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Local and Global Geometry of Directed Last-Passage Percolation Models

Zhang, Lingfu

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Local and Global Geometry of Directed Last-Passage Percolation Models

Lingfu Zhang

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Professor Allan Murray Sly

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Abstract

This thesis focuses on the study of exactly-solvable last-passage percolation (LPP) models, which are central objects in the Kardar-Parisi-Zhang universality class. Historically, most studies of these models are through algebraic formulae in integrable probability, bearing the disadvantage of being less informative and unrobust. In this thesis, we develop techniques in the direction of combining formulae and geometric arguments, and use them to solve two problems in the LPP models.

For the first part, we demonstrate the convergence of the empirical distribution of local environments along geodesics, for the explicit model of exponential LPP. We also provide explicit formulae for the limiting distributions. These results essentially enable one to compute all local statistics along the geodesic. We also show convergence of the distribution of the environment around a typical point, and the local environments are homogeneous along geodesics. Our proofs make extensive use of a correspondence with TASEP as seen from a single second-class particle, for which we prove new results concerning ergodicity and convergence to equilibrium.

For the second part, we study fractal geometry of the directed landscape, a continuous LPP model constructed recently in [42], and has since been shown to be the universal scaling limit of various exactly-solvable LPP models. We study the difference of passage times from two fixed points, introduced in [10]. Owing to geodesic geometry, it turns out that the difference process is almost surely locally constant. The set of non-constancy is connected to disjointness of geodesics and inherits remarkable fractal properties. We introduce several new ideas and show that the set of non-constancy of the 2D difference process and the 1D temporal process have Hausdorff dimensions 5/3 and 2/3 respectively. A particularly crucial ingredient in our analysis is the novel construction of a local time process for the geodesic akin to Brownian local time, supported on the "zero set" of the geodesic, and we also show that such "zero set" has Hausdorff dimension 1/3.

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Chapter 1

Overview

This thesis contains some of my studies on the planar directed last-passage percolation (LPP) models. While the microscopic details depend on the specification, in general terms they all consist of a 2D space, such as $\mathbb{R}^2$ or $\mathbb{Z}^2$, with an environment of random noise through which *directed* paths travel, accruing the integral of the noise along it — a quantity known as passage time or weight. Typically, two endpoints in the environment are fixed, and a maximization is done over the weights of all paths with these endpoints. A path which achieves the *maximum* weight is called a *geodesic*, although, traditionally, the latter is used to denote shortest paths in a metric space. The maximum weight is also called the passage time between the endpoints. Typical examples of such LPP models include the longest increasing subsequence in a uniformly random permutation, and the lattice models where i.i.d. random weights are put on $\mathbb{Z}^2$, and the directed path with maximum total weight between two fixed vertices is considered.

Such LPP models are believed to belong to the *Kardar-Parisi-Zhang (KPZ) universality class*. In probability, universality refers to the phenomenon where different probabilistic models produce the same limiting behavior under rescaling. Convergence of different random walks to Brownian motion is the most classical example of

universality, with the fluctuation exponent of 1/2. In the seminal paper [73], Kardar, Parisi, and Zhang predicted a different universal behavior for a class of planar random growth processes, characterized by height fluctuation exponent of 1/3 and spatial correlation length exponent of 2/3. Such KPZ type behaviors also widely appear in many surprising contexts, including random matrix theory, traffic flow models, polymers in random environments, sorting algorithm, and random tiling; and KPZ universality has been a major topic of interest both in statistical physics and in probability theory.

While they have been studied for decades, models in the KPZ universality class still have much to be understood mathematically. In particular, the universal behaviors are only proved for a handful of models that exhibit exactly-solvable structures. In this direction, the first breakthrough was made by Baik, Deift and Johansson [5] when they established $n^{1/3}$ fluctuations of the length of the longest increasing path from $(0, 0)$ to (n, n) in a homogeneous Poissonian field on $\mathbb{R}^2$, which is an exactly-solvable LPP model. Then Johansson proved a transversal fluctuation exponent of 2/3 for that model, and also $n^{1/3}$ fluctuations and a Tracy-Widom scaling limit for lattice LPP with i.i.d. exponential or geometric weights [72, 71]. For these models such results are obtained using exact distributional formulae from algebraic combinatorics (such as the Robinson-Schensted-Knuth bijection), random matrix theory, or queueing theory. Since then there have been tremendous developments in achieving a detailed understanding of these exactly-solvable models, with notable progress concerning scaling limits (e.g. the recent works of [80, 42]). For surveys in this direction, see e.g. [35, 89, 104].

In another related direction, there has been great interest in studying planar first passage percolation (FPP) with general weights. Such models are also conjectured to be in the KPZ universality class, but much less is known due to the lack of exactly-solvable structure. As a result, the study of FPP has relied mostly on a geometric understanding of the geodesics. Although much less is rigorously known, the connection

between understanding properties of geodesics, limit shapes and the KPZ predicted fluctuation exponents has been clear for years. See e.g. [83, 4]. More recently, a general strategy of combining geometric techniques and exactly-solvable results is proven powerful in solving problems in the KPZ universality class [15, 12, 13, 6, 14, 92, 102]. Compared to using only on formulae, this method provides more graphical information and enhances intuitive understandings. It is also more robust as formulae and algebraic structures are usually sensitive to perturbations.

In this thesis, we follow this general strategy of combining geometric and exactly-solvable techniques to study two problems in the LPP models: convergence of geodesic environments in exponential LPP, and fractal geometry of the difference profile in the directed landscape. The first part comes from the coauthored paper [79] with James B. Martin and Allan Sly, and the second part comes from the coauthored paper [60] with Shirshendu Ganguly.

For both parts, we use a common key property of the LPP models, namely, the equal in distribution between semi-infinite geodesics and competition interfaces in stationary growth processes. For the geodesic environment, we further use the connection between exponential LPP and the totally asymmetric exclusion process (TASEP), dating back to [91], to get the correspondence between a semi-infinite geodesic and the trajectory of a second-class particle, as developed in a series of works [50, 49, 87]. Then in order to understand the local environment around the geodesic, we study the stationary distribution of TASEP as seen from a second-class particle. For the fractal geometry, we decompose the difference profile into a family of competition interfaces, and use the above equal in distribution to reduce the analysis to the study of geodesics.

We next state the results, by first defining these models and setup some notations.

1.1 Setup and model definitions

Basic notations There are some basic notations used throughout this thesis.

For any $n \in \mathbb{Z}$ and $0 < \rho < 1$, we denote $\mathbf{n}^\rho := \left(\left\lfloor \frac{2(1-\rho)^2 n}{\rho^2+(1-\rho)^2} \right\rfloor , \left\lceil \frac{2\rho^2 n}{\rho^2+(1-\rho)^2} \right\rceil \right)$. We also let $\mathbf{n} = \mathbf{n}^{1/2} = (n, n)$, and in particular we have $\mathbf{0} = (0, 0)$. For each $p = (a, b) \in \mathbb{Z}^2$, denote $d(p) = a+b$ and $ad(p) = a-b$. For $n \in \mathbb{Z}$ we denote $\mathbb{L}_n = \{p \in \mathbb{Z}^2 : d(p) = 2n\}$.

For any set A, we use $|A|$ to denote its cardinality. For any $x, y \in \mathbb{R}$ we denote $x \vee y = \max\{x, y\}$, and $x \wedge y = \min\{x, y\}$, and $[\![x, y]\!]$ is the set $[x, y] \cap \mathbb{Z}$. The notation $\|\cdot\|$ denotes the l^2 norm unless otherwise stated. We use $\mathbb{H}$ to denote the upper half plane $\mathbb{R} \times \mathbb{R}_+$.

1.1.1 Exponential LPP

We formally define exponential LPP, a typical LPP model with exactly-solvable structures. It is defined on the 2D discrete lattice $\mathbb{Z}^2$, where to each vertex $p \in \mathbb{Z}^2$ we associate an independent weight $\omega(p)$ with Exp(1) distribution. For two vertices $p, q \in \mathbb{Z}^2$, we say $p \leq q$ if p is coordinate-wise less than or equal to q. For such p, q and any up-right path γ from p to q, we define the *passage time* or *weight of the path* to be

$$T(\gamma) := \sum_{w \in \gamma} \omega(w).$$

Then almost surely there is a unique up-right path from p to q that has the largest passage time. We call this path the *geodesic* $\Gamma_{p,q}$, and call $T_{p,q} := T(\Gamma_{p,q})$ the *passage time from p to p*. When $p \not\leq q$, we let $T_{p,q} = -\infty$.

For any $p \in \mathbb{Z}^2$, an infinite up-right path starting from p is called a *semi-infinite geodesic*, if for any two vertices in the path, the path between them is the geodesic between them. The existence and uniqueness of semi-infinite geodesics have been established (see e.g. [39, 50]), and we summarize the known facts here. For any $p \in \mathbb{Z}^2$, and $0 < \rho < 1$, almost surely there is a unique semi-infinite geodesic from p,

denoted as

$$\Gamma_p^\rho = \{p = (a_1, b_1), (a_2, b_2), \cdots\},$$

such that $\lim_{i\to\infty} \frac{a_i}{b_i} = \frac{\rho^2}{(1-\rho)^2}$. For fixed ρ, almost surely, for any sequence $q_n \in \mathbb{Z}^2$ that goes to infinity in the $\left(1, \frac{\rho^2}{(1-\rho)^2}\right)$ direction, Γ_{p,q_n} converges to Γ_p^ρ, in the sense that in any compact set Γ_{p,q_n} is the same as Γ_p^ρ when n is large enough. The collection of all the semi-infinite geodesics (in the $\left(1, \frac{\rho^2}{(1-\rho)^2}\right)$-direction) exhibits a tree structure. In particular, for any $p, q \in \mathbb{Z}^2$, the paths Γ_p^ρ and Γ_q^ρ are the same except for finitely many initial points.

For simplicity of notations, we also denote $\Gamma_p = \Gamma_p^{1/2}$. Below we shall always assume (the probability 1 event) that for any $p \le q \in \mathbb{Z}^2$, there is a unique geodesic $\Gamma_{p,q}$, and a unique $(1,1)$-direction semi-infinite geodesic Γ_p.

1.1.2 Directed landscape

Recent developments have confirmed the convergence of several exactly-solvable LPP models, including exponential LPP, to a well-defined scaling limit (see Section 1.4.1 below). This object was introduced in [42] and named the *directed landscape*. It is a random continuous function from the parameter space

$$\mathbb{R}^4_\uparrow = \{u = (p; q) = (x, s; y, t) \in \mathbb{R}^4 : s < t\}$$

to $\mathbb{R}$, and is usually denoted by $\mathcal{L}$. The following composition law is satisfied: for any $x, z \in \mathbb{R}$ and $r < s < t$, we have

$$\mathcal{L}(x, r; z, t) = \max_{y\in\mathbb{R}} \mathcal{L}(x, r; y, s) + \mathcal{L}(y, s; z, t). \tag{1.1}$$

For a continuous path $\pi : [s,t] \to \mathbb{R}$, its weight under the metric $\mathcal{L}$ is given by

$$\|\pi\|_{\mathcal{L}} = \inf_{k\in\mathbb{N}} \inf_{s=t_0<t_1<\cdots<t_k=t} \sum_{i=1}^{k} \mathcal{L}(\pi(t_{i-1}), t_{i-1}; \pi(t_i), t_i).$$

Such a path π is said to be a geodesic from $(\pi(s), s)$ to $(\pi(t), t)$ if

$$\|\pi\|_{\mathcal{L}} = \mathcal{L}(\pi(s), s; \pi(t), t).$$

It is shown in [42] that almost surely, there exists at least one geodesic from p to q for any $u = (p;q) = (x,s;y,t) \in \mathbb{R}^4_\uparrow$. We let $\pi_u = \pi_{(p;q)} = \pi_{(x,s;y,t)}$ denote any such geodesic from p to q. Further, for any fixed $u \in \mathbb{R}^4_\uparrow$, almost surely the geodesic π_u is unique.

One can also consider semi-infinite geodesics in the directed landscape. For each $p = (x,s) \in \mathbb{R}^2$ and $r \in \mathbb{R}$, we denote by $\pi_p^r : [s,\infty) \to \mathbb{R}$ a semi-infinite geodesic started from p in the r direction; i.e. we have $t^{-1}\pi_p^r(t) \to r$ as $t \to \infty$, and for any $s \le s_1 < s_2 < \infty$, the restriction of π_p^r on $[s_1, s_2]$ is a geodesic (from $(\pi_p^r(s_1), s_1)$ to $(\pi_p^r(s_2), s_2)$). When $r = 0$, we will drop the r dependence and simply use $\pi_p = \pi_p^0$. However the existence of such semi-infinite geodesics is not obvious and will be proved in Section 3.3. Besides, the uniqueness of such semi-infinite geodesics may not always hold, and that will also be discussed in Section 3.3.

1.2 Main results I: convergence of geodesic environments in exponential LPP

As been stated, to study general FPP models without exactly-solvable structure, the geometry of the geodesics has been an important tool. When trying to understand the behavior of finite or infinite geodesics, a well-known open question is whether the

empirical distributions of weights along geodesics converge; see e.g. [69] where it is proposed by Hoffman during a 2015 American Institute of Mathematics workshop. For non-exactly-solvable models, the best known result is by Bates [16], who showed convergence for various abstract dense families of weight distributions. His proof uses a 0-1 law and a variational formula, and does not rely on any particular exactly-solvable structure.

We, on the other hand, study the limiting local behavior for the exactly-solvable model of exponential LPP. Rather than the weights along geodesics, we consider the more general "empirical environment" around vertices, along a finite or semi-infinite geodesic, and we show that it converges to a deterministic measure. By the environment we mean the weights of nearby points, and the path of the geodesic through them. In particular, this positively answers the question of Hoffman for a first explicit model. Our approach relies on information provided by the exactly-solvable structure. In addition to proving convergence results, we also give an explicit description of the limiting distribution, which depends on the direction of the geodesic. Using this description one can compute any limiting local statistics of the geodesic, and we give some first examples below.

For each $v \in \mathbb{Z}^2$, we denote $\omega\{v\} := \{\omega(v+u)\}_{u\in\mathbb{Z}^2}$. For any (finite or semi-infinite) directed path γ we let $\gamma[i]$ be the i-th vertex in γ.

For any $u \leq v \in \mathbb{Z}^2$, and each $w \in \Gamma_{u,v}$, we regard $(\omega\{w\}, \Gamma_{u,v} - w)$ as a point in $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$, and we define the empirical distribution of the environment along $\Gamma_{u,v}$, or just empirical distribution along $\Gamma_{u,v}$, as

$$\mu_{u,v} := \frac{1}{|\Gamma_{u,v}|} \sum_{w\in\Gamma_{u,v}} \delta_{(\omega\{w\},\Gamma_{u,v}-w)}.$$

Similarly, we define the empirical distribution along a semi-infinite geodesic as

$$\mu^{\rho}_{v;r} := \frac{1}{2r+1} \sum_{i=1}^{2r+1} \delta_{(\omega\{\Gamma^{\rho}_{v}[i]\}, \Gamma^{\rho}_{v} - \Gamma^{\rho}_{v}[i])},$$

for any $v \in \mathbb{Z}^2$, $\rho \in (0,1)$, and $r \in \mathbb{Z}_{\geq 0}$. We will show that these empirical distributions converge. For each ρ, there is a limiting distribution ν^{ρ} on $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$, which is explicit and will be defined in Section 2.3.

For the next four theorems we fix any $\rho \in (0,1)$.

Theorem 1.1. *In probability $\mu_{\mathbf{0},\mathbf{n}^{\rho}} \to \nu^{\rho}$ as $n \to \infty$.*

Theorem 1.2. *Almost surely $\mu^{\rho}_{\mathbf{0};r} \to \nu^{\rho}$ as $r \to \infty$.*

We also have convergence of distributions.

Theorem 1.3. *The law of $(\omega\{\Gamma^{\rho}_{\mathbf{0}}[i]\}, \Gamma^{\rho}_{\mathbf{0}} - \Gamma^{\rho}_{\mathbf{0}}[i])$ converges to ν^{ρ} as $i \to \infty$.*

Theorem 1.4. *For each $0 < \alpha < 2$, the law of $(\omega\{\Gamma_{\mathbf{0},\mathbf{n}^{\rho}}[\lfloor \alpha n \rfloor]\}, \Gamma_{\mathbf{0},\mathbf{n}^{\rho}} - \Gamma_{\mathbf{0},\mathbf{n}^{\rho}}[\lfloor \alpha n \rfloor])$ converges to ν^{ρ} as $n \to \infty$.*

For the limiting distribution of ν^{ρ} to be defined in Section 2.3, the construction is explicit, and here we give the formula for the distribution function at the origin.

Proposition 1.5. *For $(\omega, \gamma) \sim \nu^{\rho}$, and $h \geq 0$, we have*

$$\mathbb{P}[\omega(\mathbf{0}) > h] = \left(1 + \frac{\rho(1-\rho)h}{(1-\rho)^2 + \rho^2}\right)(1 + \rho(1-\rho)h)e^{-h}.$$

The distribution of $\omega(\mathbf{0})$ given in Proposition 1.5 is a mixture of Gamma$(1,1)$, Gamma$(2,1)$ and Gamma$(3,1)$ distributions. In the case $\rho = 1/2$, for example, the weights of this mixture are $1/4$, $1/2$, and $1/4$ respectively, and the distribution of $\omega(\mathbf{0})$ can be interpreted as that of $2\min(E_1 + E_2, E_3 + BE_4)$ with $B \sim$ Bernoulli$(1/2)$ and $(E_i)_{1 \leq 1 \leq 4}$ i.i.d. $\sim$ Exp(1) independently of B. Related but slightly less simple

representations can be given for general ρ. See the discussion at the end of Section 2.3.

From the distribution ν^ρ and the convergence results, we can obtain information about the geodesics. The following result is such an example.

Proposition 1.6. *Denote by $N_{n,\rho}$ the number of "corners" along $\Gamma_{\mathbf{0},\mathbf{n}^\rho}$; that is, the number of $v \in \mathbb{Z}^2$ such that $\{v-(1,0), v, v+(0,1)\} \subset \Gamma_{\mathbf{0},\mathbf{n}^\rho}$, or $\{v-(0,1), v, v+(1,0)\} \subset \Gamma_{\mathbf{0},\mathbf{n}^\rho}$. Then we have $\frac{N_{n,\rho}}{2n} \to \frac{2\rho^2(1-\rho)^2(1+2\rho-2\rho^2)}{(1-\rho)^2+\rho^2}$ in probability, as $n \to \infty$.*

For example, for $\rho = 1/2$, the proportion of steps of the geodesic which are "corners" converges to $3/8$.

In our proofs of the above results we will use the connection between LPP and the totally asymmetric exclusion process (TASEP), which can be described as a Markov process $(\eta_t)_{t\in\mathbb{R}}$ on $\{0,1\}^{\mathbb{Z}}$, where $\eta_t(x) = 1$ means that there is a particle at site x at time t, whereas $\eta_t(x) = 0$ means that there is a hole at site x at time t. If there is a particle at site x and a hole at site $x+1$, they switch at rate 1. For each $\rho \in (0,1)$ there is a (density ρ) stationary distribution for TASEP, where for each site there is a particle with probability ρ and a hole with probability $1-\rho$ independently (i.e. i.i.d. Bernoulli). Such i.i.d. Bernoulli TASEP corresponds to a growing interface in $\mathbb{Z}^2$, which (when rotated by $\pi/4$) is a random walk at any time. Dividing the interface into two competing clusters, this gives a competition interface which corresponds to a semi-infinite geodesic; see e.g. [49, 87]. On the other hand, in TASEP such a competition interface corresponds to a "second class particle", which can switch with a hole to the right of it, or with a (normal) particle to the left of it. Thus, the environment seen from a semi-infinite geodesic corresponds to TASEP as seen from a single second class particle. Under such correspondence, we also prove a corresponding result for TASEP with a single second class particle as well, which may be of independent interest.

Theorem 1.7. $\lim_{t\to\infty} \Phi_t^\rho = \Psi^\rho$.

Here Φ_t^ρ and Ψ^ρ are measures on $\{0,1\}^{\mathbb{Z}}$ to be defined in Section 1.4.5, and we briefly describe them here. Consider TASEP with a single second class particle, where initially there is a second class particle at the origin, and any other site has a (normal) particle with probability ρ independently. Then Φ_t^ρ is the law of such TASEP at time t, as seen from the second class particle. The measure Ψ^ρ is the stationary distribution of a TASEP as seen from a second class particle with density ρ. In proving this theorem, we will also show that this stationary process is ergodic in time (Proposition 2.26).

1.3 Main results II: fractal geometry in the directed landscape

To motivate our study for the difference profile, we point out that another central object in the KPZ universality class which had been rigorously established much before $\mathcal{L}$, is known as the parabolic Airy_2 process, $\mathcal{P}_1 : \mathbb{R} \to \mathbb{R}$ [88] obtained as the projection $\mathcal{L}(\mathbf{0}; \cdot, 1)$, i.e., when the second spatial coordinate is allowed to vary while the remaining space coordinate is fixed to be 0 and the two temporal coordinates are frozen to be 0 and 1 respectively. In words, the parabolic Airy_2 process traces out the last-passage time in $\mathcal{L}$ from the origin to the points $(\cdot, 1)$.

However, by natural translation invariant properties of $\mathcal{L}$, we have the following equality in distribution (among many other symmetries, see e.g. Lemma 3.5) for any $y \in \mathbb{R}$,

$$\mathcal{L}(\mathbf{0}; \cdot, 1) \stackrel{d}{=} \mathcal{L}(y, 0; y + \cdot, 1).$$

Thus the landscape $\mathcal{L}$ provides a natural coupling of infinitely many parabolic Airy_2 processes "rooted" at different spatial points, making the coupling structure an

intriguing object of study. Such a study was initiated in [10], who introduced the following process, which, as we will soon see, encodes valuable information about the coupling.

Difference profile: Fix $y_a, y_b \in \mathbb{R}$ with $y_a < y_b$. Consider the random function $\mathcal{D} : \mathbb{H} \to \mathbb{R}$ (recall that $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$ denotes the upper half plane) given by

$$\mathcal{D}(x,t) = \mathcal{L}(y_a, 0, x, t) - \mathcal{L}(y_b, 0, x, t). \tag{1.2}$$

We will call $\mathcal{D}$ the *weight difference profile*; it is the difference in the weights of two geodesics with differing but fixed starting points and a common ending point as the latter varies. For concreteness, below we will fix $y_a = -1$ and $y_b = 1$ respectively.

Several properties of $\mathcal{D}$ have already been recorded in the literature. We list some of them below.

Owing to certain key properties of geodesics, it turns out that $\mathcal{D}$ is almost everywhere locally constant, with probability one (while this is a consequence of known arguments, this has not been completely spelled out in the literature. One of our main results will in fact imply this). Further, the following is known.

Lemma 1.8. *Almost surely, for any fixed t, $\mathcal{D}(\cdot, t)$ is a continuous non-increasing function.*

This has been proved several times in the literature, for example [42, Lemma 9.1] or [10, Theorem 1.1(2)]. Thus, in particular, Lemma 1.8 implies that, for any fixed t, at any given point x, $\mathcal{D}(x,t)$ is either constant in a neighborhood of x or decreasing at x on at least one side.

Moreover, it is an easy argument that the non-constant points of a continuous non-increasing function must form a perfect set, i.e., be closed and have no isolated points. Perfect sets must necessarily be uncountable. In particular, $-\mathcal{D}(\cdot, t)$ can be interpreted as the distribution function of a random measure supported on the

uncountable set $\overline{\mathsf{NC}}_t$ (the notation $\overline{\mathsf{NC}}_t$ is chosen to be in correspondence with the 2D and temporal counterparts NC and $\widehat{\mathsf{NC}}_x$, to be defined shortly). Canonical examples of a similar nature include the Cantor function or Brownian local time. Associated to any such set is its fractal dimension, which quantifies how "sparse" the set is. With the dimension of the mentioned two examples being classically known, one is led to ask: what is the fractal dimension of $\overline{\mathsf{NC}}_t$?

This question was originally raised and answered in [10], where it was shown that the Hausdorff dimension of $\overline{\mathsf{NC}}_t$ is $\frac{1}{2}$ almost surely. Further, in [17], it was shown that $\overline{\mathsf{NC}}_t$ is precisely the set of points x, which admit geodesics to the points $(-1, 0)$ and $(1, 0)$ respectively, which are disjoint except at their initial point (x, t). Finally, more recently, in [58], comparisons were made between $\overline{\mathsf{NC}}_t$ and Brownian local time, showing, among other things, that the former is absolutely continuous with respect to the latter in a certain precise sense.

However, in all the above works, the monotonicity of $\mathcal{D}(\cdot, t)$ was crucially used with the arguments breaking down if the temporal coordinate was varied instead, in which case the process $\mathcal{D}(x, \cdot)$ is no longer monotone, necessitating the development of new ideas. In the second part of this thesis, we will investigate the spatial-temporal behavior of the process $\mathcal{D}$ and compute new fractal dimensions associated to it.

To this end, we define $\mathsf{NC} \subset \mathbb{H}$ to be the set of all (x, t) where the difference profile $\mathcal{D}$ is not constant in a neighborhood. A one dimensional variant is $\widehat{\mathsf{NC}}_x \subset \mathbb{R}_+$ consisting of all t such that the function $t \mapsto \mathcal{D}(x, t)$ is not constant in a neighborhood of t. We note that $\widehat{\mathsf{NC}}_x \subset \{t \in \mathbb{R}_+ : (x, t) \in \mathsf{NC}\}$. As we will mostly work with the temporal non-constant set at $x = 0$, we also denote $\widehat{\mathsf{NC}} = \widehat{\mathsf{NC}}_0$.

With the above preparation, we can now state our two main results.

Theorem 1.9. *Almost surely, the set* NC *has Hausdorff dimension* $5/3$.

Theorem 1.10. *Almost surely, the set* $\widehat{\mathsf{NC}}$ *has Hausdorff dimension* $2/3$.

A key ingredient in proving these two theorems is a novel construction of a version of "Brownian local time" for the semi-infinite geodesic and analysis of its fractal properties which we include as our final main result.

Consider the semi-infinite geodesic $\pi_{\mathbf{0}}$, and its intersection with the line $\{(0,t) : t \geq 0\}$. Denote by

$$Z := \{t \geq 0 : \pi_{\mathbf{0}}(t) = 0\}, \tag{1.3}$$

the geodesic zero set.

Theorem 1.11. *Almost surely, the set Z has Hausdorff dimension* $1/3$.

To show this, we construct a non-decreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$, which we call the *geodesic local time* of $\pi_{\mathbf{0}}$, analogous to the classical Brownian local time. This induces a (random) measure μ on $\mathbb{R}_{\geq 0}$, supported on the set Z similar to how Brownian local time is supported on the zero set of Brownian motion. An important step in the construction of L is the construction of the geodesic occupation measure. We define the occupation measure ν_h for the geodesic $\pi_{\mathbf{0}}$ at height h, by setting for any measurable $A \subset \mathbb{R}$,

$$\nu_h(A) = \int_0^h \mathbb{1}[\pi_{\mathbf{0}}(t) \in A] dt. \tag{1.4}$$

Given the above, the construction of L is completed by proving the existence of the density of ν_h at 0.

Proposition 1.12. *For any height $h > 0$, almost surely $\lim_{w \searrow 0} (2w)^{-1} \nu_h([-w, w])$ exists and is finite.*

It is worth pointing out that some other, somewhat unrelated to our theme, aspects of fractal geometry in the Directed Landscape were investigated recently [37, 41]. In another direction, fractal properties often manifests in remarkable behavior of the system pertaining to "chaos and noise-sensitivity". Refer e.g. to the success story of the study of dynamical critical percolation summarized beautifully in [61]. In the KPZ universality paradigm, such a study was initiated recently in [57].

1.4 Related exactly-solvable models

In this section we review some useful objects and results from the literature. We start by explaining the connection between the exponential LPP and the directed landscape.

1.4.1 Convergence to the directed landscape

Define a metric $\mathcal{K}_n$ on $\mathbb{R}^2$ as follows. For any $(x, s; y, t) \in \mathbb{R}^4$, we let

$$\mathcal{K}_n(x, s; y, t) = 2^{-4/3} n^{-1/3} F(ns + 2^{5/3} n^{2/3} x, ns; nt + 2^{5/3} n^{2/3} y, nt),$$

where $F(p; q) = T_{p,q} - 2d(q-p) - \omega(p)$ for $p, q \in \mathbb{Z}^2$, and F is extended to be a function on $\mathbb{R}^4$ using the following rounding from [44]. Let $\mathsf{r} : \mathbb{R}^2 \to \mathbb{Z}^2$ be the function where

$$\mathsf{r}(x, y) = \begin{cases} (x, y) & x, y \in \mathbb{Z}; \\ (\lfloor x \rfloor, y) & x \notin \mathbb{Z}, y \in \mathbb{Z}; \\ (x, \lfloor y \rfloor) & x \in \mathbb{Z}, y \notin \mathbb{Z}; \\ (\lceil x \rceil, \lfloor y \rfloor) & x \notin \mathbb{Z}, y \notin \mathbb{Z}. \end{cases}$$

We then let $F(x, y; z, w) = T_{(\lceil x \rceil, \lceil y \rceil), \mathsf{r}(z,w)} - 2d(\mathsf{r}(z, w) - \mathsf{r}(x, y)) - \omega((x, y))\mathbb{1}[x, y \in \mathbb{Z}]$.

Given this, we define the geodesic sets in $\mathcal{K}_n$ as follows. For any $p, q \in \mathbb{R}^2$, a set $A \subset \mathbb{R}^2$ is called a *geodesic set* in $\mathcal{K}_n$ from p to q, if there is a total ordering $\preceq_A$ on A, such that $p \preceq_A p' \preceq_A q$ for any $p' \in A$; and for any $p_1 \preceq_A p_2 \preceq_A p_3 \in A$, there is $\mathcal{K}_n(p_1; p_2) + \mathcal{K}_n(p_2; p_3) = \mathcal{K}_n(p_1; p_3)$. Such a set A is called a *maximal geodesic set* in $\mathcal{K}_n$ if it is not contained in any other geodesic set from p to q.

Definition 1.13. For any $p = (x, y) \in \mathbb{R}^2$, let $\mathsf{R}_n(p) = (2^{-5/3} n^{-2/3}(x - y), n^{-1} y)$ and for any set $A \subset \mathbb{R}^2$ and $n \in \mathbb{N}$, let $\mathsf{R}_n(A) = \{(\mathsf{R}_n(p) : p \in A\}$. Finally, for any

continuous path $\pi : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is any subset, define its *graph* $\mathrm{Graph}(\pi)$ as the set $\{(\pi(t), t) : t \in I\}$.

Given the above, it is straightforward to check that for any $p, q \in \mathbb{Z}^2$, the set $\mathsf{R}_n(\Gamma_{p,q})$ is a maximal geodesic set from $\mathsf{R}_n(p)$ to $\mathsf{R}_n(q)$. We end with the following recent convergence result.

Theorem 1.14 ([44, Theorem 13.8(2)]). *There is a coupling of $\mathcal{L}$ with $\mathcal{K}_n$ for all $n \in \mathbb{N}$, such that the following holds almost surely. First, for any compact $K \subset \mathbb{R}^4_\uparrow$, we have $\mathcal{K}_n \to \mathcal{L}$ uniformly in K. Second, for any $(p_n; q_n) \to (p; q) \in \mathbb{R}^4_\uparrow$, and any $\mathcal{K}_n$ maximal geodesic set from p_n to q_n, denoted as π_n, they are pre-compact in Hausdorff topology, and any subsequential limit is* $\mathrm{Graph}(\pi)$*, for some $\mathcal{L}$ geodesic π from p to q.*

1.4.2 Airy line ensemble and Brownian properties

The parabolic Airy line ensemble $\mathcal{P} = \{\mathcal{P}_n\}_{n=1}^\infty$ is an ordered family of random processes on $\mathbb{R}$, constructed in [36]. We summarize some crucial properties of it that we will make use of. Define $\mathcal{A}_n(x) := \mathcal{P}_n(x) + x^2$ for any $x \in \mathbb{R}$, $n \in \mathbb{N}$.

1. Stationarity: $\mathcal{A} = \{\mathcal{A}_n\}_{n=1}^\infty$ is stationary with respect to horizontal shift.

2. Ergodicity: $\mathcal{A} = \{\mathcal{A}_n\}_{n=1}^\infty$ is ergodic with respect to horizontal shift.

3. Brownian Gibbs property: for any $m \le n \in \mathbb{N}$ and $x \le y$, let $S = [\![m, n]\!] \times [x, y]$. We can think of $\mathcal{P}$ as a (random) function on $\mathbb{N} \times \mathbb{R}$, then the process $\mathcal{P}|_S$ given $\mathcal{P}|_{S^c}$ is just $n-m+1$ Brownian bridges connecting up the points $\mathcal{P}_i(x)$ and $\mathcal{P}_i(y)$, for $m \le i \le n$, conditioned on that $\mathcal{P}_i(z) > \mathcal{P}_{i+1}(z)$ for any $1 \vee (m-1) \le i \le n$ and $z \in [x, y]$.

The first property is from its construction, and the third property is proved in [36]. The second property is proved in [38]. We note that here and throughout this paper, all Brownian motions and Brownian bridges have diffusive coefficients equal to 2.

