorem 4.8.2 in case $(**)$, we first run the proof in $\mathcal{K}_1$ and apply Lemma 4.8.5 to show that (4.8.44) and (4.8.45) with $\mathcal{Y} = \mathcal{Y}_1$, $\mathcal{Z} = \mathcal{Z}_1$, $\gamma = \alpha^2$ are true for $J = \mathcal{K}_1$. Unlike in case $(***)$, we are not able to continue these estimates all the way up to $r = 0$, so we take $\tau_0 \neq 0$ to account for the fact that the proper time variable with respect to which $\mathcal{K}_1$ is Kasner-like must be modified. Thus, we can apply Lemma 4.8.7 to deduce the Kasner-like behavior of $\mathcal{K}_1$ claimed in Theorem 4.8.2. On the other hand, Lemma 4.8.6 and Lemma 4.8.7, applied to $\mathcal{Y} = \mathcal{Y}_2$, $\mathcal{Z} = \mathcal{Z}_2$, $\gamma = \alpha^{-2}$ determine the Kasner-like behavior of $J = \mathcal{K}_2$ (where here we take $\tau = 0$).

Finally, we prove the proper time estimates (4.8.14) and (4.8.15). For the former, taking the logarithm of (4.8.53) in the context of the region $\mathcal{K}_2$ (where $\tau_0 = 0$) yields:

$$\log \tau = \frac{1}{2} \log \mathcal{Y}_2 - \frac{1}{2} \log \mathcal{Z}_2 + \frac{\alpha^{-2} + 3}{2} \log r + O(1).$$

In particular, evaluating this at $s = s_{out}$, then using Lemma 4.7.5 to estimate $r(s_{out}) = r_{out}$ yields

$$\log \tau^{-1}(s_{out}) = \frac{1}{4} \alpha^{-2} b_-^{-2} \epsilon^{-2} + \frac{\alpha^{-2} + 3}{4(1 - \alpha^2)} b_-^{-2} \epsilon^{-2} + O(\log(\epsilon^{-1})) = b_-^{-2} \frac{\alpha^{-2} + 1}{2(1 - \alpha^2)} + O(\log(\epsilon^{-1})).$$

For the final estimate (4.8.15), we shall use the fact that $\frac{d\tau}{dr} = \frac{\Omega}{-2\dot{r}}$ to find that

$$|\tau(s_{in}) - \tau(s_{out})| \leq (r_{in} - r_{out}) \cdot \max_{s \in \mathcal{K}_{bo}} \frac{\Omega}{-2\dot{r}} \leq r_{in}^{3/2} \cdot \max_{s \in \mathcal{K}_{bo}} \left(\frac{\Omega^2}{-4\dot{r}} \frac{1}{-r\dot{r}} \right)^{1/2}.$$

Now applying the estimate (4.7.48) and then (4.7.47) again (as $-r\dot{r} \sim \epsilon^2$ it is absorbed into the $\exp\big(D \log(\epsilon^{-1})\big)$ term in the next line), one obtains

$$\begin{aligned} |\tau(s_{in}) - \tau(s_{out})| &\leq (\frac{r_{in}}{r_-})^{\frac{3+(\alpha-\epsilon^2)^2}{2}} \cdot \exp\left(-\frac{1}{4} b_-^{-2} \epsilon^{-2} \right) \cdot \exp\big(D \log(\epsilon^{-1})\big), \\ &\leq \exp\left(-\frac{1}{4} b_-^{-2} \epsilon^{-2} \left[\frac{3 + \alpha^2}{1 - \alpha^2} + 1 \right] \right) \cdot \exp\big(D \log(\epsilon^{-1})\big), \end{aligned}$$

yielding (4.8.15) as required. This completes the proof of Theorem 4.8.2, and also Theorem 4.1.2.

Chapter 5

BKL bounces outside homogeneity: Gowdy symmetry

In this Chapter, we state and prove our main results concerning *BKL bounces outside homogeneity in Gowdy symmetry*, based on our recent arXiv preprint [100]. We will be making frequent references to the $\mathbb{T}^3$ Gowdy symmetric spacetimes introduced in Section 2.6.

5.1 Statement of our main results and overview

Let us recall the global existence result of Proposition 2.6.1, and the remarks following it. To start relating the general Gowdy framework to the heuristics of BKL in Section 1.2, the key issue is to understand the asymptotic behaviour of (P, Q) in the direction of the singularity, i.e. as $t \to 0$. For instance, one would like to understand the asymptotic behaviour of the functions $P(t, \theta)$ and $Q(t, \theta)$ in the limit $t \to 0$. By considering a (Fuchsian) series expansion at $t = 0$, i.e. an asymptotic expansion whose terms are powers of t with coefficients depending on θ, one could conjecture that, e.g. from [84]:

$$P(t, \theta) = -V(\theta) \log t + \Phi(\theta) + o(1), \tag{5.1.1}$$

$$Q(t, \theta) = q(\theta) + t^{2V(\theta)}[\Psi(\theta) + o(1)], \tag{5.1.2}$$

where $V, \Phi, q, \Psi : \mathbb{S}^1 \to \mathbb{R}$ are regular functions of θ. Upon substituting into the equations (2.6.2)–(2.6.3), one sees that these expansions are self-consistent, at least under the assumption that for each $\theta \in \mathbb{S}^1$, one of the following holds:

(i) The coefficient $V(\theta)$ satisfies $0 < V(\theta) < 1$, or

(ii) $V(\theta) > 0$, but $\partial_\theta q(\theta) = 0$, or

(iii) $V(\theta) < 1$, but $\Psi(\theta) = 0$.

Note: in the subclass of polarized Gowdy spacetimes from Definition 2.6.1, where the equations reduce to the single linear wave equation (2.6.7), any value of $V(\theta)$ is permitted.

Deriving asymptotics of the form (5.1.1)–(5.1.2) was a crucial step in Ringström's proof of the Strong Cosmic Censorship for Gowdy symmetric spacetimes [37, 121, 123]. In particular, for our applications we shall make use of the following, see [37, Step 1] in the polarized case and [121, Proposition 1.5] for the unpolarized case but with with the assumption $0 < V(\theta) < 1$.

Theorem 5.1.0. *Let (P, Q) be a solution to the Gowdy symmetric wave map system (2.6.2)–(2.6.3), arising from regular initial data $(P, Q, t\partial_t P, t\partial_t Q)|_{t=t_0} \in (C^{k+1})^2 \times (C^k)^2$ for some $k \geq 1$. Then for all $\theta_0 \in \mathbb{S}^1$, there exists some $V(\theta_0) \in \mathbb{R}$ such that $-t\partial_t P(t, \theta_0) \to V(\theta_0)$ as $t \to 0$, in other words $-t\partial_t P(t, \theta) \to V(\theta)$* pointwise.

Suppose moreover that either $Q \equiv 0$ i.e. the solution lies in the polarized Gowdy subclass, or we assume a priori that $0 < V(\theta) < 1$ for all $\theta \in \mathbb{S}^1$. Then $V(\theta)$ is C^{k-1} and moreover there exist C^{k-1} functions $\Phi(\theta)$, $q(\theta)$ and $r(\theta)$ such that the following limits hold uniformly in the C^{k-1} norm as $t \to 0$:

$$-t\partial_t P(t, \theta) \to V(\theta), \qquad P(t, \theta) + V(\theta) \log t \to \Phi(\theta), \tag{5.1.3}$$

$$e^{2P} t\partial_t Q(t, \theta) \to r(\theta), \qquad e^{2P}(Q(t, \theta) - q(\theta)) \to \frac{r(\theta)}{2V(\theta)}. \tag{5.1.4}$$

Note that these imply (5.1.1) and (5.1.2), given $\Psi(\theta) = \frac{r(\theta)}{2V(\theta)} e^{-2\Phi(\theta)}$.

Under the same assumptions, if $\lambda(t, \theta)$ solves the remaining equations (2.6.4)–(2.6.5), then

$$\lambda(t, \theta) = V^2(\theta) \log t + L(\theta) + o(1), \tag{5.1.5}$$

where $L(\theta)$ obeys the asymptotic momentum constraint $dL = -2V \cdot d\Phi$.

In a context where (5.1.3)–(5.1.5) hold, a straightforward change of coordinates from the gauge (2.6.1) implies that the metric behaves in a *Kasner-like fashion* near the $t = 0$ singularity and moreover one can describe the *Kasner exponents* by

$$p_1(\theta) = \frac{V^2(\theta) - 1}{V^2(\theta) + 3}, \quad p_2(\theta) = \frac{2 - 2V(\theta)}{V^2(\theta) + 3}, \quad p_3(\theta) = \frac{2 + 2V(\theta)}{V^2(\theta) + 3}. \tag{5.1.6}$$

These exponents depend on θ, but always satisfy the (vacuum) Kasner relations (1.1.4).

We remark that in [121, 123], Ringström proves much stronger statements than Theorem 5.1.0. For instance, he shows that for *any* $\theta_0 \in \mathbb{S}^1$ with $0 < V(\theta_0) < 1$, then there exists a neighborhood I of θ_0 in $\mathbb{S}^1$ such that $V(\theta)$ is C^{k-1} in $I \subset \mathbb{S}^1$ and moreover the limits (5.1.3)–(5.1.5) hold when restricted to $\theta \in I$. Furthermore, it is proven that for *generic*[1] smooth initial data, $0 < V(\theta) < 1$ for all $\theta \in \mathbb{S}^1 \setminus S$, where S is a finite set of points known as "spikes". See the review article [124] for details.

[1]Here generic means open and dense in the C^∞-topology on initial data sets $(P, Q, t\partial_t P, t\partial_Q)|_{t=t_0}$.

In this chapter of our dissertation, our main concern is not understanding Strong Cosmic Censorship in full generalirt, nor the issue of spikes, but instead to address the questions posed in Page 37. In the current section, we plan to state the precise version of Theorem 1.5.2 and Theorem 1.5.3 succeeding these questions. The first point of call is to introduce a more precise version of the three conditions (1) *Weak subcriticality*, (2) *Energy boundedness* and (3) *Closeness to singularity* introduced there.

5.1.1 Setup of the initial data

Notation and key parameters

We first introduce various parameters $\gamma, \gamma', \gamma'', \zeta, N, \delta, \dots$ that appear throughout.

- The real number $0 < \gamma < 1$ captures the size of $-t\partial_t P$ allowable in our results. Many of the subsequent parameters will depend on γ, as do the constants appearing in our quantitative estimates. We further define γ', γ'' to be related constants that satisfy

$$0 < \frac{\gamma}{2} < \gamma' < \gamma < \gamma'' < 1. \tag{5.1.7}$$

- The natural number N represents the largest number of ∂_θ-derivatives with which we commute the Gowdy system (2.6.2)–(2.6.3) in deriving energy estimates. Our choice of N depends on $\gamma, \gamma', \gamma''$, and we expect $N \to \infty$ as $\gamma \to 0$. We often use K to denote an integer with $0 \leq K \leq N$.

- The real number $\zeta > 0$ represents the maximal admissible size of initial data in the L^2 sense. See already the initial data bounds (5.1.11)–(5.1.12) below.

- The real number $C_* > 0$ is used in the bootstrap argument. We choose C_* depending on γ, γ', γ'' and ζ, though we do not make this explicit. The number $A_* > 0$ will depend on C_* and represents the rate at which our energies may blow up.

- The real number $t_* > 0$ represents how close to the $t = 0$ singularity our initial data is required to be in order to obtain our results. That is, our results apply for initial data at $t = t_0$ for $0 < t_0 \leq t_*$. Our choice of t_* will depend on γ and ζ and we expect $t_* \to 0$ as either $\gamma \to 0$ or $\zeta \to \infty$.

- Other constants, often denoted C, will be allowed to depend on all of aforementioned parameters. The notation δ will be used to represent quantities depending on all of these parameters (e.g. γ, ζ, t_*) such that $\delta \downarrow 0$ as $t_* \downarrow 0$. We often allow abuse of notation such as "$\delta + \delta = \delta$" or "$C_*\delta = \delta$" etc.

Initial L^∞ and L^2 bounds

For the Gowdy symmetric system (2.6.2)–(2.6.3), initial data is given for (P, Q) and their $t\partial_t$–derivatives at $\{t = t_0\}$ for some $0 < t_0 \leq t_*$. That is, we let $P_D, Q_D, \dot{P}_D, \dot{Q}_D : \mathbb{S}^1 \to \mathbb{R}$ with

$$(P_D, Q_D, \dot{P}_D, \dot{Q}_D) = (P, Q, t\partial_t P, t\partial_t Q)|_{t=t_0} \tag{5.1.8}$$

We characterize an open set $\mathcal{O}_{\gamma,\zeta}$ (perhaps more precisely $\mathcal{O}_{\gamma,\gamma',\gamma'',\zeta}$) to which our results apply. Initial data $(P_D, Q_D, \dot{P}_D, \dot{Q}_D)$ in this open set will obey the following pointwise bounds:

$$-\dot{P}_D \geq \gamma'', \quad (1 - \dot{P}_D)^2 + (e^{P_D} t_0 \partial_\theta Q_D)^2 \leq (1 - \gamma'')^2 \tag{5.1.9}$$

$$e^{P_D} \dot{Q}_D \leq \zeta t_0^\gamma, \quad e^{P_D} \leq \zeta t_0^\gamma, \tag{5.1.10}$$

as well as the following L^2 energy bounds at $t = t_0$, where N depends on γ and $\zeta > 0$:

$$\frac{1}{2} \sum_{K=0}^{N} \left(\int_{\mathbb{S}^1} (\partial_\theta^K \dot{P}_D)^2 + (t_0 \partial_\theta^{K+1} P_D)^2 + (t_0^\gamma \partial_\theta^K P_D)^2 \, d\theta \right) \leq \zeta, \tag{5.1.11}$$

$$\frac{1}{2} \sum_{K=0}^{N} \left(\int_{\mathbb{S}^1} e^{2P_D} \left((\partial_\theta^K \dot{Q}_D)^2 + (t_0 \partial_\theta^{K+1} Q_D)^2 \right) \, d\theta \right) \leq \zeta. \tag{5.1.12}$$

Finally, one assumes $0 < t_0 \leq t_*$, with t_* is chosen small depending on $\gamma, \zeta, \gamma', \gamma''$.

5.1.2 Theorem 5.1.1 – Stability

Theorem 5.1.1 is the precise version of our stability result, see the rough version Theorem 1.5.2.

Theorem 5.1.1 (Stability)**.** *Consider initial data $(P_D, Q_D, \dot{P}_D, \dot{Q}_D)$ for the Gowdy symmetric system (2.6.2)–(2.6.3), which obeys (5.1.9), (5.1.10), (5.1.11), (5.1.12) with $0 < t_0 \leq t_*$. Then for N sufficiently large depending on γ and for t_* chosen sufficiently small, depending on γ, N and ζ, for $0 < t \leq t_0$ we have the following L^∞ bounds:*

$$\gamma \leq -t\partial_t P \leq 2 - \gamma, \quad |e^P t\partial_\theta Q| \leq 1 - \gamma. \tag{5.1.13}$$

There exists $C = C(\gamma, \gamma', \gamma'', \zeta)$ and $A_* = A_*(\gamma)$ such that one has the L^2 energy bound:

$$\sum_{K=0}^{N} \int_{\mathbb{S}^1} (t\partial_t \partial_\theta^K P)^2 + t^2 (\partial_\theta^{K+1} P)^2 + (\partial_\theta^K P)^2 + e^{2P} (t\partial_t \partial_\theta^K Q)^2 + e^{2P} t^2 (\partial_\theta^{K+1} Q)^2 \, d\theta \leq C t^{-2A_* - 2K(1-\gamma')}. \tag{5.1.14}$$

Furthermore, there exists $V : \mathbb{S}^1 \to \mathbb{R}$ bounded so that $-t\partial_t P(t, \theta) \to V(\theta)$ pointwise as $t \to 0$. Finally for $k = 1$ whenever $\gamma > \frac{1}{3}$ and $k = 0$ otherwise, there exists a C^k function $q : \mathbb{S}^1 \to \mathbb{R}$ so that $Q(t, \theta) \to q(\theta)$ as $t \to 0$ in the C^k topology.

5.1.3 Theorem 5.1.2 – BKL bounces

In Theorem 5.1.2, we make precise our bounce result, see the rough version in Theorem 5.1.2. The idea is that certain quantities will obey a system of nonlinear ODEs, plus error terms, and that one may use properties of the ODEs to understand certain aspects of the dynamics.

Theorem 5.1.2 (Bounces). *Let (P, Q) be a solution to the Gowdy symmetric system (2.6.2)–(2.6.3), arising from initial data $(P_D, Q_D, \dot{P}_D, \dot{Q}_D)$ obeying (5.1.9)–(5.1.12) and $0 < t_0 \leq t_*$ as in Theorem 5.1.1.*

Let $\gamma : (0, t_0] \to (0, t_0] \times \mathbb{S}^1$ be a timelike curve parameterized by the t-coordinate, and let $\mathcal{P}_\gamma(t) = -t\partial_t P(\gamma(t))$ and $\mathcal{Q}_\gamma(t) = e^P t\partial_\theta Q(\gamma(t))$. Then there exist error terms $\mathcal{E}_\mathcal{P}(t), \mathcal{E}_\mathcal{Q}(t)$ depending on γ but with $|\mathcal{E}_\mathcal{P}(t)|, |\mathcal{E}_\mathcal{Q}(t)| \leq t^{\boldsymbol{\gamma}'}$ uniformly in the choice of γ, such that

$$t\frac{d}{dt}\mathcal{P}_\gamma = \mathcal{Q}_\gamma^2 + \mathcal{E}_\mathcal{P}, \qquad t\frac{d}{dt}\mathcal{Q}_\gamma = (1 - \mathcal{P}_\gamma)\mathcal{Q}_\gamma + \mathcal{E}_\mathcal{Q}. \tag{5.1.15}$$

Since γ is timelike there exists $\theta_0 \in \mathbb{S}^1$ such that $\gamma(t) \to (0, \theta_0)$ as $t \to 0$. Further, $\mathcal{Q}_\gamma(t)$ converges to 0 as $t \to 0$, while $\mathcal{P}_\gamma(t)$ converges to $\mathcal{P}_{\gamma,\infty} = V(\theta_0)$. Finally, there exists $C = C(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \boldsymbol{\gamma}'', \zeta) > 0$ such that:

(i) If the $\lim_{t\to 0} \partial_\theta Q(\gamma(t))$ does not converge to 0, then necessarily $V(\theta_0) = \mathcal{P}_{\gamma,\infty} \leq 1$ and moreover

$$|\mathcal{P}_{\gamma,\infty} - \min\{\mathcal{P}_\gamma(t_0), 2 - \mathcal{P}_\gamma(t_0)\}| \leq C \cdot (t_0^{\boldsymbol{\gamma}'} + \mathcal{Q}_\gamma^2(t_0))^{1/2}. \tag{5.1.16}$$

(ii) Let $\boldsymbol{\gamma} > \frac{1}{2}$. By Theorem 5.1.1, $\partial_\theta q(\theta_0)$ exists, and $\lim_{t\to 0} \partial_\theta Q(\gamma(t)) \to \partial_\theta q(\theta_0)$. If $\partial_\theta q(\theta_0) = 0$, then one instead has:

$$|\mathcal{P}_{\gamma,\infty} - \mathcal{P}_\gamma(t_0)| \leq C \cdot t_0^{2\boldsymbol{\gamma}'-1}. \tag{5.1.17}$$

where $\frac{1}{2} < \boldsymbol{\gamma}' < \boldsymbol{\gamma}$ is chosen appropriately.

A corollary of Theorem 5.1.2 is the following stability / instability statement regarding *nonpolarized* perturbations of a class of *polarized* Gowdy solutions.

Corollary 5.1.3 (Stability / Instability). *Let $P(t, \theta)$ be a smooth solution to the polarized Gowdy equation (2.6.7), arising from initial data given by $(P_D, \dot{P}_D)$ at $t = t_1 > 0$. By Theorem 5.1.0, there exists smooth $V, \Phi : \mathbb{S}^1 \to \mathbb{R}$ such that $P(t, \theta) = -V(\theta)\log t + \Phi(\theta) + o(1)$ as $t \to 0$.*

Suppose that $0 < V(\theta) < 2$ for all $\theta \in \mathbb{S}^1$. Then there exists $N \in \mathbb{N}$ such that for (possibly nonpolarized) perturbations of this data with $\|(\tilde{P}_D, \tilde{Q}_D, \dot{\tilde{P}}_D, \dot{\tilde{Q}}_D) - (P_D, 0, \dot{P}_D, 0)\|_{(H^{N+1})^2\times(H^N)^2} \leq \varepsilon$, then for ε sufficiently small the perturbed solution still has $-t\partial_t\tilde{P}(t, \theta) \to \tilde{V}(\theta)$ pointwise as $t \to 0$, for $\tilde{V}$ satisfying $0 < \tilde{V}(\theta) < 2$.

Next, suppose further that $\frac{1}{2} < V(\theta) < \frac{3}{2}$ for the original polarized solution. Then for perturbations as above, we moreover have that $\partial_\theta\tilde{Q}(t, \theta) \to \partial_\theta\tilde{q}(\theta)$ uniformly for some C^1 function $\tilde{q} : \mathbb{S}^1 \to \mathbb{R}$. Furthermore, for any $\tilde{\varepsilon} > 0$, ε may be chosen small enough depending on $\tilde{\varepsilon}$ such that:

(i) $|\tilde{V}(\theta) - \min\{V(\theta), 2 - V(\theta)\}| \leq \tilde{\varepsilon}$ *if* $\partial_\theta \tilde{q}(\theta) \neq 0$, *while*

(ii) $|\tilde{V}(\theta) - V(\theta)| \leq \tilde{\varepsilon}$ *if* $\partial_\theta \tilde{q}(\theta) = 0$.

We consider Corollary 5.1.3 a stability / instability result in the sense that while the perturbed spacetime retains the spacelike singularity and curvature blowup, in the case where the original $V(\theta)$ has $1 < V(\theta_0) < 2$ for some $\theta_0 \in \mathbb{S}^1$ a generic unpolarized perturbation will have a corresponding $\hat{V}(\theta_0)$ with $\hat{V}(\theta_0) \approx 2 - V(\theta_0)$, which is "far away" from $V(\theta_0)$.

Our result applies in particular when the background unperturbed spacetime is an exact Kasner spacetime with $P(\theta) = -V \log t$ where $0 < V < 2$ (and Kasner exponents given by (5.1.6)). When $1 < V < 2$, this gives a precise instability mechanism for a certain range of Kasner exponents. See also Figure 5.1. Note our methods do not allow us to access this mechanism outside this range, especially regarding spatially inhomogeneous perturbations.

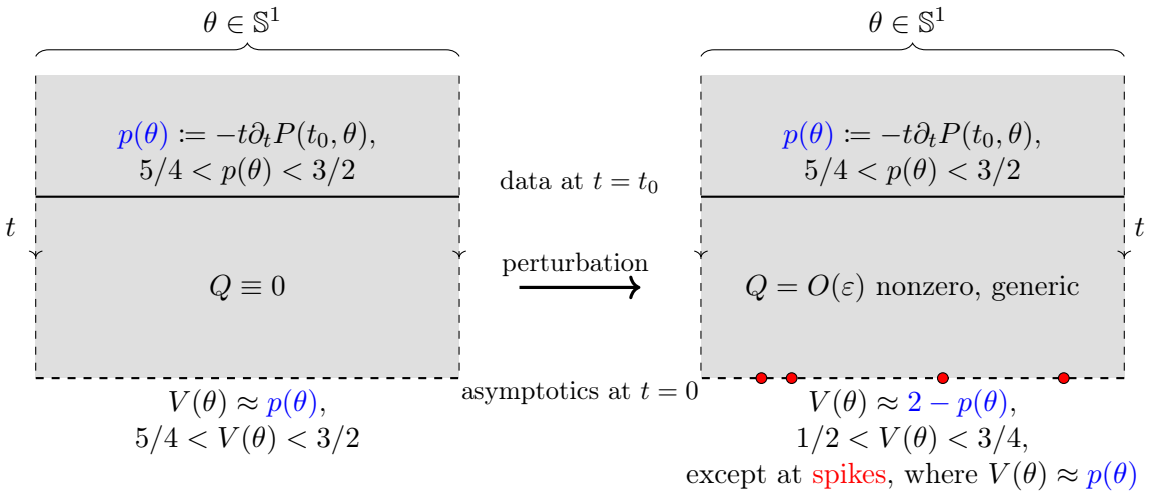


Figure 5.1: Diagram illustrating an example of a global bounce instability arising from a non-polarized perturbation of a polarized Gowdy spacetime. In the unperturbed spacetime (left) $Q \equiv 0$ and the asymptotic quantity $V(\theta)$ is close to the (inhomogeneous) data $p(\theta)$, while in the perturbed spacetime (right) with generic $Q = O(\varepsilon)$ the bounce of Theorem 5.1.2(i) applied to constant θ-curves γ means that $V(\theta)$ is instead close to $2 - p(\theta)$, except at finitely many points representing spikes.

Our instability is triggered when $\partial_\theta q(\theta) \neq 0$, while if $\partial_\theta q(\theta) = 0$ and $\frac{1}{2} < V(\theta) < \frac{3}{2}$ our result suggests that the instability is suppressed. Note that in light of [123] it is true that for an open and dense subset of perturbations, the instability is indeed triggered at all but finitely many $\theta \in \mathbb{S}^1$; at the remaining points we leave open the possibility of spikes.

5.1.4 Future research directions

Before moving to a sketch of the proof, we propose two conjectures which would go beyond a *single* BKL bounce, but remains in the 1+1-dimensional setting. The first conjecture still concerns Gowdy

symmetry but with multiple bounces. Note that this would mean exiting the low-velocity regime with $0 < V(\theta) < 2$.

Conjecture 5.1.4. *There exists a large, open class of initial data for the Gowdy symmetric system which exhibit any finite number of BKL bounces.*

One expects that this would require a different choice of gauge from (2.6.1), since the results of this paper suggest that ∂_θ may not be the correct "spatial derivative" outside of the low-velocity regime; see already Section 5.1.5. To go beyond a finite number of bounces to a possibly infinite number, one needs to move beyond Gowdy symmetry, and into the realm of more general $\mathbb{T}^2$-symmetric spacetimes, see [18] for definitions.

Conjecture 5.1.5. *One can describe initial data for the Einstein vacuum equations in $\mathbb{T}^2$ symmetry which exhibit an infinite number of BKL bounces.*

The reason to study $\mathbb{T}^2$ symmetry is that the symmetry assumption no longer enforces that any of the $\omega^I(x)$ are integrable in the sense of Frobenius, thus giving the potential for infinite bounces. Note both Conjectures 5.1.4 and 5.1.5 are supported by the heuristic and numerical investigations of [19]; this article also suggests possibilities for the ODE system and corresponding nonlinear orbits causing the bounces.

5.1.5 Sketch of the proof

The proofs of the stability result Theorem 5.1.1 and the bounce result Theorem 5.1.2 will be done in tandem and there are three major steps, which we briefly sketch here. Recalling the main evolution equations (2.6.2)–(2.6.3), we caricature these steps as follows:

Step 1: **Analysis in the spatially homogeneous case:** In this step, we ignore certain terms in (2.6.2)–(2.6.3) involving ∂_θ-derivatives which one expects to become negligible as $t \to 0$. We do, however, keep terms involving the expression $e^P t\partial_\theta Q$. The result is a nonlinear ODE system for the quantities $-t\partial_t P$ and $e^P t\partial_\theta Q$; one then analyzes this system to determine bounds for these quantities.

Step 2: **Linearization of the ODE system:** In the second step, we consider the *best possible behaviour* of the terms involving further ∂_θ-derivatives i.e. the non-spatially homogeneous corrections. We do this by taking commuting ∂_θ with the ODE system in Step 1, resulting in a new *linear* ODE system for new quantities whose coefficients are given by the solution (i.e. some orbit) of the ODE system in Step 1.

Step 3: **Energy estimates:** Finally, one must ensure that we can close our argument without loss of derivatives (necessary due to the existence of top order terms such as $\partial_\theta^2 P$ in (2.6.2).)

One achieves this via L^2 energy estimates, which are allowed to blow up but only at a controlled rate as $t \to 0$.

We now explain in a little more detail how these steps apply to the Gowdy symmetric system. For Step 1, with $\mathcal{P} = -t\partial_t P$ and $\mathcal{Q} = e^P t\partial_\theta Q$, the spatially homogeneous ODE system one gets is:

$$t\partial_t \mathcal{P} = \mathcal{Q}^2, \qquad t\partial_t \mathcal{Q} = (1-\mathcal{P})\mathcal{Q}. \tag{5.1.18}$$

(Eventually we also include another ODE variable $\mathcal{R} = e^P t\partial_t Q$ but we ignore $\mathcal{R}$ this for now.) The upshot is that one can study the 2-dimensional ODE system (5.1.18) via phase plane analysis, as we see in Figure 5.2 below. The direction of the arrows in Figure 5.2 is with respect to $t \downarrow 0$.

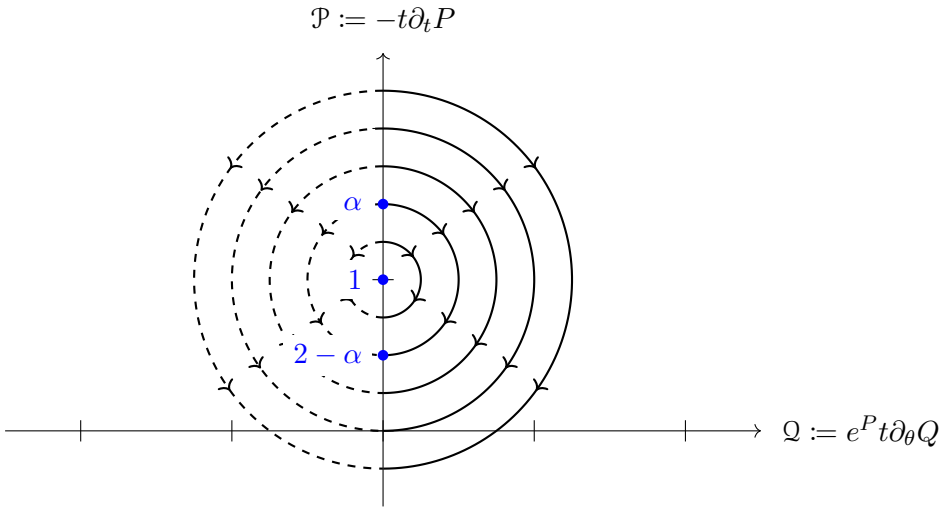


Figure 5.2: Phase portrait showing the dynamics of $\mathcal{P}$ and $\mathcal{Q}$ towards $t = 0$ in the spatially homogeneous case. A bounce is a transition from α to $2-\alpha$.

The ODE system (5.1.18) contains a line of fixed points at $\mathcal{Q} = 0$. Each of these fixed points represents an exact (vacuum) Kasner solution with exponents p_I as in (5.1.6) where $V(\theta)$ is constant and equal to the value of $\mathcal{P}$ at the fixed point. These fixed points are (orbitally) stable as $t \downarrow 0$ if $\mathcal{P} \leq 1$ and unstable otherwise.

The dynamics of the ODE system in the remaining region $\mathcal{Q} \neq 0$ are then described by *heteroclinic orbits* linking an unstable fixed point to a stable fixed point. In fact, one may solve the system exactly using a conserved quantity $\mathcal{K} = (\mathcal{P}-1)^2 + \mathcal{Q}^2$. Thus the heteroclinic orbits, which we identify as BKL bounces, link the unstable fixed point $(\mathcal{P} = \alpha, \mathcal{Q} = 0)$ to the stable fixed point $(\mathcal{P} = 2-\alpha, \mathcal{Q} = 0)$, for any $\alpha > 1$.

Now, the condition of weak subcriticality together with the conserved quantity $\mathcal{K} = (\mathcal{P}-1)^2 + \mathcal{Q}^2$ means that any $(\mathcal{P}, \mathcal{Q})$-orbit associated to $\mathcal{O}_{\gamma,\zeta}$ remains in the bounded region $\{\mathcal{K} \leq (1-\gamma)^2\}$. This will be essential in Steps 2 and 3.

In Step 2, we commute the ODEs (5.1.18) with ∂_θ. This yields:

$$t\partial_t(\partial_\theta \mathcal{P}) = 2\mathcal{Q}\partial_\theta \mathcal{Q}, \qquad t\partial_t(\partial_\theta \mathcal{Q}) = -\mathcal{Q}\partial_\theta \mathcal{P} + (1-\mathcal{P})\partial_\theta \mathcal{Q}. \tag{5.1.19}$$

We see this is a *linear* ODE system for $\partial_\theta \mathcal{P}$ and $\partial_\theta \mathcal{Q}$ whose coefficients are functions of the dynamical $\mathcal{P}$ and $\mathcal{Q}$ orbit from Step 1. The fact that the orbit remains in the bounded region $\{\mathcal{K} \leq (1-\gamma)^2\}$ allows one to prove, using the linear system (5.1.19), that $|\partial_\theta \mathcal{P}| + |\partial_\theta \mathcal{Q}| = O(t^{-(1-\gamma)})$.

So these ∂_θ-derivatives are permitted to blow up as $t \to 0$, but at a controlled rate. This blow-up rate is sharp in the following sense: if our $(\mathcal{P}, \mathcal{Q})$-orbit is a constant orbit at an unstable fixed point with $\mathcal{P} = 2 - \gamma - \varepsilon$ with ε small then generic solutions of (5.1.19) have $\partial_\theta \mathcal{Q} = O(t^{-(1-\gamma)+\varepsilon})$. This choice of $(\mathcal{P}, \mathcal{Q})$-orbit is related to spikes; it is unsurprising that near spikes certain ∂_θ-derivatives will blow up.

But in any case, the outcome of Step 2 is that heuristically, each ∂_θ-derivative comes with a loss of $t^{-(1-\gamma)}$, for instance one expects $\partial_\theta^2 P = O(t^{-2(1-\gamma)})$; this confirms the AVTD expectation that ∂_θ-derivatives cost fewer powers of t than ∂_t-derivatives (note this requires $\gamma > 0$ and is exactly where we require that our spacetimes are "low-velocity" with $0 < \mathcal{P} < 2$). This is good news, since in (2.6.2) the term $t^2 \partial_\theta^2 P$ will be $O(t^{2\gamma})$, and thus negligible as $t \to 0$. So throwing this term away to derive the first ODE of (5.1.18) in Step 1 is justifiable, at least at the heuristic level. Similar arguments hold for the other terms we threw away in Steps 1 and 2.

Step 3 concerns making the argument of the above paragraph rigorous. In particular, we must overcome the issue of *derivative loss* due to terms such as $\partial_\theta^2 P$. This is achieved via energy estimates; let $\mathcal{E}^{(K)}(t)$ represent an L^2-based energy which resembles (2.6.6) but controlling $\partial_\theta^K P$ and $\partial_\theta^K Q$ in place of P and Q. We explain the main energy estimate: commuting (2.6.2) with ∂_θ^K yields the following

$$(t\partial_t)^2 \partial_\theta^K P - (t\partial_\theta)^2 \partial_\theta^K P = \overbrace{2e^{2P}\left(t\partial_t Q \cdot t\partial_t \partial_\theta^K Q - t\partial_\theta Q \cdot t\partial_\theta^{K+1} Q\right)}^{\text{(I)}} + \underbrace{2e^{2P}\left((t\partial_t Q)^2 - (t\partial_\theta Q)^2\right)\partial_\theta^K P}_{\text{(II)}} + \underbrace{\text{lower order quantities}}_{\text{(III)}}.$$

A standard argument will then yield the differentiated energy estimate:

$$\left| t\frac{d}{dt}\mathcal{E}^{(K)}(t) \right| \leq \sqrt{2\mathcal{E}^{(K)}(t)} \cdot \left(\|\text{(I)} + \text{(II)} + \text{(III)}\|_{L^2} \right).$$

It remains to estimate each of (I), (II) and (III) in L^2. Here we need to use Step 1 and Step 2, the key observation being that $e^P t\partial_\theta Q$, represented by $\mathcal{Q}$ in Step 1, is bounded by $1 - \gamma$. It turns out that $e^P t\partial_t Q$ is similarly bounded, and thus for appropriately defined $\mathcal{E}^{(K)}(t)$ the terms (I) and (II) may be bounded by:

$$\|\text{(I)}\|_{L^2} + \|\text{(II)}\|_{L^2} \leq A_* \sqrt{2\mathcal{E}_\phi^{(K)}(r)},$$

where A_* is a constant only depending on γ.

For the remaining lower order terms (III), we will also have to use Step 2. To simplify the exposition, in this sketch we replace (III) by just the single term $c \cdot \partial_\theta(e^P t\partial_\theta Q) \cdot e^P t\partial_\theta^K Q$, where c may

depend on K. One can think of this as $c \cdot \partial_\theta \mathfrak{Q} \cdot e^P t \partial_\theta^K Q$, and thus via Step 2 is $O(t^{-(1-\gamma)}) \cdot e^P t \partial_\theta^K Q$. This seems alarming at first sight, since this $O(t^{-(1-\gamma)})$ prefactor is not integrable with respect to $\frac{dt}{t}$ as $t \to 0$. But $e^P t \partial_\theta^K Q$ is not dependent on the Kth order energy $\mathcal{E}^{(K)}(t)$, but instead $\mathcal{E}^{(K-1)}(t)$. So

$$\|(\mathrm{III})\|_{L^2} \leq C_{\gamma,K}\, t^{-(1-\gamma)} \sqrt{2\mathcal{E}^{(K-1)}(t)}.$$

Our eventual estimate for (III) will also involve $\mathcal{E}^{(k)}(t)$ for all $k < K$, see already Propositions 5.3.1 and 5.3.3.

But sticking to our simplified (III), combining all of the above, as well as combining with a similar energy estimate using the Q-wave equation (2.6.3), yields

$$\left| t \frac{d}{dt} \mathcal{E}^{(K)}(t) \right| \leq 2A_* \mathcal{E}^{(K)}(t) + 2C_{\gamma,K}\, t^{-(1-\gamma)} \sqrt{\mathcal{E}^{(K)}(t)} \cdot \sqrt{\mathcal{E}^{(K-1)}(t)}. \tag{5.1.20}$$

When $K = 0$, the last term on the right hand side is absent. Because of the $2A_*$, even the 0th order energy $\mathcal{E}^{(0)}(t)$ will blow up as t^{-2A_*} as $t \to 0$. Further, the appearance of $t^{-(1-\gamma)}$ means that as K increases the rate of blow up also increases. Indeed, one can use (5.1.20) to show:

$$\mathcal{E}^{(K)}(t) \leq D_{\gamma,K,\zeta}\, t^{-2A_* - 2K(1-\gamma)},$$

where $D_{\gamma,K,\zeta}$ depends on γ and the data, as well as the regularity index K.

It is crucial that A_* is *independent of* K. The reason for this is that one now applies an interpolation argument together with the above, much like what is done in [63], to show e.g. that $\partial_\theta^2 P = O(t^{2(1-\gamma)-\delta})$, with $\delta \to 0$ as $N \to \infty$, where N is the maximum regularity index for which one applies the energy estimate. This means that one may interpret the ignored terms in Step 1 and Step 2 as negligible errors, as claimed.

To apply Steps 1 to 3 to the nonlinear problem, one uses a standard bootstrap argument. In our proof, we shall actually perform Step 3 first. That is, one bootstraps L^∞ bounds on 0th order and 1st order quantities, see (B1)–(B4), and uses these to derive the energy estimate (5.1.20). Then one combines this estimate, an interpolation argument, and the ODE analysis of Steps 1 and 2, to improve the bootstrap assumptions. This will yield the stability result Theorem 5.1.1, while the bounce result Theorem 5.1.2 follows from more detailed ODE analysis.

5.2 Bootstrap assumptions, energies and interpolation lemmas

5.2.1 The L^∞ bootstrap assumptions

As explained in Section 5.1.1, let $C_* = C_*(\gamma, \zeta) > 0$ be a large real number to be chosen later in the argument. We often make reference to the following four low order L^∞ bootstrap assumptions:

$$|t\partial_t P| \leq C_*, \quad |e^P t\partial_\theta Q| \leq C_*, \tag{B1}$$

$$|e^P t\partial_t Q|, |e^{-P}| \leq C_* t^{\gamma'}, \tag{B2}$$

$$|\partial_\theta P|, |e^P t\partial_\theta^2 Q| \leq C_* t^{-(1-\gamma)}, \tag{B3}$$

$$|t\partial_t\partial_\theta P|, |e^P t\partial_t\partial_\theta Q| \leq C_* t^{-(1-\gamma)}. \tag{B4}$$

5.2.2 Energies

Definition 5.2.1. Let $0 \leq K \leq N$. Define the following Kth order energies at fixed t:

$$\mathcal{E}_P^{(K)}(t) \coloneqq \frac{1}{2}\int_{\mathbb{S}^1} \left((t\partial_t\partial_\theta^K P)^2 + t^2(\partial_\theta^{K+1}P)^2 + (\partial_\theta^K P)^2\right) d\theta, \tag{5.2.1}$$

$$\mathcal{E}_Q^{(K)}(t) \coloneqq \frac{1}{2}\int_{\mathbb{S}^1} e^{2P}\left((t\partial_t\partial_\theta^K Q)^2 + t^2(\partial_\theta^{K+1}Q)^2\right) d\theta, \tag{5.2.2}$$

$$\mathcal{E}^{(K)}(t) \coloneqq \mathcal{E}_P^{(K)}(t) + \mathcal{E}_Q^{(K)}(t). \tag{5.2.3}$$

To recover asymptotics for $Q(t,\theta)$ without the e^P weight we also make use of the following energy:

$$\mathcal{E}_{Q,u}^{(K)}(t) \coloneqq \frac{1}{2}\int_{\mathbb{S}^1} \left((t\partial_t\partial_\theta^K Q)^2 + t^2(\partial_\theta^{K+1}Q)^2 + (\partial_\theta^K Q)^2\right) d\theta. \tag{5.2.4}$$

5.2.3 Sobolev–type inequalities

Lemma 5.2.1 (Sobolev interpolation inequality)**.** *Let N, K be integers with $0 \leq N < K$, and let $f : \mathbb{S}^1 \to \mathbb{R}$ be such that $\partial_x^K f \in L^2(\mathbb{S}^1)$. Then the following L^∞–L^2 interpolation inequality holds:*

$$\|\partial_x^N f\|_{L^\infty(\mathbb{S}^1)} \lesssim_{N,K} \|f\|_{L^\infty(\mathbb{S}^1)}^{1-\alpha}\|\partial_x^K f\|_{L^2(\mathbb{S}^1)}^{\alpha}, \qquad \text{where } \alpha = \frac{N}{K-\frac{1}{2}}. \tag{5.2.5}$$

Proof. This is standard, see for instance [111, Lecture II]. □

Lemma 5.2.2 (Weighted Sobolev product inequality)**.** *Let $M, N \geq 0$ be integers and let $K = M+N$. Let $w : \mathbb{S}^1 \to \mathbb{R}_{>0}$ be a weight function and let $W = \log w$. Then for f, g sufficiently regular*

one has

$$\|w\,\partial_x^M f\,\partial_x^N g\|_{L^2(\mathbb{S}^1)} \lesssim \|f\|_{L^\infty(\mathbb{S}^1)} \left(\|w\,\partial_x^K g\|_{L^2(\mathbb{S}^1)} + \sum_{j=1}^{K-1} \|\partial_x W\|_{L^\infty(\mathbb{S}^1)}^j \|w\,\partial_x^{K-j} g\|_{L^2(\mathbb{S}^1)} \right) + \|g\|_{L^\infty(\mathbb{S}^1)} \left(\|w\,\partial_x^K f\|_{L^2(\mathbb{S}^1)} + \sum_{j=1}^{K-1} \|\partial_x W\|_{L^\infty(\mathbb{S}^1)}^j \|w\,\partial_x^{K-j} f\|_{L^2(\mathbb{S}^1)} \right). \quad (5.2.6)$$

Proof. This is a generalization of [122, Lemma 6.16] where $\mathbb{R}^n$ is replaced by the unit circle $\mathbb{S}^1$ and we also introduce a weight function w. A complete proof of this lemma is provided in Appendix C. □

5.3 The energy estimate hierarchy

In this section, we derive energy estimates for $\mathcal{E}^{(K)}(t)$, at orders $0 \le K \le N$, where $N = N(\gamma)$ is chosen sufficiently large, by commuting the equations (2.6.2)–(2.6.3) with up to K ∂_θ-derivatives.

To handle "error" terms in the hierarchy (where the precise coefficients arising in the commuted equations are not crucial), it is useful to introduce the following schematic notation: expressions such as

$$\sum_{k_p+k_1+\ldots+k_i=K} \partial_\theta^{k_p} f * \partial_\theta^{k_1} g * \cdots * \partial_\theta^{k_i} g$$

will represent some linear combination of products of the form $\partial_\theta^{k_p} f \cdot \partial_\theta^{k_1} g \cdots \partial_\theta^{k_i} g$ such that $i \ge 1$, $k_p \ge 1$ and $k_j \ge 1$ for all $1 \le j \le i$ and $k_p + k_1 + \ldots + k_i = K$. We emphasize that unless explicitly stated otherwise, in these schematic sums i and j will be positive integers, as are the indices k_p, k_1, k_i etc.

In the event that any index e.g. k_p is allowed to be 0, this will be explicitly stated, and similarly if there are further constraints on any index.

5.3.1 Energy estimates for P

Proposition 5.3.1. *Let (P, Q) be a solution to the Gowdy symmetric system (2.6.2)–(2.6.3) obeying the bootstrap assumptions (B1)–(B4). Then there exists a constant A_* depending only on C_*, as well as a constant $C^{(K)}$ depending on C_* and the regularity index $K \in \{0, 1, \ldots, N\}$, such that*

$$\left| t \frac{d}{dt} \mathcal{E}_P^{(K)}(t) \right| \le 2A_* \mathcal{E}^{(K)}(t) + \sum_{k=0}^{K-1} C^{(K)} t^{-2(1-\gamma)(K-k)} \mathcal{E}^{(k)}(t), \quad (5.3.1)$$

where it is understood that the final term is absent if $K = 0$.

Proof. For any $K \geq 1$, commuting the wave equation for P (2.6.2) with ∂_θ^K yields

$$(t\partial_t)^2\partial_\theta^K P - t^2\partial_\theta^2\partial_\theta^K P =$$

$$2\,\partial_\theta^K P\, e^{2P}\left((t\partial_t Q)^2 - (t\partial_\theta Q)^2\right) + 2e^{2P}(t\partial_t Q)(t\partial_t\partial_\theta^K Q) + 2e^{2P}(t\partial_\theta Q)(t\partial_\theta^{K+1} Q) \tag{5.3.2a}$$

$$+ t^2 e^{2P} \sum_{\substack{i\geq 0, k_r < K \\ k_1+\cdots+k_i+k_r+k_s=K}} \partial_\theta^{k_1} P * \cdots * \partial_\theta^{k_i} P * \partial_t\partial_\theta^{k_r} Q * \partial_t\partial_\theta^{k_s} Q \tag{5.3.2b}$$

$$+ t^2 e^{2P} \sum_{\substack{i\geq 0, k_r < K \\ k_1+\cdots+k_i+k_r+k_s=K}} \partial_\theta^{k_1} P * \cdots * \partial_\theta^{k_i} P * \partial_\theta^{k_r+1} Q * \partial_\theta^{k_s+1} Q \tag{5.3.2c}$$

Note that in the uncommuted case $K = 0$, (5.3.2a) is replaced by $e^{2P}\left((t\partial_t Q)^2 + (t\partial_\theta Q)^2\right)$, while the terms (5.3.2b)–(5.3.2c) are absent; this case will be simpler in the subsequent estimate.

The first line (5.3.2a) is the leading order contribution that gives rise to the $2A_*\mathcal{E}^{(K)}(t)$ on the RHS of (5.3.1). It remains to estimate the remaining lines (5.3.2b) and (5.3.2c) in L^2. For this purpose, we will use the product estimate Lemma 5.2.2. It will be convenient to replace the schematic expression (5.3.2a) by

$$\sum_{\substack{i\geq 0, 0\leq k_r, k_s < K \\ k_1+\cdots+k_i+k_r+k_s=K}} \partial_\theta^{k_1} P * \cdots * \partial_\theta^{k_i} P * \partial_\theta^{k_r}(e^P t\partial_t Q) * \partial_\theta^{k_s}(e^P t\partial_t Q), \tag{5.3.2b'}$$

noting these two schematic expressions are equivalent since expanding (5.3.2b') simply means that more derivatives can fall on P. We can apply Lemma 5.2.2 with $w = 1$ to this expression; since we can use (B2) to bound $|e^P t\partial_t Q| \leq C_*$ if $k_r, k_s = 0$, repeated use of this lemma yields:

$$\|(5.3.2b')\|_{L^2} \lesssim (1 + C_*^2)\sum_{k=0}^{K-1}(\|\partial_\theta^k P\|_{L^2} + \|\partial_\theta^k(e^P t\partial_t Q)\|_{L^2}) \cdot (\|\partial_\theta P\|_{L^\infty} + \|\partial_\theta(e^P t\partial_t Q)\|_{L^\infty})^{K-k}. \tag{5.3.3}$$

Note in this expression it is critical that the sum *does not include* $K = k$; these top order objects appeared instead in (5.3.2a).

To explain a little further how one arrives at this estimate, note that summands in (5.3.2b') containing $i + 1$ terms in the product (not including undifferentiated copies of $e^P t\partial_t Q$ which are estimated by (B2)), the maximum number of ∂_θ derivatives landing on either P or $e^P t\partial_t Q$ is exactly $K - i$. Then repeated use of Lemma 5.2.2 allows us to put exactly $K - i$ ∂_θ derivatives on one of these, which we estimate in L^2, while the remaining i terms are estimated by either $\|\partial_\theta P\|_{L^\infty}$ or $\|\partial_\theta(e^P t\partial_t Q)\|_{L^\infty}$. Setting $k = K - i \in \{1, \ldots, K-1\}$, we obtain the desired estimate (5.3.3).

The next step is to represent the right hand side of (5.3.3) in terms of our energies. One issue is that our energy $\mathcal{E}_Q^{(K)}(t)$ in Definition 5.2.1 controls $\|e^P(t\partial_t\partial_\theta^K Q)^2\|_{L^2}$ as opposed to $\|\partial_\theta^K(e^P t\partial_t Q)\|_{L^2}$. However, this will not be a major issue, since by Lemma 5.3.2, which we defer to after this proof,

will tell us that assuming (B2),

$$\|\partial_\theta^k(e^P t\partial_t Q)\|_{L^2} \lesssim \sum_{j=1}^{k}(\|\partial_\theta^j P\|_{L^2} + \|e^P(t\partial_t\partial_\theta^j Q)\|_{L^2}) \cdot (\|\partial_\theta P\|_{L^\infty} + \|\partial_\theta(e^P t\partial_t Q)\|_{L^\infty})^{k-j}.$$

Thereby combining this with (5.3.3) and the expression for $\mathcal{E}^{(K)}(t)$ in Definition 5.2.1 – and using that (5.3.2b) and (5.3.2b') are equivalent – and finally using the remaining bootstrap assumptions (B3)–(B4) to estimate $\|\partial_\theta P\|_{L^\infty} + \|\partial_\theta(e^P t\partial_t Q)\|_{L^\infty}$, one therefore deduces that

$$\|(5.3.2\text{b})\|_{L^2} \lesssim_{C_*,K} \sum_{k=0}^{K-1} t^{(1-\gamma)(K-k)}\sqrt{\mathcal{E}^{(k)}(t)}. \tag{5.3.4}$$

The estimate for (5.3.2c) is essentially the same; we first rewrite (5.3.2c) in the schematic form

$$\sum_{\substack{i\geq 0, 0\leq k_r,k_s<K \\ k_1+\cdots+k_i+k_r+k_s=K}} \partial_\theta^{k_1} P * \cdots * \partial_\theta^{k_i} P * \partial_\theta^{k_r}(e^P t\partial_\theta Q) * \partial_\theta^{k_s}(e^P t\partial_\theta Q), \tag{5.3.4c'}$$

then use the product estimate Lemma 5.2.2 along with the forthcoming Lemma 5.3.2 to express the L^2 norm of this in terms of familiar quantities:

$$\|(5.3.4\text{c'})\| \lesssim \sum_{k=0}^{K-1}(\|\partial_\theta^k P\|_{L^2} + \|e^P(t\partial_\theta^{k+1} Q)\|_{L^2}) \cdot (\|\partial_\theta P\|_{L^\infty} + \|\partial_\theta(e^P t\partial_\theta Q)\|_{L^\infty})^{K-k} \tag{5.3.5}$$

We then combine with the definition of the energies $\mathcal{E}^{(K)}(t)$, and use the bootstrap assumption (B3) to estimate $\|\partial_\theta P\|_{L^\infty} + \|\partial_\theta(e^P t\partial_\theta Q)\|_{L^\infty}$; one yields

$$\|(5.3.2\text{c})\|_{L^2} \lesssim_{C_*,K} \sum_{k=0}^{K-1} t^{(1-\gamma)(K-k)}\sqrt{\mathcal{E}^{(k)}(t)}. \tag{5.3.6}$$

Now, to conclude, for $0 \leq K \leq N$ we write

$$\begin{aligned} &t\partial_t\left(\frac{1}{2}\left((t\partial_t\partial_\theta^K P)^2 + t^2(\partial_\theta^{K+1}P)^2 + (\partial_\theta^K P)^2\right)\right) \\ &\qquad = (t\partial_t\partial_\theta^K P)\left[(t\partial_t)^2\partial_\theta^K P - t^2\partial_\theta^{K+2}P + \partial_\theta^K P\right] + t^2(\partial_\theta^{K+1}P)^2 + \partial_\theta\left(t\partial_t\partial_\theta^K P \cdot t^2\partial_\theta^{K+1}P\right). \end{aligned}$$

Integrating over $\theta \in \mathbb{S}^1$ so that the last term vanishes, and inserting the commuted wave equation, one has the following identity for the $t\frac{d}{dt}$ derivative of the energy:

$$\begin{aligned} &t\frac{d}{dt}\mathcal{E}_P^{(K)}(t) \\ &\qquad = \int_{\mathbb{S}^1}\left[(t\partial_t\partial_\theta^K P)\left((5.3.2\text{a}) + \partial_\theta^K P\right) + t^2(\partial_\theta^{K+1}P)^2\right]dx \\ &\qquad\qquad + \int_{\mathbb{S}^1}\left[(t\partial_t\partial_\theta^K P)\left((5.3.2\text{b}) + (5.3.2\text{c})\right)\right]dx. \end{aligned}$$

From the bootstrap assumptions (B1)–(B2), it is clear that $\|(5.3.2a)\|_{L^2} \leq 10C_*^2\sqrt{\mathcal{E}^{(K)}(t)}$. Using Cauchy–Schwarz, the first integral in the above expression can thus be bounded as:

$$\left|\int_{\mathbb{S}^1}\left[(t\partial_t\partial_\theta^K P)\left((5.3.2a)+\partial_\theta^K P\right)+t^2(\partial_\theta^{K+1}P)^2\right]\,dx\right| \leq (10C_*^2+9)\mathcal{E}^{(K)}(t).$$

On the other hand, the latter integral can be estimated using (5.3.4) and (5.3.6); one yields that for some $C^{(K)} > 0$,

$$\left|\int_{\mathbb{S}^1}\left[(t\partial_t\partial_\theta^K\phi)\left((5.3.2b)+(5.3.2c)\right)\right]\,dx\right| \leq \sqrt{C^{(K)}}\sum_{k=0}^{K-1} t^{-(1-\gamma)(K-k)}\sqrt{\mathcal{E}^{(k)}(t)}\cdot\sqrt{\mathcal{E}^{(K)}(t)}.$$

Combining these and applying Young's inequality, Proposition 5.3.1 follows, with $2A_* = 10C_*^2 + 10$. □

We end this subsection with the promised Lemma 5.3.2, used to relate $\|\partial_\theta^k(e^P t\partial_t Q)\|_{L^2}$ to $\|e^P t\partial_t\partial_\theta^k Q\|_{L^2}$ in the estimate (5.3.3), and $\|\partial_\theta^k(e^P t\partial_\theta Q)\|_{L^2}$ to $\|e^P t\partial_\theta^{k+1}Q\|_{L^2}$ in (5.3.5).

Lemma 5.3.2. *Let $f : \mathbb{S}^1 \to \mathbb{R}$ be a sufficiently regular function, with $|e^P f| \leq C_*$. Then for $P : \mathbb{S}^1 \to \mathbb{R}$ regular function, and $k \in \mathbb{N}$, one has*

$$\|\partial_\theta^k(e^P f)\|_{L^2} \lesssim (1+C_*)\sum_{j=0}^{k}(\|\partial_\theta^j P\|_{L^2} + \|e^P\partial_\theta^j f\|_{L^2})\cdot(\|\partial_\theta P\|_{L^\infty} + \|\partial_\theta(e^P f)\|_{L^\infty})^{k-j}. \tag{5.3.7}$$

Proof. We prove (5.3.7) using induction on $k \in \mathbb{N}$. The base case $k = 1$ follows from

$$|\partial_\theta(e^P f)| \leq |\partial_\theta P|\cdot|e^P f| + |e^P\partial_\theta f| \leq C_*\cdot|\partial_\theta P| + |e^P\partial_\theta f|.$$

Now, let us suppose that (5.3.7) holds for $k \leq K \in \mathbb{N}$. To go from $k = K$ to $k = K+1$, we use our schematic notation to expand $\partial_\theta^{K+1}(e^P f)$, and write

$$\partial_\theta^{K+1}(e^P f) = \partial_\theta^{K+1}P\cdot e^P f + e^P\partial_\theta^{K+1}f + \sum_{k_1+\cdots+k_j+k_f=K+1}\partial_\theta^{k_1}P * \cdots * \partial_\theta^{k_j}P * \partial_\theta^{k_f}(e^P f), \tag{5.3.8}$$

where crucially each of the $k_1, \ldots, k_j, k_f$ will not exceed K. Therefore, one may apply the product estimate Lemma 5.3.2 to find

$$\left\|\sum_{k_1+\cdots+k_j+k_f=K+1}\partial_\theta^{k_1}P * \cdots * \partial_\theta^{k_j}P * \partial_\theta^{k_f}(e^P f)\right\|_{L^2}$$
$$\lesssim \sum_{j=0}^{K}(\|\partial_\theta^j P\|_{L^2} + \|\partial_\theta^j(e^P f)\|_{L^2})\cdot(\|\partial_\theta P\|_{L^\infty} + \|\partial_\theta(e^P f)\|_{L^\infty})^{K+1-j}.$$

We now apply the induction hypothesis to change $\|\partial_\theta^j(e^P f)\|_{L^2}$ to $\|e^P \partial_\theta^j P\|_{L^2}$. We thus deduce

$$\left\| \sum_{k_1+\cdots+k_j+k_f=K+1} \partial_\theta^{k_1} P * \cdots * \partial_\theta^{k_j} P * \partial_\theta^{k_f}(e^P f) \right\|_{L^2}$$
$$\lesssim (1+C_*) \sum_{j=0}^{K} (\|\partial_\theta^j P\|_{L^2} + \|e^P \partial_\theta^j f\|_{L^2}) \cdot (\|\partial_\theta P\|_{L^\infty} + \|\partial_\theta(e^P f)\|_{L^\infty})^{K+1-j}.$$

On the other hand, the first two terms on the right hand side of (5.3.8) have L^2 norm bounded by $C_*\|\partial_\theta^{K+1} P\|_{L^2} + \|e^P \partial_\theta^{K+1} f\|_{L^2}$. This proves the estimate (5.3.7) for $k = K+1$ and completes the proof of the lemma. □

5.3.2 Energy estimates for Q

We now prove the analogous energy estimate for $\mathcal{E}_Q^{(K)}(t)$.

Proposition 5.3.3. *Let (P, Q) be a solution to the Gowdy symmetric system (2.6.2)–(2.6.3) obeying the bootstrap assumptions (B1)–(B4). Then there exists a constant A_* depending only on C_*, as well as a constant $C^{(K)}$ depending on C_* and the regularity index $K \in \{0, 1, \ldots, N\}$, such that*

$$\left| t\frac{d}{dt}\mathcal{E}_Q^{(K)}(t) \right| \leq (2A_* + 2C_* t^\gamma)\mathcal{E}^{(K)}(t) + \sum_{k=0}^{K-1} C^{(K)} t^{-2(1-\gamma)(K-k)} \mathcal{E}^{(k)}(t), \tag{5.3.9}$$

where it is understood that the final term is absent if $K = 0$.

Proof. Commuting the Q wave equation (2.6.3) with ∂_θ^K, one yields for $1 \leq K \leq N$:

$$(t\partial_t)^2 \partial_\theta^K Q - t^2 \partial_\theta^2 \partial_\theta^K Q =$$
$$- 2t\partial_t \partial_\theta^K P \cdot t\partial_t Q - 2t\partial_t P \cdot t\partial_t \partial_\theta^{K+1} Q + 2t\partial_\theta^{K+1} P \cdot t\partial_\theta Q + 2t\partial_t P \cdot t\partial_\theta^{K+1} Q \tag{5.3.10a}$$
$$+ \sum_{k_p+k_q=K} \left(t\partial_t \partial_\theta^{k_p} P * t\partial_t \partial_\theta^{k_q} Q + t\partial_\theta^{k_p+1} P * t\partial_\theta^{k_q+1} Q \right). \tag{5.3.10b}$$

The base case $K = 0$ is just the equation (2.6.3) and will be easier to deal with. Note it again crucial that neither of k_p, k_q is allowed to equal K.

In light of the e^{2P} weight appearing in $\mathcal{E}_Q^{(K)}(t)$ in Definition 5.2.1, we will be required to estimate e^P times the right hand side of this equation in L^2. For the lower order term (5.3.10b), we use a similar method to the estimate for (5.3.2b) in Proposition 5.3.1. For the first term in (5.3.10b), it is convenient to write instead

$$e^P \sum_{k_p+k_q=K} t\partial_t \partial_\theta^{k_p} P * t\partial_t \partial_\theta^{k_q} Q = \sum_{\substack{i \geq 0 \\ k_1+\cdots+k_i+k_p+k_q=K}} \partial_\theta^{k_1} P * \cdots * \partial_\theta^{k_i} P * t\partial_t \partial_\theta^{k_p} P * \partial_\theta^{k_q}(e^P t\partial_t Q)$$

Then one may estimate this in L^2 in the same way as the term (5.3.2b'), eventually getting

$$\left\| e^P \sum_{k_p+k_q=K} t\partial_t \partial_\theta^{k_p} P * t\partial_t \partial_\theta^{k_q} Q \right\|$$
$$\lesssim_{C_*} \sum_{k=0}^{K-1} (\|\partial_\theta^k P\|_{L^2} + \|t\partial_t \partial_\theta^k P\|_{L^2} + \|e^P t\partial_t \partial_\theta^k Q\|_{L^2}) \cdot$$
$$(\|\partial_\theta P\|_{L^\infty} + \|t\partial_t \partial_\theta P\|_{L^\infty} + \|\partial_\theta (e^P t\partial_t Q)\|_{L^\infty})^{K-k}$$

Thereby using Definition 5.2.1 and the bootstrap assumptions (B1)–(B4), we get

$$\left\| e^P \sum_{k_p+k_q=K} t\partial_t \partial_\theta^{k_p} P * t\partial_t \partial_\theta^{k_q} Q \right\| \lesssim_{C_*,K} \sum_{k=0}^{K-1} t^{-(1-\gamma)(K-k)} \sqrt{\mathcal{E}^{(k)}(t)}.$$

The same method allows us to estimate the second term in (5.3.10b) in the same way, thus

$$\|e^P (5.3.10\text{b})\|_{L^2} \lesssim_{C_*,K} \sum_{k=0}^{K-1} t^{-(1-\gamma)(K-k)} \sqrt{\mathcal{E}^{(k)}(t)}. \tag{5.3.11}$$

To conclude, we first write the derivative identity; for $0 \le K \le N$:

$$t\partial_t \left(\frac{1}{2} \left(e^{2P} (t\partial_t \partial_\theta^K Q)^2 + e^{2P} t^2 (\partial_\theta^{K+1} Q)^2 \right) \right)$$
$$= e^P (t\partial_t \partial_\theta^K Q)\, e^P \left[(t\partial_t)^2 \partial_\theta^K Q - t^2 \partial_\theta^{K+2} Q \right] + \partial_\theta \left(e^{2P} t\partial_t \partial_\theta^K Q \cdot t^2 \partial_\theta^{K+1} Q \right)$$
$$+ t\partial_t P \cdot e^{2P} (t\partial_t \partial_\theta^K Q)^2 + (1 + t\partial_t P) e^{2P} t^2 (\partial_\theta^{K+1} Q)^2 - 2t\partial_\theta P \cdot e^{2P} t\partial_t \partial_\theta^K Q \cdot t\partial_\theta^{K+1} Q.$$

Next, integrating this over $\theta \in \mathbb{S}^1$, one deduces that

$$t\frac{d}{dt} \mathcal{E}_Q^{(K)}(t)$$
$$= \int_{\mathbb{S}^1} \left[e^P t\partial_t \partial_\theta^K Q \left(e^P (5.3.10\text{a}) + t\partial_t P \cdot e^P t\partial_t \partial_\theta^K Q \right) + (1 + t\partial_t P) e^{2P} t^2 (\partial_\theta^{K+1} Q)^2 \right] dx$$
$$- \int_{\mathbb{S}^1} 2t\partial_\theta P \cdot e^P t\partial_t \partial_\theta^K Q \cdot e^P t\partial_\theta^{K+1} Q\, dx + \int_{\mathbb{S}^1} \left[(t\partial_t \partial_\theta^K Q) \left(e^P (5.3.10\text{b}) \right) \right] dx.$$

Finally, using the expression for (5.3.10a), the bootstrap assumptions (B1) and (B2), and the expression for $\mathcal{E}^{(K)}(t)$ in Definition 5.2.1, one may check that the first line of the right hand side is bounded by say $(10C_*^2 + 9)\mathcal{E}^{(K)}(t)$.

On the other hand, by using the bootstrap assumption (B3) for the second integral and (5.3.11) for the third, one yields:

$$\left| \int_{\mathbb{S}^1} 2t\partial_\theta P \cdot e^P t\partial_t \partial_\theta^K Q \cdot e^P t\partial_\theta^{K+1} Q\, dx \right| \le 2C_* t^\gamma \mathcal{E}^{(K)}(t),$$

$$\left| \int_{\mathbb{S}^1} \left[(t\partial_t \partial_\theta^K Q) \left(e^P (5.3.10\text{b}) \right) \right] dx \right| \lesssim \sqrt{\mathcal{E}^{(K)}(t)} \cdot \sum_{k=1}^{K-1} t^{(1-\gamma)(K-k)} \sqrt{\mathcal{E}^{(k)}(t)}.$$

Therefore applying Young's inequality, one deduces (5.3.9) for $2A_* = 10(C_*^2 + 1)$. □

5.3.3 The energy hierarchy

We now use Propositions 5.3.1, 5.3.3 together with the initial data assumption (5.1.11)–(5.1.12), to show that the total energy of order K, $\mathcal{E}^{(K)}(t)$, grows at most polynomially in t as $t \downarrow 0$, and moreover that the rate of blow-up depends linearly in K.

Proposition 5.3.4. *Let (P, Q) be a solution to the Gowdy symmetric system* (2.6.2)–(2.6.3) *in the interval $t \in [t_b, t_0]$, such that the solution obeys the bootstrap assumptions* (B1)–(B4). *Assuming also the bounds* (5.1.11)–(5.1.12) *for the initial data, then there exist a constant A_* depending only on C_*, as well as constants $C^{(K)}$ depending on C_*, K and ζ, such that for $0 \leq K \leq N$, the total energy $\mathcal{E}^{(K)}(t)$ satisfies the bound:*

$$\mathcal{E}^{(K)}(t) \leq C^{(K)} t^{-2A_* - 2K(1-\gamma)}. \tag{5.3.12}$$

Proof. Combining Propositions 5.3.1 and 5.3.3, it is straightforward to show that for some $A_* > 0$ depending only on C_* and constants $C^{(K)} > 0$ (we allow A_* to differ from the previous propositions), one has the following derivative estimate:

$$\left| t \frac{d}{dt} \mathcal{E}^{(K)}(t) \right| \leq (2A_* + 2C_* t^\gamma) \mathcal{E}^{(K)}(t) + C^{(K)} \sum_{k=0}^{K-1} r^{-(1-\gamma)(K-k)} \mathcal{E}^{(k)}(t).$$

As we integrate "backwards" i.e. towards $r = 0$, the derivative estimate we actually use is the following:

$$t \frac{d}{dt} \mathcal{E}^{(K)}(t) \geq -(2A_* + 2C_* t^\gamma) \mathcal{E}^{(K)}(t) - C^{(K)} \sum_{k=0}^{K-1} r^{-\gamma(K-k)} \mathcal{E}^{(k)}(t).$$

In fact, using the integrating factor t^{2A_*}, which is *crucially independent of K*, we write:

$$t \frac{d}{dt} \left(t^{2A_*} \mathcal{E}^{(K)}(t) \right) \geq -2C_* t^\gamma \left(t^{2A_*} \mathcal{E}^{(K)}(t) \right) - C^{(K)} \sum_{k=0}^{K-1} t^{-2(1-\gamma)(K-k)} \left(t^{2A_*} \mathcal{E}^{(k)}(t) \right). \tag{5.3.13}$$

We will now use (5.3.13) and induction on $K \in \{0, \ldots, N\}$ that

$$t^{2A_*} \mathcal{E}^{(K)}(t) \lesssim t^{-2(1-\gamma)K}, \tag{5.3.14}$$

where the implied constant is now allowed to depend on C_*, K and ζ. This is equivalent to (5.3.12). Note that the dependence on ζ comes from the fact that the initial data bound (5.1.11)–(5.1.12) implies that

$$\sum_{k=0}^{N} t_0^{2A_*} \mathcal{E}^{(k)}(t_0) \leq \zeta. \tag{5.3.15}$$

For the base case $K = 0$, one simply applies Grönwall's inequality to (5.3.13) for $K = 0$; then for $t \in [t_b, t_0]$

$$t^{2A_*}\mathcal{E}^{(0)}(t) \leq \exp(F(t_0,t)) \cdot t_0^{2A_*}\mathcal{E}^{(0)}(t_0), \quad \text{where} \quad F(s_a,s_b) = \int_{s_b}^{s_a} 2C_*\tilde{t}^{\gamma}\frac{d\tilde{t}}{\tilde{t}}.$$

Since $F(s_a, s_b)$ is uniformly bounded for $s_a, s_b \in [t_b, t_0]$, it follows from the initial data bound (5.3.15) that (5.3.14) holds for $K = 0$.

Moving onto the induction step, assume that (5.3.14) holds for $0 \leq K < \bar{K} \leq N$; we wish to prove it also holds for $K = \bar{K}$. Applying Grönwall's inequality to (5.3.13) for $K = \bar{K}$, we have that

$$t^{2A_*}\mathcal{E}^{(\bar{K})}(t) \leq \exp(F(t_0,t)) \cdot t_0^{2A_*}\mathcal{E}^{(\bar{K})}(t_0) + \int_t^{t_0} \exp\big(F(\tilde{t},t)\big) C^{(\bar{K})} \sum_{k=0}^{\bar{K}-1} \tilde{t}^{-2(1-\gamma)(\bar{K}-k)} \tilde{t}^{2A_*}\mathcal{E}^{(k)}(\tilde{t}) \frac{d\tilde{t}}{\tilde{t}}.$$

It therefore follows from the initial data bound (5.3.15) and the inductive hypothesis for $\tilde{t}^{2A_*}\mathcal{E}^{(k)}(\tilde{t})$ that

$$t^{2A_*}\mathcal{E}^{(\bar{K})}(t) \lesssim \zeta + \int_t^{t_0} \sum_{k=0}^{\bar{K}-1} \tilde{t}^{-2(1-\gamma)(\bar{K}-k)} \cdot \tilde{t}^{-2\gamma k} \frac{d\tilde{t}}{\tilde{t}} \lesssim t^{-2(1-\gamma)\bar{K}}$$

as required. This completes the proof of the proposition. □

5.3.4 An auxiliary energy estimate

In order to recover precise asymptotics for $Q(t,\theta)$ as $t \to 0$ we also make use of the following energy estimate for the energy $\mathcal{E}^{(K)}_{Q,u}(t)$, see Definition 5.2.1.

Proposition 5.3.5. *Let (P, Q) be a solution to the Gowdy symmetric system (2.6.2)–(2.6.3) obeying the bootstrap assumptions (B1)–(B4). Then there exists a constant A_* depending only on C_*, as well as a constant $C^{(K)}$ depending on C_* and the regularity index $K \in \{0, 1, \ldots, N\}$ such that*

$$\left| t\frac{d}{dt}\mathcal{E}^{(K)}_{Q,u}(t) \right| \leq 2A_*\mathcal{E}^{(K)}_{Q,u}(t) + C^{(K)}\sqrt{2\mathcal{E}^{(K)}_{Q,u}(t)} \cdot \left(t^{-(1-\gamma)}\sqrt{2\mathcal{E}^{(K-1)}_{Q,u}(t)} + t^{\gamma'}\sqrt{2\mathcal{E}^{(K)}(t)} \right). \quad (5.3.16)$$

With initial data as in Proposition 5.3.4, one can moreover show that for some $C^{(K)} > 0$,

$$\mathcal{E}^{(K)}_{Q,u}(t) \leq C^{(K)} t^{-2A_* - 2K(1-\gamma)}. \quad (5.3.17)$$

Here $C^{(K)}$ may also depend on the initial data $(Q_D, \dot{Q}_D)$.

Proof. For the auxiliary estimate we again commute the wave equation (2.6.3) with ∂_θ^K, though we shall group terms in a slightly different way to (5.3.10a) and (5.3.10b). We have:

$$(t\partial_t)^2\partial_\theta^K Q - t^2\partial_\theta^2\partial_\theta^K Q = -2t\partial_t P \cdot t\partial_t\partial_\theta^{K+1}Q + 2t\partial_\theta P \cdot t\partial_\theta^{K+1}Q \quad (5.3.18a)$$

$$+ \sum_{\substack{k_p+k_q=K \\ 0\leq k_q<K}} \left(t\partial_t\partial_\theta^{k_p}P * t\partial_t\partial_\theta^{k_q}Q + t\partial_\theta^{k_p+1}P * t\partial_\theta^{k_q+1}Q \right). \quad (5.3.18b)$$

The difference between this and (5.3.10a)–(5.3.10b) is that the top order term containing P (i.e. $k_p = K$) is now included in (5.3.18b). We shall now estimate (5.3.18b) in L^2 without the e^P weight.

Using Lemma 5.2.2 and the fact that $1 \leq k_p \leq K$, one finds

$$\begin{aligned}\|(5.3.18\text{b})\|_{L^2} \lesssim \|t\partial_t\partial_\theta^K P\|_{L^2}\|t\partial_t Q\|_{L^\infty} + \|t\partial_t\partial_\theta P\|_{L^\infty}\|t\partial_t\partial_\theta^{K-1}Q\|_{L^2} \\ + \|t\partial_\theta^{K+1}P\|_{L^2}\|t\partial_\theta Q\|_{L^\infty} + \|t\partial_\theta^2 P\|_{L^\infty}\|t\partial_\theta^K Q\|_{L^2}.\end{aligned}$$

For the first and third terms on the right hand side, the L^2 norm is controlled by $\mathcal{E}_P^{(K)}(t)$, while the L^∞ norm is controlled via the bootstrap assumptions (B1) and (B2), which together give $\|t\partial_t Q\|_{L^\infty}, \|t\partial_\theta Q\|_{L^\infty} \leq C_*^2 t^\gamma$. For the second and fourth terms, the L^2 norm is controlled by $\mathcal{E}_{Q,u}^{(K-1)}(t)$, while the L^∞ norms are controlled using (B3)–(B4).

Combining all of these will yield that

$$\|(5.3.18\text{b})\| \lesssim t^{\gamma'}\sqrt{\mathcal{E}_P^{(K)}(t)} + t^{-(1-\gamma)}\sqrt{\mathcal{E}_{Q,u}^{(K-1)}(t)}. \tag{5.3.19}$$

Continuing, we write down the derivative identity:

$$\begin{aligned}t\partial_t\left(\frac{1}{2}\left((t\partial_t\partial_\theta^K Q)^2 + t^2(\partial_\theta^{K+1}Q)^2 + (\partial_\theta^K Q)^2\right)\right) \\ = (t\partial_t\partial_\theta^K Q)\left[(t\partial_t)^2\partial_\theta^K Q - t^2\partial_\theta^{K+2}Q + \partial_\theta^K Q\right] + t^2(\partial_\theta^{K+1}Q)^2 + \partial_\theta\left(t\partial_t\partial_\theta^K Q \cdot t^2\partial_\theta^{K+1}Q\right).\end{aligned}$$

Integrating over $\theta \in \mathbb{S}^1$, one thus yields that

$$\begin{aligned}t\frac{d}{dt}\mathcal{E}_Q^{(K)}(t) \\ = \int_{\mathbb{S}^1}\left[(t\partial_t\partial_\theta^K Q)\left((5.3.18\text{a}) + \partial_\theta^K Q\right) + t^2(\partial_\theta^{K+1}Q)^2\right]dx + \int_{\mathbb{S}^1}\left[(t\partial_t\partial_\theta^K Q)\cdot(5.3.18\text{b})\right]dx.\end{aligned}$$

Using the bootstrap assumptions (B1) and (B3) to deal with (5.3.18a), and using (5.3.19) to deal with (5.3.18b), one therefore deduces the estimate (5.3.16) in a similar way to say Proposition 5.3.1.

Inserting the bound (5.3.12) into (5.3.16), we find that:

$$t\frac{d}{dt}\mathcal{E}_{Q,u}^{(K)}(t) \geq -2A_*\mathcal{E}_{Q,u}^{(K)}(t) - 2C^{(K)}\sqrt{\mathcal{E}_{Q,u}^{(K)}(t)}\cdot\left(t^{-(1-\gamma)}\sqrt{\mathcal{E}_{Q,u}^{(K-1)}(t)} + t^{\gamma'-A_*-K(1-\gamma)}\right).$$

Equivalently,

$$t\frac{d}{dt}\sqrt{t^{2A_*}\mathcal{E}_{Q,u}^{(K)}(t)} \geq -C^{(K)}\left(t^{-(1-\gamma)}\sqrt{t^{2A_*}\mathcal{E}_{Q,u}^{(K-1)}(t)} + t^{-\gamma'-A_*-K(1-\gamma)}\right).$$

The integrated estimate (5.3.17) then follows easily using induction on $K \in \{0,\ldots,N\}$. □

5.4 Derivation of ODEs

5.4.1 Low order interpolation estimates

The goal in this section is to use the energy estimates of Proposition 5.3.4 together with the Sobolev interpolation in Lemma 5.2.1 to provide L^∞ bounds for low order ∂_θ derivatives of P and Q, allowing us to treat (2.6.2)–(2.6.3) as ODEs for certain quantities without worrying about losing derivatives.

Lemma 5.4.1. *Let (P, Q) be as in Proposition 5.3.4. Given any $0 < \gamma' < \gamma$, for N chosen sufficiently large there exists a family of constants $\delta = \delta(C_*, \zeta, t_*) > 0$ with $\delta \downarrow 0$ as $t_* \downarrow 0$, such that for any $0 \leq k \leq 3$ and $t \in [t_b, t_0]$ one has*

$$\|\partial_\theta^k P(t,\cdot)\|_{L^\infty} + \|t\partial_t\partial_\theta^k P(t,\cdot)\|_{L^\infty} + \|e^P t\partial_t\partial_\theta^k Q(t,\cdot)\|_{L^\infty} + \|e^P t\partial_\theta^{k+1} Q\|_{L^\infty} \leq \delta t^{-k(1-\gamma')}. \quad (5.4.1)$$

Proof. We use the energy estimate (5.3.12) for $K = N$, alongside Lemma 5.3.2 applied to $f = t\partial_t Q$ and $f = t\partial_\theta Q$, to derive the following top order L^2 estimate:

$$\|\partial_\theta^N P\|_{L^2}^2 + \|t\partial_t\partial_\theta^N P\|_{L^2}^2 + \|\partial_\theta^N(e^P t\partial_t Q)\|_{L^2}^2 + \|\partial_\theta^N(e^P t\partial_\theta Q)\|_{L^2}^2 \leq 2C^{(N)} t^{-2A_*-2N(1-\gamma)}. \quad (5.4.2)$$

Note that while $C^{(N)}$ depends on N, the number A_* does not, and we later choose N depending on A_*.