As mentioned in Section 1.3, the first line $\mathcal{P}_1$ (of $\mathcal{P}$) is the parabolic Airy_2 process on $\mathbb{R}$, and it has the same distribution as the function $\mathcal{L}(\mathbf{0}; \cdot, 1)$. This process also has the following Brownian property. For $K \in \mathbb{R}, d > 0$, let $\mathcal{B}^{[K,K+d]}$ denote a Brownian motion on $[K, K+d]$, taking value 0 at K. Let $\mathcal{P}_1^{[K,K+d]}$ denote the random function on $[K, K+d]$ defined by

$$\mathcal{P}_1^{[K,K+d]}(x) := \mathcal{P}_1(x) - \mathcal{P}_1(K), \ \forall x \in [K, K+d].$$

Theorem 1.15 ([30, Theorem 1.1])**.** *There exists an universal constant $G > 0$ such that the following holds. For any fixed $M > 0$, there exists $a_0 = a(M)$ such that for all intervals $[K, K+d] \subset [-M, M]$ and for all measurable $A \subset C^0([K, K+d])$ with $0 < \mathbb{P}[\mathcal{B}^{[K,K+d]} \in A] = a \leq a_0$,*

$$\mathbb{P}\left(\mathcal{P}_1^{[K,K+d]} \in A\right) \leq a e^{GM(\log a^{-1})^{5/6}}.$$

Here $C^0([K, K+d])$ is the space of all continuous functions on $[K, K+d]$.

We will also need the Brownian absolute continuity for the evolution induced by the directed landscape, with some general initial conditions. Namely, take any $t > 0$ and any function $f_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, such that $f(x) \neq -\infty$ for some x, and $\frac{f_0(x) - x^2/t}{|x|} \to -\infty$ as $|x| \to \infty$. Define

$$f_t(y) := \sup_{x \in \mathbb{R}} f_0(x) + \mathcal{L}(x, 0; y, t).$$

Such evolution of functions is also called the KPZ fixed point (see [80]).

Theorem 1.16 ([93, Theorem 1.2])**.** *For any fixed $t > 0$ and f_0 satisfying the above conditions, and any $y_1 < y_2$, the law of $y \mapsto f_t(y + y_1) - f(y_1)$ for $y \in [0, y_2 - y_1]$ is absolutely continuous with respect to the law of a Brownian motion on $[0, y_2 - y_1]$.*

Finally, we quote the following result on the convergence of the exponential LPP

point-to-line profile to the parabolic Airy_2 process. It can be deduced from Theorem 1.14 and the fact that $\mathcal{P}_1$ has the same distribution as $\mathcal{L}(\mathbf{0}; \cdot, 1)$.

Theorem 1.17 ([12, Theorem 3.8]). *As $n \to \infty$, we have*

$$x \mapsto 2^{-4/3} n^{-1/3} \left(T_{\mathbf{0},(n+\lfloor x(2n)^{2/3} \rfloor, n-\lfloor x(2n)^{2/3} \rfloor)} - 4n \right) \Rightarrow \mathcal{P}_1$$

weakly in the topology of uniform convergence on compact sets.

Historically, such convergence was first proved in the sense of finite dimensional distributions [20, 24]. Using the so-called slow decorrelation phenomenon, and proving equicontinuity of the point-to-line profile, it also follows that the weak convergence holds in the topology of uniform convergence on compact sets [12, 51].

In the next a few subsections, we introduce the duality between semi-infinite geodesics and competition interfaces in exponential LPP, and TASEP with a second class particle, and explain the relations between them.

1.4.3 Busemann function

Busemann functions were originally used to study the large-scale geometry of geodesics in Riemannian manifolds, and was introduced to study general first passage percolation (FPP) models by Hoffman [67, 68]. It has also been intensively studied and used in the exponential LPP setting (see e.g. [94]), as we define now.

By the already stated tree structure of semi-infinite geodesics, for each $\rho \in (0, 1)$, and any $p, q \in \mathbb{Z}^2$, we denote $B^\rho(p, q) := T_{p,\mathbf{c}} - T_{q,\mathbf{c}}$, where $\mathbf{c} \in \mathbb{Z}^2$ is the coalescing point of Γ_p^ρ and Γ_q^ρ; i.e. $\mathbf{c}$ is the vertex in $\Gamma_p^\rho \cap \Gamma_q^\rho$ with the smallest $d(\mathbf{c})$. Such B^ρ is called the *Busemann function in direction ρ*. We also write $G^\rho(p) = B^\rho(\mathbf{0}, p)$. This Busemann function satisfies the following properties.

1. For each $p_1, p_2, p_3 \in \mathbb{Z}^2$, $B^\rho(p_1, p_2) + B^\rho(p_2, p_3) = B^\rho(p_1, p_3)$. In particular, $B^\rho(p_1, p_2) = G^\rho(p_2) - G^\rho(p_1)$.

2. For each $(a,b) \in \mathbb{Z}^2$, $G^\rho(a,b) = G^\rho(a+1,b) \wedge G^\rho(a,b+1) - \omega(a,b)$.

3. For any down-right path $\mathcal{U} = \{p_k\}_{k\in\mathbb{Z}}$, the random variables $B^\rho(p_k, p_{k-1})$ are independent. The law of $B^\rho(p_k, p_{k-1})$ is $\mathrm{Exp}(\rho)$ if $p_k = p_{k-1} - (0,1)$, and is $-\mathrm{Exp}(1-\rho)$ if $p_k = p_{k-1} + (1,0)$.

4. Denote $\omega^\vee(a,b) = G^\rho(a,b) - G^\rho(a-1,b) \vee G^\rho(a,b-1)$ for $(a,b) \in \mathbb{Z}^2$, then $\{\omega^\vee(p)\}_{p\in\mathbb{Z}^2}$ are i.i.d. $\mathrm{Exp}(1)$.

The first two properties are by definition. For the third property, a proof can be found in [95]. The last property comes from [49, Lemma 4.2] (see also [7]).

1.4.4 Competition interface and a hole-particle pair in TASEP

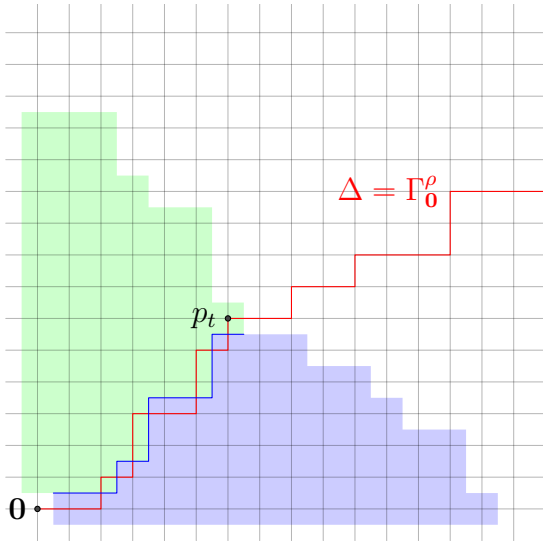


Figure 1.1: An illustration of the growing interface from LPP: the blue and green areas are the two clusters C_1 and C_2 respectively, and the red curve is the semi-infinite geodesic $\Gamma^\rho_{\mathbf{0}}$.

We next discuss the competition interface starting from the origin (see e.g. [50, 49]). We consider an up-right growing interface: for each $t \geq 0$, we say that a vertex $p \in \mathbb{Z}^2$ is occupied by time t if $G^\rho(p) \leq t$, and we denote by I_t the set of vertices occupied by time t. Then the waiting times in this growing interface are given by

$\{\omega^\vee(p)\}_{p\in\mathbb{Z}^2}$, which are i.i.d. Exp(1); thus $(I_t)_{t\in\mathbb{R}}$ is a Markov process, such that given I_t, each vertex $p \notin I_t$ with $p-(0,1), p-(1,0) \in I_t$ becomes occupied with rate 1 independently.

We next define two clusters C_1 and C_2: let $\mathbb{N}\times\{0\} \subset C_1$, and $\{0\}\times\mathbb{N} \subset C_2$. For any $(a,b) \in \mathbb{Z}^2$ with $a,b>0$, let its "parent" be either $(a-1,b)$ or $(a,b-1)$, whichever is occupied later; then (a,b) is in the same cluster as its parent. It is then straight forward to check that, the boundary between C_1 and C_2, denoted by $\Delta = \{\Delta[i]\}_{i\in\mathbb{N}}$, can be defined as follows: first we let $\Delta[1] = \mathbf{0}$, and then let $\Delta[i+1] = \Delta[i]+(1,0)$ if $G^\rho(\Delta[i]+(1,0)) \le G^\rho(\Delta[i]+(0,1))$, and $\Delta[i+1] = \Delta[i]+(0,1)$ otherwise. From this recursive construction we have that Δ consists of all $p \in \mathbb{Z}^2_{\ge 0}$, such that $p+(1,0) \in P_1$, and $p+(0,1) \in P_2$.

We also have that $\Delta = \Gamma^\rho_{\mathbf{0}}$, for the latter one satisfies the same recursive relation. In words, the competition interface defined from $\{\omega^\vee(p)\}_{p\in\mathbb{Z}^2}$ is equivalent to the semi-infinite geodesic defined from $\{\omega(p)\}_{p\in\mathbb{Z}^2}$. We also define the process $(p_t)_{t\ge 0}$, such that p_t is the last vertex in $\Gamma^\rho_{\mathbf{0}}$ with $G^\rho(p_t) \le t$ (see Figure 1.1).

We now describe the connection between LPP and TASEP (denoted as a Markov process on $\{0,1\}^{\mathbb{Z}}$). We take the following initial conditions: let $\eta_0(0) = 0$ and $\eta_0(1) = 1$, and for any other x let $\eta_0(x) = 1$ with probability ρ and $\eta_0(x) = 0$ with probability $1-\rho$ independently. We label the holes by $\mathbb{Z}$ from left to right, with the one at site 0 labeled 0; and label the particles by $\mathbb{Z}$ from right to left, with the one at site 1 labeled 0. For any $a,b \in \mathbb{Z}$ such that at time 0 the particle labeled b is to the left of the hole labeled a, we denote $L(a,b)$ as the time when the particle interchanges with the hole. Then we have $L(a,b) = G^\rho(a,b)$ in distribution jointly, see e.g. [49, Section 4.2]. We could couple this TASEP with LPP so that this equality holds almost surely.

As in [50], we keep track of a "hole-particle pair" in TASEP, which is a hole with a particle next to it in the right. At $t=0$ it is the hole at site 0 and particle at site 1.

Whenever the particle is interchanged with a hole to the right, we move this pair to the right; and whenever the hole is interchanged with a particle to the left, we move this pair to the left. Recall the process $(p_t)_{t\geq 0}$, then we have the follow lemma from [50], which says that the trajectory of this "hole-particle pair" can be mapped to the competition interface.

Lemma 1.18. *Under the above coupling, for the hole-particle pair at time t, let b_t be the label of the particle and a_t be the label of the hole. Then $p_t = (a_t, b_t)$.*

Note that a_t is also the number of times the pair moved to the right, and b_t is the number of times that pair moved to the left. Then at time t the hole-particle pair is at site $a_t - b_t$ and $a_t - b_t + 1$. We denote η'_t as η_t centered by the hole-particle pair. Specifically, we denote $\eta'_t(x) = \eta_t(x + a_t - b_t)$, where $a_t - b_t$ is the location of the hole in the pair, at time t. This process $(\eta'_t)_{t\geq 0}$ is also a Markov process, and we denote Φ^{ρ}_t as the measure of η'_t (see Figure 1.2).

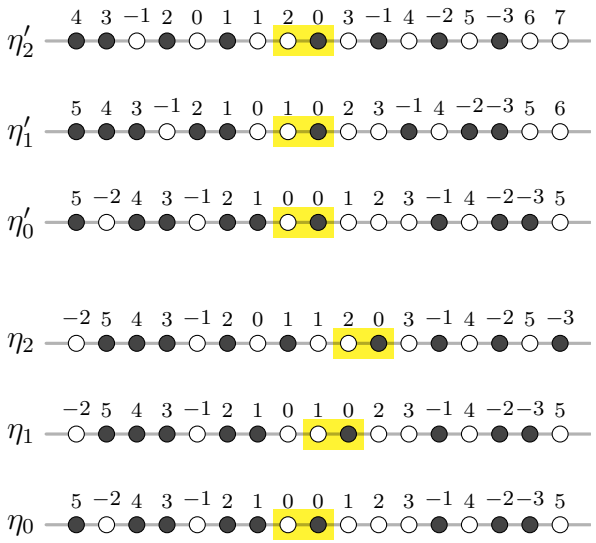


Figure 1.2: An illustration of the evolution of η_t (and the centered version η'_t): the numbers above the particles/holes are the labels, which increase from left to right for holes, and decrease from left to right for particles. The yellow boxes indicate the tracked hole-particle pairs.

1.4.5 second class particle and the stationary distribution

The "hole-particle pair" in Section 1.4.4 also plays the role of a second class particle in the TASEP, as shown by [50]. Such a TASEP with second class particles is a Markov process on $\{1, *, 0\}^{\mathbb{Z}}$, where the symbols 1, $*$, and 0 represent particles, second class particles, and holes respectively. Any second class particle can exchange places with a hole to its right, or with a (normal) particle to its left. We will see that the process $(\eta'_t)_{t\geq 0}$ described above corresponds to a TASEP as seen from an isolated second class particle.

Stationary measures for TASEP as seen from a second class particle are constructed in [47]; we now give a construction which yields one of the measures they describe (specifically, one with equal asymptotic density in both directions). Let $Y_1(x), x \geq 1$ and $Y_2(x), x \geq 1$ be independent collections of i.i.d. Bernoulli(ρ) random variables. Let $R_1(x) = \sum_{y=1}^{x} Y_1(y)$ and $R_2(x) = \sum_{y=1}^{x} Y_2(y)$.

Then we can define a symmetric random walk W by

$$W(x) = R_2(x) - R_1(x) \tag{1.5}$$

for $x \geq 0$. We define also

$$M(x) = \sup_{0\leq y\leq x} W(y), \tag{1.6}$$

and $\mathcal{E} = \{x \geq 1 : M(x) > M(x-1)\}$, and let $E(x) = \#\{y \in [\![1, x]\!] : y \in \mathcal{E}\}$.

Then we can see $M(x)-W(x)$ as a symmetric random walk with steps in $\{-1, 0, 1\}$ and reflected to stay non-negative. The points of $\mathcal{E}$, i.e. the points of increase of E, are those times where reflection occurs. We have that $x \in \mathcal{E}$ iff $M(x-1) = W(x-1)$, $Y_2(x) = 1$, and $Y_1(x) = 0$. By well-known properties of symmetric random walk, we can obtain that as $x \to \infty$, $\mathbb{P}(x \in \mathcal{E})$ decays like $x^{-1/2}$, while $|\mathcal{E} \cap [\![0, x]\!]|/x^{1/2}$ converges in distribution to a random variable supported on $(0, \infty)$.

Now we define a configuration σ on the non-negative half-line, which copies Y_1

except at points of $\mathcal{E}$. We set $\sigma(0) = *$ and, for $x \geq 1$,

$$\sigma(x) = \begin{cases} 1 & \text{if } Y_1(x) = 1 \\ 0 & \text{if } Y_1(x) = 0 \text{ and } x \notin \mathcal{E} \\ * & \text{if } Y_1(x) = 0 \text{ and } x \in \mathcal{E}. \end{cases}$$

(There is a natural interpretation involving the departure process of a discrete-time $M/M/1$ queue – see [48].) We wish to extend to give a configuration $\sigma(x), x \in \mathbb{Z}$ on the whole line. We can do this in two equivalent ways:

1. Note that $\sigma(x), x \geq 0$ is a renewal process with renewals at points x where $\sigma(x) = *$, i.e. where $x \in \mathcal{E}$. Between successive renewal points, we see an i.i.d. sequence of finite strings in $\bigcup_{n\geq 0}\{0,1\}^n$. We can extend to a renewal process on the whole line by extending this sequence of i.i.d. strings, separated by stars, leftward from the origin also.

2. Alternatively, we can exploit the symmetry of the TASEP under exchanging holes/particles and left/right. Write π_ρ for the distribution defined above on $\sigma(x), x \geq 0$. Now generate another configuration $\tilde{\sigma}(x), x \geq 0$ from $\pi_{1-\rho}$, independently of σ, and for $x \geq 1$ set

$$\sigma(-x) = \begin{cases} 1 & \text{if } \tilde{\sigma}(x) = 0 \\ 0 & \text{if } \tilde{\sigma}(x) = 1 \\ * & \text{if } \tilde{\sigma}(x) = *. \end{cases}$$

(The equivalence of these two definitions follows from the random walk construction above. If we look at the configuration between 0 and the first $*$ to the right of 0, we obtain a finite string of holes and particles whose distribution is invariant under exchanging both left/right and hole/particle; this invariance comes from the

invariance under reflection of the random walk path beginning and ending at level 0 which is used to construct the configuration.)

We also extend the definition of $\mathcal{E}$ to the whole line, by saying $x \in \mathcal{E}$ whenever $\sigma(x) = *$.

Now we have a distribution of $\sigma(x), x \in \mathbb{Z}$. Note immediately that $\sigma(x), x \geq 1$ is independent of $\sigma(x), x \leq -1$. This distribution is stationary for the TASEP with second class particles, as seen from one of the second class particles.

Proposition 1.19 ([47, Theorem 1]). *Let $(\sigma_t)_{t\geq 0}$ be TASEP with second class particles, started from $\sigma_0 = \sigma$. Suppose that at time t, the second class particle starting from the origin is at site l_t, and let σ_t^0 be σ_t shifted by l_t, i.e. $\sigma_t^0(x) = \sigma_t(x + l_t)$. Then for each $t \geq 0$, σ_t^0 has the same distribution as σ.*

Now we have a distribution of $\sigma(x), x \in \mathbb{Z}$. Note immediately that $\sigma(x), x \geq 1$ is independent of $\sigma(x), x \leq -1$.

Given a configuration σ, there are two related projections of it which involve setting all the $*$ symbols except for the one at the origin to be either 1s or 0s.

1. The simpler one consists of setting all $*$ symbols on positive sites to be 0, and all $*$ symbols on negative sites to be 1. This gives a configuration distributed according to product measure, in which the non-zero sites are i.i.d. Bernoulli(ρ).

2. Alternatively, we can follow the opposite rule of setting all $*$ symbols on positive sites to be 1 and all $*$ symbols on negative sites to be 0. Specifically, define a configuration $\overline{\eta}^*$ by $\overline{\eta}^*(0) = *$ and for $x \neq 0$,

$$\overline{\eta}^*(x) = \begin{cases} 0 & \text{if } \sigma(x) = 0, \text{ or if } \sigma(x) = * \text{ and } x < 0 \\ 1 & \text{if } \sigma(x) = 1, \text{ or if } \sigma(x) = * \text{ and } x > 0. \end{cases}$$

This gives a configuration which, compared to product measure of Bernoulli(ρ),

has a bias towards particles on positive sites and towards holes on negative sites. This bias decays as one gets further away from the origin.

Theorem 2 of [47] says that the distribution of $\overline{\eta}^*(x)$ is stationary for the TASEP as seen from an isolated second class particle. The bias above reflects the tendency created by the dynamics of the process for the second class particle to get stuck behind particles and to get stuck in front of holes.

The combination of the two projections above gives a coupling between the configuration $\overline{\eta}^*$ and a configuration with product measure in which the discrepancies are precisely the non-zero members of $\mathcal{E}$. The fact that the intersection of $\mathcal{E}$ with $[0, x]$ grows on the order of $\sqrt{x}$ can be used to show that product measure and the stationary distribution of the TASEP as seen from an isolated second class particle are mutually singular.

For later calculation, it will be useful to look at the position of the first hole to the right of the origin in $\overline{\eta}^*$ (and similarly the first particle to the left).

Let $X_+ = \min\{x \geq 1 : \overline{\eta}^*(x) = 0\}$, which is also $\min\{x \geq 1 : \sigma(x) = 0\}$. From the random walk construction of $\sigma(x), x > 0$, one gets that

$$X_+ = \min\{x \geq 1 : Y_1(x) = 0, \text{ and for some } y \in [\![1, x]\!], Y_2(y) = 0\}.$$

That is, to find X_+ we look for the first 0 in the process Y_2, and then we look for the first 0 in the process Y_1 from then on. Since all the variables $Y_1(x)$ and $Y_2(x)$ are i.i.d. Bernoulli(ρ), this gives that $X_+ + 1$ is the sum of two Geometric($1 - \rho$) random variables, and so

$$\mathbb{P}(X_+ = k) = k(1 - \rho)^2 \rho^{k-1} \tag{1.7}$$

for $k \geq 1$. Similarly if X_- is the location of the first particle to the left of the origin, then

$$\mathbb{P}(X_- = -k) = k\rho^2(1 - \rho)^{k-1}. \tag{1.8}$$

Finally we go back to the hole-particle representation of the second class particle. Take $\overline{\eta} : \mathbb{Z} \to \{0, 1\}$, such that $\overline{\eta}(0) = 0$, $\overline{\eta}(1) = 1$; for $x > 1$ let $\overline{\eta}(x) = \overline{\eta}^*(x-1)$, and for $x < 0$ let $\overline{\eta}(x) = \overline{\eta}^*(x)$. Denote by Ψ^ρ as the measure of $\overline{\eta}$. Then Ψ^ρ is the stationary measure of the Markov process of the TASEP as seen from a hole-particle pair, having the same transition probabilities as $(\eta'_t)_{t\geq 0}$ (defined at the end of Section 1.4.4). One of our results (Theorem 1.7) asserts that Ψ^ρ is the limit of Φ^ρ_t as $t \to \infty$.

1.5 Some other research projects

Besides the contents of this thesis, I have developed broad interests in probability theory during my graduate study, and conducted projects in other research directions, such as the TASEP slow bond problem, random sorting networks, Anderson model of localiation (random schördinger operator), factor of IID, and convergence of Markov chains. Some of these projects are also related to the KPZ universality class, and the rest have less connections to the topic of this thesis. In this section, we will briefly introduce my contributions in all these directions, and try to keep the discussions short and nontechnical.

1.5.1 Problems within the KPZ universality class

The slow bond problem. In the slow bond problem originally posed by Janowsky and Lebowitz [70], they ask in TASEP whether a small perturbation could affect the macroscopic current. More precisely, one reduces the jump rate across an edge $(0, 1)$ to $1-\epsilon$. This corresponds to taking $\omega(\mathbf{n})$ as $\mathrm{Exp}(1-\epsilon)$ for all $n \in \mathbb{Z}$, in the exponential LPP. We denote $T^\epsilon_{p,q}$ to be the passage time from p to q in this perturbed field. Different groups of physicists, using a range of heuristics and numerical simulations, reached opposing conclusions on whether any $\epsilon > 0$ would affect the macroscopic behavior of $T^\epsilon_{\mathbf{0},\mathbf{n}}$. Much difficulty of this problem is from that such small pertur-

bations destroy the exactly-solvable structure, and make the algebraic tools break down. Using geometric arguments, a breakthrough was made in [15], by proving that $\lim_{n\to\infty} \frac{\mathbb{E}[T^{\epsilon}_{\mathbf{0},\mathbf{n}}]}{n} > 4$ for any $\epsilon > 0$ (while $\lim_{n\to\infty} \frac{\mathbb{E}[T_{\mathbf{0},\mathbf{n}}]}{n} = 4$, from [91]). Thus the critical value of ϵ (to affect the macroscopic behavior of the flow) equals 0. In my work [92] with Sarkar and Sly, using multiscale analysis of geodesics, we do a more careful study of the effect of the perturbation and give an upper bound of it as $\epsilon \to 0$.

Theorem 1.20 ([92]). *For any $C > 0$, $\lim_{n\to\infty} \frac{\mathbb{E}[T^{\epsilon}_{\mathbf{0},\mathbf{n}}]}{n} - 4 < \epsilon^C$ when ϵ is small enough.*

This implies that the phase transition at $\epsilon = 0$ is of infinite order, and explains why it was so challenging to predict on the basis of numerical simulations.

Temporal correlation structure. As mentioned before, in verifying the KPZ behaviors of the exponential LPP model, the distribution of a single passage time is well understood using exactly-solvable methods; and so is the point-to line-profile from $\mathbf{0}$ to $\mathbb{L}_n$ (see e.g. Theorem 1.17). From the growth process point of view, $X_q = \max_{p\in\mathbb{L}_0} T_{p,q}$ is considered, as the passage time with flat initial condition. The distribution of X_q for q varying in $\mathbb{L}_n$ is well understood [21, 22].

The next question asks the temporal correlation structure, i.e. when one endpoint varies in the $(1,1)$-direction. There have been many predictions by physicists about the correlation exponents, supported by experimental or numerical studies, as stated by Ferrari and Spohn [52]. While exact expressions for the joint distributions are also available, it does not appear easy to get the correlation order out of the impressive but complicated formulas. More recently, in [9, 51] the following has been verified:

$$\lim_{n\to\infty} n^{-2/3}\mathrm{Cov}(T_{\mathbf{0},\mathbf{n}}, T_{\mathbf{0},\lfloor\tau\mathbf{n}\rfloor}) = \begin{cases} \Theta(\tau^{2/3}) & \text{as } \tau \to 0, \\ C - \Theta((1-\tau)^{2/3}) & \text{as } \tau \to 1, \end{cases}$$

for C being a constant. It is also verified that

$$\rho(\tau) = \lim_{n\to\infty} n^{-2/3}\mathrm{Cov}(X_{\mathbf{n}}, X_{\lfloor \tau \mathbf{n} \rfloor}) = C - \Theta((1-\tau)^{2/3})$$

as $\tau \to 1$. The remaining prediction of Ferrari and Spohn is on the behavior of $\rho(\tau)$ for $\tau \to 1$, which is conjectured to be in the order of $\tau^{4/3}$. This is verified by in my work [12] with Basu and Ganguly, by using geometric arguments involving coalescence of geodesics.

Theorem 1.21 ([12]). *As $\tau \to 0$ we have $\frac{\log(\rho(\tau))}{\log(\tau)} \to \frac{4}{3}$.*

Disjoint optimizers in the directed landscape. In LPP, one can take two sequences of vertices $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$, and consider directed paths from p_i to q_i that are disjoint. The maximum total weight of such disjoint paths is the *multiple-point passage time*. This is a natural object to study from the RSK bijection point of view. For a uniform random permutation, the passage time is the length of the longest increasing subsequence, and the RSK bijection maps it to the length of the first row of a Young diagram under Plancherel measure. The k-multi-point passage time is then the total length of the first k rows of the Young diagram.

In my work [45] with Dauvergne, we extend the directed landscape to multiple-point passage times, using last-passage percolation across the Airy line ensemble, which can be viewed as the limiting analog of the Young diagram. We also show that they are the scaling limits of the multi-point passage times in Brownian LPP. As one application, our construction implies that the Airy line ensemble can be constructed from the directed landscape, thus one direction of the RSK bijection holds in the limit. Our results also set up a framework to study coalescence of geodesics by analyzing last-passage percolation across the Airy line ensemble, as is used in this thesis to study fractal behaviors in the directed landscape (see Appendix C).

Colored TASEP and the Oriented Swap Process. In [103] we prove a shift-invariance property for the *colored TASEP*, which is used to verify a distributional identity conjectured by various people. This is another example of problems being difficult to solve only using algebraic approaches.

For each $x \in \mathbb{Z}$, the passage times from $(x, -x)$ in exponential LPP correspond to a TASEP with step initial condition, denoted by η^x; thus $\{\eta^x\}_{x\in\mathbb{Z}}$ is a coupling of a family of step initial TASEPs. The colored TASEP is another widely studied coupling of TASEPs. Initially, at each site $x \in \mathbb{Z}$ there is a particle with color x. For each particle at any time, if the particle right next to it has a larger color, with rate 1 these two particles swap. By considering particles with color $\leq x$, one can get a TASEP, where initially each site $\leq x$ is occupied by a particle and each site $> x$ is empty. Shift this by $-x$, we get a step initial TASEP ζ^x. It is natural to compare the joint distributions of $\{\zeta^x\}_{x\in\mathbb{Z}}$ and $\{\eta^x\}_{x\in\mathbb{Z}}$, and for this, we look at the passage times. Let $F_n(x, y) = T_{(x,-x),(n+y,n-y)}$, which is the time when the $n + y - x + 1$-th rightmost particle makes its $n - y + x + 1$-th jump, in η^x. Let $G_n(x, y)$ be the same time in ζ^x. Then $F_n(x, \cdot)$ and $G_n(x, \cdot)$ are equal in distribution for each x.

In [103], we prove the following shift-invariance for the colored TASEP. Take integers $x_1 \leq \cdots \leq x_k$, $y_1 \geq \cdots \geq y_k$, and $x'_1 \leq \cdots \leq x'_k$, $y'_1 \geq \cdots \geq y'_k$, such that each $x_i - y_i = x'_i - y'_i$. Then $\{G_n(x_i, y_i)\}_{i=1}^k$ has the same distribution as $\{G_n(x'_i, y'_i)\}_{i=1}^k$. Shift invariance of such spirit was first proved for the six-vertex model by Borodin-Gorin-Wheller and Galashin [23, 55]. For LPP, such shift invariance has been proved for F_n by Dauvergne [40]. Thus with my result, we have that $\{F_n(x_i, y_i)\}_{i=1}^k$ and $\{G_n(x_i, y_i)\}_{i=1}^k$ are equal in distribution, for any $x_1 \leq \cdots \leq x_k$ and $y_1 \geq \cdots \geq y_k$.

Another application of results in [103] is the following distributional identity. Consider the colored TASEP on a finite interval $\{1, \ldots, N\}$, which would terminate at the state where the particle at each site x has color $N + 1 - x$. This is also called the Oriented Swap Process (OSP), an incidence of the random sorting networks, for

it can be viewed as a permutation of size N evolving in time. Let $U_N(x)$ be the time of last swap between site x and $x+1$ for each $1 \le x \le N-1$. The results in [103] imply that U_N has the same distribution as $\{T_{(1,1),(x,N-x)}\}_{x=1}^{N-1}$, resolving conjectures in [27, 19]. This implies asymptotic results on the vector U_N, extending results on the finishing times of the OSP by Angel-Holroyd-Romik [3].

1.5.2 Anderson localization

Consider $H := -\Delta + \delta V$ on $\mathbb{Z}^d$, where Δ is the discrete Laplacian, $\delta > 0$ is the disorder strength, and $V : \mathbb{Z}^d \to \mathbb{R}$ is the i.i.d. potential. Named after Anderson who pioneered its study in 1958 [2], this model has been widely used to describe transport in disordered media, such as moving quantum particles in a material with impurities. A core property is the localization phenomenon, which can be stated as follows in terms of the spectrum. For the discrete Laplacian Δ on $\mathbb{Z}^d$, by Fourier transform its spectrum is $[0, 4d]$, and it is an absolutely continuous spectrum. For the operator H, however, the i.i.d. random potential V can make H have eigenvalues, whose eigenfunctions are exponentially localized. We say *Anderson localization* holds for H in an interval I, if $Hu = \lambda u$ with $u : \mathbb{Z}^d \to \mathbb{R}$ growing at most polynomially and $\lambda \in I$ implies that u decays exponentially. It implies that H has pure point spectrum in I.

The first wave of mathematical studies in understanding the localization phenomenon happened in the 1980s [74, 54, 53, 31], represented by works of Fröhlich and Spencer, with the following picture established. Fix the distribution of V, and assume that it is bounded, and its distribution function is ρ-Hölder for some $\rho > 0$. If the disorder strength δ is large enough, then Anderson localization holds in the whole spectrum of H; and for any fixed δ, the localization holds in $[0, \varepsilon]$ for some small $\varepsilon > 0$. Also, when dimension $d = 1$, Anderson localization is proven in the whole spectrum of H. Since then, the main open problem in this subject is on $d \ge 2$

and δ is small. It is conjectured that when $d = 2$, Anderson localization always holds in the whole spectrum of H; and when $d \geq 3$, Anderson localization holds only near the edge of the spectrum when δ is small.

In addition to this conjectured picture which remains wildly open, another question is to prove robustness in terms of the structure of the noise, by studying less regular distributions of V. The core particular case is when the potential is i.i.d. Bernoulli(1/2), modeling alloy type materials. For this *Anderson-Bernoulli* model, a breakthrough was made by Bourgain and Kenig [25]. Based on multi-scale analysis of Fröhlich and Spencer, [25] prove Anderson localization near the edge of spectrum for an analog model in $\mathbb{R}^d$. A key new ingredient is the *unique continuation principle* in $\mathbb{R}^d$, which roughly says that, if $u : \mathbb{R}^d \to \mathbb{R}$ satisfies $\Delta u = Vu$ for some bounded V on $\mathbb{R}^d$, and u is also bounded, then u can not be too small on any ball with positive radius. However, this approach cannot be directly applied to the lattice model, due to the lack of a discrete unique continuation principle (DUCP). More recently, Ding and Smart [46] proved Anderson-Bernoulli localization near the edge for $\mathbb{Z}^2$, by proving a quantitative DUCP on $\mathbb{Z}^2$, which holds for i.i.d. random V with high probability. The proof is inspired by polynomial arguments of Buhovsky-Logunov-Malinnikova-Sodin [28] on discrete harmonic functions. In my work [77] with Li, we settle this problem in the most physically relevant dimension $\mathbb{Z}^3$.

Theorem 1.22 ([77])**.** *There exists $\varepsilon > 0$, depending on δ, such that almost surely the following holds. For any function $u : \mathbb{Z}^3 \to \mathbb{R}$ and $\lambda \in [0, \varepsilon]$, if $Hu = \lambda u$ and $\inf_{n\geq 0} \sup_{a\in\mathbb{Z}^3} (1 + |a|)^{-n} |u(a)| < \infty$, we have $\inf_{t>0} \sup_{a\in\mathbb{Z}^3} \exp(t|a|)|u(a)| < \infty$. This implies that H has pure point spectrum in $[0, \varepsilon]$.*

The main new ingredient in [77] is the formulation and proof of a 3D quantitative DUCP. Unlike the 2D case, our 3D quantitative DUCP is for any deterministic and bounded V. Its proof uses novel geometric studies of the H eigenfunctions, decomposing the 3D lattice into 2D triangular lattices, and also the polynomial arguments

from [28] as inputs.