We now interpolate between (5.4.2) and the low-order L^∞ bootstrap assumptions (B1) and (B2). Applying Lemma 5.2.1, with $f \in \{P, t\partial_t P, e^P t\partial_t Q, e^P t\partial_\theta Q\}$ and $0 \leq k \leq 3$ one finds:

$$\|\partial_\theta^k f(t,\cdot)\|_{L^\infty} \lesssim_N \|f(t,\cdot)\|_{L^\infty}^{1-\alpha} \|\partial_\theta^N f(t,\cdot)\|_{L^2}^{\alpha}, \qquad \text{where } \alpha = \frac{k}{N - \frac{1}{2}}.$$

Inserting the bound (5.4.2) and the bootstrap assumptions (B1) and (B2), one finds

$$\|\partial_\theta^k f(t,\cdot)\|_{L^\infty} \lesssim_{N,C_*} \left(t^{-2A_*-2N(1-\gamma)}\right)^{\frac{\alpha}{2}} = t^{-k(1-\gamma)} \cdot \left(t^{-2A_*-(1-\gamma)}\right)^{\frac{\alpha}{2}}.$$

Note that as $N \to \infty$, $\alpha \to 0$. In particular, for N chosen sufficiently large (depending on A_*, γ and $\gamma' < \gamma$) one can guarantee that the second term in the product $(t^{-2A_*-(1-\gamma)})^{\frac{\alpha}{2}}$ can be bounded by say $t^{-\frac{\gamma-\gamma'}{2}}$. Thus for this choice of N, we deduce that for f as above,

$$\|\partial_\theta^k f(t,\cdot)\|_{L^\infty} \leq C_{N,C_*} t^{-k(1-\gamma)-\frac{\gamma-\gamma'}{2}} = C_{N,C_*} t^{-k(1-\gamma')} \cdot t^{(k-\frac{1}{2})(\gamma-\gamma')}. \quad (5.4.3)$$

To go from (5.4.3) to (5.4.1), there are two more steps required. The first is that we can expand out $\partial_\theta^k(e^P t\partial_t Q)$ and $\partial_\theta^k(e^P t\partial_\theta Q)$ to yield, for some suitably modified constant C_{N,C_*},

$$\|\partial_\theta^k P(t,\cdot)\|_{L^\infty} + \|t\partial_t\partial_\theta^k P(t,\cdot)\|_{L^\infty} + \|e^P t\partial_t\partial_\theta^k Q(t,\cdot)\|_{L^\infty} + \|e^P t\partial_\theta^{k+1} Q\|_{L^\infty} \leq t^{-k(1-\gamma')} \cdot (C_{N,C_*} t^{\frac{1}{2}(\gamma-\gamma')}).$$

Note we used here that $k - \frac{1}{2} \geq \frac{1}{2}$. Then recalling that $t \leq t_0 \leq t_*$, letting $\delta = C_{N,C_*} \cdot t_*^{\frac{1}{2}(\gamma-\gamma')}$ completes the proof of the lemma. □

Using Proposition 5.3.5, we also derive interpolated estimates for Q, without the e^P weight.

Lemma 5.4.2. *Let (P, Q) be as in Proposition 5.3.5. For γ, γ', N as in Lemma 5.4.2 there exists a family of constants $\delta = \delta(C_*, \zeta, t_*) > 0$ with $\delta \downarrow 0$ as $t_* \downarrow 0$, such that for any $0 \leq k \leq 3$ and $t \in [t_b, t_0]$ one has*

$$\|\partial_\theta^k Q(t,\cdot)\|_{L^\infty} + \|t\partial_t \partial_\theta^k Q(t,\cdot)\|_{L^\infty} \leq \delta t^{-k(1-\gamma')}. \tag{5.4.4}$$

Proof. The proof is identical to that of Lemma 5.4.1, using instead the energy $\mathcal{E}^{(N)}_{Q,u}(t)$ and the estimate (5.3.17). □

5.4.2 The main bounce ODEs

Corollary 5.4.3. *Let (P, Q) be as in Proposition 5.3.4. Then for N chosen sufficiently large there exists a family of constants $\delta = \delta(C_*, \zeta, t_*) > 0$, with $\delta \downarrow 0$ as $r_* \downarrow 0$, such that for all $(t, x) \in [t_b, r_0] \times \mathbb{S}^1$ and all $a \in [-1, 1]$:*

$$\left|(t\partial_t + at\partial_\theta)(-t\partial_t P) - \left((e^P t\partial_\theta Q)^2 - (e^P t\partial_t Q)^2\right)\right|(t,x) \leq \delta t^{\gamma'}, \tag{5.4.5}$$

$$\left|(t\partial_t + at\partial_\theta)(e^P t\partial_\theta Q) - (1 + t\partial_t P)(e^P t\partial_\theta Q)\right|(t,x) \leq \delta t^{\gamma'}, \tag{5.4.6}$$

$$\left|(t\partial_t + at\partial_\theta)(e^P t\partial_t Q) + (t\partial_t P)(e^P t\partial_t Q)\right|(t,x) \leq \delta t^{\gamma'}. \tag{5.4.7}$$

Proof. Each of these will follow straightforwardly from Lemma 5.4.1 and basic manipulation of the equations (2.6.2) and (2.6.3). For (5.4.5), from (2.6.2) one has:

$$(t\partial_t + at\partial_\theta)(-t\partial_t P) - \left((e^P t\partial_\theta Q)^2 - (e^P t\partial_t Q)^2\right) = -t^2\partial_\theta^2 P - at^2\partial_t\partial_\theta P. \tag{5.4.8}$$

Then applying Lemma 5.4.1, the right hand side is bounded by

$$|t^2\partial_\theta^2 P| + |at^2\partial_t\partial_\theta P| \leq \delta t^{2-2(1-\gamma')} + \delta t^{1-(1-\gamma')} \leq 2\delta t^{\gamma'},$$

and the bound (5.4.5) follows upon redefining δ appropriately.

For (5.4.6), one simply applies the product rule on $\partial_t(e^P t\partial_\theta Q)$ to get:

$$(t\partial_t + at\partial_\theta)(e^P t\partial_\theta Q) - (1 + t\partial_t P)(e^P t\partial_\theta Q) = e^P t^2\partial_t\partial_\theta Q + at\partial_\theta(e^P t\partial_\theta Q). \tag{5.4.9}$$

Applying Lemma 5.4.1 repeatedly, one bounds the right hand side of this by $\delta(2 + C_*)t^{\gamma'}$, so (5.4.6) follows upon further redefining δ.

For (5.4.7), one uses the equation (2.6.3), to derive

$$(t\partial_t + at\partial_\theta)(e^P t\partial_t Q) + (t\partial_t P)(e^P t\partial_t Q) = e^P t^2\partial_\theta^2 Q + 2e^P t\partial_\theta Q \cdot t\partial_\theta P + at\partial_\theta(e^P t\partial_t Q), \tag{5.4.10}$$

and by Lemma 5.4.1 is bounded by $\delta(2 + 3C_*)t^{\gamma'}$, thus completing the proof upon redefining δ. □

5.4.3 The equations of variation

On top of the ODEs for $-t\partial_t P$, $e^P t\partial_\theta Q$ and $e^P t\partial_t Q$ exhibited in Corollary 5.4.3, we shall also require corresponding ODEs for their ∂_θ-derivatives; that is, the linear system obtained by linearizing the ODEs of Corollary 5.4.3 around a given solution for $(-t\partial_t P, e^P t\partial_\theta Q, e^P t\partial_t Q)$.

Corollary 5.4.4. *Let (P, Q) be as in Proposition 5.3.4. Then for N chosen sufficiently large there exists a family of constants $\delta = \delta(C_*, \zeta, t_*) > 0$, with $\delta \downarrow 0$ as $r_* \downarrow 0$, such that for all $(t, x) \in [t_b, r_0] \times \mathbb{S}^1$ and all $a \in [-1, 1]$:*

$$\left|(t\partial_t + at\partial_\theta)(-t\partial_t\partial_\theta P) - 2(e^P t\partial_\theta Q)\partial_\theta(e^P t\partial_\theta Q) + 2(e^P t\partial_t Q)\partial_\theta(e^P t\partial_t Q)\right|(t,x) \leq \delta t^{-1+2\gamma'}, \tag{5.4.11}$$

$$\left|(t\partial_t + at\partial_\theta)(\partial_\theta(e^P t\partial_\theta Q)) - t\partial_t\partial_\theta P\, e^P t\partial_\theta Q - (1 + t\partial_t P)\partial_\theta(e^P t\partial_\theta Q)\right|(t,x) \leq \delta t^{-1+2\gamma'}, \tag{5.4.12}$$

$$\left|(t\partial_t + at\partial_\theta)(\partial_\theta(e^P t\partial_t Q)) + t\partial_t\partial_\theta P\, e^P t\partial_t Q + t\partial_t P\, \partial_\theta(e^P t\partial_t Q)\right|(t,x) \leq \delta t^{-1+2\gamma'}. \tag{5.4.13}$$

Proof. Each of (5.4.11)–(5.4.13) will be derived by commuting the equations (5.4.8)–(5.4.10) used in the proof of Corollary 5.4.3 with a ∂_θ-derivative, then using Lemma 5.4.1 to bound the right hand side. We demonstrate this by deriving (5.4.13); by commuting (5.4.10) with ∂_θ, one yields

$$\begin{aligned}(t\partial_t + at\partial_\theta)(\partial_\theta(e^P t\partial_t Q)) + t\partial_t\partial_\theta P\, e^P t\partial_t Q + t\partial_t P\, \partial_\theta(e^P t\partial_t Q)& \\ = 3t\partial_\theta P\, t\partial_\theta^2 Q + t\, e^P t\partial_\theta^3 Q + t\partial_\theta^2 P\, e^P t\partial_\theta Q + at\partial_\theta^2(e^P t\partial_t Q).&\end{aligned}$$

By Lemma 5.4.1, each ∂_θ-derivative on the right hand side, not including the one derivative in $e^P t\partial_\theta Q$, costs $t^{-1+\gamma'}$, thus the additional power of t on the right hand side means

$$\left|(t\partial_t + at\partial_\theta)(\partial_\theta(e^P t\partial_t Q)) + t\partial_t\partial_\theta P\, e^P t\partial_t Q + t\partial_t P\, \partial_\theta(e^P t\partial_t Q)\right| \lesssim_{C_*} \delta t^{1-2(1-\gamma')} = \delta t^{-1+2\gamma'}.$$

Redefining δ to absorb the implied constant, one deduces (5.4.13). □

5.5 Low order ODE analysis

In this section, we apply the ODEs derived in Section 5.4 to derive L^∞ bounds for 0th and 1st order quantities. Eventually, these will be used to improve the bootstrap assumptions (B1)–(B4).

5.5.1 The bounce ODE

Lemma 5.5.1. *For $t_0 \leq t_* < 1$ sufficiently small, let $\mathcal{P}, \mathcal{Q}, \mathcal{R} : [t_b, t_0] \subset \mathbb{R}_{>0} \to \mathbb{R}$ satisfy the following ODEs, where for some $0 < \gamma' < \gamma$ the error terms $\mathcal{E}_i$ obey $|\mathcal{E}_i| \leq \delta t^{\gamma'}$ for $i = 1, 2, 3$:*

$$t\partial_t \mathcal{P} = \mathcal{Q}^2 - \mathcal{R}^2 + \mathcal{E}_1, \tag{5.5.1}$$

$$t\partial_t \mathcal{Q} = (1 - \mathcal{P})\mathcal{Q} + \mathcal{E}_2, \tag{5.5.2}$$

$$t\partial_t \mathcal{R} = \mathcal{P}\mathcal{R} + \mathcal{E}_3. \tag{5.5.3}$$

Suppose furthermore that for some $\gamma'' > \gamma$, one has $(\mathcal{P} - 1)^2(t_0) + \mathcal{Q}^2(t_0) \leq (1 - \gamma'')^2$ and $|\mathcal{R}(t_0)| \leq \delta t_0^{\gamma'}$. Then for δ chosen sufficiently small (depending on all of $\gamma, \gamma', \gamma''$) the solution obeys the following bounds for $t \in [t_b, t_0]$.

$$(\mathcal{P} - 1)^2(t) + \mathcal{Q}^2(t) \leq (1 - \gamma)^2, \qquad |\mathcal{R}(t)| \leq t^{\gamma'}. \tag{5.5.4}$$

Proof. We proceed via using a continuity / bootstrap argument, with bootstrap assumption which is exactly (5.5.4). That is, we assume (5.5.4) holds on an interval $[\tilde{t}, t_0] \subset [t_b, t_0]$, and show that we may in turn improve upon (5.5.4) in this interval. For the improvement step, we use an approximate monotonicity property of the ODE system; let $\mathcal{K}$ be defined by

$$\mathcal{K} := (\mathcal{P} - 1)^2 + \mathcal{Q}^2 + \mathcal{R}^2. \tag{5.5.5}$$

Using (5.5.1)–(5.5.3), one may show that $t\partial_t \mathcal{K} = \mathcal{R}^2 + 2\mathcal{E}_1(\mathcal{P} - 1) + 2\mathcal{E}_2\mathcal{Q} + 2\mathcal{E}_3\mathcal{R}$. Thus using $\mathcal{R}^2 \geq 0$ and taking (5.5.4) as a bootstrap assumption, $t\partial_t \mathcal{K}$ may be bounded by

$$t\partial_t \mathcal{K} \geq -\left(4(1 - \gamma) + 2t^{\gamma'}\right)\delta t^{\gamma'}.$$

Thus for $\delta = \delta(\gamma, \gamma', \gamma'')$ chosen sufficiently small, one may guarantee that

$$\int_t^{t_0} \partial_t \mathcal{K}(\tilde{t}) d\tilde{t} \leq \frac{1}{2}\left((1 - \tfrac{\gamma + \gamma''}{2})^2 - (1 - \gamma'')^2\right). \tag{5.5.6}$$

Similarly using our assumptions on initial data we may choose δ sufficiently small such that

$$\mathcal{K}(t_0) \leq (1 - \gamma'')^2 + \frac{1}{2}\left((1 - \tfrac{\gamma + \gamma''}{2})^2 + (1 - \gamma'')^2\right). \tag{5.5.7}$$

Combining (5.5.6) and (5.5.7), it is clear that one has $\mathcal{K}(t) \leq (1 - \frac{\gamma + \gamma''}{2})^2 < (1 - \gamma)^2$ for $t \in [\tilde{t}, t_0]$, thereby improving the first inequality in (5.5.4).

For the second inequality in (5.5.4), from (5.5.3) and $\mathcal{P} \geq \gamma$ (which follows from the first part of (5.5.4)), one has the following differential inequality for $\mathcal{R}^2$:

$$t\partial_t(\mathcal{R}^2) \geq 2\gamma\mathcal{R}^2 + 2\mathcal{E}_3\mathcal{R}.$$

Using an integrating factor and the assumptions on $\mathcal{R}$ and $\mathcal{E}_3$, we may thus write

$$t\partial_t(t^{-2\gamma}\mathcal{R}^2) \geq 2\delta t^{-2(\gamma-\gamma')}.$$

Our assumption at $t = t_0$ means $t_0^{-2\gamma}\mathcal{R}^2(t_0) \geq \delta^2 t_0^{-2(\gamma-\gamma')}$. Thus for δ chosen sufficiently small, we integrate the above inequality in the direction of decreasing t and yield that $t^{-2\gamma}\mathcal{R}^2 < t^{-2(\gamma-\gamma')}$. Multiplying both sides by $t^{2\gamma}$, this is a strict improvement of the second inequality in (5.5.4). This completes our continuity argument, completing the proof of Lemma 5.5.1. □

5.5.2 The equations of variation

Lemma 5.5.2. *For $0 < t_0 \leq t_* < 1$, let $\mathcal{P}, \mathcal{Q}, \mathcal{R} : [t_b, t_0] \to \mathbb{R}$ satisfy the assumptions of Lemma 5.5.1. Further, let $\mathcal{L}, \mathcal{M}, \mathcal{N} : [t_b, t_0] \to \mathbb{R}$ obey the following ODEs, where $|\mathcal{E}_i| \leq \delta t^{-1+2\gamma'}$ for $i = 4, 5, 6$:*

$$t\partial_t\mathcal{L} = 2\mathcal{Q}\mathcal{M} - 2\mathcal{R}\mathcal{N} + \mathcal{E}_4, \tag{5.5.8}$$

$$t\partial_t\mathcal{M} = -\mathcal{Q}\mathcal{L} + (1-\mathcal{P})\mathcal{M} + \mathcal{E}_5, \tag{5.5.9}$$

$$t\partial_t\mathcal{N} = \mathcal{R}\mathcal{L} + \mathcal{P}\mathcal{N} + \mathcal{E}_6. \tag{5.5.10}$$

For some $\zeta > 0$, impose the following conditions at initial data: $|\mathcal{L}(t_0)| + |\mathcal{M}(t_0)| + |\mathcal{N}(t_0)| \leq \zeta t_0^{-1+\gamma}$. We further assume that $2\gamma' > \gamma$.

Then for δ chosen sufficiently small depending on $\gamma, \gamma', \gamma''$ and ζ, there exists a constant $D > 0$ depending on the same parameters such that for all $t \in [t_b, t_0]$, one has

$$|\mathcal{L}(t)| + |\mathcal{M}(t)| + |\mathcal{N}(t)| \leq D\, t^{-1+\gamma}. \tag{5.5.11}$$

Proof. We shall rewrite the system (5.5.8)–(5.5.10) in the following matrix form:

$$t\partial_t \begin{bmatrix} \mathcal{L} \\ \mathcal{M} \\ \mathcal{N} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 2\mathcal{Q} & -2\mathcal{R} \\ -\mathcal{Q} & 1-\mathcal{P} & 0 \\ \mathcal{R} & 0 & \mathcal{P} \end{bmatrix}}_{=:\mathbf{L}} \begin{bmatrix} \mathcal{L} \\ \mathcal{M} \\ \mathcal{N} \end{bmatrix} + \begin{bmatrix} \mathcal{E}_4 \\ \mathcal{E}_5 \\ \mathcal{E}_6 \end{bmatrix}. \tag{5.5.12}$$

The goal is to bound the operator norm of the matrix $\mathbf{L} = \mathbf{L}(r)$, as a function of $t \in [t_b, t_0]$, in such a way that allows one to integrate this equation. Note our operator norm will be with respect to the ℓ^2-norm on $\mathbb{R}^3$, i.e. for a 3×3 matrix $\mathbf{M}$ we write

$$\|\mathbf{M}\|_{op} \coloneqq \sup_{\mathrm{x} \in \mathbb{R}^3 \setminus \{0\}} \frac{\|\mathbf{M}\mathrm{x}\|_{\ell^2}}{\|\mathrm{x}\|_{\ell^2}}.$$

In fact, we shall actually estimate the operator norm of

$$\tilde{\mathbf{L}} = \begin{bmatrix} 0 & 2\mathcal{Q} & -2\mathcal{R} \\ -\mathcal{Q} & 1-\mathcal{P} & 0 \\ \mathcal{R} & 0 & 0 \end{bmatrix} = \mathbf{L} + \mathbf{P},$$

where we note $\mathbf{P} = \mathrm{diag}(0, 0, \mathcal{P})$ is a positive definite matrix with respect to the ℓ^2 inner product. Our strategy will be to estimate $\|\tilde{\mathbf{L}}\|_{op}$ differently depending on the size of $\mathcal{Q}(t)$. The idea is that when $\mathcal{Q}(t)$ is small, the largest matrix element of $\tilde{\mathbf{L}}$ will be $1 - \mathcal{P}$, which is bounded by $1 - \gamma$ by Lemma 5.5.1.

On the other hand, when $\mathcal{Q}(t)$ is not small, we will only have a weaker quantitative bound $\|\tilde{\mathbf{L}}\|_{op} \leq 10$; this will be mitigated using an estimate on the size of the interval for which $\mathcal{Q}$ small is not small.

Case 1: $|\mathcal{Q}(t)| \leq \frac{1}{4}(\gamma'' - \gamma)$:

Recall from the proof of Lemma 5.5.1, particularly (5.5.6) and (5.5.7), that for $t \in [t_b, t_0]$ one has

$$(\mathcal{P} - 1)^2 + \mathcal{Q}^2 \leq \left(1 - \frac{\gamma + \gamma''}{2}\right)^2.$$

In particular, the Hilbert–Schmidt[2] norm of $\tilde{\mathbf{L}}$ is given by:

$$\|\tilde{\mathbf{L}}\|_{HS}^2 = (\mathcal{P} - 1)^2 + \mathcal{Q}^2 + 4\mathcal{Q}^2 + 5\mathcal{R}^2 \leq \left(1 - \frac{\gamma + \gamma''}{2}\right)^2 + \frac{1}{4}(\gamma'' - \gamma)^2 + 5\mathcal{R}^2.$$

Without the $5\mathcal{R}^2$ term, the right hand side is strictly less than $(1 - \gamma)^2$. From Lemma 5.5.1, we have $\mathcal{R}^2 \leq t^{2\gamma'}$, and in particular for $t < t_*$ chosen small enough one has $\|\tilde{\mathbf{L}}\|_{HS} \leq 1 - \gamma$. Since for any 3×3 matrix $\mathbf{M}$ one has $\|\mathbf{M}\|_{op} \leq \|\mathbf{N}\|_{HS}$ one concludes that:

$$\|\tilde{\mathbf{L}}(t)\|_{op} \leq 1 - \gamma \qquad \text{whenever } |\mathcal{Q}(t)| \leq \frac{1}{4}(\gamma'' - \gamma). \tag{5.5.13}$$

Case 2: $|\mathcal{Q}(t)| > \frac{1}{4}(\gamma'' - \gamma)$:

If one assumes no additional smallness for $|\mathcal{Q}(t)|$ then one may only use Lemma 5.5.1 to bound the individual matrix elements of $\tilde{\mathbf{L}}(t)$. Using Lemma 5.5.1, we simply crudely bound each nonzero matrix element of $\tilde{\mathbf{L}}(t)$ by 4, then proceeding via the Hilbert–Schmidt norm as above one can show $\|\tilde{\mathbf{L}}\|_{op} \leq \sqrt{80} \leq 10$.

As mentioned previously, this will be mitigated using control on the size of the set $B = \{t \in [t_b, t_0] : |\mathcal{Q}(t)| > \frac{1}{4}(\gamma'' - \gamma)\}$, at least upon assuming sufficient smallness on δ and t_*. To justify this, suppose that $\delta < \frac{1}{64}(\gamma'' - \gamma)^2$ and that $\mathcal{R}^2 \leq t_*^{2\gamma'} \leq \frac{1}{64}(\gamma'' - \gamma)^2$. Then using (5.5.4) and the equation (5.5.1) for $t\partial_t\mathcal{P}$,

$$t\partial_t\mathcal{P} \geq \frac{1}{16}(\gamma'' - \gamma)^2 - \mathcal{R}^2 - \delta \geq \frac{1}{32}(\gamma'' - \gamma)^2$$

[2]Recall the Hilbert–Schmidt norm of a 3×3 matrix $\mathbf{A}$ is given by the ℓ^2-norm of all its matrix elements:

$$\|\mathbf{A}\|_{HS} = \left(\sum_{i,j=1}^{3} \mathbf{A}_{ij}^2\right)^{1/2}.$$

while $t \in B$. Furthermore, for $t \notin B$, from Lemma 5.5.1 one still has $t\partial_t \mathcal{P} \geq -Ct^{\gamma'}$ for some $C > 0$. But from (5.5.4), $\mathcal{P}(r)$ is certainly bounded between 0 and 2. So:

$$2 \geq \int_t^{t_0} t\partial_t P(\tilde{t}) \, \frac{d\tilde{t}}{\tilde{t}} = \int_{\tilde{t} \in B} t\partial_t P(\tilde{t}) \, \frac{d\tilde{t}}{\tilde{t}} + \int_{t \notin B} t\partial_t P(\tilde{t}) \, \frac{d\tilde{t}}{\tilde{t}} \geq \mu(B) \cdot \frac{1}{32}(\gamma'' - \gamma)^2 - C\gamma'^{-1} t_*^{\gamma'}.$$

Here $\mu(B)$ is the measure of the set B with respect to the measure $\frac{dt}{t}$. Let us choose t_* small enough so that $C\gamma'^{-1} t_*^{\gamma'} \leq 2$. Then collecting all this information,

$$\|\tilde{\mathbf{L}}(t)\|_{op} \leq 10, \qquad \text{whenever } |\mathcal{Q}| > \frac{1}{4}(\gamma'' - \gamma) \tag{5.5.14}$$

$$\text{where } B = \left\{ t \in [t_b, t_0] : |\mathcal{Q}(t)| > \frac{1}{4}(\gamma'' - \gamma) \right\} \text{ has } \mu(B) \leq 128(\gamma'' - \gamma)^{-2}.$$

We now use (5.5.13) and (5.5.14) to complete the proof of the lemma. We introduce the further notation:

$$\mathbf{x} = \begin{bmatrix} \mathcal{L} \\ \mathcal{M} \\ \mathcal{N} \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} \mathcal{E}_4 \\ \mathcal{E}_5 \\ \mathcal{E}_5 \end{bmatrix}.$$

So that we further rewrite the ODE system (5.5.12) as $t\partial_t \mathbf{x} = (\tilde{\mathbf{L}} + \mathbf{P})\mathbf{x} + \mathbf{e}$. Using the positivity of $\mathbf{P}$, one then shows that $\|\mathbf{x}\|_{\ell^2}^2 = \mathbf{x}^T\mathbf{x}$ satisfies

$$t\partial_t \|\mathbf{x}\|_{\ell^2}^2 \geq 2\mathbf{x}^T\tilde{\mathbf{L}}\mathbf{x} + 2\mathbf{x}^T\mathbf{e} \geq -2\|\tilde{\mathbf{L}}\|_{op} \|\mathbf{x}\|_{\ell^2}^2 - 2\sqrt{3}\delta t^{-1+2\gamma'} \|\mathbf{x}\|_{\ell^2}.$$

Note the final inequality follows from the definition of the operator norm and using Cauchy–Schwarz on $\mathbf{x}^T\mathbf{e}$, together with the assumed bounds on the error terms $\mathcal{E}_i$. Therefore

$$t\partial_t \|\mathbf{x}\|_{\ell^2} \geq -\|\tilde{\mathbf{L}}\|_{op} \cdot \|\mathbf{x}\|_{\ell^2} - \sqrt{3}\delta t^{-1+2\gamma'}. \tag{5.5.15}$$

The idea is now simply to apply Grönwall's inequality to this differential inequality. Recalling that we always integrate in the direction of decreasing t, the crucial bound will be the following, which follows from (5.5.13) and (5.5.14):

$$\begin{aligned}
\int_{t_2}^{t_1} \|\tilde{\mathbf{L}}(\tilde{t})\|_{op} \, \frac{d\tilde{t}}{\tilde{t}} &\leq \int_{t \in [t_2, t_1] \setminus B} \|\tilde{\mathbf{L}}\|_{op} \, \frac{d\tilde{t}}{\tilde{t}} + \int_{t \in [t_2, t_1] \cap B} \|\tilde{\mathbf{L}}\|_{op} \, \frac{d\tilde{t}}{\tilde{t}} \\
&\leq \int_{t_2}^{t_1} (1 - \gamma) \, \frac{d\tilde{t}}{\tilde{t}} + \int_{t \in B} 10 \, \frac{d\tilde{t}}{\tilde{t}} \\
&\leq (1 - \gamma) \log\left(\frac{t_1}{t_2}\right) + 10\mu(B) = (1 - \gamma) \log\left(\frac{t_1}{t_2}\right) + 1280(\gamma'' - \gamma)^{-2}.
\end{aligned} \tag{5.5.16}$$

To conclude, integrating (5.5.15) and inserting our initial data bounds yields the following integral inequality, where $\beta(t) = \sqrt{3}(\zeta t_0^{-1+\gamma} + \delta(1 - 2\gamma')^{-1} t^{-1+2\gamma'})$:

$$\|\mathbf{x}(t)\|_{\ell^2} \leq \int_t^{t_0} \|\tilde{\mathbf{L}}(\tilde{t})\|_{op} \cdot \|\mathbf{x}(\tilde{t})\|_{\ell^2} \, \frac{d\tilde{t}}{\tilde{t}} + \beta(t).$$

Thus Grönwall's inequality in integral form implies that

$$\|\mathbf{x}(t)\|_{\ell^2} \leq \beta(t) + \int_t^{t_0} \beta(\tilde{t}) \cdot \|\tilde{\mathbf{L}}(\tilde{t})\|_{op} \cdot \exp\left(\int_t^{\tilde{t}} \|\tilde{\mathbf{L}}(\tilde{\tilde{t}})\| \frac{d\tilde{\tilde{t}}}{\tilde{\tilde{t}}}\right) \frac{d\tilde{t}}{\tilde{t}}.$$

By (5.5.16) and the fact $\|\tilde{\mathbf{L}}(t)\|_{op} \leq 10$ holds everywhere, we can bound the integrand here by:

$$\beta(\tilde{t}) \cdot \|\tilde{\mathbf{L}}(\tilde{t})\|_{op} \cdot \exp\left(\int_t^{\tilde{t}} \|\tilde{\mathbf{L}}(\tilde{\tilde{t}})\| \frac{d\tilde{\tilde{t}}}{\tilde{\tilde{t}}}\right) \lesssim \beta(\tilde{t}) \cdot \left(\frac{\tilde{t}}{t}\right)^{1-\gamma} \lesssim \zeta \left(\frac{\tilde{t}}{t_0 t}\right)^{1-\gamma} + \delta \tilde{t}^{2\gamma'-\gamma} t^{-1+\gamma}.$$

Therefore, using that $2\gamma' > \gamma$, inserting this into the above yields that $\|\mathbf{x}(t)\|_{\ell^2} \lesssim t^{-1+\gamma}$. By the definition of $\mathbf{x}$, and since the ℓ^2 and ℓ^∞ norms on $\mathbb{R}^3$ are uniformly equivalent, the lemma follows. □

5.6 The stability result

5.6.1 Proof of Theorem 5.1.1

The proof of our stability result Theorem 5.1.1 follows from a boostrap argument. By local existence for the Gowdy wave map system, for initial data as given there exists some $t_b \in (0, t_0)$ such that a solution to the Gowdy system (2.6.2)–(2.6.3) exists in the interval $t \in [t_b, t_0]$, which moreover satisfies the bootstrap assumptions (B1)–(B4).

We show that assuming this, one may then improve upon the bootstrap assumptions, for instance showing the same (B1)–(B4) hold with C_* replaced by say $C_*/2$. By a standard continuity argument, t_b may then be any number in the interval $(0, t_0)$, and the corresponding solution thus obeys (improved) bootstrap assumptions in the whole of $t \in (0, t_0)$. We then apply the results of Section 5.3 and Section 5.5 to conclude.

Improving the bootstrap assumptions

Let (P, Q) be a solution of the Gowdy symmetric system in our bootstrap region $t \in [t_b, t_0]$ with assumptions on initial data as given. Then the results of Section 5.3 and Section 5.4 all apply, in particular Proposition 5.3.4, Lemma 5.4.1, Corollary 5.4.3 and Corollary 5.4.4. We now proceed in the following steps:

Step 1: ODE analysis on timelike curves

The first step will be to use Corollary 5.4.3, Corollary 5.4.4 and the results of Section 5.5 to provide L^∞ bounds for the following 6 key quantities:

$$-t\partial_t P, \quad e^P t\partial_\theta Q, \quad e^P t\partial_t Q, \quad -t\partial_t\partial_\theta P, \quad e^P t\partial_\theta^2 Q, \quad e^P t\partial_t\partial_\theta Q.$$

To do set, let $\gamma : [t_b, t_0] \to (0, +\infty) \times \mathbb{S}^1$ with $\gamma(t) = (t, \theta(t))$ be a C^1 *past-directed timelike curve*. One example is a curve of constant θ i.e. with $\theta'(t) = 0$. In view of the metric (2.6.1), the timelike character is equivalent to $|\theta'(t)| < 1$. For such a curve, γ, define

$$\mathcal{P}(t) \coloneqq -t\partial_t P(\gamma(t)), \quad \mathcal{Q}(t) \coloneqq e^P t\partial_\theta Q(\gamma(t)), \quad \mathcal{R}(t) \coloneqq e^P t\partial_t Q(\gamma(t)), \tag{5.6.1}$$

$$\mathcal{L}(t) \coloneqq -t\partial_t\partial_\theta P(\gamma(t)), \quad \mathcal{M}(t) \coloneqq \partial_\theta(e^P\partial_\theta Q)(\gamma(t)), \quad \mathcal{N}(t) \coloneqq \partial_\theta(e^P t\partial_t Q)(\gamma(t)). \tag{5.6.2}$$

The initial data assumption (5.1.9) means that $(\mathcal{P}(t_0) - 1)^2 + \mathcal{Q}^2(t_0) \leq (1 - \boldsymbol{\gamma}'')^2$ and the second assumption (5.1.10) means that for $\delta \leq \zeta t_*^{\boldsymbol{\gamma}-\boldsymbol{\gamma}'}$, one also has $\mathcal{R}(t_0) \leq \delta t_0^{\boldsymbol{\gamma}'}$. Combining these initial data bounds with Corollary 5.4.3, it is clear that along the curve γ, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ satisfy the assumptions of Lemma 5.5.1, with $a = \frac{d\theta}{dt}$. Applying this lemma yields that:

$$(1 - \mathcal{P}(t))^2 + \mathcal{Q}^2(t) \leq (1 - \boldsymbol{\gamma})^2, \quad \mathcal{R}(t) \leq t^{\boldsymbol{\gamma}'}. \tag{5.6.3}$$

We also provide bounds for $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$. To do this, we use the ODEs in Corollary 5.4.4. With error terms bounded as in this corollary, it is evident that the quantities $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ satisfy the system (5.5.8)–(5.5.9) in Lemma 5.5.2.

We also need to verify the conditions on initial data for the ODEs. To do so, we use the initial L^2 bounds (5.1.11) and (5.1.12). It follows from these, Sobolev embedding in $\mathbb{S}^1$, and also (5.1.9)–(5.1.10), that one has $|\partial_\theta \dot{P}_D(\theta)|, |\partial_\theta(e^{P_D} t_0 \partial_\theta Q_D)(\theta)|, |\partial_\theta(e^{P_D}\dot{Q}_D)(\theta)| \leq C_{\mathbb{S}^1}\zeta$, where $C_{\mathbb{S}^1}$ is a Sobolev constant that is independent of ζ and $\boldsymbol{\gamma}$. Thus one has $|\mathcal{L}(t_0)|, |\mathcal{M}(t_0)|, |\mathcal{N}(t_0)| \leq C_{\mathbb{S}^1}\zeta \leq \zeta t_0^{-1+\boldsymbol{\gamma}}$, so long as $t_0 \leq t_*$ is chosen small enough to absorb the $C_{\mathbb{S}^1}$. Thus applying Lemma 5.5.2, we deduce that for some constant $D > 0$ depending on $\boldsymbol{\gamma}$, ζ etc but not C_*, one bounds

$$|\mathcal{L}(t)| + |\mathcal{M}(t)| + |\mathcal{N}(t)| \leq Dt^{-1+\boldsymbol{\gamma}}. \tag{5.6.4}$$

Step 2: Improving the bootstrap assumptions (B1) and (B2)

Using (5.6.3), and allowing γ to vary over all timelike curves, in particular all curves with constant θ-coordinate, we find that $|t\partial_t P(t,x)| \leq 2 - \boldsymbol{\gamma}$ and $|e^P t\partial_\theta Q(t,x)| \leq 1 - \boldsymbol{\gamma}$ for all $(t,x) \in [t_b, t_0] \times \mathbb{S}^1$. Thus choosing $C_* \geq 2(2 - \boldsymbol{\gamma})$, one easily improves upon (B1). Note that since $\mathcal{P}$ cannot change sign, (5.6.3) actually yields $-t\partial_t P(t,x) \geq \boldsymbol{\gamma}$.

Similarly using (5.6.3), $|e^P t\partial_t Q(t,x)| \leq t^{\boldsymbol{\gamma}'}$, so the first part of (B2) is improved for $C_* \geq 2$. For the second part of (B2), we simply use:

$$e^{-P}(t,x) \leq e^{-P}(t_0,\theta) \cdot \exp\bigg(-\int_t^{t_0}(-t\partial_t P)(\tilde{t},\theta)\frac{d\tilde{t}}{\tilde{t}}\bigg) \leq e^{-P_D}(\theta)\exp\bigg(\boldsymbol{\gamma}\log\bigg(\frac{t}{t_0}\bigg)\bigg),$$

where in the second step we used that $-t\partial_t P(t,x) \geq \boldsymbol{\gamma}$. Using the initial data assumption (5.1.10), we thus have $e^P(t,x) \leq \zeta t^{\boldsymbol{\gamma}}$. Since $\boldsymbol{\gamma}' < \boldsymbol{\gamma}$ and $t \leq 1$, choosing $C_* \geq 2\zeta$ also improves the second part of (B2).

Step 3: Improving the bootstrap assumptions (B3) and (B4)

We shall improve these using (5.6.4). Allowing γ to vary over all timelike curves, for all $(t,\theta) \in [t_b, t_0] \times \mathbb{S}^1$:

$$|t\partial_t\partial_\theta P(t,\theta)| \leq Dt^{-(1-\gamma)}, \quad |\partial_\theta(e^P t\partial_\theta Q)(t,\theta)| \leq Dt^{-(1-\gamma)}, \quad |\partial_\theta(e^P t\partial_t Q)(t,\theta)| \leq Dt^{-(1-\gamma)}. \tag{5.6.5}$$

We now integrate the first inequality in (5.6.5). Using (5.1.11) and Sobolev embedding to bound $|\partial_\theta P(t_0,\theta)|$, we deduce that $|\partial_\theta P(t,\theta)| \leq Dt^{-(1-\gamma)}$ also, where D is modified appropriately.

We now expand the second and third inequalities in (5.6.5), yielding that

$$|e^P t\partial_\theta^2 Q| \leq Dt^{-(1-\gamma)} + |\partial_\theta P| \cdot |e^P t\partial_\theta Q|, \quad |e^P t\partial_t\partial_\theta Q| \leq Dt^{-(1-\gamma)} + |\partial_\theta P| \cdot |e^P t\partial_t Q|.$$

Now using the above bound for $|\partial_\theta P|$ and the bounds from Step 2, by modifying D appropriately we also have $|e^P t\partial_\theta^2 Q|, |e^P t\partial_t\partial_\theta Q| \leq Dt^{-(1-\gamma)}$. Choosing $C_* \geq 2D$, we thereby improve the remaining bootstrap assumptions.

Completion of the proof

To complete the proof, it remains to prove the statements contained in the final paragraph of Theorem 5.1.1. The former two bounds follow immediately from (5.6.3) and Proposition 5.3.4 respectively.

To show that $-t\partial_t P(t,\theta)$ converges pointwise as $t \to 0$, note that this is equivalent to showing that $\mathcal{P}(t)$, as defined in (5.6.1), has a limit as $t \to 0$, for curves γ of constant θ-coordinate. From Step 1, we know that $\mathcal{P}$ solves an ODE as in Lemma 5.5.1, namely

$$t\partial_t \mathcal{P} = \mathcal{Q}^2 - \mathcal{R}^2 + \mathcal{E}_1,$$

where $|\mathcal{E}_1| \lesssim t^{\gamma'}$. By (5.6.3), one also has $\mathcal{R}^2 \lesssim t^{2\gamma'}$. We can integrate the above and write:

$$\mathcal{P}(t) - \mathcal{P}(t_0) + \int_t^{t_0} (\mathcal{E}_1 - \mathcal{R}^2)\frac{d\tilde{t}}{\tilde{t}} = -\int_t^{t_0} \mathcal{Q}^2 \frac{d\tilde{t}}{\tilde{t}}.$$

By the lower bound for $\mathcal{P}(t)$ in (5.6.3) and the aforementioned bounds for $\mathcal{E}_1$ and $\mathcal{R}$ implying a lower bound for the LHS, it is clear that the right hand side attains a limit as $t \to 0$. Thus the quantity $\mathcal{P}(t)$ also attains a limit as $t \to 0$.

Finally, we show that $Q(t,\theta)$ and $\partial_\theta Q(t,\theta)$ converge *uniformly* as $t \to 0$, where for the latter statement we require $\gamma > \frac{1}{3}$. Starting with $Q(t,\theta)$, we multiply (2.6.3) by $t^{-\sigma}$, for some $\sigma > 0$ to be determined:

$$t\partial_t(t^{-\sigma} t\partial_t Q) = (-2t\partial_t P - \sigma) \cdot t^{-\sigma} t\partial_t Q + t^{2-\sigma}\partial_\theta^2 Q + 2t^{2-\sigma}\partial_\theta P \partial_\theta Q. \tag{5.6.6}$$

We shall choose $\sigma < \gamma'$. Therefore, we have $-2\partial_t P - \sigma > 0$, while by Lemma 5.4.1 and Lemma 5.4.2, we can bound the final two terms on the right hand side by

$$|t^{2-\sigma}\partial_\theta^2 Q| + |t^{2-\sigma}\partial_\theta P \partial_\theta Q| \lesssim t^{2-\sigma-2(1-\gamma')} = t^{2\gamma'-\sigma}.$$

Since $2\gamma' - \sigma \geq \gamma' > 0$, this is therefore integrable with respect to $\frac{dt}{t}$ as $t \to 0$. Therefore integrating (5.6.6) yields that $t^{-\sigma} t\partial_t Q$ is uniformly bounded in the region $0 < t \leq t_0$. This means that:

$$Q(t,\theta) = Q(t_0,\theta) - \int_t^{t_0} (t^{-\sigma} t\partial_t Q(\tilde{t},\theta))\tilde{t}^{-1+\sigma}\, d\tilde{t}.$$

indeed converges uniformly as $t \to 0$ to some $q(\theta)$, and $|Q(t,\theta) - q(\theta)| \lesssim t^\sigma$.

To prove a similar statement for $\partial_\theta Q$, we need to differentiate (5.6.6) in θ,

$$t\partial_t(t^{-\sigma} t\partial_t\partial_\theta Q) = (-2t\partial_t P - \sigma)\cdot t^{-\sigma} t\partial_t\partial_\theta Q - 2t\partial_t\partial_\theta P \cdot t^{-\sigma} t\partial_t Q \\ + t^{2-\sigma}\partial_\theta^3 Q + 2t^{2-\sigma}\partial_\theta P\partial_\theta^2 Q + 2t^{2-\sigma}\partial_\theta^2 P\partial_\theta Q. \quad (5.6.7)$$

To bound the last term on the first line, we use Step 2. In other words, we use the estimates (B2), which together yield $t\partial_t Q \lesssim t^{2\gamma'}$. Therefore, using also Lemma 5.4.1 we get that:

$$|t\partial_t\partial_\theta P \cdot t^{-\sigma} t\partial_t Q| \lesssim t^{-(1-\gamma')} \cdot t^{2\gamma'-\sigma} = t^{3\gamma'-1-\sigma}.$$

For the second line of (5.6.7), we again combine Lemma 5.4.1 and Lemma 5.4.2, to get

$$|t^{2-\sigma}\partial_\theta^3 Q| + |t^{2-\sigma}\partial_\theta P\partial_\theta^2 Q| + |t^{2-\sigma}\partial_\theta^2 P\partial_\theta Q| \lesssim t^{2-\sigma-3(1-\gamma')} = t^{3\gamma'-1-\sigma}.$$

For $\gamma > \frac{1}{3}$, we may choose $\gamma' > \frac{1}{3}$ also and then choose $\sigma < 3\gamma' - 1$. For this choice, $-2t\partial_t P - \sigma > 2\gamma' - (3\gamma' - 1) = 1 - \gamma' \geq 0$, so the first term on the right hand side of (5.6.7) can be ignored, and by integrability of $t^{3\gamma'-1-\sigma}$ with respect to $\frac{dt}{t}$ one yields that $t^{-\sigma} t\partial_t\partial_\theta Q$ is uniformly bounded in the region $0 < t \leq t_0$. One now concludes as in the previous case, and one yields:

$$|\partial_\theta Q(t,\theta) - \partial_\theta q(\theta)| \lesssim_\sigma t^\sigma, \qquad \text{for any } \sigma < 3\gamma' - 1. \quad (5.6.8)$$

5.7 BKL bounces

5.7.1 Proof of Theorem 5.1.2

Under the assumptions of Theorem 5.1.2, we may apply the stability result Theorem 5.1.1; in particular Corollary 5.4.3 applies, and the ODEs (5.1.15) follow exactly as in the proof of Theorem 5.1.2 – note that the term involving $\mathcal{R}$ can now be ignored since we have $\mathcal{R}(t) \lesssim t^{\gamma'}$ from (B2).

It remains to prove the various convergence results stated in Theorem 5.1.2. The convergence of $\mathcal{P}_\gamma(t)$ to some $\mathcal{P}_{\gamma,\infty} = V(\theta_0)$ follows in the same manner to the pointwise convergence of $-t\partial_t P(t,\theta)$

in the proof of Theorem 5.1.1. To show the convergence of $\mathcal{Q}_\gamma(t)$ to 0, we first use that from the first ODE in (5.1.15) and the convergence of $\mathcal{P}_\gamma$, the integral

$$\int_0^{t_0} \mathcal{Q}_\gamma^2(\tilde{t})\,\frac{d\tilde{t}}{\tilde{t}} \tag{5.7.1}$$

is finite, or in other words $\mathcal{Q}_\gamma^2(t)$ converges to 0 in an averaged sense. In particular there is a sequence $\{t_k\}$ with $t_k \to 0$ such that $\mathcal{Q}_\gamma^2(t_k) \to 0$ as $k \to \infty$. To upgrade this sequential convergence to convergence of $\mathcal{Q}_\gamma(t)$, we use the second equation in (5.1.15), or more precisely the ODE $t\partial_t \mathcal{Q}_\gamma^2 = 2\mathcal{Q}_\gamma^2(1-\mathcal{P}_\gamma) + 2\mathcal{E}_\mathcal{Q}\mathcal{Q}_\gamma$.

Integrating this gives that for any $0 < t \le t_k \le t_0$, we have:

$$|\mathcal{Q}_\gamma^2(t) - \mathcal{Q}_\gamma^2(t_k)| \le C\int_0^{t_k} \mathcal{Q}_\gamma^2(\tilde{t})\frac{d\tilde{t}}{\tilde{t}} + C\int_0^{t_k} \tilde{t}^{\gamma'}\frac{d\tilde{t}}{\tilde{t}}.$$

By the finiteness of the integral (5.7.1), the right hand side is finite, and in fact converges to 0 as $k \to \infty$. Since we already know $\mathcal{Q}_\gamma^2(t_k) \to 0$, this implies that $\mathcal{Q}_\gamma^2(t)$, and therefore also $\mathcal{Q}_\gamma(t)$, converges to 0 as $t \to 0$.

Moving onto (i), it will be necessary to use the conserved quantity $\mathcal{K}$ encountered in the proof of Lemma 5.5.1. Recall from there that defining $\mathcal{K}_\gamma = (\mathcal{P}_\gamma - 1)^2 + \mathcal{Q}_\gamma^2$, one can show

$$\left| t\frac{d}{dt}\mathcal{K}_\gamma \right| \lesssim t^{\gamma'}.$$

Using the convergence of $\mathcal{P}_\gamma$ and $\mathcal{Q}_\gamma$, it holds that $\mathcal{K}_\gamma(t) \to \mathcal{K}_{\gamma,\infty} := (\mathcal{P}_{\gamma,\infty} - 1)^2$ as $t \to 0$, and moreover from the above that $|\mathcal{K}_\gamma(t) - \mathcal{K}_{\gamma,\infty}| \lesssim t^{\gamma'}$. Inserting $t = t_0$, one has:

$$|(\mathcal{P}_\gamma(t_0) - 1)^2 - (\mathcal{P}_{\gamma,\infty} - 1)^2| \lesssim t_0^{\gamma'} + \mathcal{Q}_\gamma^2(t_0). \tag{5.7.2}$$

In (i), we assumed that $\partial_\theta Q(\gamma(t))$ did not converge to 0 as $t \to 0$. However, since $\mathcal{Q}_\gamma(t) = e^P t\partial_\theta Q(\gamma(t))$ does converge to 0, this implies that $e^{P(\gamma(t))}t \to 0$ as $t \to 0$. Now simply note that

$$t\frac{d}{dt}\log\Big(e^{P(\gamma(t))}t\Big) = 1 - \mathcal{P}_\gamma(t) \to 1 - \mathcal{P}_{\gamma,\infty} \quad \text{ as } t \to 0,$$

and thus in order for $e^{P(\gamma(t))}t \to 0$ we must have $\mathcal{P}_{\gamma,\infty} \le 1$. Combining this with (5.7.2) yields the estimate (5.1.16). (For instance, one uses that if $|x^2 - y^2| \le Z$ with $x, y \ge 0$ then $|x - y| \le Z^{1/2}$.)

For (ii), if $\gamma > \frac{1}{2}$ and $\partial_\theta Q(\gamma(t))$ converges to 0, it will be helpful to use (5.6.8) from the proof of Theorem 5.1.1, or rather its generalization to

$$|\partial_\theta Q(\gamma(t)) - \partial_\theta q(\theta_0)| \lesssim_\sigma t^\sigma, \qquad \text{for any } \sigma < 3\gamma' - 1.$$

Note that this generalization, where γ is allowed to be any timelike curve rather than only a constant θ-curve, is proved easily using previous methods. In (ii), we may insert $\partial_\theta q(\theta_0) = 0$, and therefore $|\partial_\theta Q(\gamma(t))| \lesssim t^\sigma$.

Furthermore, from $-t\partial_t P \leq 2-\gamma$ it holds that $e^P \leq t^{-2+\gamma}$, therefore one has $|e^P t\partial_\theta Q(\gamma(t))| \lesssim t^{-1+\gamma+\sigma}$. But by the definition of $\mathcal{Q}_\gamma(t)$, this means that $|\mathcal{Q}_\gamma(t)| \lesssim t^{-1+\gamma+\sigma}$. Note that so long as $\gamma > \frac{1}{2}$, $\gamma' < \gamma$ and $\sigma < 3\gamma' - 1$ may be chosen such that $-1+\gamma+\sigma > 0$. In the case of (ii), we therefore have from (5.1.15) that

$$|t\partial_t \mathcal{P}_\gamma(t)| \lesssim t^{2(-1+\gamma+\sigma)} + t^{\gamma'}.$$

For convenience, let use choose $\sigma = \gamma' - \gamma + \frac{1}{2}$, where $\gamma > \gamma' > \frac{3}{2} - \gamma$. Then $t^{2(-1+\gamma+\sigma)} = t^{2\gamma'-1}$, and integrating the above immediately yields (5.1.17). □

5.7.2 Proof of Corollary 5.1.3

Lastly, we shall apply our Theorems 5.1.1 and 5.1.2 to prove the stability / instability corollary. Let $P(t,\theta)$ be as stated. By Theorem 5.1.0, for all $k \in \mathbb{N}$ the convergence $-t\partial_t P(t,\theta) \to V(\theta)$ holds in the C^k norm.

Since we are assuming $0 < V(\theta) < 2$, since $\theta \in \mathbb{S}^1$ it holds that there exists $\gamma \in (0,1)$ such that $\gamma < V(\theta) < 2-\gamma$. In fact, by the above convergence we can find $\gamma, \gamma', \gamma''$ and $t_0 > 0$ such that for N chosen (depending on γ) as in Theorem 5.1.1, we have that for $0 < t \leq t_0$ and all $\theta \in \mathbb{S}^1$:

$$0 < \gamma' < \gamma < \gamma'' < -t\partial_t P(t,\theta) < 2-\gamma'' < 2-\gamma < 2-\gamma' < 2,$$

and there exists some $\zeta > 0$ so that for all $0 < t \leq t_0$:

$$\mathcal{E}(t) = \frac{1}{2}\sum_{K=0}^{N} \int_{\mathbb{S}^1} \left((t\partial_t \partial_\theta^K P)^2 + t^2(\partial_\theta^{K+1} P)^2 + t^{2\gamma}(\partial_\theta^K P)^2 \right) d\theta \leq \frac{\zeta}{2}.$$

(This follows because in fact $\mathcal{E}(t) \to \frac{1}{2}\sum_{K=0}^{N} \int (\partial_\theta^K V)^2 \, d\theta$ as $t \to 0$.)

We note that t_0 above may differ from the initial data time $t_1 > 0$ in the statement of Corollary 5.1.3. This is mitigated a standard *Cauchy stability* argument, for any $\varepsilon_0 > 0$ there exists $\varepsilon > 0$ such that a perturbation of size ε at $t = t_1$ implies a perturbation of size ε_0 at $t = t_0$ i.e. $\|(\tilde{P}_D, \tilde{Q}_D, \dot{\tilde{P}}_D, \dot{\tilde{Q}}_D) - (P_D, 0, \dot{P}_D, 0)\|_{(H^{N+1})^2 \times (H^N)^2} \leq \varepsilon$ implies $\|(\tilde{P}(t_0,\theta), \tilde{Q}(t_0,\theta), t\partial_t \tilde{P}(t_0,\theta), t\partial_t \tilde{Q}(t_0,\theta)) - (P(t_0,\theta), 0, t\partial_t P(t_0,\theta), 0)\|_{(H^{N+1})^2 \times (H^N)^2} \leq \varepsilon_0$. Due to this we may instead consider perturbations of initial data as perturbations at time $t = t_0$.

Thereby by choosing ε (and thus ε_0) small enough one can guarantee that

$$(1 + t\partial_t \tilde{P})^2(t_0,\theta) + (e^{\tilde{P}} t_0 \partial_\theta \tilde{P})^2(t_0,\theta) \leq (1-\gamma'')^2, \quad |e^{\tilde{P}} t\partial_t \tilde{Q}(t_0,\theta)| \leq \zeta t_0^\gamma, \quad e^{\tilde{P}}(t_0,\theta) \leq \zeta t_0^\gamma$$

as well as the energy bound

$$\frac{1}{2}\sum_{K=0}^{N} \int_{\mathbb{S}^1} \Big((t\partial_t \partial_\theta^K \tilde{P})^2 + t^2(\partial_\theta^{K+1}\tilde{P})^2 + t^{2\gamma}(\partial_\theta^K \tilde{P})^2 + e^{2\tilde{P}} \left[(t\partial_t \partial_\theta^K \tilde{Q})^2 + (t\partial_\theta^{K+1}\tilde{Q})^2 \right] \Big) d\theta \leq \zeta.$$

That is, the perturbed data at $t = t_0$ satisfies the assumptions (5.1.9)–(5.1.12). Moreover, t_0 can be chosen small enough so that $0 < t_0 \leq t_* = t_*(\gamma, \zeta)$, so that one may apply Theorems 5.1.1 and 5.1.2.

The remainder of the argument is then a direct application of these theorems. Applying Theorem 5.1.1 to the perturbed data, it is clear that $-t\partial_t \tilde{P}(t, \theta)$ converges to some $\tilde{V}(\theta)$ pointwise, with $\gamma \leq \tilde{V}(\theta) \leq 2 - \gamma$.

In the setting where $\frac{1}{2} < V(\theta) < \frac{3}{2}$, we are allowed to in fact choose $\gamma > \gamma' > \frac{1}{2}$. By the final statement of Theorem 5.1.1, $\partial_\theta \tilde{Q} \to \partial_\theta \tilde{q}(\theta)$ uniformly as claimed. We next apply Theorem 5.1.2. In the case that $\partial_\theta \tilde{q}(\theta_0) \neq 0$, Theorem 5.1.2(i) implies that for γ the timelike curve with constant $\theta = \theta_0$,

$$\begin{aligned}
\tilde{V}(\theta_0) = \tilde{\mathcal{P}}_{\gamma,\infty} &= \min\{-t\partial_t \tilde{P}(t_0, \theta_0), 2 + t\partial_t \tilde{P}(t_0, \theta_0)\} + O(t_0^{\gamma'/2}) + O(e^{\tilde{P}} t_0 \partial_\theta \tilde{Q}(t_0, \theta_0)) \\
&= \min\{-t\partial_t P(t_0, \theta_0), 2 + t\partial_t P(t_0, \theta_0)\} + O(t_0^{\gamma'/2}) + O(\varepsilon_0) \\
&= \min\{V(\theta_0), 2 - V(\theta_0)\} + O(t_0^{\gamma'/2}) + O(\varepsilon_0).
\end{aligned}$$

Since we have license to choose t_0 and ε_0 small depending on $\tilde{\varepsilon}$, Corollary 5.1.3(i) follows.

Similarly, in the case that $\partial_\theta \tilde{q}(\theta_0) = 0$, Theorem 5.1.2(ii)implies that for the same timelike curve γ,

$$\begin{aligned}
\tilde{V}(\theta_0) = \tilde{\mathcal{P}}_{\gamma,\infty} &= -t\partial_t \tilde{P}(t_0, \theta_0) + O(t_0^{2\gamma'-1}) \\
&= -t\partial_t P(t_0, \theta_0) + O(t_0^{2\gamma'-1}) + O(\varepsilon_0) = V(\theta_0) + O(t_0^{2\gamma'-1}) + O(\varepsilon_0).
\end{aligned}$$

Again choosing t_0 and ε_0 appropriately small yields Corollary 5.1.3(ii). □

Chapter 6

BKL bounces outside homogeneity: Surface symmetry

In this Chapter, we state and prove out main results concerning *BKL bounces outside homogeneity for the Einstein–Maxwell–scalar field system in surface symmetry*, based on the arXiv preprint [101]. We make frequent references to Chapter 2, in particular the Einstein–Maxwell–scalar field system in the Rein gauge, given in Section 2.5. The results of this Chapter also contain significant similarities to those of Chapter 5, and we make a detailed comparison in Page 248.

6.1 Statement of our main results and overview

In Section 2.5, we listed the Einstein–Maxwell–scalar field system in surface symmetry, for the gauge (2.5.1), in Proposition 2.5.1. Our goal in Chapter 6 is to provide an in-depth description of the behaviour of solutions to the Einstein–Maxwell–scalar field system (Ein)–(Max) in the vicinity of a spacelike singularity. In the gauge (2.5.1), the spacelike singularity corresponds to the boundary portion $\{r = 0\} \subset \partial\mathcal{Q}$.

A priori, it is unclear that a solution arising from reasonable initial data even reaches $\{r = 0\}$. For example, the (subextremal) Reissner-Nordström black hole interior written in the gauge (2.5.1) is given by:

$$\mathbf{g}_{RN} = -\left(-1 + \frac{2M}{r} - \frac{Q^2}{r^2}\right)^{-1} dr^2 + \left(-1 + \frac{2M}{r} - \frac{Q^2}{r^2}\right) dx^2 + r^2 d\sigma_{\mathbb{S}^2},$$

for constants M and Q obeying $0 < |Q| < M$. Hence the solution, at least in this gauge, is only well-defined for $r_- < r < r_+$, where $r_\pm = M \pm \sqrt{M^2 - Q^2}$. It is well-known that the boundary piece $\{r = r_-\}$ is a null (and completely regular) Cauchy horizon as opposed to a spacelike singularity.

Nonetheless, within this Chapter we restrict attention to scenarios where $\{r = 0\}$ is reached. Indeed, our first main result, Theorem 6.1.1, shows global existence towards $\{r = 0\}$ for suitably

chosen initial data. See also previous results of [138, 152, 118, 140] concerning the Einstein–Vlasov and Einstein–scalar field system. We also an analogy to our results in Chapter 3 concerning the Einstein–Maxwell–scalar field system instead in the *double-null gauge*.

Given existence up to $\{r = 0\}$, one should next ask how solutions behave near $r = 0$. For instance, one could consider the asymptotic behaviour of the metric and the matter quantities. This is also partially answered in [117, 140, 103] for the matter models they consider, where *under certain conditions* it is shown that the metric and scalar field behave in a Kasner–like fashion, for instance

$$\phi(r, x) = \Psi(x) \log r + \Xi(x) + o(1) \quad \text{as } r \to 0, \tag{6.1.1}$$

where the coefficients $\Psi(x)$ and $\Xi(x)$ are allowed to depend on the spatial coordinate $x \in \mathbb{S}^1$. Furthermore, the *generalized Kasner exponents* of the near-singularity spacetime are:

$$p_1(x) = \frac{\Psi(x)^2 - 1}{\Psi(x)^2 + 3}, \quad p_2(x) = p_3(x) = \frac{2}{\Psi(x)^2 + 3}. \tag{6.1.2}$$

The "*certain conditions*" alluded to above, in the context of the Einstein–Maxwell–scalar field system, are versions of (QTS) and (SKE) in this new setting. We consider two different cases: for the pure Einstein–scalar field system with $\mathbf{F} = 0$ and $\kappa \in \{0, +1\}$, there are no additional conditions; global existence towards $\{r = 0\}$ and the asymptotics (6.1.1)–(6.1.2) always apply. Furthermore, the function $\Psi(x)$ is permitted to take any value.

When either $\mathbf{F} \neq 0$ or $\kappa = -1$, two additional conditions are required:

- Assumption 1: Global existence towards $\{r = 0\}$ with $e^{2\mu} = -g(\nabla r, \nabla r) \to 0$ uniformly as $r \downarrow 0$. (This is an analogue of (QTS).)
- Assumption 2: If $\mathbf{F} \neq 0$, then assume the *subcriticality condition*, namely that for some $\alpha > 0$ and all r sufficiently small, one has $|r\partial_r \mu|(r, x) \geq \frac{1}{2}(2+\alpha)$. (This, less obviously, is an analogue of (SKE).)

In the case that these assumptions apply then asymptotics of the form (6.1.1)–(6.1.2) were found in Chapter 3, with the caveats being that Chapter 3 is limited to $\kappa = +1$ and uses the double null gauge in place of (2.5.1). One drawback of the double null gauge, as discussed in Section 3.1, is that null coordinates tend to degenerate towards $\{r = 0\}$, and one only has $C^{1,\alpha}$ regularity for say $\Psi(x)$. In our current gauge (2.5.1), the asymptotics (6.1.1)–(6.1.2) can be given to any desired degree of regularity.

Theorem 6.1.0. *Let $(\mathcal{M}, \mathbf{g}, \phi, \mathbf{F})$ be a surface symmetric solution to the Eintein–Maxwell–scalar field system of Proposition 2.5.1, arising from initial data $(\phi, \mu, \lambda, \partial_r\phi, \partial_r\mu, \partial_r\lambda)|_{r=r_0} \in (C^{k+1})^3 \times (C^k)^3$ for some $k \geq 1$. Suppose that either $\mathbf{F} = 0$ and $\kappa \in \{0, +1\}$, or that Assumptions 1 and 2*

above hold. Then there exist C^{k-1} *functions* $\Psi(x)$, $\Xi(x)$, $M(x)$ *and* $\Lambda(x)$ *on* $\mathbb{S}^1$ *such that:*

$$\phi(r,x) = \Psi(x)\log r + \Xi(x) + \mathrm{Err}_\phi(r,x), \tag{6.1.3}$$

$$\mu(r,x) = \tfrac{1}{2}(\Psi(x)^2+1)\log r + M(x) + \mathrm{Err}_\mu(r,x), \tag{6.1.4}$$

$$\lambda(r,x) = \tfrac{1}{2}(\Psi(x)^2-1)\log r + \Lambda(x) + \mathrm{Err}_\lambda(r,x), \tag{6.1.5}$$

where the error terms tend to 0 *in the* C^{k-1} *norm uniformly as* $r \to 0$. *Furthermore,* $M(x)$ *obeys the asymptotic momentum constraint equation*

$$dM(x) = \Psi(x)\cdot d\,\Xi(x). \tag{6.1.6}$$

By performing a change of coordinates to change the gauge (2.5.1) into a Kasner-like form, this confirms the asymptotics (6.1.1)–(6.1.2). To the extent of the author's knowledge, Theorem 6.1.0 is new, but is not our main focus; we therefore defer its proof to Appendix B. Our focus instead is to resolve the questions of Page 40. In the current section, we state the precise version of Theorem 1.5.5 and Theorem 1.5.6 discussed there. The remainder of this section precisely imitates the structure of Section 5.1.

6.1.1 Setup of the initial data

Notation and key parameters

- The real number $\eta \geq 2$ is a constant that captures both the maximum and minimum allowed size of the quantity $r\partial_r\phi$ (see already (6.1.10)). Though η may be chosen arbitrarily large, many of the subsequent parameters will depend on η, and all quantitative bounds will depend on the choice of η.

- The positive real number γ is a small parameter representing the admissible blow up rate for first-order derivatives, for instance we always have $\partial_x\phi = O(r^{-\gamma})$. One can use any γ with $0 < \gamma \leq \frac{1}{100}$.

- The natural number N represents the largest number of ∂_x-derivatives with which we commute our system (2.5.8)–(2.5.12) in deriving energy estimates. That is, our *top-order* energy estimate will control $r\partial_r\partial_x^N f$ and $re^{\mu-\lambda}\partial_x^{N+1} f$ for $f \in \{\phi,\mu,\lambda\}$. Our choice of N will depend on η and we expect $N \to \infty$ as $\eta \to \infty$. Also K will represent an integer with $0 \leq K \leq N$.

- The real number $\zeta > 0$ will represent the maximum admissible size of initial data, see already (6.1.9). ζ may be arbitrarily large so long as it is compensated by taking r_* small, see below.

- The real number $C_* > 0$ will be used in the bootstrap argument, see Section 6.2.1. The eventual choice of C_* will depend on η and ζ, though we do not make this explicit. The real

number $A_* > 0$ will depend on C_* and will represent the rate at which our energies may blow up.

- The real number $r_* > 0$ represents how close we require our initial data to be to $r = 0$ in order to obtain our results. That is, our main results will apply to initial data given at $r = r_0$ so long as $0 < r_0 \leq r_*$. Note that r_* will depend on η and ζ and we expect $r_* \to 0$ as $\eta, \zeta \to \infty$.
- Other constants, often denoted C, will be allowed to depend on all of the aforementioned parameters. We also use δ to represent quantities depending on all these parameters (e.g. η, ζ, r_*) such that $\delta \to 0$ as $r_* \to 0$. We often abuse notation and write for instance "$\delta + \delta = \delta$" or "$C_*\delta = \delta$" etc.

Initial L^∞ and L^2 bounds

As in Proposition 2.5.2, given $Q \in \mathbb{R}$, $\kappa \in \{-1, 0, +1\}$ and $r_0 > 0$, initial data will be given by $(f, r\partial_r f)|_{r=r_0} = (f_D, \dot{f}_D)$ for $f \in \{\phi, \lambda, \mu\}$, where the functions $\phi_D, \mu_D, \lambda_D, \dot{\phi}_D, \dot{\mu}_D, \dot{\lambda}_D$ satisfy certain constraints. Our results will apply to a large class of initial data $(f_D, \dot{f}_D) \in H^{N+1} \times H^N$ that obeys the following three conditions:

- (*Weak subcriticality*) For $x \in \mathbb{S}^1$:

$$\eta^{-1} \leq \dot{\phi}_D(x) = r\partial_r\phi(r_0, x) \leq \eta, \qquad \frac{Q^2}{r_0^2} e^{2\mu_D}(x) = \frac{Q^2}{r_0^2} e^{2\mu}(r_0, x) \leq 1, \tag{6.1.7}$$

- (*Closeness to singularity*) For $x \in \mathbb{S}^1$:

$$e^{2\mu_D}(x) = e^{2\mu}(r_0, x) \leq \zeta r_0, \qquad e^{2(\mu_D - \lambda_D)}(x) = e^{2(\mu-\lambda)}(r_0, x) \leq \zeta r_0, \tag{6.1.8}$$

- (*Energy boundedness*) The following L^2 bound holds:

$$\frac{1}{2} \sum_{f \in \{\phi, \lambda, \mu\}} \sum_{K=0}^{N} \left(\int_{\mathbb{S}^1} (\dot{f}_D^2 + r_0^2 e^{2(\mu_D - \lambda_D)} (\partial_x^{K+1} f_D)^2 + r_0^{2\gamma} (\partial_x^K f_D)^2 \right) dx \Bigg|_{r=r_0} \leq \zeta. \tag{6.1.9}$$

Furthermore, we will require r_* to be chosen sufficiently small depending on η and ζ.

6.1.2 Theorem 6.1.1: Global existence

Theorem 6.1.1 (Global existence)**.** *Consider initial data $(\phi_D, \mu_D, \lambda_D, \dot{\phi}_D, \dot{\mu}_D, \dot{\lambda}_D)$ at $r = r_0$ as in Proposition 2.5.2. Then there exists $r_* = r_*(\eta, \zeta) > 0$ such that if $0 < r_0 \leq r_*$ and the conditions* (6.1.7)–(6.1.9) *are satisfied, then the maximal interval of existence I for the solution (ϕ, μ, λ) to the surface symmetric Einstein–Maxwell–scalar field system* (2.5.8)–(2.5.12) *satisfies $(0, r_0] \subset I$.*

Furthermore, for $(r, x) \in (0, r_0] \times \mathbb{S}^1$ one has the following L^∞ bounds:

$$(4\eta)^{-1} \leq r\partial_r\phi(r, x) \leq 4\eta, \qquad \frac{Q^2}{r^2} e^{2\mu}(r, x) \leq 16\eta^2, \tag{6.1.10}$$

and for $0 \leq K \leq N = N(\eta)$, there exists $C = C(\eta, \zeta, K) > 0$ and $A_* = A_*(\eta) > 0$ such that one has the L^2 energy bounds: for fixed $r > 0$,

$$\sum_{f \in \{\phi, \lambda, \mu\}} \left(\int (r\partial_r \partial_x^K f)^2 + (re^{\mu - \lambda} \partial_x^{K+1} f)^2 + r^{-2\gamma} (\partial_x^K f)^2 \, dx \right) \leq C r^{-2A_* - 2K\gamma}. \tag{6.1.11}$$

Remark. Theorem 6.1.1 is independent of the choice of sectional curvature $\kappa \in \{-1, 0, +1\}$, as well as the choice of $Q \in \mathbb{R}$, so long as the second equation in (6.1.7) is satisfied. So Theorem 6.1.1 applies equally to the surface symmetric Einstein–scalar field system and to the surface symmetric Einstein–Maxwell–scalar field system. We make distinctions between these two models in Theorem 6.1.2.

6.1.3 Theorem 6.1.2: BKL bounces

Theorem 6.1.2 (Bounces)**.** *Let (ϕ, μ, λ) be a solution to the surface symmetric Einstein–scalar field system (2.5.8)–(2.5.12), arising from initial data $(\phi_D, \mu_D, \lambda_D, \dot{\phi}_D, \dot{\mu}_D, \dot{\lambda}_D)$ obeying (6.1.7)–(6.1.9) and $0 < r_0 \leq r_*$ as in Theorem 6.1.1.*

Let $\gamma : (0, r_0] \to \mathcal{Q}$ be any timelike curve parameterized by r, and let $\mathcal{P}_\gamma(r) = r\partial_r \phi(\gamma(r))$ and $\mathcal{Q}_\gamma(r) = \frac{Q^2}{r^2} e^{2\mu}(\gamma(r))$, note $\mathcal{Q} \equiv 0$ if and only if $Q = 0$. Then there exist error terms $\mathcal{E}_\mathcal{P}(r), \mathcal{E}_\mathcal{Q}(r)$ depending on γ but with $|\mathcal{E}_\mathcal{P}(r)|, |\mathcal{E}_\mathcal{Q}(r)| \leq r^{1/2}$ uniformly in the choice of γ, such that

$$r\frac{d}{dr}\mathcal{P}_\gamma = -\mathcal{P}_\gamma \mathcal{Q}_\gamma + \mathcal{E}_\mathcal{P}, \qquad r\frac{d}{dr}\mathcal{Q}_\gamma = \mathcal{Q}_\gamma(\mathcal{P}_\gamma^2 - 1 - \mathcal{Q}_\gamma - \mathcal{E}_\mathcal{Q}). \tag{6.1.12}$$

Furthermore, there exists $C = C(\eta, \zeta) > 0$ such that:

(i) If $Q = 0$, then $\mathcal{P}_\gamma(r)$ converges to $\mathcal{P}_{\gamma,\infty}$ as $r \to 0$ and $|\mathcal{P}_{\gamma,\infty} - \mathcal{P}_\gamma(r_0)| \leq C r_0^{1/2}$.

(ii) If $Q \neq 0$, then $\mathcal{Q}_\gamma(r)$ converges to 0 as $r \to 0$, while $\mathcal{P}_\gamma(r)$ converges to $\mathcal{P}_{\gamma,\infty}$ as $r \to 0$, necessarily satisfying $\mathcal{P}_{\gamma,\infty} \geq 1$ as well as:

$$|\mathcal{P}_{\gamma,\infty} - \max\{\mathcal{P}_\gamma(r_0), \mathcal{P}_\gamma(r_0)^{-1}\}| \leq C(\mathcal{Q}_\gamma(r_0) + r_0^{1/2}). \tag{6.1.13}$$

A corollary of Theorem 6.1.2 is the following stability and instability statement regarding *charged* perturbations of a class of *uncharged* solutions to the surface symmetric Einstein–scalar field system. We assume that $\kappa \in \{0, +1\}$ so as to be able to apply Theorem 6.1.0 for the unperturbed uncharged spacetime.

Corollary 6.1.3 (Stability / Instability)**.** *Let (ϕ, μ, λ) be a smooth (i.e. C^∞) solution to the surface symmetric Einstein–scalar field system (2.5.8)–(2.5.12) with $Q = 0$ and $\kappa \in \{0, +1\}$, arising from initial data given by $(\phi_D, \mu_D, \lambda_D, \dot{\phi}_D, \dot{\mu}_D, \dot{\lambda}_D)$ at $r = r_1 > 0$. By Theorem 6.1.0, there exists smooth $\Psi, \Xi : \mathbb{S}^1 \to \mathbb{R}$ such that $\phi(r, x) = \Psi(x) \log r + \Xi(x) + o(1)$ as $r \to 0$.*

Suppose further that $\Psi(x) > 0$ for all $x \in \mathbb{S}^1$. Then there exists $N \in \mathbb{N}$ such that for possibly charged perturbations of the initial data, i.e. $|Q| \leq \varepsilon$, and $\|(\tilde{f}_D, \dot{\tilde{f}}_D) - (f_D, \dot{f}_D)\|_{H^{N+1} \times H^N} \leq$

ε for $f \in \{\phi, \mu, \lambda\}$, then for ε sufficiently small the solution $(\tilde{\phi}, \tilde{\mu}, \tilde{\lambda})$ arising from data $(\tilde{\phi}_D, \tilde{\mu}_D, \tilde{\lambda}_D, \dot{\tilde{\phi}}_D, \dot{\tilde{\mu}}_D, \dot{\tilde{\lambda}}_D)$ and charge Q has (a) global existence towards $r = 0$, (b) is such that for any $x \in \mathbb{S}^1$ the limit $\tilde{\Psi}(x) = \lim_{r\to 0} r\partial_r\phi(r, x)$ exists and moreover for any $\tilde{\varepsilon} > 0$, if ε is chosen small enough depending on $\tilde{\varepsilon}$ then:

(i) $|\tilde{\Psi}(x) - \Psi(x)| \leq \tilde{\varepsilon}$ *if* $Q = 0$,

(ii) $|\tilde{\Psi}(x) - \max\{\Psi(x), \Psi(x)^{-1}\}| \leq \tilde{\varepsilon}$ *if* $Q \neq 0$.

Remark. We consider Corollary 6.1.3 a stability / instability result because in a very similar way to Corollary 5.1.3 in the Gowdy symmetric setting. See also Figure 6.1 below. It remains an open problem to study cases where $\Psi(x)$ is allowed to change sign; this is outside the scope of this article, given our dependence on $\eta \geq 2$ and the weak subcriticality condition (6.1.7).

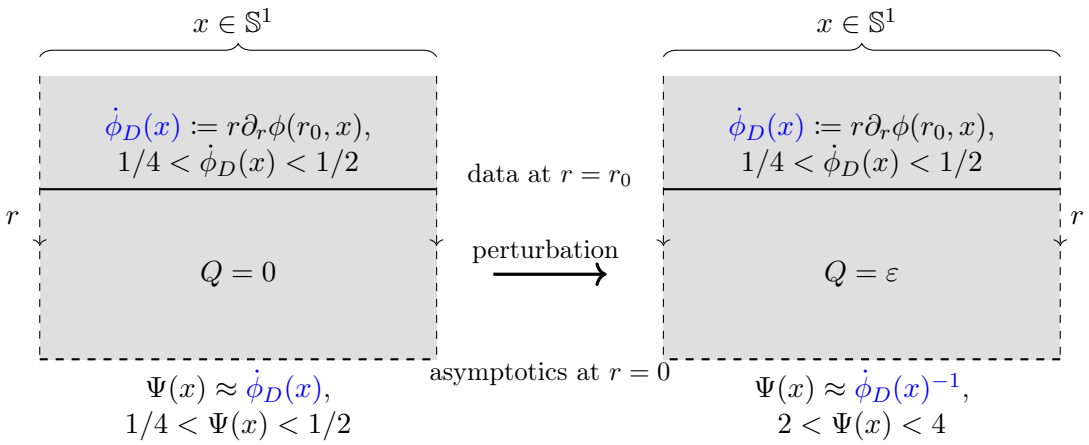


Figure 6.1: Example of a bounce instability arising from a charged perturbation of an uncharged spacetime. In the unperturbed spacetime (left) $Q = 0$ and the asymptotic quantity $\Psi(x)$ is close to the (inhomogeneous) data $\dot{\phi}_D(x)$, while in the charged spacetime (right) with $Q = \varepsilon$ the bounce of Theorem 6.1.2 applied to constant x-curves γ means that $\Psi(x)$ is instead close to $\dot{\phi}_D(x)^{-1}$.

6.1.4 Outlook: bounces outside symmetry

Before moving to a sketch of the proof, we consider a natural conjecture regarding bounces, complementary to those in Section 5.1.4, which we hope to return to in the near future. That is, to understand an analogue of this result for the full $1 + 3$-dimensional Einstein–scalar field outside of symmetry.

Conjecture 6.1.4. *There exists an open set of initial data for the Einstein scalar field equations with no symmetry assumptions such that the maximal (past) development arising from such initial data terminates in a Kasner-like spacelike singularity at $t = 0$, but such that the intermediate dynamics (between data and the singularity) exhibit one, or potentially a finite number, of BKL bounces.*

A resolution of this conjecture would present a key step towards understanding the full heuristics of BKL in $1+3$-dimensional vacuum; the latter problem is substantially more difficult since [16] suggests that in vacuum there are infinitely many BKL bounces and that the ODE dynamics are chaotic. As mentioned above, addition of the scalar field allows study of a regime where there are only finitely many bounces.

6.1.5 Sketch of the proof

The proof will proceed in a very similar manner to the Gowdy proof in Section 5.1.5, and we illustrate the three key steps using two of the equations in the Einstein–Maxwell–scalar field system. These two equations are the μ evolution equation (2.5.9) and the wave equation for ϕ (2.5.12). Letting $\kappa = 0$ for simplicity, these equations are:

$$r\partial_r \mu = \frac{1}{2}\left((r\partial_r \phi)^2 + r^2 e^{2(\mu-\lambda)} (\partial_x \phi)^2 + 1 - \frac{Q^2}{r^2} e^{2\mu} \right), \tag{6.1.14}$$

$$r\partial_r (r\partial_r \phi) = r^2 e^{2(\mu-\lambda)} \partial_x^2 \phi + r^2 e^{2(\mu-\lambda)} \partial_x(\mu-\lambda)\partial_x \phi - \frac{Q^2}{r^2} e^{2\mu}\, r\partial_r \phi. \tag{6.1.15}$$

We now describe the three major steps as follows:

Step 1: **Analysis in the spatially homogeneous case:** Here, this involves setting all terms involving a ∂_x-derivative to zero, and understanding the resulting nonlinear ODE system.

Step 2: **Linearization of the ODE system:** In the second step, we consider the *best possible behaviour* of the terms involving a ∂_x-derivative. We do this by taking commuting ∂_x with the ODE system in Step 1, resulting in a new linear ODE system for certain ∂_x derivatives whose coefficients are given by the solution (i.e. some orbit) of the ODE system in Step 1.

Step 3: **Energy estimates:** Finally, one must ensure that we can close our argument without loss of derivatives (necessary due to the existence of top order terms such as $\partial_x^2 \phi$ in (6.1.15).) One achieves this via L^2 energy estimates, which are allowed to blow up but only at a mild rate as $r \to 0$.

We now explain in more detail how each of these steps applies to our simplified system (6.1.14)–(6.1.15). For Step 1, we remove the terms with ∂_x-derivatives in these equations. Letting $\mathcal{P} = r\partial_r \phi$ and $\mathcal{Q} = \frac{Q^2}{r^2} e^{2\mu}$, upon removing these terms and rephrasing in terms of $\mathcal{P}$ and $\mathcal{Q}$ we yield the ODE system:

$$r\partial_r \mathcal{P} = -\mathcal{P}\mathcal{Q}, \qquad r\partial_r \mathcal{Q} = \mathcal{Q}(\mathcal{P}^2 - 1 - \mathcal{Q}). \tag{6.1.16}$$

We restrict attention to the region $\mathcal{P} > 0, \mathcal{Q} \geq 0$. Then this ODE system contains a line of fixed points at $\mathcal{Q} = 0$, each of which represents an exact generalized Kasner solution to the Einstein–scalar field system (with $\mathbf{F} = 0$). Moreover these fixed points are (orbitally) stable if $\mathcal{P} \geq 1$ and unstable otherwise. (Note that stability is meant in the direction $r \to 0$.)

The dynamics of the ODE system in the remaining region $\mathcal{P} > 0, \mathcal{Q} > 0$ may be described as a union of *heteroclinic orbits* linking an unstable fixed point to a stable fixed point. In fact, the dynamics can be solved exactly due to the fact that $\mathcal{K} := \mathcal{P} + \mathcal{P}^{-1} + \mathcal{Q}\mathcal{P}^{-1}$ turns out to be a conserved quantity of the system. As a consequence, these heteroclinic orbits, which we identify as the BKL bounces, link the unstable fixed point $(\mathcal{P} = \alpha, \mathcal{Q} = 0)$ to the stable fixed point $(\mathcal{P} = \alpha^{-1}, \mathcal{Q} = 0)$, for any $\alpha > 1$. The phase portrait for the ODE system is given below in Figure 6.2.

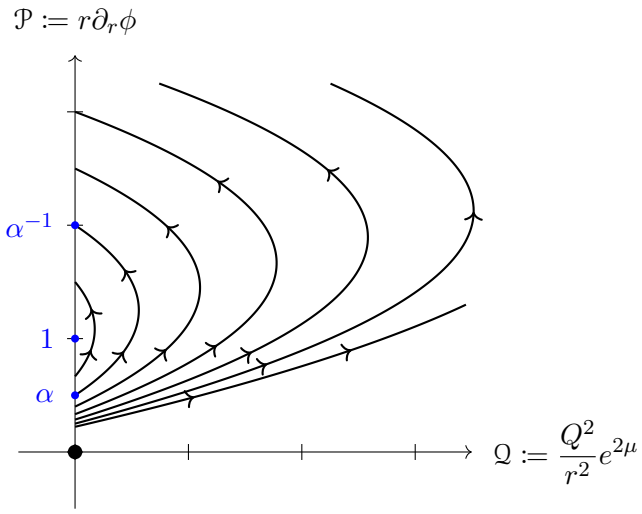


Figure 6.2: Phase portrait showing the dynamics of $\mathcal{P}$ and $\mathcal{Q}$ towards $r = 0$ in the exactly spatially homogeneous case. A bounce is a transition from α to α^{-1}.

The requirement of weak subcriticality means that even upon evolution of $\mathcal{P}$ and $\mathcal{Q}$ according to (6.1.16), the dynamics will remain in a bounded portion of the $(\mathcal{P}, \mathcal{Q})$-plane, and importantly this will remain true if we introduce small error terms in (6.1.16). This will be essential in both Step 2 and Step 3.

We now move to Step 2, which involves commuting (6.1.16) with ∂_x. In our case, it is actually more convenient instead to find evolution equations for $\mathcal{M} = \partial_x\mu = \partial_x \log \mathcal{Q}$ and $\mathcal{N} = \partial_x\mathcal{P}$. The idea is that commutation will yield a *linear* ODE system for $\mathcal{M}$ and $\mathcal{N}$ whose coefficients are functions of the (dynamical) $\mathcal{P}$ and $\mathcal{Q}$, namely the linear system

$$r\partial_r\mathcal{M} = -\mathcal{Q}\mathcal{M} + \mathcal{P}\mathcal{N}, \qquad r\partial_r\mathcal{N} = -2\mathcal{P}\mathcal{Q}\mathcal{N} - \mathcal{Q}\mathcal{N}. \tag{6.1.17}$$

One then uses (6.1.17) to find estimates for $\mathcal{M}$ and $\mathcal{N}$ that are *uniform* in the choice of dynamical orbit for $\mathcal{P}$ and $\mathcal{Q}$ (at least for orbits compatible with the boundedness property above). One finds that for any $\gamma > 0$, one has[1] $\mathcal{M}, \mathcal{N} = O(r^{-\gamma})$.

The outcome of Step 2 is that heuristically each ∂_x-derivative comes with a loss of $r^{-\gamma}$. Since one may choose $\gamma < 1$, this confirms AVTD behaviour in the sense that ∂_x-derivatives cost less than

[1] Actually, for the exact system (6.1.17), it can be shown that $\mathcal{M} = O((\log r)^2), \mathcal{N} = O(\log r)$, but since in the eventual analysis we encounter error terms it will be more straightforward to allow a little loss.

a ∂_r-derivative. For instance, one expects $\partial_x\phi = O(r^{-\gamma}), \partial_x(\mu-\lambda) = O(r^{-\gamma})$, and $\partial_x^2\phi = O(r^{-2\gamma})$ in the equations (6.1.14)–(6.1.15). Using this, as well as that $e^{\mu-\lambda} = O(r)$, one verifies that the terms involving a ∂_x-derivative, which we threw away in Step 1, are all of size $O(r^{4-2\gamma})$. So these are integrable with respect to $\frac{dr}{r}$ towards $r = 0$, and may be treated as negligible errors in the ODEs (6.1.16).

Step 3 concerns turning the above paragraph into rigorous bounds. In particular, one must overcome the issue of *derivative loss* due to terms such as $\partial_x^2\phi$. This is achieved via energy estimates; define for instance the Kth order energy $\mathcal{E}_\phi^{(K)}(r)$ as:

$$\mathcal{E}_\phi^{(K)}(r) = \frac{1}{2}\int_{\mathbb{S}^1}\left((r\partial_r\partial_x^K\phi(r,x))^2 + r^2e^{2(\mu-\lambda)}(\partial_x^{K+1}\phi(r,x))^2 + (\partial_x^K\phi(r,x))^2\right)dx.$$

To estimate $\mathcal{E}_\phi^{(K)}(r)$, we commute the wave equation (6.1.15) with ∂_x^K, yielding:

$$(r\partial_r)^2\partial_x^K\phi = r^2e^{2(\mu-\lambda)}\partial_x^{K+2}\phi - \overbrace{\frac{Q^2}{r^2}e^{2\mu}\left(r\partial_r\partial_x^K\phi + (2\partial_x^K\mu)(r\partial_r\phi)\right)}^{\text{(I)}} + \underbrace{3r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x^{K+1}\phi}_{\text{(II)}} - \underbrace{\frac{Q^2}{r^2}e^{2\mu}(2\partial_x\mu)(r\partial_r\partial_x^{K-1}\phi)}_{\text{(III)}} + \cdots.$$

There are many more terms in the $\cdots$, but the terms present here will illustrate the ideas necessary in the energy estimate. A standard argument yields the derivative estimate

$$\left|r\frac{d}{dr}\mathcal{E}_\phi^{(K)}(r)\right| \leq \sqrt{2\mathcal{E}_\phi^{(K)}(r)}\cdot\left(\|\text{(I)} + \text{(II)} + \text{(III)} + \cdots\|_{L^2}\right).$$

It thus suffices to estimate each of (I), (II) and (III) in L^2. Here we need to use Step 1 and Step 2. For the first term in (I), we use that $\frac{Q^2}{r^2}e^{2\mu}$ is less than the maximum value of $\mathcal{Q}$ over all orbits, which is bounded, while $\|r\partial_r\partial_x^K\phi\|_{L^2} \leq \sqrt{2\mathcal{E}_\phi^{(K)}(r)}$ by definition. For the second term in (I), we also use that $r\partial_r\phi$ is less than the maximum value of $\mathcal{P}$, while $\partial_x^K\mu$ is controlled by (the square root) of some other Kth order energy. We thus conclude that for some constant A_* depending only on η,

$$\|\text{(I)}\|_{L^2} \leq A_*\sqrt{2\mathcal{E}_\phi^{(K)}(r)}.$$

For the expression (II), we use that $\|re^{(\mu-\lambda)}\partial_x^{K+1}\phi\|_{L^2} \leq \sqrt{2\mathcal{E}_\phi^{(K)}(r)}$, while the prefactor $re^{\mu-\lambda}\partial_x(\mu-\lambda)$ is of size $O(r^{2-\gamma})$ by Step 2. So there is a constant $C_{\eta,\zeta,K}$ depending on η, the data and also[2] K so that

$$\|\text{(II)}\|_{L^2} \leq C_{\eta,\zeta,K}\, r^{2-\gamma}\sqrt{2\mathcal{E}_\phi^{(K)}(r)}.$$

Finally, in (III), by Step 1 and Step 2 we know that $\frac{Q^2}{r^2}e^{2\mu}(2\partial_x\mu)$ is $O(r^{-\gamma})$. This seems alarming, since this is not integrable with respect to $\frac{dr}{r}$ as $r \to 0$. But the other factor in (III), $r\partial_r\partial_x^{K-1}\phi$ is

[2]The dependence on the regularity index K is because there are secretly more terms like (II) hidden in $\cdots$.

not dependent on the Kth order energy, but instead the $(K-1)$th order energy $\sqrt{2\mathcal{E}_\phi^{(K-1)}(r)}$. So:

$$\|(\mathrm{III})\|_{L^2} \leq C_{\eta,\zeta,K}\, r^{-\gamma}\sqrt{2\mathcal{E}_\phi^{(K-1)}(r)}.$$

Combining all of the above, we obtain the following derivative estimate for $K \geq 1$:

$$\left| r\frac{d}{dr}\mathcal{E}_\phi^{(K)}(r) \right| \leq 2(A_* + C_{\eta,\zeta,K}\, r^{2-\gamma})\mathcal{E}_\phi^{(K)}(r) + 2C_{\eta,\zeta,K}\, r^{-\gamma}\sqrt{\mathcal{E}_\phi^{(K)}(r)}\cdot\sqrt{\mathcal{E}_\phi^{(K-1)}(r)} + \cdots . \quad (6.1.18)$$

When $K = 0$, the last term on the right hand side is absent. Because of the $2A_*$, even the 0th order energy $\mathcal{E}^{(0)}(r)$ will blow up as r^{-2A_*} as $r \to 0$. Furthermore, the appearance of $r^{-\gamma}$ means that as K increases the rate of blow up also increases. But in any case, one uses this expression to find an energy bound

$$\mathcal{E}^{(K)}(r) \leq D_{\eta,K,\zeta}\, r^{-2A_*-2K\gamma},$$

where $D_{\eta,K,\zeta}$ depends on η and the data, as well as the regularity index K. It is crucial that A_* is *independent of* K. This is because applying an L^2-L^∞ interpolation argument to the above bound, one shows e.g. that $\partial_x^2\phi = O(r^{-2\gamma-\delta})$, where $\delta \to 0$ as $N \to \infty$, where N is the maximum regularity index for which we perform the energy estimate.

To apply Steps 1 to 3 in the nonlinear problem, one uses a standard bootstrap argument; note we actually perform Step 3 first. That is, one first bootstraps L^∞-bounds on 0th order and 1st order quantities, see (B1)–(B4), and uses this to derive the energy estimate (6.1.18). From the energy estimate, the interpolation argument is used to control expressions such as $\partial_x^2\phi$, which are then treated as error terms in the ODE analysis of Step 1 and Step 2. The nonlinear ODE analysis then allows us to improve the bootstrap assumptions (B1)–(B4), completing both the proof of global existence (Theorem 6.1.1) and first part of the bounce theorem (Theorem 6.1.2). The remainder of Theorem 6.1.2 follows from more detailed ODE analysis.

6.1.6 Comparison to results in Gowdy symmetry

Before continuing to the proof proper, we make an explicit comparison between our results and those of Chapter 5 concerning BKL bounces in Gowdy symmetry. Referring back to Section 2.6 for details on the equations in Gowdy, we provide a "dictionary" translating between the two.

- Due to the reduction to $1+1$-dimensions, the Gowdy spacetimes have one dynamical time variable t and spatial variable $\theta \in \mathbb{S}^1$, in comparison to r and $x \in \mathbb{S}^1$ in our surface symmetric spacetimes.

- Just as our surface symmetric spacetimes may be described as Kasner–like using Theorem 6.1.0, with the exponents given using $\Psi(x) = \lim_{r\to 0} r\partial_r\phi(r,x)$ and the correspondence (6.1.2), the $\{t = 0\}$ singularity of Gowdy spacetimes can also be described as Kasner–like

using the asymptotic quantity $V(\theta) = \lim_{t\to 0}(-t\partial_t P(t,\theta))$ and the analogous correspondence (5.1.6). However, unlike $V(\theta)$, the function $\Psi(x)$ is smooth and contains no "spike points" of discontinuity.

- The *polarized Gowdy spacetimes* of Definition 2.6.1, where the asymptotic quantity $V(\theta)$ is smooth and may be prescribed freely i.e. no longer has the restriction $0 < V(\theta) < 1$, can be held in direct analogy with the *uncharged* Einstein–scalar field solutions in surface symmetry; in light of Theorem 6.1.0, at least if $\kappa \in \{0, +1\}$ then the function $\Psi(x)$ can similarly be prescribed freely. On the other hand, if $\mathbf{F} \neq 0$ then necessarily $|\Psi(x)| \geq 1$.

- Moving on to the results, Theorem 5.1.1 and Theorem 6.1.1 both characterise initial data for the (unpolarised) Gowdy system at $t = t_0 > 0$ and the Einstein–Maxwell–scalar field system at $r = r_0 > 0$ such that one reaches a singularity at $\{t = 0\}$ and $\{r = 0\}$ respectively.

  In both cases, this class of initial data is engineered to be large enough to allow for spacetimes with nontrivial intermediate dynamics i.e. between $0 < t < t_0$ or $0 < r < r_0$ suggestive of BKL bounces.

  Indeed, Theorem 5.1.2 is the analogue of our bounce result Theorem 6.1.2, proving that on any causal curve γ directed towards the singularity, certain quantities an ODE reminiscent of BKL bounces.

- There is, however, one feature of [100] that does not appear in the present article, namely the aforementioned "spikes". Spikes correspond to special orbits of the ODE system converging to unstable fixed points, and generically occur at finitely many $\theta \in \mathbb{S}^1$ in Gowdy spacetimes.

  We comment that spikes do not occur for the Einstein–Maxwell–scalar field in surface symmetry since the rigidity of the electromagnetic field means that for $Q \neq 0$ it is impossible for any orbit to converge to an unstable fixed point. If one considered the Einstein–Maxwell–*charged* scalar field system where ϕ is dynamically coupled to $\mathbf{F}$, then spikes would similarly become an issue.

6.2 Bootstrap assumptions and energies

Theorems 6.1.1 and 6.1.2 will be proven simultaneously via a bootstrap argument. That is, we suppose that a solution (ϕ, μ, λ) to the surface symmetric Einstein–Maxwell–scalar field system (2.5.8)–(2.5.12) exists in the interval $r \in [r_b, r_0]$, where $r_b > 0$ is a "bootstrap time".

We moreover suppose that the solution satisfies certain pointwise bootstrap assumptions in this interval $[r_b, r_0]$, namely (B1)–(B4) below. The strategy is to then use these bootstrap assumptions to derive energy estimates, where energies are suitable L^2 type integrals to be defined in Section 6.2.2. Though the energy estimates will blow up mildly towards $r = 0$, they will be used together with an

ODE-based argument to eventually improve our L^∞ bootstrap assumptions, completing the proof of Theorem 6.1.1. Further analysis of the ODEs will yield Theorem 6.1.2.

6.2.1 The L^∞ bootstrap assumptions

As mentioned in Section 6.1.1, $C_* = C_*(\eta, \zeta) > 0$ will represent a large real number to be chosen later in the argument. In the proof of Theorem 6.1.1, we make repeated reference to the following four low order L^∞ bootstrap assumptions:

$$\frac{Q^2}{r^2} e^{2\mu} \leq C_*, \quad e^{2\mu} + e^{2(\mu-\lambda)} \leq C_* r, \tag{B1}$$

$$|r\partial_r \phi|,\ |r\partial_r \mu|,\ |r\partial_r \lambda| \leq C_*, \tag{B2}$$

$$|\partial_x \phi|,\ |\partial_x \mu|,\ |\partial_x \lambda| \leq C_* r^{-\gamma}, \tag{B3}$$

$$|r\partial_r \partial_x \phi|,\ |r\partial_r \partial_x \mu|,\ |r\partial_r \partial_x \lambda| \leq C_* r^{-\gamma}. \tag{B4}$$

Using the assumptions on the initial data (6.1.7)–(6.1.9) and Sobolev embedding, one can choose C_* depending on ζ such that the (B1)–(B4) will hold in a neighborhood of $r = r_0$. Thus we can always find some r_b so that we can carry out our bootstrap argument in the interval $r \in [r_b, r_0]$.

6.2.2 Energies

Below, we define the L^2 type quantities that will appear in our energy estimates of Section 6.3. Since eventually have a hierarchy of energy estimates (see already Proposition 6.3.4), we define a separate energy at each order of differentiability, up to a top-order energy with N derivatives, with N eventually chosen depending on η.

Definition 6.2.1. Let $0 \leq K \leq N$. Define the following Kth order energies at fixed r:

$$\mathcal{E}_\phi^{(K)}(r) \coloneqq \frac{1}{2} \int \left((r\partial_r \partial_x^K \phi)^2 + r^2 e^{2(\mu-\lambda)} (\partial_x^{K+1} \phi)^2 + (\partial_x^K \phi)^2 \right) dx, \tag{6.2.1}$$

$$\mathcal{E}_\mu^{K)}(r) \coloneqq \frac{1}{2} \int \left((r\partial_r \partial_x^K \mu)^2 + r^2 e^{2(\mu-\lambda)} (\partial_x^{K+1} \mu)^2 + (\partial_x^K \mu)^2 \right) dx, \tag{6.2.2}$$

$$\mathcal{E}_\lambda^{(K)}(r) \coloneqq \frac{1}{2} \int \left((r\partial_r \partial_x^K \lambda)^2 + r^2 e^{2(\mu-\lambda)} (\partial_x^{K+1} \lambda)^2 + (\partial_x^K \lambda)^2 \right) dx, \tag{6.2.3}$$

$$\mathcal{E}^{(K)}(r) \coloneqq \mathcal{E}_\phi^{(K)}(r) + \mathcal{E}_\mu^{(K)}(r) + \mathcal{E}_\lambda^{(k)}(r). \tag{6.2.4}$$

Remark. Even at order $K = 0$, the energy $\mathcal{E}_\phi^{(0)}(r)$ includes the lower order term ϕ^2. Even when that Theorem 6.1.0 applies, this will blow up as $O((\log r)^2)$. However, since we allow our energies to blow up towards $r = 0$, this $O((\log r)^2)$ blow up will not be of any concern. It will be useful

to include the $(\partial_x^K \phi)^2$ term in $\mathcal{E}_\phi^{(K)}(r)$ since this is how we will eventually get pointwise control on ∂_x-derivatives of ϕ without the problematic weight $re^{\mu-\lambda}$ which vanishes as $r \to 0$.

6.3 The energy estimate hierarchy

We now derive energy estimates for $\mathcal{E}^{(K)}(r)$, at orders $0 \leq K \leq N$, where $N = N(\eta)$ is chosen sufficiently large. The derivation of such energy estimates will involve commuting the equations (2.5.8)–(2.5.12) with up to K ∂_x-derivatives. See Section 6.1.5 for an overview.

For the junk and lower order terms in the hierarchy (where the precise value of coefficients arising in the commuted equations are not crucial), we introduce the following schematic notation: expressions such as

$$\sum_{k_p+k_1+\ldots+k_i=K} \partial_x^{k_p} f * \partial_x^{k_1} g * \cdots * \partial_x^{k_i} g$$

will represent some linear combination of products of the form $\partial_x^{k_p} f \cdot \partial_x^{k_1} g \cdots \partial_x^{k_i} g$ such that $i \geq 1$, $k_p \geq 1$ and $k_j \geq 1$ for all $1 \leq j \leq i$ and $k_p + k_1 + \ldots + k_i = K$. We emphasize that unless explicitly stated otherwise, in these schematic sums i and j will be positive integers, as are the indices k_p, k_1, k_i etc. In the event that any index e.g. k_p is allowed to be 0, this will be explicitly stated, and similarly if there are further constraints on any index.

6.3.1 Energy estimates for the scalar field ϕ

Proposition 6.3.1. *Let (μ, λ, ϕ, Q) be a solution to the Einstein–Maxwell–scalar field system (2.5.8)–(2.5.12) obeying the bootstrap assumptions (B1)–(B4). Then there exists a constant A_* depending only on C_*, as well as constants $C_1^{(K)}$ and $C_2^{(K)}$ depending on C_* and the regularity index $K \in \{0, 1, \ldots, N\}$, such that*

$$\left| r \frac{d}{dr} \mathcal{E}_\phi^{(K)}(r) \right| \leq 2A_* \mathcal{E}^{(K)}(r) + C_1^{(K)} r^{1-\gamma} \mathcal{E}^{(K)}(r) + \sum_{k=0}^{K-1} C_2^{(K)} r^{-2\gamma(K-k)} \mathcal{E}^{(k)}(r), \tag{6.3.1}$$

where it is understood that the final term is absent if $K = 0$.

Proof. For any $K \geq 1$, commuting the wave equation (2.5.12) with ∂_x^K yields

$$(r\partial_r)^2 \partial_x^K \phi - r^2 e^{2(\mu-\lambda)} \partial_x^2 \partial_x^K \phi = \left(-\frac{Q^2}{r^2} + \kappa \right) e^{2\mu} \left(r\partial_r \partial_x^K \phi + 2r\partial_r \phi \, \partial_x^K \mu \right) \tag{6.3.2a}$$

$$+ r^2 e^{2(\mu-\lambda)} \sum_{k_p+k_1+\cdots+k_i=K+2} \partial_x^{k_p} \phi * \partial_x^{k_1} (\mu - \lambda) * \cdots * \partial_x^{k_i} (\mu - \lambda) \tag{6.3.2b}$$

$$+ \left(-\frac{Q^2}{r^2} + \kappa \right) e^{2\mu} \sum_{\substack{0 \leq k_p < K, k_1 < K \\ k_p+k_1+\cdots+k_i=K}} r\partial_r \partial_x^{k_p} \phi * \partial_x^{k_1} \mu * \cdots * \partial_x^{k_i} \mu. \tag{6.3.2c}$$

Note that in the uncommuted case $K = 0$, the final term on the RHS of (6.3.2a) is absent, as is the term (6.3.2c); we leave the derivation of (6.3.1) in this case to the reader.

The first line (6.3.2a) is the leading order contribution that gives rise to the $2A_*\mathcal{E}^{(K)}(r)$ on the RHS of (6.3.1). It remains to estimate the remaining lines (6.3.2b) and (6.3.2c) in L^2. We start with (6.3.2c). Using the bootstrap assumptions (B1)–(B2) and the Sobolev product estimate Lemma 5.2.2 with $w = 1$, one finds that

$$\begin{aligned}\|(6.3.2c)\|_{L^2} &\leq C_*^2 \left\| \sum_{\substack{0\leq k_p<K, k_1<K \\ k_p+k_1+\cdots+k_i=K}} r\partial_r \partial_x^{k_p}\phi * \partial_x^{k_1}\mu * \cdots * \partial_x^{k_i}\mu \right\|_{L^2} \\ &\lesssim_{C_*} \sum_{k=0}^{K-1} \left(\|r\partial_r\partial_x^k\phi\|_{L^2} + \|\partial_x^k\mu\|_{L^2} \right) \cdot \left(\|r\partial_r\partial_x\phi\|_{L^\infty} + \|\partial_x\mu\|_{L^\infty} \right)^{K-k}.\end{aligned}$$

As this is the first time we derive such an estimate, we include a more detailed explanation. In the first line, we use (B1) to estimate first $\left| -\frac{Q^2}{r^2}e^{2\mu} + \kappa e^{2\mu} \right| \leq C_*$, and if $k_p = 0$ we get another C_* from $r\partial_r\phi$ and (B2).

To get the second line, for summands such that there are $i+1$ terms in the product, note that the maximum number of ∂_x-derivatives landing on either $r\partial_r\phi$ or μ is $K - i$. Furthermore, (repeated) use of the product estimate Lemma 5.2.2 allows us to put exactly this many ∂_x-derivatives on some term in the product, which we estimate in L^2, while the remaining i terms in the product are estimated by either $\|\partial_x(r\partial_r\phi)\|_{L^\infty}$ or $\|\partial_x\mu\|_{L^\infty}$. By setting $k = K - i \in \{1, \ldots, k-1\}$, we obtain the desired estimate.

By using the remaining bootstrap assumptions (B3)–(B4) to estimate $\|r\partial_r\partial_x\phi\|_{L^\infty} + \|\partial_x\mu\|_{L^\infty}$, as well as the definition of $\mathcal{E}^{(k)}(r)$, we therefore deduce that

$$\|(6.3.2c)\|_{L^2} \lesssim_{C_*,K} \sum_{k=0}^{K-1} r^{-\gamma(K-k)} \sqrt{\mathcal{E}^{(k)}(r)}. \tag{6.3.3}$$

We move onto the L^2 estimate for (6.3.2b). For this term we first use (B1) to remove one of the factors of $e^{\mu-\lambda}$, but for the remaining factor of $e^{\mu-\lambda}$ we use Lemma 5.2.2 with $w = re^{\mu-\lambda}$. In particular, $\partial_x W = \partial_x \log w = \partial_x(\mu - \lambda)$, and one gets

$$\begin{aligned}\|(6.3.2b)\|_{L^2} &\leq rC_* \left\| re^{\mu-\lambda} \sum_{k_p+k_1+\cdots+k_i=K+2} \partial_x^{k_p}\phi * \partial_x^{k_1}(\mu-\lambda) * \cdots * \partial_x^{k_i}(\mu-\lambda) \right\|_{L^2} \\ &\lesssim_{C_*} r \sum_{k=1}^{K+1} \left(\|re^{\mu-\lambda}\partial_x^k\phi\|_{L^2} + \|re^{\mu-\lambda}\partial_x^k(\mu-\lambda)\|_{L^2} \right) \cdot \left(\|\partial_x\phi\|_{L^\infty} + \|\partial_x(\mu-\lambda)\|_{L^\infty} \right)^{K+2-k}.\end{aligned}$$

It is important we applied Lemma 5.2.2 in such a way that the top order term $\partial_x^{K+1}\phi$ has the weight $re^{\mu-\lambda}$ in front to match with the energy $\mathcal{E}^{(K)}(r)$. Fortunately, the weighted version of Lemma 5.2.2 means that any additional factors of $\|\partial_x(\mu-\lambda)\|_{L^\infty}$ reduce the number of derivatives on either $re^{\mu-\lambda}\partial_x^k\phi$ or $re^{\mu-\lambda}\partial_x^k(\mu-\lambda)$, and we obtain the above.

Using the bootstrap assumption (B3) and the definition of $\mathcal{E}^{(k)}(r)$, we thus deduce

$$\|(6.3.2\text{b})\|_{L^2} \lesssim_{C_*,K} \sum_{k=0}^{K} r^{1-\gamma(K+1-k)}\sqrt{\mathcal{E}^{(k)}(r)}. \tag{6.3.4}$$

The additional power of r means that the top order term $\sqrt{\mathcal{E}^{(K)}(r)}$ appears with a weight that is suitably integrable as $r \downarrow 0$, and thus does not contribute to the $A_*\mathcal{E}^{(K)}(r)$ on the RHS of (6.3.1).

Now, to conclude, for $0 \leq K \leq N$ we write

$$\begin{aligned} r\partial_r &\left(\frac{1}{2}\left((r\partial_r\partial_x^K\phi)^2 + r^2e^{2(\mu-\lambda)}(\partial_x^{K+1}\phi)^2 + (\partial_x^K\phi)^2\right)\right) \\ &= (r\partial_r\partial_x^K\phi)\left[(r\partial_r)^2\partial_x^K\phi - r^2e^{2(\lambda-\mu)}\partial_x^{K+2}\phi + \partial_x^K\phi\right] + r^2e^{2(\mu-\lambda)}(\partial_x^{K+1}\phi)^2 \\ &\quad + \partial_x\left(r^2e^{2(\mu-\lambda)}r\partial_r\partial_x^K\phi\cdot\partial_x^{K+1}\phi\right) - 2r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\cdot r\partial_r\partial_x^K\phi\cdot\partial_x^{K+1}\phi. \end{aligned}$$

Integrating over $x \in \mathbb{S}^1$ so that the first term in the last line vanishes, and inserting the commuted wave equation, one has the following identity for the $r\frac{d}{dr}$ derivative of the energy:

$$\begin{aligned} r\frac{d}{dr}\mathcal{E}_\phi^{(K)}(r) &= \int_{\mathbb{S}^1}\left[(r\partial_r\partial_x^K\phi)\left((6.3.2\text{a}) + \partial_x^K\phi\right) + r^2e^{2(\mu-\lambda)}(\partial_x^{K+1}\phi)^2\right]dx \\ &\quad + \int_{\mathbb{S}^1}\left[(r\partial_r\partial_x^K\phi)\left((6.3.2\text{b}) + (6.3.2\text{c}) - 2r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x^{K+1}\phi\right)\right]dx. \end{aligned}$$

From the bootstrap assumptions (B1)–(B2), one may estimate $\|(6.3.2\text{a})\|_{L^2} \leq 10C_*^2\sqrt{2\mathcal{E}^{(K)}(r)}$. Using Cauchy–Schwarz, the first integral in the above expression can thus be bounded as:

$$\left|\int_{\mathbb{S}^1}\left[(r\partial_r\partial_x^K\phi)\left((6.3.2\text{a}) + \partial_x^K\phi\right) + r^2e^{2(\mu-\lambda)}(\partial_x^{K+1}\phi)^2\right]dx\right| \leq 2(10C_*^2+9)\mathcal{E}^{(K)}(r).$$

On the other hand, the latter integral can be estimated using (6.3.4) and (6.3.3) – the expression involving $r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)$ can be bounded in the same way as (6.3.2b) – one yields that for some $C_1^{(K)}, C_2^{(K)} > 0$,

$$\begin{aligned} &\left|\int_{\mathbb{S}^1}\left[(r\partial_r\partial_x^K\phi)\left((6.3.2\text{b}) + (6.3.2\text{c}) - 2r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x^{K+1}\phi\right)\right]dx\right| \\ &\qquad \leq \left(C_1^{(K)}r^{1-\gamma}\sqrt{\mathcal{E}^{(K)}(r)} + \sqrt{C_2^{(K)}}\sum_{k=0}^{K-1} r^{-\gamma(K-k)}\sqrt{\mathcal{E}^{(k)}(r)}\right)\cdot\sqrt{\mathcal{E}^{(K)}(r)}. \end{aligned}$$

Combining these and applying Young's inequality, Proposition 6.3.1 follows, with $A_* = 10C_*^2 + 10$. □

6.3.2 Energy estimates for the metric variables μ and λ

Proposition 6.3.2. *Let (μ, λ, ϕ, Q) be a solution to the Einstein–Maxwell–scalar field system* (2.5.8)–(2.5.12) *obeying the bootstrap assumptions* (B1)–(B4). *Then there exists a constant A_**

depending only on C_*, as well as constants $C_1^{(K)}$ and $C_2^{(K)}$ depending on C_* and the regularity index $K \in \{0, 1, \dots, N\}$, such that

$$\left| r\frac{d}{dr}\mathcal{E}_\mu^{(K)}(r) \right| \leq 2A_*\mathcal{E}^{(K)}(r) + C_1^{(K)} r^{1-\gamma}\mathcal{E}^{(K)}(r) + \begin{cases} \sum_{k=0}^{K-1} C_2^{(K)} r^{-2\gamma(K-k)}\mathcal{E}^{(k)}(r), & \text{if } K \neq 0, \\ 2A_* & \text{if } K = 0. \end{cases} \tag{6.3.5}$$

Proof. We consider the equation (2.5.11) as a wave equation for μ. As in the proof of Proposition 6.3.1, we first commute (2.5.11) with ∂_x^K. Instead of doing this directly, we first rewrite (2.5.11) as:

$$(r\partial_r)^2\mu - r^2 e^{2(\mu-\lambda)}\partial_x^2\mu = \left(-\frac{Q^2}{r^2} + \kappa\right) e^{2\mu} r\partial_r(\lambda + 2\mu) + r^2 e^{2(\mu-\lambda)}\partial_x\mu\,\partial_x(\mu-\lambda) + 2r^2 e^{2(\mu-\lambda)}(\partial_x\phi)^2 + \frac{Q^2}{r^2}e^{2\mu},$$

Now commuting this equation with ∂_x^K, one yields for $1 \leq K \leq N$:

$$\begin{aligned}
&(r\partial_r)^2\partial_x^K\mu - r^2 e^{2(\mu-\lambda)}\partial_x^2\partial_x^K\mu \\
&= \left(-\frac{Q^2}{r^2} + \kappa\right) e^{2\mu}(r\partial_r\partial_x^K(\lambda + 2\mu) + 2r\partial_r(\lambda+2\mu)\,\partial_x^K\mu) + \frac{Q^2}{r^2}e^{2\mu}\,\partial_x^K\mu && \text{(6.3.6a)} \\
&\quad + r^2 e^{2(\mu-\lambda)} \sum_{k_p+k_1+\cdots+k_i=K+2} \partial_x^{k_p}\mu * \partial_x^{k_1}(\mu-\lambda) * \cdots * \partial_x^{k_i}(\mu-\lambda) && \text{(6.3.6b)} \\
&\quad + r^2 e^{2(\mu-\lambda)} \sum_{k_p+k_q+k_1+\cdots+k_i=K+2} \partial_x^{k_p}\phi * \partial_x^{k_q}\phi * \partial_x^{k_1}(\mu-\lambda) \cdots * \partial_x^{k_i}(\mu-\lambda) && \text{(6.3.6c)} \\
&\quad + \left(-\frac{Q^2}{r^2} + \kappa\right) e^{2\mu} \sum_{\substack{k_p \geq 0, k_1 < K \\ k_p+k_1+\cdots+k_i=K}} r\partial_r\partial_x^{k_p}(\lambda+2\mu) * \partial_x^{k_1}\mu * \cdots * \partial_x^{k_i}\mu && \text{(6.3.6d)} \\
&\quad + \frac{Q^2}{r^2}e^{2\mu} \sum_{\substack{1 \leq k_1 < K \\ k_1+\cdots k_i=K}} \partial_x^{k_1}\mu * \cdots * \partial_x^{k_i}\mu. && \text{(6.3.6e)}
\end{aligned}$$

Note for $K = 0$, the terms (6.3.6d) and (6.3.6e) are absent, as is the expression involving $r\partial_r(\lambda + 2\mu)\partial_x^K\mu$ in (6.3.6a). We leave the details in this case to the reader, highlighting that the (uncommuted) term $\frac{Q^2}{r^2}e^{2\mu}$ in (6.3.6a) is responsible for the final term A_* arising in (6.3.5).

We now proceed exactly as in the proof of Proposition 6.3.1. We first bound the terms (6.3.6b) and (6.3.6c). Just as we did for (6.3.2b) in the proof of Proposition 6.3.1, we apply Lemma 5.2.2 with $w = re^{\mu-\lambda}$, to get

$$\|(6.3.6b)\|_{L^2} + \|(6.3.6c)\|_{L^2} \lesssim_{C_*,K} \sum_{k=0}^{K} r^{1-\gamma(K+1-k)}\sqrt{\mathcal{E}^{(k)}(r)}. \tag{6.3.7}$$

Moving onto (6.3.6d) and (6.3.6e), we now apply Lemma 5.2.2 with $w = 1$. The result is

$$\|(6.3.6d)\|_{L^2} + \|(6.3.6e)\|_{L^2} \lesssim_{C_*,K} \sum_{k=0}^{K-1} r^{-\gamma(K-k)} \sqrt{\mathcal{E}^{(k)}(r)}. \tag{6.3.8}$$

We highlight that (6.3.8) does not have the top order term $\mathcal{E}^{(K)}(r)$ on the RHS, while (6.3.7) does feature this top order term but with a favourable weight $r^{1-\gamma}$.

Now, just as in the proof of Proposition 6.3.1 one can use the ∂_x^K-commuted wave equation for μ to derive the following derivative identity for $\mathcal{E}_\mu^{(K)}(r)$:

$$r\frac{d}{dr}\mathcal{E}_\mu^{(K)}(r) = \int_{\mathbb{S}^1} \left[(r\partial_r\partial_x^K\mu)\left((6.3.6a) + \partial_x^K\mu\right) + r^2 e^{2(\mu-\lambda)}(\partial_x^{K+1}\mu)^2 \right] dx$$

$$+ \int_{\mathbb{S}^1} (r\partial_r\partial_x^K\mu)\left((6.3.6b) + (6.3.6c) + (6.3.6d) + (6.3.6e) - 2r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x^{K+1}\mu\right) dx.$$

Using the structure of (6.3.6a) and the bootstrap assumptions (B1)–(B2), one finds $\|(6.3.6a)\|_{L^2} \leq 20C_*^2\sqrt{2\mathcal{E}^{(K}(r)}$, and therefore

$$\left| \int_{\mathbb{S}^1} \left[(r\partial_r\partial_x^K\mu)\left((6.3.6a) + \partial_x^K\mu\right) + r^2 e^{2(\mu-\lambda)}(\partial_x^{K+1}\mu)^2 \right] dx \right| \leq 2(20C_*^2 + 19)\mathcal{E}^{(K)}(r).$$

Furthermore, combining (6.3.7) and (6.3.8) yields that for some $C_1^{(K)}, C_2^{(K)} > 0$:

$$\left| \int_{\mathbb{S}^1} (r\partial_r\partial_x^K\mu)\left((6.3.6b) + (6.3.6c) + (6.3.6d) + (6.3.6e) - 2r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x^{K+1}\mu\right) dx \right|$$

$$\leq \left(C_1^{(K)} r^{1-\gamma}\sqrt{\mathcal{E}^{(K)}(r)} + \sqrt{C_2^{(K)}} \sum_{k=0}^{K-1} r^{-\gamma(K-k)}\sqrt{\mathcal{E}^{(k)}(r)} \right) \cdot \sqrt{\mathcal{E}^{(K)}(r)}.$$

Inserting these bounds into the derivative identity, we get (6.3.5) with $A_* = 20(C_*^2 + 1)$. □

It remains to prove the energy estimate for λ. In light of Proposition 6.3.2, it will be enough to prove a corresponding estimate for the modified energy $\mathcal{E}_{\mu-\lambda}^{(K)}(r)$, defined by:

$$\mathcal{E}_{\mu-\lambda}^{(K)}(r) \coloneqq \frac{1}{2}\int_{\mathbb{S}^1} \left((r\partial_r\partial_x^K(\mu-\lambda))^2 + r^2e^{2(\mu-\lambda)}(\partial_x^{K+1}(\mu-\lambda))^2 + (\partial_x^K(\mu-\lambda))^2 \right) dx.$$

With this definition, we have the following derivative estimate for the energy:

Proposition 6.3.3. *Let (μ, λ, ϕ, Q) be a solution to the Einstein–Maxwell–scalar field system* (2.5.8)–(2.5.12) *obeying the bootstrap assumptions* (B1)–(B4). *Then there exists a constant A_* depending only on C_*, as well a constant $C_2^{(K)}$ depending on C_* and the regularity index $K \in \{0, 1, \ldots, N\}$, such that*

$$\left| r\frac{d}{dr}\mathcal{E}_{\mu-\lambda}^{(K)}(r) \right| \leq 2A_*\mathcal{E}^{(K)}(r) + \begin{cases} \sum_{k=0}^{K-1} C_2^{(K)} r^{-2\gamma(K-k)}\mathcal{E}^{(k)}(r), & \text{if } K \neq 0, \\ 2A_* & \text{if } K = 0. \end{cases} \tag{6.3.9}$$

Proof. Here we instead commute the transport equation (2.5.13) with ∂_x^K, yielding for $1 \leq K \leq N$:

$$r\partial_r\partial_x^K(\mu-\lambda) = \left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}\,\partial_x^K\mu + \left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}\sum_{\substack{k_1<K\\ k_1+\cdots+k_i=K}}\partial_x^{k_1}\mu * \cdots * \partial_x^{k_i}\mu, \quad (6.3.10)$$

while for $K=0$ the final term is not present but there is an additional 1 (see the RHS of (2.5.13)).

From this equation we can derive further equations, namely

$$\begin{aligned} r\partial_r(re^{\mu-\lambda}\partial_x^{K+1}(\mu-\lambda)) = &\left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}\,re^{\mu-\lambda}\partial_x^{K+1}\mu + (1+r\partial_r(\mu-\lambda))\,re^{\mu-\lambda}\partial_x^{K+1}(\mu-\lambda)\\ &+\left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}\,re^{\mu-\lambda}\sum_{\substack{k_1<K+1\\ k_1+\cdots+k_i=K+1}}\partial_x^{k_1}\mu * \cdots * \partial_x^{k_i}\mu, \quad (6.3.11)\end{aligned}$$

as well as

$$\begin{aligned} &r\partial_r(r\partial_r\partial_x^K(\mu-\lambda)) =\\ &\left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}\,r\partial_r\partial_x^K\mu + \left(\frac{2Q^2}{r^2}e^{2\mu}+2r\partial_r\mu\left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}\right)\partial_x^K\mu + r\partial_r\partial_x^K(\mu-\lambda)\\ &\quad+\left(\frac{2Q^2}{r^2}e^{2\mu}+2r\partial_r\mu\left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}\right)\sum_{\substack{k_1<K\\ k_1+\cdots+k_i=K}}\partial_x^{k_1}\mu * \cdots * \partial_x^{k_i}\mu\\ &\quad+\left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}\sum_{\substack{k_1<K\\ k_1+\cdots+k_i=K}}r\partial_r\partial_x^{k_1}\mu * \cdots * \partial_x^{k_i}\mu, \quad (6.3.12)\end{aligned}$$

From (6.3.12), (6.3.11) and (6.3.10), repeated use of the bootstrap assumptions (B1)–(B4) and the weighted product estimate Lemma 5.2.2 yields that for some $C_2^{(K)} > 0$,

$$\begin{aligned} &\|r\partial_r\partial_x^K(\mu-\lambda)\|_{L^2} + \|r\partial_r(re^{\mu-\lambda}\partial_x^{K+1}(\mu-\lambda))\|_{L^2} + \|r\partial_r(r\partial_r\partial_x^K(\mu-\lambda))\|_{L^2}\\ &\qquad\leq (A_*-1)\sqrt{2\mathcal{E}^{(K)}(r)} + \sqrt{C_2^{(K)}}\sum_{k=0}^{K-1} r^{-\gamma(K-k)}\sqrt{\mathcal{E}^{(k)}(r)}.\end{aligned}$$

Proposition 6.3.3 then follows immediately from this and Young's inequality. □

6.3.3 The energy hierarchy

We now use Propositions 6.3.1, 6.3.2 and 6.3.3 together with the initial data bound (6.1.9), to show that the total energy of order K, $\mathcal{E}^{(K)}(r)$, grows at most polynomially in r as $r \downarrow 0$, and moreover that the rate of blow-up depends only mildly in K.

Proposition 6.3.4. *Let (μ, λ, ϕ, Q) be a solution to the Einstein–Maxwell–scalar field system (2.5.8)–(2.5.12) in the interval $r \in [r_b, r_0]$, such that the solution obeys the bootstrap assumptions (B1)–(B4). Assuming also the bound (6.1.9) for the initial data, then there exist a constant A_* depending only on C_*, as well as constants $C^{(K)}$ depending on C_*, K and ζ, such that for $0 \leq K \leq N$, the total energy $\mathcal{E}^{(K)}(r)$ satisfies the bound:*

$$\mathcal{E}^{(K)}(r) \leq C^{(K)}\, r^{-2A_*-2K\gamma}. \tag{6.3.13}$$

Proof. Combining Propositions 6.3.1, 6.3.2 and 6.3.3, it is straightforward to show that for some $A_* > 0$ depending only on C_* and constants $C_1^{(K)}, C_2^{(K)} > 0$ (which are allowed to differ from those of the previous propositions), one has the following derivative estimate:

$$\left| r\frac{d}{dr}\mathcal{E}^{(K)}(r) \right| \leq 2A_*\mathcal{E}^{(K)}(r) + C_1^{(K)} r^{1-\gamma}\mathcal{E}^{(K)}(r) + \begin{cases} C_2^{(K)} \sum_{k=0}^{K-1} r^{-2\gamma(K-k)}\mathcal{E}^{(k)}(r), & \text{if } K \neq 0, \\ 2A_*, & \text{if } K = 0. \end{cases}$$

Since we shall integrate towards $r = 0$, the derivative estimate we actually use is the following:

$$r\frac{d}{dr}\mathcal{E}^{(K)}(r) \geq -2A_*\mathcal{E}^{(K)}(r) - C_1^{(K)} r^{1-\gamma}\mathcal{E}^{(K)}(r) - \begin{cases} C_2^{(K)} \sum_{k=0}^{K-1} r^{-2\gamma(K-k)}\mathcal{E}^{(k)}(r), & \text{if } K \neq 0, \\ 2A_*, & \text{if } K = 0. \end{cases}$$

In fact, using the integrating factor r^{2A_*}, which is *crucially independent of* K, we write:

$$\begin{aligned} r\frac{d}{dr}\left(r^{2A_*}\mathcal{E}^{(K)}(r)\right) \geq \\ - C_1^{(K)} r^{1-\gamma}\left(r^{2A_*}\mathcal{E}^{(K)}(r)\right) - \begin{cases} C_2^{(K)} \sum_{k=0}^{K-1} r^{-2\gamma(K-k)}\left(r^{2A_*}\mathcal{E}^{(k)}(r)\right), & \text{if } K \neq 0, \\ r^{2A_*}A_*, & \text{if } K = 0. \end{cases} \end{aligned} \tag{6.3.14}$$

We will now use (6.3.14) and induction on $K \in \{0, \ldots, N\}$ to show that

$$r^{2A_*}\mathcal{E}^{(K)}(r) \lesssim r^{-2\gamma K}, \tag{6.3.15}$$

where the implied constant is now allowed to depend on C_*, K and ζ. This is equivalent to (6.3.13). Note that the dependence on ζ comes from the fact that the initial data bound (6.1.9) implies that for $A_* \geq \gamma$,

$$\sum_{k=0}^{N} r_0^{2A_*}\mathcal{E}^{(k)}(r_0) \leq \zeta. \tag{6.3.16}$$

We begin with the base case $K = 0$. Applying Grönwall's inequality to (6.3.14) for $K = 0$, we obtain that for any $r \in [r_b, r_0]$, one has

$$r^{2A_*}\mathcal{E}^{(0)}(r) \leq \exp\Big(F^{(0)}(r_0, r)\Big) \cdot r_0^{2A_*}\mathcal{E}^{(0)}(r_0) + \int_r^{r_0} \exp\Big(F^{(0)}(\tilde{r}, r)\Big)\tilde{r}^{2A_*} 2A_* \frac{d\tilde{r}}{\tilde{r}},$$

$$\text{where} \qquad F^{(0)}(s_a, s_b) = \int_{s_b}^{s_a} C_1^{(0)}\tilde{r}^{1-\gamma}\frac{d\tilde{r}}{\tilde{r}}.$$

Since $F^{(0)}(s_a, s_b)$ is uniformly bounded for $s_a, s_b \in [r_b, r_0]$, it follows from the initial data bound (6.3.16) that (6.3.15) holds for $K = 0$.

Moving onto the induction step, assume that (6.3.15) holds for $0 \leq K < \bar{K} \leq N$; we wish to prove it also holds for $K = \bar{K}$. Applying Grönwall's inequality to (6.3.14) for $K = \bar{K}$, we have that

$$r^{2A_*}\mathcal{E}^{(\bar{K})}(r) \leq \exp\Big(F^{(\bar{K})}(r_0, r)\Big) \cdot r_0^{2A_*}\mathcal{E}^{(\bar{K})}(r_0) + \int_r^{r_0} \exp\Big(F^{(\bar{K})}(\tilde{r}, r)\Big) C_2^{(\bar{K})} \sum_{k=0}^{\bar{K}-1} \tilde{r}^{-2\gamma(\bar{K}-k)} \tilde{r}^{2A_*}\mathcal{E}^{(k)}(\tilde{r}) \frac{d\tilde{r}}{\tilde{r}},$$

$$\text{where} \qquad F^{(\bar{K})}(s_a, s_b) = \int_{s_b}^{s_a} C_1^{(\bar{K})} \tilde{r}^{1-\gamma} \frac{d\tilde{r}}{\tilde{r}}.$$

Since $F^{(K)}(s_a, s_b)$ is uniformly bounded for $s_a, s_b \in [r_b, r_0]$, it follows from the initial data bound (6.3.16) and the inductive hypothesis for $\tilde{r}^{2A_*}\mathcal{E}^{(k)}(\tilde{r})$ that

$$r^{A_*}\mathcal{E}^{(\bar{K})}(r) \lesssim \zeta + \int_r^{r_0} \sum_{k=0}^{\bar{K}-1} \tilde{r}^{-2\gamma(\bar{K}-k)} \cdot \tilde{r}^{-2\gamma k} \frac{d\tilde{r}}{\tilde{r}} \lesssim r^{-2\gamma\bar{K}}$$

as required. This completes the proof of the proposition. □

6.3.4 Derivation of bounce ODEs

We now apply Proposition 6.3.4 together with the Sobolev interpolation of Lemma 5.2.1 to provide L^∞ bounds for low order derivatives of ϕ, μ and λ, so that we may treat certain equations in the Einstein–Maxwell–scalar field system (2.5.9)–(2.5.12) as ODEs without worrying about losing derivatives.

Lemma 6.3.5. *Let (μ, λ, ϕ, Q) be as in Proposition 6.3.4. Then for N chosen sufficiently large there exists a family of constants $\delta = \delta(C_*, \zeta, r_*) > 0$ with $\delta \downarrow 0$ as $r_* \downarrow 0$, such that for any $0 \leq k \leq 3$, $f \in \{\phi, \mu, \lambda\}$, and $r \in [r_b, r_0]$ one has*

$$\|\partial_x^k f(r, \cdot)\|_{L^\infty} + \|r\partial_r \partial_x^k f(r, \cdot)\|_{L^\infty} \leq \delta r^{-1/2}. \tag{6.3.17}$$

Proof. The energy estimate (6.3.13) for $K = N$, together with the definition of $\mathcal{E}^{(K)}(r)$ (see Definition 5.2.1), implies the following L^2-estimates for $f \in \{\phi, \mu, \lambda\}$:

$$\|\partial_x^N f\|_{L^2}^2 + \|r\partial_r \partial_x^N f\|_{L^2}^2 \leq 2C^{(N)} r^{-2A_* - 2N\gamma}. \tag{6.3.18}$$

Note that while $C^{(N)}$ depends on N, the number A_* does not, and we later choose N depending on A_*.

We now interpolate between (6.3.18) and the low-order L^∞ bootstrap assumptions (B2)–(B3). Applying Lemma 5.2.1, for $f \in \{\phi, \mu, \lambda\}$ and $0 \leq k \leq 3$ one finds:

$$\|r\partial_r \partial_x^k f(r, \cdot)\|_{L^\infty} \lesssim_N \|r\partial_r f(r, \cdot)\|_{L^\infty}^{1-\alpha} \|r\partial_r \partial_x^N f(r, \cdot)\|_{L^2}^{\alpha}, \qquad \text{where } \alpha = \frac{k}{N - \frac{1}{2}}.$$

Inserting the bound (6.3.18) and the bootstrap assumption (B2), one finds

$$\|r\partial_r\partial_x^k f(r,\cdot)\|_{L^\infty} \lesssim_{N,C_*} \left(r^{-2A_*-2N\gamma}\right)^{\frac{\alpha}{2}} = r^{-k\gamma}\cdot\left(r^{-2A_*-\gamma}\right)^{\frac{\alpha}{2}}.$$

Since we chose $\gamma < \frac{1}{100}$, we have $r^{-k\gamma} < r^{-1/8}$. Furthermore, for N chosen sufficiently large (depending on A_* and k) one can guarantee $(r^{-2A_*-\gamma})^{\frac{\alpha}{2}} < r^{-1/8}$. Thus for this choice of N there exists a constant C_{N,C_*} such that

$$\|r\partial_r\partial_x^k f(r,\cdot)\|_{L^\infty} \leq C_{N,C_*} r^{-1/4}. \tag{6.3.19}$$

To get a similar estimate for $\|\partial_x^k f(r,\cdot)\|_{L^\infty}$ we first apply the Lemma 5.2.1 to the initial data assumption (6.1.9), yielding $\|r\partial_r\partial_x^k f(r_0,\cdot)\|_{L^\infty} \lesssim \zeta r^{-\gamma} \lesssim \zeta r^{-1/4}$. Then simply integrating (6.3.19) from $r = r_0$, there exists a constant $C_{N,C_*,\zeta}$ so that

$$\|\partial_x^k f(r,\cdot)\|_{L^\infty} \leq C_{N,C_*,\zeta} r^{-1/4}. \tag{6.3.20}$$

Combining (6.3.19) and (6.3.20), and redefining the constant $C_{N,C_*,\zeta}$ appropriately, one has

$$\|\partial_x^k f(r,\cdot)\|_{L^\infty} + \|r\partial_r\partial_x^k f(r,\cdot)\|_{L^\infty} \leq (C_{N,C_*,\zeta} r^{1/4})\cdot r^{-1/2}.$$

Recalling that $r \leq r_0 \leq r_*$, letting $\delta = C_{N,C_*,\zeta}\cdot r_*^{1/4}$ completes the proof of the lemma. □

Corollary 6.3.6. *Let (μ,λ,ϕ,Q) be as in Proposition 6.3.4. Then for N chosen sufficiently large there exists a family of constants $\delta = \delta(C_*,\zeta,r_*) > 0$, with $\delta \downarrow 0$ as $r_* \downarrow 0$, such that for all $(r,x) \in [r_b,r_0]\times\mathbb{S}^1$ and all $a \in [-1,1]$, one has:*

$$\left|(r\partial_r + are^{\mu-\lambda}\partial_x)(r\partial_r\phi) + (r\partial_r\phi)\frac{Q^2}{r^2}e^{2\mu}\right|(r,x) \leq \delta r^{1/2}, \tag{6.3.21}$$

$$\left|(r\partial_r + are^{\mu-\lambda}\partial_x)\left(\frac{Q^2}{r^2}e^{2\mu}\right) - \frac{Q^2}{r^2}e^{2\mu}\left((r\partial_r\phi)^2 - 1 - \frac{Q^2}{r^2}e^{2\mu}\right)\right|(r,x) \leq \frac{Q^2}{r^2}e^{2\mu}\cdot\delta r^{1/2}. \tag{6.3.22}$$

Furthermore, for the same $\delta = \delta(C_*,\zeta,r_*) > 0$ one has:

$$\left|(r\partial_r + are^{\mu-\lambda}\partial_x)(\partial_x\mu) + \frac{Q^2}{r^2}e^{2\mu}\partial_x\mu - (r\partial_r\phi)(r\partial_r\partial_x\phi)\right|(r,x) \leq \delta r^{1/2}, \tag{6.3.23}$$

$$\left|(r\partial_r + are^{\mu-\lambda}\partial_x)(r\partial_r\partial_x\phi) + \frac{2Q^2}{r^2}e^{2\mu}(r\partial_r\phi)\partial_x\mu + \frac{Q^2}{r^2}e^{2\mu}r\partial_r\partial_x\phi\right|(r,x) \leq \delta r^{1/2}. \tag{6.3.24}$$

Proof. This follows immediately from Lemma 6.3.5 and the equations (2.5.9) and (2.5.12). We derive the equations (6.3.22) and (6.3.24), leaving the remaining equations to the reader. For (6.3.22), using (2.5.9) we derive:

$$\begin{aligned} r\partial_r\left(\frac{Q^2}{r^2}e^{2\mu}\right) &= \frac{Q^2}{r^2}e^{2\mu}\left((r\partial_r\phi)^2 + r^2e^{2(\mu-\lambda)}(\partial_x\phi)^2 + \left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu} - 2\right) \\ &= \frac{Q^2}{r^2}e^{2\mu}\left((r\partial_r\phi)^2 - 1 - \frac{Q^2}{r^2}e^{2\mu}\right) + \frac{Q^2}{r^2}e^{2\mu}\left(r^2e^{2(\mu-\lambda)}(\partial_x\phi)^2 + \kappa e^{2\mu}\right). \end{aligned}$$

Noting there is also an $are^{\mu-\lambda}\partial_x$-derivative on the left hand side of (6.3.22), it thus suffices to show that

$$r^2e^{2(\mu-\lambda)}(\partial_x\phi)^2+|\kappa e^{2\mu}|+2|are^{\mu-\lambda}\partial_x\mu|\leq \delta r^{1/2}. \tag{6.3.25}$$

Inserting the bootstrap assumptions (B1) to bound $e^{\mu-\lambda}$ and $e^{2\mu}$, and Lemma 6.3.5 to bound $\partial_x\phi$ and $\partial_x\mu$,

$$r^2e^{2(\mu-\lambda)}(\partial_x\phi)^2+|\kappa e^{2\mu}|+2|are^{\mu-\lambda}\partial_x\mu|\leq C_*^2r^3(\delta r^{-1/2})^2+C_*r+2C_*r^{3/2}(\delta r^{-1/2}).$$

Since the powers of r appearing on the right hand side all exceed $r^{1/2}$, it is clear that upon redefining δ that one has the estimate (6.3.25).

We move onto (6.3.24). For this, we commute the wave equation (2.5.12) once with ∂_x, then add an additional term of $are^{\mu-\lambda}r\partial_r\partial_x^2\phi$, to yield

$$\begin{aligned}(r\partial_r)^2\partial_x\phi+are^{\mu-\lambda}r\partial_r\partial_x^2\phi+2\frac{Q^2}{r^2}e^{2\mu}(\partial_x\mu)(r\partial_r\phi)+\frac{Q^2}{r^2}e^{2\mu}r\partial_r\partial_x\phi&\\ =r^2e^{2(\mu-\lambda)}\left(\partial_x^3\phi+3\partial_x(\mu-\lambda)\partial_x^2\phi+\partial_x^2(\mu-\lambda)\partial_x\phi\right)+\kappa e^{2\mu}\left(r\partial_r\partial_x\phi+2(\partial_x\mu)(r\partial_r\phi)\right)&\\ +are^{\mu-\lambda}r\partial_r\partial_x^2\phi.&\end{aligned}$$

Using the bootstrap assumptions (B1)–(B2) and Lemma 6.3.5, the right hand side may be bounded by

$$C_*r^2(\delta r^{-1/2}+(\delta r^{-1/2})^2)+C_*r(\delta r^{-1/2}+2C_*\delta r^{-1/2})+C_*r^2\delta r^{-1/2}.$$

Since we are allowed to redefine $\delta=C_*\delta$, this quantity is bounded by $\delta r^{1/2}$, as required. □

6.4 Low order ODE analysis

Here, we now study ODE systems of the type found in Corollary 6.3.6, with the eventual aim of improving the bootstrap assumptions (B1)–(B4). As explained in Section 6.1.5, the first step is studying a nonlinear ODE system derived from (6.3.21)–(6.3.22), and then studying the linearization of this ODE system, derived from (6.3.23)–(6.3.24).

6.4.1 The BKL bounce ODE

Lemma 6.4.1. *For $r_0\leq 1$, let $\mathcal{P},\mathcal{Q}:[r_b,r_0]\subset\mathbb{R}_{>0}\rightarrow\mathbb{R}_{\geq 0}$ satisfy the following ODEs, where the error terms $\mathcal{E}_i$ obey $|\mathcal{E}_i|\leq\delta r^{1/2}$ for $i=1,2$:*

$$r\partial_r\mathcal{P}=-\mathcal{P}\mathcal{Q}+\mathcal{E}_1, \tag{6.4.1}$$

$$r\partial_r\mathcal{Q}=\mathcal{Q}(\mathcal{P}^2-1-\mathcal{Q}+\mathcal{E}_2). \tag{6.4.2}$$

Suppose furthermore that for some $\eta > 2$*, one has* $\eta^{-1} \leq \mathcal{P}(r_0) \leq \eta$ *and* $\mathcal{Q}(r_0) \leq 1$*. Then for* δ *chosen sufficiently small (depending on* η*) the solution obeys the following bounds for* $r \in [r_b, r_0]$*.*

$$(4\eta)^{-1} \leq \mathcal{P}(r) \leq 4\eta, \qquad \mathcal{Q}(r) \leq 16\eta^2. \tag{6.4.3}$$

Proof. We proceed by using a continuity / bootstrap argument, with the bootstrap assumption being (6.4.3). So we assume (6.4.3) holds on an interval $[\tilde{r}, r_0] \subset [r_b, r_0]$, then show we may actually improve upon (6.4.3).

For the improvement step, we use an approximately conserved quantity of the ODE (which would be exactly conserved if $\mathcal{E}_i \equiv 0$); let $\mathcal{K}$ be defined by

$$\mathcal{K} := \mathcal{P} + \mathcal{P}^{-1} + \mathcal{Q}\mathcal{P}^{-1}. \tag{6.4.4}$$

From (6.4.1) and (6.4.2), one checks that $r\partial_r \mathcal{K} = \mathcal{E}_1(1 - \mathcal{P}^{-2} - \mathcal{P}^{-2}\mathcal{Q}) + \mathcal{P}^{-1}\mathcal{Q}\mathcal{E}_2$. Thus assuming the bootstrap assumption (6.4.3), $r\partial_r \mathcal{K}$ may be bounded by

$$|r\partial_r \mathcal{K}| \leq (1 + 16\eta^2 + 64\eta^3 + 256\eta^4)\delta r^{1/2}.$$

Thus for $\delta = \delta(\eta)$ chosen sufficiently small, one may guarantee that $\int_r^{r_0} |\partial_r \mathcal{K}(\tilde{r})| d\tilde{r} \leq \frac{1}{2}\eta$. Using also the initial data assumptions $\eta^{-1} \leq \mathcal{P}(r_0) \leq \eta$ and $\mathcal{Q}(r_0) \leq 1$, we have

$$\mathcal{K}(r) \leq \mathcal{K}(r_0) + \frac{1}{2}\eta = \mathcal{P}(r_0) + \mathcal{P}^{-1}(r_0) + \mathcal{Q}\mathcal{P}^{-1}(r_0) + \frac{1}{2}\eta \leq \frac{7}{2}\eta.$$

Hence for all r, one has

$$\mathcal{P}(r) + \mathcal{P}^{-1}(r) + \mathcal{Q}\mathcal{P}^{-1}(r) \leq \frac{7}{2}\eta,$$

from which one can read off $\mathcal{P}(r) + \mathcal{P}(r) \leq \frac{7}{2}\eta$ and $\mathcal{Q}(r) \leq \left(\frac{7}{2}\eta\right)^2$, improving upon the bootstrap assumption (6.4.3). By a standard continuity argument, (6.4.3) therefore holds in the entire interval $[r_b, r_0]$. □

6.4.2 The equations of variation

Lemma 6.4.2. *For* $r_0 < 1$*, let* $\mathcal{P}, \mathcal{Q} : [r_b, r_0] \to \mathbb{R}_{\geq 0}$ *satisfy the assumptions of Lemma 6.4.1. Further, let* $\mathcal{M}, \mathcal{N} : [r_b, r_0] \to \mathbb{R}$ *obey the following ODEs, where* $|\mathcal{E}_i| \leq \delta r^{1/2}$ *for* $i = 1, 2, 3, 4$*:*

$$r\partial_r \mathcal{M} = -\mathcal{Q}\mathcal{M} + \mathcal{P}\mathcal{N} + \mathcal{E}_3, \tag{6.4.5}$$

$$r\partial_r \mathcal{N} = -2\mathcal{P}\mathcal{Q}\mathcal{M} - \mathcal{Q}\mathcal{N} + \mathcal{E}_4. \tag{6.4.6}$$

For $0 < \gamma < \frac{1}{100}$ *fixed and some* $\zeta > 0$*, impose the following conditions at initial data:* $|\mathcal{M}(r_0)| + |\mathcal{N}(r_0)| \leq \zeta r_0^{-\gamma}$*. Then for* δ *chosen sufficiently small depending on* η*,* γ *and* ζ*, there exists a constant* $D > 0$ *depending on the same constants* η*,* γ *and* ζ*, such that for all* $r \in [r_b, r_0]$*, one has*

$$|\mathcal{M}(r)| + |\mathcal{N}(r)| \leq D r^{-\gamma}. \tag{6.4.7}$$

Proof. Let $b = \frac{\gamma}{10\mathfrak{y}}$ be a small parameter used for notational convenience. We rewrite the system (6.4.5)–(6.4.6) in the following matrix form, upon conjugation by a linear transformation $\mathcal{M} \mapsto b\mathcal{M}$:

$$r\partial_r \begin{bmatrix} b\mathcal{M} \\ \mathcal{N} \end{bmatrix} = \underbrace{\begin{bmatrix} -\mathcal{Q} & b\mathcal{P} \\ -2b^{-1}\mathcal{P}\mathcal{Q} & -\mathcal{Q} \end{bmatrix}}_{=:\mathbf{L}} \begin{bmatrix} b\mathcal{M} \\ \mathcal{N} \end{bmatrix} + \begin{bmatrix} b\mathcal{E}_3 \\ \mathcal{E}_4 \end{bmatrix}. \tag{6.4.8}$$

Our goal is to bound the operator norm of the matrix $\mathbf{L} = \mathbf{L}(r)$, as a function of $r \in [r_b, r_0]$. As b is chosen small, such a bound on the operator norm will depend on the size of $\mathcal{Q} = \mathcal{Q}(r)$. Note our definition of the operator norm will be with respect to the ℓ^∞-norm on $\mathbb{R}^2$, i.e. for a 2×2 matrix $\mathbf{M}$ we write

$$\|\mathbf{M}\|_{op} := \sup_{\mathrm{x} \in \mathbb{R}^2 \setminus \{0\}} \frac{\|\mathbf{M}\mathrm{x}\|_{\ell^\infty}}{\|\mathrm{x}\|_{\ell^\infty}}.$$

Case 1: $\mathcal{Q}(r) \leq \frac{1}{16} b\mathfrak{y}^{-1}$:

When this holds it can be immediately checked from this bound and (6.4.3) that each of the matrix elements of $\mathbf{L}(r)$ are bounded in absolute value by $\frac{\gamma}{2}$. Since for a 2×2 matrix $\mathbf{M}$ we have:

$$\|\mathbf{M}\|_{op} \leq 2 \max_{i,j} |\mathbf{M}_{ij}|,$$

this implies that

$$\|\mathbf{L}(r)\|_{op} \leq \gamma \qquad \text{whenever } \mathcal{Q}(r) \leq \frac{1}{16} b\mathfrak{y}^{-1}. \tag{6.4.9}$$

Case 2: $\mathcal{Q}(r) > \frac{1}{16} b\mathfrak{y}^{-1}$:

With no additional smallness for $\mathcal{Q}(r)$ the only bounds we have for $\mathbf{L}(r)$ will come from the bound (6.4.3) from Lemma 6.4.1. By bounding each of the matrix elements of $\mathbf{L}(r)$ using (6.4.3), one may deduce $\|\mathbf{L}(r)\|_{op} \leq 256 b^{-1}\mathfrak{y}^3$.

What is needed in this case is that we can control the size of the set $B = \{r \in [r_b, r_0] : \mathcal{Q}(r) > \frac{1}{16} b\mathfrak{y}^{-1}\}$, at least for δ sufficiently small. To justify this, suppose $\delta < \frac{1}{128} b\mathfrak{y}^{-2}$. Then using (6.4.3) and the equation (6.4.1) for $r\partial_r \mathcal{P}$, one has that for $r \in B$

$$r\partial_r \mathcal{P} \leq -(4\eta)^{-1} \cdot \frac{1}{16} b\mathfrak{y}^{-1} + \delta \leq -\frac{1}{128} b\mathfrak{y}^{-2}.$$

But from (6.4.3), $\mathcal{P}(r)$ is bounded between $(4\mathfrak{y})^{-1}$ and $4\mathfrak{y}$. Furthermore, for all $r \in [r_b, r_0]$, not necessarily in B, one has $r\partial_r \mathcal{P}(r) \leq \delta r^{1/2} < \frac{1}{128} b\mathfrak{y}^{-2} r^{1/2}$. So

$$\begin{aligned} 4\mathfrak{y} - (4\mathfrak{y})^{-1} \geq \int_r^{r_0} -r\partial_r \mathcal{P}(\tilde{r}) \, \frac{d\tilde{r}}{\tilde{r}} &= \int_{\tilde{r} \in B} -r\partial_r P(\tilde{r}) \, \frac{d\tilde{r}}{\tilde{r}} + \int_{r \notin B} -r\partial_r P(\tilde{r}) \, \frac{d\tilde{r}}{\tilde{r}} \\ &\geq \nu(B) \cdot \frac{1}{128} b\mathfrak{y}^{-2} - \frac{1}{64} b\mathfrak{y}^{-2} r_0^{1/2}. \end{aligned}$$

Here $\nu(B)$ is the measure of the set B with respect to the measure $\frac{dr}{r}$. Using the smallness of b, this leads to the estimate $\nu(B) \leq 512 b^{-1}\mathfrak{y}^3$.

Collecting all this information, we have that

$$\|\mathbf{L}(r)\|_{op} \leq 256 b^{-1}\eta^3 \qquad \text{whenever } \mathcal{Q} > \frac{1}{16} b\eta^{-1}, \tag{6.4.10}$$

$$\text{where } B = \left\{ r \in [r_b, r_0] : \mathcal{Q}(r) > \frac{1}{16} b\eta^{-1} \right\} \text{ has } \nu(B) \leq 512 b^{-1}\eta^3.$$

We now use (6.4.9) and (6.4.10) to complete the proof. Integrating (6.4.8) and applying $\|\cdot\|_{\ell^\infty}$ norms on vectors, one finds the integral inequality:

$$\left\| \begin{bmatrix} b\mathcal{M} \\ \mathcal{N} \end{bmatrix} \right\|_{\ell^\infty}(r) \leq \left\| \begin{bmatrix} b\mathcal{M} \\ \mathcal{N} \end{bmatrix} \right\|_{\ell^\infty}(r_0) + \int_r^{r_0} \left\| \begin{bmatrix} b\mathcal{E}_3 \\ \mathcal{E}_4 \end{bmatrix} \right\|_{\ell^\infty}(\tilde{r}) \frac{d\tilde{r}}{\tilde{r}} + \int_r^{r_0} \|\mathbf{F}(\tilde{r})\|_{op} \left\| \begin{bmatrix} b\mathcal{M} \\ \mathcal{N} \end{bmatrix} \right\|_{\ell^\infty}(\tilde{r}) \frac{d\tilde{r}}{\tilde{r}}.$$

By using the initial data assumption $|\mathcal{M}(r_0)| + |\mathcal{N}(r_0)| \leq \zeta r_0^{-\gamma}$ along with the error bound $|\mathcal{E}_i| \leq \delta r^{1/2}$, there exists some constant D_1 depending on η, γ and ζ such that

$$\left\| \begin{bmatrix} b\mathcal{M} \\ \mathcal{N} \end{bmatrix} \right\|_{\ell^\infty}(r) \leq D_1 r_0^{-\gamma} + \int_r^{r_0} \|\mathbf{F}(\tilde{r})\|_{op} \left\| \begin{bmatrix} b\mathcal{M} \\ \mathcal{N} \end{bmatrix} \right\|_{\ell^\infty}(\tilde{r}) \frac{d\tilde{r}}{\tilde{r}}.$$

Thus Grönwall's inequality in integral form implies that

$$\left\| \begin{bmatrix} a\mathcal{M} \\ \mathcal{N} \end{bmatrix} \right\|_{\ell^\infty}(r) \leq D_1 r_0^{-\gamma} \cdot \exp\left(\int_r^{r_0} \|\mathbf{F}(\tilde{r})\|_{op} \frac{d\tilde{r}}{\tilde{r}} \right).$$

We now apply (6.4.9) and (6.4.10) to estimate the integral inside the exp. Using the characterisation of the set R in the case of (6.4.10), the result is that there exists some D_2 depending on η and γ such that

$$\begin{aligned} \int_r^{r_0} \|\mathbf{F}(\tilde{r})\|_{op} \frac{d\tilde{r}}{\tilde{r}} &\leq \int_{\tilde{r}\in B} 126 b^{-1}\eta^3 \frac{d\tilde{r}}{\tilde{r}} + \int_{\tilde{r}\notin B} \gamma \frac{d\tilde{r}}{\tilde{r}} \\ &\leq 126 b^{-1}\eta^3 \nu(B) + \int_r^{r_0} \gamma \frac{d\tilde{r}}{\tilde{r}} = \gamma \log\left(\frac{r_0}{r}\right) + D_2. \end{aligned}$$

Inserting this into the above inequality and using that $b = \frac{\gamma}{10\eta}$ has explicit dependence on γ and η we get (6.4.7), as required. □

6.5 Global existence towards $r = 0$

6.5.1 Proof of Theorem 6.1.1

The proof of Theorem 6.1.1 will follow from a bootstrap argument. By local existence (Proposition 2.5.2), there exists some $r_b \in (0, r_0)$ such that a solution to the Einstein–Maxwell–scalar field system (2.5.9)–(2.5.12) exists in the interval $r \in [r_b, r_0]$, which moreover satisfies the bootstrap assumptions (B1)–(B4).

We show that assuming this, we are able to improve upon the bootstrap assumptions, for instance by showing that (B1)–(B4) hold with C_* replaced by $C_*/2$. By a standard continuity argument, one can allow r_b to be any real number in the interval $(0, r_0)$, giving global existence towards $r = 0$.

For this purpose, let (ϕ, μ, λ, Q) be a solution of the Einstein–Maxwell–scalar field system in our bootstrap region $r \in [r_b, r_0]$ with the initial data assumptions (6.1.7), (6.1.8) and (6.1.9). Then the results of Section 6.3 all apply, in particular Proposition 6.3.4, Lemma 6.3.5 and Corollary 5.4.3. For convenience, let us record the result of Lemma 6.3.5 again here: for $f \in \{\phi, \mu, \lambda\}$ and $0 \leq k \leq 3$, there exists a family of constants $\delta = \delta(C_*, \zeta, r_*)$ with $\delta \downarrow 0$ as $r_* \downarrow 0$ so that

$$\|\partial_x^k f(r, \cdot)\|_{L^\infty} + \|r\partial_r \partial_x^k f(r, \cdot)\|_{L^\infty} \leq \delta r^{-1/2}. \tag{6.5.1}$$

The improvement of the bootstrap assumptions will be proceed in the following steps:

Step 1: ODE analysis on timelike curves

The first step will be to use Corollary 5.4.3 and the results of Section 6.4 to provide L^∞ bounds for the following 4 key quantities:

$$r\partial_r \phi, \quad \frac{Q^2}{r^2} e^{2\mu}, \quad \partial_x \mu, \quad r\partial_r \partial_x \phi.$$

For this purpose, let $\gamma : [r_b, r_0] \to \mathcal{Q}$ with $\gamma(r) = (r, x(r))$ be a C^1 *past-directed timelike curve*, one example being a curve of constant x. In view of the metric (2.5.1), γ being timelike implies that $|\frac{dx}{dr}| \leq e^{\mu - \lambda}$. For such a curve $\gamma(r)$, let us define

$$\mathcal{P}(r) \coloneqq r\partial_r \phi(\gamma(r)), \quad \mathcal{Q}(r) \coloneqq \frac{Q^2}{r^2} e^{2\mu}(\gamma(r)), \tag{6.5.2}$$

$$\mathcal{M}(r) \coloneqq \partial_x \mu(\gamma(r)), \quad \mathcal{N}(r) \coloneqq r\partial_r \partial_x \phi(\gamma(r)). \tag{6.5.3}$$

Note the initial data assumption (6.1.7) implies that $\eta^{-1} \leq \mathcal{P}(r_0) \leq \eta$ and $\mathcal{Q}(r_0) \leq 1$. Furthermore, the assumption (6.1.9), together with Sobolev embedding on $\mathbb{S}^1$, implies there is a constant $C_{\mathbb{S}^1}$ independent of all parameters such that $|\mathcal{M}(r_0)|, |\mathcal{N}(r_0)| \leq C_{\mathbb{S}^1} \zeta r_0^{-\gamma}$. Furthermore, from (6.3.21)–(6.3.22) of Corollary 5.4.3 with $a = a(r) = \frac{dx}{dr} \cdot (e^{\mu - \lambda})^{-1}$, one has

$$\left| r\frac{d}{dr}\mathcal{P}(r) + \mathcal{P}(r)\mathcal{Q}(r) \right| (r, x) \leq \delta r^{1/2}, \tag{6.5.4}$$

$$\left| r\frac{d}{dr}\mathcal{Q}(r) - \mathcal{Q}(r)\left(\mathcal{P}(r)^2 - 1 - \mathcal{Q}(r)\right) \right| (r, x) \leq \mathcal{Q}(r) \cdot \delta r^{1/2}.$$

This is exactly what we need to apply the nonlinear ODE results of Lemma 6.4.1. Therefore we have $(4\eta)^{-1} \leq \mathcal{P}(r) \leq 4\eta$ and $\mathcal{Q}(r) \leq 16\eta^2$. This is true for all timelike curves in our bootstrap spacetime, including all constant x curves, therefore for all $(r, x) \in [r_b, r_0] \times \mathbb{S}^1$, one has

$$(4\eta)^{-1} \leq r\partial_r \phi(r, x) \leq 4\eta, \qquad \frac{Q^2}{r^2} e^{2\mu}(r, x) \leq 16\eta^2. \tag{6.5.5}$$

We move onto $\mathcal{M}(r)$ and $\mathcal{N}(r)$. Using now (6.3.23)–(6.3.24) from Corollary 5.4.3, we also have

$$\left| r\frac{d}{dr}\mathcal{M}(r) + \mathcal{Q}(r)\mathcal{M}(r) - \mathcal{P}(r)\mathcal{N}(r) \right| (r,x) \leq \delta r^{1/2},$$

$$\left| r\frac{d}{dr}\mathcal{N}(r) + 2\mathcal{P}(r)\mathcal{Q}(r)\mathcal{M}(r) + \mathcal{Q}(r)\mathcal{N}(r) \right| (r,x) \leq \delta r^{1/2}.$$

Together with the initial data assumption $|\mathcal{M}(r_0)|, |\mathcal{N}(r_0)| \leq C_{\mathbb{S}^1}\zeta r_0^{-\gamma}$, this is what we need to apply Lemma 6.4.2 – note the change from ζ to $C_{\mathbb{S}^1}\zeta$ is harmless. Therefore for some $D > 0$ depending on η, γ, ζ but not on C_*, we have $|\mathcal{M}(r)| + |\mathcal{N}(r)| \leq Dr^{-\gamma}$. This is true for all timelike curves $\gamma(r)$, so for all $(r,x) \in [r_b, r_0] \times \mathbb{S}^1$,

$$|\partial_x \mu(r,x)| \leq Dr^{-\gamma}, \qquad |r\partial_r\partial_x\phi(r,x)| \leq Dr^{-\gamma}. \tag{6.5.6}$$

Step 2: Improving the bootstrap assumption (B1)

By choosing $C_* \geq 32\eta^2$ and using (6.5.5), we have already improved the first half of (B1). However, in order to improve the remaining parts of (B1), it will be necessary to upgrade the pointwise bound on $\frac{Q^2}{r^2}e^{2\mu}$ to an integrated bound on timelike curves.