1.5.3 Factor of IID

Consider a *vertex-transitive graph* $G = (V, E)$, i.e., for any two vertices there is a graph automorphism mapping one to another. Examples of such graphs include the infinite d-regular tree and the lattice $\mathbb{Z}^d$. For a measurable space Ω, a measure ν on Ω^V is called a *factor of IID* if it is the ϕ-push-forward of the product measure μ_0^V, where μ_0 is a measure on some measurable space Ω_0, and $\phi : \Omega_0^V \to \Omega^V$ is a measurable function and commute with any automorphism of G.

From the ergodic theory point of view, on lattice this property is the same as being isomorphic (as dynamic systems) to a (generalized) Bernoulli shift, following Ornstein's results from the 1970s; and this is a strong property in the ergodic hierarchy. From the computer science point of view, a natural question is to design local sampling algorithms for a stochastic system, and such studies have been done for various models, such as maximum independent sets, random matching, and random coloring of graphs. If a stochastic system is proven to be a factor of IID, a local algorithm could first sample the i.i.d. process, then apply local functions that well approximate ϕ, since ϕ is measurable.

In my works [98] with Sly and [82] with Nam and Sly, we prove factor of IID results for the Voter model on the lattice, and the free Ising model on d-regular trees (also called *binary symmetric channel* or *broadcasting problem on trees*). Both models are classical in probability theory with applications in various disciplines, such as statistical mechanics, information theory, and theoretical computer science. Our proofs are by explicit constructions using Markov processes.

Voter Model Stationary Measure. In the Voter model on $\mathbb{Z}^d$, at each vertex, there is a voter with an opinion in $\{0, 1\}$. Each voter at rate 1 chooses a neighbor

at random among its $2d$ neighbors with equal probability and changes its opinion to the same as that neighbor. For any $p \in [0,1]$, if we start with i.i.d. Bernoulli p, then the distribution would converge to a stationary distribution μ_p. For $d = 1, 2$, there is $\mu_p = p\mu_1 + (1-p)\mu_0$. For $d \geq 3$, these $\{\mu_p\}_{p\in[0,1]}$ are all the extremal stationary distributions, i.e. ones that cannot be written as a linear combination of others.

Theorem 1.23 ([98]). *For $d \geq 3$ and each $p \in [0,1]$, μ_p is a factor of IID.*

Via its duality with coalescing random walks, μ_p is an example of the *generalized divide and color models*, studied by Steif and Tykesson [99]. In such models, the vertices of a graph are partitioned by a *random equivalence relation (RER)* and each equivalence class of vertices is given a random color independently. For such a model, if the RER is defined as a factor of IID, and each equivalent class has finitely many vertices, we can then assign colors as a factor of IID, by choosing a leader in each equivalent class. However, the choice of lead cannot be a factor of IID when the equivalent class is infinite. It is asked in [99] whether there exist natural examples of divide and color models, such that the equivalence classes are infinite but the coloring process is nonetheless a factor of IID. In particular, they asked whether μ_p for $d \geq 3$ would be such examples. Our result affirmatively answers these questions.

Ising model on infinite d-regular trees. Another natural class of the divide color models is the Ising and Potts models, for they are colorings of components in the FK percolation. For the FK percolation on $\mathbb{Z}^d$, depending on the temperature, there is either no infinite component or exactly one infinite component. The Ising or Potts models are factors IID in the former case, and not in the latter case.

However, for the Ising model on the infinite d-regular tree, it can be more complicated to determine whether it is a factor of IID. We note that the Gibbs measure may not be unique, and we are interested in the most natural one, with free boundary conditions. Let β be the inverse temperature. When $\tanh\beta \leq (d-1)^{-1}$, the FK

percolation has only finite components, so the Ising model is a factor of IID. On the other hand, when $\tanh\beta > (d-1)^{-\frac{1}{2}}$, it is in the reconstruction regime, and a short correlation computation by Sly proves that it is not a factor of IID. The intermediate regime $(d-1)^{-1} < \tanh\beta \leq (d-1)^{-\frac{1}{2}}$ remains open [78]. We solves this problem in [82] up to a constant factor.

Theorem 1.24 ([82]). *There exists a constant $c > 0$, such that for $d \geq 3$ and $\beta \geq 0$ with $\tanh\beta \leq c(d-1)^{-\frac{1}{2}}$, the free Ising model on the infinite d-regular tree with inverse temperature β is a factor of IID.*

This result gives another family of natural divide and color models, with infinite equivalent classes and being factors of IID. Our proof is by an innovative approach of designing an infinite system of SDEs and constructing a strong solution that converges to the free boundary Ising model.

1.5.4 Convergence of Markov Chain

Cutoff in Repeated Averages. For a finite graph with a number at each vertex, at each step one picks an edge uniformly and replaces the two numbers on the vertices by their averages. This *repeated average process* and its variants, under names such as *gossip algorithm*, *concensus algorithm*, *Deffuant model*, have been studied in different disciplines [32, 97, 86, 62, 75, 18, 1]. Consider this process in a complete graph with n vertices. Let $\mathbf{x}_k \in \mathbb{R}^n$ be the vector of numbers after step k, and $\bar{\mathbf{x}}$ be the limiting vector where each coordinate equals the average. A straight forward computation would imply that the L^2 distance $\|\mathbf{x}_k - \bar{\mathbf{x}}\|_2$ decays exponentially. The more difficult question is on the L^1 distance, and its rate of decay was asked by Bourgain around 1980. In my work with Chatterjee, Diaconis, and Sly [33], we address this problem with the initial vector $\mathbf{x}_0 = (1, 0, \ldots, 0)$, and prove a cutoff behavior around $n\log_2(n)/2$, and get Gaussian decay profile in a window of order $n\sqrt{\log(n)}$.

Theorem 1.25 ([33]). *Let $\Phi : \mathbb{R} \to [0,1]$ be the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. As $n \to \infty$ we have $\|\mathbf{x}_{\lfloor n(\log_2(n)+a\sqrt{\log_2(n)})/2 \rfloor} - \bar{\mathbf{x}}\|_1 \to 2\Phi(-a)$ in probability.*

For general initial vectors, the decay could be faster, and one can design the initial vector to get decay functions of different shapes. Using linearity of this model we show that the initial vector of $(1, 0, \ldots, 0)$ would give the slowest decay in L^1 (among all vectors with entries being non-negative and sum to 1), thus giving an upper bound for the L^1 decay for any initial vector.

For the proof, a key observation is that the repeated averaging process can initially be coupled with a fragmentation process, where numbers split exactly in two at rate $1/n$. The cutoff happens when most of the mass is in numbers of order $n^{-1+o(1)}$, and at this moment the L^1 distance is still close to 2. Then we apply the L^2 analysis to get fast decay of the L^1 distance in an additional $o(n \log(n))$ time.

Coalescing Random Walk on Random Graphs. The coalescing random walk (CRW) on a graph $\mathcal{G} = (V, E)$ model is defined as follows. Initially, each vertex $v \in V$ is occupied by one particle. Each particle performs independent continuous-time edge simple random walk (SRW); i.e. for each edge incidental to the current location of the particle, with rate 1 it jumps across that edge. Whenever two particles meet (or "collide", i.e. one particle jumps to the current location of the other one), they merge into one particle which continues to perform SRW. Whenever such collisions happen, the total number of particles in the system then decreases by 1.

We are interested in the rate of decay (as $t \to \infty$) of the probability $P_t(x)$ of some specific site x being occupied at time t, i.e., $P_t(x) := \mathbb{P}[x \in \xi_t]$ where ξ_t is the set of elements in V that are occupied at time t. This model has received a lot of attention in the literature. The asymptotics of P_t at time t on $\mathbb{Z}^d$ and on a torus $(\mathbb{Z}/n\mathbb{Z})^d$ have been well studied, see e.g. [26, 100, 101]. Some bounds on P_t and on the coalescence time on general graphs are proved for instance in [84, 85, 34]. Besides, the motivation

of studying the model also stems from its connection with voter model via duality.

The exact rate of decay is predicted to take the form $P_t \approx 1/(t\alpha_t)$, where α_t is the probability that two random walks starting from stationary distribution never meet before time t. It was originally described by a heuristic of van der Berg and Kesten and proven rigorously for $\mathbb{Z}^d$ when $d \geq 3$ in [100, 101]. It starts by considering $-\frac{\mathrm{d}}{\mathrm{d}t}\sum_x P_t(x) = \sum_{x\sim y}\mathbb{P}[x \text{ and } y \text{ are occupied at time } t]$. By reversing time and assuming independence of particles on x and y, the right hand side roughly equals $\alpha_t P_t^2$. This gives $P_t \approx 1/(t\alpha_t)$. A second heuristic predicts that particles exhibit certain mean-field behaviors where the first collision time among a remaining of i particles is roughly $\mathrm{Exp}(\binom{i}{2}/t_{\mathrm{meet}})$ distributed [84]. Here t_{meet} is the mean meeting time of two independent walkers starting from stationary distribution. This gives the prediction that $P_t \approx 2t_{\mathrm{meet}}/(t|V|)$.

In [66], we verify the two heuristics for graphs given by configuration model with degree ≥ 3. We work in the regime when

$$t \ll t_{\mathrm{coal}} := \mathbb{E}[\inf\{s : \text{There is only one particle at time } s\}].$$

We show that for $1 \ll t \ll |V|$, P_t decays in the order of t^{-1}, and $(tP_t)^{-1}$ is approximately the probability that two particles starting from the root of the corresponding unimodular Galton-Watson tree never collide after one of them leaves the root, which is also roughly $|V|/(2t_{\mathrm{meet}})$. By taking the local weak limit, for the corresponding unimodular Galton-Watson tree, we also prove convergence of tP_t as $t \to \infty$.

Chapter 2

Environment seen from geodesics in the discrete setting

2.1 A roadmap of the proofs

In this chapter we prove the results stated in Section 1.2. There are two main ingredients: geometry of the geodesics in exponential LPP, and the TASEP as seen from a second class particle.

As mentioned above, the environment seen from a semi-infinite geodesic corresponds to TASEP as seen from a single second class particle (or hole-particle pair). We will first construct the limiting distribution ν^ρ in Section 2.3, using the stationary measure Ψ^ρ of TASEP as seen from a hole-particle pair from Section 1.4.5; we then prove Proposition 1.5 and 1.6 assuming the convergence results.

For the remaining results we take the following approach. In Section 2.4 we prove Theorem 1.7, by first proving a weaker version involving convergence in an averaged sense (Proposition 2.15), and then upgrading using the connection with LPP and geometric arguments. In Section 2.5 we prove a weaker version of Theorem 1.2 involving convergence in probability (Theorem 2.24). This is done by using Theorem

1.7 and ergodicity of the stationary process, which we prove as Proposition 2.26. From then on we work completely in the LPP setting. In Section 2.6 we prove Theorem 1.1 using Theorem 2.24, by covering a finite geodesic with an infinite one.

The next few sections rely on a generalization of Theorem 1.1, which is Proposition 2.33, and which says that for a family of geodesics whose endpoints vary range along two anti-diagonal segments, their empirical distributions converge jointly (in probability). The proof is via taking a finite (i.e. whose size does not grow) dense subset of these geodesics, and showing that each geodesic in the family can be mostly covered by them. Using this result, in Section 2.8 we prove Theorem 1.3 and 1.4, by showing that the distributions at neighboring points are close to each other. In Section 2.9, by covering long or semi-infinite geodesics by short ones, we prove that the empirical distribution concentrates exponentially fast, and thus upgrade Theorem 2.24 to Theorem 1.2.

2.1.1 Some previous arguments

In the development of these results, there were some alternative arguments that was previously taken — we briefly discuss the arguments here.

Compared to the current arguments, the major difference of the other approach is that, working just from the LPP perspective, one could prove convergence of the empirical distribution for finite geodesics, i.e. Theorem 1.1, without identifying the limit. The idea is to study the passage times $T_{\mathbf{0},u}$ and $T_{v,\mathbf{n}^\rho}$, where u, v vary in the lines $x+y=\alpha n$ and $x+y=(\alpha+\varepsilon)n$ for some $\alpha \in (0,2)$ and small $\varepsilon > 0$. These two point-to-line profiles are independent, and by Theorem 1.17, each converges (after rescaling) to the Airy_2 process, which is locally like a Brownian motion. Then it can be shown that in small neighborhoods of the intersections of the geodesic with these two lines, the point-to-line profiles (after rescaling) are similar to the maximum of the sum of two independent Brownian motions, or $R-B, R+B$, where R is a Bessel_3

process and B is a Brownian motion. (In recent work [43], such behavior is observed for geodesics in the directed landscape, the limiting object of the exponential LPP model.) One can also show that the geodesic between these lines is stable with respect to small perturbations of the line-to-point profiles. From this we could deduce that, for different vertices along the geodesic, the local statistics are similar, and that the correlation decays fast when the distance between the vertices is of order n. Thus we prove concentration of the empirical distribution via decay of the variance. To show that it converges, and to prove convergence for semi-infinite geodesics, we would need to cover long geodesics with short ones, using arguments similar to those in Section 2.6–2.9. Finally, from there to get an explicit description of the limiting distribution, we still need to use the stationary distribution of TASEP with a single second class particle.

The main reason we take the current path is for simplicity of arguments. For instance, it is highly technical to establish asymptotic independence of environments around different (far away) vertices along the geodesic, and also the stability of geodesics (with respect to the point-to-line profiles). The convergence to $R-B, R+B$ would involve computations using the KPZ fixed point formulae from [80]. These arguments are now replaced by proving convergence of TASEP as seen from a second class particle, although the proof of the TASEP convergence still relies on the LPP setting and geometric arguments (see Section 2.4). It is also more natural to start by showing convergence for semi-infinite geodesics, and deduce the result for finite ones (as opposed to the other way round), given the input from TASEP as seen from a second class particle. The more geometric arguments later in Section 2.6–2.9 are more inherited and adapted from the previous approach.

2.1.2 Further applications and questions

With the limiting measure ν^ρ one can get any local information along the geodesics. Before diving into the detailed proofs, we discuss some questions, which can potentially be answered using our explicit descriptions of ν^ρ, either as direct applications or require some further analysis.

The first question is communicated to us by Alan Hammond. Given that a vertex on the geodesic has a large weight, how would the local environment behave? For a vertex with a large weight, it would force the geodesic to go through it. Thus we expect that conditional on this, weights of nearby vertices are distributed like i.i.d. exponential random variables. From the TASEP aspect, a large weight corresponds to a long waiting time between two jumps of the second class particle, and this is mostly due a "jam" in TASEP, i.e., a consecutive sequence of particles to the right of the second class particle, and a sequence of holes to the left. This resembles a "reversed" step initial condition.

A related question is about vertices near but off the geodesic. Using our formulation of ν^ρ it can be verified that, for those next to the geodesic, their weights are strictly stochastically dominated by Exp(1). It is then interesting to see if the distribution converges to Exp(1) as the distance increases to infinity.

The next question is about a slightly different setting, that of LPP with i.i.d. geometric weights. The main difference is that in that case the weights are discrete, and the geodesics are not necessarily unique. However, one could still consider the "rightmost" geodesics. Geometric LPP corresponds to discrete-time TASEP, and one can similarly construct a stationary measure for such a TASEP as seen from a single second class particle. For a correspondence with the rightmost geodesic, one takes a second class particle which is prioritized to jump to the right rather than to the left. One can similarly construct a limiting distribution, and thus get local information about the environment along the rightmost geodesics. One question that

would be interesting to study is the proportion of "unique geodesic vertices". For fixed endpoints (or for one fixed endpoint and a fixed direction), take the intersection of all the geodesics, and call the vertices in that intersection "unique geodesic vertices". Do these unique geodesic vertices asymptotically make up a positive proportion of the geodesic? Furthermore, does the proportion converge in probability, and can we compute the limit explicitly? We think such questions are related to the convergence of the environment around the rightmost geodesic, because we expect that a vertex on the path is unlikely to be "locally unique" without being a unique geodesic vertex in the sense mentioned above. Anomalous "locally but not globally unique" vertices should make up a vanishing proportion of the path in the limit.

Another direction concerns the scaling limit of the measure ν^ρ. In a recent work by Dauvergne, Sarkar, and Virág, the authors zoom in around the geodesic in the directed landscape, constructing a local environment [43, Theorem 1.1]. It is reasonable to expect that when zooming out, the measure ν^ρ would converge to the local environment constructed there. Also, once this is established, we would like to see if our explicit description of ν^ρ could be used to get some explicit information about the local environment and geodesics in the directed landscape (e.g. [43, Problem 4]).

2.2 Estimates for exponential LPP

In this section we list some useful estimates, which are mostly from the literature.

For the rest of this chapter, for any $a, b \in \mathbb{Z}$ and $\rho \in (0,1)$, we denote

$$\langle a, b\rangle_\rho = \left(\left\lfloor \frac{2(1-\rho)^2 a}{\rho^2 + (1-\rho)^2} \right\rfloor + b, \left\lceil \frac{2\rho^2 a}{\rho^2 + (1-\rho)^2} \right\rceil - b \right).$$

Then $\langle n, 0\rangle_\rho = \mathbf{n}^\rho$. We also write $\langle a, b\rangle = \langle a, b\rangle_{1/2} = (a+b, a-b)$.

We start with estimates on passage times. We have that $T_{\mathbf{0},(m,n)}$ has the same law as the largest eigenvalue of X^*X where X is an $(m+1) \times (n+1)$ matrix of

i.i.d. standard complex Gaussian entries (see [71, Proposition 1.4]). Hence we get the following one-point estimates from [76, Theorem 2].

Theorem 2.1. *There exist constants $C, c > 0$, such that for any $m \geq n \geq 1$ and $x > 0$, we have*

$$\mathbb{P}[T_{\mathbf{0},(m,n)} - (\sqrt{m} + \sqrt{n})^2 \geq x m^{1/2} n^{-1/6}] \leq C e^{-cx}. \tag{2.1}$$

In addition, for each $\psi > 1$, there exist $C', c' > 0$ depending on ψ such that if $\frac{m}{n} < \psi$, we have

$$\begin{aligned} &\mathbb{P}[T_{\mathbf{0},(m,n)} - (\sqrt{m} + \sqrt{n})^2 \geq x n^{1/3}] \leq C' e^{-c' \min\{x^{3/2}, x n^{1/3}\}}, \\ &\mathbb{P}[T_{\mathbf{0},(m,n)} - (\sqrt{m} + \sqrt{n})^2 \leq -x n^{1/3}] \leq C' e^{-c' x^3}, \end{aligned} \tag{2.2}$$

and as a consequence

$$|\mathbb{E} T_{\mathbf{0},(m,n)} - (\sqrt{m} + \sqrt{n})^2| \leq C' n^{1/3}. \tag{2.3}$$

We also have the following parallelogram estimate.

Proposition 2.2 ([12, Theorem 4.2])**.** *Let U be the parallelogram whose one pair of sides have length $2n^{2/3}$ and are aligned with $\mathbb{L}_0$ and $\mathbb{L}_n$ respectively, and let their midpoints being $(m, -m)$ and $\mathbf{n}$. Let U_1, U_2 be the part of U below $\mathbb{L}_{n/3}$ and above $\mathbb{L}_{2n/3}$ respectively. For each $\psi < 1$, there exist constants $C, c > 0$ depending only on ψ, such that when $|m| < \psi n$,*

$$\mathbb{P}\Big[\sup_{u \in U_1, v \in U_2} |T_{u,v} - \mathbb{E} T_{u,v}| \geq x n^{1/3} \Big] \leq C e^{-cx}.$$

Such a result is first proved as [15, Proposition 10.1, 10.5], in the setting of Poissonian DLPP. In the setting of exponential DLPP a proof is given in [12, Appendix C], following the ideas in [15].

We will also need the following estimate on the coalescing probability of two geodesics, for finite and semi-infinite geodesics respectively.

Proposition 2.3 ([102]). *For each $\psi \in (0,1)$, there exists $C > 0$, such that*

$$\mathbb{P}[\Gamma_{\mathbf{0},\langle n,b_1\rangle} \cap \mathbb{L}_{n-m} = \Gamma_{\mathbf{0},\langle n,b_2\rangle} \cap \mathbb{L}_{n-m}] > 1 - Cm^{-2/3}|b_1 - b_2|$$

for any $n, m \in \mathbb{N}, b_1, b_2 \in \mathbb{Z}$, such that $m < n/3$, $|b_1|, |b_2| < \psi n$.

Proposition 2.4 ([14, Theorem 2]). *For any $\rho \in (0,1)$, there is a constant $C > 0$, such that for any $r \in \mathbb{N}$, and $k > 1$, we have $\mathbb{P}[\Gamma^{\rho}_{\mathbf{0}} \cap \mathbb{L}_{\lfloor r^{3/2}k \rfloor} \neq \Gamma^{\rho}_{\langle 0,r\rangle_\rho} \cap \mathbb{L}_{\lfloor r^{3/2}k \rfloor}] < Ck^{-2/3}$.*

We next give some estimates on transversal fluctuation of geodesics. Such estimates on transversal fluctuation of geodesics have appeared and been used in several works recently (see e.g. [12, 15, 14, 56, 11].

Lemma 2.5. *For each $\psi \in (0,1)$, there exist constants C, c such that the following is true. For any $r < n \in \mathbb{N}$ large enough, and $|b| < \psi n$, let $\langle r, b'\rangle$ be the vertex in $\mathbb{L}_r$ that is closest to the straight line connecting $\mathbf{0}$ and $\langle n, b\rangle$, and let $\langle r, b''\rangle$ be the intersection of $\mathbb{L}_r$ with $\Gamma_{\mathbf{0},\langle n,b\rangle}$. Then we have $\mathbb{P}[|b' - b''| > xr^{2/3}] < Ce^{-cx}$ for any $x > 0$.*

This is a slightly generalized version of [102, Proposition 2.3], and the proof follows in exactly the same way. See also [14, Theorem 3]. We omit the proof here. We have a similar estimate for semi-infinite geodesics as well.

Lemma 2.6. *For any $\rho \in (0,1)$, there exist $C, c > 0$ such that the following is true. Let $b_r \in \mathbb{Z}$ such that $\Gamma^{\rho}_{\mathbf{0}}[2r+1] = \langle r, b_r\rangle_\rho$, then $\mathbb{P}[|b_r| > xr^{2/3}] < Ce^{-cx^3}$ for any $x > 0$.*

Proof. For simplicity of notation we denote $T^*_{u,v} = T_{u,v} - \omega(v)$ for any vertices $u \leq v$. The event $|b_r| > xr^{2/3}$ implies that there exists $b \in \mathbb{Z}$ such that $|b| > xr^{2/3}$, and

$T^*_{\mathbf{0},\langle r,b\rangle_\rho} + B^\rho(\langle r,b\rangle_\rho, \mathbf{r}^\rho) > T^*_{\mathbf{0},\mathbf{r}^\rho}$. For this we denote $L_j := \{\langle r,b\rangle_\rho : |b - 2jr^{2/3}| \le r^{2/3}\}$ for $j \in \mathbb{Z}$, $|j| \le r^{1/3}$, and we have

$$\begin{aligned}
&\mathbb{P}\big[\max_{u\in L_j} T^*_{\mathbf{0},u} + B^\rho(u,\mathbf{r}^\rho) > T^*_{\mathbf{0},\mathbf{r}^\rho}\big] \\
<&\mathbb{P}\big[\max_{u\in L_j} T^*_{\mathbf{0},u} - \mathbb{E}[T^*_{\mathbf{0},u}] > c_0 j^2 r^{1/3}\big] + \mathbb{P}\big[T^*_{\mathbf{0},\mathbf{r}^\rho} - \mathbb{E}[T^*_{\mathbf{0},\mathbf{r}^\rho}] < -c_0 j^2 r^{1/3}\big] \\
&+ \mathbb{P}\big[\max_{u\in L_j} B^\rho(u,\mathbf{r}^\rho) + \mathbb{E}[T^*_{\mathbf{0},u}] - \mathbb{E}[T^*_{\mathbf{0},\mathbf{r}^\rho}] > -2c_0 j^2 r^{1/3}\big],
\end{aligned}$$

where $c_0 > 0$ is a small enough constant. We claim that each term in the right hand side is bounded by e^{-cj^3}, for some constant $c > 0$; then by summing over $j \in \mathbb{Z}$ for $2|j| > x - 1$ we get the conclusion. For the first term, when $2|j| < 0.9r^{1/3}$ we apply Proposition 2.2; when $0.9r^{1/3} \le 2|j| \le r^{1/3}$, we use that $T^*_{\mathbf{0},u} \le T^*_{\langle -\lfloor 0.1r\rfloor,0\rangle,u}$ and then apply Proposition 2.2 to $\max_{u\in L_j} T^*_{\langle -\lfloor 0.1r\rfloor,0\rangle,u}$. For the second term we apply Theorem 2.1. For the last term, we use (2.3) to conclude that $\mathbb{E}[T^*_{\mathbf{0},u}] - \mathbb{E}[T^*_{\mathbf{0},\mathbf{r}^\rho}] \le c_1 r^{1/3} - b(\rho^{-1} - (1-\rho)^{-1})$ for some constant $c_1 > 0$; and use that $b \mapsto B^\rho(\langle r,b\rangle_\rho, \mathbf{r}^\rho) - b(\rho^{-1} - (1-\rho)^{-1})$ is a (two-sided) centered random walk. □

In addition to the above one-point bounds, we also quote the following uniform bound on transversal fluctuation.

Lemma 2.7 ([12, Proposition C.9])**.** *For each $\psi \in (0,1)$ there exist constants C, c such that the following is true. For $x > 0$, $n \in \mathbb{N}$, and $|b| < \psi n$, consider the rectangle whose one pair of opposite edges are given by the segment connecting $\langle 0, -\lfloor xn^{2/3}\rfloor\rangle$ and $\langle 0, \lfloor xn^{2/3}\rfloor\rangle$, and the segment connecting $\langle n, b - \lfloor xn^{2/3}\rfloor\rangle$ and $\langle n, b + \lfloor xn^{2/3}\rfloor\rangle$. Then the probability that the geodesic $\Gamma_{\mathbf{0},\langle n,b\rangle}$ exit this rectangle is at most Ce^{-cx^3}.*

Combining these transversal fluctuation estimates we get the following, for finite and semi-infinite geodesics respectively.

Corollary 2.8. *For each $\psi \in (0,1)$, there exist constants C, c such that the following is true. Take any $r < n \in \mathbb{N}$ large enough, $x > 0$, and $|b| < \psi n$. Let $\langle r, b'\rangle$ be the*

vertex in $\mathbb{L}_r$ that is closest to the straight line connecting $\mathbf{0}$ and $\langle n, b\rangle$. Consider the parallelogram whose one pair of opposite edges are given by the segment connecting $\langle 0, -\lfloor xr^{2/3}\rfloor\rangle$ and $\langle 0, \lfloor xr^{2/3}\rfloor\rangle$, and the segment connecting $\langle r, b' - \lfloor xr^{2/3}\rfloor\rangle$ and $\langle r, b' + \lfloor xr^{2/3}\rfloor\rangle$. Then with probability $1 - Ce^{-cx}$, the geodesic $\Gamma_{\mathbf{0},\langle n,b\rangle}$ below $\mathbb{L}_r$ is contained in that rectangle.

Corollary 2.9. *For each $\rho \in (0,1)$, there exist constants C, c such that the following is true. Take any $r \in \mathbb{N}$ large enough and any $x > 0$. Consider the parallelogram whose one pair of opposite edges are given by the segment connecting $\langle 0, -\lfloor xr^{2/3}\rfloor\rangle_\rho$ and $\langle 0, \lfloor xr^{2/3}\rfloor\rangle_\rho$, and the segment connecting $\langle r, -\lfloor xr^{2/3}\rfloor\rangle_\rho$ and $\langle r, \lfloor xr^{2/3}\rfloor\rangle_\rho$. Then with probability $1 - Ce^{-cx^3}$, the part of the geodesic $\Gamma^\rho_{\mathbf{0}}$ below $\mathbb{L}_r$ is contained in that rectangle.*

We also use the following verison, where the rectangle is replaced by a parallelogram.

Lemma 2.10. *There exist constants $C, c > 0$ such that the following is true. Take any $n \in \mathbb{N}$ large enough and any $x > 0$, and denote $m_{n,x} = xn^{2/3}$. Consider the parallelogram whose four vertices are $(-m_{n,x}, 0)$, $(m_{n,x}, 0)$, and $(n - m_{n,x}, n)$, $(n + m_{n,x}, n)$. Then with probability $1 - Ce^{-c(x^3 \wedge n^{2/3}x)}$, we have $\Gamma_{\mathbf{0}} \cap (\mathbb{Z} \times [\![0, n]\!])$ is contained in this parallelogram.*

Proof. We consider the rectangle whose four vertices are

$$(-m_{n,x}/2, m_{n,x}/2),\ (m_{n,x}/2, -m_{n,x}/2),\ (n + m_{n,x}, n),\ (n + m_{n,x}, n).$$

If the part of the geodesic $\Gamma_{\mathbf{0}}$ below $\mathbb{L}_n$ is contained in this rectangle, then $\Gamma_{\mathbf{0}} \cap (\mathbb{Z} \times [\![0, n]\!])$ is contained in the parallelogram. Thus by Corollary 2.9, the probability is at least $1 - C'e^{-c'm_{n,x}^3(n+m_{n,x})^{-2}}$ for some constants $C, c > 0$; and this implies the conclusion. □

Finally, we have the following estimate on the passage time along a semi-infinite geodesic.

Lemma 2.11. *For each $\rho \in (0,1)$, there exist constants C, c such that the following is true. Take any $l > 0$. Let u_* be the first vertex in $\Gamma^\rho_{\mathbf{0}}$ above the line $\{((1-\rho)y + (1-\rho)^2 l, -\rho y + \rho^2 l) : y \in \mathbb{R}\}$. Then $\mathbb{P}[|T_{\mathbf{0},u_*} - l| > x^2 l^{1/3}] < Ce^{-cx}$ for any $x > 0$.*

Proof. Let $C, c > 0$ denote large and small constants depending only on ρ, and below the values can change from line to line. Let U be the parallelogram whose four vertices are $\langle cl, -Cxl^{2/3}\rangle_\rho$, $\langle cl, Cxl^{2/3}\rangle_\rho$, $\langle Cl, -Cxl^{2/3}\rangle_\rho$, $\langle Cl, Cxl^{2/3}\rangle_\rho$ (round to the nearest lattice points). By Corollary 2.9, with probability at least $1 - C'e^{-cx}$ we have $u_* \in U$. Note that for any $v \in U$ that is within distance 1 to the line $\{((1-\rho)y + (1-\rho)^2 l, -\rho y + \rho^2 l) : y \in \mathbb{R}\}$, we have $|\mathbb{E}T_{\mathbf{0},v} - l| < Cx^2 l^{1/3}$. Then by covering $\mathbf{0}$ and all such U with Cx parallelograms, each of size in the order of $l \times l^{2/3}$, and applying Proposition 2.2, we get the conclusion. □

Combining the above two results we get the following.

Corollary 2.12. *For each $\rho \in (0,1)$, there exist constants C, c such that the following is true. Take any $l > 0$, and let u_* be the last vertex in $\Gamma^\rho_{\mathbf{0}}$ with $T_{\mathbf{0},u_*} < l$. Then with probability $> 1 - Ce^{-cx}$, u_* is between the line $\{((1-\rho)y + (1-\rho)^2(l - x^2 l^{1/3}), -\rho y + \rho^2(l - x^2 l^{1/3})) : y \in \mathbb{R}\}$ and $\{((1-\rho)y + (1-\rho)^2(l + x^2 l^{1/3}), -\rho y + \rho^2(l + x^2 l^{1/3})) : y \in \mathbb{R}\}$, thus below $\mathbb{L}_l$ when $x < l^{1/3}$; and $\Gamma_{\mathbf{0},u_*}$ is contained in the rectangle whose one pair of opposite edges are given by the segment connecting $\langle 0, -\lfloor xl^{2/3}\rfloor\rangle_\rho$ and $\langle 0, \lfloor xl^{2/3}\rfloor\rangle_\rho$, and the segment connecting $\langle l, -\lfloor xl^{2/3}\rfloor\rangle_\rho$ and $\langle l, \lfloor xl^{2/3}\rfloor\rangle_\rho$.*

2.3 The limiting distribution

We can now define the measures ν^ρ from Ψ^ρ. Let $(\overline{\eta}_t)_{t\in\mathbb{R}}$ be the stationary Markov process of TASEP centered by a hole-particle pair, i.e., the law of $\overline{\eta}_t$ for each $t \in \mathbb{R}$

is Ψ^ρ. Note that $(\overline{\eta}_t)_{t\in\mathbb{R}}$ and $(\eta'_t)_{t\geq 0}$ (from Section 1.4.4) have the same transition probability but different initial conditions. The idea is to construct the competition interface from $(\overline{\eta}_t)_{t\in\mathbb{R}}$, in a way as described in Section 1.4.4, then take the environment as seen around the origin. This would be the environment along the geodesic $\Gamma^\rho_{\mathbf{0}}$, as seen at a uniform time; i.e. from p_t of a uniform t, where p_t is defined as the last vertex in $\Gamma^\rho_{\mathbf{0}}$ with $G^\rho(p_t) \leq t$. To get the environment as seen from a uniform vertex ν^ρ we would do a reweighting.

We use $\hat{\mathbf{\Psi}}^\rho$ to denote the measure of $(\overline{\eta}_t)_{t\in\mathbb{R}}$. As above, we label the particles from right to left, and the holes from left to right, such that at time 0 the particle at site 1 and the hole at site 0 are both labeled 0. Let $\overline{L}(a,b)$ be the time when the particle labeled b is switched with the hole labeled a, and let $\overline{\omega}(a,b) = \overline{L}(a+1,b) \wedge \overline{L}(a,b+1) - \overline{L}(a,b)$. We define $\overline{\gamma} \subset \mathbb{Z}^2$ as the collection of all (a,b), such that there is a time when the particle labeled b is at site 1 and the hole labeled a is at site 0. We let $\hat{\nu}^\rho$ be the measure given by the law of $(\{\overline{\omega}(a,b)\}_{(a,b)\in\mathbb{Z}^2}, \overline{\gamma})$ under $\hat{\mathbf{\Psi}}^\rho$.