More precisely, let $\gamma : [r_b, r_0] \to \mathbb{R}$ be a timelike curve as in Step 1. We rewrite the ODE-type estimate (6.5.4) in the following fashion, using crucially the lower bound for $r\partial_r\phi$ in (6.5.5):

$$\left| r\frac{d}{dr}\log\mathcal{P}(r) + \mathcal{Q}(r) \right| (r,x) \leq 4\eta\delta r^{1/2}.$$

We now integrate this with respect to the differential $\frac{dr}{r}$. One finds:

$$\left| \int_r^{r_0} \mathcal{Q}(\tilde{r})\frac{d\tilde{r}}{\tilde{r}} \right| \leq \left| \left[\log\mathcal{P}(r) \right]_r^{r_0} \right| + 4\eta\delta \leq \log\left(16\eta^2\right) + 4\eta\delta,$$

where the second inequality follows from the upper and lower bounds in (6.5.5). We apply this estimate to the case where $\gamma(r) = (r,x)$ is a timelike curve of constant x; we have for $D_1 = \log\left(16\eta^2\right) + 4\eta\delta$ (which is crucially independent of C_*) the estimate

$$\left| \int_r^{r_0} \frac{Q^2}{r^2}e^{2\mu}(\tilde{r},x)\frac{d\tilde{r}}{\tilde{r}} \right| \leq D_1. \tag{6.5.7}$$

To apply this to improve bounds for $e^{2\mu}$ and $e^{2(\mu-\lambda)}$, we use the equations (2.5.8) and (2.5.9) to derive that

$$\begin{aligned} r\partial_r(r^{-1}e^{2\mu}) &= \left((r\partial_r\phi)^2 + r^2e^{2(\mu-\lambda)}(\partial_x\phi)^2 + \kappa e^{2\mu} - \frac{Q^2}{r^2}e^{2\mu} \right) r^{-1}e^{2\mu} \\ &\geq \left(-e^{2\mu} - \frac{Q^2}{r^2}e^{2\mu} \right) r^{-1}e^{2\mu}, \end{aligned} \tag{6.5.8}$$

$$r\partial_r(r^{-1}e^{2(\mu-\lambda)}) = \left(1 + 2\kappa e^{2\mu} - \frac{2Q^2}{r^2}e^{2\mu}\right) r^{-1}e^{2(\mu-\lambda)}$$
$$\geq 2\left(-e^{2\mu} - \frac{Q^2}{r^2}e^{2\mu}\right) re^{2(\mu-\lambda)}. \quad (6.5.9)$$

Using the bootstrap assumption (B1) and that $r_* \geq r_0$ may be chosen small, let us choose r_* so that alongside (6.5.7), we have that $\int_r^{r_0} e^{2\mu}(\tilde{r}, x)\frac{d\tilde{r}}{\tilde{r}} \leq 1$. Then integrating (6.5.8) and (6.5.9) along constant x-curves, as well as applying the initial data assumption (6.1.8),

$$r^{-1}e^{2\mu}(r, x) \leq e^{D_1+1}r_0^{-1}e^{2\mu}(r_0, x) \leq e^{D_1+1}\zeta,$$

$$r^{-1}e^{2(\mu-\lambda)}(r, x) \leq e^{2(D_1+1)}r_0^{-1}e^{2(\mu-\lambda)}(r_0, x) \leq e^{2(D_1+1)}\zeta.$$

Thus for C_* chosen larger than $2e^{2(D_1+1)}\zeta$, we may improve the latter part of (B1).

Step 3: Improving the bootstrap assumption (B2)

The bootstrap assumption for $|r\partial_r\phi|$ in (B2) is already improved due to (6.5.5), so long as we choose $C_* \geq 8\eta$. To improve $|r\partial_r\lambda|$ and $|r\partial_r\mu|$, we use (2.5.8) and (2.5.9). Using Lemma 6.3.5 to estimate the term involving $\partial_x\phi$, and using both (6.5.5) and $e^{2\mu} \leq e^{D_1+1}\zeta r$ and $e^{2(\mu-\lambda)} \leq e^{2(D_1+1)}\zeta r$ from Step 2,

$$|r\partial_r\lambda| \leq \frac{1}{2}\left((r\partial_r\phi)^2 + r^2e^{2(\mu-\lambda)}(\partial_x\phi)^2 + 1 + e^{2\mu} + \frac{Q^2}{r^2}e^{2\mu}\right)$$
$$\leq \frac{1}{2}\left(16\eta^2 + e^{2(D_1+1)}\zeta\delta^2 r^{3-2\gamma} + 1 + e^{D_1+1}\zeta r + 16\eta^2\right).$$

By $r \leq r_0 \leq 1$, the right hand side is bounded independently of C_*, and thus by choosing $C_* \geq 2\cdot\text{RHS}$, we have improved the bootstrap assumption for $|r\partial_r\lambda|$. The improvement for $|r\partial_r\mu|$ follows from an identical calculation.

Step 4: Improving the bootstrap assumptions (B3) and (B4)

We complete the proof by improving the remaining bootstrap assumptions (B3) and (B4). We start with (B4), noting that the bootstrap assumption for $|r\partial_r\partial_x\phi|$ is already improved due to (6.5.6), so long as $C_* \geq 2D$. To improve the bootstrap assumptions for $|r\partial_r\partial_x\lambda|$ and $|r\partial_r\partial_x\mu|$, we differentiate (2.5.8) and (2.5.9) with respect to x, then combine several earlier estimates. Differentiating (2.5.8) yields

$$r\partial_r\partial_x\lambda = (r\partial_r\phi)(r\partial_r\partial_x\phi) + r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)(\partial_x\phi)^2 + r^2e^{2(\mu-\lambda)}\partial_x\phi\,\partial_x^2\phi$$
$$- \kappa e^{2\mu}\partial_x\mu + \frac{Q^2}{r^2}e^{2\mu}\partial_x\mu.$$

Therefore using all of (6.5.5), (6.5.6), Lemma 6.3.5 and the estimates for $e^{2\mu}$ and $e^{2(\mu-\lambda)}$ in Step 2, one yields

$$|r\partial_r\partial_x\lambda| \leq (4\eta D + e^{2(D_1+1)}\zeta D^3 r^{3-2\gamma} + e^{2(D_1+1)}\zeta D^2 r^{3-\gamma} + e^{D_1+1}\zeta Dr + 16\eta^2 D)r^{-\gamma}.$$

One can carry out an identical calculation for $|r\partial_r\partial_x\mu|$. Therefore for some D_2 independent of C_*, one has

$$|r\partial_r\partial_x\phi|, |r\partial_r\partial_x\lambda|, |r\partial_r\partial_x\mu| \leq D_2 r^{-\gamma}. \tag{6.5.10}$$

Finally, to improve (B3), we integrate (6.5.10) from $r = r_0$, and use the initial data assumption (6.1.9) – together with Sobolev embedding – to get $|\partial_x\phi(r_0)| + |\partial_x\mu(r_0)| + |\partial_x\lambda(r_0)| \leq C_{\mathbb{S}^1}\zeta r_0^{-\gamma}$. Thereby integrating (6.5.10) yields:

$$|\partial_x\phi|, |r\partial_r\partial_x\lambda|, |r\partial_r\partial_x\mu| \leq (D_2\gamma^{-1} + C_{\mathbb{S}^1}\zeta)r^{-\gamma}. \tag{6.5.11}$$

Thus upon choosing $C_* \geq 2(D_2\gamma^{-1} + C_{\mathbb{S}^1}\zeta)$, (6.5.10) and (6.5.11) improve the remaining bootstrap assumptions (B4) and (B3). By the standard bootstrap argument, this concludes the proof of Theorem 6.1.1. (The two estimates (6.1.10) and (6.1.11) follow from (6.5.5) and Proposition 6.3.4 respectively.) □

6.6 BKL bounces

6.6.1 Proof of Theorem 6.1.2

The assumptions of Theorem 6.1.2 mean that our global existence result Theorem 6.1.1 applies, as well as all the results of Section 6.3. In particular Corollary 5.4.3 applies, and one derives the ODEs (6.1.12) as in the proof of Theorem 6.1.1, see e.g. (6.5.4); note here we allow $\delta = 1$.

It remains to prove the convergence results (i) and (ii). The case (i) is easy, since $Q = 0$ implies $\mathcal{Q}_\gamma \equiv 0$, and the first equation in (6.1.12) yields $|r\partial_r\mathcal{P}_\gamma| \leq r^{1/2}$. Since $r^{-1/2}$ is integrable towards $r = 0$, this immediately yields that $\mathcal{P}_\gamma(r) \to \mathcal{P}_{\gamma,\infty}$ as $r \to \infty$ with $|\mathcal{P}_\gamma(r) - \mathcal{P}_{\gamma,\infty}| \leq 2r^{1/2}$.

We move onto the more interesting (ii). We first show that $\mathcal{Q}_\gamma(r) \to 0$ as $r \to 0$. For this purpose, we rearrange the first equation in (6.1.12) to yield

$$\mathcal{Q}_\gamma = -\mathcal{P}_\gamma^{-1} \cdot r\frac{d}{dr}\mathcal{P}_\gamma(r) + \mathcal{P}_\gamma^{-1} \cdot \mathcal{E}_\mathcal{P}.$$

Using from Theorem 6.1.2 that $(4\eta)^{-1} \leq \mathcal{P}_\gamma \leq 4\eta$, one thus yields that

$$\int_r^{r_0} \mathcal{Q}_\gamma(\tilde{r})\,\frac{d\tilde{r}}{\tilde{r}} \leq \log\left(\frac{\mathcal{P}_\gamma(r)}{\mathcal{P}_\gamma(r_0)}\right) + 4\eta r_0^{1/2} \leq \log\big(16\eta^2\big) + 4\eta r_0^{1/2}. \tag{6.6.1}$$

In particular, this integral is finite, and it follows that $\mathcal{Q}_\gamma(r)$ must tend to 0 as $r \to 0$ in an averaged sense. In particular there is a sequence $\{r_k\}$ with $r_k \to 0$ such that $\mathcal{Q}_\gamma(r_k) \to 0$ as $k \to \infty$. To upgrade this sequential convergence to convergence of $\mathcal{Q}_\gamma(r)$, we use the second equation in (6.1.12). Since $\mathcal{P}_\gamma^2 - 1 - \mathcal{Q} - \mathcal{E}_\mathcal{Q}$ is bounded, there is some constant C such that upon integrating the second equation in (6.1.12), for $0 < r < r_k$,

$$|\mathcal{Q}_\gamma(r) - \mathcal{Q}_\gamma(r_k)| \leq C\int_r^{r_k} \mathcal{Q}_\gamma(\tilde{r})\frac{d\tilde{r}}{\tilde{r}}.$$

Since we have already established that the integral in (6.6.1) is finite, the right hand side of this inequality tends to 0 as $k \to \infty$. It thus follows that $\mathcal{Q}_\gamma(r) \to 0$ as $r \to 0$.

To show that $\mathcal{P}_\gamma(r)$ converges as $r \to 0$, we again use the boundedness of the integral (6.6.1). Using this alongside the boundedness of $\mathcal{P}_\gamma(r)$, one finds that the right hand side of the first equation in (6.1.12) is integrable towards $r = 0$ with respect to $\frac{dr}{r}$. Thus there exists some $\mathcal{P}_{\gamma,\infty}$ so that $\mathcal{P}_\gamma(r) \to \mathcal{P}_{\gamma,\infty}$ as $r \to 0$. To show that $\mathcal{P}_{\gamma,\infty} \geq 1$, the second equation in (6.1.12) can be rearranged to

$$\mathcal{Q}_\gamma^{-1} \cdot r\frac{d}{dr}\mathcal{Q}_\gamma(r) = \mathcal{P}_\gamma^2 - 1 - \mathcal{Q}_\gamma - \mathcal{E}_\mathcal{Q} \to \mathcal{P}_{\gamma,\infty}^2 - 1 \quad \text{as } r \to 0.$$

In particular, for $\mathcal{Q}_\gamma(r)$ to remain bounded as $r \to 0$ it must be that the right hand side of this is nonnegative i.e. $\mathcal{P}_\gamma^2 \geq 1$.

Finally, to show (6.1.13), we use the almost conserved quantity $\mathcal{K}$ encountered in the proof of Lemma 6.4.1. Recall from this proof that defining $\mathcal{K}_\gamma = \mathcal{P}_\gamma + \mathcal{P}_\gamma^{-1} + \mathcal{Q}_\gamma\mathcal{P}_\gamma^{-1}$, one can show

$$\left| r\frac{d}{dr}\mathcal{K}_\gamma \right| \lesssim r^{1/2}.$$

Using the convergence of $\mathcal{P}_\gamma$ and $\mathcal{Q}_\gamma$, it therefore holds that there exists $\mathcal{K}_{\gamma,\infty} = \mathcal{P}_{\gamma,\infty} + \mathcal{P}_{\gamma,\infty}^{-1}$ so that $\mathcal{K}_\gamma(r) \to \mathcal{K}_{\gamma,\infty}$ as $r \to 0$ and moreover,

$$|\mathcal{K}_\gamma(r) - \mathcal{K}_{\gamma,\infty}| \lesssim r^{1/2}.$$

Substituting $r = r_0$, one therefore finds that

$$\left|(\mathcal{P}_{\gamma,\infty} + \mathcal{P}_{\gamma,\infty}^{-1}) - (\mathcal{P}_\gamma(r_0) + \mathcal{P}_\gamma^{-1}(r_0) + \mathcal{Q}_\gamma(r_0)\mathcal{P}_\gamma^{-1}(r_0))\right| \lesssim r_0^{1/2}.$$

Using this and $\mathcal{P}_{\gamma,\infty} \geq 1$, the estimate (6.1.13) follows by solving a quadratic inequality. Note, of course, that (6.1.13) is most interesting in the case that $\mathcal{Q}_\gamma(r_0)$ is small. This completes the proof of Theorem 6.1.2. □

6.6.2 Proof of Corollary 5.1.3

Finally, we apply Theorems 6.1.1 and 6.1.2 to prove this stability / instability result. Let (ϕ, μ, λ, Q) be as stated, so that by Theorem 6.1.0 there exist smooth functions $\Psi(x)$, $\Xi(x)$, $M(x)$ and $\Lambda(x)$ on $\mathbb{S}^1$ so that:

$$\phi(r,x) = \Psi(x)\log r + \Xi(x) + \mathrm{Err}_\phi(r,x),$$

$$\mu(r,x) = \tfrac{1}{2}(\Psi(x)^2 + 1)\log r + M(x) + \mathrm{Err}_\mu(r,x),$$

$$\lambda(r,x) = \tfrac{1}{2}(\Psi(x)^2 - 1)\log r + \Lambda(x) + \mathrm{Err}_\lambda(r,x),$$

where the error terms and their $r\partial_r$-derivatives tend to 0 as $r \to 0$ in the C^∞ topology.

Since we also assume that $\Psi(x) > 0$ for all $x \in \mathbb{S}^1$ and $\mathbb{S}^1$ is compact, there exists some $\eta \geq 2$ such that for some $0 < \tilde{r}_1 \leq r_1$, one has $(\eta/2)^{-1} \leq r\partial_r\phi(r,x) \leq \eta/2$ for all $0 < r < \tilde{r}_1$. Let N be chosen (depending on η) as in Theorem 6.1.1, $\gamma < \frac{1}{100}$ be fixed, and consider the energy: at fixed $r > 0$

$$\mathcal{E}(r) = \frac{1}{2} \sum_{f \in \{\phi,\lambda,\mu\}} \sum_{K=0}^{N} \left(\int_{\mathbb{S}^1} (r\partial_r \partial_x^K f)^2 + r^2 e^{2(\mu-\lambda)} (\partial_x^{K+1} f)^2 + r^{2\gamma} (\partial_x^K f)^2 \, dx \right).$$

By the asymptotics of Theorem 6.1.0, one has that

$$\mathcal{E}(r) \to \frac{1}{2} \sum_{K=0}^{N} \int_{S^1} \left((\partial_x^K \Psi)^2 + (\partial_x^K (\tfrac{1}{2}(\Psi^2+1)))^2 + (\partial_x^K (\tfrac{1}{2}(\Psi^2-1)))^2 \right) dx \quad \text{as } r \to 0.$$

So letting ζ be four times this limit, and making $\tilde{r}_1$ smaller if necessary, it follows that $\mathcal{E}(r) \leq \zeta/2$ for $0 < r < \tilde{r}_1$. Similarly using Theorem 6.1.0 and increasing ζ if necessary, one can guarantee $r^{-1}e^{2(\mu-\lambda)} \leq \zeta/2$ and $r^{-1}e^{2\mu} \leq \zeta/2$ for $0 < r < \tilde{r}_1$.

By these considerations, there will exist some $0 < r_0 < \tilde{r}_1$ so that if we abuse notation and *redefine* the objects $(\phi_D, \mu_D, \lambda_D, \dot{\phi}_D, \dot{\mu}_D, \dot{\lambda}_D)$ as

$$(\phi_D, \mu_D, \lambda_D, \dot{\phi}_D, \dot{\mu}_D, \dot{\lambda}_D) = (\phi, \mu, \lambda, r\partial_r\phi, r\partial_r\mu, r\partial_r\lambda)|_{r=r_0},$$

then $(\phi_D, \mu_D, \lambda_D, \dot{\phi}_D, \dot{\mu}_D, \dot{\lambda}_D)$ will obey the three assumptions (6.1.7), (6.1.8) and (6.1.9) required for the applications of our results Theorem 6.1.1 and Theorem 6.1.2, and in fact satisfy these assumptions with η and ζ replaced by $\eta/2$ and $\zeta/2$ respectively. We also assume that r_0 is chosen small enough to satisfy $r_0 < r_*(\eta, \zeta)$.

We now prove Corollary 5.1.3 with the initial data set given by this $(\phi_D, \mu_D, \lambda_D, \dot{\phi}_D, \dot{\mu}_D, \dot{\lambda}_D)$ at $r = r_0$ rather than the $r = r_1$ initial data stated. This is not an issue, as a standard Cauchy stability argument yields that perturbations of size ε at $r = r_1$ correspond to perturbations of size $\tilde{\varepsilon}$ at $r = r_0$, with $\tilde{\varepsilon} \to 0$ as $\varepsilon \to 0$.

The remainder of the argument is then a direct application of Theorem 6.1.1 and Theorem 6.1.2. Indeed, as stated in the corollary let $|Q| \leq \varepsilon$, and $\|(\tilde{f}_D, \dot{\tilde{f}}_D) - (f_D, \dot{f}_D)\|_{H^{N+1} \times H^N} \leq \varepsilon$ for $f \in \{\phi, \mu, \lambda\}$. Then for ε sufficiently small, it follows that the perturbed data $(\tilde{\phi}_D, \tilde{\mu}_D, \tilde{\lambda}_D, \dot{\tilde{\phi}}_D, \dot{\tilde{\mu}}_D, \dot{\tilde{\lambda}}_D)$ still obeys the assumptions (6.1.7), (6.1.8) and (6.1.9) for our choice of η and ζ. (Note the unperturbed background used $\eta/2$ and $\zeta/2$ so that the perturbed spacetime satisfies the required bounds with η and ζ.)

Thus one can apply Theorem 6.1.1 to the perturbed data, showing that (a) the resulting spacetime still features global existence towards $r = 0$. For (b), we use Theorem 6.1.2, applied to timelike curves $\gamma : r \mapsto (r, x)$ with constant x-coordinate. Then $\Psi(x)$ is then just $\mathcal{P}_{\gamma,\infty}$ for the original spacetime while $\tilde{\Psi}(x)$ is the same thing for the perturbed spacetime (which still exists due to Theorem 6.1.2), denoted by $\tilde{\mathcal{P}}_{\gamma,\infty}$.

To conclude, note that when $Q = 0$, by Theorem 6.1.2(i) one has

$$\tilde{\mathcal{P}}_{\gamma,\infty} = \tilde{\dot{\phi}}_D + O(r_0^{1/2}) = \dot{\phi}_D + O(\varepsilon) + O(r_0^{1/2}) = \mathcal{P}_{\gamma,\infty} + O(\varepsilon) + O(r_0^{1/2}).$$

By the aforementioned Cauchy stability argument, we are allowed to take r_0 as small as required, in particular we may choose the $O(r_0^{1/2})$ term to be less than $\frac{1}{2}\tilde{\varepsilon}$. Then choose ε so that the $O(\varepsilon)$ term is also smaller than $\frac{1}{2}\tilde{\varepsilon}$, while maintaining that the size of the perturbation at $r = r_1$ is still less than ε. This yields (i).

When $Q \neq 0$, the argument is almost identical, except that by Theorem 6.1.2(ii), one instead has

$$\tilde{\mathcal{P}}_{\gamma,\infty} = \max\{\tilde{\dot{\phi}}_D, \tilde{\dot{\phi}}_D^{-1}\} + O(r_0^{1/2}) + O\left(\frac{Q^2}{r^2} e^{2\mu}(r_0)\right) = \max\{\mathcal{P}_{\gamma,\infty}, \mathcal{P}_{\gamma,\infty}^{-1}\} + O(\varepsilon) + O(r_0^{1/2}),$$

where we used that $|Q| \leq \varepsilon$. (ii) therefore follows from Cauchy stability considerations as before. $\square$

Chapter 7

The (linearized) Einstein equations near Kasner

From Chapter 7 onwards, we switch our attention to considering the *scattering problem towards spacelike singularities*, as introduced in Section 1.6. As mentioned there, we will consider not scattering for the Einstein equations (Ein) but instead for two model linear hyperbolic problems on an exact (generalized) Kasner background (Kas). The results presented in Chapter 7 to Chapter 9 are based on our arXiv preprint [102].

In Chapter 7, we shall first explain how these model problems, in particular the second, and much more involved, model problem, which represents the *linearized Einstein–scalar field system* (1.6.4)–(1.6.5), fits into the general discourse of the nonlinear Einstein equations (Ein), spacelike singularities, and the BKL proposal. Towards the end of this Section, we will write down the full linearized system, with variables $(\mathfrak{n}_i{}^j, \mathfrak{k}_i{}^j, \phi, \psi)$.

From Chapter 7 onwards, note that we no longer restrict our attention purely to $(1+3)$-dimensional General Relativity, and consider the generalized Kasner family in $(1+D)$-dimensions.

7.1 The ADM decomposition

To study the linearized Einstein–scalar field system in the generality we require, as well a to simply fix notation, we introduce the (nonlinear) ADM-type gauge. A standard reference to the ADM-type gauge we employ is [119, Chapter 2], though our notation differs slightly, and we also generalize to higher dimension.

For a spacetime $\mathcal{M} \cong (0,T) \times \mathbb{T}^D$ endowed with a timelike function t, the standard $(1+D)$-decomposition of the spacetime metric is as follows, where x^i are standard coordinates on $\mathbb{T}^D$.

$$\mathbf{g} = -n^2 dt^2 + \bar{g}_{ij}(dx^i + X^i dt)(dx^j + X^j dt). \tag{7.1.1}$$

We call this an *ADM-type* gauge, with $n : \mathcal{M} \to \mathbb{R}_+$ the *lapse function* and $X = X^i \partial_{x^i}$ a vector

field tangent to the surfaces Σ_t of constant t called the *shift vector field*. The Riemannian metric $\bar{g}_{ij} = \bar{g}_{ij}(t)$ on Σ_t is often called the *first fundamental form*.

While the lapse $n = (-g(\nabla t, \nabla t))^{1/2}$ is determined purely by the choice of time function t, the shift vector field X^i is determined by the choice of coordinates x^i on each spatial slice Σ_t. The (future-directed) unit normal to the foliation of $(\mathcal{M}, \mathbf{g})$ by the hypersurfaces Σ_t is given by:

$$\mathbf{N} \coloneqq n^{-1} \left(\frac{\partial}{\partial t} - X^i \frac{\partial}{\partial x^i} \right). \tag{7.1.2}$$

The following definitions are standard for the ADM-type gauge, see e,g, [119, Chapter 2].

Definition 7.1.1. The *second fundamental form* k_{ij} of the foliation of $\mathcal{M}$ by Σ_t is a symmetric $(0,2)$-tensor tangent to Σ_t, given by the following, where the symbol $\mathbf{D}$ represents the spacetime Levi-Civita connection.

$$k(\xi, \eta) = -\mathbf{g}(\xi, \mathbf{D}_\eta \mathbf{N}), \quad \forall\, \xi, \eta \text{ tangent to } \Sigma_t.$$

The *mean curvature* $\operatorname{tr} k = \operatorname{tr}_{\bar{g}} k$ of the foliation is then given by $\operatorname{tr} k = (\bar{g}^{-1})^{ij} k_{ij}$.

Definition 7.1.2. One says that the ADM-type gauge (7.1.1) is *Constant Mean Curvature* (CMC) if the mean curvature of the foliation is constant on each slice Σ_t, with mean curvature $\operatorname{tr} k = -t^{-1}$.

Definition 7.1.3. We say that the gauge (7.1.1) is *Constant Mean Curvature Transported Coordinates* (CMCTC) if it is CMC and also satisfies $X^i \equiv 0$. In this case, the coordinates x^i are transported by the normal vector field $\mathbf{N}$, i.e. $\mathbf{N} x^i \equiv 0$ for $i = 1, \dots, D$.

On the other hand, we say that the gauge (7.1.1) is *Constant Mean Curvature Spatially Harmonic* (CMCSH) if it is CMC and is such that the coordinate map $(\mathbb{T}^D, \bar{g}_{ij}(t)) \to (\mathbb{T}^D, \bar{g}_{Euc})$ is a harmonic map for all $t \in (0, T)$, where $\bar{g}_{Euc}$ represents the standard Euclidean metric on $\mathbb{T}^D$. In other words,

$$(\bar{g}^{-1})^{jk} \left(\Gamma^i_{jk}[\bar{g}(t)] - \Gamma^i_{jk}[\bar{g}_{Euc}] \right) = 0, \tag{7.1.3}$$

where $\Gamma^i_{jk}[\bar{g}]$ are the Christoffel symbols of the Levi-Civita connection associated to $\bar{g}$, and similarly for $\bar{g}_{Euc}$. (With respect to standard coordinates, $\Gamma^i_{jk}[\bar{g}_{Euc}] = 0$, but we choose to write (7.1.3) in this form to clarify that it is actually a geometric equation.)

In Proposition 7.1.1, we present the (nonlinear) Einstein–scalar field system (Ein), (Scal) with respect to the ADM-type gauge. The variables $(\bar{g}_{ij}, k_{ij}, n, X^i)$ are used to describe the metric, while the variables $(\phi, \mathbf{N}\phi)$ are used to describe the scalar field ϕ. In Proposition 7.1.1, the symbol ∇_i will always represent the Levi-Civita connection with respect to the Riemannian metric $\bar{g}_{ij} = \bar{g}_{ij}(t)$. We also raise and lower indices with respect to this metric. In particular, $k_i^{\ j}$ will always mean $(\bar{g}^{-1})^{jk} k_{ik}$.

Proposition 7.1.1 (Einstein–scalar field in ADM-type gauge)**.** *In a spacetime described by a CMC ADM-type gauge with $k = -t^{-1}$, the Einstein–scalar field system* (Ein), (Scal) *are as follows:*

1. *The metric variables* $(\bar{g}_{ij}, k_{ij})$ *obey the system*

$$\partial_t \bar{g}_{ij} = -2nk_{ij} + \bar{g}_{ik}\nabla_j X^k + \bar{g}_{jk}\nabla_i X^k, \tag{7.1.4}$$

$$\partial_t k_{ij} = -\nabla_i\nabla_j n + n\left(\mathrm{Ric}_{ij}[\bar{g}] - 2\nabla_i\phi\nabla_j\phi - 2k_i{}^\ell k_{\ell j} - \frac{1}{t}k_{ij}\right) + X^k\nabla_k k_{ij} + k_{jk}\nabla_i X^k + k_{ik}\nabla_j X^k. \tag{7.1.5}$$

Here $\mathrm{Ric}_{ij}[\bar{g}]$ *denotes the Ricci tensor of the spatial metric* $\bar{g}$*, and may be expressed as*

$$\mathrm{Ric}_{ij}[\bar{g}] = -\frac{1}{2}(\bar{g}^{-1})^{ab}\partial_a\partial_b\bar{g}_{ij} - \frac{1}{2}(\bar{g}^{-1})^{ab}\partial_i\partial_j\bar{g}_{ab} + \frac{1}{2}(\bar{g}^{-1})^{ab}\partial_a\partial_i\bar{g}_{bj} + \frac{1}{2}(\bar{g}^{-1})^{ab}\partial_a\partial_j\bar{g}_{bi} + \mathcal{N}(\bar{g}^{-1}, \partial\bar{g}), \tag{7.1.6}$$

where the expression $\mathcal{N}(\bar{g}^{-1}, \partial\bar{g})$ *is a nonlinear term quadratic in the term* $\partial\bar{g}$*.*

If the gauge is moreover spatially harmonic (i.e. satisfies (7.1.3)*), then*

$$\mathrm{Ric}_{ij}[\bar{g}] = -\frac{1}{2}(\bar{g}^{-1})^{ab}\partial_a\partial_b\bar{g}_{ij} + \widehat{\mathcal{N}}(\bar{g}^{-1}, \partial\bar{g}), \tag{7.1.7}$$

where $\widehat{\mathcal{N}}(\bar{g}^{-1}, \partial\bar{g})$ *is a modified nonlinear term, still quadratic in* $\partial\bar{g}$*.*

2. *The matter variables* $(\phi, \mathbf{N}\phi)$ *evolve as follows:*

$$\partial_t\phi = n\left(\mathbf{N}\phi + X^i\nabla_i\phi\right), \tag{7.1.8}$$

$$\partial_t(\mathbf{N}\phi) = n\left((\bar{g}^{-1})^{ij}\nabla_i\nabla_j\phi - \frac{1}{t}\mathbf{N}\phi + X^i\nabla_i(\mathbf{N}\phi)\right) + (\bar{g}^{-1})^{ij}\nabla_i n\nabla_j\phi. \tag{7.1.9}$$

3. *The lapse* n *obeys the following elliptic equation:*

$$-(\bar{g}^{-1})^{ij}\nabla_i\nabla_j n + \frac{1}{t^2}(n-1) = -n\left(\mathrm{R}[\bar{g}] - 2(\bar{g}^{-1})^{ij}\nabla_i\phi\nabla_j\phi\right). \tag{7.1.10}$$

Here $\mathrm{R}[\bar{g}] = \mathrm{tr}\,\mathrm{Ric}_{ij}[\bar{g}]$ *is the scalar curvature of* $\bar{g}$*. If the gauge is moreover chosen to be spatially harmonic, then the shift vector* X^i *also obeys an elliptic equation as follows.*

$$(\bar{g}^{-1})^{ab}\nabla_a\nabla_b X^i + (\bar{g}^{-1})^{ij}\mathrm{Ric}_{jk}[\bar{g}]X^k = (\bar{g}^{-1})^{ij}\left(2k_j{}^a\nabla_a n + \frac{1}{t}\nabla_j n - 4n\,\mathbf{N}\phi\,\nabla_j\phi\right) - 2(\bar{g}^{-1})^{ab}\left(nk_a{}^c + \nabla_a X^c\right)\left(\Gamma^i_{bc}[\bar{g}] - \Gamma^i_{bc}[g_{Euc}]\right). \tag{7.1.11}$$

4. *Finally, the following Hamiltonian and momentum constraint equations are satisfied:*

$$\mathrm{R}[\bar{g}] - k_a{}^b k_b{}^a + \frac{1}{t^2} = 2(\mathbf{N}\phi)^2 + 2(\bar{g}^{-1})^{ij}\nabla_i\phi\nabla_j\phi, \tag{7.1.12}$$

$$\nabla_b\, k_a{}^b = -2\,\mathbf{N}\phi\,\nabla_a\phi. \tag{7.1.13}$$

Proof. The ADM evolution equations (7.1.4)–(7.1.5) and the constraints (7.1.12)–(7.1.13) are standard and follow from a higher dimensional analogue of [119, Section 2.3], upon substitution of the energy momentum tensor (1.1.5) and the gauge condition $\operatorname{tr} k = -t^{-1}$. One obtains the elliptic equation (7.1.10) by taking the trace of (7.1.5) and again using the CMC condition. The equation (7.1.8) is a consequence of the expression (7.1.2) for the normal vector field $\mathbf{N}$, while (7.1.9) follows from the wave equation (Scal) written in the gauge (7.1.1).

In the case of CMCSH gauge, the relevant equations can be found in [8], at least in the vacuum case. See also [7, 57] for the addition of matter described by the scalar field ϕ with energy momentum tensor (1.1.5). □

Remark. From the PDE viewpoint, it is important to have a local well-posedness theory for the equations of Proposition 7.1.1. There are several ways to do this, each of which involves picking a specific spatial gauge. One of these is the CMCSH gauge, where local well-posedness is shown in [8]. For a more general overview we refer the reader to [66].

We conclude Section 7.1 by revisiting the Kasner spacetimes with respect to the ADM decomposition. The (generalized) Kasner spacetimes $(\mathcal{M}_{Kas}, \mathbf{g}_{Kas}, \phi_{Kas})$ with (Kas), $\phi_{Kas} = p_\phi \log t$ and the Kasner relations (1.1.6), are naturally written in the ADM-type gauge, with:

$$\mathring{n} \equiv 1, \quad \mathring{X}^i \equiv 0, \quad \mathring{\bar{g}}_{ij} = t^{2p_i}\delta_{ij}, \quad \mathring{\phi} = p_\phi \log t, \tag{7.1.14}$$

$$\mathring{k}_{ij} = -\frac{p_i}{t}\mathring{\bar{g}}_{ij}, \quad \mathrm{Ric}_{ij}[\mathring{g}] \equiv 0, \quad \mathring{\mathbf{N}}\mathring{\phi} = \partial_t \mathring{\phi} = \frac{p_\phi}{t}. \tag{7.1.15}$$

The Kasner spacetimes are spatially homogeneous, in the sense that the coordinate vector fields ∂_{x^i} are Killing fields of g_{Kas}, and solve to the Einstein–scalar field equations as stated in Proposition 7.1.1. With $\mathring{n}$ and $\mathring{X}^i$ as above, the gauge is moreover both CMCTC and CMCSH. Thus in studying perturbations of the Kasner spacetimes, it is permissible to use either of these gauges.

7.2 The linearized Einstein equations

In this section, we derive the linearized Einstein–scalar field system around Kasner, to which our Theorem 7.3.1, scattering for the model of linearized gravity in Kasner, applies. After presenting the equations, we then comment upon several features of this system, including gauge invariance (see Lemma 7.2.3) and well posedness (see Proposition 7.2.4), and remark upon how to interpret the system near $t = 0$ in Section 7.2.2.

After finishing this discussion, we project this system in frequency space using Fourier series (see Appendix D), in preparation for the energy estimates of Chapter 9. Finally, we explain how different gauges and energy estimates will be applied in our proof.

Our aim is to present the linearized Einstein–scalar field system (1.6.4)–(1.6.5) around a fixed Kasner background $(\mathcal{M}_{Kas}, \mathbf{g}_{Kas}, \phi_{Kas})$. To make this concrete, we first define several variables

which we deem *linearly small* in Definition 7.2.1. We then derive the linearized system by Taylor expanding (to first order) the system (7.1.4)–(7.1.13) around a fixed Kasner background with

$$(\bar{g}_{ij}, k_{ij}, \phi, \mathbf{N}\phi, n, X^i) = (\mathring{\bar{g}}_{ij}, \mathring{k}_{ij}, p_\phi \log t, p_\phi, 1, 0),$$

and removing all terms which are quadratic or higher order in the linearly small quantities.

Since the Kasner solution is itself a solution to the nonlinear system (7.1.4)–(7.1.13), the outcome of the procedure described above is a homogeneous linear system of PDEs where the unknowns are the linearly small quantities and the coefficients are related to the background Kasner variables. This system of PDEs is what is presented in Proposition 7.2.2.

For a further justification of the linearization procedure, we refer the reader to Section 3 of [129]. We now define our linearly small variables, which differ slightly from those of [129].

Definition 7.2.1. We define the following *linearly small* quantities:

$$\eta_i^{\ j} \coloneqq -\frac{1}{2}(\mathring{\bar{g}}^{-1})^{jk}(\bar{g}_{ik} - \mathring{\bar{g}}_{ik}), \quad \kappa_i^{\ j} \coloneqq t(\bar{g}^{-1})^{jk}k_{ik} - t(\mathring{\bar{g}}^{-1})^{jk}\mathring{k}_{ik} = t(\bar{g}^{-1})^{jk}k_{ik} + p_{\underline{i}}\delta_{\underline{i}}^{\ j}. \tag{7.2.1}$$

$$\phi \coloneqq \phi - \mathring{\phi}, \quad \psi \coloneqq t(\mathbf{N}\phi - \mathring{\mathbf{N}}\mathring{\phi}) = t\,\mathbf{N}\phi - p_\phi, \tag{7.2.2}$$

$$\nu \coloneqq n - \mathring{n} = n - 1, \quad \chi^i \coloneqq tX^i. \tag{7.2.3}$$

We also let $\operatorname{tr}\eta = \eta_i^{\ i}$ and $\operatorname{tr}\kappa = \kappa_i^{\ i}$ be the usual traces.

Lemma 7.2.1. *A consequence of the CMC gauge condition* $\operatorname{tr}_{\bar{g}} k = -t^{-1}$ *is that:*

$$\operatorname{tr}\kappa = 0. \tag{7.2.4}$$

On the other hand, the linearized version of the spatially harmonic condition (7.1.3)*, which we often denote the linearized CMCSH gauge*[1]*, is the following:*

$$2\partial_j \hat{\eta}_i^{\ j} = \partial_i \operatorname{tr}\hat{\eta}. \tag{7.2.5}$$

Proof. The equation (7.2.4) follows immediately by taking the trace of (7.2.1) and using that $\operatorname{tr}_{\bar{g}} k = \operatorname{tr}_{\mathring{\bar{g}}} \mathring{k} = -t^{-1}$. In the case of spatially harmonic gauge, note that in the usual Euclidean coordinates, the condition (7.1.3) may be written as $(\bar{g}^{-1})^{ab}\Gamma^i_{ab}[\bar{g}] = 0$, or as

$$\frac{1}{2}(\bar{g}^{-1})^{ab}(\bar{g}^{-1})^{ij}\left(\partial_a \bar{g}_{jb} + \partial_b \bar{g}_{ja} - \partial_j \bar{g}_{ab}\right) = 0.$$

As the background $\mathring{\bar{g}}_{ij}$ is spatially homogeneous, $\partial_i \bar{g}_{jk} = \partial_i(\bar{g}_{jk} - \mathring{\bar{g}}_{jk})$ is linearly small. Writing $\bar{h}_{jk} = \bar{g}_{jk} - \mathring{\bar{g}}_{jk}$, one may then Taylor expand the inverse metric $(\bar{g}^{-1})^{ab}$ to write the above equation as

$$\frac{1}{2}(\mathring{g}^{-1})^{ab}(\mathring{g}^{-1})^{ij}\left(\partial_a \bar{h}_{jb} + \partial_b \bar{h}_{ja} - \partial_j \bar{h}_{ab}\right) = \mathcal{Q}(h, \partial h),$$

[1]We always demarcate when we are using a CMCSH gauge by writing the $(1,1)$ tensors $\hat{\eta}_i^{\ j}$ and $\hat{\kappa}_i^{\ j}$ with a hat.

where $\mathcal{Q}(h, \partial h)$ is quadratic and higher order in the linearly small quantity h. Ignoring this term, and using $\eta_i{}^j = -\frac{1}{2}(\mathring{g}^{-1})^{jk}\bar{h}_{ik}$, this above equation becomes

$$-2(\mathring{g}^{-1})^{ij}\partial_a \eta_j{}^a + (\mathring{g}^{-1})^{ij}\partial_j \eta_a{}^a = 0.$$

Up to multiplication by $\mathring{g}_{ik}$ and relabelling indices, this is exactly (7.1.3). □

Proposition 7.2.2 (Linearized Einstein–scalar field system)**.** *The linear system of PDEs for the variables* $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ *obtained by linearizing the ADM system* (7.1.4)–(7.1.13) *around a background Kasner solution, is as follows.*

1. *The metric variables* $(\eta_i{}^j, \kappa_i{}^j)$ *obey the evolution equations*

$$t\partial_t \eta_i{}^j = \kappa_i{}^j + (2t\mathring{k}_p{}^j)\eta_i{}^p - (2t\mathring{k}_i{}^p)\eta_p{}^j + (t\mathring{k}_i{}^j)\nu - \frac{1}{2}\partial_i \chi^j - \frac{1}{2}\mathring{g}_{ip}\mathring{g}^{jq}\partial_q \chi^p, \tag{7.2.6}$$

$$t\partial_t \kappa_i{}^j = t^2 \widetilde{\mathrm{Ric}}_i{}^j[\eta] - t^2\mathring{g}^{jk}\partial_i\partial_k \nu - (t\mathring{k}_i{}^j)\nu + (t\mathring{k}_a{}^j)\partial_i \chi^a - (t\mathring{k}_i{}^a)\partial_a \chi^j. \tag{7.2.7}$$

Here, the expression $\widetilde{\mathrm{Ric}}[\eta]$ *is the linearized Ricci tensor, and is defined as*

$$\widetilde{\mathrm{Ric}}_i{}^j[\eta] \coloneqq \mathring{g}^{ab}\partial_a\partial_b \eta_i{}^j + \mathring{g}^{ja}\partial_i\partial_a(\mathrm{tr}\,\eta) - \mathring{g}^{ab}\partial_i\partial_b \eta_a{}^j - \mathring{g}^{ja}\partial_a\partial_b \eta_i{}^b. \tag{7.2.8}$$

In particular, if $\hat{\eta}_i{}^j$ *moreover satisfies the spatially harmonic condition* (7.2.5)*, then*

$$\widetilde{\mathrm{Ric}}_i{}^j[\hat{\eta}] = \mathring{g}^{ab}\partial_a\partial_b \hat{\eta}_i{}^j. \tag{7.2.9}$$

2. *The matter variables* (ϕ, ψ) *obey*

$$t\partial_t \phi = \psi + p_\phi \nu, \tag{7.2.10}$$

$$t\partial_t \psi = t^2\mathring{g}^{ab}\partial_a\partial_b \phi - p_\phi \nu. \tag{7.2.11}$$

3. *The linearized lapse* ν *obeys the elliptic equation*

$$\left(1 - t^2\mathring{g}^{ab}\partial_a\partial_b\right)\nu = -2t^2\mathring{g}^{ab}\partial_a\partial_b(\mathrm{tr}\,\eta) + 2t^2\mathring{g}^{ab}\partial_i\partial_a \eta_b{}^i. \tag{7.2.12}$$

If $\hat{\eta}_i{}^j$ *moreover satisfies the spatially harmonic condition* (7.2.5)*, this equation simplifies to*

$$\left(1 - t^2\mathring{g}^{ab}\partial_a\partial_b\right)\nu = -t^2\mathring{g}^{ab}\partial_a\partial_b(\mathrm{tr}\,\hat{\eta}). \tag{7.2.13}$$

Furthermore, if (7.2.5) *holds we have an additional elliptic equation for the shift:*

$$-t^2\mathring{g}^{ab}\partial_a\partial_b \chi^j = -t^2\mathring{g}^{ij}\partial_i \nu - 2t^2\mathring{g}^{ij}(t\mathring{k}_i{}^k)\partial_k \nu - 4t^2\mathring{g}^{pq}(t\mathring{k}_p{}^i)\partial_q \hat{\eta}_i{}^j + 2t^2\mathring{g}^{ij}\partial_i[t\mathring{k}_b{}^a\,\hat{\eta}_a{}^b + 2p_\phi\phi]. \tag{7.2.14}$$

4. As well as the linearized CMC condition (7.2.4)*, the following constraints are satisfied:*

$$-2t^2\mathring{g}^{ab}\partial_a\partial_b(\operatorname{tr}\eta) + 2t^2\mathring{g}^{ab}\partial_i\partial_a\eta_b{}^i + 2t\mathring{k}_b{}^a\,\kappa_a{}^b + 4p_\phi\psi = 0. \tag{7.2.15}$$

$$\partial_j\kappa_i{}^j + \partial_i[t\mathring{k}_b{}^a\,\eta_a{}^b + 2p_\phi\phi] - t\mathring{k}_i{}^j\partial_j(\operatorname{tr}\eta) = 0. \tag{7.2.16}$$

$$\mathring{g}^{ab}\partial_a\kappa_b{}^c + \mathring{g}^{cd}\partial_d[t\mathring{k}_b{}^a\,\eta_a{}^b + 2p_\phi\phi] - 2\mathring{g}^{ab}\,t\mathring{k}_b{}^d\,\partial_a\eta_d{}^c = 0. \tag{7.2.17}$$

As mentioned the solution is in CMCSH gauge if (7.2.5) *is also satisfied. In this case,* (7.2.15) *simplifies to*

$$-t^2\mathring{g}^{ab}\partial_a\partial_b(\operatorname{tr}\hat{\eta}) + 2t\mathring{k}_b{}^a\,\hat{\kappa}_a{}^b + 4p_\phi\psi = 0. \tag{7.2.18}$$

5. Finally, there are linearized symmetry conditions for $\eta_i{}^j$ *and* $\kappa_i{}^j$*:*

$$\mathring{g}^{ab}\eta_b{}^c = \mathring{g}^{cb}\eta_b{}^a, \tag{7.2.19}$$

$$\mathring{g}^{ab}\left(\kappa_b{}^c - 2t\mathring{k}_b{}^d\,\eta_d{}^c\right) = \mathring{g}^{cb}\left(\kappa_b{}^a - 2t\mathring{k}_b{}^d\,\eta_d{}^a\right). \tag{7.2.20}$$

Remark. We outline some of the main differences from the linearized system derived in [129, Proposition 3.2]. The first difference is that we use the linearly small quantity $\eta_i{}^j$, while [129] directly uses the metric perturbation $\bar{h}_{ij} = \bar{g}_{ij} - \mathring{g}_{ij}$. Secondly, we do not insist on transported coordinates, and therefore allow a linearized shift vector χ^j. Thirdly, we use $\psi = \mathbf{N}\phi - p_\phi$ as our linearly small quantity, in place of $\partial_t\phi - \frac{p_\phi}{t}$. ([129] do mention our ψ as an 'alternative approach'.)

Proof. We shall provide detailed proofs of four of the equations in Proposition 7.2.2, namely the evolution equations (7.2.6) and (7.2.7), the elliptic equation (7.2.14) for χ^j in CMCSH gauge, and the symmetry condition (7.2.20). The remaining equations may be derived by similar methods and are left to the reader.

As in [129, Proposition 3.2], the overall strategy will be to consider the equations in Proposition 7.1.1 and Taylor expand them around the background Kasner solution, eventually discarding terms which are quadratic or higher in the linearly small quantities of Definition 7.2.1. We also make use of $\bar{h}_{ij} = \bar{g}_{ij} - \mathring{g}_{ij}$, and as in [129], at least within this proof we shall write $(\mathring{g}^{-1})^{jk}$ and $(\bar{g}^{-1})^{jk}$ to denote the inverse metrics.

We start with the derivation of (7.2.6). For this, the first step is to multiply (7.1.4) by $t(\mathring{g}^{-1})^{jk}$. Using the fact that $t\partial_t(\mathring{g}^{-1})^{jk} = 2t\mathring{k}_\ell{}^j(\mathring{g}^{-1})^{k\ell}$, and relabelling indices, we deduce that

$$-\frac{1}{2}t\partial_t((\mathring{g}^{-1})^{jk}\bar{g}_{ik}) = -t\mathring{k}_\ell{}^j(\mathring{g}^{-1})^{k\ell}\bar{g}_{ik} + n(\mathring{g}^{-1})^{jk}tk_{ik} - \frac{1}{2}(\mathring{g}^{-1})^{jk}\bar{g}_{i\ell}\nabla_k(tX^\ell) - \frac{1}{2}(\mathring{g}^{-1})^{jk}\bar{g}_{k\ell}\nabla_i(tX^\ell).$$

We now begin to Taylor expand. Writing $\bar{g}_{ab} = \mathring{g}_{ab} + \bar{h}_{ab}$ in the first, third and fourth terms of the right hand side, while using $n = 1 + \nu$ and $(\mathring{g}^{-1})^{jk} = (\bar{g}^{-1})^{jk} + (\mathring{g}^{-1})^{ja}\bar{h}_{ab}(\bar{g}^{-1})^{bk}$ in the second term, we have

$$t\partial_t\left(-\frac{1}{2}(\mathring{g}^{-1})^{jk}\bar{g}_{ik}\right) = -t\mathring{k}_i{}^j - t\mathring{k}_\ell{}^j(\mathring{g}^{-1})^{k\ell}\bar{h}_{ik} + (1+\nu)\left(t(\bar{g}^{-1})^{jk} + (\mathring{g}^{-1})^{ja}\bar{h}_{ab}\, t(\bar{g}^{-1})^{bk}\right)k_{ik}$$
$$- \frac{1}{2}(\mathring{g}^{-1})^{jk}(\mathring{g}_{i\ell} + \bar{h}_{i\ell})\nabla_k\chi^\ell - \frac{1}{2}(\mathring{g}^{-1})^{jk}(\mathring{g}_{k\ell} + \bar{h}_{k\ell})\nabla_i\chi^\ell.$$

We now simply use that $\eta_i{}^j = -\frac{1}{2}(\mathring{g}^{-1})^{jk}\bar{h}_{ik}$, as well as the fact that modulo quadratic error terms, ∇_a can be replaced by ∂_a since the Christoffel symbols and the renormalized shift χ^ℓ are both linearly small. Therefore for $\mathcal{Q}_\eta = \mathcal{Q}_\eta(\bar{h}, \partial\bar{h}, \nu, \chi, \partial\chi)$ a quadratic and higher order nonlinearity,

$$t\partial_t\eta_i{}^j = t(\bar{g}^{-1})^{jk}k_{ik} - t\mathring{k}_i{}^j + 2t\mathring{k}_\ell{}^j\eta_i{}^\ell - 2t(\bar{g}^{-1})^{bk}k_{ik}\eta_b{}^j + t(\bar{g}^{-1})^{jk}k_{ik}\nu$$
$$- \frac{1}{2}(\mathring{g}^{-1})^{jk}\mathring{g}_{i\ell}\partial_k\chi^\ell - \frac{1}{2}\partial_i\chi^j + \mathcal{Q}_\eta.$$

Thereby upon inserting the definition of the linearly small quantity $\kappa_i{}^j = t(\bar{g}^{-1})^{jk}k_{ik} - t\mathring{k}_i{}^j$, and removing all quadratic or higher order terms in the linearly small quantities, the equation (7.2.6) follows.

The next equation we derive is (7.2.7). For this equation, we firstly combine the two equations (7.1.4) and (7.1.5) to deduce

$$t\partial_t\left(t(\bar{g}^{-1})^{jk}k_{ik}\right) = -t^2(\bar{g}^{-1})^{jk}\nabla_i\nabla_k n + nt^2(\bar{g}^{-1})^{jk}\mathrm{Ric}_{ik}[\bar{g}]$$
$$- 2nt^2(\bar{g}^{-1})^{jk}\nabla_i\phi\nabla_k\phi - (n-1)t(\bar{g}^{-1})^{jk}k_{ik}$$
$$+ t(\bar{g}^{-1})^{jk}(tX^\ell)\nabla_\ell k_{ik} + t(\bar{g}^{-1})^{jk}k_{k\ell}\nabla_k(tX^\ell) - t(\bar{g}^{-1})^{k\ell}k_{ik}\nabla_\ell(tX^j).$$

We now simply insert the definitions of all the linearly small quantities, and replace all ∇_a derivatives with ∂_a derivatives modulo quadratic error terms, to deduce that for some term $\mathcal{Q}_\kappa$ quadratic in the linearly small quantities η, $\partial\eta$, $\partial^2\eta$, ν, $\partial\nu$, $\partial^2\nu$, κ, $\partial\kappa$, χ, $\partial\chi$ and $\partial\phi$,

$$t\partial_t\kappa_i{}^j = t^2(\mathring{g}^{-1})^{jk}\mathrm{Ric}_{ik}[\bar{g}] - t^2(\mathring{g}^{-1})^{jk}\partial_i\partial_k\nu - t\mathring{k}_i{}^j\nu + t\mathring{k}_\ell{}^j\partial_k\chi^\ell - t\mathring{k}_i{}^\ell\partial_\ell\chi^j + \mathcal{Q}_\kappa.$$

Upon using (7.1.6) to express $\mathrm{Ric}_{ik}[\bar{g}]$ and removing all quadratic or higher order error terms, the equation (7.2.7) follows. (If instead we used (7.1.7) we get the desired result in the case of CMCSH gauge.)

The next order of business is the elliptic equation (7.2.14), assuming the CMCSH gauge condition (7.1.3). We shall derive this equation by using (7.1.11), which we multiply by t^3 and rearrange to get

$$t^2(\bar{g}^{-1})^{ab}\nabla_a\nabla_b(tX^i) = t^2(\bar{g}^{-1})^{ij}\left(2tk_j{}^a\nabla_a n + \nabla_j n - 4np_\phi\nabla_j\phi\right) - 2nt^2(\bar{g}^{-1})^{ab}(tk_a{}^c)\Gamma^i_{bc}[\bar{g}]$$
$$- 2t^2(\bar{g}^{-1})^{ab}\nabla_a(tX^c)\Gamma^i_{bc}[\bar{g}] - t^2(\bar{g}^{-1})^{ij}\mathrm{Ric}_{jk}[\bar{g}]tX^k - 4nt^2(\bar{g}^{-1})^{ij}(t\mathbf{N}\phi - p_\phi)\nabla_j\phi.$$

Now, we observe that the second line of this expression is already quadratic or higher order in our linearly small variables $(\mathfrak{\eta}_i{}^j, \mathfrak{\kappa}_i{}^j, \phi, \psi, \nu, \chi^j)$ and their derivatives, and we therefore discard it. Similarly, we may Taylor expand all but the final term on the first line, and we deduce that for some $\mathcal{Q}_\chi$ which is quadratic or higher order in the linearly small quantities, one has:

$$t^2(\mathring{g}^{-1})^{ab}\partial_a\partial_b\chi^i = t^2(\mathring{g}^{-1})^{ij}(2t\mathring{k}_j{}^a\partial_a\nu + \partial_j\nu - 4p_\phi\partial_j\phi) - 2t^2(\bar{g}^{-1})^{ab}(tk_a{}^c)\Gamma^i_{bc}[\bar{g}] + \mathcal{Q}_\chi.$$

It remains to expand the expression $t^2(\bar{g}^{-1})^{ab}(tk_a{}^c)\Gamma^i_{bc}[\bar{g}]$. Using the local coordinate form of Christoffel symbols, and the fact that $\partial_i\bar{g}_{jk} = \partial_i\bar{h}_{jk}$, we have

$$-2t^2(\bar{g}^{-1})^{ab}(tk_a{}^c)\Gamma^i_{bc}[\bar{g}] = -t^2(\bar{g}^{-1})^{ab}(\bar{g}^{-1})^{ij}(tk_a{}^c)\left(\partial_b\bar{h}_{jc} + \partial_c\bar{h}_{jb} - \partial_j\bar{h}_{cb}\right).$$

Since the final parenthesized expression is linearly small, we may replace $(\bar{g}^{-1})^{ij}$ and $(tk_i{}^j)$ by their background Kasner analogues up to a quadratic error. Using also that $tk_i{}^j$ is self-adjoint with respect to $\bar{g}_{ij}$, we find that up to a quadratic or higher order error we call $\mathcal{Q}_\Gamma$, one has

$$\begin{aligned}
&-2t^2(\bar{g}^{-1})^{ab}(tk_a{}^c)\Gamma^i_{bc}[\bar{g}] \\
&\quad= -2t^2(\mathring{g}^{-1})^{ab}(t\mathring{k}_a{}^c)\partial_b((\mathring{g}^{-1})^{ij}\bar{h}_{jc}) + \partial_a\bar{h}_{jb} + t^2(\mathring{g}^{-1})^{ij}(t\mathring{k}_a{}^c)\partial_j((\mathring{g}^{-1})^{ab}\bar{h}_{bc}) + \mathcal{Q}_\Gamma \\
&\quad= 4t^2(\mathring{g}^{-1})^{ab}(t\mathring{k}_a{}^c)\partial_b\hat{\mathfrak{n}}_c{}^i - 2t^2(\mathring{g}^{-1})^{ij}(t\mathring{k}_a{}^c)\partial_j\hat{\mathfrak{n}}_c{}^a + \mathcal{Q}_\Gamma.
\end{aligned}$$

Combining this with the expression for $t^2(\mathring{g}^{-1})^{ab}\partial_a\partial_b\chi^i$ above, and discarding the nonlinearities $\mathcal{Q}_\chi$ and $\mathcal{Q}_\Gamma$, we get the elliptic equation (7.2.14), as required.

The final equation we derive in detail is the symmetry constraint (7.2.20) for $\mathfrak{\kappa}_i{}^j$ and $\mathfrak{\eta}_i{}^j$. This arises from the symmetry of the second fundamental form k_{ij}, which we write in the following way (as in the elliptic equation for shift, the $(1,1)$-tensor $k_i{}^j$ means $(\bar{g}^{-1})^{jk}k_{ik}$.)

$$(\bar{g}^{-1})^{ab}(tk_b{}^c) = (\bar{g}^{-1})^{cb}(tk_b{}^a).$$

Expanding the various terms, we therefore have

$$((\mathring{g}^{-1})^{ab} - (\mathring{g}^{-1})^{ad}\bar{h}_{de}(\bar{g}^{-1})^{eb})(t\mathring{k}_b{}^c + \mathfrak{\kappa}_b{}^c) = ((\mathring{g}^{-1})^{cb} - (\mathring{g}^{-1})^{cd}\bar{h}_{de}(\bar{g}^{-1})^{eb})(t\mathring{k}_b{}^a + \mathfrak{\kappa}_b{}^a).$$

The equation (7.2.20) then follows by inserting the definition of $\mathfrak{\eta}_i{}^j$, rearranging, cancelling $(\mathring{g}^{-1})^{ab}(t\mathring{k}_b{}^c) = (\mathring{g}^{-1})^{cb}(t\mathring{k}_b{}^a)$ from both sides, and finally removing quadratic and higher order terms. □

7.2.1 Spatial gauge and well-posedness

Though we have fixed our foliation via the CMC condition $\mathrm{tr}_{\bar{g}}\, k = -t^{-1}$, there is still gauge freedom in the choice of linearized quantities $(\mathfrak{\eta}_i{}^j, \mathfrak{\kappa}_i{}^j, \chi^j)$ in Definition 7.2.1 and in Proposition 7.2.2 due to freedom in the choice of spatial coordinates, which we may view as diffeomorphisms of the spatial manifold $\mathbb{T}^D$ that we allow to be time-dependent.

While one way of removing this freedom by imposing some gauge on initial data as well as in evolution, for instance by specifying a shift vector χ^j or alternatively by imposing the linearized CMCSH condition (7.2.5), it will be useful in our context of this article to allow all such gauge choices, as well as how to change between them. At the infinitesimal or linearized level, this is achieved via a time-dependent spatial vector field ξ^j on $\mathbb{T}^D$, as in the following lemma.

Lemma 7.2.3. *Let $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ be a solution to the linearized Einstein–scalar field system in Proposition 7.2.2 (not necessarily obeying the linearized CMCSH condition (7.2.5)). Let ξ^j be a possibly time-dependent vector field on $\mathbb{T}^D$. Then the system (7.2.6)–(7.2.20) is invariant under the gauge transformation:*

$$\eta_i{}^j \mapsto \eta_i{}^j + \frac{1}{2}\partial_i\xi^j + \frac{1}{2}\mathring{g}_{ip}\mathring{g}^{jq}\partial_q\xi^p, \tag{7.2.21}$$

$$\kappa_i{}^j \mapsto \kappa_i{}^j + (t\mathring{k}_i{}^p)\partial_p\xi^j - (t\mathring{k}_p{}^j)\partial_i\xi^p, \tag{7.2.22}$$

$$\chi^j \mapsto \chi^j - t\partial_t\xi^j, \tag{7.2.23}$$

while the scalar variables ϕ, ψ, ν are left unchanged. In particular,

(i) For a solution $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ as above, then applying the above transformation using the choice of vector field ξ^j given by inverting the elliptic equation

$$t^2\mathring{g}^{ab}\partial_a\partial_b\xi^j = t^2\mathring{g}^{jk}\left(\partial_k \operatorname{tr}\eta - 2\partial_\ell\eta_k{}^\ell\right)$$

yields a solution in linearized CMCSH gauge, via the transformation (7.2.21)–(7.2.23).

(ii) Let $\chi_\circ^j$ be a fixed (possibly time-dependent) vector field. For any solution $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ as above, then for any choice of $T > 0$ choose the vector field ξ^j to satisfy

$$\xi^j = \int_T^t \left(\chi^j(s) - \chi_\circ^j(s)\right)\frac{ds}{s}.$$

Then after applying the transformation (7.2.21)–(7.2.23) to our solution, the transformed solution has $\chi^j \equiv \chi_\circ^j$. In particular, by choosing $\chi_\circ^j \equiv 0$, we can always transform to a solution with zero shift. We say that such solutions are in linearized CMCTC gauge.

(Sketch) Proof. We leave the proof of invariance of the system (7.2.6)–(7.2.20) under the transformation (7.2.21)–(7.2.23) to the reader, and only mention at a heuristic level how the transformation (7.2.21)–(7.2.23) is derived. Essentially, we consider coordinate transformations of the form $\tilde{x}^j = x^j + \xi^j$. Then the ADM-type metric (7.1.1) may be rewritten as

$$\begin{aligned} g &= -n^2dt^2 + \bar{g}_{ij}(dx^i + X^idt)(dx^j + X^jdt) \\ &= -n^2dt^2 + \bar{g}_{ij}\left(d\tilde{x}^i - \frac{\partial\xi^i}{\partial x^k}dx^k - \frac{\partial\xi^i}{\partial t}dt + X^idt\right)\left(d\tilde{x}^j - \frac{\partial\xi^j}{\partial x^\ell}dx^\ell - \frac{\partial\xi^j}{\partial t}dt + X^idt\right) \\ &= -n^2 + \tilde{\bar{g}}_{ij}(d\tilde{x}^i + \tilde{X}^idt)(d\tilde{x}^j + \tilde{X}^j) + \mathcal{Q}_\xi(\xi, \partial\xi). \end{aligned}$$

In this final line, $\mathcal{Q}_\chi(\xi, \partial\xi)$ is a quadratic nonlinearity, while the modified first fundamental form $\tilde{\bar{g}}_{ij}$ and shift vector $\tilde{X}^j$ are given by

$$\tilde{\bar{g}}_{ij} = \bar{g}_{ij} - \bar{g}_{ik}\frac{\partial \xi^k}{\partial x^j} - \bar{g}_{jk}\frac{\partial \xi^k}{\partial x^i}, \qquad \tilde{X}^j = X^j - \frac{\partial \xi^j}{\partial t}.$$

One now inserts $\tilde{\bar{g}}_{ij}$ and $\tilde{X}^j$ in place of $\bar{g}_{ij}$ and X^j in the expressions of the linearly small quantities of Definition 7.2.1. (One can also suitably modify the second fundamental form). Discarding terms which are quadratic or higher in the original linearly small quantities and ξ^j, we are left with (7.2.21)–(7.2.23).

Leaving the remaining details to the reader, we move on to (i) and (ii). For (i), suppose that the change of gauge vector field ξ^j satisfies the elliptic equation $t^2 \mathring{g}^{ab}\partial_a\partial_b\xi^j = t^2\mathring{g}^{jk}(\partial_k \operatorname{tr}\eta - 2\partial_\ell \eta_k{}^l)$, and consider the transformed version of $\eta_i{}^j$, which from (7.2.21) we know to be

$$\hat{\eta}_i{}^j = \eta_i{}^j + \frac{1}{2}\partial_i\xi^j + \frac{1}{2}\mathring{g}_{ip}\mathring{g}^{jq}\partial_q\xi^p.$$

To check this is now in CMCSH gauge, we want to check that $2\partial_j\hat{\eta}_i{}^j = \partial_i \operatorname{tr}\hat{\eta}$. For this we compute:

$$\begin{aligned} t^2\mathring{g}^{ij}(2\partial_k\hat{\eta}_i{}^k - \partial_i \operatorname{tr}\hat{\eta}) &= t^2\mathring{g}^{ij}(2\partial_k\eta_i{}^k - \partial_i \operatorname{tr}\eta) + t^2\mathring{g}^{ij}\partial_k\partial_i\xi^k + t^2\mathring{g}^{ik}\partial_i\partial_k\xi^j - t^2\mathring{g}^{ij}\partial_i\partial_k\xi^k \\ &= t^2\mathring{g}^{ij}(2\partial_k\eta_i{}^k + t^2\mathring{g}^{ik}\partial_i\partial_k\xi^j = 0. \end{aligned}$$

The final equality followed exactly due to the choice of ξ^j. This completes the proof of (i). The proof of (ii) is more straightforward; one simply uses the transformation formula for the shift (7.2.23). □

Remark. In Lemma 7.2.3, one may in particular choose a vector field ξ^j which is *time-independent*. This has the effect of transforming $\hat{\eta}_i{}^j$ and $\hat{\kappa}_i{}^j$ but leaving χ^j unchanged. In particular, even in the class of solutions with $\chi^j = 0$ such gauge changes are still permitted

Upon fixing the spatial gauge, by either fixing the shift vector χ^j or by insisting upon the linearized CMCSH condition (7.2.5), we can appeal to the following well-posedness result.

Proposition 7.2.4 (Well-posedness for the linearized Einstein–scalar field system)**.** *Let* $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C)$ *be Cauchy data (for instance at* $t = 1$*) that verifies the constraint equations* (7.2.4) *and* (7.2.15)–(7.2.20)*, with regularity* $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C) \in H^{s+1} \times H^s \times H^{s+1} \times H^s$*. Then both of the following well-posedness statements hold.*

(i) Suppose furthermore that $(\hat{\eta}_C)_i{}^j$ *verifies the linearized CMCSH condition* (7.2.5)*. Then there exists a unique CMCSH solution* $(\hat{\eta}_i{}^j(t), \hat{\kappa}_i{}^j(t), \phi(t), \psi(t), \nu(t), \chi^j(t))$ *to the system* (7.2.6)–

(7.2.20) *for* $t \in (0, +\infty)$*, where* χ^j *has zero average*[2]*, and with regularity:*

$$\left(\eta_i{}^j, \kappa_i{}^j\right) \in C^0((0,+\infty), H^{s+1} \times H^s) \cap C^1((0,+\infty), H^s \times H^{s-1}),$$

$$(\phi, \psi) \in C^0((0,+\infty), H^{s+1} \times H^s) \cap C^1((0,+\infty), H^s \times H^{s-1}),$$

$$\nu \in C^0((0,+\infty), H^{s+2}), \qquad \chi^j \in C^0((0,+\infty), H^{s+2}).$$

(ii) Alternatively, let $\chi^j_\circ$ *be any (possibly time-dependent) vector field on* $\mathbb{T}^D$*, with* $\chi^j_\circ \in C^0((0,+\infty), H^{s+2})$*. Then there exists a unique solution* $(\eta_i{}^j(t), \kappa_i{}^j(t), \phi(t), \psi(t), \nu(t))$ *to the system* (7.2.6)–(7.2.20) *for* $t \in (0, +\infty)$*, such that* $\chi^j \equiv \chi^j_\circ$ *at all times, with the same regularity as above.*

Proof. Let us start with (i). We first use the equations (7.2.6), (7.2.7), (7.2.9), (7.2.10), (7.2.11), (7.2.12), (7.2.14), which is a linear elliptic-hyperbolic system in the background $(\mathcal{M}_{Kas}, g_{Kas})$. (Note it is crucial that we used (7.2.9) in place of (7.2.8) since only then does $\widetilde{\mathrm{Ric}}[\hat{\eta}]$ appearing in 7.2.7 become a top order term of hyperbolic character.) Therefore we may apply standard elliptic-hyperbolic theory, e.g. that of [8], to deduce that there is a solution to these equations (7.2.6), (7.2.7), (7.2.9), (7.2.10), (7.2.11), (7.2.12), (7.2.14) for $t \in (0, +\infty)$. Moreover, this solution has the regularity claimed in (i).

It remains to check that the constraint equations (7.2.15)–(7.2.20) and the gauge conditions (7.2.4) and (7.1.3) also remain true for $t \in (0, +\infty)$. The propagation of constraints is left to Proposition F.0.1. Finally, uniqueness follows from the elliptic-hyperbolic theory, so long as we fix χ^j to have zero average in order to have uniqueness for solutions of the elliptic equation (7.2.14).

We now move on to (ii). In order to show existence, the first step will be to apply a gauge transformation to the Cauchy data $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C)$ using a vector field ξ^j_C which solves the elliptic equation

$$\mathring{g}^{ab}(1)\partial_a\partial_b\xi^j_C = \mathring{g}^{jk}(1)(\partial_k \operatorname{tr}\eta_C - 2\partial_\ell(\eta_C)_k{}^\ell).$$

As in Lemma 7.2.3, this has the effect of putting the transformed $(\hat{\eta}_C)_i{}^j = (\eta_C)_i{}^j + \frac{1}{2}\partial_i\xi^j + \frac{1}{2}\mathring{g}_{ip}(1)\mathring{g}^{jq}(1)\partial_q\xi^p$ in CMCSH gauge, at least at $t = 1$. This allows us to use the existence result in (i), and we generate a CMCSH solution $(\hat{\eta}_i{}^j(t), \hat{\kappa}_i{}^j(t), \phi(t), \psi(t), \nu(t), \chi^j(t))$ for $t > 0$ which achieves the Cauchy data $((\hat{\eta}_C)_i{}^j, (\hat{\kappa}_C)_i{}^j, \phi_C, \psi_C)$.

We now perform another change of gauge to recover the solution we desire. Note that the shift vector $\chi^j(t)$ was generated in evolution in CMCSH gauge, and thus cannot be expected to equal $\chi^j_\circ$, and also that our CMCSH solution does not match our initial choice of Cauchy data $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C)$. We resolve both these issues by using yet another gauge transformation

[2]In this paper, the vector fields χ^j and ξ^j will always have zero average, meaning their $\lambda = 0$ Fourier coefficients χ^j_0 and ξ^j_0 always vanish. Note the zero modes have no affect on how $\eta_i{}^j$ and $\kappa_i{}^j$ transform.

vector field coming from Lemma 7.2.3(ii), namely

$$\xi^j = -\xi^j_C + \int_1^t \left(\chi^j(s) - \chi^j_\circ(s)\right) \frac{ds}{s}.$$

After applying this gauge transformation to our CMCSH solution $(\hat{\eta}_i{}^j(t), \hat{\kappa}_i{}^j(t), \phi(t), \psi(t), \nu(t), \chi^j(t))$, it is immediate to see that we obtain a solution satisfying the requirements of (ii).

The regularity is also immediate, and we finally turn to uniqueness. By taking the difference of two such solutions, uniqueness is equivalent to the statement that the only solution to the linearized Einstein–scalar field system with trivial data $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C) = (0,0,0,0)$ and vanishing shift $\chi^j(t) = 0$ being the zero solution. Suppose, on the contrary, that there exists a non-trivial such solution $((\eta_\circ)_i{}^j(t), (\kappa_\circ)_i{}^j(t), \phi_\circ(t), \psi_\circ(t))$.

By uniqueness of CMCSH solutions, we know that this solution must not be CMCSH, hence there exists some time $T > 0$ such that $2\partial_j(\eta_\circ)_i{}^j(T) \neq \partial_i \operatorname{tr}\eta_\circ(T)$. Now we apply Lemma 7.2.3(i) to this solution – this generates, a new CMCSH solution $((\hat{\eta}_\circ)_i{}^j(t), (\hat{\kappa}_\circ)_i{}^j(t), \phi_\circ(t), \psi_\circ(t), \nu_\circ(t), \chi^j_\circ(t))$, via a transformation associated to the (zero average) vector field $\xi^j_\circ$ which solves the elliptic equation

$$t^2 \mathring{g}^{ab}\partial_a\partial_b\xi^j_\circ = t^2\mathring{g}^{jk}(\partial_j \operatorname{tr}\eta_\circ - 2\partial_\ell((\eta_\circ)_k{}^\ell).$$

Since $(\eta_\circ)_i{}^j = 0$ at $t = 1$, we must have $\xi^j_\circ(1) = 0$. On the other hand, by definition of $T > 0$, we must have $\xi^j_\circ(T) \neq 0$. As a result, there must exist some $\hat{T} > 0$ such that $\partial_t\xi^j_\circ(\hat{T}) \neq 0$, and therefore by the transformation rule (7.2.23) and the fact that we initially had vanishing shift, we must have $\chi^j_\circ(\hat{T}) \neq 0$.

However, by $\xi^j_\circ(1) = 0$ the solution $((\hat{\eta}_\circ)_i{}^j(t), (\hat{\kappa}_\circ)_i{}^j(t), \phi_\circ(t), \psi_\circ(t), \nu_\circ(t), \chi^j_\circ(t))$ is a CMCSH solution with trivial data, and hence by uniqueness in the CMCSH gauge we must have $\chi^j_\circ(t) = 0$ for all $t > 0$. This contradicts the existence of a nontrivial solution with $\chi^j = 0$, and completes the proof of (ii). □

7.2.2 Scattering data at $t = 0$

The linearly small quantities $\eta_i{}^j$ and ϕ generically blow up towards $t = 0$, and are thus unsuitable choices for scattering data at $t = 0$ itself. Therefore, we introduce renormalized quantities $\Upsilon_i{}^j$ and φ, given by:

$$\Upsilon_i{}^j \coloneqq \eta_i{}^j + \int_t^1 \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\frac{ds}{s} \cdot \kappa_q{}^p, \tag{7.2.24}$$

$$\varphi \coloneqq \phi - \psi \log t. \tag{7.2.25}$$

In fact, we often entertain a more general form of $\Upsilon_i{}^j$ and φ, where the time $t = 1$ is replaced by $t = T > 0$. That is, we introduce the quantities $\tilde{\Upsilon}_i{}^j$ and $\tilde{\varphi}$ as

$$\tilde{\Upsilon}_i{}^j \coloneqq \eta_i{}^j + \int_t^T \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\,\frac{ds}{s} \cdot \kappa_q{}^p, \qquad \tilde{\varphi} \coloneqq \phi - \psi \log\left(\frac{t}{T}\right). \tag{7.2.26}$$

These more generally defined quantities will play a role in the following two lemmas.

Part of Theorem 7.3.1 is that $\Upsilon_i{}^j$ and φ remain bounded near $t = 0$, and in fact tend to limits $(\Upsilon_\infty)_i{}^j$ and φ_∞ as $t \to 0$. This will also be true for $\kappa_i{}^j$ and ψ, which will tend to limits $(\kappa_\infty)_i{}^j$ and ψ_∞ as $t \to 0$. In this section, we prove two important properties of these renormalized quantities, or more generally the corresponding quantities $((\kappa_\infty)_i{}^j, (\tilde{\Upsilon}_\infty)_i{}^j, \psi_\infty, \tilde{\varphi}_\infty)$ for any $T > 0$.

Firstly, in Lemma 7.2.5, we show that these limiting quantities $((\kappa_\infty)_i{}^j, (\tilde{\Upsilon}_\infty)_i{}^j, \psi_\infty, \tilde{\varphi}_\infty)$ also obey constraint equations, analogous to those of (7.2.15)–(7.2.20), which we often denote the *asymptotic constraint equations*. Then in Lemma 7.2.6, we determine how $\tilde{\Upsilon}_i{}^j$ transforms under the (linearized) change of gauge discussed in Section 7.2.1.

Lemma 7.2.5. *Let s be sufficiently large, and let $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ be a solution to the linearized Einstein–scalar field system in Proposition 7.2.2, with regularity as in Proposition 7.2.4, where the Kasner exponents satisfy the subcriticality condition* (1.6.11)*. For arbitrary $T > 0$, suppose furthermore that the quantities $\kappa_i{}^j$, $\tilde{\Upsilon}_i{}^j$, ψ and $\tilde{\varphi}$ remain bounded towards $t = 0$, and moreover tend to a limit:*

$$(\kappa_i{}^j, \tilde{\Upsilon}_i{}^j, \psi, \tilde{\varphi}) \to ((\kappa_\infty)_i{}^j, (\tilde{\Upsilon}_\infty)_i{}^j, \psi_\infty, \tilde{\varphi}_\infty) \quad \text{in } C^3 \text{ as } t \to 0.$$

Then the following asymptotic constraint equations hold:

$$t\mathring{k}_b{}^a (\kappa_\infty)_a{}^b + 2p_\phi \psi_\infty = 0, \tag{7.2.27}$$

$$\partial_j (\kappa_\infty)_i{}^j + \partial_i [t\mathring{k}_b{}^a (\tilde{\Upsilon}_\infty)_a{}^b + 2p_\phi \tilde{\varphi}_\infty] - t\mathring{k}_i{}^j \partial_j (\operatorname{tr} \tilde{\Upsilon}_\infty) = 0, \tag{7.2.28}$$

$$\mathring{g}^{ab}(T)\partial_a (\kappa_\infty)_b{}^c + \mathring{g}^{cd}(T)\partial_d [t\mathring{k}_b{}^a (\tilde{\Upsilon}_\infty)_a{}^b + 2p_\phi \tilde{\varphi}_\infty] - 2\mathring{g}^{ab}(T)\, t\mathring{k}_b{}^d\, \partial_a (\tilde{\Upsilon}_\infty)_d{}^c = 0, \tag{7.2.29}$$

as well as the following symmetry conditions:

$$\mathring{g}^{ab}(T)(\tilde{\Upsilon}_\infty)_b{}^c = \mathring{g}^{cb}(T)(\tilde{\Upsilon}_\infty)_b{}^a, \tag{7.2.30}$$

$$\mathring{g}^{ab}(T)\left((\kappa_\infty)_b{}^c - 2t\mathring{k}_b{}^d (\tilde{\Upsilon}_\infty)_d{}^c\right) = \mathring{g}^{cb}(T)\left((\kappa_\infty)_b{}^a - 2t\mathring{k}_b{}^d (\tilde{\Upsilon}_\infty)_d{}^a\right). \tag{7.2.31}$$

Finally, we have the linearized CMC condition $\operatorname{tr} \kappa_\infty = 0$.

Proof. For (7.2.27) and (7.2.28), the idea is to simply take the usual constraints (7.2.15) and (7.2.16), rewrite them in terms of the renormalized variables $\tilde{\Upsilon}_i{}^j$, $\tilde{\varphi}$, and then finally take the limit $t \to 0$. For instance, we rewrite the Hamiltonian constraint (7.2.15) using (7.2.24) as

$$-2t^2 \mathring{g}^{ab}\partial_a\partial_b \operatorname{tr} \tilde{\Upsilon} + 2t^2 \mathring{g}^{ab}\partial_i\partial_a \tilde{\Upsilon}_b{}^i - 2t^2 \mathring{g}^{ab} \int_t^T \mathring{g}_{bp}(s)\mathring{g}^{iq}(s)\frac{ds}{s} \cdot \partial_i\partial_a \kappa_q{}^p + 2(t\mathring{k}_b{}^a)\kappa_a{}^b + 4p_\phi\psi = 0.$$

We now take the limit of this equation as $t \to 0$. The last two terms will give us the desired (7.2.27), so it suffices to show that the remaining terms tend to 0.

As $\partial_a\partial_b\tilde{\Upsilon}_i{}^j$ and $\partial_a\partial_b\kappa_i{}^j$ are bounded towards $t \to 0$ by assumption, it suffices to show that as $t \to 0$,

$$t^2\mathring{g}^{ab} \to 0 \quad \text{and} \quad t^2\mathring{g}^{ab}\int_t^1 \mathring{g}_{bp}(s)\mathring{g}^{iq}(s)\frac{ds}{s} = 0.$$

In fact both of these will follow by the non-degeneracy of the Kasner exponents; using a modification of Lemma 9.4.3 (with $t_{\lambda *}$ replaced by 1 in the lemma) to estimate the integral above, one shows these terms are bounded by $\max_i t^{2-2p_i}(1 + |\log t|)$ and thus tend to 0 as $t \to 0$.