We then let $\mathbf{\Psi}^\rho$ be the measure $\hat{\mathbf{\Psi}}^\rho$ conditional on $\overline{L}(\mathbf{0}) = 0$, i.e. we let $d\mathbf{\Psi}^\rho = \lim_{\epsilon\to 0_+} \frac{\mathbb{1}[\overline{L}(\mathbf{0})>-\epsilon]d\hat{\mathbf{\Psi}}^\rho}{\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{L}(\mathbf{0})>-\epsilon]}$. As $(\overline{\eta}_t)_{t\in\mathbb{R}}$ under $\hat{\mathbf{\Psi}}^\rho$ is a Markov process, the limit could be computed as $\hat{\mathbf{\Psi}}^\rho$ conditional on that there is a jump of the hole particle pair at time 0; i.e. first reweight $\hat{\mathbf{\Psi}}^\rho$ by $\mathbb{1}[\overline{\eta}_{0_-}(2) = 0] + \mathbb{1}[\overline{\eta}_{0_-}(-1) = 1]$, the events where a jump is allowed, then let the jump happen at time 0. More precisely, we can describe the limit as follows. We have

$$\mathbf{\Psi}^\rho = \frac{\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{\eta}_{0_-}(2) = 0]\mathbf{\Psi}^{\rho,1} + \mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{\eta}_{0_-}(-1) = 1]\mathbf{\Psi}^{\rho,2}}{\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{\eta}_{0_-}(2) = 0] + \mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{\eta}_{0_-}(-1) = 1]} = \frac{(1-\rho)^2\mathbf{\Psi}^{\rho,1} + \rho^2\mathbf{\Psi}^{\rho,2}}{(1-\rho)^2+\rho^2},$$

where $\mathbf{\Psi}^{\rho,1}$ (resp. $\mathbf{\Psi}^{\rho,2}$) is $\hat{\mathbf{\Psi}}^\rho$ conditional on that a jump of the hole-particle pair to the left (resp. to the right) happens at time 0. More precisely, we define these measures as follows. Let $(\overline{\eta}_t)_{t\in\mathbb{R}} \sim \mathbf{\Psi}^{\rho,1}$, then the process $(\overline{\eta}_t)_{t<0}$ has distribution

given by

$$\frac{\mathbb{1}[\overline{\eta}_{0_-}(2)=0]d\hat{\mathbf{\Psi}}^{\rho}}{\mathbb{P}_{\hat{\mathbf{\Psi}}^{\rho}}[\overline{\eta}_{0_-}(2)=0]};$$

and given $\overline{\eta}_{0_-}$ we let $\overline{\eta}_0$ be that $\overline{\eta}_0(-1)=\overline{\eta}_0(0)=0$, $\overline{\eta}_0(1)=1$, and $\overline{\eta}_0(x)=\overline{\eta}_{0_-}(x+1)$ for any $x\notin\{-1,0,1\}$; and $(\overline{\eta}_t)_{t\geq 0}$ is the Markov process of TASEP as seen from a hole-particle pair. Similarly, for $(\overline{\eta}_t)_{t\in\mathbb{R}}\sim\mathbf{\Psi}^{\rho,2}$, then the process $(\overline{\eta}_t)_{t<0}$ has distribution given by

$$\frac{\mathbb{1}[\overline{\eta}_{0_-}(-1)=1]d\hat{\mathbf{\Psi}}^{\rho}}{\mathbb{P}_{\hat{\mathbf{\Psi}}^{\rho}}[\overline{\eta}_{0_-}(-1)=1]};$$

and given $\overline{\eta}_{0_-}$, we have $\overline{\eta}_0(0)=0$, $\overline{\eta}_0(1)=\overline{\eta}_0(2)=1$, and $\overline{\eta}_0(x)=\overline{\eta}_{0_-}(x-1)$ for any $x\notin\{0,1,2\}$; and $(\overline{\eta}_t)_{t\geq 0}$ is the Markov process of TASEP as seen from a hole-particle pair.

From this construction, the laws of $\overline{\eta}_0$ under $\mathbf{\Psi}^{\rho,1}$ and $\mathbf{\Psi}^{\rho,2}$ can also be described as follows. Let Ψ^{ρ}_{+} be the law of $\{\overline{\eta}_0(x+1)\}_{x\in\mathbb{N}}$ and Ψ^{ρ}_{-} be the law of $\{\overline{\eta}_0(-x)\}_{x\in\mathbb{N}}$, for $\overline{\eta}_0\sim\Psi^{\rho}$. Under $\mathbf{\Psi}^{\rho,1}$, there is $\overline{\eta}_0(-1)=\overline{\eta}_0(0)=0$, $\overline{\eta}_0(1)=1$, and $\{\overline{\eta}_0(x+1)\}_{x\in\mathbb{N}}\sim\Psi^{\rho}_{+}$ and $\{\overline{\eta}_0(-x-1)\}_{x\in\mathbb{N}}\sim\Psi^{\rho}_{-}$, and are independent. Under $\mathbf{\Psi}^{\rho,2}$, there is $\overline{\eta}_0(0)=0$, $\overline{\eta}_0(1)=\overline{\eta}_0(2)=1$, and $\{\overline{\eta}_0(x+2)\}_{x\in\mathbb{N}}\sim\Psi^{\rho}_{+}$, $\{\overline{\eta}_0(-x)\}_{x\in\mathbb{N}}\sim\Psi^{\rho}_{-}$ are independent.

We let ν^{ρ} be the measure given by the law of $(\{\overline{\omega}(a,b)\}_{(a,b)\in\mathbb{Z}^2},\overline{\gamma})$ under $\mathbf{\Psi}^{\rho}$. By Lemma 2.14 below we can see that $\overline{\omega}(\mathbf{0})$ has exponential tail under ν^{ρ}, so $\mathbb{E}_{\nu^{\rho}}[\overline{\omega}(\mathbf{0})]<\infty$. We then show that $\hat{\nu}^{\rho}$ is ν^{ρ} reweighted by $\overline{\omega}(\mathbf{0})$.

Lemma 2.13. *We have* $d\hat{\nu}^{\rho}=\frac{\overline{\omega}(\mathbf{0})d\nu^{\rho}}{\mathbb{E}_{\nu^{\rho}}[\overline{\omega}(\mathbf{0})]}$.

Let's explain why such a reweighting is expected. Consider the sequence of times $\{\overline{L}(u)\}_{u\in\overline{\gamma}}$ when the hole-particle pair jumps; this is a stationary point process in $\mathbb{R}$. Then ν^{ρ} corresponds to the environment as seen from the hole-particle at a typical jump-time. On the other hand, $\hat{\nu}^{\rho}$ corresponds to the environment as seen from the hole-particle at its position during the interval containing time 0. Because of the "inspection effect", this is not the position at a typical jump – rather it is biased by the length of the interval in the point process containing time 0, which is $\overline{\omega}(\mathbf{0})$.

Proof of Lemma 2.13. For each $s < 0$, we let $\mathbf{\Psi}^\rho_s$ be the measure of $\hat{\mathbf{\Psi}}^\rho$ conditional on $\overline{L}(\mathbf{0}) = s$, i.e., let $d\mathbf{\Psi}^\rho_s = \lim_{\epsilon\to 0_+} \frac{\mathbb{1}[s-\epsilon<\overline{L}(\mathbf{0})<s]d\hat{\mathbf{\Psi}}^\rho}{\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[s-\epsilon<\overline{L}(\mathbf{0})<s]}$. Since $\hat{\mathbf{\Psi}}^\rho$ is stationary, we have $\mathbb{1}[s-\epsilon<\overline{L}(\mathbf{0})<s]d\hat{\mathbf{\Psi}}^\rho = (\mathbb{1}[\overline{L}(\mathbf{0})>-\epsilon]\mathbb{1}[\overline{\omega}(\mathbf{0})>-s]d\hat{\mathbf{\Psi}}^\rho)\circ\mathscr{T}_s$, where $\mathscr{T}_s$ is the translation operator: for any process $P=(P_w)_{w\in\mathbb{R}}$, we denote $\mathscr{T}_sP$ as the process $(P_{s+w})_{w\in\mathbb{R}}$. This implies that

$$\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{L}(\mathbf{0})=s]d\mathbf{\Psi}^\rho_s = \mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{L}(\mathbf{0})=0](\mathbb{1}[\overline{\omega}(\mathbf{0})>-s]d\mathbf{\Psi}^\rho)\circ\mathscr{T}_s,$$

where $\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{L}(\mathbf{0})=s] = \lim_{\epsilon\to 0}\epsilon^{-1}\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[s-\epsilon<\overline{L}(\mathbf{0})<s]$ and $\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{L}(\mathbf{0})=0] = \lim_{\epsilon\to 0}\epsilon^{-1}\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{L}(\mathbf{0})>-\epsilon]$ are the probability densities. By integrating over $s<0$, the left hand side is $d\hat{\mathbf{\Psi}}^\rho$, under which the law of $(\{\overline{\omega}(a,b)\}_{(a,b)\in\mathbb{Z}^2},\overline{\gamma})$ is $\hat{\nu}^\rho$. For the right hand side, we note that the laws of $(\{\overline{\omega}(a,b)\}_{(a,b)\in\mathbb{Z}^2},\overline{\gamma})$ are the same under $(\mathbb{1}[\overline{\omega}(\mathbf{0})>-s]d\mathbf{\Psi}^\rho)\circ\mathscr{T}_s$ or $\mathbb{1}[\overline{\omega}(\mathbf{0})>-s]d\mathbf{\Psi}^\rho$. So by integrating over $s<0$ and considering the law of $(\{\overline{\omega}(a,b)\}_{(a,b)\in\mathbb{Z}^2},\overline{\gamma})$, we get $\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{L}(\mathbf{0})=0]\overline{\omega}(\mathbf{0})d\nu^\rho$. Thus we conclude that $d\hat{\nu}^\rho = \mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{L}(\mathbf{0})=0]\overline{\omega}(\mathbf{0})d\nu^\rho$. Since $\hat{\nu}^\rho$ and ν^ρ are probability measures, by integrating both sides we get that $\mathbb{P}_{\hat{\mathbf{\Psi}}^\rho}[\overline{L}(\mathbf{0})=0]\mathbb{E}_{\nu^\rho}[\overline{\omega}(\mathbf{0})]=1$, and the conclusion follows. □

We now use the above construction of ν^ρ to compute local statistics of the geodesics (assuming the convergence results). We start with the following computations on the next jump-times.

Lemma 2.14. *For any $h\geq 0$ we have*

$$\begin{aligned}
\mathbb{P}_{\mathbf{\Psi}^{\rho,1}}[\overline{L}(1,0)>h] &= (1+(1-\rho)\rho h)e^{-(1-\rho)h},\\
\mathbb{P}_{\mathbf{\Psi}^{\rho,1}}[\overline{L}(0,1)>h] &= (1+\rho h)e^{-\rho h},\\
\mathbb{P}_{\mathbf{\Psi}^{\rho,2}}[\overline{L}(1,0)>h] &= (1+(1-\rho)h)e^{-(1-\rho)h},\\
\mathbb{P}_{\mathbf{\Psi}^{\rho,2}}[\overline{L}(0,1)>h] &= (1+(1-\rho)\rho h)e^{-\rho h}.
\end{aligned}$$

Proof. Let $D_+ = \min\{x \geq 1 : \overline{\eta}_0(x+1) = 0\}$, the number of particles between the origin and the leftmost hole at a positive site, Similarly let $D_- = \min\{x \geq 1 : \overline{\eta}_0(-x) = 1\}$, the number of holes to the right of the rightmost particle at a negative site, up to and including the origin.

The distribution of D_+ under $\mathbf{\Psi}^{\rho,1}$ is that of X_+ given by (1.7), while the distribution of D_+ under $\mathbf{\Psi}^{\rho,2}$ is that of $X_+ + 1$ (which is the distribution of the sum of two independent Geometric$(1-\rho)$ random variables).

Similarly the distribution of D_- under $\mathbf{\Psi}^{\rho,2}$ is that of X_- at (1.8), while the distribution of D_- under $\mathbf{\Psi}^{\rho,1}$ is that of $X_- + 1$.

In order for the particle which is at site 1 at time 0 to jump, the hole starting at site $D_+ + 1$ must exchange places with each of the D_+ particles starting in $[\![1, D_+ - 1]\!]$. So given D_+, the distribution of $\overline{L}(1,0)$ is the sum of D_+ independent Exp(1) random variables; that is, a Gamma$(D_+, 1)$ distribution. A random variable V with Gamma(k) distribution has $\mathbb{E}[e^{-sV}] = (1+s)^{-k}$, and from this we obtain, for any $s > -1 + \rho$,

$$\mathbb{E}_{\mathbf{\Psi}^{\rho,1}}[e^{-s(\overline{L}(1,0))}] = \sum_{k=1}^{\infty} k(1-\rho)^2\rho^{k-1}(1+s)^{-k} = \frac{(1+s)(1-\rho)^2}{(1+s-\rho)^2},$$

which can be shown to match the expression for $\mathbb{P}_{\mathbf{\Psi}^{\rho,1}}[\overline{L}(1,0) > h]$ given in the statement.

Similarly, in order for the hole which is at site 0 at time 0 to jump, the particle starting at site $-D_-$ must exchange places with each of the D_- holes starting in $[\![-D_- + 1, 0]\!]$. One obtains

$$\mathbb{E}_{\mathbf{\Psi}^{\rho,1}}[e^{-s(\overline{L}(0,1))}] = \sum_{k=1}^{\infty} k\rho^2(1-\rho)^{k-1}(1+s)^{-(k+1)} = \frac{\rho^2}{(\rho+s)^2},$$

which matches the desired expression for $\mathbb{P}_{\mathbf{\Psi}^{\rho,1}}[\overline{L}(0,1) > h]$.

Analogous calculations give the probabilities under $\mathbf{\Psi}^{\rho,2}$. □

Now we compute the law of $\omega(\mathbf{0})$ for $\omega \sim \nu^\rho$.

Proof of Proposition 1.5. It suffices to compute the law of $\overline{L}(1,0) \wedge \overline{L}(0,1)$, under the measure $\mathbf{\Psi}^\rho = \frac{(1-\rho)^2 \mathbf{\Psi}^{\rho,1} + \rho^2 \mathbf{\Psi}^{\rho,2}}{(1-\rho)^2+\rho^2}$. Note that under either $\mathbf{\Psi}^{\rho,1}$ or $\mathbf{\Psi}^{\rho,2}$, the random variables $\overline{L}(1,0)$ and $\overline{L}(0,1)$ are independent. Thus by Lemma 2.14 we get that

$$\mathbb{P}_{\mathbf{\Psi}^{\rho,1}}[\overline{L}(1,0) \wedge \overline{L}(0,1) > h] = (1+\rho h)(1+\rho(1-\rho)h)e^{-h},$$

and

$$\mathbb{P}_{\mathbf{\Psi}^{\rho,2}}[\overline{L}(1,0) \wedge \overline{L}(0,1) > h] = (1+(1-\rho)h)(1+\rho(1-\rho)h)e^{-h}.$$

Thus the conclusion follows. □

Assuming Theorem 1.1, we can also compute the proportion of "corners" in geodesics.

Proof of Proposition 1.6. Assuming Theorem 1.1, we have

$$\frac{N_{n,\rho}}{2n} \to \mathbb{P}_{\nu^\rho}[\{(0,0),(0,1),(-1,0)\} \subset \overline{\gamma}] + \mathbb{P}_{\nu^\rho}[\{(0,0),(0,-1),(1,0)\} \subset \overline{\gamma}],$$

as $n \to \infty$. From the construction of ν^ρ, this equals

$$\frac{(1-\rho)^2 \mathbb{P}_{\mathbf{\Psi}^{\rho,1}}[\overline{L}(1,0) > \overline{L}(0,1)] + \rho^2 \mathbb{P}_{\mathbf{\Psi}^{\rho,2}}[\overline{L}(1,0) < \overline{L}(0,1)]}{(1-\rho)^2+\rho^2}.$$

Using that $\overline{L}(1,0) - \overline{L}(0,0)$ and $\overline{L}(0,1) - \overline{L}(0,0)$ are independent under either $\mathbf{\Psi}^{\rho,1}$ or $\mathbf{\Psi}^{\rho,2}$, by Lemma 2.14 we have

$$\mathbb{P}_{\mathbf{\Psi}^{\rho,1}}[\overline{L}(1,0) > \overline{L}(0,1)] = \rho^2(1+2\rho-2\rho^2),$$
$$\mathbb{P}_{\mathbf{\Psi}^{\rho,2}}[\overline{L}(1,0) < \overline{L}(0,1)] = (1-\rho)^2(1+2\rho-2\rho^2).$$

Thus the conclusion follows. □

We finish this section by giving an outline of alternative derivation of the formulas

in Proposition 1.5 and Proposition 1.6, which also leads to representations of the type mentioned after the statement of Proposition 1.5.

Note that under $\mathbf{\Psi}^{\rho,2}$, D_+ takes values in $\{2, 3, \dots\}$ and has the distribution of the sum of two independent Geometric random variables with parameter $1-\rho$. Given D_+, the random variable $\overline{L}(1,0)$ is the sum of D_+ independent Exp(1) random variables. From this, $\overline{L}(1,0)$ has the same distribution as the sum of two Exp$(1-\rho)$ random variables, or equivalently of $\frac{1}{1-\rho}(E_1 + E_2)$ for E_1, E_2 i.i.d. $\sim$ Exp(1).

Meanwhile under $\mathbf{\Psi}^{\rho,2}$, D_- takes values in $\{1, 2, \dots\}$ and has the distribution of 1 less than the sum of two independent Geometric(ρ) random variables. Note that if $X \sim \text{Geom}(\rho)$, then $X - 1 \stackrel{d}{=} BX$ where $B \sim \text{Ber}(\rho)$ independently of X. We obtain that $\overline{L}(0,1)$ has the distribution of $\frac{1}{\rho}(E_3 + BE_4)$, for $B \sim \text{Ber}(\rho)$ and E_3, E_4 i.i.d. $\sim$ Exp(1) independently of B.

Note $\overline{L}(0,1)$ and $\overline{L}(1,0)$ are independent under $\mathbf{\Psi}^{\rho,2}$. So we can combine the previous two paragraphs to get that the distribution of $\overline{\omega}(\mathbf{0}) = \min(\overline{L}(0,1), \overline{L}(1,0))$ under $\mathbf{\Psi}^{\rho,2}$ is that of

$$\min\left\{\frac{1}{1-\rho}(E_1 + E_2), \frac{1}{\rho}(E_3 + BE_4)\right\}$$

for $B \sim \text{Ber}(\rho)$ and $(E_i)_{1 \le 1 \le 4}$ i.i.d. $\sim$ Exp(1) independently of B.

We continue in the particular case $\rho = 1/2$. Then the distribution of $\overline{\omega}(\mathbf{0})$ is the same under $\mathbf{\Psi}^{\rho,1}$ as under $\mathbf{\Psi}^{\rho,2}$, and so its distribution under $\mathbf{\Psi}^{\rho}$ is again the same, that of $2\min\{(E_1 + E_2), (E_3 + BE_4)\}$ for $B \sim \text{Ber}(1/2)$ and $(E_i)_{1 \le 1 \le 4}$ i.i.d. $\sim$ Exp(1) independently of B.

By elementary arguments involving the memoryless property of exponentials, this distribution can be seen to be a $(1/4, 1/2, 1/4)$ mixture of Gamma$(1,1)$, Gamma$(2,1)$ and Gamma$(3,1)$ distributions.

A similar but slightly more involved argument can be made for the case of gen-

eral ρ, to give that the distribution of $\overline{\omega}(\mathbf{0})$ is again a mixture of Gamma$(1,1)$, Gamma$(2,1)$ and Gamma$(3,1)$ distributions, now with weights

$$\left(\frac{\rho^4+(1-\rho)^4}{\rho^2+(1-\rho)^2}, 2\rho(1-\rho), \frac{2\rho^2(1-\rho)^2}{\rho^2+(1-\rho)^2}\right).$$

As a function of $\rho \in [0,1]$, this distribution is stochastically increasing on $[0,1/2]$, and symmetric around $1/2$.

2.4 Convergence of TASEP as seen from a single second class particle

As in the previous section, we take $(\eta_t)_{t\geq 0}$ as TASEP with initial condition $\eta_0(0) = 0, \eta_0(1) = 1$, and $\eta_0(x)$ being i.i.d. Bernoulli ρ for any $x \in \mathbb{Z} \setminus \{0,1\}$. Let $(\eta'_t)_{t\geq 0}$ be η_t as seen from the hole-particle pair, and let Φ^ρ_t be the measure of η'_t. Also let Ψ^ρ be the stationary measure of TASEP as seen from a single hole-particle pair. The goal of this section is to show the convergence of Φ^ρ_t to Ψ^ρ as $t \to \infty$, i.e., to prove Theorem 1.7.

2.4.1 TASEP convergence in the average sense

In this subsection we prove the convergence of Φ^ρ_t to Ψ^ρ in the averaged sense.

Proposition 2.15. *We have $T^{-1}\int_0^T \Phi^\rho_t dt \to \Psi^\rho$ in distribution, as $T \to \infty$.*

Recall the stationary measure for TASEP as seem from a second class particle σ, constructed in Section 1.4.5. Also recall that we have the following two projections of σ: first, if we set all $*$ symbols on positive sites to be 0, and all $*$ symbols on negative sites to be 1, we get i.i.d. Bernoulli(ρ) on all non-zero sites; second, if we set all $*$ symbols on positive sites to be 1, and all $*$ symbols on negative sites to be 0, we get a

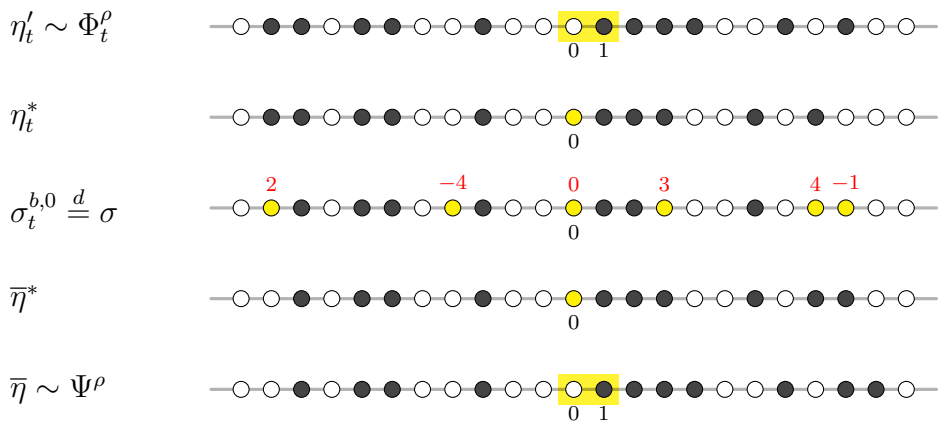


Figure 2.1: A coupling between Ψ^ρ and Φ^ρ_t via $\sigma^{b,0}_t$. The red numbers are labels of second class particles. Here $\overline{\eta}$ and η'_t are the same on $[-9, 9]$.

distribution which is stationary for the TASEP as seen from an isolated second class particle.

Now recall TASEP $(\sigma_t)_{t\geq 0}$ with (infinitely many) second class particles, and starting from $\sigma_0 = \sigma$. At time 0, we label all the second class particles with $\mathbb{Z}$ from right to left, such that the one at the origin is labeled 0. We consider two ways where the labels evolve.

- Rule (a): for all second class particles, the labels never change.
- Rule (b): two second class particles labeled $i > j$, if they are at sites x and $x + 1$, then with rate 1 they exchange their labels.

We note that when forgetting the labels, the dynamic is unchanged. For each $i \in \mathbb{Z}$ and $t \geq 0$, we denote $l^{a,i}_t$ as the site of the second class particle labeled by i at time t, under Rule (a). Then for each $i \in \mathbb{Z}$ we have $l^{a,i}_t > l^{a,i+1}_t$, and there is no other second class particle between sites $l^{a,i}_t$ and $l^{a,i+1}_t$. We also denote $l^{b,i}_t$ as the site of the second class particle labeled by i at time t, under Rule (b).

Define $\sigma^{a,i}_t, \sigma^{b,i}_t : \mathbb{Z} \to \{0, 1, *\}$ as $\sigma^{a,i}_t(x) = \sigma_t(x + l^{a,i}_t)$ and $\sigma^{b,i}_t(x) = \sigma_t(x + l^{b,i}_t)$, which is σ_t as seen from the second class particle labeled by i, under each rule. As σ is

a renewal process, and that σ is stationary, we have that $\sigma_t^{a,i}$ has the same distribution as σ. We next show that the same is true for $\sigma_t^{b,i}$.

Lemma 2.16. *For each $i \in \mathbb{Z}$ and $t \geq 0$, $\sigma_t^{b,i}$ has the same distribution as σ.*

Proof. Take any measurable set $B \subset \{0, 1, *\}^{\mathbb{Z}}$, it suffices to show that $\mathbb{P}[\sigma_t^{b,i} \in B] = \mathbb{P}[\sigma \in B]$.

We fix $t \geq 0$. Since for each $i \in \mathbb{Z}$, $\sigma_t^{a,i}$ has the same distribution as σ, we have

$$\lim_{N\to\infty} \frac{1}{2N+1}\mathbb{E}[|\{-N \leq i \leq N : \sigma_t^{a,i} \in B\}|] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} \mathbb{P}[\sigma_t^{a,i} \in B] = \mathbb{P}[\sigma \in B].$$

As each second class particle jumps with rate at most 1, for any $\epsilon > 0$ we can find $M > 0$, such that $\mathbb{P}[|l_t^{a,i} - l_t^{b,i}| > M] < \epsilon$ for any $i \in \mathbb{Z}$. For each i with $|i| < N - M$, if $|l_t^{b,i} - l_t^{a,i}| \leq M$, we must have $l_t^{b,i} \in \{l_t^{a,j} : -N \leq j \leq N\}$, since this set contains all locations of second class particles in $[\![l_t^{a,i} - M, l_t^{a,i} + M]\!]$. We then have that

$$\mathbb{E}[|\{l_t^{b,i} : -N \leq i \leq N\} \setminus \{l_t^{a,i} : -N \leq i \leq N\}|] < 2M + \epsilon(2N + 1).$$

Thus since ϵ is arbitrarily taken, we have

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} \mathbb{P}[\sigma_t^{b,i} \in B] = \lim_{N\to\infty} \frac{1}{2N+1}\mathbb{E}[|\{-N \leq i \leq N : \sigma_t^{b,i} \in B\}|] = \mathbb{P}[\sigma \in B].$$

Now that σ is a renewal process, $\sigma_0^{b,i}$, thus $\sigma_t^{b,i}$, has the same distribution for all i. From the above equation the conclusion follows. □

We define $\eta_t^{*,i} : \mathbb{Z} \to \{0, 1, *\}$ from $\sigma_t^{b,i}$, by identifying all second class particles whose labels are $< i$ with holes, and all second class particles whose labels are $> i$ with particles. Formally, we let $\eta_t^{*,i}(0) = *$, and $\eta_t^{*,i}(x) = 1$ for any x such that $\sigma_t(x+l_t^{b,i}) = 1$, or $x = l_t^{b,j} - l_t^{b,i}$ for some $j > i$; and $\eta_t^{*,i}(x) = 0$ such that $\sigma_t(x+l_t^{b,i}) = 0$, or $x = l_t^{b,j} - l_t^{b,i}$ for some $j < i$.

From such construction, $(\eta_t^{*,0})_{t\geq 0}$ is TASEP starting from i.i.d. Bernoulli(ρ) on all non-zero sites, as seen from the only second class particle. Thus we immediately get the following.

Lemma 2.17. *Take $\eta'_t : \mathbb{Z} \to \{0,1\}$ such that $\eta'_t(0) = 0$, $\eta'_t(1) = 1$, and $\eta'_t(x) = \eta_t^{*,0}(x)$ for $x < 0$, $\eta'_t(x) = \eta_t^{*,0}(x-1)$ for $x > 1$. Then the law of η'_t is given by Φ_t^ρ.*

Now we finish the proof of Proposition 2.15, via a coupling between η'_t and $\overline{\eta} \sim \Psi^\rho$.

Proof of Proposition 2.15. It suffices to show that for any finitely supported set $B \subset \{0,1\}^{\mathbb{Z}}$, we have $T^{-1}\int_0^T \Phi_t^\rho(B)dt \to \Psi^\rho(B)$. Below we fix B and assume that it is supported on $[\![-L, L+1]\!]$.

Recall the construction of Ψ^ρ from σ. By Lemma 2.16, from $\sigma_t^{b,0}$, we get $\overline{\eta}^*$ (the stationary for the TASEP as seen from an isolated second class particle, from Section 1.4.5) by identifying all $*$ with 1 in $\mathbb{Z}_+$ and all $*$ with 0 in $\mathbb{Z}_-$. Next, by replacing the $*$ at the origin by a $0-1$ pair we get $\overline{\eta}$, with distribution Ψ^ρ. Thus by Lemma 2.17, and comparing the procedures of getting Ψ^ρ and Φ_t^ρ from $\sigma_t^{b,0}$, we have

$$|\Phi_t^\rho(B) - \Psi^\rho(B)| \leq \mathbb{P}[\{l_t^{b,i} - l_t^{b,0} : i > 0\} \cap [\![-L, 0]\!] = \{l_t^{b,i} - l_t^{b,0} : i < 0\} \cap [\![0, L]\!] = \emptyset].$$

Then we have

$$\int_0^T |\Phi_t^\rho(B) - \Psi^\rho(B)|dt \leq \sum_{i\in\mathbb{N}} \int_0^T \mathbb{P}[l_t^{b,i} - l_t^{b,0} \in [\![-L, 0]\!]]dt + \sum_{i\in\mathbb{Z}_-} \int_0^T \mathbb{P}[l_t^{b,i} - l_t^{b,0} \in [\![0, L]\!]]dt.$$

For each $i \in \mathbb{N}$ we recursively define a sequence of stopping times: let $T_{i,1} = \inf\{t \geq 0, l_t^{b,i} - l_t^{b,0} \in [\![-L, 0]\!]\} \cup \{\infty\}$; and given $T_{i,n} < \infty$, let $T_{i,n+1} = \inf\{t \geq T_{i,n} + 1, l_t^{b,i} - l_t^{b,0} \in [\![-L, 0]\!]\} \cup \{\infty\}$. Note that there exists $\delta > 0$ depending only on L, such that $\mathbb{P}[l^{b,i}_{T_{i,n}+1} > l^{b,0}_{T_{i,n}+1} \mid T_{i,n} < \infty] > \delta$; and if $l_{t_0}^{b,i} > l_{t_0}^{b,0}$ for some $t_0 \geq 0$, then $l_t^{b,i} > l_t^{b,0}$ for

any $t > t_0$. Then we have $\mathbb{P}[T_{i,n+1} \geq T \mid T_{i,n} < T] > \delta$, and

$$\int_0^T \mathbb{P}[l_t^{b,i} - l_t^{b,0} \in [\![-L, 0]\!]]dt \leq \sum_{n=1}^{\infty} \mathbb{P}[T_{i,n} < T] \leq \sum_{n=1}^{\infty} (1-\delta)^{n-1} \mathbb{P}[T_{i,1} < T]$$
$$= \delta^{-1} \mathbb{P}[T_{i,1} < T].$$

Next we bound $\sum_{i \in \mathbb{N}} \mathbb{P}[T_{i,1} < T]$. Take any $\epsilon > 0$. From the renewal construction of σ, we have that $l_0^{b,0} - l_0^{b,i}$ is the sum of i i.i.d. random variables, each with infinite expectation. Thus we have

$$\lim_{T \to \infty} \mathbb{P}[l_0^{b,0} - l_0^{b,\lceil \epsilon T \rceil} < 3T] = 0.$$

Since each label moves with rate at most 1, given $\{l_0^{b,0}\}_{i \in \mathbb{Z}}$ satisfying $l_0^{b,0} - l_0^{b,\lceil \epsilon T \rceil} \geq 3T$, for each $j \in \mathbb{Z}_{\geq 0}$ the probability that $T_{\lceil \epsilon T \rceil + j, 1} < T$ is bounded by the probability that the sum of $\lceil 3T \rceil + j - L$ independent $\mathrm{Exp}(1/2)$ random variables is less than T. Summing up such probabilities for all j we get

$$\lim_{T \to \infty} \sum_{i > \epsilon T} \mathbb{P}[T_{i,1} < \infty] = 0.$$

From these we get

$$\limsup_{T \to \infty} \sum_{i \in \mathbb{N}} \int_0^T \mathbb{P}[l_t^{b,i} - l_t^{b,0} \in [\![-L, 0]\!]]dt - \delta^{-1} \epsilon T \leq 0.$$

Similarly we have

$$\limsup_{T \to \infty} \sum_{i \in \mathbb{Z}_-} \int_0^T \mathbb{P}[l_t^{b,i} - l_t^{b,0} \in [\![0, L]\!]]dt - \delta^{-1} \epsilon T \leq 0.$$

Adding them up we get

$$\limsup_{T\to\infty} T^{-1}\int_0^T |\Phi_t^\rho(B) - \Psi^\rho(B)|dt \le \delta^{-1}\epsilon.$$

Since $\epsilon > 0$ is arbitrarily taken, the conclusion follows. □

2.4.2 Convergence to the stationary distribution

In the next two subsections we upgrade Proposition 2.15 to Theorem 1.7. The general idea is to show that Φ_t^ρ and Φ_{t+s}^ρ are close when s is much smaller than t.

Proposition 2.18. *For any $N \in \mathbb{N}$ and $\rho \in (0,1)$, there is a constant $C > 0$ such that the following is true. Take any $s, t > C$ with $t < s^{1.01}$, and any function $f : \{0,1\}^{[\![-N,N]\!]} \to [0,1]$, regarded as a function on $\{0,1\}^{\mathbb{Z}}$, we have $|\Phi_t^\rho(f) - \Phi_{t+s}^\rho(f)| < C(s/t)^{1/30}$.*

Using this we can deduce Theorem 1.7.

Proof of Theorem 1.7. Take any $N \in \mathbb{N}$ and $f : \{0,1\}^{[\![-N,N]\!]} \to [0,1]$, regarded as a function on $\{0,1\}^{\mathbb{Z}}$, it suffices to show that

$$\lim_{t\to\infty} \Phi_t^\rho(f) = \Psi^\rho(f). \tag{2.4}$$

Take any $\delta > 0$. By Proposition 2.15 we have that $(\delta t)^{-1}\int_0^{\delta t} \Phi_{t+s}^\rho(f)ds \to \Psi^\rho(f)$ as $t \to \infty$. By Proposition 2.18 we have for any $t > C$,

$$\left|\Phi_t^\rho(f) - (\delta t)^{-1}\int_0^{\delta t} \Phi_{t+s}^\rho(f)ds\right| < C\delta^{1/30} + (C \vee t^{1/1.01})(\delta t)^{-1},$$

where C depends only on ρ and N. Thus $\limsup_{t\to\infty} |\Phi_t^\rho(f) - \Psi^\rho(f)| \le C\delta^{1/30}$, and by sending $\delta \to 0$ we get (2.4). □

The idea of proving Proposition 2.18 is to couple two copies of $(\eta'_t)_{t\geq 0}$. For this we recall the set up in Section 1.4.4.

Let $(\eta_t^-)_{t\geq 0}$ and $(\eta_t^+)_{t\geq 0}$ be two copies of TASEP, with $\eta_0^-(0) = \eta_0^+(0) = 0$ and $\eta_0^-(1) = \eta_0^+(1) = 1$; and all $\eta_0^-(x), \eta_0^+(x)$ for $x \in \mathbb{Z}\setminus\{0,1\}$ are i.i.d. Bernoulli ρ. In both copies, we label the holes by $\mathbb{Z}$ from left to right, with the hole at site 0 at time 0 labeled 0; and label the particles by $\mathbb{Z}$ from right to left, with the particle at site 1 at time 0 labeled 0. Keeping track of the hole-particle pair, we denote $p_t^- = (a_t^-, b_t^-)$ and $p_t^+ = (a_t^+, b_t^+)$ as the label of the tracked particle and hole at time t (in $(\eta_t^-)_{t\geq 0}$ and $(\eta_t^+)_{t\geq 0}$ respectively), and set $\eta_t^{-\prime}(x) = \eta_t^-(x + a_t^- - b_t^-)$, $\eta_t^{+\prime}(x) = \eta_t^+(x + a_t^+ - b_t^+)$ for any $t \geq 0, x \in \mathbb{Z}$. Then $\eta_t^{-\prime}, \eta_t^{+\prime}$ has distribution Φ_t^ρ.

For any $a, b \in \mathbb{Z}$, if in $\eta_0^{-\prime}$ (resp. $\eta_0^{+\prime}$) the particle with label b is to the left of the hole with label a, we denote $L^-(a,b)$ (resp. $L^+(a,b)$) as the time when they switch; otherwise we set $L^-(a,b) = 0$ (resp. $L^+(a,b) = 0$). For each $t \geq 0$ denote

$$
\begin{aligned}
I_t^+ &:= \{u \in \mathbb{Z}^2 : L^+(u) \leq t\},\\
I_t^- &:= \{u \in \mathbb{Z}^2 : L^-(u) \leq t\},\\
\partial I_t^+ &:= \{u \in I_t^+ : L^+(u + (1,0)) \vee L^+(u + (0,1)) > t\},\\
\partial I_t^- &:= \{u \in I_t^- : L^-(u + (1,0)) \vee L^-(u + (0,1)) > t\}.
\end{aligned}
$$

For the rest of this subsection we fix $s > 0$. Take any $r \in \mathbb{N}$. For any coupling between $\eta_0^{-\prime}$ and $\eta_s^{+\prime}$, we denote $\mathcal{A}$ as the event where

$$
\eta_0^{-\prime}(x) = \eta_s^{+\prime}(x), \forall x \in \mathbb{Z}, |x| > r; \quad \sum_{x=-r}^{r} \eta_0^{-\prime}(x) = \sum_{x=-r}^{r} \eta_s^{+\prime}(x).
$$

Under $\mathcal{A}$, we can find a (unique) $p^* = (a^*, b^*) \in \mathbb{Z}^2$, such that $I_0^- \cap \{u \in \mathbb{Z}^2 : |ad(u)| > r\} = (I_s^+ - p^*) \cap \{u \in \mathbb{Z}^2 : |ad(u)| > r\}$; in particular, the sets I_0^- and $I_s^+ - p^*$ differ by a finite number of vertices.

Lemma 2.19. *There is a coupling of $\eta_0^{-\prime}$ and $\eta_s^{+\prime}$ such that $\mathbb{P}[\mathcal{A}] > 1 - C(rs^{-2/3})^{-1/10}$*

when $Cs^{2/3} < r < s^{2/3+0.01}$ *and* $s > C$*, where* $C > 0$ *is a constant depending only on* ρ*.*

We leave the construction of this coupling to the next subsection, and we now couple $(\eta_t^{-\prime})_{t\geq 0}$ and $(\eta_t^{+\prime})_{t\geq 0}$ assuming such coupling between $\eta_0^{-\prime}$ and $\eta_s^{+\prime}$.

The construction of the coupling for these two copies of TASEP is via the connection with LPP. For any $u \notin I_0^-$ we denote $\omega^{-,\vee}(u) = L^-(u) - L^-(u-(1,0)) \vee L^-(u-(0,1))$; and for any $u \notin I_0^+$ we denote $\omega^{+,\vee}(u) = L^+(u) - L^+(u-(1,0)) \vee L^+(u-(0,1))$. In addition, for $u \notin I_s^+$ we denote $\omega^{+,\vee,s}(u) = L^+(u) - L^+(u-(1,0)) \vee L^+(u-(0,1)) \vee s$. Note that for any $u \notin I_s^+$, we have $\omega^{+,\vee}(u) = \omega^{+,\vee,s}(u)$ unless both $u-(1,0), u-(0,1) \in I_s^+$. Given $\eta_0^{-\prime}$ and $\eta_s^{+\prime}$ under the coupling from Lemma 2.19, if $\mathcal{A}$ does not hold, we just couple $(\eta_t^{-\prime})_{t\geq 0}$ and $(\eta_t^{+\prime})_{t\geq 0}$ arbitrarily. If $\mathcal{A}$ holds, then for any $u \in \mathbb{Z}^2$ such that $u \notin I_0^-$ and $u + p^* \notin I_s^+$, we let $\omega^{-,\vee}(u) = \omega^{+,\vee,s}(u+p^*)$. Such coupling exists because conditioned on I_0^-, the random variables $\omega^{-,\vee}(u)$ are i.i.d. Exp(1) for all $u \notin I_0^-$; and conditioned on I_s^+, the random variables $\omega^{+,\vee,s}(u)$ are i.i.d. Exp(1) for all $u \notin I_s^+$. We shall bound the total variation distance between Φ_t^ρ and Φ_{t+s}^ρ using this coupling.

As discussed in Section 1.4.4, L^- (resp. L^+) has the same law as a Busemann function in LPP restricted to $\mathbb{Z}^2 \setminus I_0^-$ (resp. $\mathbb{Z}^2 \setminus I_0^+$). From this we can construct the semi-infinite geodesics. For any $u \in (\mathbb{Z}^2 \setminus I_0^-) \cup \partial I_0^-$, we define the semi-infinite geodesic Γ_u^- recursively, by letting $\Gamma_u^-[1] = u$, and for each $i \in \mathbb{N}$ letting $\Gamma_u^-[i+1] = \operatorname{argmin}_{v \in \{\Gamma_u^-[i]+(1,0), \Gamma_u^-[i]+(0,1)\}} L^-(v)$; and similarly we define Γ_u^+ for any $u \in (\mathbb{Z}^2 \setminus I_0^+) \cup \partial I_0^+$.

We also define the "downward semi-infinite geodesics". For any $u \in \mathbb{Z}^2 \setminus I_0^-$, we define $\Gamma_u^{-,\vee}$ by letting $\Gamma_u^{-,\vee}[1] = u$, and for each $i \in \mathbb{N}$ letting $\Gamma_u^{-,\vee}[i+1] = \operatorname{argmax}_{v \in \{\Gamma_u^{-,\vee}[i]-(1,0), \Gamma_u^{-,\vee}[i]-(0,1)\}} L^-(v)$, until $\Gamma_u^{-,\vee}[i] - (1,0)), \Gamma_u^{-,\vee}[i] - (0,1) \notin I_0^-$. Equivalently and by induction, we could also define such downward semi-infinite geodesics as following. For any $u, v \notin I_0^-$ with $u \leq v$, we let $T_{u,v}^{-,\vee}$ and $\Gamma_{u,v}^{-,\vee}$ as the

maximum passage time and geodesic from u to v, using the weights $\omega^{-,\vee}$. Then for $v \notin I_0^-$ we have $L^-(v) = \max_{u \notin I_0^-, u \le v} T_{u,v}^{-,\vee}$, and $\Gamma_v^{-,\vee}$ equals $\Gamma_{u,v}^{-,\vee}$ for u where the maximum is achieved. Similarly we define $\Gamma_u^{+,\vee}$ for any $u \notin I_0^+$, and $T_{u,v}^{+,\vee}$ and $\Gamma_{u,v}^{+,\vee}$ for any $u, v \notin I_0^+$ with $u \le v$, using the weights $\omega^{+,\vee}$. In addition, we also define $T_{u,v}^{+,\vee,s}$ and $\Gamma_{u,v}^{+,\vee,s}$ for $u, v \notin I_s^+$ with $u \le v$, using the weights $\omega^{+,\vee,s}$. For any $v \notin I_s^+$ we have $L^+(v) - s = \max_{u \notin I_s^+, u \le v} T_{u,v}^{+,\vee,s}$, and $\Gamma_v^{+,\vee} \setminus I_s^+$ is $\Gamma_{u,v}^{+,\vee,s}$ for u where the maximum is achieved.

A quick observation is that almost surely there is a "non-crossing" property for semi-infinite geodesics and downward semi-infinite geodesics: for any Γ_u^- and $\Gamma_v^{-,\vee}$, we cannot find $w \in \mathbb{Z}^2$ with $w, w-(1,0) \in \Gamma_u^-$ and $w, w+(0,1) \in \Gamma_v^{-,\vee}$ simultaneously. This is because, from the construction of Γ_u^- and $\Gamma_v^{-,\vee}$, the event $w, w-(1,0) \in \Gamma_u^-$ implies that $L^-(w) \le L^-(w+(-1,1))$, while $w, w+(0,1) \in \Gamma_v^{-,\vee}$ implies that $L^-(w) \ge L^-(w+(-1,1))$. Similarly we cannot find $w \in \mathbb{Z}^2$ such that $w, w-(0,1) \in \Gamma_u^-$ and $w, w+(1,0) \in \Gamma_v^{-,\vee}$ simultaneously. The same is true for the semi-infinite geodesics and downward semi-infinite geodesics defined using L^+.

Take $m \in \mathbb{N}$, $m > r$, we now define events depending on the parameters m and r. Let $\mathcal{B}_-$ be the event where

$$\exists u_{-,1}, u_{-,2} \in \partial I_0^-, \quad ad(u_{-,1}) < -r, ad(u_{-,2}) > r, \Gamma_{u_{-,1}}^- \cap \mathbb{L}_m = \Gamma_{u_{-,2}}^- \cap \mathbb{L}_m;$$

and let $\mathcal{B}_+ \subset \mathcal{A}$ be the event where

$$\exists u_{+,1}, u_{+,2} \in \partial I_s^+, \quad ad(u_{+,1}) < ad(p^*) - r, ad(u_{+,2}) > ad(p^*) + r, \Gamma_{u_{+,1}}^- \cap \mathbb{L}_m = \Gamma_{u_{+,2}}^- \cap \mathbb{L}_m.$$

See Figure 2.2 for the events.

Lemma 2.20. *Under $\mathcal{B}_- \cap \mathcal{B}_+$, almost surely we have that $L^-(u) = L^+(u + p^*)$ for any $u \in \mathbb{Z}^2$ with $d(u) > 2m$, and $p_t^- = p_{t+s}^+ - p^*$ for any $t > 0$ with $d(p_t^-) > 2m$.*

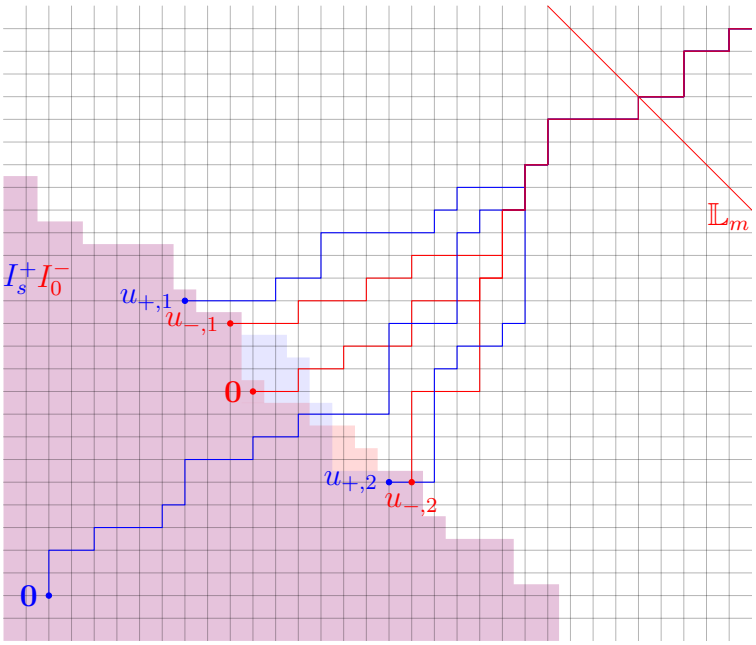


Figure 2.2: A illustration of the proof of Lemma 2.20. The red objects are those constructed from $(\eta_t^{-\prime})_{t\geq 0}$, and the blue objects are those constructed from $(\eta_t^{+\prime})_{t\geq 0}$. The difference between the red $\mathbf{0}$ and blue $\mathbf{0}$ is p^*.

Proof. Since $r < m$, under $\mathcal{A}$ we have $\{u \in \mathbb{Z}^2 : d(u) > 2m\} \cap I_0^- = \{u \in \mathbb{Z}^2 : d(u) > 2m\} \cap (I_s^+ - p^*)$. Define $U := \mathbb{Z}^2 \setminus (I_0^- \cup (I_s^+ - p^*))$. By the above "non-crossing" property, under $\mathcal{A} \cap \mathcal{B}_-$ we have $\Gamma_u^{-,\vee} \subset U$ for any $u \in U$ with $d(u) > 2m$; and under $\mathcal{A} \cap \mathcal{B}_+$ we have $\Gamma_u^{+,\vee} - p^* \subset U$ for any $u \in U + p^*$ with $d(u) > 2m$.

Now take any $u \in U$. We claim that if $\Gamma_u^{-,\vee} \subset U$ and $\Gamma_{u+p^*}^{+,\vee} \setminus I_s^+ - p^* \subset U$, we must have that $\Gamma_u^{-,\vee} = \Gamma_{u+p^*}^{+,\vee} \setminus I_s^+ - p^*$ and $L^-(u) = L^+(u+p^*) - s$. Indeed, from the coupling we have we have $\omega^{-,\vee}(v) = \omega^{+,\vee,s}(v + p^*)$ for any $v \in U$, so both $\Gamma_u^{-,\vee}$ and $\Gamma_{u+p^*}^{+,\vee} \setminus I_s^+ - p^*$ are the path in U ending at u with the maximum total weight. Then almost surely these paths must be the same with the same weights. From the claim the first conclusion follows.

For any $t > 0$ we have that p_t^- is the last vertex in $\Gamma_{\mathbf{0}}^- \cap I_t^-$ (Lemma 1.18). Similarly we have that p_{t+s}^+ is the last vertex in $\Gamma_{\mathbf{0}}^+ \cap I_{t+s}^+$. It then suffices to show that

$$\Gamma_{\mathbf{0}}^- \cap \{u \in \mathbb{Z}^2 : d(u) > 2m\} + p^* = \Gamma_{\mathbf{0}}^+ \cap \{u \in \mathbb{Z}^2 : d(u) > 2m + d(p^*)\}. \tag{2.5}$$

Indeed, by the non-crossing property, $\Gamma^-_{\mathbf{0}} \cap \{u \in \mathbb{Z}^2 : d(u) > 2m\}$ is determined by $\Gamma^{-,\vee}_u$ for all u with $d(u) > 2m$ and $L^-(u) > 0$; more precisely, $\Gamma^-_{\mathbf{0}} \cap \{u \in \mathbb{Z}^2 : d(u) > 2m\} + (1/2, 1/2)$ is the boundary of $u \in \mathbb{Z}^2$ with $d(u) > 2m + 1$, with lower end of $\Gamma^{-,\vee}_u$ to the upper-left or lower-right of $\mathbf{0}$. For the same reason, under $\mathcal{B}_+$, $\Gamma^+_{\mathbf{0}} \cap \{u \in \mathbb{Z}^2 : d(u) > 2m + d(p^*) + (1/2, 1/2)$ is the the boundary of $u \in \mathbb{Z}^2$ with $d(u) > 2m + d(p^*) + 1$, with lower end of $\Gamma^{+,\vee}_u \setminus I^+_s$ to the upper-left or lower-right of p^*. Thus by the above claim we get (2.5). □

For the proof of Proposition 2.18, we set up some additional notation (which will be used in the next subsection as well). For each $x \in \mathbb{R}$ we denote

$$\mathbb{V}_x := \{((1-\rho)^2 y + x, \rho^2 y - x) : y \in \mathbb{R}\},$$

and

$$\mathbb{U}_x := \{((1-\rho) y + (1-\rho)^2 x, -\rho y + \rho^2 x) : y \in \mathbb{R}\}.$$

For any set $A \subset \mathbb{R}$ we denote $\mathbb{V}_A := \cup_{x \in A} \mathbb{V}_x$ and $\mathbb{U}_A := \cup_{x \in A} \mathbb{U}_x$. Besides, since L^+ (resp. L^-) has the same law as a Busemann function in LPP, we can also define Γ^+_u (resp. Γ^-_u) for $u \notin I^+_0 \cup \partial I^+_0$ (resp. $u \notin I^-_0 \cup \partial I^-_0$) as the semi-infinite geodesics in the coupled LPP.

Proof of Proposition 2.18. In this proof we let $C, c > 0$ be large and small constants which depend only on ρ and N, and the values can change from line to line.

Take $m = \lfloor t/10 \rfloor$ and $r = \lfloor s^{1/3} t^{1/3} \rfloor$. We show that $\eta^{-\prime}_t$ equals $\eta^{+\ \prime}_{t+s}$ in $[\![-N, N]\!]$ with probability $> 1 - C(s/t)^{1/30}$, when $t, s > C$. We could assume that t/s is large enough (depending on ρ), since other wise we would have $1 - C(s/t)^{1/30} < 0$.

We first lower bound $\mathbb{P}[\mathcal{B}_-]$ and $\mathbb{P}[\mathcal{B}_+]$. Without loss of generality we assume that $\rho \le 1/2$. Denote $v_1 = (-\lfloor 4r\rho^{-2} \rfloor, 0)$ and $v_2 = (0, -\lfloor 4r\rho^{-2} \rfloor)$, and we take $u_{-,1}$ to be the last vertex in $\Gamma^-_{v_1} \cap I^-_0$, and $u_{-,2}$ to be the last vertex in $\Gamma^-_{v_2} \cap I^-_0$. Then

$u_{-,1} \in \{(a,b) \in \mathbb{Z}^2 : a < 0, b \geq 0\}$, and $u_{-,2} \in \{(a,b) \in \mathbb{Z}^2 : a \geq 0, b < 0\}$. Then by Corollary 2.9, we have $\mathbb{P}[ad(u_{-,1}) < -r], \mathbb{P}[ad(u_{-,2}) > r] > 1 - Ce^{-cr^{1/3}}$. By Proposition 2.4 we have $\mathbb{P}[\Gamma^-_{v_1} \cap \mathbb{L}_m = \Gamma^-_{v_2} \cap \mathbb{L}_m] > 1 - Crm^{-2/3}$. Thus we conclude that $\mathbb{P}[\mathcal{B}_-] > 1 - Crm^{-2/3}$.

For $\mathbb{P}[\mathcal{B}_+]$, we take $u_{+,1}$ to be the last vertex in $\Gamma^-_{v_1} \cap I^+_s$, and $u_{+,2}$ to be the last vertex in $\Gamma^-_{v_2} \cap I^+_s$. Since $|ad(p^*) - ad(p^+_s)| \leq r$ under $\mathcal{A}$, the events $ad(u_{+,1}) < ad(p^*) - r$ and $ad(u_{+,2}) > ad(p^*) + r$ are implied by $ad(u_{+,1}) < ad(p^+_s) - 2r$ and $ad(u_{+,2}) > ad(p^+_s) + 2r$, respectively.

Let $A = \mathbb{V}_{(-r,r)} \cap \mathbb{U}_{(s-r^2/s, s+r^2/s)}$. By Corollary 2.12 we have $\mathbb{P}[p^+_s \in A] > 1 - Ce^{-crs^{-2/3}}$. When $p^+_s \in A$, we must have $ad(u_{+,1}) < ad(p^+_s) - 2r$ and $ad(u_{+,2}) > ad(p^+_s) + 2r$, unless $u_{+,1} \in A'$ or $u_{+,2} \in A'$, where

$$A' = A + \{u \in \mathbb{R}^2 : |d(u)|, |ad(u)| \leq 2r\}.$$

By Corollary 2.9, we have $\mathbb{P}[u_{+,1} \in A'], \mathbb{P}[u_{+,2} \in A'] > 1 - Ce^{-crs^{-2/3}}$. Thus we conclude that $\mathbb{P}[ad(u_{+,1}) < ad(p^*) - r], \mathbb{P}[ad(u_{+,2}) > ad(p^*) + r] > 1 - Ce^{-crs^{-2/3}}$. By Proposition 2.4 we have $\mathbb{P}[\Gamma^+_{v_1} \cap \mathbb{L}_m = \Gamma^+_{v_2} \cap \mathbb{L}_m] > 1 - Crm^{-2/3}$. Thus we conclude that $\mathbb{P}[\mathcal{A} \setminus \mathcal{B}_+] < Crm^{-2/3}$.

By Corollary 2.12 we have $\mathbb{P}[d(p^-_t) > 2m + N] > 1 - Ce^{-ct}$. By Lemma 2.19, 2.20, and the above estimates, we have that $\eta^{-\prime}_t$ equals ${\eta^+_{t+s}}'$ in $[\![-N, N]\!]$ with probability $> 1 - Crm^{-2/3} - C(rs^{-2/3})^{-1/10}$. Thus the conclusion follows. □

2.4.3 The initial step coupling

This subsection is devoted to proving Lemma 2.19.

We define $(\tau_t)_{t \in \mathbb{R}}$ as the process of a stationary TASEP with density ρ. Then for any $t \in \mathbb{R}$, we have $\tau_t(x)$ being Bernoulli ρ for each $x \in \mathbb{Z}$ independently. Our strategy is to construct a coupling between $(\eta^{+\prime}_t)_{t \geq 0}$ and $(\tau_t)_{t \geq 0}$, where (with high

probability) $\eta_s^{+\prime}$ and τ_s are identical outside $[\![-r,r]\!]$, and have the same number of particles in $[\![-r,r]\!]$. It would be straight forward to couple $\eta_0^{-\prime}$ and τ_s since both are Bernoulli ρ on $\mathbb{Z}\setminus\{0,1\}$.

We denote $\alpha = (rs^{-2/3})^{1/5}$, and $r_i = \alpha^i s^{2/3}$, for $i = 1,2,3,4$. Below we assume that α and s are large enough depending on ρ, and also $\alpha < r^{0.01}$.

Recall (from the previous subsection) the definitions of L^+, $\omega^{+,\vee}$, I_t^+ for $t \geq 0$, the geodesics Γ_u^+ for $u \in I_0^+ \cup \partial I_0^+$ and $\Gamma_u^{+,\vee}$ for $u \notin I_0^+$; and also $T_{u,v}^{+,\vee}$ and $\Gamma_{u,v}^{+,\vee}$ for any $u, v \notin I_0^+$ with $u \leq v$, using the weights $\omega^{+,\vee}$.

We will use the following extensions of these constructions. First, as in the previous subsection, we define Γ_u^+ for $u \notin I_0^+ \cup \partial I_0^+$, using that L^+ has the same distribution as a Busemann function. Second, we extend $\omega^{+,\vee}$ to I_0^+, so that $\omega^{+,\vee}(u)$ is i.i.d. Exp(1) for each $u \in \mathbb{Z}^2$, and is independent of I_0^+. Using the weights $\omega^{+,\vee}$ on $\mathbb{Z}^2$ we can define $T_{u,v}^{+,\vee}$ and $\Gamma_{u,v}^{+,\vee}$ for any $u \leq v$. Note that these two extensions are not related and we would not use the joint law of them; so the joint law is taken arbitrarily.

A straight forward way of coupling $(\eta_t^{+\prime})_{t\geq 0}$ and $(\tau_t)_{t\geq 0}$ would be first to couple $\eta_0^{+\prime}$ with τ_0, and then to let them evolve using the same set of exponential waiting times. However, this would not give the desired construction, since $(\eta_t^{+\prime})_{t\geq 0}$ is centered at the hole-particle pair. Thus we need to first shift $\eta_0^{+\prime}$ by $ad(p_s^+)$ and then couple with τ_0. However, the vertex p_s^+ depends on the evolution, although mostly rely on the evolution around the hole-particle pair. Thus we take the following approach: we construct the coupling by first sample the evolution of $(\eta_t^{+\prime})_{t\geq 0}$ around the pair, then get a good approximation of p_s^+. Using that we could shift τ_0, and couple the rest waiting times together.

Recall that the vertex p_s^+ is also the last vertex in $\Gamma_{\mathbf{0}}^+ \cap I_s^+$. Denote $P = \mathbb{V}_{(-r_1,r_1)}$. Using $I_0^+ \cap P$ and $\{\omega^\vee(u)\}_{u\in P\setminus I_0^+}$ we defined a "restricted version" of p_s^+ as following. First we define L^P, by letting $L^P(u) = 0$ for $u \in I_0^+ \cup (\mathbb{Z}^2 \setminus P)$, and setting $L^P(u) = L^P(u-(1,0)) \vee L^P(u-(0,1)) + \omega^{+,\vee}(u)$ recursively for each $u \in P \setminus I_0^+$. We then

define $\Gamma_{\mathbf{0}}^P$, by letting $\Gamma_{\mathbf{0}}^P[1] = \mathbf{0}$, and

$$\Gamma_{\mathbf{0}}^P[i+1] = \mathrm{argmin}_{v \in \{\Gamma_{\mathbf{0}}^P[i]+(1,0), \Gamma_{\mathbf{0}}^P[i]+(0,1)\} \cap P} L^P(v)$$

for each $i \in \mathbb{N}$. We let p^P be the last vertex in $\{u \in \Gamma_{\mathbf{0}}^P : L^P(u) \le s\}$. Denote $M = ad(p^P)$. Then M is determined by $I_0^+ \cap P$ and $\{\omega^{+,\vee}(u)\}_{u \in P \setminus I_0^+}$

Lemma 2.21. *$\mathbb{P}[p^P = p_s^+] > 1 - Ce^{-c\alpha}$ for some constants $c, C > 0$ depending only on ρ.*

Proof. Consider the event $\mathcal{D}_1$ where

$$\exists u_1, u_2 \in \partial I_0^+, \quad \Gamma_{u_1}^+ \cap I_s^+ \subset \mathbb{V}_{(-r_1,-r_1/2)},\ \Gamma_{u_2}^+ \cap I_s^+ \subset \mathbb{V}_{(r_1/2,r_1)}.$$

Assume that $\mathcal{D}_1$ holds, let S be the set consisting of vertices in $I_s^+ \setminus I_0^+$ between $\Gamma_{u_1}^+$ and $\Gamma_{u_2}^+$. We note that for each $u \in S$, there is

$$\max_{v \in \{u-(1,0), u-(0,1)\}} L^+(v) = \max_{v \in \{u-(1,0), u-(0,1)\} \cap S} L^+(v).$$

Thus by induction we have $L^+(u) = L^P(u)$ for each $u \in S$. Let $\mathcal{D}_2$ be the event where $\Gamma_{\mathbf{0}}^+ \cap I_s^+ \subset \mathbb{V}_{(-r_1/2,r_1/2)}$. Then under $\mathcal{D}_1 \cap \mathcal{D}_2$ we have $(\Gamma_{\mathbf{0}}^+ \cap I_s^+) \setminus I_0^+ \subset S$, so by induction we have $\Gamma_{\mathbf{0}}^+ \cap I_s^+ = \{u \in \Gamma_{\mathbf{0}}^P : L^P(u) \le s\}$, and in particular $p^P = p_s^+$.

Denote

$$u_1' = \left(\left\lfloor -\frac{3((1-\rho)^2+\rho^2)r_1}{4\rho^2} \right\rfloor, 0 \right), \quad u_2' = \left(0, \left\lfloor -\frac{3((1-\rho)^2+\rho^2)r_1}{4(1-\rho)^2} \right\rfloor \right),$$

and take u_1 as any vertex in $\Gamma_{u_1'}^+ \cap \partial I_0^+$ and u_2 as any vertex in $\Gamma_{u_2'}^+ \cap \partial I_0^+$. We let $\mathcal{D}_3$ be the event where $d(p_s^+) < 2s$, let $\mathcal{D}_4$ be the event where $\Gamma_{u_1'}^+ \cap \{u : d(u) < 4s\} \subset \mathbb{V}_{(-r_1,-r_1/2)}$ and $\Gamma_{u_2'}^+ \cap \{u : d(u) < 4s\} \subset \mathbb{V}_{(r_1/2,r_1)}$. Since u_1, u_2 cannot be greater than p_s^+ in each coordinate, under $\mathcal{D}_2 \cap \mathcal{D}_3 \cap \mathcal{D}_4$ we have $d(u_1), d(u_2) < 4s$ and $\mathcal{D}_1$ holds.

Thus we conclude that $\mathbb{P}[p^P = p_s^+] \geq \mathbb{P}[\mathcal{D}_1 \cap \mathcal{D}_2] \geq \mathbb{P}[\mathcal{D}_2 \cap \mathcal{D}_3 \cap \mathcal{D}_4]$. By Corollary 2.12 we have $\mathbb{P}[\mathcal{D}_2 \cap \mathcal{D}_3] > 1 - Ce^{-cr_1 s^{-2/3}}$; by Corollary 2.9 we have $\mathbb{P}[\mathcal{D}_4] > 1 - Ce^{-cr_1 s^{-2/3}}$. Thus the conclusion follows. □

Now we couple $(\tau_t)_{t\geq 0}$ with $(\eta_t^{+\prime})_{t\geq 0}$ using the following steps.

1. First take $\eta_0^{+\prime}(x)$ being i.i.d. Bernoulli ρ, for each $x \in [\![-r_2, r_2]\!] \setminus \{0, 1\}$. Then by our choice of r_1, r_2, the set $I_0^+ \cap P$ is determined. We next sample $\omega^{+,\vee}(u)$ for $u \in P$, and then the number $M = ad(p^P)$ is determined.

2. Let $\tau_0(x - M)$ be i.i.d. Bernoulli ρ for each $x \in [\![-r_2, r_2]\!]$.

3. For each $x = \lfloor -r_2 \rfloor, \lfloor -r_2 \rfloor - 1, \ldots$, we take $\eta_0^{+\prime}(x)$ and $\tau_0(x-M)$ being Bernoulli ρ independently, until for some $x_* \in \mathbb{Z}$ there is $\sum_{x=x_*}^{0} \eta_0^{+\prime}(x) - \tau_0(x - M) = 0$. Then for each $x < x_*$ we take $\eta_0^{+\prime}(x) = \tau_0(x - M)$ being Bernoulli ρ independently.

4. For each $x = \lceil r_2 \rceil, \lceil r_2 \rceil + 1, \ldots$, we take $\eta_0^{+\prime}(x)$ and $\tau_0(x - M)$ being Bernoulli ρ independently, until for some $x^* \in \mathbb{Z}$ there is $\sum_{x=1}^{x^*} \eta_0^{+\prime}(x) - \tau_0(x-M) = 0$. Then for each $x > x^*$ we take $\eta_0^{+\prime}(x) = \tau_0(x - M)$ being Bernoulli ρ independently.

5. Take $\omega^{+,\vee}(u)$ to be i.i.d. Exp(1) for $u \notin P$.

Up to now we have constructed a joint law of τ_0 and $(\eta_t^{+\prime})_{t\geq 0}$, each with the desired marginal distribution. Also τ_0, M, and $\{\omega^{+,\vee}(u)\}_{u\in\mathbb{Z}^2\setminus P}$ are mutually independent.