Moving to the momentum constraint, we rewrite (7.2.16) as follows:

$$\partial_i\tilde{\Upsilon}_i{}^j + \partial_i[(t\mathring{k}_b{}^a)\tilde{\Upsilon}_a{}^b + 2p_\phi\tilde{\varphi}] - (t\mathring{k}_i{}^j)\partial_j \operatorname{tr}\tilde{\Upsilon} = -\partial_i[(t\mathring{k}_b{}^a)\kappa_a{}^b + 2p_\phi\psi]\log t.$$

Note that we used the fact that $t\mathring{k}_b{}^a$ is non-zero only when $a = b$, hence the integral appearing in (7.2.24) ends up being $\log t$. The right hand side of this rewritten momentum constraint appears exactly in the rewritten Hamiltonian constraint above, and (given that third derivatives of $\tilde{\Upsilon}_i{}^j$ and $\kappa_i{}^j$ are also bounded) is therefore bounded by $\max_i t^{2-2p_i}(1 + |\log t|^2)$. Thus taking the limit $t \to 0$ yields (7.2.28).

We next move to the symmetry conditions (7.2.30) and (7.2.31). We shall actually prove that for all times $t > 0$, we have the symmetry conditions

$$\mathring{g}^{ab}(T)\tilde{\Upsilon}_b{}^c(t) = \mathring{g}^{cb}(T)\tilde{\Upsilon}_b{}^a(t), \tag{7.2.32}$$

$$\mathring{g}^{ab}(T)\left(\kappa_b{}^c(t) - 2t\mathring{k}_b{}^d\,\tilde{\Upsilon}_d{}^c(t)\right) = \mathring{g}^{cb}(T)\left(\kappa_b{}^a(t) - 2t\mathring{k}_b{}^d\,\tilde{\Upsilon}_d{}^a(t)\right), \tag{7.2.33}$$

and simply take the limit of these as $t \to 0$. Note that (7.2.32) is equivalent to $\mathring{g}^{ac}(T)\mathring{g}_{bd}(T)\tilde{\Upsilon}_c{}^d = \tilde{\Upsilon}_b{}^a$, or

$$\mathring{g}^{ac}(T)\mathring{g}_{bd}(T)\,\eta_c{}^d + \mathring{g}^{ac}(T)\mathring{g}_{bd}(T)\int_t^T \mathring{g}_{cp}(s)\mathring{g}^{dq}(s)\frac{ds}{s}\cdot\kappa_q{}^p = \eta_b{}^a + \int_t^T \mathring{g}_{bp}(s)\mathring{g}^{aq}(s)\frac{ds}{s}\cdot\kappa_q{}^p. \tag{7.2.34}$$

We now make the following observation. Using the explicit form of $\mathring{g}_{ij}$ in (7.1.14), one has the identity:

$$\mathring{g}^{ac}(T)\mathring{g}_{bd}(T)\int_t^T \mathring{g}_{cp}(s)\mathring{g}^{dq}(s)\frac{ds}{s} = \mathring{g}_{cp}(t)\mathring{g}^{dq}(t)\int_t^T \mathring{g}^{ac}(s)\mathring{g}_{bd}(s)\frac{ds}{s}.$$

One obtains this identity using a substitution of the form $s \mapsto \tilde{s} = \frac{Tt}{s}$ in the integral.

Using this identity, one computes the second term on the left hand side of (7.2.34) to be

$$\begin{aligned}\mathring{g}^{ac}(T)\mathring{g}_{bd}(T)\int_t^T \mathring{g}_{cp}(s)\mathring{g}^{dq}(s)\frac{ds}{s}\cdot\kappa_q{}^p &= \int_t^T \mathring{g}^{ac}(s)\mathring{g}_{bd}(s)\frac{ds}{s}\cdot\mathring{g}_{cp}(t)\mathring{g}^{dq}(t)\,\kappa_q{}^p(t)\\ &= \int_t^T \mathring{g}^{ac}(s)\mathring{g}_{bd}(s)\frac{ds}{s}\cdot\left(\kappa_c{}^d + (2t\mathring{k}_p{}^d)\eta_c{}^p - (2t\mathring{k}_c{}^p)\eta_p{}^d\right),\end{aligned}$$

where we used the symmetry condition (7.2.20) in the second step. We next observe that we can identify the latter two terms on the right hand side in the following way:

$$\mathring{g}^{ac}(T)\mathring{g}_{bd}(T)\int_t^T \mathring{g}_{cp}(s)\mathring{g}^{dq}(s)\frac{ds}{s}\cdot\kappa_q{}^p = \int_t^T \mathring{g}^{ac}(s)\mathring{g}_{bd}(s)\frac{ds}{s}\cdot\kappa_c{}^d - \int_t^T s\partial_s(\mathring{g}^{ac}\mathring{g}_{bd})\frac{ds}{s}\cdot\eta_c{}^d.$$

Therefore we obtain (7.2.34) by using the fundamental theorem of calculus, and simply relabelling indices. This completes the proof of (7.2.32). For (7.2.33), we firstly observe that

$$\mathring{k}_b{}^c - (2t\mathring{k}_b{}^d)\tilde{\Upsilon}_d{}^c = \mathring{k}_b{}^c - (2t\mathring{k}_b{}^d)\eta_d{}^c + \int_t^T s\partial_s(\mathring{g}_{bp}(s))\mathring{g}^{cq}\frac{ds}{s}\cdot\mathring{k}_q{}^p.$$

Therefore integrating by parts and using (7.2.20) in a similar way to before, one eventually finds that

$$\mathring{k}_b{}^c - (2t\mathring{k}_b{}^d)\tilde{\Upsilon}_d{}^c = \mathring{g}_{bp}(T)\mathring{g}^{cq}(T)\,\mathring{k}_q{}^p - (2t\mathring{k}_d{}^c)\tilde{\Upsilon}_b{}^d.$$

From this identity, (7.2.33) follows from (7.2.32). We also deduce (7.2.30) and (7.2.31) by taking $t\to 0$.

Finally, the second momentum constraint (7.2.29) follows from combining (7.2.28) with (7.2.30) and (7.2.31). □

Lemma 7.2.6. *Consider the renormalized quantity $\tilde{\Upsilon}_i{}^j$ defined in* (7.2.26)*, for some $T>0$. Then the change of gauge associated the vector field ξ^j, which transforms $\eta_i{}^j$ and $\mathring{k}_i{}^j$ as in* (7.2.21) *and* (7.2.22)*, then transforms the expression $\tilde{\Upsilon}_i{}^j$ as follows:*

$$\tilde{\Upsilon}_i{}^j \mapsto \tilde{\Upsilon}_i{}^j + \frac{1}{2}\partial_i\xi^j + \frac{1}{2}\mathring{g}_{ip}(T)\mathring{g}^{jq}(T)\partial_q\xi^p. \tag{7.2.35}$$

In particular, choosing ξ^j to be a solution of the elliptic equation

$$\mathring{g}^{ab}(T)\partial_a\partial_b\xi^j = \mathring{g}^{jk}(T)\left(\partial_k \operatorname{tr}\tilde{\Upsilon} - 2\partial_\ell\tilde{\Upsilon}_k{}^\ell\right), \tag{7.2.36}$$

one can transform $\tilde{\Upsilon}_i{}^j$ in such a way that the transformed version satisfies the differential relation

$$\partial_i \operatorname{tr}\tilde{\Upsilon} - 2\partial_j\tilde{\Upsilon}_i{}^j = 0.$$

Proof. From the definition (7.2.26), the transformation associated to ξ^j transforms $\tilde{\Upsilon}_i{}^j$ into the following expression

$$\left(\eta_i{}^j + \frac{1}{2}\partial_i\xi^j + \frac{1}{2}\mathring{g}_{ip}(t)\mathring{g}^{jq}(t)\partial_q\xi^p\right) + \int_t^T \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\frac{ds}{s}\cdot\left(\mathring{k}_q{}^p + (t\mathring{k}_q{}^s)\partial_s\xi^p - (t\mathring{k}_s{}^p)\partial_q\xi^s\right)$$
$$= \tilde{\Upsilon}_i{}^j + \frac{1}{2}\partial_i\xi^j + \frac{1}{2}\mathring{g}_{ip}(t) + \mathring{g}^{jq}(t)\partial_q\xi^p + \int_t^T s\partial_s(\mathring{g}_{ip}(s)\mathring{g}^{jq}(s))\frac{ds}{s}\cdot\partial_q\xi^p.$$

The transformation formula (7.2.35) then follows by applying the fundamental theorem of calculus.

The second part of the lemma, concerning the change of gauge associated to the vector field ξ^j solving the elliptic equation (7.2.36), follows from the same argument as Lemma 7.2.3(i). □

7.3 Statement of Theorem 7.3.1

With the notation setup, we now state a slightly imprecise, but still rather detailed, version of the scattering theorem for the linearized Einstein–scalar field system around a Kasner spacetime obeying the subcriticality relation (1.2.11). We prove Theorem 7.3.1 in Chapter 9. The proof will actually be via a scattering theorem in another gauge, Theorem 9.5.1. Recall also the symbol $\mathcal{T}_*$ from Definition 1.6.1.

It turns out it will be most convenient to choose a slightly odd choice of gauge condition for the quantity $\Upsilon_i{}^j$ at $t = 0$, as follows.

$$2\partial_j \left((\Upsilon_\infty)_i{}^j + \int_{\mathcal{T}_*}^1 \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\frac{ds}{s} \cdot (\kappa_\infty)_q{}^p \right) = 2\partial_i \left(\operatorname{tr} \Upsilon_\infty \right). \tag{7.3.1}$$

We later show that we can always transform suitable asymptotic data into this gauge, which we often call the *canonical asymptotic gauge condition*.

Theorem 7.3.1 (Scattering for the linearized Einstein–scalar field system in subcritical Kasner spacetimes)**.** *Let $(\mathcal{M}_{Kas}, g_{Kas}, \phi_{Kas})$ be a (generalized) Kasner spacetime whose exponents satisfy the subcriticality condition* (1.2.11)*. Consider solutions $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi)$ to the linearized Einstein–scalar field equations around $(\mathcal{M}_{Kas}, g_{Kas}, \phi_{Kas})$, written as the first-order elliptic-hyperbolic system* (7.2.6)–(7.2.20) *and satisfying suitable constraints. The scattering theory is as follows:*

(i) (Existence of the scattering operator) Let s be sufficiently large. Then given Cauchy data $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C) \in H^{s+1} \times H^s \times H^{s+1} \times H^s$ at $t = 1$ satisfying suitable constraints as well as the spatially harmonic gauge condition (7.1.3)*, there exists a unique (classical) solution $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi)$ to the system* (7.2.6)–(7.2.20) *in $\mathcal{M}_{Kas}$ attaining the Cauchy data*

$$\eta_i{}^j(1, \cdot) = (\eta_C)_i{}^j(\cdot), \quad \kappa_i{}^j(1, \cdot) = (\kappa_C)_i{}^j(\cdot), \quad \phi(1, \cdot) = \phi_C(\cdot), \quad \psi(1, \cdot) = \psi_C(\cdot).$$

Let $\Upsilon_i{}^j$ and φ be as in (7.2.24)–(7.2.25)*. Then there exist unique $(1,1)$ tensors $(\kappa_\infty)_i{}^j(x)$ and $(\Upsilon_\infty)_i{}^j(x)$ and unique functions $\psi_\infty(x)$ and $\varphi_\infty(x)$ such that*

$$\kappa_i{}^j(t, \cdot) \to (\kappa_\infty)_i{}^j(\cdot) \text{ in } C^3, \qquad \Upsilon_i{}^j(t, \cdot) \to (\Upsilon_\infty)_i{}^j(\cdot) \text{ in } C^3 \quad \text{strongly as } t \to 0, \tag{7.3.2}$$

$$\psi(t, \cdot) \to \psi_\infty(\cdot) \text{ in } H^s, \qquad \varphi(t, \cdot) \to \varphi_\infty(\cdot) \text{ in } H^s \quad \text{strongly as } t \to 0. \tag{7.3.3}$$

Furthermore, there exists a change of gauge vector field $\xi_\infty^j \in H^{s+\frac{1}{2}}$ such that after applying the transformations (7.2.21)–(7.2.35)*, the transformed versions of $((\kappa_\infty)_i{}^j, (\Upsilon_\infty)_i{}^j, \psi_\infty, \varphi_\infty)$ satisfy suitable asymptotic constraints as well as the canonical asymptotic gauge condition* (7.3.1)*, and have the following regularity:*

$$\mathcal{T}_*^{-p_{\underline{i}}+p_{\underline{j}}}(\kappa_\infty)_{\underline{i}}{}^{\underline{j}} \in H^{s+\frac{1}{2}}, \qquad \mathcal{T}_*^{-p_{\underline{i}}+p_{\underline{j}}} \left((\Upsilon_\infty)_{\underline{i}}{}^{\underline{j}} + \int_{\mathcal{T}_*}^1 \mathring{g}_{\underline{i}p}(s)\mathring{g}^{\underline{j}q}(s)\frac{ds}{s} \cdot (\kappa_\infty)_q{}^p \right) \in H^{s+\frac{1}{2}}, \tag{7.3.4}$$

$$\psi_\infty \in H^{s+\frac{1}{2}}, \qquad \varphi_\infty + \log(\mathcal{T}_*)\psi_\infty \in H^{s+\frac{1}{2}}. \tag{7.3.5}$$

(ii) (Asymptotic completeness) Let s be sufficiently large, and let $(\mathfrak{k}_\infty)_i{}^j$, $(\Upsilon_\infty)_i{}^j$ be $(1,1)$ tensors in $\mathbb{T}^D$, and ψ_∞, φ_∞ be functions in $\mathbb{T}^D$ with Sobolev regularity as in (7.3.4)–(7.3.5), and satisfying suitable constraints as well as the canonical asymptotic gauge condition (7.3.1). Then there exists a unique (classical) solution $(\mathfrak{n}_i{}^j, \mathfrak{k}_i{}^j, \phi, \psi)$ to the system (7.2.6)–(7.2.20) in $\mathcal{M}_{Kas}$, with regularity

$$\left(\mathfrak{n}_i{}^j, \mathfrak{k}_i{}^j\right) \in C^0((0,+\infty), H^{s+1} \times H^s) \cap C^1((0,+\infty), H^s \times H^{s-1}),$$

$$(\phi, \psi) \in C^0((0,+\infty), H^{s+1} \times H^s) \cap C^1((0,+\infty), H^s \times H^{s-1}),$$

as well as a change of gauge vector field $\xi^j \in H^{s-1}$, such that:

1. *The quantity $\mathfrak{n}_i{}^j$ satisfies the spatially harmonic condition (7.1.3) at $t = 1$.*
2. *If $\Upsilon_i{}^j$ and φ are defined as in (7.2.24)–(7.2.25), then after applying the transformation (7.2.21)–(7.2.35) associated to the vector field ξ^j, the resulting transformed quantities obey the asymptotics (7.3.2) and (7.3.3), strongly as $t \to 0$.*

(iii) (Scattering isomorphism) For $s \in \mathbb{R}$, define the following Hilbert spaces together with their norms:

$$\mathcal{H}^s_C = \left\{ \left(\mathfrak{n}_i{}^j, \mathfrak{k}_i{}^j, \phi, \psi\right) : \mathfrak{n}_i{}^j \in H^{s+1}, \mathfrak{k}_i{}^j \in H^s, \phi \in H^{s+1}, \psi \in H^s \right\}, \tag{7.3.6}$$

$$\left\| \left(\mathfrak{n}_i{}^j, \mathfrak{k}_i{}^j, \phi, \psi\right) \right\|^2_{\mathcal{H}^s_C} := \sum_{i,j=1}^D \left(\|\mathfrak{n}_i{}^j\|^2_{H^{s+1}} + \|\mathfrak{k}_i{}^j\|^2_{H^s} \right) + \|\phi\|^2_{H^{s+1}} + \|\psi\|^2_{H^s}, \tag{7.3.7}$$

$$\begin{aligned} \mathcal{H}^s_\infty = \Big\{ \Big(& \mathfrak{k}_i{}^j, \Upsilon_i{}^j, \psi, \varphi \Big) : \\ & \mathcal{T}_*^{-p_i+p_j} \mathfrak{k}_i{}^j \in H^{s+\frac{1}{2}}, \mathcal{T}_*^{-p_i+p_j} \left(\Upsilon_i{}^j - \int_{\mathcal{T}_*}^1 \mathring{g}_{ip}(s) \mathring{g}^{jq}(s) \frac{ds}{s} \cdot \mathfrak{k}_q{}^p \right) \in H^{s+\frac{1}{2}}, \\ & \psi \in H^{s+\frac{1}{2}}, \varphi + \log(\mathcal{T}_*)\psi \in H^{s+\frac{1}{2}} \Big\}, \end{aligned} \tag{7.3.8}$$

$$\begin{aligned} & \left\| \left(\mathfrak{k}_i{}^j, \Upsilon_i{}^j, \psi, \varphi\right) \right\|^2_{\mathcal{H}^s_\infty} \\ & := \sum_{i,j=1}^D \left(\left\| \mathcal{T}_*^{-p_i+p_j} \mathfrak{k}_i{}^j \right\|_{H^{s+\frac{1}{2}}} + \left\| \mathcal{T}_*^{-p_i+p_j} \left(\Upsilon_i{}^j - \int_{\mathcal{T}_*}^1 \mathring{g}_{ip}(s) \mathring{g}^{jq}(s) \frac{ds}{s} \cdot \mathfrak{k}_q{}^p \right) \right\|_{H^{s+\frac{1}{2}}} \right) \\ & \qquad + \|\psi\|^2_{H^{s+\frac{1}{2}}} + \|\varphi + \log(\mathcal{T}_*)\psi\|^2_{H^{s+\frac{1}{2}}}. \end{aligned} \tag{7.3.9}$$

Further let $\mathcal{H}^s_{C,c}$ be the subspace of $\mathcal{H}^s_C$ obeying suitable constraints as well as the spatially harmonic gauge condition (7.1.3), and similarly let $\mathcal{H}^s_{\infty,c}$ be the subspace of $\mathcal{H}^s_\infty$ obeying the appropriate asymptotic constraints as well as the canonical asymptotic gauge condition (7.3.1).

From (i), let the operator $\mathcal{S}_\downarrow$ map the Cauchy data $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C)$ to the asymptotic quantities $((\kappa_\infty)_i{}^j, (\Upsilon_\infty)_i{}^j, \psi_\infty, \varphi_\infty)$, where the asymptotic quantities are taken <u>after</u> modification by the gauge transformation associated to the vector field ξ^j in (i). Then this map can be extended to a bounded, injective homomorphism from $\mathcal{H}^s_{C,c}$ to $\mathcal{H}^s_{\infty,c}$.

Conversely, from (ii), let the operator $\mathcal{S}_\uparrow$ map the asymptotic data $((\kappa_\infty)_i{}^j, (\Upsilon_\infty)_i{}^j, \psi_\infty, \varphi_\infty)$ satisfying (7.3.1) *to $(\eta_i{}^j(1,\cdot), \kappa_i{}^j(1,\cdot), \phi(1,\cdot), \psi(1,\cdot))$, noting that $\eta_i{}^j(1,\cdot)$ is chosen to satisfy* (7.1.3) *after modification by the change of gauge vector field ξ^j. Then this map $\mathcal{S}_\uparrow$ may be extended to a bounded, injective homomorphism from $\mathcal{H}^s_{\infty,c}$ to $\mathcal{H}^s_{C,c}$.*

Finally, $\mathcal{H}^s_{\infty,c}$ is the image of $\mathcal{S}_\downarrow$ acting on $\mathcal{H}^s_{C,c}$, and we may view $\mathcal{S}_\downarrow : \mathcal{H}^s_{C,c} \to \mathcal{H}^s_{\infty,c}$ as a Hilbert space isomorphism with inverse $\mathcal{S}_\uparrow$.

Remark. The subcriticality condition (1.2.11) is necessary in order to conclude asymptotics of the type given in Theorem 7.3.1(i), due to heuristics of [95]. However, even if one violates (1.2.11), it may be possible to prove a version of Theorem 7.3.1 if we suppress the linearly unstable directions.

For instance, if one makes the additional assumption that for any i, j, k such that $-p_i+p_j+p_k \geq 1$,

$$\frac{\partial_k \kappa_{\underline{j}}{}^{\underline{i}}(t)}{p_{\underline{i}} - p_{\underline{j}}} - \frac{\partial_j \kappa_{\underline{k}}{}^{\underline{i}}(t)}{p_{\underline{i}} - p_{\underline{k}}} = 0 \tag{7.3.10}$$

for all times $t > 0$, then our proof shows that a version of Theorem 7.3.1 still holds. (7.3.10) is true in certain symmetry classes, for instance $\mathbb{T}^2$ symmetry or polarized $U(1)$ symmetry in $1+3$ dimensions. This complements known nonlinear stability results in these classes [2, 63]. In fact one would expect a version of Theorem 7.3.1 to still hold if (7.3.10) is only valid *asymptotically as* $t \to 0$.

To make this more precise, one would like to consider the case that for any p_i, p_j, p_k violating (1.6.11), the following limit holds for the linearized Weingarten map $\kappa_i{}^j$:

$$\lim_{t\to 0} \left(\frac{\partial_k \kappa_{\underline{j}}{}^{\underline{i}}(t)}{p_{\underline{i}} - p_{\underline{j}}} - \frac{\partial_j \kappa_{\underline{k}}{}^{\underline{i}}(t)}{p_{\underline{i}} - p_{\underline{k}}} \right) = 0. \tag{7.3.11}$$

This suggests that some version of our scattering result Theorem 7.3.1 should hold in cases where the above limit holds. We could interpret this hope as a codimensional[3] stability and scattering statement for linearized gravity around Kasner spacetimes violating (1.6.11).

In fact, one could hope to prove a version of Proposition 9.4.4 where one only assumes (7.3.11), which would involve a detailed analyis of the rate at which the expression in (7.3.11) tends to 0. We formulate this resolution of codimensional stability and scattering in the following conjecture.

Conjecture 7.3.2. *Let $(\mathcal{M}_{Kas}, g_{Kas}, \phi_{Kas})$ be a (generalized) non-degenerate Kasner spacetime <u>not necessarily</u> satisfying the subcriticality relation* (1.6.11). *Consider solutions $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi)$ to*

[3]Since (7.3.11) must hold at every spatial point $x \in \mathbb{T}^D$, this is an infinite codimension statement.

the linearized Einstein–scalar field equations around $(\mathcal{M}_{Kas}, g_{Kas}, \phi_{Kas})$*, written as the first-order elliptic-hyperbolic system* (7.2.6)–(7.2.12) *and satisfying suitable constraints. Then we have the following:*

(i) (Existence of the scattering operator) Let s *be sufficiently large. Suppose that the Cauchy data* $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C)$*, satisfying suitable constraints and the spatially harmonic gauge condition* (7.1.3)*, launches a solution to* (7.2.6)–(7.2.12) *such that for any Kasner exponents* p_i, p_j, p_k *violating* (1.6.11)*, the limit* (7.3.11) *holds.*

Then the conclusions of Theorem 7.3.1(i) still hold, and the limiting quantity $(\kappa_\infty)_i{}^j$ *is such that for any Kasner exponents* p_i, p_j, p_k *violating* (1.6.11)*,*

$$\frac{\partial_k (\kappa_\infty)_{\underline{j}}{}^{\underline{i}}(t)}{p_{\underline{i}} - p_{\underline{j}}} - \frac{\partial_j (\kappa_\infty)_{\underline{k}}{}^{\underline{i}}(t)}{p_{\underline{i}} - p_{\underline{k}}} = 0. \tag{7.3.12}$$

(ii) (Asymptotic completeness) Let s *be sufficiently large, and let and let* $(\kappa_\infty)_i{}^j$*,* $(\Upsilon_\infty)_i{}^j$ *be* $(1,1)$ *tensors in* $\mathbb{T}^D$*, and* ψ_∞*,* φ_∞ *be functions in* $\mathbb{T}^D$ *with Sobolev regularity as in* (7.3.4)–(7.3.5)*, and satisfying suitable constraints as well as the canonical asymptotic gauge condition* (7.3.1)*.*

Suppose furthermore that $(\kappa_\infty)_i{}^j$ *is such that for any Kasner exponents* p_i, p_j, p_k *violating* (1.6.11)*, the identity* (7.3.12) *holds. Then the conclusions of Theorem 7.3.1(ii) still hold.*

(iii) (Scattering isomorphism) For $s \in \mathbb{R}$*, recall the Hilbert spaces* $\mathcal{H}^s_{C,c}$ *and* $\mathcal{H}^s_{\infty,c}$ *from Theorem 7.3.1(iii). Further define* $\mathcal{H}^s_{C,AVTD} \subset \mathcal{H}^s_{C,c}$ *be the subspace such that the corresponding solution* $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi)$ *to the linearized Einstein–scalar field system satisfies* (7.3.11)[4] *for Kasner exponents* p_i, p_j, p_k *violating* (1.6.11)*. Also define the subspace* $\mathcal{H}^s_{\infty,AVTD} \subset \mathcal{H}^s_{\infty,C}$ *to be the subspace such that for Kasner exponents* p_i, p_j, p_k *violating* (1.6.11)*, the identity* (7.3.12) *holds.*

In light of (i) and (ii), define $\mathcal{S}_\downarrow$ *and* $\mathcal{S}_\uparrow$ *as in Theorem 7.3.1(iii). Then* $\mathcal{S}_\downarrow : \mathcal{H}^s_{C,AVTD} \to \mathcal{H}^s_{\infty,AVTD}$ *is a Hilbert space isomorphism with inverse* $\mathcal{S}_\uparrow$*.*

7.4 Applications to the nonlinear Einstein–scalar field system

We conclude this Chapter with comments on how our ideas could potentially be applied to the study of Kasner-like singularities for the fully nonlinear Einstein–scalar field system (Ein), (Scal) and (1.1.5). For instance, it would be desirable to formulate a scattering theory relating Cauchy data and asymptotic data in the context of Kasner-like spacetimes. In light of the nonlinear stability result [63], one could conjecture that there exists a scattering map between Cauchy data that is

[4]For s small, the convergence in (7.3.11) may hold in a distributional sense.

a small perturbation Kasner data at $t = 1$, and suitable asymptotic data close to the associated Kasner data at $t = 0$.

Many additional difficulties arise in addressing the nonlinear problem. In fact, some of these difficulties arise already in *linear* hyperbolic problems where the background Kasner-like spacetime is not exactly $(\mathcal{M}_{Kas}, g_{Kas})$, but a non-homogeneous perturbation thereof, since our proof is reliant on a Fourier decomposition on $\mathbb{T}^D$, and outside of the homogeneous setting one must what consider carefully what operators such as $\log \mathcal{T}_*$ and $\mathcal{T}_*^{-p_i+p_j}$ mean. It is possible we must consider microlocal operators with both physical space and Fourier space dependence.

We now comment upon several features of our proof to discuss how they could be adapted to the nonlinear problem. (Or in some cases, linear hyperbolic problems on inhomogeneous background spacetimes.)

Renormalized quantities in the nonlinear problem

In Theorem 7.3.1, the Cauchy data at $t = 1$ uses the variables $(\mathring{\eta}_i{}^j, \mathring{\kappa}_i{}^j, \mathring{\phi}, \mathring{\psi})$ while our asymptotic data at $t = 0$ uses instead the renormalized variables $(\mathring{\kappa}_i{}^j, \mathring{\Upsilon}_i{}^j, \mathring{\psi}, \mathring{\varphi})$, since it is these renormalized variables that remain bounded (and convergent) as $t \to 0$.

In the nonlinear problem, one must determine the correct renormalized variables at $t = 0$, or in other words the variables we prescribe as "scattering data". One proposal is the following:

$$K_i{}^j(t) = (\bar{g}^{-1}(t))^{jk}(tk_{ik}(t)), \quad M_{ij}(t) = \bar{g}_{jk}(t) \cdot \exp(2K(t) \ln t)_i{}^k,$$
$$\psi(t) = t\mathbf{N}\phi(t), \quad \varphi(t) = \phi(t) - \psi(t) \log t.$$

Here exp denotes matrix exponentiation. A consequence of [63, Theorem 6.1] that that the quantities $K_i{}^j$, ψ and φ are bounded and convergent as $t \to 0$ for perturbations of Kasner spacetimes obeying the subcriticality condition (1.6.11). Assuming the same is true for M_{ij} (which does not quite follow from [63]), a candidate nonlinear scattering map could be a map between Cauchy data $(\bar{g}_{ij}(1), k_{ij}(1), \phi(1), \psi(1))$ and the asymptotic quantities $((K_\infty)_i{}^j, (M_\infty)_{ij}, \psi_\infty, \varphi_\infty) = \lim_{t\to 0}(K_i{}^j(t), M_{ij}(t), \psi(t), \varphi(t))$.

Another subtlety regarding how we use renormalized quantities in our energy estimates is that our low-frequency energy estimates really apply to $(\mathring{\kappa}_i{}^j, \tilde{\mathring{\Upsilon}}_i{}^j, \mathring{\psi}, \tilde{\mathring{\varphi}})$, with the tildes signifying that we use $T = \mathcal{T}_*$ in the definitions (7.2.26). Thus in a background which is not exactly Kasner, one must also understand how to adapt the renormalized variables in a manner akin to choosing $T = \mathcal{T}_*$.

Gauge in the nonlinear problem

In contrast to [63] where the authors use Fermi-propagated frames, our scattering result Theorem 7.3.1 employs a transported coordinate ADM-type gauge. In reality, the proof does not really

use a transported coordinate gauge but instead a *frequency adapted* gauge, see already Section 9.5 and Theorem 9.5.1.

A quick way of describing this gauge is that it is CMCSH at high-frequency and essentially CMCTC at low-frequency, see Section 9.1 for why this is the case. Both the CMCSH gauge and the CMCTC gauge exist at the nonlinear level, and our method suggests it is reasonable to use a mixture of the two gauges in the nonlinear setting, so long as one makes sense of the frequency-dependent transition between the gauges.

In particular, this has the effect that the spatial Ricci tensor $\mathrm{Ric}_{ij}[g]$ can be viewed as an elliptic operator acting on $\bar{g}_{ij}$ at high-frequency (see (7.1.7)), but that there is no shift vector X^i at low-frequency and thus the equation for $\partial_t K_i{}^j = \partial_t((\bar{g}^{-1})^{jk}(tk_i{}^j))$ will remain suitably integrable towards $t = 0$.

Norms and energies in the nonlinear problem

In Theorem 7.3.1, the Hilbert space $\mathcal{H}^s_\infty$ associated to the scattering data $((\kappa_\infty)_i{}^j, (\Upsilon_\infty)_i{}^j, \psi_\infty, \varphi_\infty)$ also features the symbol $\mathcal{T}_*$ extensively; this arises due to the way that one compares the high-frequency energy and the low-frequency energy and the relevant metrics used to control the tensorial norms. The idea is that high-frequencies use the metric $\mathring{g}_{ij}(t)$, while low-frequencies use $\mathring{g}_{ij}(t_{\lambda *})$.

As a result, the symbol $\mathcal{T}_*^{-p_i+p_j}$ appears, especially when acting on off-diagonal components of tensors. The dependence of $\mathcal{T}_*^{-p_i+p_j}$ on the Kasner exponents means that if one were instead to consider perturbations of Kasner have spatially dependent exponents, one must be careful about the norms and energies used to control the asymptotic data at $t = 0$.

This issue is perhaps easier to resolve in the direction where one solves "forwards" from asymptotic data at $t = 0$, since in this context the perturbed Kasner exponents, and thus potentially the nonlinear analogues of $\mathcal{T}_*^{-p_i+p_j}$, are given. In the other direction, one has to understand how we can generate the correct such operators in evolution.

The nonlinear terms in the evolution equations

If one can resolve the previous issues (which already play a role in studying linearized gravity around perturbations of Kasner), it remains to control the 'semilinear' nonlinear terms in the evolution equations (7.1.4)–(7.1.9). For instance, there are the $\mathcal{N}(g^{-1}, \partial g)$ terms appearing in (7.1.7), and terms involving the shift vector X^i.

Given our time-dependent frequency decomposition, this would involve an extremely careful analysis of the different types of interaction (i.e. high-high, high-low, and low-low) both in terms of regularity and in terms of behaviour in t. Similarly, one would have to understand the interaction of commutator terms that arise, including commutation with the suggested microlocal operator $\mathcal{T}_*$.

Chapter 8

Scattering towards the singularity in Kasner: The wave equation

In Chapter 8, we prove Theorem 1.6.2, stated in Section 1.6.

8.1 Overview of the proof

We illustrate several ideas involved in the proofs of Theorem 1.6.2, many of which will also apply in our later proof of Theorem 7.3.1. For now, we restrict attention to the isotropic Kasner spacetime with $p_I = \frac{1}{D}$ for $I = 1, \dots, D$, and mention only briefly the differences in the anisotropic case. We also rescale time in such a way that (1.6.1)–(1.6.2) can be rewritten as the following first order hyperbolic system, where $\Delta = (g_{Euc}^{-1})^{ij}\partial_i\partial_j$ is the standard Laplacian on $\mathbb{T}^D$.

$$t\partial_t\phi = \psi, \qquad t\partial_t\psi = t^2\Delta\phi. \tag{8.1.1}$$

Our linear scattering results will always follow upon decomposing our PDEs in frequency space using Fourier series, and performing ODE analysis for each Fourier mode. Taking the Fourier decomposition of (8.1.1), we have for $|\lambda|^2 = \sum_{i=1}^D \lambda_i^2$ the system

$$t\partial_t\phi_\lambda = \psi_\lambda, \qquad t\partial_t\psi_\lambda = -|\lambda|^2\phi_\lambda. \tag{8.1.2}$$

Remarkably, (8.1.2) can be solved explicitly. This is because (8.1.2) is Bessel's equation (of order 0) written in first order form. Using [54] as a reference regarding Bessel's equation and the Bessel functions J_ν and Y_ν, solutions to (8.1.2) can be written as the following linear combinations of Bessel functions:

$$\phi_\lambda = c_J(\lambda)J_0(|\lambda|t) + c_Y(\lambda)Y_0(|\lambda|t), \qquad \psi_\lambda = -c_J(\lambda)(|\lambda|t)J_1(|\lambda|t) - c_Y(\lambda)(|\lambda|t)Y_1(|\lambda|t). \tag{8.1.3}$$

In particular, the asymptotics of Bessel functions in [54, Chapter 10.7] imply the following behaviour for ϕ_λ and ψ_λ in the regimes $|\lambda|t \to 0$ and $|\lambda|t \to +\infty$.

$$\phi_\lambda(t) = \begin{cases} \frac{2}{\pi} c_Y(\lambda) \log(|\lambda|t) + c_J(\lambda) + \cdots & \text{as } |\lambda|t \to 0, \\ \sqrt{\frac{2}{\pi|\lambda|t}} \left(c_J(\lambda) \cos\left(|\lambda|t - \frac{\pi}{4}\right) + c_Y(\lambda) \sin\left(|\lambda|t - \frac{\pi}{4}\right)\right) + \cdots & \text{as } |\lambda|t \to +\infty, \end{cases}$$

$$\psi_\lambda(t) = \begin{cases} \frac{2}{\pi} c_Y(\lambda) + \cdots & \text{as } |\lambda|t \to 0, \\ \sqrt{\frac{2|\lambda|t}{\pi}} \left(-c_J(\lambda) \sin\left(|\lambda|t - \frac{\pi}{4}\right) + c_Y(\lambda) \cos\left(|\lambda|t - \frac{\pi}{4}\right)\right) + \cdots & \text{as } |\lambda|t \to +\infty. \end{cases}$$

One may read off from these asymptotics that as $t \to 0$, we expect an expansion $\phi = \psi_\infty \log t + \varphi_\infty + \cdots$ as in Theorem 1.6.1, and moreover that the Fourier coefficients of ψ_∞ and φ_∞ can be read off from the coefficients $c_J(\lambda)$ and $c_Y(\lambda)$. Using the characterization of Sobolev spaces H^s in Fourier space, we may then translate these asymptotics into statements about Sobolev regularity:

1. Having $\phi(1, \cdot) \in H^{s+1}$ and $\psi(1, \cdot) \in H^s$ is equivalent to the sum $\sum_\lambda |\lambda|^{2s+1}(|c_J(\lambda)|^2 + |c_Y(\lambda)|^2)$ being finite. This is due to the powers of $|\lambda|t$ arising in the $|\lambda|t \to \infty$ regime.

2. At $t = 0$, we observe that $(\psi_\infty)_\lambda = \frac{2}{\pi} c_Y(\lambda)$ and $(\varphi_\infty)_\lambda = c_J(\lambda) - \frac{2}{\pi} c_Y(\lambda) \log|\lambda| = c_J(\lambda) - (\psi_\infty)_\lambda \log|\lambda|$. Therefore, the sum $\sum_\lambda |\lambda|^{2s+1}(|c_J(\lambda)|^2 + |c_Y(\lambda)|^2)$ being finite is equivalent to ψ_∞ and the modified $\varphi_\infty + (\log|\nabla|)\psi_\infty$ being in $H^{s+\frac{1}{2}}$.

Combining these two observations yields the scattering result of Theorem 1.6.2, at least after detailed analysis of the error terms in the Bessel asymptotics.

However, since we wish to generalize to anisotropic Kasner spacetimes, and also to the study of the linearized Einstein–scalar field system, it is desirable to find a method more robust than using the explicit Bessel functions. That is, we wish to capture Bessel-like asymptotics in both the $|\lambda|t \to +\infty$ and the $|\lambda|t \to 0$ regimes, which we denote the high-frequency and low-frequency regimes respectively.

It turns out we can achieve this via energy estimates for each Fourier mode $(\phi_\lambda, \psi_\lambda)$. In our two regimes, we shall require two different energy estimates, which we explain as follows:

1. In the high-frequency regime $|\lambda|t \gg 1$, we use a high-frequency energy which is given by

$$\mathcal{E}_{\lambda,high}(t) = \frac{\psi_\lambda^2}{|\lambda|t} + \frac{\psi_\lambda \phi_\lambda}{|\lambda|t} + \frac{\phi_\lambda^2}{2|\lambda|t} + |\lambda| t \phi_\lambda^2.$$

   Using (8.1.2), a computation involving crucial top order cancellations yields that for some $C > 0$,

$$|t\partial_t \mathcal{E}_{\lambda,high}(t)| \leq C(\lambda t)^{-2} \mathcal{E}_{\lambda,high}(t).$$

   Therefore, Grönwall's inequality implies that for a fixed Fourier mode $\lambda \in \mathbb{Z}^D \setminus \{0\}$, the high-frequency energy remains well controlled in the region $|\lambda|t \geq 1$.

2. In the low-frequency regime $|\lambda|t \ll 1$, we first define $\tilde{\varphi}_\lambda = \phi_\lambda - \psi_\lambda \log(|\lambda|t)$ so that $\phi_\lambda = \tilde{\varphi}(|\lambda|t)$ exactly when $|\lambda|t = 1$. We then use a low-frequency energy quantity which is simply

$$\mathcal{E}_{\lambda,low}(t) = \psi_\lambda^2 + \tilde{\varphi}_\lambda^2.$$

This is uniformly equivalent to the high-frequency energy $\mathcal{E}_{\lambda,high}(t)$ at the transitional time $t = |\lambda|^{-1}$, and one may show from the system (8.1.2) that

$$|t\partial_t \mathcal{E}_{\lambda,low}(t)| \leq C(\lambda t)^2(1 + |\log(\lambda t)|^2)\mathcal{E}_{\lambda,low}(t).$$

Therefore, Grönwall's inequality implies that for a fixed Fourier mode $\lambda \in \mathbb{Z}^D \setminus \{0\}$, the low-frequency energy remains well controlled in the region $0 < |\lambda|t \leq 1$.

Combining these two energy estimates and translating into a Sobolev regularity statement yields a proof of Theorem 1.6.2. Furthermore, this energy method is robust and may be applied more widely. In particular, it can be generalized to apply to the linearized Einstein equations in Chapter 9.

More imminently, it will also apply to *anisotropic* Kasner spacetimes. We highlight some of the additional difficulties in that case. For the scalar wave, the most prominent new issue is that one must tailor the high-frequency energy in a way that sees the anisotropy. This is done by introducing frequency-dependent coefficients $\zeta = \zeta_\lambda(t)$ into the definition of $\mathcal{E}_{\lambda,high}$, see already Lemma 8.2.1 and Definition 8.3.1.

8.2 Fourier decomposition and preliminary lemmas

We now begin the full proof of Theorem 1.6.2, concerning scattering between Cauchy data and asymptotic data for solutions of the scalar wave equation in a non-degenerate Kasner spacetime $(\mathcal{M}_{Kas}, g_{Kas})$. As outlined in Section 8.1, the proof relies on a detailed ODE analysis for each Fourier mode $(\phi_\lambda, \psi_\lambda)$, $\lambda \in \mathbb{Z}^d$.

Projecting the first order system (1.6.1)–(1.6.2) onto each Fourier mode $\lambda \in \mathbb{Z}^D$, we find the following first order system of ODEs for the Fourier coefficients ϕ_λ and ψ_λ.

$$t\partial_t \phi_\lambda = \psi_\lambda, \tag{8.2.1}$$

$$t\partial_t \psi_\lambda = -\tau^2 \phi_\lambda, . \tag{8.2.2}$$

Here the expression $\tau^2 = \tau_\lambda^2(t)$ is the following function[1] of $\lambda \in \mathbb{Z}^D$ and $t > 0$.

$$\tau^2 = \sum_{i=1}^{D} t^{2-2p_i} \lambda_i^2. \tag{8.2.3}$$

[1]When clear from context, we often denote τ without its subscript τ_λ, primarily to shorten the exposition, and also to make it clear that τ_λ is not a Fourier coefficient. Similarly for ζ and ζ_λ.

Our aim is to find energy estimates for ϕ_λ, ψ_λ and $\varphi_\lambda = \phi_\lambda - \psi_\lambda \log t$ which are uniform in frequency $\lambda \in \mathbb{Z}^D$. The energies appearing in such energy estimates will often feature τ and derivatives thereof, and thus much of our analysis boils down to understanding properties of τ. We often appeal to the following two lemmas:

Lemma 8.2.1. *For $\lambda \in \mathbb{Z}^D \setminus \{0\}$, let $\zeta = \zeta_\lambda(t)$ be the following logarithmic ∂_t-derivative of τ:*

$$\zeta = \zeta_\lambda(t) \coloneqq \left(t \frac{d}{dt} \log \tau \right)^{-1} = \frac{\tau}{t} \cdot \left(\frac{d\tau}{dt} \right). \tag{8.2.4}$$

Then the following upper and lower bounds for $\zeta = \zeta_\lambda(t)$ hold, uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $t > 0$:

$$\min_{1 \le i \le D} \frac{1}{1-p_i} \le \zeta_\lambda(t) \le \max_{1 \le i \le D} \frac{1}{1-p_i}. \tag{8.2.5}$$

Furthermore, $\zeta_\lambda(t)$ is nonincreasing in t, and one has the integral estimate

$$\int_0^1 |\zeta_\lambda'(s)| \, ds \le \max_{1 \le i \le D} \frac{1}{1-p_i} - \min_{1 \le i \le D} \frac{1}{1-p_i}. \tag{8.2.6}$$

Proof. We differentiate the expression (8.2.3) for τ with respect to t. This yields

$$2t\tau \cdot \frac{d\tau}{dt} = \sum_{i=1}^D (2-2p_i) t^{2-2p_i} \lambda_i^2.$$

We then use the definition (8.2.4) and rearrange to find

$$\zeta(t) = \frac{2\tau^2}{\sum_{i=1}^D (2-2p_i) t^{2-2p_i} \lambda_i^2} = \frac{2 \sum_{i=1}^D t^{2-2p_i} \lambda_i^2}{\sum_{i=1}^D (2-2p_i) t^{2-2p_i} \lambda_i^2}.$$

For $\lambda \in \mathbb{Z}^D \setminus \{0\}$, at least one summand appearing in both the numerator and denominator is positive, and from this the bounds (8.2.5) are immediate.

We now show that $\zeta(t)$ is nonincreasing. Using the above, we compute the following logarithmic derivative

$$\begin{aligned} t \frac{d}{dt} \log(\zeta(t)) &= \frac{\sum_{i=1}^D (2-2p_i) t^{2-2p_i} \lambda_i^2}{\sum_{i=1}^D t^{2-2p_i} \lambda_i^2} - \frac{\sum_{i=1}^D (2-2p_i)^2 t^{2-2p_i} \lambda_i^2}{\sum_{i=1}^D (2-2p_i) t^{2-2p_i} \lambda_i^2} \\ &= \frac{\left(\sum_{i=1}^D (2-2p_i) t^{2-2p_i} \lambda_i^2 \right)^2 - \left(\sum_{i=1}^D t^{2-2p_i} \lambda_i^2 \right) \left(\sum_{i=1}^D (2-2p_i)^2 t^{2-2p_i} \lambda_i^2 \right)}{\left(\sum_{i=1}^D t^{2-2p_i} \lambda_i^2 \right) \left(\sum_{i=1}^D (2-2p_i) t^{2-2p_i} \lambda_i^2 \right)}. \end{aligned}$$

The result now follows from the Cauchy-Schwarz inequality in $\mathbb{R}^D$, i.e. that

$$\left(\sum_{i=1}^D x_i^2 \right) \left(\sum_{i=1}^D y_i \right)^2 \ge \left(\sum_{i=1}^D x_i y_i \right)^2,$$

with $x_i = t^{1-p_i} \lambda_i$ and $y_i = (1-p_i) t^{1-p_i} \lambda_i$. This shows that the numerator above is nonpositive, and thus that $\zeta(t)$ is nonincreasing. This monotonicity and the upper and lower bounds (8.2.5) for $\zeta(t)$ immediately yields (8.2.6). $\square$

We use Lemma 8.2.1, particularly (8.2.5), to estimate integrals involving powers of τ.

Lemma 8.2.2. *As in Definition 1.6.1, for each $\lambda \in \mathbb{Z}^D \setminus \{0\}$, let $t_{\lambda *} \in (0,1]$ be such that $\tau_\lambda^2(t_{\lambda *}) = 1$. Then for $\alpha > 0$, we have the following integral estimate corresponding to $\tau \geq 1$:*

$$\int_{t_{\lambda *}}^1 \tau^{-\alpha}(s)\, \frac{ds}{s} \lesssim \alpha^{-1}. \tag{8.2.7}$$

One also has the following integral estimate corresponding to $\tau \leq 1$. For $0 < t \leq t_{\lambda_*}$,

$$\int_0^t \tau^2(s) \left(1 + \log\left(\frac{t_{\lambda *}}{s}\right)^2\right) \frac{ds}{s} \lesssim \tau^2(t) \left(1 + \log\left(\frac{t_{\lambda *}}{t}\right)^2\right) \lesssim 1. \tag{8.2.8}$$

Proof. The key to these integral estimates is that we may perform a change of variables $t \mapsto \tau$, and that

$$\frac{dt}{t} = \zeta \frac{d\tau}{\tau}.$$

For instance, for (8.2.7), upon applying this change of variables the upper bound in (8.2.5) yields

$$\int_{t_{\lambda *}}^1 \tau^{-\alpha}(s)\, \frac{ds}{s} \leq \int_1^{+\infty} \tau^{-\alpha}\, \zeta \frac{d\tau}{\tau} \leq \alpha^{-1} \cdot \max_{1 \leq i \leq D} \frac{1}{1 - p_i}.$$

For (8.2.8) we start by expressing $\log\left(\frac{t_{\lambda *}}{t}\right)$ in terms of τ. Using the same change of variables, we find

$$\log\left(\frac{t_{\lambda *}}{t}\right) = \int_t^{t_{\lambda *}} \frac{dt}{t} = \int_{\tau(t)}^1 \zeta \frac{d\tau}{\tau}.$$

Using again the upper and lower bounds (8.2.5) on $\zeta(t)$, we thus have that for $0 < t \leq t_{\lambda *}$,

$$\log\left(\frac{t_{\lambda *}}{t}\right) \asymp \log \tau^{-1}. \tag{8.2.9}$$

Therefore, the integral in (8.2.8) is estimated as follows.

$$\begin{aligned} \int_0^t \tau^2(s) \left(1 + \log\left(\frac{t_{\lambda *}}{s}\right)^2\right) \frac{ds}{s} &\lesssim \int_0^{\tau(t)} \tau^2 \left(1 + (\log \tau^{-1})^2\right) \frac{d\tau}{\tau} \\ &\lesssim \tau^2 \left(1 + (\log \tau^{-1})^2\right). \end{aligned}$$

The estimate (8.2.8) then follows from using (8.2.9) again and that $\tau^2 \left(1 + (\log \tau^{-1})^2\right) \lesssim 1$ for $\tau \leq 1$. □

8.3 The high-frequency energy estimate

We commence the study of the first order ODE system (8.2.1)–(8.2.2) for $\lambda \in \mathbb{Z}^D \setminus \{0\}$. In the high-frequency regime $\tau = \tau_\lambda(t) \geq 1$, we make use of the following energy quantity.

Definition 8.3.1. The *high-frequency energy quantity* $\mathcal{E}_{\lambda,high}(t)$, used for $\lambda \in \mathbb{Z}^D \backslash \{0\}$ and $\tau_\lambda(t) \geq 1$, is given by

$$\mathcal{E}_{\lambda,high}(t) = \zeta^2 \frac{\psi_\lambda^2}{\tau} + \zeta \frac{\psi_\lambda \phi_\lambda}{\tau} + \frac{\phi_\lambda^2}{2\tau} + \zeta^2 \tau \phi_\lambda^2. \tag{8.3.1}$$

The bounds for $\zeta = \zeta_\lambda(t)$ obtained in Lemma 8.2.1 immediately yield the following coercivity of the energy, which will be especially useful to compare $\mathcal{E}_{\lambda,high}(\cdot)$ to Sobolev norms at Σ_1.

Corollary 8.3.1. *One has that uniformly in frequency $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $t \geq t_{\lambda *}$,*

$$\mathcal{E}_{\lambda,high}(t) \asymp \frac{\psi_\lambda^2(t)}{\langle \tau(t) \rangle} + \langle \tau(t) \rangle \phi_\lambda^2(t).$$

In particular, at $t = 1$ one has $\mathcal{E}_{\lambda,high}(1) \asymp \langle \lambda \rangle^{-1} \psi_\lambda^2(1) + \langle \lambda \rangle \, \phi_\lambda^2(1)$.

The main result of the current section is the following high-frequency energy estimate.

Proposition 8.3.2. *Let $(\phi_\lambda(t), \psi_\lambda(t))$ be a solution to (8.2.1)–(8.2.2). Then the ∂_t-derivative of the high-frequency energy quantity obeys the following bound, uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $t \geq t_{\lambda *}$:*

$$\left| t \frac{d}{dt} \mathcal{E}_{\lambda,high}(t) \right| \lesssim \left(t|\zeta'(t)| + \frac{1}{\tau^2} \right) \mathcal{E}_{\lambda,high}(t). \tag{8.3.2}$$

There exists some $C_{high} > 0$, depending only on the background Kasner spacetime, such that for $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $t_{\lambda *} \leq t \leq 1$, one has

$$C_{high}^{-1} \mathcal{E}_{\lambda,high}(t) \leq \mathcal{E}_{\lambda,high}(1) \leq C_{high} \, \mathcal{E}_{\lambda,high}(t). \tag{8.3.3}$$

Proof. We take the $t \frac{d}{dt}$ derivative of each expression appearing in (8.3.1), using the equations (8.2.1)–(8.2.2) as well as $t \frac{d}{dt} \tau = \zeta^{-1} \tau$. Isolating the terms where the derivative hits $\zeta(t)$, one has

$$\begin{aligned}
t \frac{d}{dt} \mathcal{E}_{\lambda,high} = {} & \zeta^2 \frac{2\psi_\lambda(-\tau^2 \phi_\lambda)}{\tau} - \zeta \frac{\psi_\lambda^2}{\tau} && \text{(8.3.4a)} \\
& + \zeta \frac{\phi_\lambda(-\tau^2 \phi_\lambda)}{\tau} + \zeta \frac{\psi_\lambda^2}{\tau} - \frac{\psi_\lambda \phi_\lambda}{\tau} && \text{(8.3.4b)} \\
& + \frac{\phi_\lambda \psi_\lambda}{\tau} - \zeta^{-1} \frac{\phi_\lambda^2}{2\tau} && \text{(8.3.4c)} \\
& + 2\zeta^2 \tau \phi_\lambda \psi_\lambda + \zeta \tau \phi_\lambda^2 && \text{(8.3.4d)} \\
& + t\zeta'(t) \left(2\zeta \frac{\psi_\lambda^2}{\tau} + \frac{\psi_\lambda \phi_\lambda}{\tau} + 2\zeta \tau \phi_\lambda^2 \right). && \text{(8.3.4e)}
\end{aligned}$$

We shall exploit crucial cancellations in the above expression. We see that

- The first term in (8.3.4a) cancels the first term in (8.3.4d).
- The second term in (8.3.4a) cancels the second term in (8.3.4b).

- The first term in (8.3.4b) cancels the second term in (8.3.4d).
- The third term in (8.3.4b) cancels the first term in (8.3.4c).

Upon applying these cancellations, we are left with the following:

$$t\frac{d}{dt}\mathcal{E}_{\lambda,high} = -\zeta^{-1}\frac{\phi_\lambda^2}{2\tau} + (8.3.4e).$$

The proof of the derivative bound (8.3.2) then follows straightforwardly, after applying Corollary 8.3.1 and the upper and lower bounds on $\zeta(t)$ from (8.2.5).

We shall deduce (8.3.3) by applying Grönwall argument to (8.3.2). For any $t_{\lambda_*} \leq t_1 \leq t_2 \leq 1$ and C the implied constant in (8.3.2), Grönwall's inequality yields that

$$\mathcal{E}_{\lambda,high}(t_2) \leq \exp\left(C\int_{t_1}^{t_2} |\zeta'(t)|\,dt + C\int_{t_1}^{t_2} \tau^{-2}(t)\,\frac{dt}{t}\right)\mathcal{E}_{\lambda,high}(t_1).$$

In fact, by applying Grönwall in the opposite direction, the reverse inequality also holds:

$$\mathcal{E}_{\lambda,high}(t_1) \leq \exp\left(C\int_{t_1}^{t_2} |\zeta'(t)|\,dt + C\int_{t_1}^{t_2} \tau^{-2}(t)\,\frac{dt}{t}\right)\mathcal{E}_{\lambda,high}(t_2).$$

Now, from Lemmas 8.2.1 and 8.2.2, the exponent

$$C\int_{t_1}^{t_2} |\zeta'(t)|\,dt + C\int_{t_1}^{t_2} \tau^{-2}(t)\,\frac{dt}{t} \leq C\int_{t_{\lambda*}}^{1} |\zeta'(t)|\,dt + C\int_{t_{\lambda*}}^{1} \tau^{-2}(t)\,\frac{dt}{t}$$

is bounded, uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$. The energy estimate (8.3.3) then follows upon taking $t_2 = 1$. □

8.4 The low-frequency energy estimates

In the low-frequency regime where $\tau = \tau_\lambda(t) \leq 1$, we need to use a different energy estimate. Since $\phi_\lambda(t)$ generally blows up as $t \to 0$, we first introduce a renormalized scalar field:

$$\tilde{\varphi}_\lambda(t) = \phi_\lambda(t) - \psi_\lambda(t)\log\left(\frac{t}{t_{\lambda*}}\right). \tag{8.4.1}$$

Note that $\tilde{\varphi}_\lambda(t)$ is chosen to coincide with $\phi_\lambda(t)$ exactly at $t = t_{\lambda*}$, and differs from the (Fourier coefficients) of the quantity $\varphi(t)$ defined in (1.6.6) by a frequency dependent object. Nevertheless, it will be the correct renormalized quantity for our energy estimates. We use the following low-frequency energy quantity.

Definition 8.4.1. The *low-frequency energy quantity* $\mathcal{E}_{\lambda,low}(t)$, used for $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $0 < \tau_\lambda(t) \leq 1$, is given by

$$\mathcal{E}_{\lambda,low}(t) = \psi_\lambda^2 + \tilde{\varphi}_\lambda^2. \tag{8.4.2}$$

Using Corollary 8.3.1, the low-frequency energy quantity is such that $\mathcal{E}_{\lambda,low}(t_{\lambda*}) \asymp \mathcal{E}_{\lambda,high}(t_{\lambda*})$. The main result of this section is the following low-frequency energy estimate.

Proposition 8.4.1. *Let $(\phi_\lambda(t), \psi_\lambda(t))$ be a solution to (8.2.1)–(8.2.2). Then the ∂_t-derivative of the low-frequency energy quantity obeys the following bound, uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $0 < t \leq t_{\lambda*}$:*

$$\left| t \frac{d}{dt} \mathcal{E}_{\lambda,low}(t) \right| \lesssim \tau^2 \left(1 + \log\left(\frac{t_{\lambda*}}{t} \right)^2 \right) \mathcal{E}_{\lambda,low}(t). \tag{8.4.3}$$

There exists some $C_{low} > 0$, depending only on the background Kasner spacetime, such that for $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $0 < t \leq t_{\lambda*}$, one has

$$C_{low}^{-1} \mathcal{E}_{\lambda,low}(t) \leq \mathcal{E}_{\lambda,low}(t_{\lambda*}) \leq C_{low}\, \mathcal{E}_{\lambda,low}(t). \tag{8.4.4}$$

Proof. In the low-frequency regime, our fundamental quantities are now $\psi_\lambda(t)$ and $\tilde{\varphi}_\lambda(t)$. We start by deriving appropriate evolution equations for these quantities. From (8.2.1)–(8.2.2) and (8.4.2) above we have

$$t \partial_t \psi_\lambda = -\tau^2 \tilde{\varphi}_\lambda + \tau^2 \log\left(\frac{t_{\lambda*}}{t} \right) \psi_\lambda, \tag{8.4.5}$$

$$t \partial_t \tilde{\varphi}_\lambda = -\tau^2 \log\left(\frac{t_{\lambda*}}{t} \right) \tilde{\varphi}_\lambda + \tau^2 \log\left(\frac{t_{\lambda*}}{t} \right)^2 \psi_\lambda. \tag{8.4.6}$$

From these equations, we easily compute

$$t \frac{d}{dt} \mathcal{E}_{\lambda,low}(t) = 2\tau^2 \log\left(\frac{t_{\lambda*}}{t} \right) \psi_\lambda^2 + 2\tau^2 \left(\log\left(\frac{t_{\lambda*}}{t}^2 \right) - 1 \right) \psi_\lambda \tilde{\varphi}_\lambda - 2\tau^2 \log\left(\frac{t_{\lambda*}}{t} \right) \tilde{\varphi}_\lambda^2.$$

Therefore the derivative bound (8.4.3) is immediate.

The energy estimate (8.4.4) follows as in the proof of Proposition 8.3.2. Letting C be the implied constant in (8.4.3), Grönwall's inequality tells us that for any $0 < t_1 \leq t_2 \leq t_{\lambda*}$, we have

$$\mathcal{E}_{\lambda,low}(t_2) \leq \exp\left(C \int_{t_1}^{t_2} \tau^2(t) \left(1 + \log\left(\frac{t_{\lambda*}}{t} \right)^2 \right) \frac{dt}{t} \right) \mathcal{E}_{\lambda,low}(t_1),$$

$$\mathcal{E}_{\lambda,low}(t_1) \leq \exp\left(C \int_{t_1}^{t_2} \tau^2(t) \left(1 + \log\left(\frac{t_{\lambda*}}{t} \right)^2 \right) \frac{dt}{t} \right) \mathcal{E}_{\lambda,low}(t_2).$$

By Lemma 8.2.2, the integral in the exponent is bounded, uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $0 < t_1 \leq t_2 \leq t_{\lambda*}$. This completes the proof of the energy estimate. □

In order to justify the existence of the scattering states $(\psi_\infty, \tilde{\varphi}_\infty)$, one must also show that the functions $\psi_\lambda(t)$ and $\tilde{\varphi}_\lambda(t)$ attain a limit as $t \to 0$. Furthermore, for the purpose of asymptotic completeness we need to show that we can prescribe the limits $(\psi_\infty)_\lambda$ and $(\tilde{\varphi}_\infty)_\lambda$ to launch a solution to the ODEs. This is achieved in the following proposition.

Proposition 8.4.2. *For some fixed $\lambda \in \mathbb{Z}^D \backslash \{0\}$, and $C_{low} > 0$ as in Proposition 8.4.1, the following hold.*

(i) Let $(\phi_\lambda(t), \psi_\lambda(t))$ be a solution to the system (8.2.1)–(8.2.2) for $0 < t \leq t_{\lambda *}$. Then there exist real numbers $(\psi_\infty)_\lambda$ and $(\tilde{\varphi}_\infty)_\lambda$ such that $\psi_\lambda(t)$ and $\tilde{\varphi}_\lambda(t)$ attain these as limits as $t \to 0$, at the following rate

$$|\psi_\lambda(t) - (\psi_\infty)_\lambda| + |\tilde{\varphi}_\lambda(t) - (\tilde{\varphi}_\infty)_\lambda| \lesssim \tau^2 \left(1 + \log\left(\frac{t_{\lambda *}}{t}\right)^2\right) \mathcal{E}^{1/2}_{\lambda,low}(t_{\lambda *}) \lesssim \tau \, \mathcal{E}^{1/2}_{\lambda,low}(t_{\lambda *}). \quad (8.4.7)$$

Furthermore, one has that $(\psi_\infty)^2_\lambda + (\tilde{\varphi}_\infty)^2_\lambda \leq C_{low} \, \mathcal{E}_{\lambda,low}(t_{\lambda *})$.

(ii) Conversely, let $(\psi_\infty)_\lambda$ and $(\tilde{\varphi}_\infty)_\lambda$ be any real numbers. Then there exists a unique solution to $(\phi_\lambda(t), \psi_\lambda(t))$ to the system (8.2.1)–(8.2.2) for $0 < t \leq t_{\lambda *}$ such that $\psi_\lambda(t)$ and $\tilde{\varphi}_\lambda(t)$ obey the asymptotics (8.4.7).

For this solution, one has that $\mathcal{E}_{\lambda,low}(t_{\lambda *}) \leq C_{low} \left((\psi_\infty)^2_\lambda + (\tilde{\varphi}_\infty)^2_\lambda\right)$.

Proof. We first prove (i). By Proposition 8.4.1, we have that the low-frequency energy $\mathcal{E}_{\lambda,low}(t)$ is uniformly bounded for $0 < t \leq t_{\lambda *}$ by its value at $t = t_{\lambda *}$. Therefore, one sees from the equations (8.4.5)–(8.4.6) that the following derivative bounds hold:

$$\left|t \frac{d}{dt} \psi_\lambda(t)\right| + \left|t \frac{d}{dt} \tilde{\varphi}_\lambda(t)\right| \lesssim \tau^2 \left(1 + \log\left(\frac{t_{\lambda *}}{t}\right)^2\right) \mathcal{E}^{1/2}_{\lambda,low}(t_{\lambda *}).$$

This derivative estimate, along with Lemma 8.2.2, shows that $\partial_t \psi_\lambda(t)$ and $\partial_t \tilde{\varphi}_\lambda(t)$ are integrable towards $t = 0$. Therefore the limits $(\psi_\infty)_\lambda$ and $(\tilde{\varphi}_\infty)_\lambda$ exist, and moreover satisfy the bounds (8.4.7). The final estimate $(\psi_\infty)^2_\lambda + (\tilde{\varphi}_\infty)^2_\lambda \leq C_{low} \, \mathcal{E}_{\lambda,low}(t_{\lambda *})$ follows from the upper bound in (8.4.4) and taking the limit $t \to 0$.

For the converse statement (ii), we require a local existence statement for the ODE system (8.4.5)–(8.4.6) given initial data at $t = 0$. Since these ODEs are singular at $t = 0$, we cannot apply the usual Picard–Lindelöf theorem. Nonetheless, local existence follows, using the Fuchsian theory of first-order ODEs in Lemma E.0.1.

We apply Lemma E.0.1(ii) with $\mathbf{V} = (\psi_\lambda, \tilde{\varphi}_\lambda)$ and $A = 0$. The derivative bounds above imply that we can apply this lemma, with initial data $\mathbf{v} = ((\psi_\infty)_\lambda, (\tilde{\varphi}_\infty)_\lambda)$, and thereby generate a local solution $(\psi_\lambda(t), \tilde{\varphi}_\lambda(t))$ to these ODEs such that $(\psi_\lambda(t), \tilde{\varphi}_\lambda(t)) \to ((\psi_\infty)_\lambda, (\tilde{\varphi}_\infty)_\lambda)$ as $t \to 0$, and that this solution is the unique solution attaining the desired limit.

Since (8.4.5)–(8.4.6) are regular for $t > 0$ are linear in $(\psi_\lambda, \tilde{\varphi}_\lambda)$, we may extend the solution to $0 < t \leq t_{\lambda *}$, indeed one may extend to the whole spacetime $t > 0$. The asymptotics (8.4.7) follow as in (i), and the final estimate $\mathcal{E}_{\lambda,low}(t_{\lambda *}) \leq C_{low} \left((\psi_\infty)^2_\lambda + (\tilde{\varphi}_\infty)^2_\lambda\right)$ follows from the lower bound in (8.4.4) □

8.5 Construction of the scattering isomorphism

We now complete the proof of Theorem 1.6.2 using our energy estimates in Sections 8.3 and 8.4. We shall first assume smoothness for the solution $(\psi(t), \phi(t))$ and derive a priori estimates relating Sobolev spaces defined at Σ_1 and Sobolev spaces defined at the $t = 0$ singularity. From these a priori estimates, the density of smooth functions in $H^s(\mathbb{T}^D)$ may be used recover the full scattering theory.

There are two directions for our scattering theory. First we show that in Section 8.5.1 that smooth Cauchy data (ϕ_C, ψ_C) at Σ_1 gives rise to a smooth solution to (1.6.1)–(1.6.2) for $t > 0$, achieving suitable (smooth) asymptotics at $t = 0$ with Sobolev bounds, and conversely that one may prescribe smooth asymptotic data $(\phi_\infty, \varphi_\infty)$ at $t = 0$ which launches a smooth solution to (1.6.1)–(1.6.2) in the same region, and whose restriction to Σ_1 is smooth and satisfies good Sobolev bounds.

Before proceeding, note that our results in Sections 8.3 and 8.4 do not apply to the zero frequency $\lambda = 0$, since $\tau(t)$ defined in (8.2.3) vanishes. However, the projection of (1.6.1)–(1.6.2) on zero frequency is trivial, as evident from the following lemma.

Lemma 8.5.1. *Let $(\phi(t), \psi(t))$ be a solution of the equations (1.6.1)–(1.6.2). Then defining $\varphi(t)$ as in (1.6.6), the 0-frequency Fourier coefficients $\psi_0(t)$ and $\varphi_0(t)$ are constant, in particular for all $t > 0$:*

$$\psi_0(t) = \psi_0(1), \qquad \varphi_0(t) = \varphi_0(1) = \phi_0(1). \tag{8.5.1}$$

Proof. This is immediate from (1.6.1) and (1.6.2) in the case that $\lambda = 0$, since in this case $\tau_0(t) = 0$ also. □

8.5.1 From Cauchy data to asymptotic data

Proposition 8.5.2. *Let ϕ_C and ψ_C be smooth functions on $\mathbb{T}^D$. Then there exists a unique smooth solution $(\phi(t), \psi(t))$ to the system (1.6.1)–(1.6.2) achieving the initial data in the sense*

$$\phi(1, \cdot) = \phi_C(\cdot), \qquad \psi(1, \cdot) = \psi_C(\cdot).$$

Moreover, there exist smooth functions $\psi_\infty, \varphi_\infty \in C^\infty(\mathbb{T}^D)$ such that for any $s \in \mathbb{R}$ and $\varphi(t)$ as in (1.6.6), the following (strong) convergence holds:

$$\psi(t) \to \psi_\infty \text{ in } H^s(\mathbb{T}^D), \quad \varphi(t) \to \varphi_\infty \text{ in } H^s(\mathbb{T}^D) \quad \text{as } t \to 0. \tag{8.5.2}$$

Finally, for the symbol $\log(\mathcal{T}_*)$ defined in Definition 1.6.1, one has

$$\|\psi_\infty\|^2_{H^{s+\frac{1}{2}}} + \|\varphi_\infty + (\log \mathcal{T}_*)\psi_\infty\|^2_{H^{s+\frac{1}{2}}} \lesssim \|\phi_C\|^2_{H^{s+1}} + \|\psi_C\|^2_{H^s}. \tag{8.5.3}$$

Proof. The existence and uniqueness of a smooth solution $(\phi(t), \psi(t))$ for $t > 0$, which attains the initial data $\phi(1, \cdot) = \phi_C(\cdot), \psi(1, \cdot) = \psi_C(\cdot)$, follows from standard theory for first-order linear hyperbolic systems. We then take the Fourier decomposition.

Since ϕ_C and ψ_C are smooth, it follows from the definition of the high-frequency energy quantity (8.3.1) and Corollary 8.3.1 that the energy quantities and Sobolev norms can be related in the following way:

$$\phi_0^2 + \psi_0^2 + \sum_{\lambda \in \mathbb{Z}^D \setminus \{0\}} \langle \lambda \rangle^{2s+1} \mathcal{E}_{\lambda, high}(1) \lesssim \sum_{\lambda \in \mathbb{Z}^D} \big(\langle \lambda \rangle^{2s+2} \phi_\lambda^2(t) + \langle \lambda \rangle^{2s} \psi_\lambda^2(t) \big) = \|\phi_C\|_{H^{s+1}}^2 + \|\psi_C\|_{H^s}^2. \quad (8.5.4)$$

The next step is to produce the functions ψ_∞ and φ_∞. For this, we consider the Fourier coefficients $\phi_\lambda(t)$ and $\psi_\lambda(t)$, which evolve according to (8.2.1)–(8.2.2) when $\lambda \neq 0$. By Proposition 8.3.2 and Corollary 8.3.1, we know that for $t_{\lambda *} \leq t < 1$, one has

$$\frac{\psi_\lambda^2(t)}{\tau_\lambda(t)} + \tau_\lambda(t) \phi_\lambda^2(t) \lesssim \mathcal{E}_{\lambda, high}(1). \quad (8.5.5)$$

Combining this with Proposition 8.4.1 and the fact that $\mathcal{E}_{\lambda, high}(t_{\lambda *})) \asymp \mathcal{E}_{\lambda, low}(t_{\lambda *})$, we find that for $0 < t < t_{\lambda *}$, one has

$$\psi_\lambda^2(t) + \tilde{\varphi}_\lambda^2(t) \lesssim \mathcal{E}_{\lambda, high}(1). \quad (8.5.6)$$

Here $\tilde{\varphi}_\lambda(t)$ is defined in (8.4.2), and is related to Fourier coefficients of $\varphi(t)$ via $\tilde{\varphi}_\lambda(t) = \varphi_\lambda(t) + \psi_\lambda(t) \log t_{\lambda *}$.

Recall now that Proposition 8.4.2(i) asserts the existence of limiting coefficients $(\psi_\infty)_\lambda$ and $(\tilde{\varphi}_\infty)_\lambda$ such that $\psi_\lambda(t) \to (\psi_\infty)_\lambda$ and $\varphi_\lambda(t) + \psi_\lambda(t) \log t_{\lambda *} = \tilde{\varphi}_\lambda(t) \to (\tilde{\varphi}_\infty)_\lambda$ as $t \to 0$. Recalling also Lemma 8.5.1, we thus define the functions ψ_∞ and φ_∞ as follows:

$$\begin{aligned} \psi_\infty(x) &= \psi_0(1) + \sum_{\lambda \in \mathbb{Z}^D \setminus \{0\}} (\psi_\infty)_\lambda \, e^{i \lambda \cdot x}, \\ \varphi_\infty(x) &= \varphi_0(1) + \sum_{\lambda \in \mathbb{Z}^D \setminus \{0\}} \big((\tilde{\varphi}_\infty)_\lambda - (\psi_\infty)_\lambda \log t_{\lambda *} \big) \, e^{i \lambda \cdot x}. \end{aligned} \quad (8.5.7)$$

Note that by definition, the function $\tilde{\varphi}_\infty(x) = \varphi_\infty(x) + (\log \mathcal{T}_*) \psi_\infty(x)$ has Fourier decomposition

$$\tilde{\varphi}_\infty(x) = \varphi_0(1) + \sum_{\lambda \in \mathbb{Z}^D \setminus \{0\}} (\tilde{\varphi}_\infty)_\lambda \, e^{i \lambda \cdot x}. \quad (8.5.8)$$

We next prove the regularity of ψ_∞ and φ_∞. For any $s \in \mathbb{R}$, letting $t \to 0$ in (8.5.6), multiplying

by $\langle\lambda\rangle^{2s+1}$ and summing over $\lambda \in \mathbb{Z}^D$ yields

$$\left(\psi_0^2(1) + \sum_{\lambda\in\mathbb{Z}^D\setminus\{0\}} \langle\lambda\rangle^{2s+1}(\psi_\infty)_\lambda^2\right) + \left(\varphi_0^2(1) + \sum_{\lambda\in\mathbb{Z}^D\setminus\{0\}} \langle\lambda\rangle^{2s+1}(\tilde{\varphi}_\infty)_\lambda^2\right) \lesssim \phi_0^2(1) + \psi_0^2(1) + \sum_{\lambda\in\mathbb{Z}^D\setminus\{0\}} \langle\lambda\rangle^{2s+1}\mathcal{E}_{\lambda,high}(1).$$

By (8.5.4), this tells us that ψ_∞ and $\tilde{\varphi}_\infty = \varphi_\infty + (\log \mathcal{T}_*)\psi_\infty$ are both in $H^{s+\frac{1}{2}}$, and in fact (8.5.3) holds. Since this is true for all $s \in \mathbb{R}$, and $\log \mathcal{T}_* : H^{s+\frac{1}{2}} \to H^s$ is bounded for every $s \in \mathbb{R}$ (see Lemma D.0.1), we have that ψ_∞ and φ_∞ are smooth, as required.

The final step is to show the convergence (8.5.2). We write the H^s-norm of $\psi(t) - \psi_\infty$ in the following manner:

$$\|\psi(t) - \psi_\infty\|_{H^s}^2 = \sum_{\substack{\lambda\in\mathbb{Z}^D\setminus\{0\}\\ \tau_\lambda(t)>1}} \langle\lambda\rangle^{2s} \left|\psi_\lambda(t) - (\psi_\infty)_\lambda\right|^2 + \sum_{\substack{\lambda\in\mathbb{Z}^D\setminus\{0\}\\ \tau_\lambda(t)\le 1}} \langle\lambda\rangle^{2s} \left|\psi_\lambda(t) - (\psi_\infty)_\lambda\right|^2.$$

We estimate the first sum on the right hand side by expanding crudely and using (8.5.5), while for the second sum we use Proposition 8.4.2, particularly the estimate (8.4.7). We find the following:

$$\|\psi(t) - \psi_\infty\|_{H^s}^2 \lesssim \sum_{\substack{\lambda\in\mathbb{Z}^D\setminus\{0\}\\ \tau_\lambda(t)>1}} \left(\tau_\lambda(t)\langle\lambda\rangle^{2s}\mathcal{E}_{\lambda,high}(1) + \langle\lambda\rangle^{2s}(\psi_\infty)_\lambda^2\right) + \sum_{\substack{\lambda\in\mathbb{Z}^D\setminus\{0\}\\ \tau_\lambda(t)\le 1}} \tau_\lambda^2(t)\langle\lambda\rangle^{2s}\mathcal{E}_{\lambda,low}(t_{\lambda*}).$$

Using again Propositions 8.3.2 and 8.4.1 to deduce $\mathcal{E}_{\lambda,low}(t_{\lambda*}) \lesssim \mathcal{E}_{\lambda,high}(1) \lesssim \langle\lambda\rangle^{-1}(\psi_C)_\lambda^2 + \langle\lambda\rangle(\phi_C)_\lambda^2$, as well as the fact that $\tau_\lambda(t) \le \langle\lambda\rangle t^{1-p}$ for $p = \max\{p_i\}$, we therefore have that

$$\|\psi(t) - \psi_\infty\|_{H^s}^2 \lesssim t^{1-p}\left(\|\phi_C\|_{H^{s+1}}^2 + \|\psi_C\|_{H^s}^2\right) + \sum_{\substack{\lambda\in\mathbb{Z}^D\setminus\{0\}\\ \tau_\lambda(t)>1}} \langle\lambda\rangle^{2s}(\psi_\infty)_\lambda^2. \tag{8.5.9}$$

Since we have $\psi_\infty \in H^s$ and $\min\{|\lambda| : \tau_\lambda(t) < 1\} \to \infty$ as $t \to 0$, we conclude that $\psi_\infty(t) \to \psi_\infty$ in H^s as $t \to 0$.

The convergence of $\varphi(t)$ is more complicated due to the additional log factors. Nevertheless, a similar but slightly more involved computation will yield that

$$\|\varphi(t) - \varphi_\infty\|_{H^s}^2 \lesssim \sum_{\substack{\lambda\in\mathbb{Z}^D\setminus\{0\}\\ \tau_\lambda(t)>1}} \left(\tau_\lambda(t)(1 + (\log t^{-1})^2)\langle\lambda\rangle^{2s}\mathcal{E}_{\lambda,high}(1) + \langle\lambda\rangle^{2s}(\varphi_\infty)_\lambda^2\right) + \sum_{\substack{\lambda\in\mathbb{Z}^D\setminus\{0\}\\ \tau_\lambda(t)\le 1}} \tau_\lambda^2(t)(1 + (\log t_{\lambda*}^{-1})^2)\langle\lambda\rangle^{2s}\mathcal{E}_{\lambda,low}(t_{\lambda*}).$$

Using the additional fact (8.2.9), one may show that when $\tau_\lambda(t) \le 1$, one has $\tau_\lambda(1 + (\log t_{\lambda^*}^{-1})^2) \lesssim (1 + (\log t^{-1})^2)$. Therefore the analogue of (8.5.9) is that

$$\|\varphi(t) - \varphi_\infty\|_{H^s}^2 \lesssim t^{1-p}\,(1 + (\log t^{-1})^2)\left(\|\phi_C\|_{H^{s+1}}^2 + \|\psi_C\|_{H^s}^2\right) + \sum_{\substack{\lambda\in\mathbb{Z}^D\setminus\{0\}\\ \tau_\lambda(t)>1}} \langle\lambda\rangle^{2s}(\varphi_\infty)_\lambda^2. \tag{8.5.10}$$

Since φ_∞ also lies in H^s, the right hand side converges to 0 as $t \to 0$. This concludes the proof of the proposition. □

8.5.2 From asymptotic data to Cauchy data

Proposition 8.5.3. *Let ψ_∞ and φ_∞ be smooth functions on $\mathbb{T}^D$. Then there exists a unique smooth solution $(\phi(t), \psi(t))$ to the system (1.6.1)–(1.6.2) for $t > 0$, which achieves the asymptotic data in the sense that for any $s \in \mathbb{R}$, one has the (strong) convergence:*

$$\psi(t) \to \psi_\infty \text{ in } H^s(\mathbb{T}^D), \quad \varphi(t) \to \varphi_\infty \text{ in } H^s(\mathbb{T}^D) \quad \text{as } t \to 0. \tag{8.5.11}$$

Moreover, for any $s \in \mathbb{R}$ one has the following bound between Sobolev spaces:

$$\|\phi(1,\cdot)\|^2_{H^{s+1}} + \|\psi(1,\cdot)\|^2_{H^s} \lesssim \|\psi_\infty\|^2_{H^{s+\frac{1}{2}}} + \|\varphi_\infty + (\log \mathcal{T}_*)\psi_\infty\|^2_{H^{s+\frac{1}{2}}}. \tag{8.5.12}$$

Proof. For existence, we first decompose $(\psi_\infty, \varphi_\infty)$ into their Fourier modes, and define also for $\lambda \neq 0$,

$$(\tilde{\varphi}_\infty)_\lambda = (\varphi_\infty)_\lambda + (\psi_\infty)_\lambda \log t_{\lambda *}.$$

By definition of the Sobolev norm and the operator $\log \mathcal{T}_*$, one has

$$(\psi_\infty)_0^2 + (\varphi_\infty)_0^2 + \sum_{\lambda \in \mathbb{Z}^D \setminus \{0\}} \langle \lambda \rangle^{2s+1} \left((\psi_\infty)_\lambda^2 + (\tilde{\varphi}_\infty)_\lambda^2 \right) = \|\psi_\infty\|^2_{H^{s+\frac{1}{2}}} + \|\varphi_\infty + (\log \mathcal{T}_*)\psi_\infty\|^2_{H^{s+\frac{1}{2}}}. \tag{8.5.13}$$

We now appeal to Proposition 8.4.2(ii), which states that for all $\lambda \in \mathbb{Z}^D \setminus \{0\}$, we can find a unique solution to the ODE system (8.4.5)–(8.4.6) which achieves the limits $(\psi_\infty)_\lambda$ and $(\tilde{\varphi}_\infty)_\lambda$ as $t \to 0$, at the rate (8.4.7). Transforming back to $(\phi_\lambda(t), \psi_\lambda(t))$, we can find a solution to (8.2.1)–(8.2.2) in the whole domain $t > 0$. Furthermore, Propositions 8.4.1 and 8.4.2 imply that for $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $0 < t \leq t_{\lambda *}$, one has

$$\mathcal{E}_{\lambda, low}(t) \lesssim (\psi_\infty)_\lambda^2 + (\tilde{\varphi}_\infty)_\lambda^2. \tag{8.5.14}$$

Combining this with Proposition 8.3.2, Corollary 8.3.1 and that $\mathcal{E}_{\lambda, high}(t_{\lambda *}) \asymp \mathcal{E}_{\lambda, low}(t_{\lambda *})$, one has for $t_{\lambda *} \leq t \leq 1$ the estimate

$$\frac{\psi_\lambda^2(t)}{\tau_\lambda(t)} + \tau_\lambda(t)\phi_\lambda^2(t) \lesssim (\psi_\infty)_\lambda^2 + (\tilde{\varphi}_\infty)_\lambda^2. \tag{8.5.15}$$

We now define our solution $(\phi(t), \psi(t))$ to the equations (1.6.1)–(1.6.2) as the sum.

$$\phi(t,x) = (\varphi_\infty)_0 + \sum_{\lambda \in \mathbb{Z}^D \setminus \{0\}} \phi_\lambda(t) e^{i\lambda \cdot x}, \qquad \psi(t,x) = (\psi_\infty)_0 + \sum_{\lambda \in \mathbb{Z}^D \setminus \{0\}} \psi_\lambda(t) e^{i\lambda \cdot x}. \tag{8.5.16}$$

Let us verify the smoothness of $(\phi(t), \psi(t))$ at $t = 1$. Using (8.5.15),(8.5.13) and $\tau_\lambda(1) \asymp \langle\lambda\rangle$, we get

$$(\varphi_\infty)_0^2 + \sum_{\lambda\in\mathbb{Z}^D\setminus\{0\}} \langle\lambda\rangle^{2s+2}\phi_\lambda^2(1) + (\psi_\infty)_0^2 + \sum_{\lambda\in\mathbb{Z}^D\setminus\{0\}} \langle\lambda\rangle^{2s}\psi_\lambda^2(1)$$
$$\lesssim \|\psi_\infty\|^2_{H^{s+\frac{1}{2}}} + \|\varphi_\infty + (\log\mathcal{T}_*)\psi_\infty\|^2_{H^{s+\frac{1}{2}}}.$$

Therefore $\phi(1,\cdot) \in H^{s+1}$, $\psi(1,\cdot) \in H^s$, and the estimate (8.5.12) holds. Since this is true for all $s \in \mathbb{R}$, these functions are smooth. Furthermore, since the Fourier modes $(\phi_\lambda(t), \psi_\lambda(t))$ satisfy the ODEs (8.2.1)–(8.2.2) for $\lambda \neq 0$, and we may appeal to Lemma 8.5.1 for the zero mode, $(\phi(t), \psi(t))$ is a smooth solution to the first order system (1.6.1)–(1.6.2), as required.

By the construction of $(\phi(t), \psi(t))$ and the proof of Proposition 8.5.3, $\psi(t)$ and $\varphi(t)$ also obey the asymptotics (8.5.11). Finally, the fact that $(\phi(t), \psi(t))$ is the *unique* smooth solution to (1.6.1)–(1.6.2) obeying these asymptotics follows by studying each Fourier mode for $\phi_\lambda(t), \psi_\lambda(t)$ and appealing to the uniqueness statement in Proposition 8.4.2 for $\lambda \in \mathbb{Z}^D \setminus \{0\}$, and Lemma 8.5.1 for $\lambda = 0$. □

8.5.3 Completion of the proof of Theorem 1.6.2

All that remains is to extend Proposition 8.5.2 and Proposition 8.5.3 to the setting where the functions $\phi(t)$ and $\psi(t)$ have finite regularity. For the sake of completeness, we provide the standard density argument here:

(i) (Existence of the scattering operator) By Proposition 8.5.2, we are already done in the case that (ϕ_C, ψ_C) are smooth. Now, for $s > \frac{D}{2} + 1$ suppose instead that $(\phi_C, \psi_C) \in H^{s+1} \times H^s$. By the density of smooth functions in Sobolev spaces, there exist smooth functions $(\phi_C^{(n)}, \psi_C^{(n)})$ such that

$$(\phi_C^{(n)}, \psi_C^{(n)}) \to (\phi_C, \psi_C) \text{ in } H^{s+1} \times H^s \text{ as } n \to \infty.$$

Applying Proposition 8.5.2 to each of $(\phi_C^{(n)}, \psi_C^{(n)})$, we construct a sequence of solutions $(\phi^{(n)}(t), \phi^{(n)}(t))$ to the system (1.6.1)–(1.6.2) and a sequence of functions $(\psi_\infty^{(n)}, \varphi_\infty^{(n)})$ such that

$$\|\psi_\infty^{(n)}\|^2_{H^{s+\frac{1}{2}}} + \|\varphi_\infty^{(n)} - (\log\mathcal{T}_*)\psi_\infty^{(n)}\|^2_{H^{s+\frac{1}{2}}} \lesssim \|\phi_C^{(n)}\|^2_{H^{s+1}} + \|\psi_C^{(n)}\|^2_{H^s},$$

and we have the (strong convergence)

$$\psi^{(n)}(t) \to \psi_\infty^{(n)} \text{ in } H^s, \quad \varphi^{(n)}(t) \to \varphi_\infty^{(n)} \text{ in } H^s \quad \text{ as } t \to 0.$$

Furthermore, since (1.6.1)–(1.6.2) is linear we also have the difference estimates of the type

$$\|\psi_\infty^{(n)} - \psi_\infty^{(m)}\|^2_{H^{s+\frac{1}{2}}} + \|(\varphi_\infty^{(n)} - (\log\mathcal{T}_*)\psi_\infty^{(n)}) - (\varphi_\infty^{(m)} - (\log\mathcal{T}_*)\psi_\infty^{(m)})\|^2_{H^{s+\frac{1}{2}}}$$
$$\lesssim \|\phi_C^{(n)} - \phi_C^{(m)}\|^2_{H^{s+1}} + \|\psi_C^{(n)} - \psi_C^{(m)}\|^2_{H^s}.$$

Therefore, by construction of the sequence $(\phi_C^{(n)}, \psi_C^{(n)})$, it is clear we can obtain $(\psi_\infty, \varphi_\infty)$ such that $(\psi_\infty^{(n)}, \varphi_\infty^{(n)}) \to (\psi_\infty, \varphi_\infty)$ in the space $\mathcal{H}^s_\infty$ defined in (1.6.10), and by similar difference estimates applied to the proof of Proposition 8.5.2 we obtain a limiting solution

$$(\phi(t), \psi(t)) \in C^0((0, +\infty), H^{s+1} \times H^s) \cap C^1((0, +\infty), H^s \times H^{s-1}),$$

which attain the asymptotics (1.6.7). Since $s > \frac{D}{2} + 1$, Sobolev embedding implies that $H^{s-1} \subset C^0$ and therefore this solution is classical.

(ii) (Asymptotic completeness) By Proposition 8.5.3, we are done in the case that $(\psi_\infty, \varphi_\infty)$ are smooth. If instead we only have $(\psi_\infty, \varphi_\infty)$ with regularity as in (1.6.8), i.e. we have $(\psi_\infty, \varphi_\infty) \in \mathcal{H}^s_\infty$, then first pick a sequence $(\psi_\infty^{(n)}, \varphi_\infty^{(n)})$ of smooth functions such that

$$(\psi_\infty^{(n)}, \varphi_\infty^{(n)}) \to (\psi_\infty, \varphi_\infty) \text{ in } \mathcal{H}^s_\infty \text{ as } n \to \infty.$$

Now apply Proposition 8.5.3 to each of $(\psi_\infty^{(n)}, \varphi_\infty^{(n)})$. We obtain a sequence $(\phi^{(n)}(t), \psi^{(n)}(t))$ of solutions of (1.6.1)–(1.6.2) which moreover obey the difference estimate

$$\|\phi^{(n)}(1, \cdot) - \phi^{(m)}(1, \cdot)\|^2_{H^{s+1}} + \|\psi^{(n)}(1, \cdot) - \psi^{(m)}(1, \cdot)\|^2_{H^s} \lesssim \|(\psi_\infty^{(n)} - \psi_\infty^{(m)}, \varphi_\infty^{(n)} - \varphi_\infty^{(m)})\|^2_{\mathcal{H}^s_\infty}.$$

Therefore, by construction of the sequence $(\phi_\infty^{(n)}, \psi_\infty^{(n)})$, we construct firstly a limit $(\phi^{(n)}(1, \cdot), \psi^{(n)}(1, \cdot)) \to (\phi(1, \cdot), \psi(1, \cdot))$ in the space $\mathcal{H}^s_C = H^{s+1} \times H^s$, and then by applying (i) again, we get a limiting solution $(\phi(t), \psi(t))$ to the system (1.6.1)–(1.6.2) as desired.

(iii) (Scattering isomorphism) By Proposition 8.5.2 and Proposition 8.5.3, we know that the two maps

$$\mathcal{S}_\downarrow : (\phi_C, \psi_C) \mapsto (\psi_\infty, \varphi_\infty) \text{ from (i)},$$

$$\mathcal{S}_\uparrow : (\psi_\infty, \varphi_\infty) \mapsto (\phi(1, \cdot), \psi(1, \cdot)) \text{ from (ii)},$$

upon restriction to the space of smooth functions, are well-defined and are such that both $\mathcal{S}_\downarrow \circ \mathcal{S}_\uparrow$ and $\mathcal{S}_\uparrow \circ \mathcal{S}_\downarrow$ are the identity map. Furthermore, the bounds (8.5.3) and (8.5.12), together with the density of $C^\infty(\mathbb{T}^D)^2$ in both the Hilbert spaces $\mathcal{H}^\infty_C$ and $\mathcal{H}^s_\infty$, imply that these maps may indeed be extended to be Hilbert space isomorphisms, as required. □

Chapter 9

Scattering towards the singularity in Kasner: Linearised gravity

In Chapter 9, we prove Theorem 7.3.1, which we stated in Section 7.3 after setting up the appropriate notation in Chapter 7. We encourage readers to familiarize themselves with the notation set up in this chapter before moving to the proof.

9.1 Overview

We highlight some new aspects of the proof, building upon the proof outlined for the case of scattering for the wave equation in Section 8.1. The key new ideas will be evident if we focus only on the linearized metric perturbations; for this purpose, we write the linearized Einstein equations (in the isotropic Kasner – as in Section 8.1 – and with a rescaled time variable) in the following schematic form, where η and κ are our main dynamical variables:

$$t\partial_t\eta = \kappa + \nu + \partial\chi, \qquad t\partial_t\kappa = t^2\Delta\eta + t^2\partial^2 * \eta + \nu + \partial\chi. \tag{9.1.1}$$

There are three major differences from the case of the scalar wave equation. Firstly, η and κ are $(1,1)$-tensors rather than scalars. In particular, we see terms of the kind $t^2\partial^2 * \eta$, which schematically represent linear combinations of contractions of the derivatives and the tensor indices.

Secondly, the equations (9.1.1) are coupled to an elliptic equation for ν, which is schematically

$$(1 - t^2\Delta)\nu = t^2\partial^2 * \eta. \tag{9.1.2}$$

Finally, we see another variable χ, which corresponds to a gauge choice (see Section 7.2.1). We single out two such gauge choices: one choice is we set $\chi = 0$, or alternatively we may choose a gauge, called the CMCSH gauge (see Definition 7.1.3), such that χ will also solve an elliptic equation of the schematic form

$$t^2\Delta\partial\chi = t^2\partial^2 * \eta. \tag{9.1.3}$$

The benefit is that in CMCSH gauge, the term $t^2\partial^2 * \eta$ in (9.1.1) vanishes. Thus in CMCSH gauge (9.1.1), coupled to (9.1.2)–(9.1.3), can be viewed as a first-order elliptic-hyperbolic system.

Let us now see how the high- and low-frequency energy estimates apply to this system.

1. At high-frequencies $|\lambda|t \gg 1$, we wish to use a similar looking energy of the form

$$\mathcal{E}_{\lambda,high}(t) = \frac{|\kappa_\lambda|_g^2}{|\lambda|t} + \frac{\langle \kappa_\lambda, \eta_\lambda\rangle_g}{|\lambda|t} + \frac{|\eta_\lambda|_g^2}{2|\lambda|t} + |\lambda|t \cdot |\eta_\lambda|_g^2.$$

   Here the norm $|\cdot|_g^2$ and the inner product $\langle\cdot,\cdot\rangle_g$ indicate one needs a metric to define norms on tensors. As in the wave equation case, we require crucial top order cancellations in order to obtain a good high-frequency energy estimate. In order to achieve the same cancellations in this case, it is essential that we use the CMCSH gauge, in order to remove the top order term $t^2\partial^2 * \eta$ in (9.1.1).

   In this gauge, we get the desired top order cancellations from the η and κ terms. However, we still need to control the contributions from ν and $\partial\chi$. The elliptic equation (9.1.2) suggests that in the $|\lambda t| \geq 1$ regime, ν_λ is of the same size as η_λ, and thus the ν appearing in the first equation in (9.1.1) seems to produce a top order contribution not conducive to using Grönwall towards $t = 0$.

   Fortunately, the Hamiltonian constraint means that we may rewrite the elliptic equation in a different schematic form, namely $(1 - t^2\Delta)\nu = \kappa$. For $|\lambda|t \geq 1$, this tells us that ν is of the same size as $(|\lambda|t)^{-2}\kappa$, which in our definition of $\mathcal{E}_{\lambda,high}(t)$ behaves better than η, and allows us to Grönwall terms involving ν.

   Similar considerations apply for $\partial\chi$. Though the elliptic equation (9.1.3) suggests $\partial\chi$ is of the same size as η, upon taking the derivative $t\partial_t\mathcal{E}_{\lambda,high}(t)$, we can apply the CMCSH gauge condition and the momentum constraint to close the energy estimate.

2. At low-frequencies $|\lambda|t \ll 1$, we first define a quantity akin to $\tilde{\varphi}$ from the wave equation case which remains bounded as $t \to 0$. In the isotropic case, it will suffice to take $\tilde{\Upsilon} = \eta - \kappa\log(\lambda t)$ (in the anisotropic setting renormalized quantities are as in (7.2.24)). We employ an energy of the type

$$\mathcal{E}_{\lambda,low}(t) = |\kappa_\lambda|_{g_*}^2 + |\tilde{\Upsilon}_\lambda|_{g_*}^2.$$

   Note that we use a different norm $|\cdot|_{g_*}^2$ to emphasize that different metrics are used to quantify tensorial objects in the high-frequency and the low-frequency estimates.

   Ignoring this subtlety for now, we consider how the different terms in (9.1.1) affect our low-frequency energy estimates. It seems that $\partial\chi$ will be extremely problematic. This is since if one uses CMCSH gauge and the elliptic equation (9.1.3), we expect $\partial\chi$ to have the same size as η. However, since it is not η but instead $\tilde{\Upsilon}$ that is bounded, a term of order η really has a term of order $\kappa\log(|\lambda|t)$, even in the isotropic case. Therefore this term is not even bounded as $|\lambda|t \to 0$.

We resolve this by using a different gauge, namely the gauge where $\chi \equiv 0$, called CMCTC in Definition 7.1.3. Of course, this means that the term $t^2\partial^2 * \eta$ in (9.1.1) will return. Fortunately, the low-frequency energy estimate did not require top order cancellations for the main terms involving κ and η, and thus the term $t^2\partial^2 * \eta$, which goes to 0 as $|\lambda|t \to 0$, is of no issue.

For terms involving ν, from (9.1.2), we see ν is also of order $t^2\partial^2 * \eta$ and presents no further issue in the energy estimates. Thus at least in the isotropic case, we can derive the same estimate $t\partial_t \mathcal{E}_{\lambda,low}(t) \leq C(|\lambda|t)^2(1 + |\log(\lambda t)|^2)$, and thereby close the argument.

Translating into statements about Sobolev spaces, we can recover a scattering result for the linearized Einstein equations. Note that different gauges are used in the high-frequency and low-frequency regimes; this is exactly the reason that we must allow a change of gauge (associated to a vector field ξ^j) in Theorem 7.3.1.

In the case that the background metric is *anisotropic*, another difficulty is associated to the tensorial norms $|\cdot|_g$ and $|\cdot|_{g_*}$ introduced in the high-frequency and low-frequency energies above. In order to close the estimates, $|\cdot|_g$ must be associated to the time-dependent metric $\mathring{g}_{ij}(t)$, where $\mathring{g}_{ij}$ represents the background Kasner metric. On the other hand $|\cdot|_{g_*}$ is associated to the metric $\mathring{g}_{ij}(t_{\lambda *})$ where $t_{\lambda *}$ is as in Definition 1.6.1. This is the source of the symbols $\mathcal{T}_*^{-p_i+p_j}$ appearing in Theorem 7.3.1.

Yet another difficulty arises from the fact that our renormalized quantity $\tilde{\Upsilon}$ is not just $\eta - \kappa\log(|\lambda|t)$, but a more complicated expression involving powers of t. Therefore, the low-frequency energy estimate involves carefully tracking the powers of t that arise, in order to ensure that $\partial_t \mathcal{E}_{\lambda,low}(t)$ is integrable towards $t = 0$; this is exactly where we require the subcriticality condition (1.2.11). This concludes our expository discussion of the proof.

9.2 Fourier decomposition of linearized Einstein–scalar field

The proof of the scattering results will involve a Fourier decomposition and detailed ODE analysis for each Fourier mode. For this purpose, we will project the whole linearized Einstein–scalar field system of Proposition 7.2.2 onto the Fourier mode corresponding to $\lambda \in \mathbb{Z}^D$, as below.

Proposition 9.2.1 (Linearized Einstein–scalar field in Fourier space)**.** *For $\lambda \in \mathbb{Z}^D$, suppose that $((\eta_\lambda)_i{}^j, (\kappa_\lambda)_i{}^j, \phi_\lambda, \psi_\lambda, \nu_\lambda, \xi_\lambda^j)$ is the projection of a solution to the linearized Einstein–scalar field system* (7.2.6)–(7.2.20) *onto the λ-Fourier mode.*

Our solutions will always be CMC in the sense that $\operatorname{tr}\kappa = 0$, and we say that the λ-Fourier mode is (linearly) spatially harmonic if the following holds (again the hat denotes CMCSH):

$$2\lambda_j(\hat{\eta}_\lambda)_i{}^j = \lambda_i \operatorname{tr}\hat{\eta}_\lambda. \tag{9.2.1}$$

As before, we demarcate the occasions when we assume the CMCSH gauge (9.2.1) *using a hat. Then the projected variables $((\eta_\lambda)_i{}^j, (\kappa_\lambda)_i{}^j, \phi_\lambda, \psi_\lambda, \nu_\lambda, \xi_\lambda^j)$ obey the following equations.*

1. *The metric variables* $((\eta_\lambda)_i{}^j, (\kappa_\lambda)_i{}^j)$ *obey the evolution equations:*

$$t\partial_t(\eta_\lambda)_i{}^j = (\kappa_\lambda)_i{}^j + (2t\mathring{k}_p{}^j)(\eta_\lambda)_i{}^p - (2t\mathring{k}_i{}^p)(\eta_\lambda)_p{}^j + (t\mathring{k}_i{}^j)\nu_\lambda + \frac{\mathrm{i}}{2}\lambda_i\chi_\lambda^j + \frac{\mathrm{i}}{2}\mathring{g}_{ip}\mathring{g}^{jq}\lambda_q\chi_\lambda^p, \quad (9.2.2)$$

$$t\partial_t(\kappa_\lambda)_i{}^j = t^2\,(\widetilde{\mathrm{Ric}}_\lambda)_i{}^j[\eta_\lambda] + t^2\mathring{g}^{jk}\lambda_i\lambda_k\nu_\lambda - (t\mathring{k}_i{}^j)\nu_\lambda - \mathrm{i}(t\mathring{k}_a{}^j)\lambda_i\chi_\lambda^a + \mathrm{i}(t\mathring{k}_i{}^a)\lambda_a\chi_\lambda^j. \quad (9.2.3)$$

Here, the expression $(\widetilde{\mathrm{Ric}}_\lambda)[\eta_\lambda]$ *is defined by*

$$t^2\widetilde{\mathrm{Ric}}_i{}^j[(\eta_\lambda)] \coloneqq -\tau^2(\eta_\lambda)_i{}^j - t^2\mathring{g}^{ja}\lambda_i\lambda_a(\mathrm{tr}\,\eta_\lambda) + t^2\mathring{g}^{ab}\lambda_i\lambda_b(\eta_\lambda)_a{}^j + t^2\mathring{g}^{ja}\lambda_a\lambda_b(\eta_\lambda)_i{}^b. \quad (9.2.4)$$

In particular, if $(\hat{\eta}_\lambda)_i{}^j$ *satisfies the spatially harmonic condition* (7.2.5)*, then*

$$t^2(\widetilde{\mathrm{Ric}}_\lambda)_i{}^j[\hat{\eta}_\lambda] = -\tau^2(\hat{\eta}_\lambda)_i{}^j. \quad (9.2.5)$$

2. *The matter variables* $(\phi_\lambda, \psi_\lambda)$ *obey*

$$t\partial_t\phi_\lambda = \psi_\lambda + p_\phi\nu_\lambda, \quad (9.2.6)$$

$$t\partial_t\psi_\lambda = -\tau^2\phi_\lambda - p_\phi\nu_\lambda. \quad (9.2.7)$$

3. *The linearized lapse* ν_λ *obeys the elliptic equation*

$$\left(1+\tau^2\right)\nu_\lambda = 2\tau^2(\mathrm{tr}\,\eta_\lambda) - 2t^2\tilde{g}^{ab}\lambda_i\lambda_a(\eta_\lambda)_b{}^i. \quad (9.2.8)$$

If $(\hat{\eta}_\lambda)_i{}^j$ *moreover satisfies the spatially harmonic condition* (7.2.5)*, this equation simplifies to*

$$\left(1+\tau^2\right)\nu_\lambda = \tau^2(\mathrm{tr}\,\hat{\eta}_\lambda). \quad (9.2.9)$$

Furthermore, if (9.2.1) *holds we have an additional elliptic equation for the shift:*

$$\tau^2\chi_\lambda^j = \mathrm{i}t^2\mathring{g}^{ij}\lambda_i\nu_\lambda + 2\mathrm{i}t^2\mathring{g}^{ij}(t\mathring{k}_i{}^k)\lambda_k\nu_\lambda + 4\mathrm{i}t^2\mathring{g}^{pq}(t\mathring{k}_p{}^i)\lambda_q(\hat{\eta}_\lambda)_i{}^j \\ - 2\mathrm{i}t^2\mathring{g}^{ij}\lambda_i[t\mathring{k}_b{}^a\,(\hat{\eta}_\lambda)_a{}^b + 2p_\phi\phi_\lambda]. \quad (9.2.10)$$

4. *As well as the linearized CMC condition* $\mathrm{tr}\,\kappa_\lambda = 0$*, the following constraints are satisfied:*

$$2\tau^2(\mathrm{tr}\,\eta_\lambda) - 2t^2\mathring{g}^{ab}\lambda_i\lambda_a(\eta_\lambda)_b{}^i + 2t\mathring{k}_b{}^a\,(\kappa_\lambda)_a{}^b + 4p_\phi\psi_\lambda = 0. \quad (9.2.11)$$

$$\lambda_j(\kappa_\lambda)_i{}^j + \lambda_i[t\mathring{k}_b{}^a\,(\eta_\lambda)_a{}^b + 2p_\phi\phi_\lambda] - t\mathring{k}_i{}^j\lambda_j(\mathrm{tr}\,\eta_\lambda) = 0. \quad (9.2.12)$$

$$\mathring{g}^{ab}\lambda_a(\kappa_\lambda)_b{}^c + \mathring{g}^{cd}\lambda_d[t\mathring{k}_b{}^a\,(\eta_\lambda)_a{}^b + 2p_\phi\phi_\lambda] - 2\mathring{g}^{ab}\,t\mathring{k}_b{}^d\,\lambda_a(\eta_\lambda)_d{}^c = 0. \quad (9.2.13)$$

As mentioned the solution is in CMCSH gauge if (9.2.1) *is also satisfied. In this case,* (9.2.11) *simplifies to*

$$\tau^2(\mathrm{tr}\,\hat{\eta}_\lambda) + 2t\mathring{k}_b{}^a\,(\hat{\kappa}_\lambda)_a{}^b + 4p_\phi\psi_\lambda = 0. \quad (9.2.14)$$

5. Finally, there are linearized symmetry conditions for $\eta_i{}^j$ *and* $\kappa_i{}^j$*, which are also propagated:*

$$\mathring{g}^{ab}(\eta_\lambda)_b{}^c = \mathring{g}^{cb}(\eta_\lambda)_b{}^a, \tag{9.2.15}$$

$$\mathring{g}^{ab}\left((\kappa_\lambda)_b{}^c - 2t\mathring{k}_b{}^d\,(\eta_\lambda)_d{}^c\right) = \mathring{g}^{cb}\left((\kappa_\lambda)_b{}^a - 2t\mathring{k}_b{}^d\,(\eta_\lambda)_d{}^a\right). \tag{9.2.16}$$

Proof. The proof follows immediately simply by replacing any spatial derivative ∂_i appearing in the systen (7.2.6)–(7.2.20) by the symbol $-\mathrm{i}\lambda_i$, and simplifying accordingly. □

It is also possible to translate Lemma 7.2.3, Lemma 7.2.5 and Lemma 7.2.6 into frequency space. To keep things brief, of these we only write the Fourier projections of the asymptotic constraints (7.2.27)–(7.2.31).

Lemma 9.2.2. *Let* $((\kappa_\infty)_i{}^j, (\tilde{\Upsilon}_\infty)_i{}^j, \psi_\infty, \tilde{\varphi}_\infty)$ *be as in Lemma 7.2.5, for some* $T > 0$*. Then for* $\lambda > 0$*, the following asymptotic constraints hold between the* λ*-Fourier projections* $(((\kappa_\infty)_\lambda)_i{}^j, ((\tilde{\Upsilon}_\infty)_\lambda)_i{}^j, (\psi_\infty)_\lambda, (\tilde{\varphi}_\infty)_\lambda)$*:*

$$t\mathring{k}_b{}^a\,((\kappa_\infty)_\lambda)_a{}^b + 2p_\phi(\psi_\infty)_\lambda = 0, \tag{9.2.17}$$

$$\lambda_j((\kappa_\infty)_\lambda)_i{}^j + \lambda_i[t\mathring{k}_b{}^a\,((\tilde{\Upsilon}_\infty)_\lambda)_a{}^b + 2p_\phi(\tilde{\varphi}_\infty)_\lambda] - t\mathring{k}_i{}^j\lambda_j(\operatorname{tr}\tilde{\Upsilon}_\infty)_\lambda = 0, \tag{9.2.18}$$

$$\mathring{g}^{ab}(T)\lambda_a((\kappa_\infty)_\lambda)_b{}^c + \mathring{g}^{cd}(T)\lambda_d[t\mathring{k}_b{}^a\,((\tilde{\Upsilon}_\infty)_\lambda)_a{}^b + 2p_\phi(\tilde{\varphi}_\infty)_\lambda] - 2\mathring{g}^{ab}(T)\,t\mathring{k}_b{}^d\,\lambda_a((\tilde{\Upsilon}_\infty)_\lambda)_d{}^c = 0, \tag{9.2.19}$$

$$\mathring{g}^{ab}(T)((\tilde{\Upsilon}_\infty)_\lambda)_b{}^c = \mathring{g}^{cb}(T)((\tilde{\Upsilon}_\infty)_\lambda)_b{}^a, \tag{9.2.20}$$

$$\mathring{g}^{ab}(T)\left(((\kappa_\infty)_\lambda)_b{}^c - 2t\mathring{k}_b{}^d\,((\tilde{\Upsilon}_\infty)_\lambda)_d{}^c\right) = \mathring{g}^{cb}(T)\left(((\kappa_\infty)_\lambda)_b{}^a - 2t\mathring{k}_b{}^d\,((\tilde{\Upsilon}_\infty)_\lambda)_d{}^a\right). \tag{9.2.21}$$

Proof. Again just replace ∂_i by $-\mathrm{i}\lambda_i$ in Lemma 7.2.5. □

9.3 The high-frequency energy estimate

In this section, we derive a high-frequency energy estimate similar to that of Proposition 8.3.2, for the Fourier modes of the linearized Einstein–scalar field system given in Proposition 9.2.1. The high-frequency energy estimates of this section will *always use the CMCSH gauge*, in particular the condition (9.2.1), and we make this clear by denoting $\hat{\eta}_i{}^j$ and $\hat{\kappa}_i{}^j$ with a hat.

9.3.1 The high-frequency energy quantity

Definition 9.3.1. The *geometric high-frequency energy quantity* $\mathcal{E}_{\lambda,\eta,high}(t)$, used for $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $\tau_\lambda(t) \geq 1$ large, is given by the following:

$$\mathcal{E}_{\lambda,\eta,high}(t) \coloneqq \frac{\zeta^2}{\tau}(\hat{\kappa}_\lambda)_i{}^j(\hat{\kappa}_\lambda)_j{}^i + \frac{\zeta}{\tau}(\hat{\kappa}_\lambda)_i{}^j(\hat{\eta}_\lambda)_j{}^i + \left(\frac{1}{2\tau} + \zeta^2\tau\right)(\hat{\eta}_\lambda)_i{}^j(\hat{\eta}_\lambda)_j{}^i. \tag{9.3.1}$$

We also define the *matter high-frequency energy quantity* $\mathcal{E}_{\lambda,\phi,high}(t)$ as follows:

$$\mathcal{E}_{\lambda,\phi,high}(t) \coloneqq \frac{\zeta^2}{\tau}\psi_\lambda^2 + \frac{\zeta}{\tau}\psi_\lambda\phi_\lambda + \left(\frac{1}{2\tau} + \zeta^2\tau\right)\phi_\lambda^2. \tag{9.3.2}$$

Finally, define the *total high-frequency energy quantity* as the sum $\mathcal{E}_{\lambda,high}(t) \coloneqq \mathcal{E}_{\lambda,\eta,high}(t) + \mathcal{E}_{\lambda,\phi,high}(t)$.

Remark. Since $\hat{\kappa}_i{}^j$ is not self-adjoint with respect to $\mathring{g}_{ij}(t)$, it is not clear a priori that $\mathcal{E}_{\lambda,\hat{\eta},high}(t)$ is positive definite. The following lemma shows that $\mathcal{E}_{\lambda,high}(t)$ is still uniformly coercive, as required for the energy estimates, at least for $\tau_\lambda(t) \geq \mathring{\tau}$, where $\mathring{\tau} \geq 1$ is a constant independent of $\lambda \in \mathbb{Z}^D \setminus \{0\}$.

Lemma 9.3.1. *For $\mathcal{E}_{\lambda,\eta,high}(t)$ as defined in* (9.3.1)*, and suppose that $(\hat{\eta}_\lambda)_i{}^j, (\hat{\kappa}_\lambda)_i{}^j$ satisfy the symmetry properties* (9.2.15)–(9.2.16)*. Then there exists some $\mathring{\tau} \geq 1$, depending only on the Kasner exponents p_i, such that for $\tau(t) = \tau_\lambda(t) \geq \mathring{\tau}$, one has the following energy coercivity, uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$:*

$$\mathcal{E}_{\lambda,\eta,high}(t) \asymp \frac{1}{\langle\tau\rangle}\mathring{g}^{ac}(t)\mathring{g}_{bd}(t)\,(\hat{\kappa}_\lambda)_a{}^b(\hat{\kappa}_\lambda)_c{}^d + \langle\tau\rangle\mathring{g}^{ac}(t)\mathring{g}_{bd}(t)\,(\hat{\eta}_\lambda)_a{}^b(\hat{\eta}_\lambda)_c{}^d. \tag{9.3.3}$$

Similarly, for $\mathcal{E}_{\lambda,\phi,high}(t)$, one has the following energy coercivity, uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $\tau_\lambda(t) \geq \mathring{\tau}$:

$$\mathcal{E}_{\lambda,\phi,high}(t) \asymp \frac{1}{\langle\tau\rangle}\psi_\lambda^2 + \langle\tau\rangle\phi_\lambda^2. \tag{9.3.4}$$

Proof. Using the (Fourier projections) of the symmetry constraints (7.2.19)–(7.2.20), one may instead write the geometric high-frequency energy quantity $\mathcal{E}_{\lambda,\eta,high}$ as:

$$\begin{aligned}\mathcal{E}_{\lambda,\eta,high}(t) = \frac{\zeta^2}{\tau}\mathring{g}^{ab}\mathring{g}_{cd}(\hat{\kappa}_\lambda)_a{}^c(\hat{\kappa}_\lambda)_b{}^d + \left(\frac{1}{2\tau} + \zeta^2\tau\right)\mathring{g}^{ab}\mathring{g}_{cd}(\hat{\eta}_\lambda)_a{}^c(\hat{\eta}_\lambda)_b{}^d \\ + \frac{\zeta}{\tau}\left[\mathring{g}^{ab}\mathring{g}_{cd} + \zeta\,\mathring{g}^{ab}\mathring{g}_{ce}(2t\mathring{k}_d{}^e) - \zeta\,\mathring{g}^{ae}\mathring{g}_{cd}(2t\mathring{k}_e{}^b)\right](\hat{\kappa}_\lambda)_a{}^c(\hat{\eta}_\lambda)_b{}^d.\end{aligned}$$

Using that $t\mathring{k}_i{}^j$ is simply the diagonal matrix $-p_{\underline{i}}\delta_{\underline{i}}{}^j$, we thus have

$$\begin{aligned}\mathcal{E}_{\lambda,\eta,high}(t) = \sum_{i,j=1}^{D}\mathring{g}^{ii}\mathring{g}_{jj}\Bigg[\frac{\zeta^2}{\tau}\left((\hat{\kappa}_\lambda)_i{}^j\right)^2 + \left(\frac{1}{2\tau} + \zeta^2\tau\right)\left((\hat{\eta}_\lambda)_i{}^j\right)^2 \\ + \frac{\zeta}{\tau}\big(1 + (2p_i - 2p_j)\zeta\big)\left((\hat{\kappa}_\lambda)_i{}^j\right)\left((\hat{\eta}_\lambda)_i{}^j\right)\Bigg].\end{aligned}$$

Using Young's inequality, and the upper bound for ζ in (8.2.5), we find that there exists some $\mathring{\tau} \geq 1$ such that for $\tau \geq \tau_0$, we have the inequality:

$$\frac{\zeta^2}{2\tau}\left((\hat{\kappa}_\lambda)_i{}^j\right)^2 + \frac{1}{2}\zeta^2\tau\left((\hat{\eta}_\lambda)_i{}^j\right)^2 \geq \left|\frac{\zeta}{\tau}\big(1 + (2p_i - 2p_j)\zeta\big)\left((\hat{\kappa}_\lambda)_i{}^j\right)\left((\hat{\eta}_\lambda)_i{}^j\right)\right|.$$

Combining all these inequalities as well as (8.2.5) then gives the lower bound in (9.3.3). (We use also that $\tau \asymp \langle\tau\rangle$ for $\lambda \neq 0$.) The upper bound in (8.2.5) is more straightforward, as is the coercivity (9.3.4) for $\mathcal{E}_{\lambda,\phi,high}(t)$. □

The aim of this section will be to prove the following high-frequency energy estimate. As we see shortly, the energy estimate in Proposition 9.3.2 will follow by summing two derivative estimates for $\mathcal{E}_{\lambda,\eta,high}(t)$ and $\mathcal{E}_{\lambda,\phi,high}(t)$ which will be obtained in Proposition 9.3.7 and Proposition 9.3.6 respectively.

Proposition 9.3.2. *Let $((\hat{\eta}_\lambda)_i^{\ j}, (\hat{\kappa}_\lambda)_i^{\ j}, \phi_\lambda, \psi_\lambda, \nu_\lambda, \chi_\lambda^j)$ be a solution to the linearized Einstein–scalar field system* (9.2.2)–(9.2.16) *in* CMCSH gauge. *Then the ∂_t-derivative of the total high-frequency energy quantity obeys the following bound, uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $\tau_\lambda(t) \geq \mathring{\tau}$:*

$$\left| t\frac{d}{dt}\mathcal{E}_{\lambda,high}(t) \right| \lesssim \left(t|\zeta'(t)| + \frac{1}{\tau} \right) \mathcal{E}_{\lambda,high}(t). \tag{9.3.5}$$

There exists some $\mathring{C}_{high}$, depending only on the background Kasner spacetime, such that for $\lambda \in \mathbb{Z}^D \setminus \{0\}$ obeying[1] *$\tau_\lambda(1) \geq \mathring{\tau}$, it is true for any t with $\tau_\lambda(t) \geq \mathring{\tau}$ that*

$$\mathring{C}_{high}^{-1}\mathcal{E}_{\lambda,high}(t) \leq \mathcal{E}_{\lambda,high}(1) \leq \mathring{C}_{high}\,\mathcal{E}_{\lambda,high}(t). \tag{9.3.6}$$

Proof (assuming Propositions 9.3.6 and 9.3.7). The derivative estimate (9.3.5) will follow from summing (9.3.9) and (9.3.11) from Propositions 9.3.6 and 9.3.7 respectively. We deduce (9.3.6) using a Grönwall argument similar to that of the proof of Proposition 8.3.3. By Lemmas 8.2.1 and 8.2.2 (in particular (8.2.7) with $\alpha = 1$), we have

$$\int_{t_{\lambda *}}^{1} \left(s|\zeta'(s)| + \frac{1}{\tau(s)} \right) \frac{ds}{s} \lesssim 1.$$

Therefore, applying Grönwall's inequality to (9.3.5), in both the forwards and backwards directions, yields that there exists some $\mathring{C}_{high} > 0$ such that for any t_1, t_2 with $\mathring{\tau} \leq \tau(t_1) \leq \tau(t_2) \leq \tau(1)$, one has

$$\mathring{C}_{high}^{-1}\mathcal{E}_{\lambda,high}(t_1) \leq \mathcal{E}_{\lambda,high}(t_2) \leq \mathring{C}_{high}\,\mathcal{E}_{\lambda,high}(t_1).$$

To deduce (9.3.6), simply set $t_2 = 1$. □

The remainder of Section 9.3 is dedicated to the Propositions 9.3.6 and 9.3.7.

9.3.2 The $\mathring{g}$-norm and elliptic estimates

For our energy estimates, due to the coercivity in Lemma 9.3.1, it will simplify the exposition somewhat to define an inner product on (constant-valued) tensors adapted to the Cauchy slices $(\Sigma_t, \mathring{g}_{ij}(t))$.

[1] $\tau_\lambda^2(1) = \langle\lambda\rangle^2$ is greater than $\mathring{\tau}^2$ for sufficiently large $|\lambda|$. For smaller λ see the mid-frequency energy estimate of Section 9.4.4.

Definition 9.3.2. Let $\alpha = \alpha_{ij\cdots k}{}^{pq\cdots r}$ be a (constant complex valued) tensor on Σ_t. We define the $\mathring{g}$-norm, $|\cdot|_{\mathring{g}}$, of α as follows:

$$|\alpha|^2_{\mathring{g}} = |\alpha_{ij\cdots k}{}^{pq\cdots r}|^2_{\mathring{g}} \coloneqq \mathring{g}^{i_1 i_2}\cdots\mathring{g}^{k_1 k_2}\ \mathring{g}_{p_1 p_2}\cdots\mathring{g}_{r_1 r_2}\ \alpha_{i_1 j_1\cdots k_1}{}^{p_1 q_1\cdots r_1}\overline{\alpha}_{i_2 j_2\cdots k_2}{}^{p_2 q_2\cdots r_2},$$

where $\mathring{g}_{ij} = \mathring{g}_{ij}(t)$ and its inverse $\mathring{g}^{ij}$ have t-dependence as in (Kas). If α is a scalar, then $|\alpha|_{\mathring{g}} = |\alpha|$.

The following lemma is standard from the theory of tensors in inner product spaces.

Lemma 9.3.3. *Let $\alpha = \alpha_{ij\cdots k}{}^{pq\cdots r}$ and $\beta = \beta_{lm\cdots n}{}^{st\cdots u}$ be (constant) tensors on Σ_t. We define the tensor product $\alpha\otimes\beta$ and the contraction $\operatorname{tr}\alpha$ as the tensors:*

$$(\alpha\otimes\beta)_{ij\cdots klm\cdots n}{}^{pq\cdots rst\ldots u} = \alpha_{ij\cdots k}{}^{pq\cdots r}\beta_{lm\cdots n}{}^{st\cdots u}, \quad (\operatorname{tr}\alpha)_{j\cdots k}{}^{q\cdots r} = \alpha_{aj\cdots k}{}^{aq\cdots r}.$$

Then the following inequalities hold regarding the $\mathring{g}$-norms:

$$|\alpha\otimes\beta|_{\mathring{g}} \le |\alpha|_{\mathring{g}}|\beta|_{\mathring{g}}, \quad |\operatorname{tr}\alpha|_{\mathring{g}} \le \sqrt{D}|\alpha|_{\mathring{g}}.$$

On the other hand, this next lemma follows from the definition (8.2.3) of τ:

Lemma 9.3.4. *Consider λ_i as a covector in Σ_t. Then the $\mathring{g}$-norms of λ_i and $t\mathring{k}_i{}^j$ are as follows:*

$$|\lambda_i|_{\mathring{g}} = \frac{\tau}{t}, \qquad |t\mathring{k}_i{}^j|_{\mathring{g}} = \left(\sum_{i=1}^{D} p_i^2\right)^{1/2} \le 1.$$

The $\mathring{g}$-norm notation will prove useful in following elliptic estimates for $\operatorname{tr}\hat{\eta}_\lambda$, ν_λ and χ^j_λ.

Proposition 9.3.5. *For $\lambda\in\mathbb{Z}^D\setminus\{0\}$ and $\tau(t)\ge\mathring{\tau}$, the following bounds hold for $\operatorname{tr}\hat{\eta}_\lambda$ and ν_λ.*

$$|\operatorname{tr}\hat{\eta}_\lambda| + |\nu_\lambda| \lesssim \frac{1}{\tau^{\frac{3}{2}}}\mathcal{E}^{1/2}_{\lambda,high}(t). \tag{9.3.7}$$

For the shift χ^j_λ, we have the following:

$$|\chi^j_\lambda|_{\mathring{g}} \lesssim \frac{t}{\tau^{\frac{3}{2}}}\mathcal{E}^{1/2}_{\lambda,high}(t). \tag{9.3.8}$$

Proof. We start with the $\operatorname{tr}\hat{\eta}_\lambda$ estimate, for which we use the linearized Hamiltonian constraint (9.2.11). To illustrate how we use the $\mathring{g}$-norm estimates above, one applies the linearized CMCSH gauge (9.2.1) and rewrites this equation tensorially as

$$\tau^2(\operatorname{tr}\hat{\eta}_\lambda) = -2\operatorname{tr}\Big(t\mathring{k}\otimes\hat{\kappa}_\lambda\Big) - 4p_\phi\psi_\lambda.$$

Therefore, upon application of Lemmas 9.3.3 and 9.3.4 one has

$$\tau^2|\operatorname{tr}\hat{\eta}_\lambda| = \tau^2|\operatorname{tr}\hat{\eta}_\lambda|_{\mathring{g}} \lesssim |\hat{\kappa}_\lambda|_{\mathring{g}} + |\psi_\lambda|_{\mathring{g}}.$$

By the definition of the $\mathring{g}$-norm and the energy coercivity in Lemma 9.3.1, one has $|\hat{\kappa}|_{\mathring{g}} + |\psi|_{\mathring{g}} \lesssim \tau^{\frac{1}{2}}\mathcal{E}^{1/2}_{\lambda,high}(t)$ for $\tau \geq \mathring{\tau}$, thus the bound $|\operatorname{tr}\hat{\eta}_\lambda| \lesssim \tau^{-\frac{3}{2}}\mathcal{E}^{1/2}_{\lambda,high}(t)$ follows.

The same estimate for $|\nu_\lambda|$ follows from the elliptic equation for the lapse (9.2.8); in the $\tau \geq \mathring{\tau}$ regime this equation tells us that $|\nu_\lambda| \leq |\operatorname{tr}\hat{\eta}_\lambda|$. This completes the proof of (9.3.7).

For (9.3.8), we write (7.2.14) in the tensorial form.

$$\tau^2\chi^j_\lambda = t^2\mathring{g}^{ij}\left[\mathrm{i}\lambda_i + 2\mathrm{i}(t\mathring{k}\otimes\lambda)_i\right]\nu_\lambda + 4t^2\mathring{g}^{pq}(\mathrm{i}\lambda)_q(t\mathring{k}\otimes\hat{\eta}_\lambda)_p{}^j - 2t^2\mathring{g}^{ij}(\mathrm{i}\lambda)_i\cdot\left[\operatorname{tr}\left(t\mathring{k}\otimes\hat{\eta}_\lambda\right) + 2p_\phi\phi_\lambda\right].$$

Using Lemmas 9.3.3 and 9.3.4, we therefore find that

$$\tau^2|\chi^j_\lambda|_{\mathring{g}} \lesssim \tau t\left(|\nu_\lambda| + |\hat{\eta}_\lambda|_{\mathring{g}} + |\phi_\lambda|\right).$$

Using (9.3.7) to estimate $|\nu_\lambda|$ and Lemma 9.3.1 to get $|\hat{\eta}_\lambda|_{\mathring{g}} + |\psi_\lambda| \lesssim \tau^{-\frac{1}{2}}\mathcal{E}^{1/2}_{\lambda,high}$, the bound (9.3.8) follows. This completes the high-frequency elliptic estimates. □

9.3.3 Estimates for the matter terms

Here, we prove the high-frequency energy estimate for the matter, which is analogous to the proof of Proposition 8.3.2, except for an argument to deal with the lapse ν_λ.

Proposition 9.3.6. *For $((\hat{\eta}_\lambda)_i{}^j, (\hat{\kappa}_\lambda)_i{}^j, \phi_\lambda, \psi_\lambda, \nu_\lambda, \chi_\lambda)$ as in Proposition 9.3.2, the following derivative estimate holds for $\mathcal{E}_{\lambda,\phi,high}(t)$, uniformly in $\lambda \in \mathbb{Z}^D$ and $\tau(t) \geq \mathring{\tau}$:*

$$\left|t\frac{d}{dt}\mathcal{E}_{\lambda,\phi,high}(t)\right| \lesssim \left(t|\zeta'(t)| + \frac{1}{\tau}\right)\mathcal{E}_{\lambda,high}(t). \tag{9.3.9}$$

Proof. Following the proof of Proposition 8.3.2, we compute $t\frac{d}{dt}\mathcal{E}_{\lambda,\phi,high}$ using (9.2.2)–(9.2.16) as follows:

$$t\frac{d}{dt}\mathcal{E}_{\lambda,\phi,high} = \zeta^2\frac{2\psi_\lambda(-\tau^2\phi_\lambda)}{\tau} - \zeta^2 p_\phi\frac{2\psi_\lambda\nu_\lambda}{\tau} - \zeta\frac{\psi_\lambda^2}{\tau} \tag{9.3.10a}$$

$$+ \zeta\frac{\phi_\lambda(-\tau^2\phi_\lambda)}{\tau} + \zeta\frac{\psi_\lambda^2}{\tau} + \zeta p_\phi\frac{(\psi_\lambda-\phi_\lambda)\nu_\lambda}{\tau} - \frac{\psi_\lambda\phi_\lambda}{\tau} \tag{9.3.10b}$$

$$+ \frac{\phi_\lambda\psi_\lambda}{\tau} + p_\phi\frac{\phi_\lambda\nu_\lambda}{\tau} - \zeta^{-1}\frac{\phi_\lambda^2}{2\tau} \tag{9.3.10c}$$

$$+ 2\zeta^2\tau\phi_\lambda\psi_\lambda + 2\zeta^2 p_\phi\tau\phi_\lambda\nu_\lambda + \zeta\tau\phi_\lambda^2 \tag{9.3.10d}$$

$$+ t\zeta'(t)\left(2\zeta\frac{\psi_\lambda^2}{\tau} + \frac{\psi_\lambda\phi_\lambda}{\tau} + 2\zeta\tau\phi_\lambda^2\right). \tag{9.3.10e}$$

We now exploit the same cancellations as in the proof of Proposition 8.3.2. The difference is that we now have some additional terms related to the linearized lapse ν_λ. After the cancellations, one has

$$t\frac{d}{dt}\mathcal{E}_{\lambda,high} = -\zeta^{-1}\frac{\phi_\lambda^2}{2\tau} + (9.3.10\mathrm{e}) + \mathfrak{E}_\nu, \qquad |\mathfrak{E}_\nu| \lesssim \frac{|\psi_\lambda\nu_\lambda|}{\tau} + \tau|\phi_\lambda\nu_\lambda|.$$

As before, it is straightforward to bound (9.3.10e) by $t|\zeta'(t)|\mathcal{E}_{\lambda,high}$, while Proposition 9.3.5 and Lemma 9.3.1 allow us to estimate the final term $\mathfrak{E}_\nu$ by

$$|\mathfrak{E}_\nu| \lesssim \frac{1}{\tau}\left(\frac{\psi_\lambda^2}{\tau} + \tau\phi_\lambda^2\right) + \tau^2\nu_\lambda^2 \lesssim \frac{1}{\tau}\mathcal{E}_{\lambda,high}.$$

Combining all of these, we deduce the derivative estimate (9.3.9). □

9.3.4 Estimates for the metric terms

Next, we prove the high frequency energy estimate for the metric quantities $(\hat{\eta}_i{}^j, \hat{\kappa}_i{}^j)$, which is similar to Proposition 9.3.2 but crucially uses the CMCSH gauge and cancellations due to symmetry via (9.2.15)–(9.2.16).

Proposition 9.3.7. *For $((\hat{\eta}_\lambda)_i{}^j, (\hat{\kappa}_\lambda)_i{}^j, \phi_\lambda, \psi_\lambda, \nu_\lambda, \chi_\lambda)$ as in Proposition 9.3.2, the following derivative estimate holds for $\mathcal{E}_{\lambda,\eta,high}(t)$, uniformly in $\lambda \in \mathbb{Z}^D$ and $\tau(t) \geq \mathring{\tau}$:*

$$\left|t\frac{d}{dt}\mathcal{E}_{\lambda,\eta,high}(t)\right| \lesssim \left(t|\zeta'(t)| + \frac{1}{\tau}\right)\mathcal{E}_{\lambda,high}(t). \tag{9.3.11}$$

Proof. Using the system (9.2.2)–(9.2.16), we calculate the derivative $t\frac{d}{dt}\mathcal{E}_{\lambda,\eta,high}$ as follows, where we group separately the main terms involving $((\hat{\eta}_\lambda)_i{}^j, (\hat{\kappa}_\lambda)_i{}^j)$, the terms involving the lapse ν_λ and the shift χ_λ^j, and finally the terms where the derivative falls on ζ.

$$t\frac{d}{dt}\mathcal{E}_{\lambda,\eta,high} = \frac{2\zeta^2}{\tau}(\hat{\kappa}_\lambda)_j{}^i(-\tau^2(\hat{\eta}_\lambda)_i{}^j) - \frac{\zeta}{\tau}(\hat{\kappa}_\lambda)_j{}^i(\hat{\kappa}_\lambda)_i{}^j \tag{9.3.12a}$$

$$+ \frac{2\zeta^2}{\tau}(\hat{\kappa}_\lambda)_j{}^i\left[t^2\mathring{g}^{jk}\lambda_i\lambda_k - t\mathring{k}_i{}^j\right]\nu_\lambda + \frac{2\zeta^2}{\tau}(\hat{\kappa}_\lambda)_j{}^i\left[-\mathrm{i}(t\mathring{k}_a{}^j)\lambda_i\chi_\lambda^a + \mathrm{i}(t\mathring{k}_i{}^a)\lambda_a\chi_\lambda^j\right] \tag{9.3.12b}$$

$$+ \frac{\zeta}{\tau}(\hat{\eta}_\lambda)_j{}^i(-\tau^2(\hat{\eta}_\lambda)_i{}^j) - \frac{1}{\tau}(\hat{\kappa}_\lambda)_j{}^i(\hat{\eta}_\lambda)_i{}^j \tag{9.3.12c}$$

$$+ \frac{\zeta}{\tau}(\hat{\eta}_\lambda)_j{}^i\left[t^2\mathring{g}^{jk}\lambda_i\lambda_k - t\mathring{k}_i{}^j\right]\nu_\lambda + \frac{\zeta}{\tau}(\hat{\eta}_\lambda)_j{}^i\left[-\mathrm{i}(t\mathring{k}_a{}^j)\lambda_i\chi_\lambda^a + \mathrm{i}(t\mathring{k}_i{}^a)\lambda_a\chi_\lambda^j\right] \tag{9.3.12d}$$

$$+ \frac{\zeta}{\tau}(\hat{\kappa}_\lambda)_j{}^i(\hat{\kappa}_\lambda)_i{}^j + \frac{\zeta}{\tau}(\hat{\kappa}_\lambda)_j{}^i\left[(2t\mathring{k}_p{}^j)(\hat{\eta}_\lambda)_i{}^p - (2t\mathring{k}_i{}^p)(\hat{\eta}_\lambda)_p{}^j\right] \tag{9.3.12e}$$

$$+ \frac{\zeta}{\tau}(\hat{\kappa}_\lambda)_j{}^i(t\mathring{k}_i{}^j)\nu_\lambda + \frac{\zeta}{\tau}(\hat{\kappa}_\lambda)_j{}^i\left[\frac{\mathrm{i}}{2}\lambda_i\chi_\lambda^j + \frac{\mathrm{i}}{2}\mathring{g}_{ip}\mathring{g}^{jq}\lambda_q\chi_\lambda^p\right] \tag{9.3.12f}$$

$$+ \left(\frac{1}{\tau} + 2\zeta^2\tau\right)(\hat{\eta}_\lambda)_j{}^i(\hat{\kappa}_\lambda)_i{}^j + \left(\frac{1}{\tau} + 2\zeta^2\tau\right)(\hat{\eta}_\lambda)_j{}^i\left[(2t\mathring{k}_p{}^j)(\hat{\eta}_\lambda)_i{}^p - (2t\mathring{k}_i{}^p)(\hat{\eta}_\lambda)_p{}^j\right] \tag{9.3.12g}$$

$$+ \left(-\frac{1}{2\zeta\tau} + \zeta\tau\right)(\hat{\eta}_\lambda)_j{}^i(\hat{\eta}_\lambda)_i{}^j \tag{9.3.12h}$$

$$+ \left(\frac{1}{\tau} + 2\zeta^2\tau\right)(\hat{\eta}_\lambda)_j{}^i(t\mathring{k}_i{}^j)\nu_\lambda + \left(\frac{1}{\tau} + 2\zeta^2\tau\right)(\hat{\eta}_\lambda)_j{}^i\left[\frac{\mathrm{i}}{2}\lambda_i\chi_\lambda^j + \frac{\mathrm{i}}{2}\mathring{g}_{ip}\mathring{g}^{jq}\lambda_q\chi_\lambda^p\right] \tag{9.3.12i}$$

$$+ t\zeta'\left[\frac{2\zeta}{\tau}(\hat{\kappa}_\lambda)_j{}^i(\hat{\kappa}_\lambda)_i{}^j + \frac{1}{\tau}(\hat{\kappa}_\lambda)_j{}^i(\hat{\eta}_\lambda)_i{}^j + 2\zeta\tau(\hat{\eta}_\lambda)_j{}^i(\hat{\eta}_\lambda)_i{}^j\right]. \tag{9.3.12j}$$

The terms (9.3.12a), (9.3.12c), (9.3.12e), (9.3.12g) and (9.3.12h) are what we deem the *main terms*. They feature the following crucial cancellations:

- The first term in (9.3.12a) cancels the part of the first term in (9.3.12g) featuring $2\zeta^2\tau$.
- The second term in (9.3.12a) cancels the first term in (9.3.12e).
- The first term in (9.3.12c) cancels the part of (9.3.12h) featuring $2\zeta^2\tau$.
- The second term in (9.3.12c) cancels the part of the first term in (9.3.12g) featuring $\frac{1}{\tau}$.
- We leave the remaining term in (9.3.12e) alone, while we notice upon relabelling indices, the remaining term in (9.3.12g) cancels itself. In other words,

$$(\hat{\eta}_\lambda)_j{}^i\left[(2t\mathring{k}_p{}^j)(\hat{\eta}_\lambda)_i{}^p-(2t\mathring{k}_i{}^p)(\hat{\eta}_\lambda)_p{}^j\right]=\mathrm{tr}\Big(\hat{\eta}_\lambda\cdot 2t\mathring{k}\cdot\hat{\eta}_\lambda\Big)-\mathrm{tr}\Big(2t\mathring{k}\cdot\hat{\eta}_\lambda\cdot\hat{\eta}_\lambda\Big)=0$$

Upon realizing these cancellations, we rewrite the sum of the main terms as follows:

$$\begin{aligned}&(9.3.12a)+(9.3.12c)+(9.3.12e)+(9.3.12g)+(9.3.12h)\\&\qquad=-\frac{1}{2\zeta\tau}(\hat{\eta}_\lambda)_j{}^i(\hat{\eta}_\lambda)_i{}^j+\frac{\zeta}{\tau}(\hat{\kappa}_\lambda)_j{}^i\left[(2t\mathring{k}_p{}^j)(\hat{\eta}_\lambda)_i{}^p-(2t\mathring{k}_i{}^p)(\hat{\eta}_\lambda)_p{}^j\right]\end{aligned}$$

and hence using Lemmas 9.3.3 and 9.3.4 as well as the energy coercivity in Lemma 9.3.1, one has

$$|(9.3.12a)+(9.3.12c)+(9.3.12e)+(9.3.12g)+(9.3.12h)|\lesssim\frac{1}{\tau}\mathcal{E}_{\lambda,\mathfrak{n},high}.\tag{9.3.13}$$

We now move onto the terms (9.3.12b), (9.3.12d), (9.3.12f) and (9.3.12i), which are the *terms involving the lapse and the shift*. In order to estimate these terms, we will have to use the linearized momentum constraint (9.2.12) as well as the linearized CMCSH gauge condition (9.2.1).

- We start with the first term of (9.3.12b). Using the linearized momentum constraint (9.2.12),

$$\frac{1}{\tau}(\hat{\kappa}_\lambda)_j{}^i t^2\mathring{g}^{jk}\lambda_i\lambda_k\nu_\lambda=-\tau[t\mathring{k}_b{}^a(\hat{\eta}_\lambda)_a{}^b+2p_\phi\phi_\lambda]\nu_\lambda+\frac{1}{\tau}t^2\mathring{g}^{jk}(t\mathring{k}_i{}^j)\lambda_j\lambda_k(\mathrm{tr}\,\hat{\eta}_\lambda)\nu_\lambda.$$

  We can now estimate this using Lemmas 9.3.3, 9.3.4 and the elliptic estimates in Proposition 9.3.5. The result is the following:

$$\left|\frac{2\zeta^2}{\tau}(\hat{\kappa}_\lambda)_j{}^i t^2\mathring{g}^{jk}\lambda_i\lambda_k\nu_\lambda\right|\lesssim\tau\cdot(|(\hat{\eta}_\lambda)_i{}^j|_{\mathring{g}}+|\phi|)\cdot|\nu_\lambda|\lesssim\frac{1}{\tau}\mathcal{E}_{\lambda,high}.$$

  The remaining terms of (9.3.12b) can be estimated immediately using the $\mathring{g}$-norm and the elliptic estimates of Proposition 9.3.5. As an illustrative example, we have

$$\left|\frac{2\zeta^2}{\tau}(\hat{\kappa}_\lambda)_j{}^i\mathrm{i}(t\mathring{k}_a{}^j)\lambda_i\chi_\lambda^a\right|\lesssim\frac{1}{\tau}\cdot|(\hat{\kappa}_\lambda)_i{}^j|_{\mathring{g}}\cdot|\lambda_i|_{\mathring{g}}\cdot|\chi_\lambda^j|_{\mathring{g}}\lesssim\frac{1}{\tau}\mathcal{E}_{\lambda,high}.$$

  At the end, we find the estimate $|(9.3.12b)|\lesssim\frac{1}{\tau}\mathcal{E}_{\lambda,high}$.

- The expressions (9.3.12d) and (9.3.12f) can be estimated immediately using the $\mathring{g}$-norm lemmas and Proposition 9.3.5. Omitting the details, one finds that

$$|(9.3.12\text{d})| + |(9.3.12\text{f})| \lesssim \frac{1}{\tau}\mathcal{E}_{\lambda,high}.$$

- The first term in (9.3.12i) can be estimated immediately using

$$\left|\left(\frac{1}{\tau} + 2\zeta^2\tau\right)(\hat{\eta}_\lambda)_j{}^i(t\mathring{k}_i{}^j)\nu_\lambda\right| \lesssim \tau \cdot |(\hat{\eta}_\lambda)_i{}^j|_{\mathring{g}} \cdot |\nu_\lambda| \lesssim \frac{1}{\tau}\mathcal{E}_{\lambda,high}.$$

  For the second term in (9.3.12i), we use the symmetry property (9.2.15) and the linearized CMCSH condition (9.2.1) to rewrite this term as

$$\left(\frac{1}{\tau} + 2\zeta^2\tau\right)(\hat{\eta}_\lambda)_j{}^i\left[\frac{\mathrm{i}}{2}\lambda_i\chi_\lambda^j + \frac{\mathrm{i}}{2}\mathring{g}_{ip}\mathring{g}^{jq}\lambda_q\chi_\lambda^p\right] = \left(\frac{1}{\tau} + 2\zeta^2\tau\right)\frac{\mathrm{i}}{2}\lambda_j(\operatorname{tr}\hat{\eta}_\lambda)\chi_\lambda^j.$$

  The insight is that from (9.3.7), the trace $\operatorname{tr}\hat{\eta}_\lambda$, behaves better than $|(\hat{\eta}_\lambda)_i{}^j|_{\mathring{g}}$, indeed

$$\left|\left(\frac{1}{\tau} + 2\zeta^2\tau\right)\frac{\mathrm{i}}{2}\lambda_j(\operatorname{tr}\hat{\eta}_\lambda)\chi_\lambda^j\right| \lesssim \tau \cdot |\lambda_i|_{\mathring{g}} \cdot |\chi_\lambda^j|_{\mathring{g}} \cdot |\operatorname{tr}\hat{\eta}_\lambda| \lesssim \frac{1}{\tau}\mathcal{E}_{\lambda,high}.$$

  The conclusion is that $|(9.3.12\text{i})| \lesssim \frac{1}{\tau}\mathcal{E}_{\lambda,high}$.

Combining all of the above, we estimate the lapse and shift terms using

$$|(9.3.12\text{b}) + (9.3.12\text{d}) + (9.3.12\text{f}) + (9.3.12\text{i})| \lesssim \frac{1}{\tau}\mathcal{E}_{\lambda,high}. \tag{9.3.14}$$

Finally, the final term (9.3.12j) is estimated trivially using the bounds (8.2.5) and Lemma 9.3.1:

$$|(9.3.12\text{i})| \lesssim t|\zeta'(t)|\mathcal{E}_{\lambda,\eta,high}. \tag{9.3.15}$$

To complete the proof, simply combine the 3 estimates (9.3.13), (9.3.14) and (9.3.15). □

9.4 The low-frequency energy estimate

In this section, we prove a low-frequency energy estimate analogous to that derived in Proposition 8.4.1 for the Fourier modes of the linearized Einstein–scalar field system derived in Proposition 9.2.1. Since $\eta_i{}^j$ and ϕ generically blow up as $t \to 0$, at low frequency we instead consider (Fourier modes of) renormalized quantities $\tilde{\Upsilon}_i{}^j$ and $\tilde{\varphi}$, as defined in (7.2.26).

To be more precise, we pick $T = t_{\lambda *}$ in (7.2.26), and thus define the renormalized quantities $(\tilde{\Upsilon}_\lambda)_i{}^j$ and $\tilde{\varphi}_\lambda$ in the following way:

$$(\tilde{\Upsilon}_\lambda)_i{}^j(t) = (\eta_\lambda)_i{}^j + G_{ip}{}^{jq}(t;t_{\lambda *})\cdot(\kappa_\lambda)_q{}^p, \quad G_{ip}{}^{jq}(t;t_{\lambda *}) \coloneqq \int_t^{t_{\lambda *}} \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\frac{ds}{s}, \tag{9.4.1}$$

$$\tilde{\varphi}_\lambda = \phi_\lambda + \log\left(\frac{t_{\lambda *}}{t}\right)\cdot\psi_\lambda. \tag{9.4.2}$$

Moreover, in contrast to the CMCSH gauge used in the high-frequency energy estimates of Section 9.3, for the low-frequency estimates we shall *use a CMCTC* gauge, i.e. a gauge where the linearized shift χ^j vanishes. We firstly rewrite the evolution equations (9.2.2)–(9.2.7) with respect to our renormalized $\tilde{\Upsilon}_i^{\ j}$, $\tilde{\varphi}$ and assuming the gauge.

Lemma 9.4.1. *In a CMCTC gauge, the $t\partial_t$-derivatives of the quantities $((\kappa_\lambda)_i^{\ j}, (\tilde{\Upsilon}_\lambda)_i^{\ j}, \psi_\lambda, \tilde{\varphi}_\lambda)$ can be rewritten as follows:*

$$\begin{aligned} t\partial_t(\kappa_\lambda)_i^{\ j} = &-t^2\mathring{g}^{ab}\lambda_a\lambda_b\left((\tilde{\Upsilon}_\lambda)_i^{\ j} - G_{ip}^{\ \ jq}(\kappa_\lambda)_q^{\ p}\right) - t^2\mathring{g}^{ja}\lambda_i\lambda_a(\operatorname{tr}\tilde{\Upsilon}_\lambda) + t^2\mathring{g}^{jk}\lambda_i\lambda_k\nu_\lambda - (t\mathring{k}_i^{\ j})\nu_\lambda \\ &+t^2\mathring{g}^{ab}\lambda_i\lambda_b\left((\tilde{\Upsilon}_\lambda)_a^{\ j} - G_{ap}^{\ \ jq}(\kappa_\lambda)_q^{\ p}\right) + t^2\mathring{g}^{ja}\lambda_a\lambda_b\left((\tilde{\Upsilon}_\lambda)_i^{\ b} - G_{ip}^{\ \ bq}(\kappa_\lambda)_q^{\ p}\right), \end{aligned} \tag{9.4.3}$$

$$t\partial_t(\tilde{\Upsilon}_\lambda)_i^{\ j} = G_{ip}^{\ \ jq}\cdot t\partial_t(\kappa_\lambda)_q^{\ p} + (t\mathring{k}_i^{\ j})\nu_\lambda, \tag{9.4.4}$$

$$t\partial_t\psi_\lambda = -t^2\mathring{g}^{ab}\lambda_a\lambda_b\left(\tilde{\varphi}_\lambda - \log\left(\frac{t_{\lambda*}}{t}\right)\psi_\lambda\right) - p_\phi\nu_\lambda, \tag{9.4.5}$$

$$t\partial_t\tilde{\varphi}_\lambda = \log\left(\frac{t_{\lambda*}}{t}\right)\cdot t\partial_t\psi_\lambda + p_\phi\nu_\lambda. \tag{9.4.6}$$

Proof. This follows from direct computation using the equations (9.2.2)–(9.2.7), as well as the symmetry conditions (9.2.15)–(9.2.16). □

9.4.1 The low-frequency energy quantity

Definition 9.4.1. The *geometric low-frequency energy quantity* $\mathcal{E}_{\lambda,\mathfrak{n},low}(t)$, used for $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $\tau_\lambda(t) \leq 1$, is given by the following:

$$\mathcal{E}_{\lambda,\mathfrak{n},low}(t) \coloneqq \mathring{g}^{ac}(t_{\lambda*})\mathring{g}_{bd}(t_{\lambda*})\,(\kappa_\lambda)_a^{\ b}(\kappa_\lambda)_c^{\ d} + \mathring{g}^{ac}(t_{\lambda*})\mathring{g}_{bd}(t_{\lambda*})\,(\tilde{\Upsilon}_\lambda)_a^{\ b}(\tilde{\Upsilon}_\lambda)_c^{\ d}. \tag{9.4.7}$$

We also define the *matter low-frequency energy quantity* $\mathcal{E}_{\lambda,\mathfrak{h},low}(t)$, as follows:

$$\mathcal{E}_{\lambda,\mathfrak{h},low}(t) \coloneqq \psi_\lambda^2 + \tilde{\varphi}_\lambda^2. \tag{9.4.8}$$

Finally, define the *total low-frequency energy quantity* as the sum $\mathcal{E}_{\lambda,low}(t) \coloneqq \mathcal{E}_{\lambda,\mathfrak{n},low}(t) + \mathcal{E}_{\lambda,\mathfrak{h},low}(t)$.

Remark. Unlike the geometric high-frequency energy quantity defined in (9.3.1), the metric $\mathring{g}_{ij}$ used as the inner product on the tensors $(\kappa_\lambda)_i^{\ j}$ and $(\tilde{\Upsilon}_\infty)_i^{\ j}$ is $\mathring{g}_{ij}(t_{\lambda*})$, rather than $\mathring{g}_{ij}(t)$. The fact that this inner product is no longer t-dependent is crucial in order to show that the tensors $(\kappa_\lambda)_i^{\ j}$ and $(\tilde{\Upsilon}_\lambda)_i^{\ j}$ are indeed bounded towards $t = 0$. This also indicates the operators $\mathcal{T}_*^{-p_i+p_j}$ arise in Theorem 7.3.1.

Another remark is that though the high-frequency energy is defined for $\tau_\lambda(t) \geq 1$, the relevant energy estimates are only valid for $\tau_\lambda(t) \geq \mathring{\tau} \geq 1$, see Lemma 9.3.1 and Proposition 9.3.2. In order to eventually compare the high-frequency energy to the low-frequency energy of Definition 9.4.1, we will require *mid-frequency energy estimates*, valid for $1 \leq \tau_\lambda(t) \leq \mathring{\tau}$, see Section 9.4.4.

The aim of this section will be to prove the following low-frequency energy estimate, see the analogous Proposition 8.4.1 in the case of the wave equation. Unlike the high-frequency energy estimate Proposition 9.3.2, the following will depend crucially on the subcriticality condition (1.2.11).

Proposition 9.4.2. *Let $((\eta_\lambda)_i^{\ j}, (\kappa_\lambda)_i^{\ j}, \phi_\lambda, \psi_\lambda, \nu_\lambda)$ be a solution to the linearized Einstein–scalar field system* (9.2.2)–(9.2.16) *in CMCTC gauge. Suppose the background Kasner exponents p_i obey the subcriticality condition* (1.2.11) *for some $\delta > 0$. Then the ∂_t-derivative of the total low-frequency energy quantity obeys the following bound, uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $0 < \tau_\lambda(t) \leq 1$:*

$$\left| t \frac{d}{dt} \mathcal{E}_{\lambda,low}(t) \right| \lesssim \left(\frac{t}{t_{\lambda *}} \right)^{\delta} \mathcal{E}_{\lambda,low}(t). \tag{9.4.9}$$

There exists some $\mathring{C}_{low}$, depending only on the background Kasner spacetime, such that for $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $0 < \tau_\lambda(t) \leq 1$, one has

$$\mathring{C}_{low}^{-1} \mathcal{E}_{\lambda,low}(t) \leq \mathcal{E}_{\lambda,low}(t_{\lambda *}) \leq \mathring{C}_{low}\, \mathcal{E}_{\lambda,low}(t). \tag{9.4.10}$$

Proof (assuming Proposition 9.4.4). The derivative estimate (9.4.9) follows immediately from Proposition 9.4.4 and the definition of $\mathcal{E}_{\lambda,low}(t)$ upon insertion of the expressions $\mathring{g}_{ij}(t_{\lambda *}) = t_{\lambda *}^{2p_{\underline{i}}} \delta_{\underline{i}j}$ and $\mathring{g}^{ij}(t_{\lambda *}) = t_{\lambda *}^{-2p_{\underline{i}}} \delta^{\underline{i}j}$.

To deduce (9.4.10), we make the simple observation that

$$\int_0^{t_{\lambda *}} \left(\frac{s}{t_{\lambda *}} \right)^{\delta} \frac{ds}{s} \leq \delta^{-1},$$

thus applying Grönwall's inequality to (9.4.9) in both the forwards and backwards directions yields that for an $0 < t_1 \leq t_2 \leq t_{\lambda *}$, we have (where C is the implied constant in (9.4.9)):

$$\exp\bigl(-C\delta^{-1}\bigr) \mathcal{E}_{\lambda,low}(t_1) \leq \mathcal{E}_{\lambda,low}(t_2) \leq \exp\bigl(C\delta^{-1}\bigr) \mathcal{E}_{\lambda,low}(t_1).$$

The energy estimate (9.4.10) then follows upon choosing $t_2 = t_{\lambda *}$ and $\mathring{C}_{low} = C\delta^{-1}$. □

In Section 9.4.2, we prove Proposition 9.4.4. We shall also use Proposition 9.4.4 to produce limiting quantities $((\kappa_\infty)_\lambda)_i^{\ j}$, $((\tilde{\Upsilon}_\infty)_\lambda)_i^{\ j}$, $(\psi_\infty)_\lambda$, $(\tilde{\varphi}_\infty)_\lambda)$ in Proposition 9.4.5.

9.4.2 Derivative estimates using the subcriticality condition

In order to prove the energy estimate in Proposition 9.4.2, we will have to estimate the powers of t arising in the evolution equations of Lemma 9.4.1. The first step will be to estimate the integral quantity $G_{ip}^{\ \ jq}$.

Lemma 9.4.3. *Consider the tensor $G_{ip}^{\ \ jq}$ defined in* (9.4.1). *It satisfies the bound*

$$\begin{aligned} \left| G_{ip}^{\ \ jq}(t; t_{\lambda *}) \right| &\leq \max\left\{ \mathring{g}_{ip}(t) \mathring{g}^{jq}(t), \mathring{g}_{ip}(t_{\lambda *}) \mathring{g}^{jq}(t_{\lambda *}) \right\} \cdot \log\left(\frac{t_{\lambda *}}{t} \right) \\ &= \max\left\{ \left(\frac{t}{t_{\lambda *}} \right)^{2p_{\underline{i}} - 2p_{\underline{j}}}, 1 \right\} \cdot t_{\lambda *}^{2p_{\underline{i}} - 2p_{\underline{j}}} \cdot \log\left(\frac{t_{\lambda *}}{t} \right) \cdot \delta_{\underline{i}p}\, \delta^{\underline{j}q}. \end{aligned} \tag{9.4.11}$$

Proof. One simply notes that for any i,p,j,q, the quantity $\mathring{g}_{ip}(s)\mathring{g}^{jq}(s)$ is either monotonically increasing or monotonically decreasing in $s \in [t, t_{\lambda *}]$, depending on whether $p_i \leq p_j$ or $p_i \geq p_j$. Therefore

$$G_{ip}{}^{jq} = \int_t^{t_{\lambda *}} \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\frac{ds}{s} \leq \max\left\{\mathring{g}_{ip}(t)\mathring{g}^{jq}(t), \mathring{g}_{ip}(t_{\lambda *})\mathring{g}^{jq}(t_{\lambda *})\right\} \cdot \int_t^{t_{\lambda *}} \frac{ds}{s},$$

from which the first line in (9.4.11) follows immediately. The second line follows upon inserting (Kas). □

We now use this lemma, along with the evolution equations in Lemma 9.4.1 to show that the time derivatives of our matter and metric quantities are integrable towards $t = 0$. The subcriticality condition (1.2.11) will be crucial in order to produce a power of t that is integrable.