In τ_0, we label the holes by $\mathbb{Z}$ from left to right, and the particles by $\mathbb{Z}$ from right to left, such that for $|x|$ large enough, the particle (or hole) at site $x - M$ has the same label as the particle (or hole) at site x in $\eta_0^{+\prime}$. Let $L^\tau(a, b)$ be the time when particle with label b switches with label a, if in τ_0 this particle is to the left of this label, and let $L^\tau(a, b) = 0$ otherwise. For each $t \geq 0$ denote $I_t^\tau := \{u \in \mathbb{Z}^2 : L^\tau(u) \leq t\}$, and $\partial I_t^\tau := \{u \in I_t^\tau : L^\tau(u + (1, 0)) \vee L^\tau(u + (0, 1)) > t\}$. From the above construction, we have I_0^τ and I_0^+ differ by at most finitely many vertices. For any $u \in (\mathbb{Z}^2 \setminus I_0^\tau) \cup \partial I_0^\tau$,

we also define the "semi-infinite geodesic" Γ^τ_u recursively, by letting $\Gamma^\tau_u[1] = u$, and for each $i \in \mathbb{N}$ letting $\Gamma^\tau_u[i+1] = \text{argmin}_{v\in\{\Gamma^\tau_u[i]+(1,0),\Gamma^\tau_u[i]+(0,1)\}} L^\tau(v)$. Note that unlike L^+, there is no coupling between L^τ with a Busemann function in LPP, thus these Γ^τ_u are not actual geodesics.

6. We couple the evolution of τ with $(\eta^{+\prime}_t)_{t\geq 0}$, by letting $L^\tau(u) - L^\tau(u-(1,0)) \vee L^\tau(u-(0,1)) = \omega^{+,\vee}(u)$ for any $u \in \mathbb{Z}^2 \setminus P$ with $L^\tau(u) > 0$; and $L^\tau(u) - L^\tau(u-(1,0)) \vee L^\tau(u-(0,1))$ for $u \in P$ be i.i.d. Exp(1), independent of $\omega^{+,\vee}$.

Now we have a coupling between $(\eta^{+\prime}_t)_{t\geq 0}$ and $(\tau_t)_{t\geq 0}$.

We denote $\mathcal{E}_1$ as the event where for any $x < -r$, $\tau_s(x) = \eta^{+\prime}_s(x)$, and the particle or hole at site x has the same label for τ_s and $\eta^{+\prime}_s$; denote $\mathcal{E}_2$ as the event where the same is true for any $x > r$. We shall lower bound $\mathbb{P}[\mathcal{E}_1]$ and $\mathbb{P}[\mathcal{E}_2]$.

For this we consider the following events. We denote $\mathcal{E}_3$ as the event where there exists a vertex $u_+ \in \partial I^+_0$, such that $ad(u_+) < x_*$ and $\Gamma^+_{u_+} \cap I^+_s \subset \mathbb{V}_{(-r_4,-r_1)}$, and $a'_+ > (1-\rho)^2 s - r_4$ for $u'_+ = (a'_+, b'_+)$ being the last vertex in $\Gamma^+_{u_+} \cap I^+_s$. We also denote $\mathcal{E}_4$ to be the event such that for each $u = (a,b) \in \partial I^+_s$ with $ad(u) < M - r + 1$, we have $a < (1-\rho)^2 s - r_4$, and $u \in \mathbb{V}_{(-\infty,-r_4)}$. Analogly we denote $\mathcal{E}^\tau_3$ as the event where there exists a vertex $u_\tau \in \partial I^\tau_0$ with $ad(u_\tau) < x_*$ and $\Gamma^\tau_{u_\tau} \cap I^\tau_s \subset \mathbb{V}_{(-r_4,-r_1)}$, and $a'_\tau > (1-\rho)^2 s - r_4$ for $u'_\tau = (a'_\tau, b'_\tau)$ being the last vertex in $\Gamma^+_{u_\tau} \cap I^\tau_s$. We denote $\mathcal{E}^\tau_4$ as the event where for each $u = (a,b) \in \partial I^\tau_s$ with $ad(u) < M - r + 1$, we have $a < (1-\rho)^2 s - r_4$, and $u \in \mathbb{V}_{(-\infty,-r_4)}$. See Figure 2.3 for an illustration of these events.

Lemma 2.22. $\{p^P = p^+_s\} \cap \mathcal{E}_3 \cap \mathcal{E}_4 \cap \mathcal{E}^\tau_3 \cap \mathcal{E}^\tau_4 \subset \mathcal{E}_1$.

Proof. Under $\mathcal{E}_3$, let S_+ be the set consisting of vertices $u = (a,b) \in \mathbb{Z}^2 \setminus I^+_0$ and above $\Gamma^+_{u_+}$, with $a < (1-\rho)^2 s - r_4$. We then have

$$\max_{v\in\{u-(1,0),u-(0,1)\}} L^+(v) = \max_{v\in\{u-(1,0),u-(0,1)\}\cap S_+} L^+(v), \quad \forall u \in S_+,$$

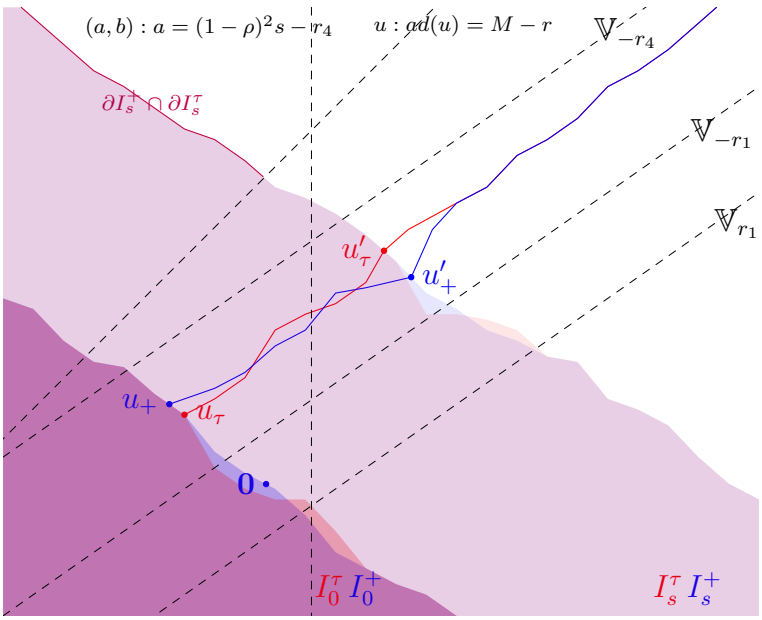


Figure 2.3: The events $\mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_3^\tau, \mathcal{E}_4^\tau$, assuming $p^P = p_s^+$.

and thus by induction we have $L^+(u) \le L^\tau(u)$ for each $u \in S_+$.

Under $\mathcal{E}_3^\tau$, let S_τ be the set consisting of vertices $u = (a,b) \in \mathbb{Z}^2 \setminus I_0^\tau$ and above Γ_{u_τ} with $a < (1-\rho)^2 s - r_4$. Then similarly, by induction we have $L^+(u) \ge L^\tau(u)$ for each $u \in S_\tau$.

Suppose that $\mathcal{E}_3 \cap \mathcal{E}_4 \cap \mathcal{E}_3^\tau \cap \mathcal{E}_4^\tau$ holds and $p^P = p_s^+$. Take any $x \in \mathbb{Z}$ with $x < -r$, and assume that $\eta_s^{+\prime}(x) = 1$. Then there is $v \in \partial I_s^+$ such that $ad(v) = ad(p_s^+) + x - 1$ and $v + (1,0) \notin I_s^+$. By $p^P = p_s^+$ we have $M = ad(p_s^+)$, so $ad(v) = M + x - 1$. The second coordinate of v is the label of the particle at site x (in $\eta_s^{+\prime}$). From $\mathcal{E}_3 \cap \mathcal{E}_4$ we have $v, v + (1,0) \in S_+$, thus

$$L^\tau(v + (1,0)) \ge L^+(v + (1,0)) > s. \tag{2.6}$$

If $\tau_s(x) = 0$, then there is $v^\tau \in \partial I_s^\tau$ such that $ad(v^\tau) = M + x$ and $v^\tau + (0,1) \notin I_s^\tau$. Then $ad(v + (1,0)) = ad(v^\tau)$. By (2.6) and $L^\tau(v^\tau) \le s$ we have $d(v) > d(v^\tau)$. From $\mathcal{E}_3^\tau \cap \mathcal{E}_4^\tau$ we have $v^\tau, v^\tau + (0,1) \in S_\tau$, then $L^+(v^\tau + (0,1)) \ge L^\tau(v^\tau + (0,1)) > s \ge L^+(v)$, implying that $d(v) < d(v^\tau)$ since $ad(v^\tau + (0,1)) = ad(v)$. This is a contradiction.

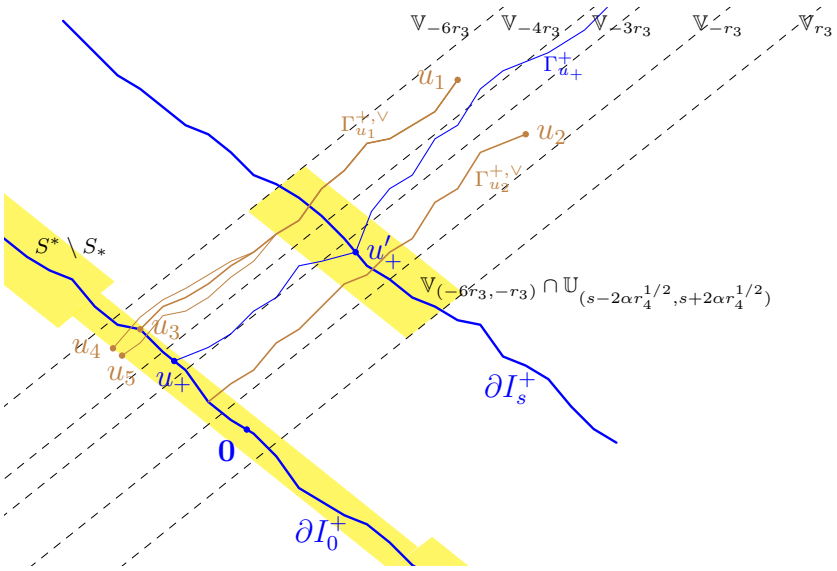


Figure 2.4: The events to lower bound $\mathbb{P}[\mathcal{E}_3]$.

Thus we must have $\tau_s(x) = 1$, and there is $v^\tau \in \partial I^\tau_s$ such that $ad(v^\tau) = M + x - 1$ and $v^\tau + (1,0) \notin I^\tau_s$. The second coordinate of v^τ is the label of the particle at site x (in τ_s). From $\mathcal{E}^\tau_3 \cap \mathcal{E}^\tau_4$ we have $v^\tau, v^\tau + (1,0) \in S_\tau$, so

$$L^+(v^\tau + (1,0)) \geq L^\tau(v^\tau + (1,0)) > s \geq L^+(v). \tag{2.7}$$

Since $ad(v) = ad(v^\tau)$, from (2.7) we have $d(v^\tau) \geq d(v)$; and from (2.6) and $L^\tau(v^\tau) \leq s$ we have $d(v^\tau) \leq d(v)$. Then we must have $d(v^\tau) = d(v)$ and $v^\tau = v$, so the label of the particle at site x is the same for $\eta^{+\prime}_s$ and τ_s.

By similar arguments, when $\eta^{+\prime}_s(x) = 0$ we must have $\tau_s(x) = 0$, and the label of the hole at site x is the same for $\eta^{+\prime}_s$ and τ_s. Thus we have that the event $\mathcal{E}_1$ holds. □

Lemma 2.23. *$\mathbb{P}[\mathcal{E}_3], \mathbb{P}[\mathcal{E}^\tau_3] > 1 - C\alpha^{-1/2}$, and $\mathbb{P}[\mathcal{E}_4], \mathbb{P}[\mathcal{E}^\tau_4] > 1 - Ce^{-c\alpha}$, for constants $c, C > 0$ depending only on ρ.*

Proof. We shall write the proof for the estimates of $\mathbb{P}[\mathcal{E}_3]$ and $\mathbb{P}[\mathcal{E}_4]$, and the approach we take here applies to the estimates for $\mathbb{P}[\mathcal{E}^\tau_3]$ and $\mathbb{P}[\mathcal{E}^\tau_4]$, essentially verbatim. We will use $c, C > 0$ to denote small and large enough constants depending only on ρ,

and their values can change from line to line.

We consider the following events (see Figure 2.4).

$\mathcal{E}_5$: $x_* > -r_3$.

$\mathcal{E}_6$: $\mathbb{V}_{(-jr_4,jr_4)} \cap \partial I_0^+ \subset \mathbb{U}_{(-j\alpha r_4^{1/2},j\alpha r_4^{1/2})}$ for each $j \in \mathbb{N}$.

$\mathcal{E}_7$: $\mathbb{V}_{(-6r_3,-r_3)} \cap \partial I_s^+ \subset \mathbb{U}_{(s-2\alpha r_4^{1/2},s+2\alpha r_4^{1/2})}$.

$\mathcal{E}_8$: Let $u_1 = 2s((1-\rho)^2,\rho^2)+(-5(1-\rho)r_3, 5\rho r_3)$ and $u_2 = 2s((1-\rho)^2,\rho^2)+(-2(1-\rho)r_3, 2\rho r_3)$ (round to the nearest lattice point). Then $\Gamma_{u_1}^{+,\vee} \subset \mathbb{V}_{(-6r_3,-4r_3)}$, and $\Gamma_{u_2}^{+,\vee} \subset \mathbb{V}_{(-3r_3,-r_3)}$.

We have $\mathcal{E}_5 \cap \mathcal{E}_6 \cap \mathcal{E}_7 \cap \mathcal{E}_8 \subset \mathcal{E}_3$. Indeed, we take any $u_+ \in \partial I_0^+ \cap \mathbb{V}_{(-4r_3,-3r_3)}$, and let $u'_+ = (a'_+, b'_+)$ be the last vertex in $\Gamma_{u_+}^+ \cap I_s^+$. By $\mathcal{E}_6$, and note that $r_3 > C\alpha r_4^{1/2}$ by our choice of the parameters, we have $ad(u_+) < -r_3$. So under $\mathcal{E}_5 \cap \mathcal{E}_6$ we have $ad(u_+) < x_*$. Since $\Gamma_{u_+}^+$ does not cross $\Gamma_{u_1}^{+,\vee}$ or $\Gamma_{u_2}^{+,\vee}$, by $\mathcal{E}_7 \cap \mathcal{E}_8$ we have that $\Gamma_{u_+}^+ \cap I_s^+ \subset \mathbb{V}_{(-6r_3,-r_3)}$ and $u'_+ \in \mathbb{U}_{(s-2\alpha r_4^{1/2},s+2\alpha r_4^{1/2})} \cap \mathbb{V}_{(-6r_3,-r_3)}$. Thus we get $a'_+ > (1-\rho)^2 s - r_4$. It remains to estimate the probabilities of these events and take a union bound.

The number $-x_*$ is just the time of a symmetric random walk hitting 0 after r_2. Thus $\mathbb{P}[\mathcal{E}_5] \geq 1 - Cr_2^{1/2} r_3^{-1/2} = 1 - C\alpha^{-1/2}$.

The event $\mathcal{E}_6$ is again an estimate on the hitting probability of a random walk. For each $j \in \mathbb{N}$ we have $\mathbb{P}[\mathbb{V}_{(-jr_4,jr_4)} \cap I_0^+ \subset \mathbb{U}_{(-j\alpha r_4^{1/2},j\alpha r_4^{1/2})}] > 1 - Ce^{-cj\alpha}$; so when $\alpha > C$ we have $\mathbb{P}[\mathcal{E}_6] \geq 1 - Ce^{-c\alpha}$.

We let $S_1 = \mathbb{V}_{(-r_4,r_4)} \cap \mathbb{U}_{(-\alpha r_4^{1/2},\infty)}$ and $S^1 = \mathbb{V}_{(-r_4,r_4)} \cap \mathbb{U}_{(\alpha r_4^{1/2},\infty)}$, and for each $j \geq 2$ we let

$$S_j = \mathbb{V}_{(-jr_4,-(j-1)r_4]\cup[(j-1)r_4,jr_4)} \cap \mathbb{U}_{(-j\alpha r_4^{1/2},\infty)},$$

$$S^j = \mathbb{V}_{(-jr_4,-(j-1)r_4]\cup[(j-1)r_4,jr_4)} \cap \mathbb{U}_{(j\alpha r_4^{1/2},\infty)}.$$

Let $S_* = \cup_{j\in\mathbb{N}} S_j$ and $S^* = \cup_{j\in\mathbb{N}} S^j$. For $\mathcal{E}_7$, we consider the following events:

$\mathcal{E}_7'$: $T_{u,v}^{+,\vee} < s$ for any $v \in \mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{U}_{(-\infty,s-2\alpha r_4^{1/2})}$ and $u \in S_*$, $u, v \in \mathbb{Z}^2$.

$\mathcal{E}_7''$: For any $v \in \mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{U}_{(s+2\alpha r_4^{1/2},\infty)} \cap \mathbb{Z}^2$, there exists $u \in \mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{U}_{(\alpha r_4^{1/2},\infty)} \cap \mathbb{Z}^2$ such that $T_{u,v}^{+,\vee} > s$.

Then under $\mathcal{E}_7' \cap \mathcal{E}_7'' \cap \mathcal{E}_6$, we have that $\mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{U}_{(-\infty,s-2\alpha r_4^{1/2})} \cap \mathbb{Z}^2 \subset I_s^+$, and $\mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{U}_{(s+2\alpha r_4^{1/2},\infty)} \cap I_s^+ = \emptyset$, thus $\mathcal{E}_7$ holds.

To lower bound $\mathbb{P}[\mathcal{E}_7']$, we just need to consider $T_{u,v}^{+,\vee}$, for $v \in \mathbb{Z}^2$ within distance 1 from $\mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{U}_{s-2\alpha r_4^{1/4}}$; and $u \in \mathbb{Z}^2$ within distance 1 from

$$\mathbb{V}_{(-jr_4,-(j-1)r_4]\cup[(j-1)r_4,jr_4)} \cap \mathbb{U}_{-j\alpha r_4^{1/2}},$$

for each $j \in \mathbb{N}$. By (2.3) we have $\mathbb{E}T_{u,v}^{+,\vee} < s - c\alpha r_4^{1/2} - c(j-1)^2 r_4^2 s^{-1}$. Then if the slope of the line connecting u, v is between $\frac{\rho^2}{10(1-\rho)^2}$ and $\frac{(1-\rho)^2}{10\rho^2}$, we apply Proposition 2.2 by covering these pairs with parallelogram of size $Cs \times Cs^{2/3}$; and for all other such pairs we apply Theorem 2.1. We conclude that $\mathbb{P}[\mathcal{E}_7'] > 1 - C(r_3 s^{-2/3})(r_4 s^{-2/3})e^{-c\alpha}$ when $\alpha > C$.

For $\mathbb{P}[\mathcal{E}_7'']$, we need to consider $T_{u,v}^{+,\vee}$, for $v \in \mathbb{Z}^2$ within distance 1 from $\mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{U}_{s+2\alpha r_4^{1/4}}$, and $u = v - (s + \alpha r_4^{1/4}/2)((1-\rho)^2, \rho^2)$ (round up to the nearest lattice point). By (2.3) we have $\mathbb{E}T_{u,v}^{+,\vee} > s + c\alpha r_4^{1/2}$. Again we apply Proposition 2.2 by covering these u, v with parallelogram of size $Cs \times Cs^{2/3}$, and we conclude that $\mathbb{P}[\mathcal{E}_7''] > 1 - Cr_3 s^{-2/3} e^{-c\alpha}$ when $\alpha > C$.

For $\mathcal{E}_8$, we denote u_3 as the lower end point of $\Gamma_{u_1}^{+,\vee}$. Consider the event $\mathcal{E}_8'$, where for any $u \notin S_* \cup \mathbb{V}_{(-5r_3-r_2,-5r_3+r_2)}$, there is $T_{u,u_1}^{+,\vee} < 2s - 4\alpha r_4^{1/2}$; and $T_{u_1',u_1}^{+,\vee} > 2s - 4\alpha r_4^{1/2}$, where $u_1' = u_1 - (2s - 2\alpha r_4^{1/2})((1-\rho)^2, \rho^2)$ (round to the nearest lattice point). Under $\mathcal{E}_8' \cap \mathcal{E}_6$ we must have $u_3 \in \mathbb{V}_{(-5r_3-r_2,-5r_3+r_2)}$. We next lower bound $\mathbb{P}[\mathcal{E}_8']$. By (2.3) we have $\mathbb{E}T_{u_1',u_1}^{+,\vee} > 2s - 3\alpha r_4^{1/2}$. For $T_{u,u_1}^{+,\vee}$ with $u \notin S_* \cup \mathbb{V}_{(-5r_3-r_2,-5r_3+r_2)}$, again we just need to consider $u \in \mathbb{Z}^2$ within distance 1 from $\mathbb{V}_{(-jr_4,-(j-1)r_4]\cup[(j-1)r_4,jr_4)} \cap \mathbb{U}_{-j\alpha r_4^{1/2}}$,

for each $j \in \mathbb{N}$. By (2.3), when $j \geq 2$ we have

$$\mathbb{E}T^{+,\vee}_{u,u_1} < 2s - c(j-1)^2 r_4^2 s^{-1} < 2s - 5\alpha r_4^{1/2} - c(j-1)^2 r_4^2 s^{-1}/2;$$

and when $j = 1$ we have

$$\mathbb{E}T^{+,\vee}_{u,u_1} < 2s - c r_2^2 s^{-1} < 2s - 5\alpha r_4^{1/2}$$

for $u \notin \mathbb{V}_{(-5r_3-r_2,-5r_3+r_2)}$. Here we used that $r_4^2 s^{-1}, r_2^2 s^{-1} > C\alpha r_4^{1/2}$, due to our choice of the parameters. Then (similar to lower bounding $\mathbb{P}[\mathcal{E}_7']$ and $\mathbb{P}[\mathcal{E}_7'']$) we can lower bound $\mathbb{P}[\mathcal{E}_8']$ by covering these u and u_1 with parallelograms of size $Cs \times Cs^{2/3}$, and applying Proposition 2.2 and Theorem 2.1. We conclude that $\mathbb{P}[\mathcal{E}_8'] > 1 - Cr_4 s^{-2/3} e^{-c\alpha}$ when $\alpha > C$.

Now we take u_4, u_5 as the intersection of $\mathbb{U}_{-\alpha r_4^{1/2}}$ with $\mathbb{V}_{-5r_3-2r_2}$ and $\mathbb{V}_{-5r_3+2r_2}$, respectively (round to the nearest lattice point). By Lemma 2.7 and Corollary 2.8, we have

$$\mathbb{P}[\Gamma^{+,\vee}_{u_4,u_1} \cap \mathbb{U}_{(-\alpha r_4^{1/2}, \alpha r_4^{1/2})} \subset \mathbb{V}_{(-5r_3-3r_2,-5r_3-r_2)}] > 1 - Ce^{-cr_2\alpha^{-2/3} r_4^{-1/3}},$$

$$\mathbb{P}[\Gamma^{+,\vee}_{u_5,u_1} \cap \mathbb{U}_{(-\alpha r_4^{1/2}, \alpha r_4^{1/2})} \subset \mathbb{V}_{(-5r_3+r_2,-5r_3+3r_2)}] > 1 - Ce^{-cr_2\alpha^{-2/3} r_4^{-1/3}},$$

and

$$\mathbb{P}[\Gamma^{+,\vee}_{u_4,u_1} \subset \mathbb{V}_{(-6r_3,-4r_3)}], \mathbb{P}[\Gamma^{+,\vee}_{u_5,u_1} \subset \mathbb{V}_{(-6r_3,-4r_3)}] > 1 - Ce^{-cr_3 s^{-2/3}}.$$

Note that when the above four events happen, and $u_3 \in \mathbb{V}_{(-5r_3-r_2,-5r_3+r_2)}$, we have that $\Gamma^{+,\vee}_{u_1} = \Gamma^{+,\vee}_{u_3,u_1}$ is between $\Gamma^{+,\vee}_{u_4,u_1}$ and $\Gamma^{+,\vee}_{u_5,u_1}$ by ordering of geodesics, and $\Gamma^{+,\vee}_{u_1} \subset \mathbb{V}_{(-6r_3,-4r_3)}$. We can argue similarly for $\Gamma^{+,\vee}_{u_2}$. Then with $\mathbb{P}[\mathcal{E}_8'] > 1 - Ce^{-c\alpha}$ we conclude that $\mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_8] < C(e^{-c\alpha} + e^{-cr_2\alpha^{-2/3} r_4^{-1/3}} + e^{-cr_3 s^{-2/3}}) < Ce^{-c\alpha}$.

Taking together the lower bounds for $\mathbb{P}[\mathcal{E}_5], \mathbb{P}[\mathcal{E}_6], \mathbb{P}[\mathcal{E}_7], \mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_8]$, we conclude that

$\mathbb{P}[\mathcal{E}_3] > 1 - C\alpha^{-1/2}$.

We next bound $\mathbb{P}[\mathcal{E}_4]$. For this we define two more events.

$\mathcal{E}_9 : |M - (1-2\rho)s| < r_2$.

$\mathcal{E}_{10}$: For any $u = (a, b)$ with $ad(u) < (1-2\rho)s + r_2 - r + 1$ and $a \geq (1-\rho)^2 s - r_4$, we have $u \notin I_s^+$.

Obviously we have $\mathcal{E}_9 \cap \mathcal{E}_{10} \subset \mathcal{E}_4$.

By Corollary 2.12 we have $\mathbb{P}[\mathcal{E}_9] > 1 - Ce^{-cr_2 s^{-2/3}}$. For $\mathcal{E}_{10}$, we take $u_* = (a_*, b_*)$ where $a_* = \lceil (1-\rho)^2 s - r_4 \rceil$ and $b_* = a_* - \lceil (1-2\rho)s + r_2 - r \rceil$, and $\mathcal{E}_{10}$ is equivalent to $L^+(u_*) > s$. Denote $u'_* = (\lfloor r_4 \rfloor, \lfloor r_4 \rfloor) \notin S^*$, so under $\mathcal{E}_6$ we have $u'_* \notin I_0^+$. Thus under $\mathcal{E}_6 \setminus \mathcal{E}_{10}$ we have $T^{+,\vee}_{u'_*, u_*} \leq s$; then $\mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_{10}] \leq \mathbb{P}[T^{+,\vee}_{u'_*, u_*} \leq s] < Ce^{-crs^{-2/3}}$, where the last inequality is by Theorem 2.1. Thus we conclude that $\mathbb{P}[\mathcal{E}_4] > 1 - Ce^{-c\alpha}$. $\square$

Proof of Lemma 2.19. Let $C > 0$ be a constant depending only on ρ. By Lemma 2.21, 2.22 and 2.23, we have $\mathbb{P}[\mathcal{E}_1] > 1 - C\alpha^{-1/2}$; and by symmetry we also have $\mathbb{P}[\mathcal{E}_2] > 1 - C\alpha^{-1/2}$. Under $\mathcal{E}_1 \cap \mathcal{E}_2$ we have $\tau_s(x) = \eta_s^{+\prime}(x)$ for any $|x| > r$, and $\sum_{x=-r}^{r} \tau_s(x) = \sum_{x=-r}^{r} \eta_s^{+\prime}(x)$.

We couple τ_s with $\eta_0^{-\prime}$ as following. For each $x = 1, 2, \ldots$, we take $\eta_0^{-\prime}(x)$ and $\tau_s(x)$ being Bernoulli ρ independently, until for some $\overline{x}^* \in \mathbb{Z}$ there is $\sum_{x=1}^{\overline{x}^*} \eta_0^{-\prime}(x) - \tau_s(x) = 0$. Then for each $x > \overline{x}^*$ we take $\eta_0^{-\prime}(x) = \tau_s(x)$ being Bernoulli ρ independently. For each $x = 0, -1, \ldots$, we take $\eta_0^{-\prime}(x)$ and $\tau_s(x)$ being Bernoulli ρ independently, until for some $\overline{x}_* \in \mathbb{Z}$ there is $\sum_{x=\overline{x}_*}^{0} \eta_0^{-\prime}(x) - \tau_s(x) = 0$. Then for each $x < \overline{x}_*$ we take $\eta_0^{-\prime}(x) = \tau_s(x)$ being Bernoulli ρ independently.

Under this coupling, when $\overline{x}_* > -r$ and $\overline{x}^* < r$ we have $\tau_s(x) = \eta_0^{-\prime}(x)$ for any $|x| > r$, and $\sum_{x=-r}^{r} \tau_s(x) = \sum_{x=-r}^{r} \eta_0^{-\prime}(x)$. Also, since partial sums of $\eta_0^{-\prime}(x) - \tau_s(x)$ are symmetric ranomd walks, we have $\mathbb{P}[\overline{x}_* \leq -r], \mathbb{P}[\overline{x}^* \geq r] < Cr^{-1/2}$. Thus we have

$$\mathbb{P}[\mathcal{A}^c] < \mathbb{P}[\mathcal{E}_1^c] + \mathbb{P}[\mathcal{E}_2^c] + \mathbb{P}[\overline{x}_* \leq -r] + \mathbb{P}[\overline{x}^* \geq r] < C\alpha^{-1/2} + Cr^{-1/2},$$

and the conclusion follows. □

2.5 Convergence along semi-infinite geodesics: ergodicity of the stationary distribution

In this section we prove convergence in law along semi-infinite geodesics, which is a weaker version of Theorem 1.2.

Theorem 2.24. *For each $\rho \in (0,1)$, we have in probability $\mu^{\rho}_{\mathbf{0};r} \to \nu^{\rho}$, as $r \to \infty$.*

As before, we take $(\eta_t)_{t\geq 0}$ as the TASEP with density ρ, conditioned on $\eta_0(0) = 0, \eta_0(1) = 1$, and let $(\eta'_t)_{t\geq 0}$ be η_t as seen from the hole-particle pair (corresponding to a second class particle). Let $\overline{\boldsymbol{\eta}} = (\overline{\eta}_t)_{t\in\mathbb{R}}$ be the stationary process of TASEP as seen from a hole-particle pair, and $\hat{\boldsymbol{\Psi}}^{\rho}$ be its measure. Also let Ψ^{ρ} be the measure of $\overline{\eta}_0$.

For any process $P = (P_w)_{w\in\mathbb{R}}$ and $t \in \mathbb{R}$, we denote $\mathscr{T}_t P$ as the process $(P_{t+w})_{w\in\mathbb{R}}$. By Lemma 1.18 and 2.13, we can deduce Theorem 2.24 from the following result. To make it well-defined, we extend $(\eta'_t)_{t\geq 0}$ to $\boldsymbol{\eta}' = (\eta'_t)_{t\in\mathbb{R}}$, such that $\eta'_t = \eta'_0$ for each $t < 0$.

Proposition 2.25. *For any function f on $\{0,1\}^{\mathbb{Z}\times\mathbb{R}}$ that is bounded and measurable, we have*

$$T^{-1}\int_0^T f(\mathscr{T}_t\boldsymbol{\eta}')dt \to \hat{\boldsymbol{\Psi}}^{\rho}(f)$$

in probability.

By Birkhoff's Theorem, this proposition follows from Theorem 1.7, and the following ergodicity result.

Proposition 2.26. *The process $\overline{\boldsymbol{\eta}} = (\overline{\eta}_t)_{t\in\mathbb{R}}$ is ergodic in time.*

Assuming this we prove Proposition 2.25 now.

Proof of Proposition 2.25. Without loss of generality we assume that $0 \leq f \leq 1$, and for some $s > 0$, it is $\mathscr{F}_s$-measurable, where $\mathscr{F}_s$ is the σ-algebra generated by $A \times \{0,1\}^{\mathbb{Z}\times(-\infty,-s)\cup(s,\infty)}$ for all measurable $A \subset \{0,1\}^{\mathbb{Z}\times[-s,s]}$. Take any $\delta > 0$, then by Birkoff's Theorem and Proposition 2.26, we can find r large enough such that $\mathbb{P}\left[\left|r^{-1}\int_0^r f(\mathscr{T}_t\overline{\boldsymbol{\eta}})dt - \hat{\boldsymbol{\Psi}}^\rho(f)\right| > \delta\right] < \delta$. For each $t > 0$, denote $\chi_t = \mathbb{1}\left[\left|r^{-1}\int_{t+s}^{t+s+r} f(\mathscr{T}_w\boldsymbol{\eta}')dw - \hat{\boldsymbol{\Psi}}^\rho(f)\right| > \delta\right]$. Then by Theorem 1.7 we have

$$\lim_{N\to\infty} N^{-1}\sum_{i=0}^{N-1}\mathbb{E}[\chi_{ir}] = \mathbb{P}\left[\left|r^{-1}\int_0^r f(\mathscr{T}_t\overline{\boldsymbol{\eta}})dt - \hat{\boldsymbol{\Psi}}^\rho(f)\right| > \delta\right] < \delta.$$

This implies that for any N large enough, we have $\mathbb{P}[\sum_{i=0}^{N-1}\chi_{ir} > \sqrt{\delta}N] < \sqrt{\delta}$, thus

$$\mathbb{P}\left[\left|(Nr)^{-1}\int_s^{Nr+s} f(\mathscr{T}_t\boldsymbol{\eta}')dt - \hat{\boldsymbol{\Psi}}^\rho(f)\right| > \sqrt{\delta}+\delta\right] < \delta,$$

which implies our conclusion since $\delta > 0$ is arbitrary. □

It now remains to prove Proposition 2.26. Our key step is the following coupling of two copies of Ψ^ρ.

Lemma 2.27. *For any $L \in \mathbb{N}$ and $\epsilon > 0$, there exist an integer $M > L$, and a coupling between Ψ^ρ and itself, such that the following is true. Let $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ be sampled from this coupling, then*

1. *Restricted to $[\![-L, L]\!]$, $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ are independent.*
2. *With probability $> 1-\epsilon$, $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ have the same number of particles in $[\![-M, 0]\!]$ and in $[\![0, M]\!]$, and $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ are identical on $\mathbb{Z}\setminus[\![-M, M]\!]$.*

Recall that we defined the configuration $\overline{\eta}^*(x)$ for $x > 0$ in terms of two independent collections of i.i.d. Bernoulli(ρ) random variables $Y_1(x), x \geq 1$ and $Y_2(x), x \geq 1$.