Proposition 9.4.4. *For $((\eta_\lambda)_i{}^j, (\kappa_\lambda)_i{}^j, \phi_\lambda, \psi_\lambda, \nu_\lambda)$ as in Proposition 9.4.2, define $(\tilde{\Upsilon}_\lambda)_i{}^j$ and $\tilde{\varphi}_\lambda$ as in* (9.4.1), (9.4.2). *Then given the subcriticality condition* (1.2.11) *on the background Kasner exponents, we have the following derivative estimates uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $0 < \tau_\lambda(t) \leq 1$.*

$$t_{\lambda *}^{-p_i+p_j}|t\partial_t(\kappa_\lambda)_i{}^j| + t_{\lambda *}^{-p_i+p_j}|t\partial_t(\tilde{\Upsilon}_\lambda)_i{}^j| + |t\partial_t\psi_\lambda| + |t\partial_t\tilde{\varphi}_\lambda| \lesssim \left(\frac{t}{t_{\lambda *}}\right)^\delta \cdot \mathcal{E}_{\lambda,low}^{1/2}(t). \tag{9.4.12}$$

Proof. Let us start with $|t\partial_t(\kappa_\lambda)_i{}^j|$. We will bound this by grouping the terms in (9.4.3) as follows:

$$|t\partial_t(\kappa_\lambda)_i{}^j| = -t^2\mathring{g}^{ab}\lambda_a\lambda_b(\tilde{\Upsilon}_\lambda)_i{}^j + t^2\mathring{g}^{ab}\lambda_i\lambda_b(\tilde{\Upsilon}_\lambda)_a{}^j + t^2\mathring{g}^{ja}\lambda_a\lambda_b(\tilde{\Upsilon}_\lambda)_i{}^b \tag{9.4.13a}$$

$$+ G_{ip}{}^{jq}t^2\mathring{g}^{ab}\lambda_a\lambda_b(\kappa_\lambda)_q{}^p - G_{ap}{}^{jq}t^2\mathring{g}^{ab}\lambda_i\lambda_b(\kappa_\lambda)_q{}^p - G_{ip}{}^{bq}t^2\mathring{g}^{ja}\lambda_a\lambda_b(\kappa_\lambda)_q{}^p \tag{9.4.13b}$$

$$- t^2\mathring{g}^{ja}\lambda_i\lambda_a(\operatorname{tr}\tilde{\Upsilon}) + t^2\mathring{g}^{jk}\lambda_i\lambda_k\nu_\lambda - (t\mathring{k}_i{}^j)\nu_\lambda. \tag{9.4.13c}$$

We shall estimate the terms occuring in (9.4.13a)–(9.4.13c) in terms of the low-frequency energy $\mathcal{E}_{\lambda,low}(t)$. By the definition of the low-frequency energy, we have that

$$\left|(\kappa_\lambda)_i{}^j\right| \lesssim t_{\lambda *}^{p_i-p_j}\mathcal{E}_{\lambda,low}^{1/2}, \qquad \left|(\tilde{\Upsilon}_\lambda)_i{}^j\right| \lesssim t_{\lambda *}^{p_i-p_j}\mathcal{E}_{\lambda,low}^{1/2}, \qquad \left|\operatorname{tr}\tilde{\Upsilon}\right| \lesssim \mathcal{E}_{\lambda,low}^{1/2}. \tag{9.4.14}$$

On the other hand, we estimate λ_i using (8.2.3) and $0 < \tau^2 = \sum_i t^{2-2p_i}\lambda_i^2 \leq 1$ as follows

$$|\lambda_i| \leq t_{\lambda *}^{p_i-1}. \tag{9.4.15}$$

Using these estimates and (Kas), it is straightforward to then estimate (9.4.13a) in the following manner.

$$\begin{aligned} t_{\lambda *}^{-p_i+p_j} \cdot |(9.4.13a)| &\lesssim \sum_{a=1}^D \left(\frac{t}{t_{\lambda *}}\right)^{2-2p_a} \cdot \left[t_{\lambda *}^{-p_i+p_j}\left|(\tilde{\Upsilon}_\lambda)_i{}^j\right| + t_{\lambda *}^{-p_a+p_j}\left|(\tilde{\Upsilon}_\lambda)_a{}^j\right| + t_{\lambda *}^{-p_i+p_a}\left|(\tilde{\Upsilon}_\lambda)_i{}^a\right|\right] \\ &\lesssim \max_{a=1,\ldots,D}\left\{\left(\frac{t}{t_{\lambda *}}\right)^{2-2p_a}\right\} \cdot \mathcal{E}_{\lambda,low}^{1/2}. \end{aligned} \tag{9.4.16}$$

For the second line (9.4.13b), we first exploit a crucial cancellation in the first and third terms of (9.4.13b) when $b = j$. In particular, from the cancellation and (9.4.11) one has

$$|G_{ip}{}^{jq}t^2\mathring{g}^{ab} - G_{ip}{}^{bq}t^2\mathring{g}^{ja}| \lesssim \max_{i,j\neq b}\left\{\left(\frac{t}{t_{\lambda*}}\right)^{2+2p_i-2p_j-2p_b}\right\}\log\left(\frac{t_{\lambda*}}{t}\right)\cdot t_{\lambda*}^{2-p_a-p_b-p_q+p_p}.$$

To get this estimate, there are really two cases depending on whether $p_i > p_j$ or $p_i \leq p_j$; mnote that the case where $p_i > p_j$ in (9.4.11) is covered by allowing $i = j$ in the above maximum. Estimating the second term in (9.4.13b) using (9.4.11) is more straightforward – in that the power of t ends up being t^{2-2p_a} or t^{2-2p_j} rather than featuring more complicated linear combinations of Kasner exponents – and one eventually yields

$$t_{\lambda*}^{-p_i+p_j}\cdot|(9.4.13b)| \lesssim \max_{j\neq b}\left\{\left(\frac{t}{t_{\lambda*}}\right)^{2+2p_i-2p_j-2p_{\underline{b}}}\right\}\log\left(\frac{t_{\lambda*}}{t}\right)\cdot\mathcal{E}_{\lambda,low}^{1/2}. \tag{9.4.17}$$

To deal with (9.4.13c), the $\operatorname{tr}\tilde{\Upsilon}_\lambda$ term is straightforward, while for the ν_λ term we may use the elliptic equation (9.2.8) to get the following preliminary estimate, recalling that now $\tau^2 \leq 1$:

$$\begin{aligned}|\nu_\lambda| &\lesssim t^2\mathring{g}^{ab}\lambda_a\lambda_b\left|\operatorname{tr}\tilde{\Upsilon}_\lambda\right| + t^2\mathring{g}^{ab}\lambda_i\lambda_a\left|(\tilde{\Upsilon}_\lambda)_b{}^i\right| + G_{bp}{}^{iq}t^2\mathring{g}^{ab}\lambda_i\lambda_a\left|(\kappa_\lambda)_q{}^p\right| \\ &\lesssim \max_{a=1,\dots,D}\left\{\left(\frac{t}{t_{\lambda*}}\right)^{2-2p_a}\right\}\left(1+\log\left(\frac{t_{\lambda*}}{t}\right)\right)\cdot\mathcal{E}_{\lambda,low}^{1/2}.\end{aligned} \tag{9.4.18}$$

As the final term of (9.4.13c) contributes only when $i = j$, and $t\mathring{k}_i{}^j$ is bounded by $\max p_i$, we therefore have

$$t_{\lambda*}^{-p_i+p_j}\cdot|(9.4.13c)| \lesssim \max_{a=1,\dots,D}\left\{\left(\frac{t}{t_{\lambda*}}\right)^{2-2p_a}\right\}\left(1+\log\left(\frac{t_{\lambda*}}{t}\right)\right)\cdot\mathcal{E}_{\lambda,low}^{1/2}. \tag{9.4.19}$$

Combining all of (9.4.16), (9.4.17) and (9.4.19), we therefore have

$$t_{\lambda*}^{-p_i+p_j}|t\partial_t(\kappa_\lambda)_i{}^j| \lesssim \max_{b\neq c}\left\{\left(\frac{t}{t_{\lambda*}}\right)^{2+2p_a-2p_b-2p_c}\right\}\left(1+\log\left(\frac{t_{\lambda*}}{t}\right)\right)\cdot\mathcal{E}_{\lambda,low}^{1/2}. \tag{9.4.20}$$

We next move onto $|t\partial_t(\tilde{\Upsilon}_\lambda)_i{}^j|$. Using (9.4.4) and the above, we write this as follows:

$$t\partial_t(\tilde{\Upsilon}_\lambda)_i{}^j = -G_{ir}{}^{js}t^2\mathring{g}^{ab}\lambda_a\lambda_b(\tilde{\Upsilon}_\lambda)_s{}^r + G_{ir}{}^{ja}t^2\mathring{g}^{sb}\lambda_a\lambda_b(\tilde{\Upsilon}_\lambda)_s{}^r + G_{ir}{}^{js}t^2\mathring{g}^{ra}\lambda_a\lambda_b(\tilde{\Upsilon}_\lambda)_s{}^b \tag{9.4.21a}$$

$$+ G_{ir}{}^{js}G_{sp}{}^{rq}t^2\mathring{g}^{ab}\lambda_a\lambda_b(\kappa_\lambda)_q{}^p - G_{ir}{}^{ja}G_{sp}{}^{rq}t^2\mathring{g}^{sb}\lambda_a\lambda_b(\kappa_\lambda)_q{}^p - G_{ir}{}^{js}G_{sp}{}^{bq}t^2\mathring{g}^{ra}\lambda_a\lambda_b(\kappa_\lambda)_q{}^p \tag{9.4.21b}$$

$$- G_{ir}{}^{js}t^2\mathring{g}^{ra}\lambda_s\lambda_a(\operatorname{tr}\tilde{\Upsilon}) + G_{ir}{}^{js}t^2\mathring{g}^{rk}\lambda_s\lambda_k\nu_\lambda - G_{ip}{}^{jq}(t\mathring{k}_q{}^p)\nu_\lambda + (t\mathring{k}_i{}^j)\nu_\lambda. \tag{9.4.21c}$$

To estimate (9.4.21a), note that there is a cancellation between the first two terms when $a = s$, in a similar fashion to the cancellation in the estimate of (9.4.13c) above. Tracking the powers of t and $t_{\lambda*}$ carefully, and using (9.4.14) and (9.4.15) as before, one yields

$$t_{\lambda*}^{-p_i+p_j}\cdot|(9.4.21a)| \lesssim \max_{a,b\neq c}\left\{\left(\frac{t}{t_{\lambda*}}\right)^{2+2p_a-2p_b-2p_c}\right\}\left(1+\log\left(\frac{t_{\lambda*}}{t}\right)\right)\cdot\mathcal{E}_{\lambda,low}^{1/2}. \tag{9.4.22}$$

For (9.4.21b), there are two cases depending on whether $p_i \leq p_j$ or $p_i \geq p_j$. If $p_i \leq p_j$, there is a cancellation between the first and the second expression when $a = s$ while the contribution of the third expression is fine. Alternatively if $p_i \geq p_j$, there is a cancellation between the first and the third when $b = r$ while the contribution of the second is fine. Omitting the details, we end up at the following estimate

$$t_{\lambda*}^{-p_i+p_j} \cdot |(9.4.21b)| \lesssim \max_{a,b\neq c} \left\{ \left(\frac{t}{t_{\lambda*}} \right)^{2+2p_a-2p_b-2p_c} \right\} \left(1 + \log\left(\frac{t_{\lambda*}}{t} \right) \right)^2 \cdot \mathcal{E}_{\lambda,low}^{1/2}. \tag{9.4.23}$$

The term involving $\log\left(\frac{t_{\lambda*}}{t}\right)$ appears with a square since (9.4.21b) is quadratic in the ${G_{ab}}^{cd}$ tensor.

Finally, for (9.4.21c), we use again (9.4.18) and the fact that the third and fourth terms are nonzero if and only if $i = j$. Therefore

$$t_{\lambda*}^{-p_i+p_j} \cdot |(9.4.21c)| \lesssim \max_{a=1,\dots,D} \left\{ \left(\frac{t}{t_{\lambda*}} \right)^{2-2p_a} \right\} \left(1 + \log\left(\frac{t_{\lambda*}}{t} \right) \right)^2 \cdot \mathcal{E}_{\lambda,low}^{1/2}. \tag{9.4.24}$$

The square appears here because we must use (9.4.11) on to of (9.4.18) to estimate the third term ${G_{ip}}^{jq}(t\mathring{k}_q{}^p)\nu_\lambda$. Combining the three estimates (9.4.21a), (9.4.21b), (9.4.21c) then yields

$$t_{\lambda*}^{-p_i+p_j} |t\partial_t(\tilde{\Upsilon}_\lambda)_i{}^j| \lesssim \max_{a,b\neq c} \left\{ \left(\frac{t}{t_{\lambda*}} \right)^{2+2p_a-2p_b-2p_c} \right\} \left(1 + \log\left(\frac{t_{\lambda*}}{t} \right) \right)^2 \cdot \mathcal{E}_{\lambda,low}^{1/2}. \tag{9.4.25}$$

(We reiterate that powers of t such as $\left(\frac{t}{t_{\lambda*}}\right)^{2-2p_c}$ are included in (9.4.25) upon taking $a = b$ in the max.)

For the matter quantities ψ_λ and $\tilde{\varphi}_\lambda$, we use (9.4.5) and (9.4.6). Here, it is straightforward to use (9.4.18) as well as the trivial bounds from the definition of the low-frequency energy:

$$|\psi_\lambda| + |\tilde{\varphi}_\lambda| \lesssim \mathcal{E}_{\lambda,low}^{1/2}.$$

Inserting all these bounds into (9.4.5) and (9.4.6), one deduces

$$|t\partial_t\psi_\lambda| + |t\partial_t\tilde{\varphi}_\lambda| \lesssim \max_{a=1,\dots,D} \left\{ \left(\frac{t}{t_{\lambda*}} \right)^{2-2p_a} \right\} \left(1 + \log\left(\frac{t_{\lambda*}}{t} \right) \right)^2 \cdot \mathcal{E}_{\lambda,low}^{1/2}. \tag{9.4.26}$$

Combining all of (9.4.20), (9.4.25) and (9.4.26), one deduces the inequality.

$$t_{\lambda*}^{-p_i+p_j} \left(|t\partial_t(\kappa_\lambda)_i{}^j| + |t\partial_t(\tilde{\Upsilon}_\lambda)_i{}^j| \right) + |t\partial_t\psi_\lambda| + |t\partial_t\tilde{\varphi}_\lambda| \\ \lesssim \max_{a,b\neq c} \left\{ \left(\frac{t}{t_{\lambda*}} \right)^{2+2p_a-2p_b-2p_c} \right\} \left(1 + \log\left(\frac{t_{\lambda*}}{t} \right) \right)^2 \mathcal{E}_{\lambda,low}^{1/2}.$$

At this point that we apply the subcriticality condition (1.2.11). Since, for $b \neq c$, this condition tells us $2 + 2p_a - 2p_b - 2p_c \geq 2\delta$, we may apply the simple inequality $x^\delta(1 + \log x^{-1})^2 \lesssim 1$ for $x = \frac{t}{t_{\lambda*}}$, to yield the desired derivative bound (9.4.12). □

Remark. A more careful study of the proof shows that the only terms which give a contribution of $\left(\frac{t}{t_{\lambda *}}\right)^{2+2p_a-2p_b-2p_c}$ rather than just $\left(\frac{t}{t_{\lambda *}}\right)^{2-2p_c}$ are terms that involve the expression

$$\left(G_{ap}{}^{bq}t^2\mathring{g}^{cd} - G_{ap}{}^{cq}t^2\mathring{g}^{bd}\right)\lambda_c(\mathsf{k}_\lambda)_q{}^p. \tag{9.4.27}$$

Upon relabelling indices, this arises in (9.4.13b), and in the case that $p_i \geq p_j$, we also see it in (9.4.21b). (One may use the symmetry constraint $\mathring{g}^{ab}(t_{\lambda *})\tilde{\Upsilon}_b{}^c = \mathring{g}^{cb}(t_{\lambda *})\tilde{\Upsilon}_b{}^a$ to reduce to this case.)

Therefore, even if the subcriticality condition (1.2.11) fails, it is possible to close the argument if the expression (9.4.27) can be shown to converge to 0 sufficiently fast as $t \to 0$. Upon expanding out $G_{ip}{}^{jq}$ in (9.4.27), one may show that (9.4.27) converging to 0 requires (7.3.11); note however to close the argument one must find the rate of convergence in (7.3.11).

9.4.3 Convergence to a limit

To find the scattering states $((\mathsf{k}_\infty)_i{}^j, (\tilde{\Upsilon}_\infty)_i{}^j, \psi_\infty, \tilde{\varphi}_\infty)$, one must show that $(\mathsf{k}_i{}^j(t), \tilde{\Upsilon}_i{}^j(t), \psi(t), \tilde{\varphi}(t))$ converges as $t \to 0$. Conversely, for asymptotic completeness we also need to show that one can prescribe these limits and launch a solution to the system (9.2.2)–(9.2.16). We achieve this in the following proposition.

Proposition 9.4.5. *For fixed $\lambda \in \mathbb{Z}^D \setminus \{0\}$, and $\mathring{C}_{low} > 0$ as in Proposition 9.4.2, the following hold.*

(i) Let $((\mathsf{p}_\lambda)_i{}^j, (\mathsf{k}_\lambda)_i{}^j, \phi_\lambda, \psi_\lambda, \nu_\lambda)$ be a solution to the system (9.2.2)–(9.2.16) for $0 < \tau(t) \leq 1$ in CMCTC gauge. Then there exist constant $(1,1)$ tensors $((\mathsf{k}_\infty)_\lambda)_i{}^j$, $((\tilde{\Upsilon}_\infty)_\lambda)_i{}^j$ and real numbers $(\psi_\infty)_\lambda$, $(\tilde{\varphi}_\infty)_\lambda$ such that $(\mathsf{k}_\lambda)_i{}^j(t)$, $(\tilde{\Upsilon}_\lambda)_i{}^j(t)$, $\psi_\lambda(t)$ and $\tilde{\varphi}_\lambda(t)$ attain these as limits as $t \to 0$, at the following rate

$$|(\mathsf{k}_\lambda)_i{}^j(t) - ((\mathsf{k}_\infty)_\lambda)_i{}^j| + |(\tilde{\Upsilon}_\lambda)_i{}^j(t) - ((\tilde{\Upsilon}_\infty)_\lambda)_i{}^j| \lesssim t_{\lambda *}^{p_i - p_j} \cdot \left(\frac{t}{t_{\lambda *}}\right)^\delta \mathcal{E}^{1/2}_{\lambda,low}(t_{\lambda *}), \tag{9.4.28}$$

$$|\psi_\lambda(t) - (\psi_\infty)_\lambda| + |\tilde{\varphi}_\lambda(t) - (\tilde{\varphi}_\infty)_\lambda| \lesssim \left(\frac{t}{t_{\lambda *}}\right)^\delta \mathcal{E}^{1/2}_{\lambda,low}(t_{\lambda *}). \tag{9.4.29}$$

Furthermore, one has the following estimate on the limiting quantities:

$$\mathring{g}^{ac}(t_{\lambda *})\mathring{g}_{bd}(t_{\lambda *})\left[((\mathsf{k}_\infty)_\lambda)_a{}^b((\mathsf{k}_\infty)_\lambda)_c{}^d + ((\tilde{\Upsilon}_\infty)_\lambda)_a{}^b((\tilde{\Upsilon}_\infty)_\lambda)_c{}^d\right] + (\psi_\infty)_\lambda^2 + (\tilde{\varphi}_\infty)_\lambda^2 \leq \mathring{C}_{low}\mathcal{E}_{\lambda,low}(t_{\lambda *}). \tag{9.4.30}$$

(ii) Conversely, let $((\mathsf{k}_\infty)_\lambda)_i{}^j$, $((\tilde{\Upsilon}_\infty)_\lambda)_i{}^j$ be constant $(1,1)$ tensors and $(\psi_\infty)_\lambda$ and $(\tilde{\varphi}_\infty)_\lambda$ be real numbers, satisfying the λ-Fourier projections of the asymptotic constraints (7.2.27)–(7.2.31), for $T = t_{\lambda *}$. Then there exists a unique solution $((\mathsf{p}_\lambda)_i{}^j, (\mathsf{k}_\lambda)_i{}^j, \phi_\lambda, \psi_\lambda, \nu_\lambda)$ to the system

(9.2.2)–(9.2.16) *for* $0 < \tau(t) \leq 1$ *in CMCTC gauge such that the solution obeys the asymptotics* (9.4.28)–(9.4.29).

For this solution, one has that

$$\mathcal{E}_{\lambda,low}(t_{\lambda*}) \leq \mathring{C}_{low}\left(\mathring{g}^{ac}(t_{\lambda*})\mathring{g}_{bd}(t_{\lambda*})\left[((\mathsf{\kappa}_\infty)_\lambda)_a{}^b((\mathsf{\kappa}_\infty)_\lambda)_c{}^d + ((\tilde{\Upsilon}_\infty)_\lambda)_a{}^b((\tilde{\Upsilon}_\infty)_\lambda)_c{}^d\right] + (\psi_\infty)_\lambda^2 + (\tilde{\varphi}_\infty)_\lambda^2\right). \quad (9.4.31)$$

Proof. We first prove (i). This follows straightforwardly upon combining Proposition 9.4.2 with Proposition 9.4.4. More specifically, combining (9.4.10) with (9.4.12) yields that for a solution $((\eta_\lambda)_i{}^j, (\mathsf{\kappa}_\lambda)_i{}^j, \phi_\lambda, \psi_\lambda, \nu_\lambda)$ to the system (9.2.2)–(9.2.16) in the range $0 < \tau(t) \leq 1$,

$$t_{\lambda*}^{-p_i+p_j}|t\partial_t(\mathsf{\kappa}_\lambda)_i{}^j| + t_{\lambda*}^{-p_i+p_j}|t\partial_t(\tilde{\Upsilon}_\lambda)_i{}^j| + |t\partial_t\psi_\lambda| + |t\partial_t\tilde{\varphi}_\lambda| \lesssim \left(\frac{t}{t_{\lambda*}}\right)^\delta \cdot \mathcal{E}_{\lambda,low}^{1/2}(t_{\lambda*}).$$

For $\delta > 0$, $t^{\delta-1}$ is integrable towards $t = 0$, thus there is a limit $((\mathsf{\kappa}_\lambda)_i{}^j, (\tilde{\Upsilon}_\lambda)_i{}^j, \psi_\lambda, \tilde{\varphi}_\lambda) \to (((\mathsf{\kappa}_\infty)_\lambda)_i{}^j, ((\tilde{\Upsilon}_\infty)_\lambda)_i{}^j, (\psi_\infty)_\lambda, (\tilde{\varphi}_\infty)_\lambda)$ as $t \to 0$, and moreover that the estimates (9.4.28) and (9.4.29) are satisfied. The final estimate in (i) then follows straightforwardly from Proposition 9.4.2.

Moving onto (ii), we apply the Fuchsian theory of first-order ODEs. In order to do so, we firstly reinterpret Proposition 9.4.4 as follows. This proposition tells us that solutions $\mathbf{V} = (t_{\lambda*}^{-p_{\underline{i}}+p_{\underline{j}}}(\mathsf{\kappa}_\lambda)_{\underline{i}}{}^{\underline{j}}, t_{\lambda*}^{-p_{\underline{i}}+p_{\underline{j}}}(\tilde{\Upsilon}_\lambda)_{\underline{i}}{}^{\underline{j}}, \psi_\lambda, \tilde{\varphi}_\lambda)$ to the first-order hyperbolic system (9.4.3)–(9.4.6) coupled to the elliptic equation (9.2.8) satisfy the following Fuchsian ODE:

$$t\frac{d\mathbf{V}}{dt} = \left(\frac{t}{t_{\lambda*}}\right)^\delta F\left[\frac{t}{t_{\lambda*}}, \mathbf{V}\right],$$

where F is bounded and continuous in both its variables and linear in $\mathbf{V}$.

Therefore, we may apply Lemma E.0.1(b) to the above ODE, with $A = 0$ and initial data

$$\mathbf{v} = (t_{\lambda*}^{-p_{\underline{i}}+p_{\underline{j}}}((\mathsf{\kappa}_\infty)_\lambda)_{\underline{i}}{}^{\underline{j}}, t_{\lambda*}^{-p_{\underline{i}}+p_{\underline{j}}}((\tilde{\Upsilon}_\infty)_\lambda)_{\underline{i}}{}^{\underline{j}}, (\psi_\infty)_\lambda, (\tilde{\varphi}_\infty)_\lambda).$$

The lemma generates a solution $((\eta_\lambda)_i{}^j, (\mathsf{\kappa}_\lambda)_i{}^j, \phi_\lambda, \psi_\lambda, \nu_\lambda)$ to the first-order elliptic hyperbolic system above, which moreover attains the asymptotics (9.4.28) and (9.4.29). Moreover, the solution is unique in the whole domain $0 < t \leq t_{\lambda*}$. (Note Lemma E.0.1(b) only gives local existence and uniqueness, but since the system (9.4.3)–(9.4.6) and (9.2.8) are linear one may extend to the whole domain $0 < t \leq t_{\lambda*}$.)

So far, the argument (including Proposition 9.4.4) has *not* used the constraint equations (9.2.11)–(9.2.16). It remains to recover these constraints for the solution we generated; we leave this to Proposition F.0.2. It is only here that we require that $(((\mathsf{\kappa}_\infty)_\lambda)_i{}^j, ((\tilde{\Upsilon}_\infty)_\lambda)_i{}^j, (\psi_\infty)_\lambda, (\tilde{\varphi}_\infty)_\lambda)$ satisfy the asymptotic constraints (9.2.17)–(9.2.21). This completes the proof of the proposition. □

9.4.4 Connecting the high and low frequency energies

We shall eventually show that in a well-chosen gauge, the high-frequency energy quantity $\mathcal{E}_{\lambda,high}(t)$ of Definition 9.3.1 and the low-frequency energy quantity $\mathcal{E}_{\lambda,low}(t)$ of Definition 9.4.1 are comparable. One subtlety is that $\mathcal{E}_{\lambda,high}(t)$ only has good coercivity properties for $\tau_\lambda(t) \geq \mathring{\tau} \geq 1$, while $\mathcal{E}_{\lambda,low}(t)$ is defined only for $\tau_\lambda(t) \leq 1$. To fill in the gap $1 \leq \tau_\lambda(t) \leq \mathring{\tau}$, we rely on the following *mid-frequency energy estimate*. Like the high-frequency energy estimate, this will *always use the CMCSH gauge* (9.2.1).

Definition 9.4.2. The *geometric mid-frequency energy quantity* $\mathcal{E}_{\lambda,\mathfrak{n},mid}(t)$, used for $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $1 \leq \tau_\lambda(t) \leq \mathring{\tau}$ and solutions in the CMCSH gauge, is given by

$$\mathcal{E}_{\lambda,\mathfrak{n},mid}(t) \coloneqq \mathring{g}^{ac}(t)\mathring{g}_{bd}(t)\,(\hat{\kappa}_\lambda)_a{}^b(\hat{\kappa}_\lambda)_c{}^d + \mathring{g}^{ac}(t)\mathring{g}_{bd}(t)\,(\hat{\mathfrak{n}}_\lambda)_a{}^b(\hat{\mathfrak{n}}_\lambda)_c{}^d. \tag{9.4.32}$$

Similarly, the *matter mid-frequency energy quantity* is defined in the same regime, and is given by

$$\mathcal{E}_{\lambda,\phi,mid}(t) \coloneqq \mathfrak{psi}_\lambda^2 + \phi_\lambda^2. \tag{9.4.33}$$

Finally, define the *total mid-frequency energy quantity* as the sum $\mathcal{E}_{\lambda,low}(t) \coloneqq \mathcal{E}_{\lambda,\mathfrak{n},low}(t) + \mathcal{E}_{\lambda,\phi,low}(t)$.

Proposition 9.4.6. *Let $((\hat{\mathfrak{n}}_\lambda)_i{}^j, (\hat{\kappa}_\lambda)_i{}^j, \phi_\lambda, \mathfrak{psi}_\lambda, \mathfrak{nu}_\lambda, \chi_\lambda^j)$ be a solution to the linearized Einstein–scalar field system (9.2.2)–(9.2.16) in CMCSH gauge. Then there exists some $\mathring{C}_{mid}$, depending only on the background Kasner spacetime, such that for $\lambda \in \mathbb{Z}^D \setminus \{0\}$ and $1 \leq \tau_\lambda(t) \leq \mathring{\tau}$, one has*

$$\mathring{C}_{mid}^{-1}\,\mathcal{E}_{\lambda,mid}(t) \leq \mathcal{E}_{\lambda,mid}(\tau_\lambda^{-1}(\mathring{\tau})) \leq \mathring{C}_{mid}\,\mathcal{E}_{\lambda,mid}(t). \tag{9.4.34}$$

Enlarging the constant $\mathring{C}_{mid}$ if necessary, one has the following equivalence between the high-frequency energy $\mathcal{E}_{\lambda,high}(t)$ at $t = \tau_\lambda^{-1}(\mathring{\tau})$ and the low-frequency energy $\mathcal{E}_{\lambda,low}(t)$ at $t = t_{\lambda*} = \tau_\lambda^{-1}(1)$.

$$\mathring{C}_{mid}^{-1}\,\mathcal{E}_{\lambda,low}(t_{\lambda*}) \leq \mathcal{E}_{\lambda,high}(\tau_\lambda^{-1}(\mathring{\tau})) \leq \mathring{C}_{mid}\,\mathcal{E}_{\lambda,low}(t_{\lambda*}). \tag{9.4.35}$$

Proof. The mid-frequency energy estimate (9.4.34) will be much more straightforward to show than either of the high- or low-frequency energy estimates, since we are working in a bounded domain $\log\tau \in [0, \log\mathring{\tau}]$, and therefore, by (8.2.9), a bounded domain of $\log\left(\frac{t}{t_{\lambda*}}\right) \subset [0, C\log\mathring{\tau}]$. As a result, it suffices to show a derivative estimate of the form:

$$\left| t\frac{d}{dt}\mathcal{E}_{\lambda,mid}(t) \right| \lesssim \mathcal{E}_{\lambda,mid}(t). \tag{9.4.36}$$

To prove (9.4.36), we use the same $\mathring{g}$-norm that was introduced in Section 9.3.2; then by definition of $\mathcal{E}_{\lambda,mid}$,

$$\mathcal{E}_{\lambda,mid}(t) = |\hat{\mathfrak{n}}_\lambda|_{\mathring{g}}^2 + |\hat{\kappa}_\lambda|_{\mathring{g}}^2 + \phi_\lambda^2 + \mathfrak{psi}_\lambda^2.$$

Further, using the same proof as Proposition 9.3.5 it is straightforward to show that $|\mathfrak{nu}_\lambda|^2 \lesssim \mathcal{E}_{\lambda,mid}(t)$ and $|\chi_\lambda|_{\mathring{g}}^2 \lesssim t^2\mathcal{E}_{\lambda,mid}(t)$. From this point, (9.4.36) follows immediately upon differentiating $\mathcal{E}_{\lambda,mid}(t)$ term-by-term using (9.2.2)–(9.2.7), and applying Cauchy-Schwarz.

Using the fact that $\mathcal{E}_{\lambda,mid}(t_{\lambda*}) = \mathcal{E}_{\lambda,low}(t_{\lambda*})$, while $\mathcal{E}_{\lambda,mid}(\tau^{-1}(\mathring{\tau})) \asymp \mathcal{E}_{\lambda,high}(\tau^{-1}(\mathring{\tau}))$ by Lemma 9.3.1, the final estimate (9.4.35) follows immediately from (9.4.34). □

9.5 Scattering for the linearized Einstein–scalar field equations

In Section 9.5, we prove our scattering result for the linearized Einstein–scalar field system around Kasner spacetimes satisfying the subcriticality condition, Theorem 7.3.1. Recall that the gauge used in Theorem 7.3.1 is a CMCTC gauge with vanishing shift, $\chi^j = 0$, though we do permit changes of gauge associated to a time-independent vector field ξ^j using (7.2.21)–(7.2.35).

On the other hand, our energy estimates use various different gauges; though the low-frequency energy estimate of Section 9.4 used a CMCTC gauge, our high-frequency energy estimate of Section 9.3 crucially used a CMCSH gauge (as did the mid-frequency estimate of Section 9.4.4, though this was not as crucial).

As a result, we instead first prove a version of Theorem 7.3.1 not in the CMCTC gauge, but in a gauge called the *Constant Mean Curvature Frequency Adapted* (CMCFA) gauge which we introduce in Section 9.5.1. This reinterpreted scattering result, Theorem 9.5.1, is stated in Section 9.5.2 and its proof will be in Section 9.5.3.

In Section 9.5.4, we understand how to translate back to CMCTC gauge, and thereby complete the proofs of the remaining theorems regarding the linearized Einstein–scalar field system, that is the scattering result Theorem 7.3.1.

9.5.1 Frequency adapted norms and gauges

As well as the gauge issue mentioned above, we make the following comments about Theorem 7.3.1; the idea being that these comments will inform how our second version, Theorem 9.5.1, will differ:

(I) The variables $(\mathring{\eta}_i{}^j, \mathring{\kappa}_i{}^j, \phi, \psi)$ used to describe Cauchy data at $t = 1$ differ from the variables $(\mathring{\kappa}_i{}^j, \mathring{\Upsilon}_i{}^j, \psi, \varphi)$ used to describe the asymptotic data at $t = 0$.

(II) In retrieving the optimal regularity of (7.3.4), we require a gauge transformation associated to a vector field ξ^j. (Furthermore, the regularity of ξ^j cannot be controlled particularly well.)

(III) The regularity statements (7.3.4)–(7.3.5), and the spaces $\mathcal{H}^s_\infty$, feature a correction by a term involving the symbol $\mathcal{T}_*$; in particular $\mathcal{T}_*^{-p_i+p_j}$ often acts on tensorial quantities such as $\mathring{\kappa}_i{}^j$.

These issues are inevitable, in light of the energy estimates presented in Section 9.3 and Section 9.4. This is since both the relevant gauge and the metrics $\mathring{g}_{ij}(t)$ and $\mathring{g}_{ij}(t_{\lambda *})$ used to define the energies differ between the high-frequency and the low-frequency regimes.

Thus, in Theorem 9.5.1, we first present a version of Theorem 7.3.1 that will follow more cleanly from our energy estimates. In preparation, we make a series of definitions corresponding to (I)–(III) above. Each of theses definitions will involve the symbol $\mathcal{T}_*$, see Definition 1.6.1. We start by defining new renormalized quantities in order to deal with (I).

Definition 9.5.1. We define the following *frequency adapted renormalized quantity* ${}^{(F)}\Upsilon_i{}^j$:

$$ {}^{(F)}\Upsilon_i{}^j \coloneqq \eta_i{}^j + \int_{\min\{t,\mathcal{T}_*\}}^{\mathcal{T}_*} \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\,\frac{ds}{s}\cdot \kappa_q{}^p = \Upsilon_i{}^j - \int_{\max\{t,\mathcal{T}_*\}}^{1} \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\,\frac{ds}{s}\cdot \kappa_q{}^p, \tag{9.5.1} $$

as well as the following *frequency adapted scalar field quantity* ${}^{(F)}\varphi$:

$$ {}^{(F)}\varphi \coloneqq \phi + \max\left\{0, \log\left(\frac{\mathcal{T}_*}{t}\right)\right\}\cdot\psi = \varphi + \max\{\log(\mathcal{T}_*), \log t\}\cdot\psi. \tag{9.5.2} $$

Remark. The point is that the Fourier coefficients $({}^{(F)}\Upsilon_\lambda)_i{}^j$ coincide with $(\eta_\lambda)_i{}^j$ in the high-frequency regime $t \geq t_{\lambda *}$, while $({}^{(F)}\Upsilon_\lambda)_i{}^j$ coincides with $(\tilde{\Upsilon}_\lambda)_i{}^j$ in the low-frequency regime $t \leq t_{\lambda *}$; here $(\tilde{\Upsilon}_\lambda)_i{}^j$ is defined in (9.5.1). Similarly, ${}^{(F)}\varphi_\lambda$ coincides with ϕ_λ at high frequency and with $\tilde{\varphi}_\lambda$ at low frequency.

Next, we define a new (linearized) gauge condition used to deal with (II).

Definition 9.5.2. We say that a solution $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ solving the linearized Einstein–scalar field system (7.2.6)–(7.2.20) is in a linearized *Constant Mean Curvature Frequency Adapted* (CMCFA) gauge if frequency adapted renormalized quantity ${}^{(F)}\Upsilon_i{}^j$ satisfies the differential relation

$$ \partial_i \operatorname{tr}{}^{(F)}\Upsilon - 2\partial_j {}^{(F)}\Upsilon_i{}^j = 0. \tag{9.5.3} $$

Finally, we define a norm for $(1,1)$ tensor fields to deal with (III). We shall often apply this to the $(1,1)$ tensor fields ${}^{(F)}\Upsilon_i{}^j$, $\kappa_i{}^j$ and also the derivative of the shift, $\partial_i\chi^j$, viewed as a $(1,1)$ tensor field.

Definition 9.5.3. Let $A = A_i{}^j$ be a $(1,1)$ tensor field on $\mathbb{T}^D$. Define the *frequency adapted L^2 norm* of A, thought to live on the boundary $\{t=0\}$ of $\mathcal{M}_{Kas}$, via the following:

$$ \|A_i{}^j\|^2_{L^{2,F}(\Sigma_0)} \coloneqq \sum_{i,j=1}^{D} \|\mathcal{T}_*^{-p_i+p_j} A_i{}^j\|^2_{L^2} = \mathring{g}^{ac}(\mathcal{T}_*)\mathring{g}_{bd}(\mathcal{T}_*)\cdot A_a{}^b A_c{}^d. $$

Equivalently, one has the frequency space definition:

$$ \|A_i{}^j\|^2_{L^{2,F}(\Sigma_0)} \coloneqq \sum_{\lambda\in\mathbb{Z}^D} \mathring{g}^{ac}(t_{\lambda *})\mathring{g}_{bd}(t_{\lambda *})(A_\lambda)_a{}^b (A_\lambda)_c{}^d. $$

One further defines the *frequency adapted H^s norm* of A using:

$$ \|A_i{}^j\|^2_{H^{s,F}(\Sigma_0)} \coloneqq \sum_{\lambda\in\mathbb{Z}^D} \langle\lambda\rangle^{2s}\mathring{g}^{ac}(t_{\lambda *})\mathring{g}_{bd}(t_{\lambda *})(A_\lambda)_a{}^b (A_\lambda)_c{}^d. \tag{9.5.4} $$

9.5.2 Statement of scattering for the linearized Einstein–scalar field system, v2

Theorem 9.5.1 (Scattering for the linearized Einstein–scalar field system in subcritical Kasner spacetimes, v2)**.** *Let $(\mathcal{M}_{Kas}, g_{Kas}, \phi_{Kas})$ be a (generalized) Kasner spacetime whose exponents satisfy the subcriticality condition* (1.2.11) *with $\delta > 0$. Consider solutions $(\mathring{\eta}_i{}^j, \mathring{\kappa}_i{}^j, \phi, \psi, \nu, \chi^j)$ to the linearized Einstein–scalar field system of Proposition 7.2.2 around $(\mathcal{M}_{Kas}, g_{Kas}, \phi_{Kas})$, with the solution such that the CMCFA gauge condition* (9.5.3) *holds. Then the scattering theory is as follows:*

(i) (Existence of the scattering operator) Let s be sufficiently large. Then given Cauchy data $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C) \in H^{s+1} \times H^s \times H^{s+1} \times H^s$ at $t = 1$ satisfying the constraint equations (7.2.15)–(7.2.20) *and gauge conditions* (9.5.3) *and $\operatorname{tr} \kappa_C = 0$, there exists a unique (classical) solution $(\mathring{\eta}_i{}^j, \mathring{\kappa}_i{}^j, \phi, \psi)$ to the system* (7.2.6)–(7.2.20) *satisfying $\operatorname{tr} \kappa = 0$ and the CMCFA condition* (9.5.3)*, which attains the Cauchy data:*

$$ {}^{(F)}\mathring{\Upsilon}_i{}^j(1,\cdot) = \mathring{\eta}_i{}^j(1,\cdot) = (\eta_C)_i{}^j(\cdot), \quad \mathring{\kappa}_i{}^j(1,\cdot) = (\kappa_C)_i{}^j(\cdot), $$

$$ {}^{(F)}\varphi(1,\cdot) = \phi(1,\cdot) = \phi_C(\cdot), \quad \psi(1,\cdot) = \psi_C(\cdot). $$

Furthermore, for some $\beta > 0$ depending only on the background $(\mathcal{M}_{Kas}, g_{Kas})$, there exist unique $(1,1)$ tensors $(\kappa_\infty)_i{}^j$ and $({}^{(F)}\Upsilon_\infty)_i{}^j$, and functions ψ_∞ and ${}^{(F)}\varphi_\infty$ on $\mathbb{T}^D$, such that

$$ \mathring{\kappa}_i{}^j(t,\cdot) \to (\kappa_\infty)_i{}^j(\cdot), \quad {}^{(F)}\mathring{\Upsilon}_i{}^j(t,\cdot) \to ({}^{(F)}\Upsilon_\infty)_i{}^j(\cdot) \quad \text{in } H^{s-\beta} \text{ strongly as } t \to 0, \tag{9.5.5} $$

$$ \psi(t,\cdot) \to \psi_\infty(\cdot), \quad {}^{(F)}\varphi(t,\cdot) \to {}^{(F)}\varphi_\infty(\cdot) \quad \text{in } H^s \text{ strongly as } t \to 0. \tag{9.5.6} $$

These limiting objects $(\kappa_\infty)_i{}^j$, $(\tilde{\Upsilon}_\infty)_i{}^j$, ψ_∞ and ϕ_∞ obey the asymptotic constraints (7.2.27)–(7.2.31) *(with $\tilde{\Upsilon}_i{}^j$ and $\tilde{\varphi}$ replaced by ${}^{(F)}\Upsilon_i{}^j$ and ${}^{(F)}\varphi$ respectively, and T replaced by the symbol $\mathcal{T}_*$) as well as the limiting frequency adapted gauge condition:*

$$ \partial_i \operatorname{tr} {}^{(F)}\Upsilon_\infty - 2\partial_j ({}^{(F)}\Upsilon_\infty)_i{}^j = 0. \tag{9.5.7} $$

Finally, for the frequency adapted Sobolev spaces $H^{s,F}(\Sigma_0)$ in Definition 9.5.3, one has

$$ (\kappa_\infty)_i{}^j, ({}^{(F)}\Upsilon_\infty)_i{}^j \in H^{s+\frac{1}{2},F}(\Sigma_0), \qquad \psi_\infty, {}^{(F)}\varphi_\infty \in H^{s+\frac{1}{2}}. \tag{9.5.8} $$

(ii) (Asymptotic completeness) Let s be sufficiently large, and let $(\kappa_\infty)_i{}^j$, $({}^{(F)}\Upsilon_\infty)_i{}^j$ be $(1,1)$ tensor fields and ψ_∞, ${}^{(F)}\varphi_\infty$ be functions on $\mathbb{T}^D$ verifying the constraints (7.2.27)–(7.2.31) *(modified as in (i)), with regularity as in* (9.5.8)*, and obeying the gauge condition* (9.5.7)*. Then there exists a unique (classical) solution $(\mathring{\eta}_i{}^j, \mathring{\kappa}_i{}^j, \phi, \psi, \nu, \chi^j)$ to the system* (7.2.6)–(7.2.20) *satisfying $\operatorname{tr} \kappa = 0$ and the CMCFA condition* (9.5.3)*, such that as $t \to 0$, the asymptotics* (9.5.5)–(9.5.6) *hold.*

The regularity of the solution will be as follows:

$$(\eta_i{}^j, \kappa_i{}^j) \in C^0((0,+\infty), H^{s+1} \times H^s) \cap C^1((0,+\infty), H^s \times H^{s-1}),$$
$$(\phi, \psi) \in C^0((0,+\infty), H^{s+1} \times H^s) \cap C^1((0,+\infty), H^s \times H^{s-1}),$$
$$(\nu, \chi^j) \in C^0((0,+\infty), H^{s+2} \times H^{s+2}).$$

(iii) (Scattering isomorphism) For $s \in \mathbb{R}$, we define the following Hilbert spaces together with their norms:

$$\mathcal{H}^s_C = \left\{ \left(\eta_i{}^j, \kappa_i{}^j, \phi, \psi \right) : \eta_i{}^j \in H^{s+1}, \kappa_i{}^j \in H^s, \phi \in H^{s+1}, \psi \in H^s \right\}, \tag{9.5.9}$$

$$\left\| \left(\eta_i{}^j, \kappa_i{}^j, \phi, \psi \right) \right\|^2_{\mathcal{H}^s_C} := \|\eta_i{}^j\|^2_{H^{s+1}} + \|\kappa_i{}^j\|^2_{H^s} + \|\phi\|^2_{H^{s+1}} + \|\psi\|^2_{H^s}, \tag{9.5.10}$$

$$\begin{aligned} {}^{(F)}\mathcal{H}^s_\infty = \Big\{ \Big(& \kappa_i{}^j, \Upsilon_i{}^j, \psi, \varphi \Big) : \\ & \kappa_i{}^j \in H^{s+\frac{1}{2},F}(\Sigma_0), \Upsilon_i{}^j \in H^{s+\frac{1}{2},F}(\Sigma_0), \psi \in H^{s+\frac{1}{2}}, \varphi \in H^{s+\frac{1}{2}} \Big\}, \end{aligned} \tag{9.5.11}$$

$$\left\| \left(\kappa_i{}^j, \Upsilon_i{}^j, \psi, \varphi \right) \right\|^2_{{}^{(F)}\mathcal{H}^s_\infty} = \|\kappa_i{}^j\|^2_{H^{s+\frac{1}{2},F}(\Sigma_0)} + \|\Upsilon_i{}^j\|^2_{H^{s+\frac{1}{2},F}(\Sigma_0)} + \|\psi\|^2_{H^{s+\frac{1}{2}}} + \|\varphi\|^2_{H^{s+\frac{1}{2}}}. \tag{9.5.12}$$

Further, let $\mathcal{H}^s_{C,c}$ be the subspace of $\mathcal{H}_C$ obeying the constraint equations (7.2.15)–(7.2.20) *as well as the gauge conditions* $\operatorname{tr}\kappa = 0$ *and* (9.5.3)*, and ${}^{(F)}\mathcal{H}^s_{\infty,c}$ be the subspace of ${}^{(F)}\mathcal{H}^s_\infty$ obeying the asymptotic constraints* (9.2.17)–(9.2.21)*, modified as in (i), and the gauge conditions* $\operatorname{tr}\kappa = 0$ *and* (9.5.7)*.*

Then the map ${}^{(F)}\mathcal{S}_\downarrow$, which is defined to map $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C)$ from (i) to the limiting quantities $((\kappa_\infty)_i{}^j, ({}^{(F)}\Upsilon_\infty)_i{}^j, \psi_\infty, {}^{(F)}\varphi_\infty)$, may be extended to a Hilbert space isomorphism ${}^{(F)}\mathcal{S}_\downarrow : \mathcal{H}^s_{C,c} \to {}^{(F)}\mathcal{H}^s_{\infty,c}$, whose inverse ${}^{(F)}\mathcal{S}_\uparrow = {}^{(F)}\mathcal{S}_\downarrow^{-1}$ is exactly given by (a suitable extension of) the map taking $((\kappa_\infty)_i{}^j, ({}^{(F)}\Upsilon_\infty)_i{}^j, \psi_\infty, {}^{(F)}\varphi_\infty)$ in (ii) to $(\eta_i{}^j(1,\cdot), \kappa_i{}^j(1,\cdot), \phi(1,\cdot), \psi(1,\cdot))$.

9.5.3 Proof of scattering for the linearized Einstein–scalar field system, v2

As in Section 8.5 for the case of the wave equation, we first assume smoothness for all our variables and derive a priori estimates between Sobolev spaces, then use the density of smooth functions to recover the full scattering theorem. As in Section 8.5, we have two propositions, providing the a priori estimates in both directions of the scattering theory. Before proceeding, we determine what happens at zero frequency.

Lemma 9.5.2. *Let $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ be a solution of the linearized Einstein–scalar field system of Proposition 7.2.2, in any gauge. Then with $\Upsilon_i{}^j$ and φ as in* (7.2.24)–(7.2.25)*, the 0-frequency*

Fourier coefficients
$((\kappa_0)_i{}^j, (\Upsilon_0)_i{}^j, \psi_0, \varphi_0)$ *are constant, in particular for all* $t > 0$*:*

$$(\kappa_0)_i{}^j(t) = (\kappa_0)_i{}^j(0), \quad (\Upsilon_0)_i{}^j(t) = (\Upsilon_0)_i{}^j(0), \quad \psi_0(t) = \psi_0(1), \qquad \varphi_0(t) = \varphi_0(1). \tag{9.5.13}$$

Proof. This follows immediately from Proposition 9.2.1 in the case that $\lambda = 0$. □

From Cauchy data to asymptotic data

Proposition 9.5.3. *Let* $(\eta_C)_i{}^j$ *and* $(\kappa_C)_i{}^j$ *be smooth* $(1,1)$ *tensors on* $\mathbb{T}^D$*, and* ϕ_C *and* ψ_C *be smooth functions on* $\mathbb{T}^D$*, obeying the constraint equations* (7.2.15)–(7.2.20) *with* $t = 1$*, as well as* (9.5.3) *and* $\operatorname{tr}\kappa_C = 0$*. Then there exists a unique*[2] *smooth solution* $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ *to the system* (7.2.6)–(7.2.20) *for* $t > 0$*, satisfying* $\operatorname{tr}\kappa = 0$ *and the CMCFA gauge condition* (9.5.3)*, that achieves the initial data.*

Moreover, for some $\beta > 0$*, there exist smooth* $(1,1)$*-tensors* $(\kappa_\infty)_i{}^j$ *and* $({}^{(F)}\Upsilon_\infty)_i{}^j$*, and smooth functions* ψ_∞ *and* ${}^{(F)}\varphi_\infty$*, such that for* $s \in \mathbb{R}$ *and the solution as above, the following strong convergence holds as* $t \to 0$*, where the frequency adapted quantities* ${}^{(F)}\Upsilon_i{}^j$ *and* ${}^{(F)}\varphi$ *are given in Definition 9.5.1:*

$$\kappa_i{}^j(t) \to (\kappa_\infty)_i{}^j, \ {}^{(F)}\Upsilon_i{}^j(t) \to ({}^{(F)}\Upsilon_\infty)_i{}^j \text{ in } H^{s-\beta}, \ \psi(t) \to \psi_\infty, \ {}^{(F)}\varphi(t) \to {}^{(F)}\varphi_\infty \text{ in } H^s. \tag{9.5.14}$$

For the $H^{s,F}(\Sigma_0)$ *norms of Definition 9.5.3, one has*

$$\|(\kappa_\infty)_i{}^j\|^2_{H^{s+\frac{1}{2},F}(\Sigma_0)} + \|({}^{(F)}\Upsilon_\infty)_i{}^j\|^2_{H^{s+\frac{1}{2},F}(\Sigma_0)} + \|\phi_\infty\|^2_{H^{s+\frac{1}{2}}} + \|{}^{(F)}\varphi_\infty\|^2_{H^{s+\frac{1}{2}}}$$
$$\lesssim \|(\eta_C)_i{}^j\|^2_{H^{s+1}} + \|(\kappa_C)_i{}^j\|^2_{H^s} + \|\phi_C\|^2_{H^{s+1}} + \|\psi_C\|^2_{H^s}. \tag{9.5.15}$$

Finally, the quantities $(\kappa_\infty)_i{}^j$*,* $({}^{(F)}\Upsilon_\infty)_i{}^j$*,* ψ_∞ *and* ${}^{(F)}\varphi_\infty$ *satisfy the constraints* (7.2.27)–(7.2.31)*, with* $T = \mathcal{T}_*$*.*

Proof. We start with existence and uniqueness of a smooth solution for $t > 0$. For this purpose, we firstly apply Proposition 7.2.4(ii) with $\chi^j_\circ = 0$, which gives us a unique smooth solution $(\tilde\eta_i{}^j, \tilde\kappa_i{}^j, \phi, \psi, \nu)$ in the CMCTC gauge (i.e. with vanishing χ^j).

However, we would like existence and uniqueness in the CMCFA gauge instead. In light of Lemma 7.2.3 and Lemma 7.2.6, this may be achieved via the change of gauge associated to the vector field ξ^j, with

$$\mathcal{T}_M^2 \mathring{g}^{ab}(\mathcal{T}_M)\partial_a\partial_b\xi^j = \mathcal{T}_M^2\mathring{g}^{jk}(\mathcal{T}_M)(\partial_k \operatorname{tr}\tilde\Upsilon - 2\partial_\ell\tilde\Upsilon_k{}^\ell), \quad \text{where } \mathcal{T}_M = \max\{t, \mathcal{T}_*\}. \tag{9.5.16}$$

The symbol $\mathcal{T}_M = \max\{t, \mathcal{T}_*\}$ arises due to the definition of ${}^{(F)}\Upsilon_i{}^j$. Using this change of gauge, one has existence in the CMCFA gauge. Uniqueness also follows by a similar argument to uniqueness in Proposition 7.2.4(b) – essentially reverse this change of gauge and use uniqueness in CMCTC.

[2]For uniqueness, one requires as always that χ^j has zero average.

Given our solution, we now take the Fourier decomposition. From this point, for $\lambda \in \mathbb{Z}^D \setminus \{0\}$, we shall always work in the variables $((\mathfrak{k}_\lambda)_i{}^j, ({}^{(F)}\mathfrak{T}_\lambda)_i{}^j, \mathfrak{p}_\lambda, {}^{(F)}\mathfrak{f}_\lambda, \mathfrak{n}_\lambda, \mathfrak{X}_\lambda^j)$, and that by definition of the CMCFA gauge, we have the following description in the high-frequency regime $t_{\lambda *} \geq 1$ and the low-frequency regime $0 < t_{\lambda *} \leq 1$: for the high-frequency (and mid-frequency) regime $t_{\lambda *} \geq 1$,

$$((\mathfrak{k}_\lambda)_i{}^j, ({}^{(F)}\mathfrak{T}_\lambda)_i{}^j, \mathfrak{p}_\lambda, {}^{(F)}\mathfrak{f}_\lambda, \mathfrak{n}_\lambda, \mathfrak{X}_\lambda^j) = ((\hat{\mathfrak{k}}_\lambda)_i{}^j, (\hat{\mathfrak{n}}_\lambda)_i{}^j, \mathfrak{p}_\lambda, \mathfrak{h}_\lambda, \mathfrak{n}_\lambda, \mathfrak{X}_\lambda^j)$$

and the λ-Fourier projections in this range are in the CMCSH gauge, i.e. they satisfy (9.2.1). This means that we can apply our high-frequency energy estimates of Section 9.3.

On the other hand, in the low-frequency regime $0 < t_{\lambda *} \leq 1$, we have

$$((\mathfrak{k}_\lambda)_i{}^j, ({}^{(F)}\mathfrak{T}_\lambda)_i{}^j, \mathfrak{p}_\lambda, {}^{(F)}\mathfrak{f}_\lambda, \mathfrak{n}_\lambda, \mathfrak{X}_\lambda^j) = ((\mathfrak{k}_\lambda)_i{}^j, (\tilde{\mathfrak{T}}_\lambda)_i{}^j, \mathfrak{p}_\lambda, \tilde{\mathfrak{f}}_\lambda, \mathfrak{n}_\lambda, \mathfrak{X}_\lambda^j),$$

with $(\tilde{\mathfrak{T}}_\lambda)_i{}^j$ and $\tilde{\mathfrak{f}}_\lambda$ as in (9.4.1) and (9.4.2). Here, we do not quite have $\mathfrak{X}_\lambda^j = 0$ as in the CMCTC gauge used in Section 9.4, and instead satisfy the CMCFA gauge condition $2\lambda_j {}^{(F)}\mathfrak{T}_i{}^j = \lambda_i \operatorname{tr} {}^{(F)}\mathfrak{T}$. Nevertheless, as we see shortly it is not a major difficulty to go back and forth from CMCFA gauge to CMCTC gauge in the low-frequency regime, and one may still apply the low-frequency energy estimates of Section 9.4.

We first relate our energies to the initial data $((\mathfrak{n}_C)_i{}^j, (\mathfrak{k}_C)_i{}^j, \mathfrak{p}_C, \mathfrak{h}_C)$. Splitting into high-frequencies and mid-frequences depending on $\tau_\lambda(1)$, and using Lemma 9.3.1 in the high-frequency case, we relate the energies to Sobolev norms of initial data:

$$\begin{aligned} |(\mathfrak{k}_0)_i{}^j|^2 + |({}^{(F)}\mathfrak{T}_0)_i{}^j|^2 + \mathfrak{p}_0^2 + {}^{(F)}\mathfrak{f}_0^2 + \sum_{\substack{\lambda \in \mathbb{Z}^D \setminus \{0\} \\ \tau_\lambda(1) < \mathring{\tau}}} \langle \lambda \rangle^{2s+1} \mathcal{E}_{\lambda, mid}(1) + \sum_{\substack{\lambda \in \mathbb{Z}^D \setminus \{0\} \\ \tau_\lambda(1) \geq \mathring{\tau}}} \langle \lambda \rangle^{2s+1} \mathcal{E}_{\lambda, high}(1) \\ \asymp \|(\mathfrak{n}_C)_i{}^j\|_{H^{s+1}}^2 + \|(\mathfrak{k}_C)_i{}^j\|_{H^s}^2 + \|\mathfrak{h}_C\|_{H^{s+1}}^2 + \|\mathfrak{p}_C\|_{H^s}^2. \quad (9.5.17) \end{aligned}$$

Applying the high-frequency energy estimate Proposition 9.3.2, as well as the mid-frequency energy estimate Proposition 9.3.2 (which both use the CMCSH as in our case for $t \geq t_{\lambda *}$), we deduce that

$$\begin{aligned} |(\mathfrak{k}_0)_i{}^j|^2 + |({}^{(F)}\mathfrak{T}_0)_i{}^j|^2 + \mathfrak{p}_0^2 + {}^{(F)}\mathfrak{f}_0^2 + \sum_{\lambda \in \mathbb{Z}^D \setminus \{0\}} \langle \lambda \rangle^{2s+1} \mathcal{E}_{\lambda, low}(t_{\lambda *}) \\ \lesssim \|(\mathfrak{n}_C)_i{}^j\|_{H^{s+1}}^2 + \|(\mathfrak{k}_C)_i{}^j\|_{H^s}^2 + \|\mathfrak{h}_C\|_{H^{s+1}}^2 + \|\mathfrak{p}_C\|_{H^s}^2. \quad (9.5.18) \end{aligned}$$

We next apply the low-frequency energy estimate Proposition 9.4.2. However, since this proposition assumes the CMCTC gauge, we need to find a way to translate the energy estimate. Our method is as follows: we first generate another solution $(({}^{(T)}\mathfrak{k}_\lambda)_i{}^j, ({}^{(T)}\tilde{\mathfrak{T}}_\lambda)_i{}^j, \mathfrak{p}_\lambda, \tilde{\mathfrak{f}}_\lambda, \mathfrak{n}_\lambda)$ which is in the CMCTC gauge and which *agrees with our solution* at $t = t_{\lambda *}$, and thus to which Proposition 9.4.2 and Proposition 9.4.4 apply, then transform into CMCFA gauge using some change of gauge vector field $\mathfrak{X}_\lambda^j$.

This change of gauge vector field agrees exactly with (9.5.16) above. Furthermore, projecting (9.5.16) onto low frequencies with $0 < t \leq t_{\lambda *}$ where by definition, $\tau^2(t_{\lambda *}) = t_{\lambda *}^2 \mathring{g}^{ab}(t_{\lambda *})\lambda_a\lambda_b = 1$, ξ_λ^j is exactly given by

$$\xi_\lambda^j = t_{\lambda *}^2 \mathring{g}^{jk}(t_{\lambda *})[\lambda_k \operatorname{tr}{}^{(T)}\tilde{\Upsilon}_\lambda - 2\lambda_\ell ({}^{(T)}\tilde{\Upsilon}_\lambda)_k{}^\ell]. \tag{9.5.19}$$

Upon checking the $t_{\lambda *}$ weights carefully, Propositions 9.4.2 and 9.4.4 tell us that

$$|t_{\lambda *}^{p_j - 1}\xi_\lambda^j| \lesssim \mathcal{E}_{\lambda,low}^{1/2}(t_{\lambda *}), \qquad \left| t\frac{t}{dt}(t_{\lambda *}^{p_j-1}\xi_\lambda^j)\right| \lesssim \left(\frac{t}{t_{\lambda *}}\right)^\delta \mathcal{E}_{\lambda,low}^{1/2}(t_{\lambda *}). \tag{9.5.20}$$

Next, the change of gauge vector field ξ_λ^j relates the two gauges by (see (7.2.22), (7.2.35) and (7.2.23)):

$$\begin{aligned}
(\kappa_\lambda)_i{}^j &= ({}^{(T)}\kappa_\lambda)_i{}^j - \mathrm{i}(t\mathring{k}_i{}^p)\lambda_p\xi_\lambda^j + \mathrm{i}(t\mathring{k}_p{}^j)\lambda_i\xi_\lambda^p,\\
({}^{(F)}\Upsilon_\lambda)_i{}^j &= ({}^{(T)}\tilde{\Upsilon}_\lambda)_i{}^j - \frac{\mathrm{i}}{2}\lambda_i\xi_\lambda^j - \frac{\mathrm{i}}{2}\mathring{g}_{ip}(t_{\lambda *})\mathring{g}^{jq}(t_{\lambda *})\lambda_q\xi_\lambda^p,\\
\chi_\lambda^j &= -t\frac{t}{dt}\xi_\lambda^j.
\end{aligned}$$

By (9.5.20), $\lambda_i\xi_\lambda^j$ satisfies the same estimates as either of $({}^{(T)}\kappa_\lambda)_i{}^j$ or $({}^{(T)}\tilde{\Upsilon}_\lambda)_i{}^j$. So all the low-frequency energy estimates of Section 9.4 apply equally well to $((\kappa_\lambda)_i{}^j, ({}^{(F)}\Upsilon_\lambda)_i{}^j)$ in our CMCFA gauge.

In particular, from the second estimate in (9.5.20) we deduce an analogue of Proposition 9.4.5(i), in other words we have unique tensors $((\kappa_\infty)_\lambda)_i{}^j$ and $(({}^{(F)}\Upsilon_\infty)_\lambda)_i{}^j$, and real numbers $(\psi_\infty)_\lambda$ and $({}^{(F)}\varphi_\infty)_\lambda$ with the CMCFA analogue of the estimates (9.4.28)–(9.4.29). We now simply define

$$\begin{aligned}
(\kappa_\infty)_i{}^j(x) &= (\kappa_0)_i{}^j + \sum_{\lambda\in\mathbb{Z}^D\setminus\{0\}} ((\kappa_\infty)_\lambda)_i{}^j\, e^{i\lambda\cdot x},\\
({}^{(F)}\Upsilon_\infty)_i{}^j(x) &= ({}^{(F)}\Upsilon_0)_i{}^j + \sum_{\lambda\in\mathbb{Z}^D\setminus\{0\}} (({}^{(F)}\Upsilon_\infty)_\lambda)_i{}^j\, e^{i\lambda\cdot x},\\
\psi_\infty(x) &= \psi_0 + \sum_{\lambda\in\mathbb{Z}^D\setminus\{0\}} (\psi_\infty)_\lambda\, e^{i\lambda\cdot x},\\
{}^{(F)}\varphi_\infty(x) &= {}^{(F)}\varphi_0 + \sum_{\lambda\in\mathbb{Z}^D\setminus\{0\}} ({}^{(F)}\varphi_\infty)_\lambda\, e^{i\lambda\cdot x}.
\end{aligned}$$

The final estimate in Proposition 9.4.5(i), combined with (9.5.18) and the definition of $H^{s,F}(\Sigma_0)$ then yields (9.5.15). Since this is true for all $s \in \mathbb{R}$, and $H^{s,F}(\Sigma_0) \subset H^{s-\beta}$ for some $\beta > 0$ independent of s, this tells us that our limiting tensors and functions are smooth.

It remains to show the convergence (9.5.14). Let us first take some $r \in \mathbb{R}$, and estimate

$$\|\kappa_i{}^j - (\kappa_\infty)_i{}^j\|_{H^r}^2 = \sum_{\lambda\in\mathbb{Z}^d} \langle\lambda\rangle^{2r} |(\kappa_\lambda)_i{}^j - ((\kappa_\infty)_\lambda)_i{}^j|^2.$$

We estimate this by splitting into low and high frequencies, and then applying the energy estimates. At low frequencies, we shall use the convergence (9.4.28), while at high and mid frequencies we only use boundedness. For convenience, let us define:

$$\mathcal{E}_{\lambda,hm}(1) = \begin{cases} \mathcal{E}_{\lambda,high}(1) & \text{if } \tau_\lambda(1) \geq \mathring{\tau}, \\ \mathcal{E}_{\lambda,mid}(1) & \text{if } 1 \leq \tau_\lambda(1) < \mathring{\tau}. \end{cases} \tag{9.5.21}$$

Using the energy estimates, one then finds:

$$\begin{aligned} &\|\mathsf{k}_i{}^j(t) - (\mathsf{k}_\infty)_i{}^j\|_{H^r}^2 \\ &\qquad \leq \sum_{\substack{\lambda\in\mathbb{Z}^D,\\ \tau_\lambda(t)\leq 1}} \langle\lambda\rangle^{2r} |(\mathsf{k}_\lambda)_i{}^j(t) - ((\mathsf{k}_\infty)_\lambda)_i{}^j|^2 + \sum_{\substack{\lambda\in\mathbb{Z}^D,\\ \tau_\lambda(t)>1}} \langle\lambda\rangle^{2r} (|(\mathsf{k}_\lambda)_i{}^j|^2 + |((\mathsf{k}_\infty)_\lambda)_i{}^j|^2) \\ &\qquad \lesssim \sum_{\substack{\lambda\in\mathbb{Z}^D,\\ \tau_\lambda(t)\leq 1}} \langle\lambda\rangle^{2r} t_{\lambda *}^{2p_i-2p_j} \left(\frac{t}{t_{\lambda *}}\right)^{2\delta} \mathcal{E}_{\lambda,hm}(1) + \sum_{\substack{\lambda\in\mathbb{Z}^D\\ \tau_\lambda(t)>1}} \langle\lambda\rangle^{2r} (\tau t^{2p_i-2p_j} + t_{\lambda *}^{2p_i-2p_j}) \mathcal{E}_{\lambda,hm}(1). \end{aligned}$$

To finally show convergence, we select an $N > 0$ such that the following quantities are bounded, uniformly in $\lambda \in \mathbb{Z}^D \setminus \{0\}$:

$$\langle\lambda\rangle^{-N} t_{\lambda *}^{2p_i-2p_j-2\delta} + \langle\lambda\rangle^{-N} \tau \max_{t\geq t_{\lambda *}} \{t^{2p_i-2p_j}\} \lesssim 1.$$

This is always possible, since $\tau \leq \langle\lambda\rangle$, while $t_{\lambda *}^{-1}$ grows polynomially in $\langle\lambda\rangle$. Substituting this into the above, we therefore find that

$$\|\mathsf{k}_i{}^j - (\mathsf{k}_\infty)_i{}^j\|_{H^r}^2 = \sum_{\substack{\lambda\in\mathbb{Z}^D,\\ \tau_\lambda(t)\leq 1}} t^{2\delta} \langle\lambda\rangle^{2r-N} \mathcal{E}_{\lambda,hm}(1) + \sum_{\substack{\lambda\in\mathbb{Z}^D,\\ \tau_\lambda(t)>1}} \langle\lambda\rangle^{2r-N} \mathcal{E}_{\lambda,hm}(1).$$

Selecting $2r - N = 2s + 1$, we therefore deduce the $\mathsf{k}_i{}^j$ convergence in (9.5.14) for $\beta = \frac{1+N}{2}$. The ${}^{(F)}\Upsilon_i{}^j$ convergence follows similarly (the additional difficulty being that we need to incorporate the bound for $G_{ip}{}^{jq}$ in Lemma 9.4.3, and thus may have to increase β further), while the ψ and ${}^{(F)}\varphi$ convergence follow from similar considerations to Proposition 8.5.2.

Since $H^{s-\beta} \subset C^3$ for s sufficiently large, the fact that our limiting quantities satisfy the relevant constraints follows from a version of Lemma 7.2.5 with $T = \mathcal{T}_*$. This completes the proof of the proposition. □

From asymptotic data to Cauchy data

Proposition 9.5.4. *Let $(\mathsf{k}_\infty)_i{}^j, ({}^{(F)}\Upsilon_\infty)_i{}^j$ be smooth $(1,1)$ tensors and $\psi_\infty, {}^{(F)}\varphi_\infty$ be smooth functions on $\mathbb{T}^D$ satisfying the asymptotic constraints (7.2.27)–(7.2.31) with $T = \mathcal{T}_*$. Then there exists a unique smooth solution $(\mathsf{n}_i{}^j, \mathsf{k}_i{}^j, \phi, \psi, \nu, \chi^j)$ to the system (7.2.6)–(7.2.20) for $t > 0$ in CMCFA*

gauge, which achieves the asymptotic data in the sense that for any $s \in \mathbb{R}$ and β as in Proposition 9.5.3, one has the (strong) convergence:

$$\kappa_i{}^j(t) \to (\kappa_\infty)_i{}^j,\ {}^{(F)}\Upsilon_i{}^j(t) \to ({}^{(F)}\Upsilon_\infty)_i{}^j \text{ in } H^{s-\beta},\ \psi(t) \to \psi_\infty,\ {}^{(F)}\varphi(t) \to {}^{(F)}\varphi_\infty \text{ in } H^s. \quad (9.5.22)$$

Moreover, for any $s \in \mathbb{R}$ one has the following bound between Sobolev spaces:

$$\begin{aligned}\|\eta_i{}^j(1,\cdot)\|_{H^{s+1}} + \|\kappa_i{}^j(1,\cdot)\|_{H^s}^2 + \|\phi(1,\cdot)\|_{H^{s+1}}^2 + \|\psi(1,\cdot)\|_{H^s}^2 \\ \lesssim \|(\kappa_\infty)_i{}^j\|_{H^{s+\frac{1}{2},F}(\Sigma_0)}^2 + \|({}^{(F)}\Upsilon_\infty)_i{}^j\|_{H^{s+\frac{1}{2},F}(\Sigma_0)}^2 + \|\phi_\infty\|_{H^{s+\frac{1}{2}}}^2 + \|{}^{(F)}\varphi_\infty\|_{H^{s+\frac{1}{2}}}^2. \quad (9.5.23)\end{aligned}$$

Proof. For existence, we decompose the scattering data $((\kappa_\infty)_i{}^j, ({}^{(F)}\Upsilon_\infty)_i{}^j, \psi_\infty, {}^{(F)}\phi_\infty)$ into their Fourier modes. For any $\lambda \neq 0$, we now appeal to Proposition 9.4.5(ii), which tells us that there exists a solution to the ODE system (9.4.3)–(9.4.6) in the range $t \in (0, t_{\lambda *})$, which achieves the relevant limits as $t \to 0$, at the rate (9.4.28)–(9.4.29).

However, the ODE system (9.4.3)–(9.4.6) is in the CMCTC gauge, as opposed to the CMCFA gauge as required. Nevertheless, as in the proof of Proposition 9.5.3, we may use a change of gauge associated to the vector field ξ_λ^j in (9.5.19) to generate another solution in the CMCFA gauge. As in that proof, we still have the estimates (9.5.20), and as a result the energy estimates of Section 9.4 will all remain valid in this gauge. We denote this solution in CMCFA gauge to be $((\kappa_\lambda)_i{}^j, ({}^{(F)}\Upsilon_\lambda)_i{}^j, \psi_\lambda, {}^{(F)}\varphi_\lambda, \nu_\lambda, \chi_\lambda^j)$ as before.

We extend this solution to the range $t > t_{\lambda *}$ by using instead the CMCSH gauge evolution; this is valid since the CMCSH gauge and the CMCFA gauge coincide for $t \geq t_{\lambda *}$. Then combining all of the low-frequency estimates in Proposition 9.4.5, the mid-frequency estimate in Proposition 9.4.6 and the high-frequency estimate of Proposition 9.3.2, we find that for $\lambda \in \mathbb{Z}^D$ and $\mathcal{E}_{\lambda,hm}(1)$ as in (9.5.21), our solution obeys:

$$\begin{aligned}\mathcal{E}_{\lambda,hm}(1) \lesssim \mathring{g}^{ac}(t_{\lambda *})\mathring{g}_{bd}(t_{\lambda *}) \Big[((\kappa_\infty)_\lambda)_a{}^b((\kappa_\infty)_\lambda)_c{}^d + (({}^{(F)}\Upsilon_\infty)_\lambda)_a{}^b(({}^{(F)}\Upsilon_\infty)_\lambda)_c{}^d\Big] \\ + (\psi_\infty)_\lambda^2 + ({}^{(F)}\varphi_\infty)_\lambda^2. \quad (9.5.24)\end{aligned}$$

Using Lemma 9.5.2 to deal with $\lambda = 0$, and using the coercivity of the high-frequency energy in Lemma 9.3.1, we therefore have the estimate

$$\begin{aligned}\langle\lambda\rangle^{2s}|(\kappa_\lambda)_i{}^j(1)|^2 + \langle\lambda\rangle^{2s+2}|({}^{(F)}\Upsilon_\lambda)_i{}^j(1)|^2 + \langle\lambda\rangle^{2s}|\psi_\lambda(1)|^2 + \langle\lambda\rangle^{2s+2}|{}^{(F)}\varphi_\lambda(1)|^2 \\ \lesssim \langle\lambda\rangle^{2s+1}\Big[\mathring{g}^{ac}(t_{\lambda *})\mathring{g}_{bd}(t_{\lambda *})\Big[((\kappa_\infty)_\lambda)_a{}^b((\kappa_\infty)_\lambda)_c{}^d + (({}^{(F)}\Upsilon_\infty)_\lambda)_a{}^b(({}^{(F)}\Upsilon_\infty)_\lambda)_c{}^d\Big] \\ +(\psi_\infty)_\lambda^2 + ({}^{(F)}\varphi_\infty)_\lambda^2\Big].\end{aligned}$$

By smoothness of the scattering data, the right-hand side is summable in $\lambda \in \mathbb{Z}^D$ for all s, and doing this sum yields the estimate (9.5.23).

Now, upon applying Fourier inversion we recover a smooth solution $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ to the system (7.2.6)–(7.2.20) for $t > 0$ in CMCFA gauge, which moreover achieves the asymptotics (9.5.22) using the same proof as in Proposition 9.5.3. We leave the details to the reader.

All that remains is to show uniqueness. Since the system is linear it suffices to show uniqueness in the case of trivial scattering data $(\kappa_\infty)_i{}^j = 0$, $({}^{(F)}\Upsilon_\infty)_i{}^j = 0$, $\psi_\infty = 0$ and ${}^{(F)}\varphi_\infty = 0$. Suppose that we have a smooth solution $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \xi^j)$ to (7.2.6)–(7.2.20) in the CMCFA gauge, achieving these trivial asymptotics in the sense of (9.5.22). To show that this solution vanishes, it will suffice to show each Fourier mode vanishes.

The 0-Fourier mode must vanish by Lemma 9.5.2, so now take any $\lambda \in \mathbb{Z}^D \setminus \{0\}$, and focus on the low-frequency regime with $0 < t \leq t_{\lambda *}$. Recall that Proposition 9.4.5(ii) gives us uniqueness in CMCTC gauge, and for uniqueness in CMCFA gauge we look for a gauge transformation that retains the trivial asymptotic data but so that the transformed solution is in CMCTC gauge.

The idea is that any λ-mode projected solution $((\kappa_\lambda)_i{}^j, ({}^{(F)}\Upsilon_\lambda)_i{}^j, \psi_\lambda, {}^{(F)}\varphi_\lambda, \nu_\lambda, \chi_\lambda^j)$ in CMCFA gauge must obey $|\chi_\lambda^j(t)| \leq Dt^\delta$ for some constant D that may depend on data but does not depend on t. This follows from (9.5.19) and the description of χ_λ^j in the proof of Proposition 9.5.3. (In other words, that any CMCFA solution may be constructed from another CMCTC solution, though this latter CMCTC solution will not necessarily have trivial asymptotic data.)

We find a CMCTC solution with trivial asymptotic data via the gauge transformation associated to the vector field ${}^{(T)}\xi_\lambda^j(t) = \int_0^t \chi_\lambda^j \frac{ds}{s}$. Since $|\chi_\lambda^j(t)| \leq Dt^\delta$, ${}^{(T)}\xi_\lambda^j(t)$ is well-defined and vanishes at $t = 0$. Further, by Lemma 7.2.3(ii) with $T = 0$, it transforms our CMCFA solution to a CMCTC solution which still has trivial asymptotic data. Thus by uniqueness of CMCTC solutions from asymptotic data, the transformed solution must vanish. By (7.2.35) with $T = t_{\lambda *}$, this tells us that

$$({}^{(F)}\Upsilon_\lambda)_i{}^j = \frac{\mathrm{i}}{2}\lambda_i {}^{(T)}\xi_\lambda^j + \frac{\mathrm{i}}{2}\mathring{g}_{ip}(t_{\lambda *})\mathring{g}^{jq}(t_{\lambda *})\lambda_q {}^{(T)}\xi_\lambda^q.$$

We now simply insert this into our frequency adapted gauge condition (9.5.3). Eventually we obtain $\mathring{g}^{ab}(t_{\lambda *})\lambda_a\lambda_b {}^{(T)}\xi_\lambda^j = 0$. In other words, the gauge transformation vector field has ${}^{(T)}\xi_\lambda^j = 0$, and our solution must have been trivial in the first place. This concludes the proof of the proposition. □

Completion of the proof of Theorem 9.5.1

All the remains is to extend Proposition 9.5.3 and Proposition 9.5.4 to the setting where $(\kappa_i{}^j, {}^{(F)}\Upsilon_i{}^j, \psi, {}^{(F)}\varphi)$ have finite regularity, by using the density of smooth functions in H^s and in $H^{s,F}(\Sigma_0)$. This is a slightly more complicated version of what was done for the wave equation in Section 8.5.3, and therefore we leave the details to the reader, with only 2 comments:

– In both directions, the data $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C)$ or $((\kappa_\infty)_i{}^j, ({}^{(F)}\Upsilon_\infty)_i{}^j, \psi_\infty, {}^{(F)}\varphi_\infty)$ is not completely free, and must be arranged to satisfy various constraint equations and gauge con-

ditions. Thus in approximating finite regularity data by smooth data, the approximating data must also satisfy these equations.

However, since in our case the constraint equations and gauge conditions are linear, it is easy to smoothly approximate data while still satisfying these equations, for instance by convolution with standard mollifiers.

– In order to insist both that our solutions are classical and to make sure that the convergence (9.5.5) is also in C^3 in order to satisfy the assumptions of Lemma 7.2.5, we must choose $s > \frac{D}{2} + 3 + \beta$. This is what we mean by requiring s to be sufficiently large in (i) and (ii). For (iii), the Hilbert space isomorphism remains true for all $s \in \mathbb{R}$.

This concludes our proof of Theorem 9.5.1. □

9.5.4 Proof of scattering in transported coordinates

In this section, we prove the original version of the scattering result, Theorem 7.3.1, in a CMCTC gauge. To go from CMCFA gauge to CMCTC gauge, we use Lemma 7.2.3(ii); thus it remains to understand χ^j in CMCFA gauge, or more precisely the related change of gauge vector field

$$\xi^j(t) := -\int_t^1 \chi^j(s)\,\frac{ds}{s}. \tag{9.5.25}$$

Lemma 9.5.5. *Let $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ be a solution to the linearized Einstein–scalar field system (7.2.6)–(7.2.20) in CMCFA gauge, with regularity as in either of (i) or (ii) of Theorem 9.5.1. Then the change of gauge vector field ξ^j defined in (9.5.25) satisfies*

$$\xi^j \in C^0((0,+\infty) : H^{s+\frac{1}{2}}).$$

Moreover there exists a unique vector field $\xi^j_\infty \in H^{s+\frac{1}{2}}$ such that $\xi^j(t) \to \xi^j_\infty$ in $H^{s+\frac{1}{2}}$ as $t \to 0$.

Proof. Let us first work at the level of each Fourier mode $((\eta_\lambda)_i{}^j, (\kappa_\lambda)_i{}^j, \phi_\lambda, \psi_\lambda, \nu_\lambda, \chi^j_\lambda)$, where $\lambda \in \mathbb{Z}^D \setminus \{0\}$ (recall that χ^j and thus ξ^j have zero average). Using (9.3.8) from Lemma 9.3.5, and unpacking the $\mathring{g}$-norm, we find that for $\tau_\lambda(t) \geq \mathring{\tau}$,

$$|\chi^j_\lambda(t)| \lesssim \frac{t^{1-p_j}}{\tau^{\frac{3}{2}}} \mathcal{E}^{1/2}_{\lambda,hm}(1). \tag{9.5.26}$$

Here we have used the high-frequency energy estimate Proposition 9.3.2 to allow the right-hand side to be $\mathcal{E}_{\lambda,hm}(1)$, as defined in (9.5.21), in place of $\mathcal{E}_{\lambda,high}(t)$. Using also the mid-frequency energy estimate Proposition 9.4.6 and the associated elliptic estimates, (9.5.26) also holds in the mid-frequency range, and thus for the whole range $\tau_\lambda(t) \geq 1$.

In the low-frequency regime, we estimate $|\chi_\lambda^j|$ by referring to the proof of Proposition 9.5.3, in particular (9.5.20) and the description of χ_λ^j immediately below it. Combining all the energy estimates, one deduces that in the low-frequency range $0 < \tau_\lambda(t) \leq 1$, one has

$$|\chi_\lambda^j(t)| \lesssim t_{\lambda *}^{1-p_j} \cdot \left(\frac{t}{t_{\lambda *}}\right)^\delta \mathcal{E}_{\lambda,hm}^{1/2}(1). \tag{9.5.27}$$

Upon using that $t \leq 1$ and $t_{\lambda *} \leq 1$ in our setting, we combine (9.5.26) and (9.5.27) to get that for all $t > 0$,

$$|\chi_\lambda^j(t)| \lesssim \min\left\{\tau^{-\frac{3}{2}}, \left(\frac{t}{t_{\lambda *}}\right)^\delta\right\} \cdot \mathcal{E}_{\lambda,hm}^{1/2}(1). \tag{9.5.28}$$

(9.5.28) implies that $|\chi_\lambda^j(t)|$ is integrable (with respect to the differential $\frac{dt}{t}$) both towards $\tau \to \infty$ and towards $t \to 0$. Indeed, using also Lemma 8.2.2, it is straightforward to see that $\xi_\lambda^j(t)$, defined in (9.5.25), has a limit $(\xi_\infty)_\lambda^j$ as $t \to 0$, and we have

$$|(\xi_\infty)_\lambda^j| \lesssim \mathcal{E}_{\lambda,high}^{1/2}(1), \qquad |\xi_\lambda^j(t) - (\xi_\infty)_\lambda^j| \lesssim \max\left\{\left(\frac{t}{t_{\lambda *}}\right)^\delta, 1\right\} \cdot \mathcal{E}_{\lambda,high}^{1/2}(1). \tag{9.5.29}$$

To complete the proof, we recall that from Theorem 9.5.1(i) and (ii), it follows that $\sum_{\lambda \in \mathbb{Z}^D \setminus \{0\}} \langle \lambda \rangle^{2s+1} \mathcal{E}_{\lambda,hm}(1) < +\infty$. The lemma then follows straightforwardly from (9.5.29). □

With Lemma 9.5.5, we now deduce Theorem 7.3.1 from Theorem 9.5.1.