For $x > 0$, define

$$\overline{Y}_1(x) = \overline{\eta}^*(x) = \begin{cases} 1, & Y_1(x) = 1 \text{ or } x \in \mathcal{E} \\ 0, & Y_1(x) = 0 \text{ and } x \notin \mathcal{E} \end{cases}$$

$$\overline{Y}_2(x) = \begin{cases} 0, & Y_2(x) = 0 \text{ or } x \in \mathcal{E} \\ 1, & Y_2(x) = 1 \text{ and } x \notin \mathcal{E} \end{cases}.$$

We also define

$$\overline{R}_1(x) = \sum_{y=1}^{x} \overline{Y}_1(x) = R_1(x) + E(x)$$
$$\overline{R}_2(x) = \sum_{y=1}^{x} \overline{Y}_2(x) = R_2(x) - E(x).$$

We have that $\overline{R}_1(x) - \overline{R}_2(x) = 2M(x) - W(x)$, where W and M are defined in (1.5) and (1.6). In particular note that $\overline{R}_1(x) \geq \overline{R}_2(x)$ for all x.

Note that $\overline{R}_1(x)$ is the number of particles of $\overline{\eta}^*(x)$ in the interval $[\![1, x]\!]$. The process $\overline{R}_1$ is certainly not Markov. However, we will exploit the fact that the process $(\overline{R}_1(x), \overline{R}_2(x)), x \geq 0$ is a Markov chain.

Consider the transition function $\mathbf{T} : \mathbb{Z}^2_{\geq} \times \mathbb{Z}^2_{\geq} \to [0, 1]$ defined by

$$\begin{aligned} \mathbf{T}((a,b),(a+1,b+1)) &= \rho^2, \\ \mathbf{T}((a,b),(a+1,b)) &= \rho(1-\rho)\frac{a-b+2}{a-b+1}, \\ \mathbf{T}((a,b),(a,b+1)) &= \rho(1-\rho)\frac{a-b}{a-b+1}, \\ \mathbf{T}((a,b),(a,b)) &= (1-\rho)^2. \end{aligned}$$

Lemma 2.28. *The process $(\overline{R}_1, \overline{R}_2)$ is a Markov chain in $\mathbb{Z}^2_{\geq}$ with transition probability $\mathbf{T}$.*

Proof. For any $x \geq 0$, we show that

$$\mathbb{P}\big[\{\overline{R}_1(y)\}_{y=0}^{x} = \{r_1(y)\}_{y=0}^{x}, \{\overline{R}_2(y)\}_{y=0}^{x} = \{r_2(y)\}_{y=0}^{x}, E(x) = h\big] \\ = \rho^{r_1(x)+r_2(x)}(1-\rho)^{2x-r_1(x)-r_2(x)} \quad (2.8)$$

for any integers $\{r_1(y)\}_{y=0}^{x}$, $\{r_2(y)\}_{y=0}^{x}$ and h such that

1. $r_1(0) = r_2(0) = 0$,
2. $r_1(y) - r_1(y-1), r_2(y) - r_2(y-1) \in \{0, 1\}$, and $r_1(y) \geq r_2(y)$ for any $1 \leq y \leq x$,
3. $0 \leq h \leq r_1(x) - r_2(x)$.

We prove this by induction on x. The base case is trivial, and now we assume that it is true for x, and consider $x+1$.

Note that we have $x+1 \in \mathcal{E}$ if the following three conditions all hold: (i) $M(x) = W(x)$ (i.e. $\overline{R}_1(x) - \overline{R}_2(x) = E(x)$); (ii) $Y_1(x+1) = 0$; (iii) $Y_2(x+1) = 1$. In that case we have $\overline{R}_1(x+1) = \overline{R}_1(x) + 1$, $\overline{R}_2(x+1) = \overline{R}_2(x)$, and $E(x+1) = E(x) + 1$. In any other case we have $\overline{R}_1(x+1) = \overline{R}_1(x) + Y_1(x+1)$, $\overline{R}_2(x+1) = \overline{R}_2(x) + Y_2(x+1)$, and $E(x+1) = E(x)$.

Denote $y_1(x+1) = r_1(x+1) - r_1(x)$ and $y_2(x+1) = r_2(x+1) - r_2(x)$. From the above transition we have that when $h \leq r_1(x) - r_2(x)$,

$$\begin{aligned} &\mathbb{P}[\{\overline{R}_1(y)\}_{y=0}^{x+1} = \{r_1(y)\}_{y=0}^{x+1}, \{\overline{R}_2(y)\}_{y=0}^{x+1} = \{r_2(y)\}_{y=0}^{x+1}, E(x+1) = h] \\ =&\mathbb{P}[\{\overline{R}_1(y)\}_{y=0}^{x} = \{r_1(y)\}_{y=0}^{x}, \{\overline{R}_2(y)\}_{y=0}^{x} = \{r_2(y)\}_{y=0}^{x}, E(x) = h] \\ &\times \mathbb{P}[Y_1(x+1) = y_1(x+1), Y_2(x+1) = y_2(x+1)], \end{aligned}$$

where the last line equals $\rho^{y_1(x+1)+y_2(x+1)}(1-\rho)^{2-y_1(x+1)-y_2(x+1)}$. When $h > r_1(x) - r_2(x)$, we must have that $h = r_1(x) - r_2(x) + 1$ and $y_1(x+1) = 1$, $y_2(x+1) = 0$, and

that

$$\begin{aligned}&\mathbb{P}[\{\overline{R}_1(y)\}_{y=0}^{x+1} = \{r_1(y)\}_{y=0}^{x+1}, \{\overline{R}_2(y)\}_{y=0}^{x+1} = \{r_2(y)\}_{y=0}^{x+1}, E(x+1) = h]\\ =&\mathbb{P}[\{\overline{R}_1(y)\}_{y=0}^{x} = \{r_1(y)\}_{y=0}^{x}, \{\overline{R}_2(y)\}_{y=0}^{x} = \{r_2(y)\}_{y=0}^{x}, E(x) = h-1]\\ &\times \mathbb{P}[Y_1(x+1) = 0, Y_2(x+1) = 1],\end{aligned}$$

where the second probability on the right hand-side equals $\rho(1-\rho)$, which also equals $\rho^{y_1(x+1)+y_2(x+1)}(1-\rho)^{2-y_1(x+1)-y_2(x+1)}$. Thus by the induction hypothesis ((2.8)) for x, we get (2.8) for $x+1$.

Finally, by summing over all h, we conclude that

$$\begin{aligned}&\mathbb{P}[\{\overline{R}_1(y)\}_{y=0}^{x} = \{r_1(y)\}_{y=0}^{x}, \{\overline{R}_2(y)\}_{y=0}^{x} = \{r_2(y)\}_{y=0}^{x}]\\ &\qquad = (r_1(x) - r_2(x) + 1)\rho^{r_1(x)+r_2(x)}(1-\rho)^{2x-r_1(x)-r_2(x)}.\end{aligned}$$

Using this we conclude that

$$\begin{aligned}&\mathbb{P}[\overline{R}_1(x+1) = r_1(x+1), \overline{R}_2(x+1) = r_2(x+1)\\ &\qquad \mid \{\overline{R}_1(y)\}_{y=0}^{x} = \{r_1(y)\}_{y=0}^{x}, \{\overline{R}_2(y)\}_{y=0}^{x} = \{r_2(y)\}_{y=0}^{x}]\\ &\qquad\qquad = \frac{r_1(x+1) - r_2(x+1) + 1}{r_1(x) - r_2(x) + 1}\rho^{y_1(x)+y_2(x)}(1-\rho)^{2-y_1(x)-y_2(x)},\end{aligned}$$

which implies the conclusion. □

We have the following mixing property of this Markov chain.

Lemma 2.29. *For any $u, v \in \mathbb{Z}^2_{\geq}$, we have $\lim_{n\to\infty} \|\mathbf{T}^n(u,\cdot) - \mathbf{T}^n(v,\cdot)\|_1 = 0$.*

Proof. Let $(\overline{R}_1^{(1)}, \overline{R}_2^{(1)})$ and $(\overline{R}_1^{(2)}, \overline{R}_2^{(2)})$ be two copies of the Markov chain with transition probabilities $\mathbf{T}$, starting from u and v respectively. For $i = 1, 2$, denote

$A^{(i)} = \overline{R}_1^{(i)} + \overline{R}_2^{(i)} - x$ and $B^{(i)} = \overline{R}_1^{(i)} - \overline{R}_2^{(i)})/2$. Then for any $x \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned}
&\mathbb{P}[A^{(i)}(x+1) = A^{(i)}(x) + 1, B^{(i)}(x+1) = B^{(i)}(x) \mid A^{(i)}(x), B^{(i)}(x)] = \rho^2 \\
&\mathbb{P}[A^{(i)}(x+1) = A^{(i)}(x) - 1, B^{(i)}(x+1) = B^{(i)}(x) \mid A^{(i)}(x), B^{(i)}(x)] = (1-\rho)^2, \\
&\mathbb{P}[A^{(i)}(x+1) = A^{(i)}(x), B^{(i)}(x+1) = B^{(i)}(x) + 1 \mid A^{(i)}(x), B^{(i)}(x)] \\
&\qquad\qquad = \rho(1-\rho)\frac{B^{(i)}(x)+2}{B^{(i)}(x)+1}, \\
&\mathbb{P}[A^{(i)}(x+1) = A^{(i)}(x), B^{(i)}(x+1) = B^{(i)}(x) - 1 \mid A^{(i)}(x), B^{(i)}(x)] \\
&\qquad\qquad = \rho(1-\rho)\frac{B^{(i)}(x)}{B^{(i)}(x)+1}.
\end{aligned}$$

Notice that at each step, for each $i = 1, 2$, exactly one of $A^{(i)}$ and $B^{(i)}$ changes.

We now couple these processes inductively in the following way. Let $N \in \mathbb{N}$. Suppose that we are given $A^{(1)}(x), A^{(2)}(x)$ and $B^{(1)}(x), B^{(2)}(x)$, we choose $A^{(1)}(x+1), A^{(2)}(x+1)$ and $B^{(1)}(x+1), B^{(2)}(x+1)$ in the following way. First, we let $B^{(1)}(x+1) = B^{(1)}(x)$ if and only if $B^{(2)}(x+1) = B^{(2)}(x)$. In the case where $B^{(1)}(x+1) = B^{(1)}(x)$, if $A^{(1)}(x) \neq A^{(2)}(x)$ we take $A^{(1)}(x+1), A^{(2)}(x+1)$ independently; and if $A^{(1)}(x) = A^{(2)}(x)$ we always take $A^{(1)}(x+1) = A^{(2)}(x+1)$. In the case where $B^{(1)}(x+1) \neq B^{(1)}(x)$, if $B^{(1)}(x) = B^{(2)}(x)$ we take $B^{(1)}(x+1) = B^{(2)}(x+1)$; if $B^{(1)}(x) \neq B^{(2)}(x)$ and $\max_{0 \leq y \leq x} B^{(1)}(y) < N$, we couple $B^{(1)}(x+1)$ and $B^{(2)}(x+1)$ such that almost surely $|B^{(1)}(x+1) - B^{(2)}(x+1)| \leq |B^{(1)}(x) - B^{(2)}(x+1)|$; if $B^{(1)}(x) \neq B^{(2)}(x)$ and $\max_{0 \leq y \leq x} B^{(1)}(y) \geq N$, we take $B^{(1)}(x+1)$ and $B^{(2)}(x+1)$ independently.

Under this coupling, we see that $A^{(1)}(x) = A^{(2)}(x)$ for all large enough x, since $A^{(1)}$ and $A^{(2)}$ are independent random walks until they are equal.

Take any $\epsilon > 0$. We claim that when N is large enough depending on u, v, ϵ, with probability at least $1 - \epsilon$ we have $B^{(1)}(x) = B^{(2)}(x)$ for some large enough x, thus for all large x. Let $x_0 = \min\{x \in \mathbb{Z}_{\geq 0} : B^{(1)}(x) = N\}$. We have $x_0 < \infty$ almost surely, since $B^{(1)}$ dominates a simple random walk. From the coupling we must have

$|B^{(1)}(x_0) - B^{(2)}(x_0)| \le |B^{(1)}(0) - B^{(2)}(0)| \le \|u - v\|_1$. For $i = 1, 2$, let $V^{(i)} : \mathbb{Z}_{\ge 0} \to \mathbb{Z}$ be a random walk satisfying $V^{(i)}(0) = B^{(i)}(x_0)$, and

$$\begin{aligned}
\mathbb{P}[V^{(i)}(x+1) = V^{(i)}(x) \mid V^{(i)}(x)] &= \rho^2 + (1-\rho)^2, \\
\mathbb{P}[V^{(i)}(x+1) = V^{(i)}(x) + 1 \mid V^{(i)}(x)] &= \rho(1-\rho), \\
\mathbb{P}[V^{(i)}(x+1) = V^{(i)}(x) - 1 \mid V^{(i)}(x)] &= \rho(1-\rho).
\end{aligned}$$

Also we let $V^{(1)}$ and $V^{(2)}$ be independent, until $V^{(1)}(x_1) = V^{(2)}(x_1)$ for some $x_1 > 0$, and let $V^{(1)}(x) = V^{(2)}(x)$ for all $x > x_1$. For some N_1 large enough (depending on u, v, ϵ) we have $\mathbb{P}[x_1 < N_1] > 1 - \epsilon/2$. We next take N large enough (depending on N_1, ϵ) such that with probability $> 1 - \epsilon/2$ we can couple $V^{(1)}, V^{(2)}$ with $B^{(1)}, B^{(2)}$, so that $V^{(1)}(x) = B^{(1)}(x_0 + x)$ and $V^{(2)}(x) = B^{(2)}(x_0 + x)$ for $0 \le x \le N_1$. Thus the claim the follows. Since ϵ is arbitrarily taken, we have that the conclusion follows. □

For the process $(\overline{R}_1, \overline{R}_2)$ starting from $(0, 0)$, denote $\mathscr{S}$ as its measure. From the above lemma we could construct a coupling between $\mathscr{S}$ and itself, as follows.

Lemma 2.30. *For any $L \in \mathbb{N}$ and $\epsilon > 0$, there exist an integer $M > L$, and a coupling between $\mathscr{S}$ and itself, such that the following is true. Let $(\overline{R}_1^{(1)}, \overline{R}_2^{(1)})$ and $(\overline{R}_1^{(2)}, \overline{R}_2^{(2)})$ be sampled from this coupling, then*

1. *Restricted to $[\![0, L]\!]$, $(\overline{R}_1^{(1)}, \overline{R}_2^{(1)})$ and $(\overline{R}_1^{(2)}, \overline{R}_2^{(2)})$ are independent.*
2. *$\mathbb{P}[\overline{R}_1^{(1)}(M) = \overline{R}_1^{(2)}(M), \overline{R}_2^{(1)}(M) = \overline{R}_2^{(2)}(M)] > 1 - \epsilon$.*

Proof. We construct the coupling by first allowing $(\overline{R}_1^{(1)}, \overline{R}_2^{(1)})$ and $(\overline{R}_1^{(2)}, \overline{R}_2^{(2)})$ to evolve independently for the first L steps. Then conditioned on $(\overline{R}_1^{(1)}(L), \overline{R}_2^{(1)}(L))$ and $(\overline{R}_1^{(2)}(L), \overline{R}_2^{(2)}(L))$, we couple $(\overline{R}_1^{(1)}(M), \overline{R}_2^{(1)}(M))$ and $(\overline{R}_1^{(2)}(M), \overline{R}_2^{(2)}(M))$ to maximize the probability that they coincide. The conclusion follows from Lemma 2.29 by taking M large enough, since there are only finitely many possible values of $(\overline{R}_1^{(1)}(L), \overline{R}_2^{(1)}(L))$ and $(\overline{R}_1^{(2)}(L), \overline{R}_2^{(2)}(L))$. □

Proof of Lemma 2.27. From the coupling in Lemma 2.30, we get the desired coupling. The configurations of particles to the right and to the left of the origin are independent of each other under Ψ^ρ, and the configuration of particles to the right of the origin is a function of $\overline{R}_1$; we can apply an analogous construction for the configuration to the left. □

Proof of Proposition 2.26. We assume the contrary, i.e., $\overline{\boldsymbol{\eta}}$ is not ergodic. Then there is a measurable set $B \subset \{0,1\}^{\mathbb{Z}}$ invariant under the Markov process, with $0 < \Psi^\rho(B) < 1$. Let $\overline{\eta} \sim \Psi^\rho$. Take $L \in \mathbb{N}$ and we consider the random variable $\chi_L(\overline{\eta}) = \mathbb{P}[\overline{\eta} \in B \mid \{\overline{\eta}(x)\}_{x\in[\![-L,L]\!]}]$. Note that this a martingale in L, and almost surely converges to $\mathbb{1}[\overline{\eta} \in B]$. Thus for any $\epsilon > 0$, we can take L large enough, such that $\mathbb{P}[|\chi_L(\overline{\eta}) - \mathbb{1}[\overline{\eta} \in B]| > \epsilon] < \epsilon$.

For the above L and ϵ, we take M and the coupling as given by Lemma 2.27. Suppose that $\overline{\eta}^{(1)}, \overline{\eta}^{(2)}$ are sampled from this coupling. By the first property of the coupling, and that χ_L only depends on the configuration in $[\![-L, L]\!]$, we have

$$\mathbb{P}[\chi_L(\overline{\eta}^{(1)}) > 1-\epsilon,\ \chi_L(\overline{\eta}^{(2)}) < \epsilon] > \Psi^\rho(B)(1-\Psi^\rho(B)) - 2\epsilon.$$

Since that $\Psi^\rho(\{\eta : \chi_L(\eta) < \epsilon\} \cap B) < \epsilon$ and $\Psi^\rho(B \setminus \{\eta : \chi_L(\eta) > 1-\epsilon\}) < \epsilon$, we have

$$\mathbb{P}[\overline{\eta}^{(1)} \in B,\ \overline{\eta}^{(2)} \notin B] > \Psi^\rho(B)(1-\Psi^\rho(B)) - 4\epsilon.$$

Using the second property of the coupling, and by taking ϵ small enough, we have that with probability $> \Psi^\rho(B)(1-\Psi^\rho(B)) - 5\epsilon > 0$, $\overline{\eta}^{(1)} \in B$ and $\overline{\eta}^{(2)} \notin B$, and $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ are identical on $\mathbb{Z} \setminus [\![-M, M]\!]$.

We next couple two TASEP starting from $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ at time 0, such that interchanges happen between neighboring sites with the same Poisson clock. With positive probability the following happens: from time 0 to time 1, no exchange happens between sites $-M-1$, $-M$, and between sites M, $M+1$; and exchanges happen between

sites x, $x+1$, sequentially for $x = -M, \cdots, M-1$, and repeat this for $2M$ times. Conditioned on $\overline{\eta}^{(1)} \in B$ and $\overline{\eta}^{(2)} \notin B$, and $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ are identical on $\mathbb{Z} \setminus [\![-M, M]\!]$, and the above event on the exchanging clocks, at time 1 the two processes starting from $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ would be identical. However, as B and $\{0,1\}^{\mathbb{Z}} \setminus B$ are assumed to be invariant under the Markov process (of TASEP as seen from a hole-particle pair), we get a subset of $\{0,1\}^{\mathbb{Z}}$ with positive Ψ^{ρ} measure, and is contained (up to a measure zero set) in both B and $\{0,1\}^{\mathbb{Z}} \setminus B$. This is a contradiction. □

2.6 From semi-infinite geodesics to finite geodesics

In this section we fix $\rho \in (0,1)$. From the convergence along semi-infinite geodesics (Theorem 2.24), we deduce the following result, which is an extension of Theorem 1.1.

Theorem 2.31. *Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of integers such that $\limsup_{n\to\infty} n^{-2/3}|b_n| < \infty$. Then in probability we have $\mu_{\mathbf{0},\langle n,b_n\rangle_\rho} \to \nu^{\rho}$ as $n \to \infty$.*

The idea is to construct semi-infinite geodesics which largely cover the finite ones.

Recall G^{ρ}, the Busemann function in the direction of ρ. We consider the following event: for any $b \in \mathbb{Z}$ with $h^{-1}n^{2/3} < |b| < hn^{2/3}$, there is $G^{\rho}(\langle 0, b\rangle) + b(\rho^{-1} - (1-\rho)^{-1}) > hn^{1/3}$; and for $b \in \mathbb{Z}$ with $|b| \geq hn^{2/3}$, there is $G^{\rho}(\langle 0, b\rangle) + b(\rho^{-1} - (1-\rho)^{-1}) > -bn^{-1/3}$. Denote this event by $\mathcal{E}_{h,n}$, we show that its probability is lower bounded uniformly in n,

Lemma 2.32. *For any $h > 0$, there is $\delta > 0$ such that $\mathbb{P}(\mathcal{E}_{h,n}) > \delta$ for all n large enough.*

Proof. Denote $F(b) = -G^{\rho}(\langle 0, b\rangle) - b(\rho^{-1} - (1-\rho)^{-1})$, then F is a (two-sided) random walk, where each step is centered with exponential tail. By independence of all the

steps, we have

$$\begin{aligned}\mathbb{P}[\mathcal{E}_{h,n}] \geq & \mathbb{P}\left[\max_{h^{-1}n^{2/3}<|b|<hn^{2/3}} F(b) < -hn^{1/3}\right] \\ & \times \mathbb{P}\left[\max_{b\geq hn^{2/3}} F(b) - F(\lfloor hn^{2/3}\rfloor) - bn^{-1/3} < hn^{1/3}\right] \\ & \times \mathbb{P}\left[\max_{b\leq -hn^{2/3}} F(b) - F(-\lfloor hn^{2/3}\rfloor) + bn^{-1/3} < hn^{1/3}\right].\end{aligned}$$

As the process converges to Brownian motion (weakly in the uniform topology) in compact sets, the first factor in the right hand side is lower bounded by a constant. We next lower bound the factor in the second line, and the third line could be lower bounded in a similar way. It is at least

$$\begin{aligned}\mathbb{P}\left[\max_{b\in\mathbb{N}} F(b) - bn^{-1/3} < hn^{1/3}\right] \geq & \mathbb{P}\left[\max_{0\leq b\leq In^{2/3}} F(b) - bn^{-1/3} < hn^{1/3}\right] \\ & - \sum_{i=I}^{\infty} \mathbb{P}\left[\max_{in^{2/3}<b\leq(i+1)n^{2/3}} F(b) \geq (i+h)n^{1/3}\right].\end{aligned}$$

where I is a large constant. As $n\to\infty$, the first term in the right hand side converges to the probability that a Brownian motion is bounded below a (sloped) line in $[0, I]$, and such probability is lower bounded uniformly in I. For the summation in the second line, the term for each i is upper bounded by

$$\begin{aligned}& \mathbb{P}\left[F(\lceil in^{2/3}\rceil) \geq (i+h)n^{1/3}/2\right] + \mathbb{P}\left[\max_{0\leq b\leq\lceil n^{2/3}\rceil} F(b) \geq (i+h)n^{1/3}/2\right] \\ \leq & \mathbb{P}\left[F(\lceil in^{2/3}\rceil) \geq (i+h)n^{1/3}/2\right] + 2\mathbb{P}\left[F(\lceil n^{2/3}\rceil) \geq (i+h)n^{1/3}/2\right],\end{aligned}$$

where the inequality is by reflection principle. By a Bernstein type estimate for sum of independent variables with exponential tails, this could be bounded by Ce^{-ci} for some $C, c > 0$, independent of n. Thus by taking I large enough the conclusion follows. □

Proof of Theorem 2.31. Take any $s \in \mathbb{N}$ and $f : \mathbb{R}^{[\![-s,s]\!]^2} \to \mathbb{R}$ with $0 \leq f \leq 1$,

regarded as a function on $\mathbb{R}^{\mathbb{Z}^2}$, we shall prove that $\mu_{\mathbf{0},\langle n,b_n\rangle_\rho}(f) \to \nu^\rho(f)$ in probability.

In this proof we use C, c to denote large or small universal positive constants, whose value may change from line to line. We then have that $|b_n| < Cn^{2/3}$ for any $n \in \mathbb{N}$. For simplicity of notation we denote $T^*_{u,v} = T_{u,v} - \omega(v)$ for any vertices $u \leq v$.

We denote $\mathcal{A}_{h,n}$ as the following event: for any $b \in \mathbb{Z}$, we have

- $T^*_{\mathbf{0},\langle n,b\rangle_\rho} + b(\rho^{-1} - (1-\rho)^{-1}) > \mathbb{E}T_{\mathbf{0},\langle n,0\rangle_\rho} - hn^{1/3}/2$, if $|b - b_n| \leq h^{-1}n^{2/3}$;
- $T^*_{\mathbf{0},\langle n,b\rangle_\rho} + b(\rho^{-1} - (1-\rho)^{-1}) < \mathbb{E}T_{\mathbf{0},\langle n,0\rangle_\rho} + hn^{1/3}/2$, if $h^{-1}n^{2/3} < |b - b_n| < hn^{2/3}$;
- $T^*_{\mathbf{0},\langle n,b\rangle_\rho} + b(\rho^{-1} - (1-\rho)^{-1}) < \mathbb{E}T_{\mathbf{0},\langle n,0\rangle_\rho} + hn^{1/3}/2 - bn^{-1/3}$, if $|b - b_n| \geq hn^{2/3}$.

By splitting $\mathbb{L}_n$ into segments of length $n^{2/3}$, and using Proposition 2.2 and Theorem 2.1, we have $\mathbb{P}[\mathcal{A}_{h,n}] > 1 - e^{-ch}$, for n and h large enough.

We also denote $\mathcal{B}_{h,n}$ as the following event:

$$\Gamma_{\mathbf{0},\langle n,\lfloor b_n - h^{-1}n^{2/3}\rfloor\rangle_\rho} \cap \mathbb{L}_{(1-h^{-1})n} = \Gamma_{\mathbf{0},\langle n,\lceil b_n + h^{-1}n^{2/3}\rceil\rangle_\rho} \cap \mathbb{L}_{(1-h^{-1})n}.$$

By Proposition 2.3, we have $\mathbb{P}[\mathcal{B}_{h,n}] > 1 - Ch^{-1/3}$, for $h < cn^{2/3}$ and h, n large enough.

Denote $\langle n, b'_n\rangle_\rho$ as the intersection of $\Gamma^\rho_{\mathbf{0}}$ with $\mathbb{L}_n$. By Theorem 2.24 we have $\mu_{\mathbf{0},\langle n,b'_n\rangle_\rho} \to \nu^\rho$ in probability; i.e. we can find a sequence $\{\epsilon_n\}_{n\in\mathbb{Z}_{\geq 0}}$, such that $\epsilon_n \to 0$, and

$$\mathbb{P}[|\mu_{\mathbf{0},\langle n,b'_n\rangle_\rho}(f) - \nu^\rho(f)| < \epsilon_n] \to 1,$$

We denote $\mathcal{E}'_{h,n}$ as $\mathcal{E}_{h,n}$ translated by $\langle n, b_n\rangle_\rho$. Let $\{h_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence with $h_n \to \infty$. By Lemma 2.32, we can choose this sequence to make $\mathbb{P}[\mathcal{E}'_{h_n,n}] = \mathbb{P}[\mathcal{E}_{h_n,n}]$ decay to zero slowly enough, such that

$$\mathbb{P}[|\mu_{\mathbf{0},\langle n,b'_n\rangle_\rho}(f) - \nu^\rho(f)| < \epsilon_n \mid \mathcal{E}'_{h_n,n}] \to 1.$$

Since $\mathbb{P}[\mathcal{A}_{h_n,n}] > 1 - e^{-ch_n}$, $\mathbb{P}[\mathcal{B}_{h_n,n}] > 1 - Ch_n^{-1/3}$, and $\mathcal{A}_{h_n,n}, \mathcal{B}_{h_n,n}$ are independent

of $\mathcal{E}'_{h_n,n}$, we have

$$\mathbb{P}[\mathcal{A}_{h_n,n}, \mathcal{B}_{h_n,n}, |\mu_{\mathbf{0},\langle n,b'_n\rangle_\rho}(f) - \nu^\rho(f)| < \epsilon_n \mid \mathcal{E}'_{h_n,n}] \to 1.$$

However, under $\mathcal{A}_{h_n,n} \cap \mathcal{B}_{h_n,n} \cap \mathcal{E}'_{h_n,n}$, there must be $|b_n - b'_n| < h_n^{-1} n^{2/3}$, and $\Gamma_{\mathbf{0},\langle n,b_n\rangle_\rho} \cap \mathbb{L}_{(1-h_n^{-1})n} = \Gamma_{\mathbf{0},\langle n,b'_n\rangle_\rho} \cap \mathbb{L}_{(1-h_n^{-1})n}$, by monotonicity of geodesics. Thus $|\mu_{\mathbf{0},\langle n,b'_n\rangle_\rho}(f) - \nu^\rho(f)| < \epsilon_n$ implies that

$$|\mu_{\mathbf{0},\langle n,b_n\rangle_\rho}(f) - \nu^\rho(f)| < \epsilon_n + h_n^{-1}.$$

So we have

$$\mathbb{P}[|\mu_{\mathbf{0},\langle n,b_n\rangle_\rho}(f) - \nu^\rho(f)| < \epsilon_n + h_n^{-1} \mid \mathcal{E}'_{h_n,n}] \to 1.$$

Note that for $v \in \Gamma_{\mathbf{0},\langle n,b_n\rangle_\rho}$ with $d(v) < n - 4s$, $f(v)$ is independent of $\mathcal{E}'_{h_n,n}$. Thus we conclude that $\mathbb{P}[|\mu_{\mathbf{0},\langle n,b_n\rangle_\rho}(f) - \nu^\rho(f)| < \epsilon_n + h_n^{-1} + 2s/n] \to 1$. This implies that $\mu_{\mathbf{0},\langle n,b_n\rangle_\rho}(f) \to \nu^\rho(f)$ in probability. □

2.7 Parallelogram uniform covering

The goal of this section is to prove the following upgraded version of Theorem 2.31. It will be the key input for the next two sections.

Proposition 2.33. *For any $\rho \in (0,1)$, $h > 0$, $s \in \mathbb{N}$, and any bounded function $f : \mathbb{R}^{[\![-s,s]\!]^2} \to \mathbb{R}$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2}$, we have*

$$\max_{a,b\in\mathbb{Z},|a|,|b|<hn^{2/3}} \mu_{\langle 0,a\rangle_\rho,\langle n,b\rangle_\rho}(f), \min_{a,b\in\mathbb{Z},|a|,|b|<hn^{2/3}} \mu_{\langle 0,a\rangle_\rho,\langle n,b\rangle_\rho}(f) \to \nu^\rho(f),$$

in probability.

The idea of the proof of this proposition is to take a finite family of geodesics, and show that for any geodesic from some $\langle 0,a\rangle_\rho$ to $\langle n,b\rangle_\rho$ with $|a|,|b| < hn^{2/3}$, it could

be covered by one of the geodesics in this family. For simplicity of notation, below we write the proof for $\rho = 1/2$, while the more general case follows essentially verbatim.

We collect some ingredients in the proof. The first one concerns continuity of the function $(a, b) \mapsto T_{\langle 0,a\rangle,\langle n,b\rangle}$.

Lemma 2.34. *There exist constants C, c such that the following is true. For $h > 0$, $\theta < 1 < t$, we have*

$$\mathbb{P}\left[\max_{\substack{|a|,|a'|,|b|,|b'|<hn^{2/3}\\ |a-a'|,|b-b'|<\theta n^{2/3}}} |T_{\langle 0,a\rangle,\langle n,b\rangle} - T_{\langle 0,a'\rangle,\langle n,b'\rangle}| > t\theta^{1/2-0.01}n^{1/3} + Ch\theta n^{1/3}\right] < Che^{-ct}$$

when n is large enough (depending on h, θ, t).

We next bound the transversal fluctuation of geodesics. Recall the anti-diagonal distance $ad(u) = a - b$, for any vertex $u = (a, b)$. For vertices $u < v$, and $0 \le l \le d(v) - d(u)$, $t > 1$, let $\mathcal{T}_{l,t}^{u,v}$ be the following event: there exists $m \in \mathbb{Z}$ with $d(u) \le 2m \le d(u) + 2l$, such that if w denotes the intersection of $\mathbb{L}_m$ with $\Gamma_{u,v}$, then $|ad(w) - ad(u)| \ge 2tl^{2/3}$; or there exists $m \in \mathbb{Z}$ with $d(v) - 2l \le 2m \le d(v)$, and $|ad(w) - ad(v)| \ge 2tl^{2/3}$ where w denotes the intersection of $\mathbb{L}_m$ with $\Gamma_{u,v}$ (see Figure 2.5).

Lemma 2.35. *For $h > 0$, there exist constants C, c such that the following is true. For any $0 \le l \le n$ large enough, and $|b| < hn^{2/3}$, $t > 1$, we have $\mathbb{P}[\mathcal{T}_{l,t}^{\mathbf{0},\langle n,b\rangle}] < Ce^{-ct}$.*

The proof of the lemma is by applying Corollary 2.8 twice, and we omit the proof here.

Our next lemma wishes to establish that, for a geodesic and a path with a good weight, it is unlikely for them being disjoint and staying together for a while.