Proof of Theorem 7.3.1. We prove (i), (ii) and (iii) as follows:

(i) Let $((\eta_C)_i{}^j, (\kappa_C)_i{}^j, \phi_C, \psi_C)$ be Cauchy data with regularity as given and satisfying the constraints (7.2.15)–(7.2.20). We now apply Theorem 9.5.1, which gives us a classical solution $(\eta_i{}^j, \kappa_i{}^j, \phi, \psi, \nu, \chi^j)$ of (7.2.6)–(7.2.20) in the CMCFA gauge. There are two steps to get to the setup of Theorem 7.3.1.

Firstly, even before changing gauge, we upgrade the convergence of the frequency adapted quantities ${}^{(F)}\Upsilon_i{}^j$ and ${}^{(F)}\varphi$ to convergence of $\Upsilon_i{}^j$ and φ. From Definition 9.5.1, we have

$$\Upsilon_i{}^j = {}^{(F)}\Upsilon_i{}^j + \int_{\max\{t,\mathcal{T}_*\}}^1 \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\,\frac{ds}{s} \cdot \kappa_q{}^p, \quad \varphi = {}^{(F)}\varphi - \max\{\log(\mathcal{T}_*), \log t\} \cdot \psi. \tag{9.5.30}$$

Here, the object involving integral in (9.5.30) is a time-dependent symbol which, in some symbol class S^{β_0} for $\beta > 0$, converges (strongly) as follows:

$$\int_{\max\{t,\mathcal{T}_*\}}^1 \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\,\frac{ds}{s} \to \int_{\mathcal{T}_*}^1 \mathring{g}_{ip}(s)\mathring{g}^{jq}(s)\,\frac{ds}{s}$$

Therefore, for s sufficiently large, the convergence (9.5.5) together with Sobolev embedding implies the existence of $(\Upsilon_\infty)_i{}^j$ such that the convergence (7.3.2) also holds, at least in CMCFA

gauge. A similar result holds for φ, and here we can get convergence in H^s by the same proof as Proposition 8.5.2.

To transform back to CMCTC gauge, we use the change of gauge associated to the vector field ξ^j defined in (9.5.25). By (7.2.22) and (7.2.35), $\kappa_i{}^j$ and $\Upsilon_i{}^j$ transform as:

$$\kappa_i{}^j \mapsto \kappa_i{}^j + (t\mathring{k}_i{}^p)\partial_p\xi^j - (t\mathring{k}_p{}^j)\partial_i\xi^p, \quad \Upsilon_i{}^j \mapsto \Upsilon_i{}^j + \frac{1}{2}\partial_i\xi^j + \frac{1}{2}\mathring{g}_{ip}(1)\mathring{g}^{jq}(1)\partial_q\xi^p.$$

Since, by Lemma 9.5.5, the vector field ξ^j also converges to ξ^j_∞ in $H^{s+\frac{1}{2}}$ as $t \to 0$, Sobolev embedding implies that we also have the desired convergence in CMCTC gauge.

Finally, the regularity statement (7.3.4) follows from (9.5.8), upon using (9.5.30) and unraveling the $H^{s,F}(\Sigma_0)$ norm from Definition 9.5.3. The remainder of Theorem 7.3.1(i) is straightforward.

(ii) The proof of Theorem 7.3.1(ii) follows from an identical manner to the above; we first apply Theorem 9.5.1(ii) to find the solution in CMCFA gauge and then transform back using ξ^j from (9.5.25).

(iii) For the scattering isomorphism in Theorem 7.3.1(iii), one simply notes that the spaces $\mathcal{H}^s_{C,c}$ in Theorem 7.3.1 and Theorem 9.5.1 are identical, while the difference between $\mathcal{H}^s_{\infty,c}$ and ${}^{(F)}\mathcal{H}^s_{\infty,c}$ exactly encodes going from the frequency adapted quantities ${}^{(F)}\Upsilon_i{}^j$ and ${}^{(F)}\varphi$ back to $\Upsilon_i{}^j$ to φ. Thus Theorem 7.3.1(iii) follows immediately from Theorem 9.5.1(iii). □

Appendix A

Bessel functions

In this appendix, we record several basic facts about Bessel functions, which are used widely in Chapter 4, particularly Section 4.4. Bessel functions are also slightly relevant to the discussion in Chapter 8. For further details, refer to Chapter 10 of [54]. We start by recalling *Bessel's equation of order* $\nu \in \mathbb{C}$:

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - \nu^2) f = 0. \tag{A.0.1}$$

Solutions to this equation are known as *Bessel functions of order* ν. Within the appendix, we shall always assume ν to be a nonnegative integer, and our applications will always be in the cases $\nu = 0, 1$.

As (A.0.1) is a second order linear homogeneous ODE, there are two linearly independent solutions, denoted $J_\nu(z)$ and $Y_\nu(z)$. These have the following properties.

Fact A.0.1. *The function $J_\nu(z)$, known as the Bessel function of the first kind, is an entire function given by the following Taylor expansion:*

$$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\tfrac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu + k + 1)}. \tag{A.0.2}$$

In particular, as $z \to 0$, $J_0(z) \to 1$ and $z^{-1} J_1(z) \to 1/2$. Furthermore, for $0 < z \leq 1$, one has $z|J_0(z)| \leq 1$.

Fact A.0.2. *The function $Y_\nu(z)$, known as the Bessel function of the second kind, can be defined via the following Fuchsian asymptotic expansion for $\nu = n \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{R}^+$:*

$$\begin{aligned} Y_n(z) = -\frac{(\frac{1}{2}z)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\tfrac{1}{4}z^2\right)^k + \frac{2}{\pi} \ln\left(\tfrac{1}{2}z\right) J_n(z) \\ - \frac{(\frac{1}{2}z)^n}{\pi} \sum_{k=0}^{\infty} (\psi(k+1) + \psi(n+k+1)) \frac{(-\frac{1}{4}z^2)^k}{k!(n+k)!}, \end{aligned} \tag{A.0.3}$$

where $\psi(z) = \Gamma'(x)/\Gamma(x)$ and $\gamma \approx 0.577$ is the Euler-Mascheroni constant. In particular, one has the following asymptotics for $Y_0(z)$ and $Y_1(z)$:

$$\left|Y_0(z) - \frac{2}{\pi}\left(\log\left(\tfrac{1}{2}z\right) + \gamma\right)\right| \lesssim z^2, \quad \left|Y_1(z) + \frac{2}{\pi z}\right| \lesssim |z \log z|. \tag{A.0.4}$$

Furthermore, for $0 < z \leq 1$, one has the quantitative estimate $z|Y_0(z)| \leq 1$.

Fact A.0.3. *By studying the expansions (A.0.2) and (A.0.3),*

$$J_0'(z) = -J_1(z) \quad \text{and} \quad Y_0'(z) = -Y_1(z). \tag{A.0.5}$$

Fact A.0.4. *As $z \to \infty$, we have the following asymptotics for $J_\nu(z)$ and $Y_\nu(z)$:*

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left(\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(z^{-1})\right), \tag{A.0.6}$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}}\left(\sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(z^{-1})\right). \tag{A.0.7}$$

We finish this appendix with two lemmas concerning Bessel's equation (A.0.1). These will be useful when studying Bessel-type equations with inhomogeneous error terms, as in Section 4.4.

Lemma A.0.5. *Consider Bessel's equation (A.0.1), where ν is a nonnegative integer. We define the usual Wronskian, normalized by the Bessel functions of the first and second kind, by*

$$\mathcal{W}_\nu(z) \coloneqq \det \begin{vmatrix} J_\nu(z) & Y_\nu(z) \\ J_\nu'(z) & Y_\nu'(z) \end{vmatrix} = J_\nu(z)Y_\nu'(z) - Y_\nu(z)J_\nu'(z). \tag{A.0.8}$$

Then

$$\mathcal{W}_\nu(z) = \frac{2}{\pi z}. \tag{A.0.9}$$

Proof. A standard manipulation of the second order ODE (A.0.1) gives

$$\frac{d}{dz}\mathcal{W}_\nu(z) = -\frac{1}{z}\mathcal{W}_\nu(z).$$

Integrating for $z \in \mathbb{R}^+$ therefore yields $\mathcal{W}_\nu(z) = Cz^{-1}$ for some constant of integration C.

To determine C, simply use the asymptotics of Facts A.0.1 and A.0.2. This yields $C = \frac{2}{\pi}$, as required. (This computation is more straightforward in our case $\nu = 0$.) □

For the next lemma, we consider Bessel's equation of order 0 as a first-order system in $(f_1, f_2) = (f, \frac{df}{dz})$:

$$\frac{d}{dz}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_2 \\ -f_1 - \frac{1}{z}f_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{z} \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{A.0.10}$$

So for any $z_0, z_1 > 0$, we may define the solution operator[1] $\mathbf{S}(z_1; z_0)$ as a linear operator $\mathbf{S}(z_1; z_0) : \mathbb{R}^2 \to \mathbb{R}^2$ in the following manner: if $(f_1(z_0), f_2(z_0))$ is considered as data for the linear ODE (A.0.10), then $\mathbf{S}(z_1; z_0)$ maps this data $(f_1(z_0), f_2(z_0))$ to the value of the solution at $z = z_1$, namely $(f_1(z_1), f_2(z_1))$. As an equation:

$$\begin{bmatrix} f_1(z_1) \\ f_2(z_1) \end{bmatrix} = \mathbf{S}(z_1; z_0) \begin{bmatrix} f_1(z_0) \\ f_2(z_0) \end{bmatrix}. \tag{A.0.11}$$

The following lemma then asserts how to use this linear operator in solving Bessel's equation with inhomogeneous terms, as well as an explicit formula and estimates for the operator $\mathbf{S}(z_1; z_0)$.

Lemma A.0.6. *Consider Bessel's equation (A.0.1) with $\nu = 0$, but with an inhomogeneous term $F(z)$, i.e.*

$$\frac{d^2 f}{dz^2} + \frac{1}{z}\frac{df}{dz} + f = F. \tag{A.0.12}$$

Then, for $z_0, z_1 \in \mathbb{R}^+$ and the solution operator $\mathbf{S}(z; w)$ defined as before, we have the following expression:

$$\begin{bmatrix} f(z_1) \\ \frac{df}{dz}(z_1) \end{bmatrix} = \mathbf{S}(z_1; z_0) \begin{bmatrix} f(z_0) \\ \frac{df}{dz}(z_0) \end{bmatrix} + \int_{z_0}^{z_1} \mathbf{S}(z_1; \tilde{z}) \begin{bmatrix} 0 \\ F(\tilde{z}) \end{bmatrix} d\tilde{z}. \tag{A.0.13}$$

Furthermore, we have the following results regarding the linear operator $\mathbf{S}(z_1; z_0)$:

(1) In terms of the Bessel functions $J_\nu(z)$, $Y_\nu(z)$, one may write

$$\mathbf{S}(z_1; z_0) = \begin{bmatrix} J_0(z_1) & Y_0(z_1) \\ -J_1(z_1) & -Y_1(z_1) \end{bmatrix} \cdot \begin{bmatrix} J_0(z_0) & Y_0(z_0) \\ -J_1(z_0) & -Y_1(z_0) \end{bmatrix}^{-1}, \tag{A.0.14}$$

$$= \frac{\pi z_0}{2} \begin{bmatrix} J_0(z_1) & Y_0(z_1) \\ -J_1(z_1) & -Y_1(z_1) \end{bmatrix} \cdot \begin{bmatrix} -Y_1(z_0) & -Y_0(z_0) \\ J_1(z_0) & J_0(z_0) \end{bmatrix}. \tag{A.0.15}$$

(2) Using the usual l^2 norm on $\mathbb{R}^2$, defined as $\|(x, y)\|_{l^2} = \sqrt{x^2 + y^2}$, and defining $\|\mathbf{A}\|_{l^2 \to l^2} \coloneqq \sup_{x \in \mathbb{R}^2, \|x\|_{l^2} = 1} \|\mathbf{A}x\|_{l^2}$ to be the l^2 operator norm of a linear map $\mathbf{A}$, we have that, for all $z_0, z_1 > 0$,

$$\|\mathbf{S}(z_1; z_0)\|_{l^2 \to l^2} \leq \max\left\{\frac{z_0}{z_1}, 1\right\}. \tag{A.0.16}$$

Proof. Once again letting $(f_1, f_2) = (f, \frac{df}{dz})$, we have now the first-order system:

$$\frac{d}{dz}\begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{z} \end{bmatrix} \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} + \begin{bmatrix} 0 \\ F(z) \end{bmatrix}.$$

[1] This can be viewed as a "non-autonomous" semigroup, also commonly known as a propagator. In particular, it follows that $\mathbf{S}(z_2; z_1) \circ \mathbf{S}(z_1; z_0) = \mathbf{S}(z_2; z_0)$.

Then the expression (A.0.13) follows from the standard theory of first-order systems and Duhamel's principle.

To get (1), we recall that in light of Fact A.0.3, the most general solution to the first-order system with no inhomogeneous term is:

$$\begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} = \begin{bmatrix} c_J J_0(z) + c_Y Y_0(z) \\ -c_J J_1(z) - c_Y Y_1(z) \end{bmatrix} = \begin{bmatrix} J_0(z) & Y_0(z) \\ -J_1(z) & -Y_1(z) \end{bmatrix} \begin{bmatrix} c_J \\ c_Y \end{bmatrix}, \tag{A.0.17}$$

where c_J, c_Y are coefficients in $\mathbb{R}$. Plugging this into (A.0.11) and allowing c_J, c_Y to vary will yield (A.0.14). The expression (A.0.15) then follows by using Lemma A.0.5 to compute the inverse.

Finally, for (2), suppose $f(z)$ is a solution to the homogeneous Bessel's equation of order 0. Then we directly compute

$$\frac{d}{dz}(f(z)^2 + f'(z)^2) = -\frac{2}{z} f'(z)^2.$$

Therefore, if $z_1 \geq z_0$, then it is clear that $f(z_1)^2 + f'(z_1)^2 \leq f(z_0)^2 + f'(z_0)^2$, while if $z_1 < z_0$, then Grönwall's inequality gives

$$f(z_1)^2 + f'(z_1)^2 \leq \exp\left(\int_{z_0}^{z_1} -\frac{2}{z}\, dz\right)(f(z_0)^2 + f'(z_0)^2) = \left(\frac{z_0}{z_1}\right)^2 (f(z_0)^2 + f'(z_0)^2).$$

This yields exactly the required estimate on $\|\mathbf{S}(z_1; z_0)\|_{l^2 \to l^2}$. □

In this paper, we will often encounter rescaled versions of Bessel's equation: let $\chi > 0$ be a constant, then consider the equation:

$$\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} + \chi^2 f = F(x). \tag{A.0.18}$$

Using Lemma A.0.6, it is then straightforward to deduce the following corollary.

Corollary A.0.7. *Let $f = f(x)$ be a solution to (A.0.18). Then, if we denote by $\mathbf{Q}_\chi$ the scaling matrix*

$$\mathbf{Q}_\chi \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & \chi \end{bmatrix},$$

and define $\mathbf{S}_\chi(x_1; x_0) = \mathbf{Q}_\chi \circ \mathbf{S}(\chi x_1; \chi x_0) \circ \mathbf{Q}_\chi^{-1}$, then for any $x_0, x_1 > 0$ we have

$$\begin{bmatrix} f(x_1) \\ \frac{df}{dx}(x_1) \end{bmatrix} = \mathbf{S}_\chi(x_1; x_0) \begin{bmatrix} f(x_0) \\ \frac{df}{dx}(x_0) \end{bmatrix} + \int_{x_0}^{x_1} \mathbf{S}_\chi(x_1; \tilde{x}) \begin{bmatrix} 0 \\ F(\tilde{x}) \end{bmatrix} d\tilde{x}. \tag{A.0.19}$$

Furthermore, we have the following results regarding $\mathbf{S}_\chi(x_1; x_0)$:

(1) An explicit formula is given by:

$$\mathbf{S}_\chi(x_1; x_0) = \begin{bmatrix} J_0(\chi x_1) & Y_0(\chi x_1) \\ -\chi J_1(\chi x_1) & -\chi Y_1(x_1) \end{bmatrix} \cdot \begin{bmatrix} J_0(\chi x_0) & Y_0(\chi x_0) \\ -\chi J_1(\chi x_0) & -\chi Y_1(\chi x_0) \end{bmatrix}^{-1}, \tag{A.0.20}$$

$$= \frac{\pi x_0}{2} \begin{bmatrix} J_0(\chi x_1) & Y_0(\chi x_1) \\ -\chi J_0(\chi x_1) & -\chi Y_1(\chi x_1) \end{bmatrix} \cdot \begin{bmatrix} -\chi Y_1(\chi x_0) & -Y_0(\chi x_0) \\ \chi J_1(\chi x_0) & J_0(\chi x_0) \end{bmatrix}. \tag{A.0.21}$$

(2) The following estimate holds on the operator norm of $\mathbf{S}_\chi(x_1; x_0)$*:*

$$\|\mathbf{S}_\chi(x_1; x_0)\|_{l^2 \to l^2} \leq \max\{\chi, \chi^{-1}\} \cdot \max\left\{1, \frac{x_0}{x_1}\right\}. \tag{A.0.22}$$

Proof. Simply use the substitution $z = \chi x$ in (A.0.18), and apply Lemma A.0.6. Details are left to the reader. □

Appendix B

Einstein–Maxwell–scalar field asymptotics in Rein gauge

In Appendix B, we prove Theorem 5.1.0, regarding asymptotics of (ϕ, μ, λ, Q) solving the surface symmetric Einstein–Maxwell–scalar system (2.5.9)–(2.5.12) near $\{r = 0\}$. Recall that we either take the case $Q = 0$ and $\kappa \in \{0, +1\}$, or take Assumption 1 and Assumption 2 as given. We proceed in a series of lemmas.

Lemma B.0.1. *In either of the cases considered, there exist constants $C > 0$ and $\alpha > 0$ such that μ satisfies*

$$\frac{Q^2}{r^2} e^{2\mu} + e^{2\mu} \leq C r^{\alpha}. \tag{B.0.1}$$

In particular, by the contnuation criterion Lemma 2.5.3 one has global existence towards $\{r = 0\}$.

Furthermore, we also have the following estimates for $e^{\mu+\lambda}$:

$$\frac{Q^2}{r^2} e^{\mu+\lambda} + e^{\mu+\lambda} \leq C r^{\alpha-1}. \tag{B.0.2}$$

Proof. If $Q = 0$, by (2.5.9) we have $r\partial_r \mu \geq \frac{1}{2}(1 + \kappa e^{2\mu})$. Thus, by integrating in the direction $r \downarrow 0$, if $\kappa \in \{0, +1\}$, $e^{2\mu} \lesssim r$ is immediate. If instead $\kappa = -1$ we use Assumption 1 which tells us that $e^{2\mu}$ tends to 0 uniformly in r. Therefore we can still show $e^{2\mu} \lesssim r^{1/2}$. So (B.0.1) holds with $\alpha = \frac{1}{2}$.

If, on the other hand $Q \neq 0$, then we use Assumption 2, which says that $r\partial_r \mu \geq \frac{1}{2}(2 + \alpha)$. Directly integrating this towards $r \downarrow 0$, the estimate (B.0.1) follows for α as in Assumption 2.

The second estimate, (B.0.2), comes from combining the negative of (2.5.13), i.e.

$$r\partial_r(\lambda - \mu) = -1 - \kappa e^{2\mu} + \frac{Q^2}{r^2} e^{2\mu},$$

with (B.0.1) to get $e^{\lambda-\mu} \lesssim r^{-1}$. We then multiply this with (B.0.1) to get (B.0.2). □

Lemma B.0.2 (Wave estimates)**.** *Let f obey the following equation, for F some inhomogeneity*

$$(r\partial_r)^2 f - r^2 e^{2(\mu-\lambda)}\partial_x^2 f = \left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu} r\partial_r f + r^2 e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x f + F. \tag{B.0.3}$$

Then for any $\gamma > 0$ satisfying $\gamma < \min\{\frac{\alpha}{2}, \frac{1}{2}\}$ where $\alpha > 0$ is as in Lemma B.0.1, one has:

$$\begin{aligned}&|(r\partial_r + re^{\mu-\lambda}\partial_x)f|(r,x) + |(r\partial_r - re^{\mu-\lambda}\partial_x)f|(r,x) + r^\gamma|f|(r,x)\\ &\quad\lesssim r_0 \sup_{x\in\mathbb{S}^1}|\partial_r f(r_0,x)| + r_0\sup_{x\in\mathbb{S}^1}|e^{\mu-\lambda}\partial_x f(r_0,x)| + r_0^\gamma \sup_{x\in\mathbb{S}^1}|f(r_0,x)| + \int_r^{r_0}\sup_{\substack{\tilde r\le s\le r_0\\ x\in\mathbb{S}^1}}|F(s,x)|\,\frac{d\tilde r}{\tilde r}.\end{aligned} \tag{B.0.4}$$

Proof. We introduce the notation $\sup_{H_{\tilde r}}|f|$ to denote $\sup_{x\in\mathbb{S}^1}|f(\tilde r,x)|$, and define

$$A(\tilde r) = \max\left\{\sup_{H_{\tilde r}}|(\partial_r + e^{\mu-\lambda}\partial_x)f|, \sup_{H_{\tilde r}}|(\partial_r - e^{\mu-\lambda}\partial_x)f|, \sup_{H_{\tilde r}} r^{-1+\gamma}|f|\right\}$$

We aim to derive an integral inequality for $A(r)$ to which we may apply Grönwall's inequality. Using the wave equation (B.0.3) for f, as well as the transport equation (2.5.13), one may derive the following:

$$\begin{aligned}&(r\partial_r - re^{\mu-\lambda}\partial_x)(\partial_r + e^{\mu-\lambda}\partial_x)f\\ &\qquad = -(\partial_r - e^{\mu-\lambda}\partial_x)f + \left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}(\partial_r + e^{\mu-\lambda}\partial_x)f + \frac{1}{r}F.\end{aligned}$$

Therefore using Lemma B.0.1, one has

$$|(r\partial_r - re^{\mu-\lambda}\partial_x)(\partial_r + e^{\mu-\lambda}\partial_x)f(\tilde r,x)| \ge -A(\tilde r) - C\tilde r^\alpha A(\tilde r) - \frac{1}{\tilde r}\sup_{H_{\tilde r}}|F|.$$

The idea is now simply to integrate this towards $r = 0$, via the characteristic (null) vector field $\underline{L} = r\partial_r - re^{\mu-\lambda}\partial_x$. Indeed, let $\gamma_{\underline{L}} : (0, r_0] \to \mathcal{Q}$ be a curve whose tangent vector is $\underline{L}$ and which is parameterized by r. That is, $\gamma_{\underline{L}}(r) = (r, X(r))$ with $\frac{dX}{dr} = -e^{\mu-\lambda}$. Then integrating the above yields

$$\begin{aligned}&|((\partial_r + e^{\mu-\lambda}\partial_x)f)(\gamma_{\underline{L}}(r))|\\ &\qquad\le |((\partial_r + e^{\mu-\lambda}\partial_x)f)(\gamma_{\underline{L}}(r_0))| + \int_r^{r_0}(1+C\tilde r^\alpha)A(\tilde r)\,\frac{d\tilde r}{\tilde r} + \int_r^{r_0}\frac{1}{\tilde r^2}\sup_{H_{\tilde r}}|F|\,d\tilde r.\end{aligned}$$

Since this is true for all curves $\gamma_{\underline{L}}$ which are integral curves of $\underline{L}$, we therefore have that for all $x \in \mathbb{S}^1$:

$$|(\partial_r + e^{\mu-\lambda}\partial_x)f)(r,x)| \le \sup_{H_{r_0}}|(\partial_r + e^{\mu-\lambda}\partial_x)f| + \int_r^{r_0}(1+C\tilde r^\alpha)A(\tilde r)\,\frac{d\tilde r}{\tilde r} + \int_r^{r_0}\frac{1}{\tilde r^2}\sup_{H_{\tilde r}}|F|\,d\tilde r.$$

On the other hand, a similar computation involving integral curves of $L = r\partial_r + re^{\mu-\lambda}\partial_x$ yields:

$$|(\partial_r - e^{\mu-\lambda}\partial_x)f)(r,x)| \leq \sup_{H_{r_0}} |(\partial_r - e^{\mu-\lambda}\partial_x)f| + \int_r^{r_0} (1 + C\tilde{r}^\alpha)A(\tilde{r})\,\frac{d\tilde{r}}{\tilde{r}} + \int_r^{r_0} \frac{1}{\tilde{r}^2} \sup_{H_{\tilde{r}}} |F|\,d\tilde{r}.$$

Finally, since $\partial_r(r^{-1+\gamma}f) = -(1-\gamma)r^{-2+\gamma}f + \frac{1}{2}r^{-1+\gamma}(\partial_r + e^{\mu-\lambda}\partial_x)f + \frac{1}{2}r^{-1+\gamma}(\partial_r - e^{\mu-\lambda}\partial_x)f$,

$$r^{-1+\gamma}|f|(r,x) \leq \sup_{H_{r_0}}(r^{-1+\gamma}|f|) + \int_r^{r_0} (1 - \gamma + \tilde{r}^\gamma)\,A(\tilde{r})\,\frac{d\tilde{r}}{\tilde{r}}.$$

Combining all of these, we deduce that

$$\begin{aligned} A(r) &\leq A(r_0) + \int_r^{r_0} (1 + C\tilde{r}^\alpha + \tilde{r}^\gamma)A(\tilde{r})\frac{d\tilde{r}}{\tilde{r}} + \int_r^{r_0} \frac{1}{\tilde{r}^2} \sup_{H_{\tilde{r}}} |F|\,d\tilde{r} \\ &\leq A(r_0) + \int_r^{r_0} (1 + C\tilde{r}^\alpha + \tilde{r}^\gamma)A(\tilde{r})\frac{d\tilde{r}}{\tilde{r}} + \frac{1}{r} \sup_{\substack{\tilde{r}\leq s\leq r_0 \\ x\in\mathbb{S}^1}} |F(s,x)|. \end{aligned}$$

One now applies Grönwall's inequality together with $\int_r^{r_0}(1 + C\tilde{r}^\alpha + \tilde{r}^\gamma)\frac{d\tilde{r}}{\tilde{r}} \leq \log\left(\frac{r_0}{r}\right) + \log D$ where $D > 0$ is some constant, to find that

$$rA(r) \lesssim r_0 A(r_0) + \int_r^{r_0} \frac{1}{\tilde{r}} \sup_{\substack{\tilde{r}\leq s\leq r_0 \\ x\in\mathbb{S}^1}} |F(s,x)|\,d\tilde{r},$$

which is equivalent to (B.0.4). □

Lemma B.0.3 (Upper bounds at all orders)**.** *For (ϕ, λ, μ) as in Theorem 5.1.0, for all $0 \leq j \leq k$ there exists a constant C_j such that*

$$\max_{f\in\{\phi,\mu,\lambda\}} |r\partial_r\partial_x^j f| + \max_{f\in\{\phi,\mu,\lambda\}} |re^{\mu-\lambda}\partial_x^{j+1} f| \leq C_j. \tag{B.0.5}$$

Moreover C_j depends only on up to j ∂_x-derivatives of $r\partial_r\phi$, $r\partial_r\mu$ and $r\partial_r\lambda$ and up to $j+1$ ∂_x-derivatives of ϕ, μ and λ at data i.e. at $r = r_0$.

Proof. For the sake of brevity, we will focus on proving the part of (B.0.5) corresponding to $f = \phi$, and mention at the end how one gets the corresponding estimates for μ and λ.

Due to (2.5.12), ϕ satisfies the wave equation (B.0.3) with $F = 0$. Therefore Lemma B.0.2 immediately yields that for some $C_0 > 0$ depending only on the C^0 norm of $r\partial_r\phi$ and $\partial_x\phi$ at data, one has

$$|r\partial_r\phi| + |re^{\mu-\lambda}\partial_x\phi| \leq C_0. \tag{B.0.6}$$

The most challenging step is to get the estimate (B.0.5) for $j = 1$. For convenience, we further introduce the notation $\sup_{\mathcal{D}_{\tilde{r},r_0}} |f| = \sup_{\tilde{r}\leq s\leq r_0, x\in\mathbb{S}^1} |F(s,x)|$, and define the quantity

$$B(\tilde{r}) = \max\left\{ \sup_{\mathcal{D}_{\tilde{r},r_0}} |r\partial_r\partial_x\phi|,\ \sup_{\mathcal{D}_{\tilde{r},r_0}} |re^{\mu-\lambda}\partial_x^2\phi|,\ \sup_{\mathcal{D}_{\tilde{r},r_0}} r^\gamma|\partial_x\phi| \right\}.$$

It will be necessary to first find some preliminary estimates for $\partial_x(\mu-\lambda)$ and $\partial_x^2(\mu-\lambda)$. To do so, we use the transport equation (2.5.13) for $\mu-\lambda$ and differentiate in x, to get:

$$r\partial_r\partial_x(\mu-\lambda) = \left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}(2\partial_x\mu) = \left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}2(r\partial_r\phi)(\partial_x\phi). \tag{B.0.7}$$

Note that the second equality followed from the constraint equation (2.5.10).

We then estimate the right hand side of (B.0.7) by:

$$\begin{aligned}\left|\left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}2(r\partial_r\phi)(\partial_x\phi)\right| &\leq \left|r^{-1}\left(\frac{Q^2}{r^2}+1\right)e^{\mu+\lambda}\cdot 2r\partial_r\phi\cdot re^{\mu-\lambda}\partial_x\phi\right| \\ &\leq r^{-1}2C_0^2\cdot\left(\frac{Q^2}{r^2}+1\right)e^{\mu+\lambda},\end{aligned}$$

for C_0 as in (B.0.6). Then using (B.0.2) in Lemma B.0.1, we have that $|r\partial_r\partial_x(\mu-\lambda)|\lesssim r^{-2+\alpha}$. Integrating, we get

$$|\partial_x(\mu-\lambda)|\lesssim r^{-2+\alpha}. \tag{B.0.8}$$

Differentiating (B.0.7) once again in x, we get

$$r\partial_r\partial_x^2(\mu-\lambda) = \left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}\left[4(r\partial_r\phi)^2(\partial_x\phi)^2+2(r\partial_r\partial_x\phi)(\partial_x\phi)+2(r\partial_r\phi)(\partial_x^2\phi)\right].$$

Using the definition of $B(r)$ and similar methods to that of the estimate for (B.0.8), we can estimate $|r\partial_r\partial_x^2(\mu-\lambda)|\lesssim r^{-2+\alpha}[r^{-\gamma}B(r)+B(r)]$. Since $B(r)$ increases as $r\to 0$, we integrate this towards $r=0$, and get

$$|\partial_x^2(\mu-\lambda)|\lesssim r^{-2+\alpha-\gamma}B(\tilde{r})+\sup_{H_{r_0}}|\partial_x^2(\mu-\lambda)|. \tag{B.0.9}$$

We now commute the wave equation (2.5.12) with ∂_x to get that

$$\begin{aligned}(r\partial_r)^2\partial_x\phi+r^2e^{2(\mu-\lambda)}\partial_x^3\phi &= \left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}(r\partial_r\partial_x\phi)+r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x^2\phi \\ &+2r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x^2\phi+r^2e^{2(\mu-\lambda)}\partial_x^2(\mu-\lambda)\partial_x\phi+2r^2e^{2(\mu-\lambda)}(\partial_x(\mu-\lambda))^2\partial_x\phi \\ &+\left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}(2\partial_x\mu)(r\partial_r\phi).\end{aligned}$$

Denote the latter two lines of this commuted equation by $F^{(1)}$. To estimate $F^{(1)}$, we will use (B.0.8), (B.0.9), and finally also $e^{\mu-\lambda}\lesssim r$ (which follows easily from (2.5.13) and (B.0.1)). We have therefore

$$\begin{aligned}|r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x^2\phi|&\lesssim r^{1+\alpha}e^{\mu-\lambda}|\partial_x^2\phi|\lesssim r^\alpha B(r),\\ |r^2e^{2(\mu-\lambda)}(\partial_x(\mu-\lambda))^2\partial_x\phi|&\lesssim r^4\cdot r^{2(-2+\alpha)}|\partial_x\phi|\lesssim r^{2\alpha-\gamma}B(r),\\ |r^2e^{2(\mu-\lambda)}\partial_x^2(\mu-\lambda)\partial_x\phi|&=|re^{\mu-\lambda}\partial_x^2(\mu-\lambda)|\cdot|re^{\mu-\lambda}\partial_x\phi|\lesssim r^{\alpha-\gamma}B(r),\end{aligned}$$

while by Lemma B.0.1, the constraint equation (2.5.9) and (B.0.6) we have:

$$\left|\left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}(2\partial_x\mu)(r\partial_r\phi)\right|\lesssim\left(\frac{Q^2}{r^2}+1\right)e^{2\mu}\cdot|\partial_x\phi|\lesssim r^{\alpha-\gamma}B(r).$$

The upshot is that $|F^{(1)}(r)| \lesssim r^{\alpha-\gamma}B(r)$. Therefore using the commuted equation and Lemma B.0.2, one shows that $B(r) \lesssim B(r_0) + \int_{r_0}^{r} \tilde{r}^{\alpha-\gamma-1}B(\tilde{r})d\tilde{r}$. Since $0 < \gamma < \alpha$, one may apply Grönwall's inequality to show that $B(r)$ is uniformly bounded by some constant C_1, thereby (B.0.5) holds for $j = 1$ and $f = \phi$.

There are two things left to do: get estimates for μ and λ, and then get estimates for higher derivatives i.e. $j \geq 1$. For μ, one uses the wave equation for μ, equation (2.5.11). Even for the uncommuted equation (2.5.11), there is already an inhomogeneity in the application of Lemma B.0.2. Fortunately, this is easily dealt with, either using Lemma B.0.1, or using $\partial_x\phi \leq C_1 r^{-\gamma}$. Commuting the equation (2.5.11) once with ∂_x and applying similar methods to before, we obtain the desired estimates at orders $j = 0, 1$ for μ.

Next, for λ, we actually already have estimates for $\mu - \lambda$, see for instance (B.0.7), (B.0.8) and (B.0.9). These can easily be used to get the desired estimates for λ at orders $j = 0, 1$. Finally, for the higher derivatives one simply has to commute the equations (2.5.12), (2.5.11) and (2.5.13) with further ∂_x-derivatives, and carry out the same approach. We remark that the commuted equations already appear in Section 6.3, and that in the context of Theorem 5.1.0 all inhomogeneous terms that appear can be easily controlled given we already have the estimates at orders $j = 0, 1$. Details are left to the reader. □

Lemma B.0.4 (Completion of Theorem 5.1.0)**.** *For (ϕ, λ, μ) as in Theorem 5.1.0, there exists a function $\Psi : \mathbb{S}^1 \to \mathbb{R}$ such that the following convergence holds strongly as $r \to 0$,*

$$r\partial_r\phi(r,\cdot) \to \Psi(\cdot), \quad r\partial_r\mu(r,\cdot) \to \frac{1}{2}(\Psi(\cdot)^2 + 1), \quad r\partial_r\lambda(r,\cdot) \to \frac{1}{2}(\Psi(\cdot)^2 - 1) \quad \text{in } C^{k-1}. \tag{B.0.10}$$

Moreover, there exist further functions $\Xi, M, \Lambda : \mathbb{S}^1 \to \mathbb{R}$ such that

$$\phi(r,\cdot) - r\partial_r\phi(r,\cdot)\log r \to \Xi(\cdot), \tag{B.0.11}$$

$$\mu(r,\cdot) - r\partial_r\mu(r,\cdot)\log r \to M(\cdot), \tag{B.0.12}$$

$$\lambda(r,\cdot) - r\partial_r\lambda(r,\cdot)\log r \to \Lambda(\cdot) \quad \text{in } C^{k-1}. \tag{B.0.13}$$

Finally, the asymptotic constraint equation $dM = \Psi d\,\Xi$ holds.

Proof. Let us first study the convergence of $r\partial_r\phi$. To do so, use the wave equation (2.5.12), which says

$$(r\partial_r)(r\partial_r\phi) = r^2e^{2(\mu-\lambda)}\partial_x^2\phi + r^2e^{2(\mu-\lambda)}\partial_x(\mu-\lambda)\partial_x\phi + \left(-\frac{Q^2}{r^2} + \kappa\right)e^{2\mu}r\partial_r\phi. \tag{B.0.14}$$

By Lemma B.0.3 for $j = 1$, noting that integrating the bound for $|r\partial_r\partial_x f|$ in (B.0.5) yields that $|\partial_x f| \lesssim 1 + |\log r|$, and also using Lemma B.0.1 as well as the bound $e^{\mu-\lambda} \lesssim r$ (found by integrating

(2.5.13) and Lemma B.0.1),

$$|(r\partial_r)(r\partial_r\phi)| \leq r^2 e^{2(\mu-\lambda)}|\partial_x^2\phi| + r^2 e^{2(\mu-\lambda)}|\partial_x(\mu-\lambda)||\partial_x\phi| + \left(\frac{Q^2}{r^2}+1\right)e^{2\mu}|r\partial_r\phi|$$
$$\lesssim r^2 + r^4(1+|\log r|)^2 + r^\alpha.$$

Crucially, the right hand side is integrable with respect to the differential $\frac{dr}{r}$ as $r \to 0$, thus $r\partial_r\phi(r,\cdot)$ converges to some function $\Psi(\cdot)$ as $r \to 0$, at least in the C^0 norm.

To upgrade C^0 convergence to C^{k-1} convergence, one differentiates (B.0.14) $k-1$ times with ∂_x. Using Lemma B.0.3 for all orders, and noting that extra powers of $(1+|\log r|)$ will arise when ∂_x hits either $e^{2(\mu-\lambda)}$ or $e^{2\mu}$, one obtains:

$$|(r\partial_r)(r\partial_r\partial_x^k\phi)| \lesssim (r^2 + r^4(1+|\log r|)^2 + r^\alpha)(1+|\log r|)^{k-1}.$$

The right hand side remains integrable with respect to $\frac{dr}{r}$, hence the convergence of $r\partial_r\phi(r,\cdot)$ to $\Psi(r,\cdot)$ is actually in C^k, as required.

Next, we move to the convergence of $r\partial_r\mu$ and $r\partial_r\lambda$. From (2.5.9)

$$r\partial_r\mu = \frac{1}{2}\left((r\partial_r\phi)^2+1\right) + \frac{1}{2}r^2e^{2(\mu-\lambda)}(\partial_x\phi)^2 + \frac{1}{2}\left(-\frac{Q^2}{r^2}+\kappa\right)e^{2\mu}.$$

By Lemma B.0.3 for $j=1$ and Lemma B.0.1 respectively, the latter two terms on the right hand side decay to 0 as $r^4(1+|\log r|)^2$ and r^α as $r \to 0$. This remains true even after one takes j ∂_x-derivatives, for $0 \leq j \leq k-1$ (though as before we introduce an additional factor of $(1+|\log r|)^j$. Since we already have convergence of $r\partial_r\phi$ to Ψ, the convergence for $r\partial_r\mu$ follows immediately. The convergence of $r\partial_r\lambda$ is analogous, using (2.5.8).

To get (B.0.11)–(B.0.13), we use that

$$(r\partial_r)(\phi - (r\partial_r\phi)\log r) = (r\partial_r)(r\partial_r\phi)\cdot\log r.$$

But we can compute the right hand side from (B.0.14), and therefore we can bound $|(r\partial_r)[\partial_x^j(\phi - (r\partial_r\phi))]|$ by $(r^2+r^4(1+|\log r|)^2+r^\alpha)(1+|\log r|)^j\log r$ for all $0 \leq j \leq k-1$. This remains integrable with respect to $\frac{dr}{r}$ towards $r=0$, hence there is some $\Xi(\cdot)$ so that $\phi(r,\cdot) - r\partial_r\phi(r,\cdot)\log r \to \Xi(\cdot)$ in C^{k-1}.

The analogous convergence for $\mu - r\partial_r\mu\log r$ and $\lambda - r\partial_r\lambda\log r$ is similar, though perhaps a little more involved, since it involves taking an $r\partial_r$-derivative of (2.5.8)–(2.5.9) in order to bound $(r\partial_r)(r\partial_r\mu)\cdot\log r$ and $(r\partial_r)(r\partial_r\lambda)\cdot\log r$. We leave the details to the reader.

Finally, we explain how to get the asymptotic momentum constraint $dM = \Psi d\,\Xi$. Using the constraint equation (2.5.10) for $\partial_x\mu$, we have:

$$\partial_x\left(\mu - (r\partial_r\mu)\log r\right) = (r\partial_r\phi)(\partial_x\phi) - (r\partial_r)[(r\partial_r\phi)(\partial_x\phi)]\cdot\log r$$
$$= (r\partial_r\phi)\cdot\partial_x\left(\phi - (r\partial_r\phi)\log r\right) - (r\partial_r)(r\partial_r\phi)\cdot(\partial_x\phi)\log r.$$

By the C^1 convergence in (B.0.10), the left hand side and the first term on the right hand side converge to $\partial_x M$ and $\Psi\partial_x\Xi$ respectively as $r \to 0$. Finally, by (B.0.14) and its bounds the final term is bounded by $(r^2+r^4(1+|\log r|)^2+r^\alpha)(\log r)^2$, which converges to 0 as $r \to 0$. This completes the proof of the asymptotic momentum constraint. $\square$

Appendix C

A weighted Sobolev product estimate

In Appendix C we provide a proof of the weighted L^2 product estimate Lemma 5.2.2. By a simple density argument, it suffices to assume that f and g are smooth, and for convenience we work in L^p–spaces with respect to the measure w^2dx i.e. with

$$\|f\|_{L^p(w^2dx)} = \left(\int_{\mathbb{S}^1} |f(x)|^p w^2(x)\, dx\right)^{\frac{1}{p}}$$

For $0 \leq k \leq K$, we shall moreover define $p_k = \frac{2K}{k}$ (where $p_0 = \infty$) and the quantities F_k and G_k as:

$$F_0 = \|f\|_{L^\infty(w^2dx)}, \qquad F_k = \sum_{i=0}^{k-1} \|\partial_x^{k-i} f\|_{L^{p_k}(w^2dx)} \|\partial_x W\|^i_{L^\infty(w^2dx)} \text{ for } 1 \leq k \leq K,$$

$$G_0 = \|g\|_{L^\infty(w^2dx)}, \qquad G_k = \sum_{i=0}^{k-1} \|\partial_x^{k-i} f\|_{L^{p_k}(w^2dx)} \|\partial_x W\|^i_{L^\infty(w^2dx)} \text{ for } 1 \leq k \leq K.$$

With this notation, Lemma 5.2.2 is reduced to proving that

$$\|\partial_x^M f\, \partial_x^N g\|_{L^2(w^2dx)} \lesssim_{M,N} F_0 G_k + G_0 F_k. \tag{C.0.1}$$

In order to prove such an estimate, our first goal will be to show that for $0 \leq k \leq K$, one has

$$F_k \lesssim F_0^{\frac{K-k}{K}} F_K^{\frac{k}{K}}. \tag{C.0.2}$$

We do this in several steps. Firstly, for $0 < k < K$ one may use the identity

$$\begin{aligned}\partial_x(w^2|\partial_x^k f|^{p_k-2}\, \partial_x^k f \cdot \partial_x^{k-1} f) &= w^2|\partial_x^k f|^{p_k} \\ &\quad + w^2(p_k-1)|\partial_x^k f|^{p_k-3}\, \partial_x^{k+1} f \cdot \partial_x^k f \cdot \partial_x^{k-1} f + 2w^2 \partial_x W \cdot |\partial_x^k f|^{p_k-2} \partial_x^k f \cdot \partial_x^{k-1} f.\end{aligned}$$

Integrating this identity over $x \in \mathbb{S}^1$, one thereby deduces that

$$\int_{\mathbb{S}^1} |\partial_x^k f|^{p_k} w^2\, dx \lesssim \int_{\mathbb{S}^1} |\partial_x^k f|^{p_k-2}\, |\partial_x^{k+1} f|\, |\partial_x^{k-1} f| w^2\, dx + \int_{\mathbb{S}^1} |\partial_x^k f|^{p_k-1}\, |\partial_x^{k-1} f|\, |\partial_x W| w^2\, dx,$$

and an application of Hölders inequality to the two integrals on the right hand side yields (all L^p spaces are with respect to the measure w^2dx):

$$\|\partial_x^k f\|_{L^{p_k}}^{p_k} \lesssim \|\partial_x^k f\|_{L^{p_k}}^{p_k-2} \|\partial_x^{k-1} f\|_{L^{p_{k-1}}} \left(\|\partial_x^{k+1} f\|_{L^{p_{k+1}}} + \|\partial_x^k f\|_{L^{p_{k+1}}} \|\partial_x W\|_{L^\infty} \right).$$

Thus by the definition of F_k one concludes that

$$\|\partial_x^k f\|_{L^{p_k}}^2 \lesssim F_{k-1} F_{k+1}.$$

On the other hand, a standard L^p interpolation estimate (where the implied constants are importantly independent of the measure associated to the L^p space), one has that for $1 \leq j \leq k-1$,

$$\|\partial_x^{k-j} f\|_{L^{p_k}}^2 \|\partial_x W\|_{L^\infty}^{2j} \lesssim \|\partial_x^{k-j} f\|_{L^{p_{k-1}}} \|\partial_x W\|_{L^\infty}^{j-1} \cdot \|\partial_x^{k-j} f\|_{L^{p_{k+1}}} \|\partial_x W\|_{L^\infty}^{j+1} \leq F_{k-1} F_{k+1}.$$

Combining the above two equations, one therefore has that $F_k^2 \lesssim F_{k-1} F_{k+1}$.

It is straightforward to go from this to (C.0.2). For instance, assuming that all the F_j are nonzero the inequality $F_j^2 \lesssim F_{j-1} F_{j+1}$ implies there is some constant $C \in \mathbb{R}$ such that

$$\begin{aligned} &- \frac{K-k}{K} \log F_0 - \frac{k}{K} \log F_K + F_k \\ &\qquad = \sum_{j=1}^{K-1} \left(- \log \frac{1}{2} F_{j-1} - \frac{1}{2} \log F_{j+1} + \log F_j \right) \cdot \left(\frac{\min\{j(K-k), (K-j)k\}}{K} \right) \leq C, \end{aligned}$$

which immediately yields (C.0.2).

Finally, in order to deduce (C.0.1) from (C.0.2) (and the analogous inequality for the G_ks) we simply apply Hölder's inequality followed by (C.0.2), as follows:

$$\|\partial_x^M f \, \partial_x^N g\|_{L^2} \leq \|\partial_x^M f\|_{L^{p_M}} \|\partial_x^N g\|_{L^{p_N}} \leq F_M G_N \leq (F_0 G_K)^{\frac{N}{K}} (F_K G_0)^{\frac{M}{K}}.$$

The conclusion thus follows from Young's inequality. □

Appendix D

Fourier decomposition and ΨDOs

In Appendix D, we review the Fourier decomposition and Sobolev spaces H^s on $\mathbb{T}^D = [-\pi, \pi]^D$. We also review pseudodifferential operators on $\mathbb{T}^D$ and their symbol calculus, and then mention some properties of the symbol $\mathcal{T}_*$ defined in Definition 1.6.1 in Lemma D.0.1.

Definition D.0.1. Let $f : \mathbb{T}^D \to \mathbb{R}$ be a sufficiently regular[1] function. The *standard Fourier decomposition*, or *Fourier series*, $(f_\lambda)_{\lambda \in \mathbb{Z}^d}$, is defined as follows. For $\lambda = (\lambda_1, \dots, \lambda_D) \in \mathbb{Z}^D$, we have

$$f_\lambda = \frac{1}{(2\pi)^D} \int_{\mathbb{T}^D} f(x) e^{-i\lambda \cdot x} \, dx = \frac{1}{(2\pi)^D} \int_{x \in [-\pi,\pi]^D} f(x) e^{-i\lambda_k x^k} \, dx^1 \cdots dx^D. \tag{D.0.1}$$

A similar decomposition holds for tensors in $\mathbb{T}^D$, e.g. a $(1,1)$ tensor $A = A_i^{\ j}$ on $\mathbb{T}^D$ has Fourier decomposition:

$$(A_\lambda)_i^{\ j} = \frac{1}{(2\pi)^D} \int_{\mathbb{T}^D} A_i^{\ j}(x) e^{-i\lambda \cdot x} \, dx.$$

Remark. For sufficiently regular functions f, the Fourier inversion formula holds:

$$f(x) = \sum_{\lambda \in \mathbb{Z}^D} f_\lambda \, e^{i\lambda \cdot x}, \tag{D.0.2}$$

hence one is able to recover a function f on $\mathbb{T}^D$ from its Fourier series, and similarly for tensors on $\mathbb{T}^D$.

Definition D.0.2. For $s \in \mathbb{R}$, a function $f : \mathbb{T}^D \to \mathbb{R}$ lies in the *Sobolev space* $H^s = H^s(\mathbb{T}^D)$ if the following squared norm is finite, where $\langle \cdot \rangle$ is the usual Japanese bracket; $\langle x \rangle^2 \coloneqq 1 + |x|^2$.

$$\|f\|_{H^s}^2 \coloneqq \sum_{\lambda \in \mathbb{Z}^D} \langle \lambda \rangle^{2s} |f_\lambda|^2. \tag{D.0.3}$$

[1]From the standard Fourier theory, $f \in L^1(\mathbb{T}^D)$ will be sufficient, however are eventually more interested in the Sobolev spaces H^s. As we also entertain $s < 0$ in some cases, one may even consider the case where f is simply a distribution.

Similarly, a $(1,1)$ tensor $A = A_i^{\ j}$ lies in the Sobolev space H^s if

$$\|A_i^{\ j}\|_{H^s}^2 \coloneqq \sum_{i,j=1}^{D} \sum_{\lambda \in \mathbb{Z}^D} \langle \lambda \rangle^{2s} \left| (A_\lambda)_i^{\ j} \right|^2 < +\infty. \tag{D.0.4}$$

Definition D.0.3. Let $L : \mathbb{Z}^D \to \mathbb{R}$ be a function that grows no faster than polynomially. We define the *symbol* $\mathcal{L}$ associated to L, as follows: For $f \in C^\infty(\mathbb{T}^D)$ smooth, let $\mathcal{L}f$ be the (smooth) function whose Fourier coefficients are given by the following expression

$$(\mathcal{L}f)_\lambda = \begin{cases} L(\lambda) f_\lambda & \text{if } \lambda \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Next, following the symbol calculus of [80], for $F : \mathbb{R} \to \mathbb{R}$ such that $F \circ T : \mathbb{Z}^D \to \mathbb{R}$ grows no faster than polynomially, define the operator $F(\mathcal{L}) : C^\infty(\mathbb{T}^D) \to C^\infty(\mathbb{T}^D)$ to be such that

$$(F(\mathcal{L})f)_\lambda = \begin{cases} F(\mathcal{L})\,(F \circ L(\lambda)) f_\lambda & \text{if } \lambda \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $m \in \mathbb{R}$, we say that $\mathcal{L}$ is in the *symbol class* S^m if $L(\lambda)$ is bounded by a constant multiple of $\langle \lambda \rangle^m$. If $\mathcal{L}$ is in the symbol class S^m, then for any $s \in \mathbb{R}$, $\mathcal{L}$ may be extended as a bounded operator from H^s to H^{s-m}.

As promised, we end this section with a brief discussion of the properties of the symbol $\mathcal{T}_*$, which is defined in Definition 1.6.1.

Lemma D.0.1. *Let $\mathcal{T}_*$ be as in Definition 1.6.1. Then:*

(1) The symbol $\log(\mathcal{T}_*)$ lies in the symbol class S^ε if and only if $\varepsilon > 0$.

(2) For $p = \max p_i$, the symbol $\mathcal{T}_*^{-1}$ lies in the symbol class S^N if and only if $N \geq \frac{1}{1-p}$.

(3) For $P = \min p_i$, the symbol $\mathcal{T}_*$ lies in the symbol class S^N if and only if $N \geq -\frac{1}{1-P}$.

Proof. Using Definition 1.6.1, the lemma will follow from an understanding of the growth rates of the function $t_* : \mathbb{Z}^D \to (0, +\infty)$ such that $t_*(\lambda) = t_{\lambda *}$ satisfies $t_*(0) = 1$ and

$$\sum_{i=1}^{D} \lambda_i^2 t_{\lambda *}^{2-2p_i} = 1, \text{ for } \lambda \neq 0.$$

For each $\lambda \neq 0$, it is immediate from this definition that for all i, one has $t_{\lambda *} \leq \lambda_i^{-\frac{1}{1-p_i}}$, and that this is inequality is sharp if λ_i is the only nonvanishing component of λ. This proves (3) immediately.

To show (2), note that for $\lambda \neq 0$, there must be some $i \in \{1, \dots D\}$ such that $\lambda_i^2 t_{\lambda *}^{2-2p_i} \geq \frac{1}{D}$. Therefore for this i, we have that $t_{\lambda *}^{-1} \lesssim \lambda_i^{\frac{1}{1-p_i}} \leq \langle \lambda \rangle^{\frac{1}{1-p}}$. Thus if $N \geq \frac{1}{1-p}$, the symbol $\mathcal{T}_*$ lies in the

symbol class S^N. Conversely, if without loss of generality $p = p_1$, then considering $\lambda = (\lambda_1, 0, \dots, 0)$ proves that we cannot have $t_{\lambda *}^{-1} \lesssim \langle \lambda \rangle^N$ for any smaller N.

Finally, since we have shown both that $t_{\lambda *} \leq \langle \lambda \rangle^{-\frac{1}{1-P}}$ and that $t_{\lambda *}^{-1} \leq \langle \lambda \rangle^{\frac{1}{1-p}}$, it is clear that $\log t_{\lambda *}^{-1} \asymp \log \langle \lambda \rangle$. From this, the claim (1) is also straightforward. $\square$

Appendix E

Fuchsian ODE systems

A *Fuchsian PDE* is an equation of the following form, where $\mathbf{V} : (0,T) \times \mathbb{R}^k \to \mathbb{R}^N$, A is a suitable matrix-valued operator, and f vanishes suitably as $t \to 0$.

$$\left(t\frac{\partial}{\partial t} + A\right)\mathbf{V} = f[t, \mathbf{V}].$$

Such equations originally arose in the "Fuchs-Frobenius" study of special functions and their singularities in the complex plane. We refer the reader to the introduction to [96] for details.

The above definition of Fuchsian PDEs is somewhat imprecise, and in this article we only make concrete definitions in the case of ODEs. In this case, the definition will be as follows.

Definition E.0.1 (Fuchsian ODEs)**.** Let $\mathbf{V} : [0,T) \subset \mathbb{R} \to \mathbb{R}^N$ be a continuous vector-valued function, and let $\varepsilon > 0$. We say that $\mathbf{V}$ satisfies a *Fuchsian ODE with t^ε weight* if for $t \in (0,t)$, the following equation holds:

$$\left(t\frac{d}{dt} + A\right)\mathbf{V} = t^\varepsilon F[t, \mathbf{V}] \tag{E.0.1}$$

where A is a constant $N \times N$ matrix, and $F : [0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous in both arguments and locally Lipschitz in its second argument.

Remark. One can make more general definitions where the RHS of (E.0.1) tends to 0 at a slower rate than t^ϵ, or where the regularity assumptions are somewhat relaxed. However we do not need to do this here, and we refer the interested reader to [96, Chapter 5].

We would like a local existence and uniqueness theory for $\mathbf{V}$ satisfying Fuchsian ODEs of the form (E.0.1), with initial data given at the 'singularity' at $t = 0$. The result, which parallels the usual Picard–Lindelöf theorem for regular ODEs, is as follows:

Lemma E.0.1. *Consider a Fuchsian ODE of the form* (E.0.1)*, and suppose moreover that the matrix A is such that (the operator norm of) the matrix $\sigma^A = \exp(A \log \sigma)$ is uniformly bounded for $\sigma \in (0,1)$. Then*

(a) Consider the initial data $\mathbf{V}(0) = 0$. *Then for some* $T_* > 0$, *there exists a unique continuous* $\mathbf{V} : [0, T_*) \to \mathbb{R}^N$ *which has* $V(0) = 0$ *and satisfies* (E.0.1) *for* $t \in (0, T_*)$. *Further, the solution satisfies the bound* $|\mathbf{V}(t)| \lesssim t^{\varepsilon}$.

(b) Now let $\mathbf{v} \in \mathbb{R}^N$ *be in the kernel of* A, *and consider initial data* $\mathbf{V}(0) = \mathbf{v}$. *Then for some* $T_* > 0$, *there exists a unique continuous* $\mathbf{V} : [0, T_*) \to \mathbb{R}^N$ *which has* $V(0) = \mathbf{V}$ *and satisfies* (E.0.1) *for* $t \in (0, T_*)$. *Further, the solution satisfies the bound* $|\mathbf{V}(t) - \mathbf{v}| \lesssim t^{\epsilon}$.

Proof. For part (a), we employ a contraction mapping argument in the following subset of $C([0, T_*), \mathbb{R}^N)$:

$$X = \{\mathbf{W} \in C([0, T_*), \mathbb{R}^N) : \mathbf{W}(0) = 0,\ |\mathbf{W}(t)| \leq M\},$$

where $T_* > 0$ and $M > 0$ will be fixed later. The idea is to use the following map $S : X \to X$.

$$S(\mathbf{W})(0) = 0, \quad S(\mathbf{W})(t) = t^{\varepsilon} \int_0^1 \sigma^{A-1+\varepsilon} F[\sigma t, \mathbf{W}(\sigma t)]\, d\sigma \qquad \forall t \in (0, T_*). \tag{E.0.2}$$

Firstly, we note that $\mathbf{V}$ satisfies the Fuchsian ODE (E.0.1) with $\mathbf{V}(0) = 0$ if and only if $S(\mathbf{V}) = \mathbf{V}$. This is because the integral in (E.0.2) can be rewritten as:

$$\begin{aligned} t^{\varepsilon} \int_0^1 \sigma^{A-1+\varepsilon} F[\sigma t, \mathbf{W}(\sigma t)]\, d\sigma &= t^{-A} \int_0^1 (\sigma t)^{A+\varepsilon} F[\sigma t, \mathbf{W}(\sigma t)]\, \frac{d\sigma}{\sigma} \\ &= t^{-A} \int_0^t \tilde{\sigma}^{A+\varepsilon} F[\tilde{\sigma}, \mathbf{W}(\tilde{\sigma})]\, \frac{d\tilde{\sigma}}{\tilde{\sigma}}. \end{aligned}$$

Therefore $S(\mathbf{V}) = \mathbf{V}$ if and only if we have that

$$t^A \mathbf{V}(t) = \int_0^t \tilde{\sigma}^{A+\epsilon} F[\tilde{\sigma}, \mathbf{V}(\tilde{\sigma})]\, \frac{d\tilde{\sigma}}{\tilde{\sigma}}, \qquad \forall t \in (0, T_*)$$

which is obviously equivalent to $\mathbf{V}$ satisfying (E.0.1) for $t \in (0, T_*)$.

We next show that for correctly chosen M and T_*, the map S sends the domain X to itself. The fact that $S(\mathbf{W})$ is continuous on $[0, T_*)$ is easily checked, while for the boundedness let us first suppose that K is the Lipschitz constant of F with respect to its second argument (since this lemma is local in nature, we assume without loss of generality that K is a global Lipschitz constant). Then we have

$$\begin{aligned} |S(\mathbf{W})(t)| &\leq t^{\varepsilon} \int_0^1 \|\sigma^A\|_{op}\, \sigma^{-1+\epsilon} |F[\sigma t, \mathbf{W}(\sigma t)]|\, d\sigma \\ &\leq t^{\epsilon} \sup_{\sigma \in (0,1)} \|\sigma^A\|_{op} \cdot \left(\sup_{s \in (0,t)} F[s, 0] + KM \right) \cdot \int_0^1 \sigma^{-1+\epsilon}\, d\sigma \\ &= \epsilon^{-1} t^{\epsilon} \sup_{\sigma \in (0,1)} \|\sigma^A\|_{op} \cdot \left(\sup_{s \in (0,t)} F[s, 0] + KM \right). \end{aligned}$$

Therefore for any $M > 0$, for T_* chosen sufficiently small we have $|S(\mathbf{W})(t)| \leq M$, hence $S : X \to X$ as required. A similar computation shows that S is a contraction mapping with respect to the C^0 norm. Indeed,

$$\begin{aligned}
|S(\mathbf{W}_1)(t) - S(\mathbf{W}_2)(t)| &\leq t^\epsilon \int_0^1 \|\sigma^A\|_{op}\, \sigma^{-1+\epsilon} |F[\sigma t, \mathbf{W}_1(\sigma t)] - F[\sigma t, \mathbf{W}_2(\sigma t)]|\, d\sigma \\
&\leq t^\epsilon \sup_{\sigma\in(0,1)} \|\sigma^A\|_{op} \cdot \int_0^1 \sigma^{-1+\epsilon} \cdot K |\mathbf{W}_1(\sigma t) - \mathbf{W}_2(\sigma t)|\, d\sigma \\
&\leq K\epsilon^{-1} t^\epsilon \sup_{\sigma\in(0,1)} \|\sigma^A\|_{op} \cdot \sup_{s\in(0,T_*)} |\mathbf{W}_1(s) - \mathbf{W}_2(s)|.
\end{aligned}$$

Therefore S is a contraction mapping so long as T_* is chosen such that $K\epsilon^{-1} T_*^\epsilon \sup_{\sigma\in(0,1)} \|\sigma^A\|_{op} < 1$.

Hence by the contraction mapping theorem, there is a unique fixed point $\mathbf{V}$ of S in the set X. This completes the existence part of the proof. For uniqueness, first observe that the contraction argument shows that two solutions $\mathbf{V}_1$ and $\mathbf{V}_2$ to (E.0.1) agree on an interval $[0, T_-] \subset [0, T_*)$ so long as they remain bounded by M. Moreover, by the above bounds they must have $|\mathbf{V}_1(t)|, |\mathbf{V}_2(t)| \lesssim t^\epsilon$ for $t \in [0, T_-]$.

Therefore, by a straightforward continuity argument, they must agree in the whole interval $[0, T_*)$, and satisfy the bound $|\mathbf{V}_1(t)| = |\mathbf{V}_2(t)| \lesssim t^\epsilon$ everywhere in this interval of existence. This concludes the proof of (a).

Finally, part (b) follows immediately from part (a) simply by taking $\tilde{\mathbf{V}} = \mathbf{V} - \mathbf{v}$ and, upon using the fact that $A\mathbf{v} = 0$, rewriting the Fuchsian ODE (E.0.1) as

$$\left(t\frac{d}{dt}\tilde{\mathbf{V}} + A\right) = t^\epsilon \tilde{F}[t, \tilde{\mathbf{V}}], \qquad \tilde{F}[t, \mathbf{W}] = F[t, \mathbf{W} + \mathbf{v}].$$

Applying part (a) to the above equation, this concludes the proof of (b). □

Appendix F

Propagation of constraints for the linearized Einstein equations

In Appendix F, we state and prove two propositions explaining how the constraint equations (and gauge conditions) are propagated in evolution. The first, Proposition F.0.1, shows that in the CMCSH evolution of Proposition 7.2.4(i), the constraints (7.2.15)–(7.2.20) and the gauge conditions (7.2.4) and (7.1.3), remain true on all Cauchy slices Σ_t. The second, Proposition F.0.2, starts with the asymptotic constraint equations of Lemma 7.2.5 and shows that the solution produced by Proposition 9.5.4 solves the actual constraint equations (7.2.15)–(7.2.20) as well as the CMC condition $\operatorname{tr}\kappa = 0$.

Proposition F.0.1. *Let* $(\hat{\eta}_i{}^j(t), \hat{\kappa}_i{}^j(t), \phi(t), \psi(t), \nu(t), \chi^j(t))$ *be a solution to the elliptic-hyperbolic system* (7.2.6), (7.2.7), (7.2.9), (7.2.10), (7.2.11), (7.2.12), (7.2.14) *for* $t > 0$. *Suppose furthermore that for some time* $T > 0$, *the constraint equations* (7.2.15), (7.2.16), (7.2.19) *and* (7.2.20) *hold at* $t = T$, *and also that the CMCSH gauge constraints* $\operatorname{tr}\hat{\kappa} = 0$ *and* (7.1.3) *hold at* $t = T$. *Then the whole system* (7.2.6)–(7.2.20) *holds for* $t > 0$.

Proof. Let us start with the symmetry constraints (7.2.19) and (7.2.20). For this purpose, we define

$$\mathcal{A}^{ac} \coloneqq \mathring{g}^{ab}\hat{\eta}_b{}^c - \mathring{g}^{cb}\hat{\eta}_b{}^a, \quad \mathcal{B}^{ac} \coloneqq \mathring{g}^{ab}(\hat{\kappa}_b{}^c - (2t\mathring{k}_b{}^d)\hat{\eta}_d{}^c) - \mathring{g}^{cb}(\hat{\kappa}_b{}^a - (2t\mathring{k}_b{}^d)\hat{\eta}_d{}^a).$$

Using only (7.2.6) and (7.2.7), one may derive evolution equations for $\mathcal{A}^{ac}$ and $\mathcal{B}^{ac}$:

$$t\partial_t\mathcal{A}^{ac} = \mathcal{B}^{ac} + (2t\mathring{k}_b{}^a)\mathcal{A}^{bc} + (2t\mathring{k}_b{}^c)\mathcal{A}^{ab},\ t\partial_t\mathcal{B}^{ac} = t^2\mathring{g}^{ij}\partial_i\partial_j\mathcal{A}^{ac} - t^2\mathring{g}^{ab}\partial_b\partial_d\mathcal{A}^{dc} - t^2\mathring{g}^{cb}\partial_b\partial_d\mathcal{A}^{ad}.$$

Assuming the spatially harmonic gauge condition (7.1.3) holds, the expression $t^2\mathring{g}^{ab}\partial_b\partial_d\mathcal{A}^{dc} + t^2\mathring{g}^{cb}\partial_b\partial_d\mathcal{A}^{ad}$ will vanish. Unfortunately we have not yet shown that (7.1.3) is true for all times, but our strategy will be first apply a gauge transformation, as in Proposition 7.2.4(ii), to create a new solution to the elliptic-hyperbolic system (7.2.6), (7.2.7), (7.2.9), (7.2.10), (7.2.11), (7.2.12), (7.2.14) that is spatially harmonic.

Then since $\mathcal{A}^{ac}$ and $\mathcal{B}^{ac}$ are gauge invariant under the transformations (7.2.21) and (7.2.22), it still suffices to deal with the case where we assume (7.1.3), in which case our equations reduce to

$$t\partial_t \mathcal{A}^{ac} = \mathcal{B}^{ac} + (2t\mathring{k}_b{}^a)\mathcal{A}^{bc} + (2t\mathring{k}_b{}^c)\mathcal{A}^{ab}, \quad t\partial_t \mathcal{B}^{ac} = t^2 \mathring{g}^{ij}\partial_i\partial_j \mathcal{A}^{ac}.$$

Hence $\mathcal{A}^{ac}$ and $\mathcal{B}^{ac}$ obey a linear and homogeneous first order hyperbolic system, and therefore their vanishing at $t = T$ implies their vanishing everywhere. To be explicit, one can use the following energy estimate:

$$\text{if } \mathcal{E}_{\mathcal{AB}}(t) = \int_{\mathbb{T}^D} \mathring{g}_{ab}\mathring{g}_{cd}(t^2\mathring{g}^{ij}\partial_i\mathcal{A}^{ac}\partial_j\mathcal{A}^{bd} + \mathcal{B}^{ac}\mathcal{B}^{bd} + \mathcal{A}^{ac}\mathcal{A}^{bd})\,dx, \text{ then } t\partial_t\mathcal{E}_{\mathcal{AB}}(t) \lesssim \mathcal{E}_{\mathcal{AB}}(t).$$

Grönwall's inequality therefore shows that $\mathcal{A}^{ac} = \mathcal{B}^{ac} = 0$ for all $t > 0$, proving (7.2.19) and (7.2.20).

We next move onto the Hamiltonian and momentum constraints (7.2.15) and (7.2.16). Instead of dealing with these equations directly, we instead define:

$$\mathcal{H} = -t^2\mathring{g}^{ab}\partial_a\partial_b \operatorname{tr}\hat{\eta} + t^2\mathring{g}^{ab}\partial_i\partial_a\hat{\eta}_b{}^i + (t\mathring{k}_b{}^a)\hat{\kappa}_a{}^b + 2p_\phi\psi,$$
$$\mathcal{P}_i = \partial_j\hat{\kappa}_i{}^j + \partial_i((t\mathring{k}_b{}^a)\hat{\eta}_a{}^b + 2p_\phi\phi) - (t\mathring{k}_i{}^j)\partial_j \operatorname{tr}\hat{\eta}.$$

Then using (7.2.6), (7.2.7), (7.2.9) and the elliptic equation (7.2.12), one derives the evolution equations:

$$t\partial_t\mathcal{H} = t^2\mathring{g}^{ij}\partial_i\mathcal{P}_j, \quad t\partial_t\mathcal{P}_i = \partial_i\mathcal{H}.$$

Note that the constraints being satisfied at $t = T$ imply that $\mathcal{H} = 0$ and $\mathcal{P}_i = 0$ at $t = T$. Now

$$\text{if } \mathcal{E}_{\mathcal{HP}}(t) = \int_{\mathbb{T}^D} (t^2\mathring{g}^{ij}\mathcal{P}_i\mathcal{P}_j + \mathcal{H}^2)\,dx, \text{ then } t\partial_t\mathcal{E}_{\mathcal{HP}}(t) \lesssim \mathcal{E}_{\mathcal{HP}}(t).$$

Therefore, $\mathcal{H} = 0$ and $\mathcal{P}_i = 0$ for all times $t > 0$. Furthermore, using also the elliptic equations (7.2.12) and (7.2.14) one derives that

$$t\partial_t \operatorname{tr}\hat{\kappa} = 0, \quad t\partial_t(2\partial_j\hat{\eta}_i{}^j - \partial_i \operatorname{tr}\hat{\eta}) = 2\mathcal{P}_i.$$

Since we already have $\mathcal{P}_i = 0$, this implies the gauge conditions (7.2.4) and (7.1.3) remain true in evolution. Combining this with $\mathcal{H} = 0$ and $\mathcal{P}_i = 0$, we also deduce (7.2.15) and (7.2.16). Finally, (7.2.17) may be derived as a consequence of (7.2.16), (7.2.19) and (7.2.20). □

Proposition F.0.2. *As in Proposition 9.4.5, let* $((\kappa_\lambda)_i{}^j, (\tilde{\Upsilon}_\lambda)_i{}^j, \psi_\lambda, \tilde{\varphi}_\lambda, \nu_\lambda)$ *be a solution to the ODE system* (9.4.3)–(9.4.6) *together with* (9.2.8)*, is such that* $((\kappa_\lambda)_i{}^j, (\tilde{\Upsilon}_\lambda)_i{}^j, \psi_\lambda, \tilde{\varphi}_\lambda) \to (((\kappa_\infty)_\lambda)_i{}^j, ((\tilde{\Upsilon}_\infty)_\lambda)_i{}^j, (\psi_\infty)_\lambda, (\tilde{\varphi}_\infty)_\lambda)$ *as* $t \to 0$.

Suppose moreover that these limiting quantities satisfy the asymptotic constraints (9.2.17)–(9.2.21)*, with* $T = \mathcal{T}_*$*. Then the solution* $((\kappa_\lambda)_i{}^j, (\tilde{\Upsilon}_\lambda)_i{}^j, \psi_\lambda, \tilde{\varphi}_\lambda, \nu_\lambda)$ *obeys the constraint equations* (9.2.11)–(9.2.16)*. (Here* $(\tilde{\Upsilon}_\lambda)_i{}^j$ *and* $\tilde{\varphi}_\lambda$ *are related to* $(\eta_\lambda)_i{}^j$ *and* ϕ_λ *by* (9.4.1) *and* (9.4.2)*.)*

Proof. The strategy is to combine the constraint propagation method of Proposition F.0.1 with some Fuchsian analysis. We start with the symmetry constraints (9.2.15) and (9.2.16). Instead of dealing with $\eta_i{}^j$ directly, we first define

$$(\mathcal{C}_\lambda)^{ac} \coloneqq \mathring{g}^{ab}(t_{\lambda *})(\tilde{\Upsilon}_\lambda)_b{}^c - \mathring{g}^{cb}(t_{\lambda *})(\tilde{\Upsilon}_\lambda)_b{}^a,$$

$$(\mathcal{D}_\lambda)^{ac} \coloneqq \mathring{g}^{ab}(t_{\lambda *})((\kappa_\lambda)_b{}^c - (2t\mathring{k}_b{}^d)(\tilde{\Upsilon}_\lambda)_d{}^c) - \mathring{g}^{cb}(t_{\lambda *})((\kappa_\lambda)_b{}^a - (2t\mathring{k}_b{}^d)(\tilde{\Upsilon}_\lambda)_d{}^a).$$

We use that by the proof of Lemma 7.2.5, $((\mathcal{C}_\lambda)^{ac}, (\mathcal{D}_\lambda)^{ac}) = (0,0)$ if and only $((\mathcal{A}_\lambda)^{ac}, (\mathcal{B}_\lambda)^{ac}) = (0,0)$, where $\mathcal{A}^{ac}$ and $\mathcal{B}^{ac}$ were defined in the proof of Proposition F.0.1.

By the assumptions of Proposition F.0.2, $((\mathcal{C}_\lambda)^{ac}, (\mathcal{D}_\lambda)^{ac})$ are bounded and tend to $(0,0)$ as $t \to 0$. Thus by the Fuchsian analysis of Lemma E.0.1, particularly the uniqueness statement, one may deduce that $((\mathcal{C}_\lambda)^{ac}, (\mathcal{D}_\lambda)^{ac})$, and therefore also $((\mathcal{A}_\lambda)^{ac}, (\mathcal{B}_\lambda)^{ac})$, vanish identically if we can prove that for the vector $\mathbf{C} = ((\mathcal{C}_\lambda)^{ac}, (\mathcal{D}_\lambda)^{ac} : a, c = 1, \dots, D)$, there is some bounded (time-dependent) matrix $A_{\mathcal{CD}}$ such that

$$t\frac{d}{dt}\mathbf{C} = t^\delta A_{\mathcal{CD}} \cdot \mathbf{C}. \tag{F.0.1}$$

One may verify (F.0.1) directly by using the system (9.4.3), (9.4.4), (9.2.8) and going through a lengthy calculation. We propose an alternative approach using linear algebra. Firstly, by Proposition 9.4.4 (ignoring all the appearances of $t_{\lambda *}$), for a solution of the system (9.4.3)–(9.4.6) and (9.2.8) there does exist a bounded (time-dependent) matrix $B_{\mathcal{CD}}$ such that for $\mathbf{V} = ((\kappa_\lambda)_i{}^j, (\tilde{\Upsilon}_\lambda)_i{}^j, \psi_\lambda, \tilde{\varphi}_\lambda)$, such that we verify the Fuchsian ODE:

$$t\frac{d}{dt}\mathbf{C} = t^\delta B_{\mathcal{CD}} \cdot \mathbf{V}.$$

On the other hand there is the natural linear map $L : ((\kappa_\lambda)_i{}^j, (\tilde{\Upsilon}_\lambda)_i{}^j, \psi_\lambda, \tilde{\varphi}_\lambda) \mapsto ((\mathcal{C}_\lambda)^{ac}, (\mathcal{D}_\lambda)^{ac})$, which is surjective onto the $D(D-1)$-dimensional space $(\mathrm{Skew}_{D\times D})^2$, meaning two copies of the space of skew-symmetric matrices.

Thus (F.0.1) holds if $B_{\mathcal{CD}}$ factorizes as $B_{\mathcal{CD}} = A_{\mathcal{CD}} \circ L$ for some bounded $A_{\mathcal{CD}}$. The key observation is that $\ker L \subset \ker B_{\mathcal{CD}}$; this follows since by Proposition F.0.1, for $t > 0$ we know that $t\frac{d}{dt}((\mathcal{A}_\lambda)^{ac}, (\mathcal{B}_\lambda)^{ac}) = 0$ if $((\mathcal{A}_\lambda)^{ac}, (\mathcal{B}_\lambda)^{ac}) = 0$ and thus the same holds true for $((\mathcal{C}_\lambda)^{ac}, \mathcal{D}_\lambda)^{ac})$. Therefore, one may apply the linear algebra fact that since $\ker L \subset \ker B_{\mathcal{CD}}$ and L is surjective onto $(\mathrm{Skew}_{D\times D})^2$, there must exist some map $A_{\mathcal{CD}} : (\mathrm{Skew}_{D\times D})^2 \to (\mathrm{Skew}_{D\times D})^2$ such that $B_{\mathcal{CD}}$ factorizes as $B_{\mathcal{CD}} = A_{\mathcal{CD}} \circ L$.

In fact, one may apply the linear algebra fact in such a way that $A_{\mathcal{CD}}$ remains uniformly bounded in time. Therefore the Fuchsian ODE (F.0.1), and thus the propagation of the symmetry constraints follows. We summarize the strategy as suitably renormalized constraints (i.e. $((\mathcal{C}_\lambda)^{ac}, (\mathcal{D}_\lambda)^{ac})$) + good derivative bounds (i.e. Proposition 9.4.4) + constraint propagation at $t > 0$ (i.e. Proposition F.0.1) implies constraint propagation from asymptotic data.

One can play the same game for the Hamiltonian and momentum constraints. Define

$$\mathcal{H}_\lambda = t^2 \mathring{g}^{ab}\lambda_a\lambda_b \operatorname{tr}\tilde{\Upsilon}_\lambda - t^2\mathring{g}^{ab}\lambda_i\lambda_a(\tilde{\Upsilon}_\lambda)_b{}^i + t^2\mathring{g}^{ab}\lambda_i\lambda_a\int_t^{t_{\lambda *}} \mathring{g}_{bp}(s)\mathring{g}^{iq}(s)\frac{ds}{s}\cdot(\kappa_\lambda)_q{}^p + (t\mathring{k}_b{}^a)(\kappa_\lambda)_a{}^b + 2p_\phi\phi_\lambda,$$

$$(\mathcal{Q}_\lambda)_i = -\mathrm{i}\lambda_j\kappa_i{}^j - \mathrm{i}\lambda_i((t\mathring{k}_b{}^a)(\tilde{\Upsilon}_\lambda)_a{}^b + 2p_\phi\tilde{\varphi}_\lambda) + \mathrm{i}(t\mathring{k}_i{}^j)\lambda_j\operatorname{tr}\tilde{\Upsilon} + \mathrm{i}\lambda_i\log\left(\frac{t}{t_{\lambda *}}\right)\Big(t^2\mathring{g}^{ab}\lambda_a\lambda_b\operatorname{tr}\tilde{\Upsilon}_\lambda - t^2\mathring{g}^{ab}\lambda_i\lambda_a(\tilde{\Upsilon}_\lambda)_b{}^i + t^2\mathring{g}^{ab}\lambda_i\lambda_a\int_t^{t_{\lambda *}}\mathring{g}_{bp}(s)\mathring{g}^{iq}(s)\frac{ds}{s}\cdot(\kappa_\lambda)_q{}^p\Big).$$

It is straightforward to check that given the assumptions of the proposition, $(\mathcal{H}_\lambda, (\mathcal{Q}_\lambda)_i)$ are bounded and tend to $(0,0)$ as $t\to 0$. Since $(\mathcal{Q}_\lambda)_i = (\mathcal{P}_\lambda)_i + \mathrm{i}\lambda_i\log\left(\frac{t}{t_{\lambda *}}\right)\mathcal{H}_\lambda$, for $\mathcal{P}_i$ as in the proof of Proposition F.0.1, we also have constraint propagation at $t>0$. Finally, from Proposition 9.4.4 one may prove good derivative bounds in the sense that for some bounded (time-dependent) matrix $B_{\mathcal{HQ}}$ and $\mathbf{V}$ as before,

$$t\frac{d}{dt}\begin{pmatrix}\mathcal{H}_\lambda\\ (\mathcal{Q}_\lambda)_i\end{pmatrix} = t^\delta B_{\mathcal{HQ}}\cdot\mathbf{V}.$$

Therefore following the same program as before we deduce that $\mathcal{H}_\lambda$ and $(\mathcal{Q}_\lambda)_i$ vanish for all $t>0$, and therefore the constraints (9.2.11) and (9.2.12) are satisfied.

As always, (9.2.13) follows from (9.2.12) and (9.2.15)–(9.2.16). This concludes the proof of the proposition. □

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