For any vertices $u < v$, and $M, l \in \mathbb{N}, m \in \mathbb{Z}$ with $d(u) \le 2m < 2m + 2Ml \le d(v)$, and a small enough parameter $c_0 > 0$, we denote $\mathcal{D}_{M,l,m}^{u,v}$ as the following event (see Figure 2.6): (1) for the intersections of $\Gamma_{u,v}$ with $\mathbb{L}_m$ and $\mathbb{L}_{m+Ml}$, denoted as u', v',

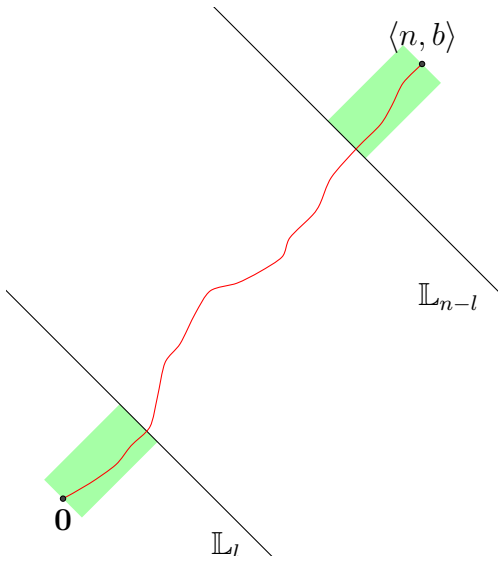


Figure 2.5: The complement of the event $\mathcal{T}_{l,t}^{\mathbf{0},\langle n,b\rangle}$: the geodesic $\Gamma_{\mathbf{0},\langle n,b\rangle}$ is restricted within the green boxes with width $tl^{2/3}$, below $\mathbb{L}_l$ or above $\mathbb{L}_{n-l}$.

we have $|ad(u') - ad(v')| < 2M^{5/6}l^{2/3}$; (2) there exists a directed path γ from $\mathbb{L}_m$ to $\mathbb{L}_{m+Ml}$, such that

- γ is disjoint from $\Gamma_{u,v}$.
- The weight of γ (i.e. $T(\gamma)$) is at least $4Ml - c_0Ml^{1/3}$.
- For each $i = 0, 1, \ldots, M$, $\|\Gamma_{u,v} \cap \mathbb{L}_{m+il} - \gamma \cap \mathbb{L}_{m+il}\|_1 < 2c_0l^{2/3}$.

Lemma 2.36. *For $h > 0$, there exist constants C, c such that the following is true. If $l > C$, $c_0 < c$, $|b| < hn^{2/3}$, and $0 \le m < m + Ml \le n$, we have $\mathbb{P}[\mathcal{D}_{M,l,m}^{\mathbf{0},\langle n,b\rangle}] < Ce^{-cM}$.*

The last ingredient we need is to bound the probability of multiple peaks in the sum of two independent point-to-lint profiles.

We denote $T_{u,v}^* = T_{u,v} - \omega(v)$ for any vertices $u \le v$; i.e. removing the weight of the last vertex in the geodesic $\Gamma_{u,v}$. For any vertices $u < v$, and $m \in \mathbb{Z}$ with $d(u) \le 2m \le d(v)$, and $\eta, t > 0$, we denote $\mathcal{M}_{\eta,t,m,g}^{u,v}$ as the following event: there exist $-g \le b_1 < b_2 < b_3 < b_4 < b_5 < b_6 \le g$, with $b_2 - b_1, b_3 - b_2, b_4 - b_3, b_5 - b_4, b_6 - b_5 \ge \eta$,

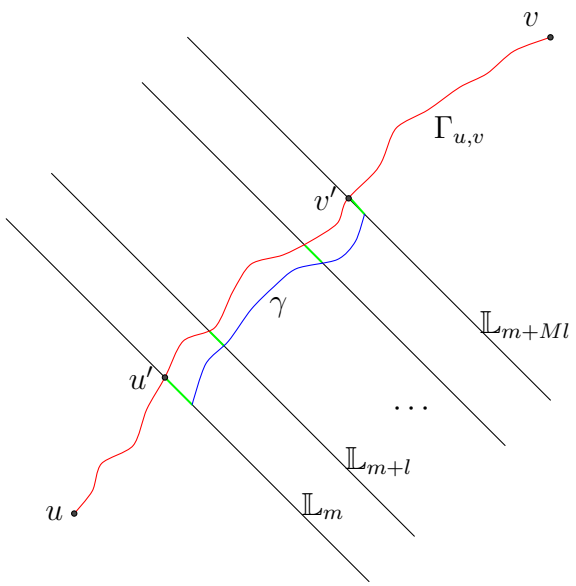


Figure 2.6: The event $\mathcal{D}^{u,v}_{M,l,m}$: each read segment has length $< c_0 l^{2/3}$, and $T(\gamma) \geq 4Ml - c_0 M l^{1/3}$.

such that $T_{u,v} = T^*_{u,\langle m,b_1\rangle} + T_{\langle m,b_1\rangle,v}$, and

$$T_{u,v} < T^*_{u,\langle m,b_i\rangle} + T_{\langle m,b_i\rangle,v} + t\eta^{1/2}, \quad \forall i = 2,3,4,5,6.$$

Lemma 2.37. *For $h > 0$, there exist constants C, c such that the following is true. For any $\theta > 0$, $0 < t < 1$, $h^{-1} < \alpha < 1 - h^{-1}$, $|\beta| < h$, we have $\mathbb{P}[\mathcal{M}^{\mathbf{0},\langle n,\beta n^{2/3}\rangle}_{\theta n^{2/3},t,\lfloor \alpha n\rfloor,2hn^{2/3}}] < Ct^{5-0.01}$, for n large enough depending on $h, \theta, t, \alpha, \beta$.*

We assume these lemmas and prove Proposition 2.33. We shall take a large finite family of geodesics, and assume there is a geodesic that is not covered by them. Our strategy is to show that this has one of the following two implications: either there are two geodesics stay close for a while; or there are several (almost) geodesics, whose end points are close, but their intersections with some line $\mathbb{L}_m$ are far from each other, implying "multiple peaks". Each of these two scenarios is unlikely to happen, using the lemmas above.

We set up the events to be used in the proof of Proposition 2.33. From now

on we take parameters $\delta_0 < \delta_1 < \delta_2 < \delta_3 < \delta_4 < \delta_5 < \delta$, whose values are to be determined. We fix the parameter h, and let n be large enough depending on all these parameters. Without lose of generality we assume that (by our choice of parameters and n) $\delta_0 n, \delta_0 n^{2/3}, hn^{2/3}, h^{-1}n, \delta^{-1}$, and each $\delta_i^{-1}\delta_{i+1}$ for $i \in \{0,1,2,3,4\}$ are integers.

We take two families of vertices $\mathfrak{P}_1 := \{\langle i\delta_0 n, j\delta_0 n^{2/3}\rangle : i,j \in \mathbb{Z}, 0 \le i \le \delta_0^{-1}/3, |j| < 4h\delta_0^{-1}\}$; and $\mathfrak{P}_2 := \mathbf{n} - \mathfrak{P}_1$. We shall show that with high probability, each geodesic $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ for some $|a|, |b| < hn^{2/3}$ is largely covered by a geodesic between vertices in $\mathfrak{P}_1$ and $\mathfrak{P}_2$.

Below we use $C, c > 0$ to denote large and small constants, which can only depend on h, and the values may change from line to line. Consider the following events.

- Let $\mathcal{T}$ be the union of $\mathcal{T}^{u,v}_{l,\delta^{-1}}$, for all $u \in \mathfrak{P}_1, v \in \mathfrak{P}_2$, and $l \in \delta_0 n\mathbb{Z}$, $0 \le l < d(v) - d(u)$. By Lemma 2.35 we have $\mathbb{P}[\mathcal{T}] < C\delta_0^{-5}e^{-c\delta^{-1}}$.

- Let

$$\mathcal{T}' = \mathcal{T}^{\langle 0,hn^{2/3}\rangle,\langle n,hn^{2/3}\rangle}_{n,h} \cup \mathcal{T}^{\langle 0,-hn^{2/3}\rangle,\langle n,-hn^{2/3}\rangle}_{n,h} \cup \mathcal{T}^{\langle 0,3hn^{2/3}\rangle,\langle n,3hn^{2/3}\rangle}_{n,h} \cup \mathcal{T}^{\langle 0,-3hn^{2/3}\rangle,\langle n,-3hn^{2/3}\rangle}_{n,h}.$$

  Then by Lemma 2.35, we can make $\mathbb{P}[\mathcal{T}']$ arbitrarily small by taking h large.

- Let $\mathcal{F}$ be the event where

$$|T^*_{\langle i\delta_1 n,a\rangle,\langle j\delta_1 n,b\rangle} - T^*_{\langle i\delta_1 n,a'\rangle,\langle j\delta_1 n,b'\rangle}| > \delta_0^{1/2-0.02} n^{1/3}$$

  for some integers $0 \le i < j \le \delta_1^{-1}$, and $|a|, |a'|, |b|, |b'| \le 2hn^{2/3}$ with $|a-a'|, |b-b'| \le \delta_0 n^{2/3}$. By Lemma 2.34, we have $\mathbb{P}[\mathcal{F}] < C\delta_1^{-2-2/3}e^{-\delta_0^{-0.01}}$.

- Let $\mathcal{D}$ be the union of $\mathcal{D}^{u,v}_{\delta^{-7},l,m}$ for all $u \in \mathfrak{P}_1$, $v \in \mathfrak{P}_2$, $l \in \{\delta_i n : i = 1,2,3,4,5\}$, $m \in \delta_0 n\mathbb{Z}$, such that $d(u) \le 2m < 2m + 2\delta^{-7}l \le d(v)$. By Lemma 2.36, we have $\mathbb{P}[\mathcal{D}] < C\delta_0^{-5}e^{-c\delta^{-7}}$.

- Let $\mathcal{H}$ denote the event where there exists some $m \in \delta_0 n\mathbb{Z}$, $0 \le m \le n$, and $l \in \{\delta_i n : i = 1, 2, 3, 4, 5\}$, $|a|, |b| < 4hn^{2/3}$, $|a - b| < \delta^{-6} l^{2/3}$, such that

$$T_{\langle m,a\rangle,\langle m+\delta^{-7}l,b\rangle} < 4\delta^{-7}l - c_0\delta^{-6}l^{1/3},$$

  where c_0 is the same as in the event $\mathcal{D}$. By applying Lemma 2.2 via splitting the lines $\mathbb{L}_m$ and $\mathbb{L}_{m+\delta^{-7}l}$ into segments of length $\delta_0 n^{2/3}$, we have $\mathbb{P}[\mathcal{H}] < C\delta_0^{-3} e^{-c\delta^{-11/3}}$.

- Let $\mathcal{M}$ be the union of $\mathcal{M}^{u,v}_{c_0(\delta_1 n)^{2/3},\delta_0^{1/2-0.03}\delta_1^{-1/3},\alpha n,4hn^{2/3}}$, for all $u \in \mathfrak{P}_1 \cap \mathbb{L}_0$, $v \in \mathfrak{P}_2 \cap \mathbb{L}_n$, and $\alpha \in \delta_1\mathbb{Z}$ with $h^{-1} < \alpha < 1 - h^{-1}$, and c_0 is the same as in the event $\mathcal{D}$. By Lemma 2.37, we have $\mathbb{P}[\mathcal{M}] < C\delta_0^{-2}\delta_1^{-1}\delta_0^{5/2-0.2}\delta_1^{-5/3} = C\delta_0^{0.3}\delta_1^{-8/3}$.

Proof of Proposition 2.33. As stated above we write the proof for $\rho = 1/2$ for simplicity of notation. We also assume that $h \in \mathbb{N}$, since the result for a larger h implies it for a smaller h.

We denote $\mathcal{E} := \mathcal{T}^c \cap \mathcal{T}'^c \cap \mathcal{F}^c \cap \mathcal{D}^c \cap \mathcal{H}^c \cap \mathcal{M}^c$.

Claim. *Under $\mathcal{E}$ the following holds: for any $|a|, |b| < hn^{2/3}$, there exist $u \in \mathfrak{P}_1$ and $v \in \mathfrak{P}_2$, with $d(u) < 4h^{-1}n$ and $d(v) > (1 - 4h^{-1})n$, such that*

$$\mathbb{L}_{2h^{-1}n} \cap \Gamma_{u,v} = \mathbb{L}_{2h^{-1}n} \cap \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}, \quad \mathbb{L}_{(1-2h^{-1})n} \cap \Gamma_{u,v} = \mathbb{L}_{(1-2h^{-1})n} \cap \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}.$$

We first deduce the conclusion assuming this claim. For any fixed set of parameters, by Theorem 2.31 and sending $n \to \infty$, we have in probability

$$\max_{u\in\mathfrak{P}_1, v\in\mathfrak{P}_2} |\mu_{u,v}(f) - \nu^{1/2}(f)| \to 0.$$

Thus since f is a function on $\mathbb{R}^{[\![-s,s]\!]^2}$, for n such that $\delta_0 n, \delta_0 n^{2/3}, hn^{2/3}, h^{-1}n \in \mathbb{Z}$, as

$n \to \infty$ we have

$$\mathbb{P}\left[\mathcal{E}, \max_{a,b\in\mathbb{Z},|a|,|b|<hn^{2/3}} |\mu_{\langle 0,a\rangle,\langle n,b\rangle}(f) - \nu^{1/2}(f)| > 10h^{-1}\|f\|_\infty\right] \to 0. \tag{2.9}$$

For all $n \in \mathbb{N}$, denote $n' \le n$ be the largest number such that $\delta_0 n', \delta_0 n'^{2/3}, hn'^{2/3}, h^{-1}n' \in \mathbb{Z}$, and $\mathcal{E}'$ as $\mathcal{E}$ for n' instead of n. Then as $n \to \infty$ we have $(n-n')/n \to 1$, and

$$\mathbb{P}\Big[\mathcal{E}', \Big(\mathcal{T}^{\langle 0,\lceil hn^{2/3}/2\rceil\rangle,\langle n,\lceil hn^{2/3}/2\rceil\rangle}_{n-n',hn^{2/3}(n-n')^{-2/3}/3} \cup \mathcal{T}^{\langle 0,-\lceil hn^{2/3}/2\rceil\rangle,\langle n,-\lceil hn^{2/3}/2\rceil\rangle}_{n-n',hn^{2/3}(n-n')^{-2/3}/3}\Big)^c, \max_{a,b\in\mathbb{Z},|a|,|b|<hn^{2/3}/2} |\mu_{\langle 0,a\rangle,\langle n,b\rangle}(f) - \nu^{1/2}(f)| > 11h^{-1}\|f\|_\infty\Big] \to 0. \tag{2.10}$$

This is because when n is large enough, using the second event in the first line and monotonicity of geodesics, for each $|a|, |b| < hn^{2/3}/2$, the geodesic $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ must intersect $\mathbb{L}_{n'}$ at $\langle n', b'\rangle$ for some b' satisfying $|b'| < hn'^{2/3}$. Then using (2.9) we get (2.10).

We next choose the parameters. We take $L \in \mathbb{N}$, and let $\delta = L^{-1}$, and $\delta_i = L^{-100^{6-i}}$ for $i = 0, 1, 2, 3, 4, 5$. From the above discussions, by first taking h large enough, then taking L large (depending on h), we could make $\mathbb{P}[\mathcal{E}^c]$ arbitrarily small. Using this with Lemma 2.35, from (2.10), we get for any $\epsilon > 0$ we could take h large enough, such that

$$\limsup_{n\to\infty} \mathbb{P}\Big[\max_{a,b\in\mathbb{Z},|a|,|b|<hn^{2/3}/2} |\mu_{\langle 0,a\rangle,\langle n,b\rangle}(f) - \nu^{1/2}(f)| > 11h^{-1}\|f\|_\infty\Big] < \epsilon,$$

which implies the conclusion.

Now we prove the above claim. Assume that $\mathcal{E}$ holds, and fix a, b such that $|a|, |b| < hn^{2/3}$. Then by $\mathcal{T}'^c$, and monotonicity of geodesics, the geodesic $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ is contained in the rectangle whose four vertices are $\langle 0, 2hn^{2/3}\rangle$, $\langle 0, -2hn^{2/3}\rangle$, $\langle n, 2hn^{2/3}\rangle$, $\langle n, -2hn^{2/3}\rangle$.

Let b^+ be the smallest number with $b^+ \in \delta_0 n^{2/3}\mathbb{Z}$ and $b^+ \ge b$. We first show that

we can find $u^* \in \mathfrak{P}_1$ with $d(u^*) < 4h^{-1}n$, such that there exists $u \in \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ with $d(u) = d(u^*)$ and $ad(u) \le ad(u^*) \le ad(u) + 2\delta_0 n^{2/3}$, and $\Gamma_{u^*,\langle n,b^+\rangle}$ intersects $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ before $\mathbb{L}_{2h^{-1}n}$.

For this we argue by contradiction, and assume that no such u^* exists. We take α_5 as the smallest number such that $\alpha_5 \in \delta_5\mathbb{Z}$ and $\alpha_5 > h^{-1}$, and take $u_5 \in \mathfrak{P}_1 \cap \mathbb{L}_{\alpha_5 n}$, being the first one on or to the right of $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$. In other words, we take $u_5 = \langle \alpha_5 n, a_5\rangle$, where a_5 is the smallest number in $\delta_0 n^{2/3}\mathbb{Z}$ such that $2a_5 \ge ad(\Gamma_{\langle 0,a\rangle,\langle n,b\rangle} \cap \mathbb{L}_{\alpha_5 n})$. Then we have that $|a_5| \le 2hn^{2/3}$. Consider the path $\Gamma_{u_5,\langle n,b^+\rangle}$. Again by $\mathcal{T}'^c$ and monotonicity of geodesics, this path is restricted within the rectangle whose four vertices are $\langle 0, 4hn^{2/3}\rangle$, $\langle 0, -4hn^{2/3}\rangle$, $\langle n, 4hn^{2/3}\rangle$, $\langle n, -4hn^{2/3}\rangle$. Suppose it intersects $\mathbb{L}_{(\alpha_5+\delta^{-7}\delta_5)n}$ at vertex $w_5 = \langle (\alpha_5 + \delta^{-7}\delta_5)n, a_5'\rangle$, then $|a_5'| \le 4hn^{2/3}$. By $\mathcal{T}^c$ we have that

$$|a_5 - a_5'| < \delta^{-1}(\delta^{-7}\delta_5 n)^{2/3} < (\delta^{-7})^{5/6}(\delta_5 n)^{2/3} - 2c_0(\delta_5 n)^{2/3} < \delta^{-6}(\delta_5 n)^{2/3}.$$

Then we have $T_{u_5,w_5} \ge 4\delta^{-7}\delta_5 n - c_0\delta^{-6}(\delta_5 n)^{1/3}$ by $\mathcal{H}^c$. Note that for the path $\Gamma_{u_5,\langle n,b^+\rangle}$, by the assumption above, it is disjoint from $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ before $\mathbb{L}_{2h^{-1}n}$. As $\alpha_5 + \delta^{-7}\delta_5 < 2h^{-1}$, we have that Γ_{u_5,w_5} is disjoint from $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$. By $\mathcal{D}^c$, the event $\mathcal{D}^{\langle 0,a\rangle,\langle n,b\rangle}_{\delta^{-7},\delta_5 n,\alpha_5 n}$ does not happen, so there must exist $0 \le j_5 \le \delta^{-7}$, such that for the intersections of $\mathbb{L}_{(\alpha_5+j_5\delta_5)n}$ with $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and $\Gamma_{u_5,w_5} \subset \Gamma_{u_5,\langle n,b^+\rangle}$, their $\|\cdot\|_1$ distance is at least $2c_0(\delta_5 n)^{2/3}$.

We next take $\alpha_4 = \alpha_5 + j_5\delta_5$, and take $u_4 \in \mathfrak{P}_1 \cap \mathbb{L}_{\alpha_4 n}$, being the first one on or to the right of $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$. Using the same arguments we could find $0 \le j_4 \le \delta^{-7}$, such that for the intersections of $\mathbb{L}_{(\alpha_4+j_4\delta_4)n}$ with $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and $\Gamma_{u_4,\langle n,b^+\rangle}$, their $\|\cdot\|_1$ distance is at least $2c_0(\delta_4 n)^{2/3}$. Similarly we can find $0 \le j_2, j_3 \le \delta^{-7}$, and $\alpha_3 = \alpha_4 + j_4\delta_4$, $\alpha_2 = \alpha_3 + j_3\delta_3$, $\alpha_1 = \alpha_2 + j_2\delta_2$, and vertices u_3, u_2, u_1. For the intersections of $\mathbb{L}_{(\alpha_i+j_i\delta_i)n}$ with $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and $\Gamma_{u_i,\langle n,b^+\rangle}$, their $\|\cdot\|_1$ distance is at least $2c_0(\delta_i n)^{2/3}$, for

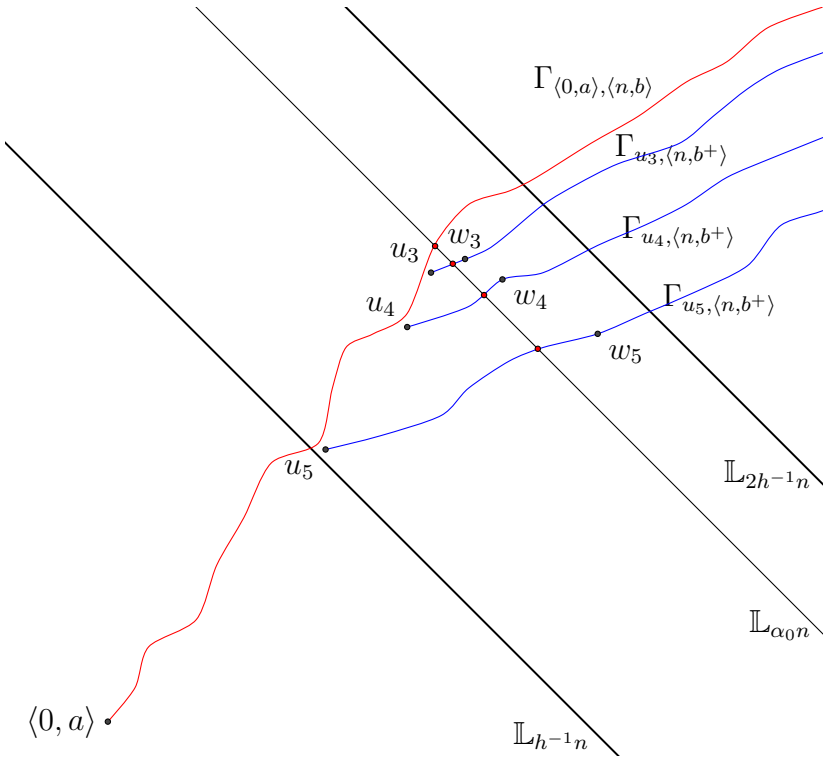


Figure 2.7: An illustration of the geodesics $\Gamma_{u_i,\langle n,b^+\rangle}$ for $i = 5, 4, 3$. Their intersections with $\mathbb{L}_{\alpha_0 n}$ are separated by $c_0(\delta_1 n)^{2/3}$.

$i = 1, 2, 3$ (see Figure 2.7).

Take $\alpha_0 = \alpha_1 + j_1\delta_1$. Suppose that the intersections of $\mathbb{L}_{\alpha_0 n}$ with $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and $\Gamma_{u_i,\langle n,b^+\rangle}$ are $\langle \alpha_0 n, b_0\rangle$ and $\langle \alpha_0 n, b_i\rangle$, for $i = 1, 2, 3, 4, 5$. By $\mathcal{T}^c$, and considering $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ above $\mathbb{L}_{\alpha_i n}$ and $\Gamma_{u_i,\langle n,b^+\rangle}$, we have $b_i - b_0 \le \delta_0 n^{2/3} + 2\delta^{-1}(\alpha_0 - \alpha_i)^{2/3} n^{2/3}$; and considering $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and $\Gamma_{u_i,\langle n,b^+\rangle}$ above $\mathbb{L}_{\alpha_{i-1} n}$, we have $b_i - b_0 \ge c_0(\delta_i n)^{2/3} - 2\delta^{-1}(\alpha_0 - \alpha_{i-1})^{2/3} n^{2/3}$. By the choice of our parameters we have $\alpha_0 - \alpha_i \le \delta^{-7}\sum_{i'=1}^{i} \delta_{i'} < 2\delta^{-7}\delta_i$. Then we have

$$c_0(\delta_i n)^{2/3} - 2\delta^{-1}(2\delta^{-7}\delta_{i-1} n)^{2/3} \le b_i - b_0 \le \delta_0 n^{2/3} + 2\delta^{-1}(2\delta^{-7}\delta_i n)^{2/3},$$

and we get that $b_i - b_{i-1} \ge c_0(\delta_1 n)^{2/3}$, for each $i = 1, 2, 3, 4, 5$. As $|b_0| \le 2hn^{2/3}$ by $\mathcal{T}'^c$, we have that $-2hn^{2/3} \le b_0 < b_5 \le 4hn^{2/3}$.

Let a^-, b^- be the largest numbers with $a^-, b^- \in \delta_0 n^{2/3}\mathbb{Z}$ and $a^- \le a$, $b^- \le b$. By

using $\mathcal{F}^c$ repeatedly, we have

$$\begin{aligned}
&T^*_{\langle 0,a^-\rangle,\langle \alpha_0 n,b_0\rangle} + T^*_{\langle \alpha_0 n,b_0\rangle,\langle n,b^-\rangle}\\
\geq &T^*_{\langle 0,a\rangle,\langle \alpha_0 n,b_0\rangle} + T^*_{\langle \alpha_0 n,b_0\rangle,\langle n,b\rangle} - 2\delta_0^{1/2-0.02}n^{1/3}\\
= &T^*_{\langle 0,a\rangle,\langle n,b\rangle} - 2\delta_0^{1/2-0.02}n^{1/3}\\
\geq &T^*_{\langle 0,a^-\rangle,\langle n,b^-\rangle} - 3\delta_0^{1/2-0.02}n^{1/3};
\end{aligned}$$

and for each $i = 1, 2, 3, 4, 5$, let u'_i be the intersection of $\mathbb{L}_{\alpha_i n}$ with $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$, we have

$$\begin{aligned}
&T^*_{\langle 0,a^-\rangle,\langle \alpha_0 n,b_i\rangle} + T^*_{\langle \alpha_0 n,b_i\rangle,\langle n,b^-\rangle}\\
\geq &T^*_{\langle 0,a\rangle,\langle \alpha_0 n,b_i\rangle} + T^*_{\langle \alpha_0 n,b_i\rangle,\langle n,b^+\rangle} - 2\delta_0^{1/2-0.02}n^{1/3}\\
= &T^*_{\langle 0,a\rangle,u'_i} + T^*_{u'_i,\langle \alpha_0 n,b_i\rangle} + T^*_{\langle \alpha_0 n,b_i\rangle,\langle n,b^+\rangle} - 2\delta_0^{1/2-0.02}n^{1/3}\\
\geq &T^*_{\langle 0,a\rangle,u'_i} + T^*_{u_i,\langle \alpha_0 n,b_i\rangle} + T^*_{\langle \alpha_0 n,b_i\rangle,\langle n,b^+\rangle} - 3\delta_0^{1/2-0.02}n^{1/3}\\
= &T^*_{\langle 0,a\rangle,u'_i} + T^*_{u_i,\langle n,b^+\rangle} - 3\delta_0^{1/2-0.02}n^{1/3}\\
\geq &T^*_{\langle 0,a\rangle,u'_i} + T^*_{u'_i,\langle n,b^+\rangle} - 4\delta_0^{1/2-0.02}n^{1/3}\\
= &T^*_{\langle 0,a\rangle,\langle n,b^+\rangle} - 4\delta_0^{1/2-0.02}n^{1/3}\\
\geq &T^*_{\langle 0,a^-\rangle,\langle n,b^-\rangle} - 5\delta_0^{1/2-0.02}n^{1/3}.
\end{aligned}$$

Note that if $\langle \alpha_0 n, b_0^-\rangle$ is the intersection of $\Gamma_{\langle 0,a^-\rangle,\langle n,b^-\rangle}$ with $\mathbb{L}_{\alpha_0 n}$, then $b_0^- \leq b_0$. Also note that $\alpha_0 \geq \alpha_5 > h^{-1}$, and $\alpha_0 \leq \alpha_5 + 2\delta^{-7}\delta_5 \leq h^{-1} + \delta_5 + 2\delta^{-7}\delta_5 < 2h^{-1}$. Thus the above inequalities contradict with $\mathcal{M}^c$. Thus we conclude that there exists $u^* \in \mathfrak{P}_1$ with $d(u^*) < 4h^{-1}n$, such that there is $u \in \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ with $d(u) = d(u^*)$ and $ad(u) \leq ad(u^*) \leq ad(u) + 2\delta_0 n^{2/3}$, and $\Gamma_{u^*,\langle n,b^+\rangle}$ intersects $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ before $\mathbb{L}_{2h^{-1}n}$.

Using the same arguments, there exists $v^* \in \mathfrak{P}_1$ with $d(v^*) > (2 - 4h^{-1})n$, such that there is $v \in \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ with $d(v) = d(v^*)$ and $ad(v) \leq ad(v^*) \leq ad(v) + 2\delta_0 n^{2/3}$, and Γ_{u^*,v^*} intersects $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ after $\mathbb{L}_{(1-2h^{-1})n}$. If Γ_{u^*,v^*} also intersects $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ before $\mathbb{L}_{2h^{-1}n}$, we must have Γ_{u^*,v^*} coincides with $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ between $\mathbb{L}_{2h^{-1}n}$ and $\mathbb{L}_{(1-2h^{-1})n}$, so

the claim follows; otherwise, Γ_{u^*,v^*} is disjoint from $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ before $\mathbb{L}_{2h^{-1}n}$, we must have that $\Gamma_{u^*,\langle n,b^+\rangle}$ intersects $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ after $\mathbb{L}_{(1-2h^{-1})n}$, since otherwise Γ_{u^*,v^*} and $\Gamma_{u^*,\langle n,b^+\rangle}$ would intersect twice, so we conclude that $\Gamma_{u^*,\langle n,b^+\rangle}$ coincides with $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ between $\mathbb{L}_{2h^{-1}n}$ and $\mathbb{L}_{(1-2h^{-1})n}$, and the claim also follows. □

For the next a few subsections we prove the lemmas used in the proof of Proposition 2.33.

2.7.1 Continuity of passage times and multiple peaks

In this subsection we prove Lemma 2.34 and 2.37. We start with the following estimate on deviations when moving one end point.

Lemma 2.38. *There are constants C, c such that the following holds. For any $h \in \mathbb{R}$, $\theta < 1 < t$, we have*

$$\mathbb{P}\left[\max_{hn^{2/3}<b,b'<(h+1)n^{2/3},|b-b'|<\theta n^{2/3}} |T_{\mathbf{0},\langle n,b\rangle} - T_{\mathbf{0},\langle n,b'\rangle}| > t\theta^{1/2-0.01}n^{1/3} + C(|h|+1)\theta n^{1/3}\right] < Ce^{-ct}$$

for n large enough (depending on h, θ, t).

Proof. By Theorem 1.17, it suffices to bound

$$\mathbb{P}\left[\max_{2^{-2/3}h<x,x'<2^{-2/3}(h+1),|x-x'|<2^{-2/3}\theta} |\mathcal{P}_1(x) - \mathcal{P}_1(x')| > 2^{-4/3}t\theta^{1/2-0.01} + 2^{-4/3}C(|h|+1)\theta\right].$$

When $C > 2$ we have $|x^2 - x'^2| < 2^{-4/3}C(|h|+1)\theta$ for all x, x' we take the max over. Then by stationarity of $\mathcal{A}_1$, we can bound this probability by

$$\mathbb{P}\left[\max_{0<x,x'<2^{-2/3},|x-x'|<2^{-2/3}\theta} |\mathcal{P}_1(x) - \mathcal{P}_1(x')| > 2^{-4/3}t\theta^{1/2-0.01}\right].$$

Note that the event now only relies on $\mathcal{P}_1 - \mathcal{P}_1(0)$ in the interval $[0, 2^{-2/3}]$. Using modulus of continuity for Brownian motions, and Theorem 1.15, we can bound this by Ce^{-ct} as desired. □

Proof of Lemma 2.34. First, note that we have the following inequality for passage times:

$$T_{\langle 0,a\rangle,\langle n,b\rangle} - T_{\langle 0,a\rangle,\langle n,b'\rangle} \geq T_{\langle 0,a'\rangle,\langle n,b\rangle} - T_{\langle 0,a'\rangle,\langle n,b'\rangle}$$

for any $a \leq a', b \leq b'$. Indeed, if we take the geodesics $\Gamma_{\langle 0,a\rangle,\langle n,b'\rangle}$ and $\Gamma_{\langle 0,a'\rangle,\langle n,b\rangle}$, then they must intersect. By switching the paths after their first intersection, we get two directed paths, between $\langle 0, a\rangle, \langle n, b\rangle$ and $\langle 0, a'\rangle, \langle n, b'\rangle$, and their total weight remains unchanged and equals $T_{\langle 0,a'\rangle,\langle n,b\rangle} + T_{\langle 0,a\rangle,\langle n,b'\rangle}$. Thus we get the above inequality from the definition of last-passage times.

From this, for any $|a|, |a'|, |b|, |b'| < hn^{2/3}$ we have

$$\begin{aligned}
|T_{\langle 0,a\rangle,\langle n,b\rangle} - T_{\langle 0,a'\rangle,\langle n,b'\rangle}| <& |T_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,b\rangle} - T_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,b'\rangle}| \\
&+ |T_{\langle 0,\lceil hn^{2/3}\rceil\rangle,\langle n,b\rangle} - T_{\langle 0,\lceil hn^{2/3}\rceil\rangle,\langle n,b'\rangle}| \\
&+ |T_{\langle 0,a\rangle,\langle n,-\lceil hn^{2/3}\rceil\rangle} - T_{\langle 0,a'\rangle,\langle n,-\lceil hn^{2/3}\rceil\rangle}| \\
&+ |T_{\langle 0,a\rangle,\langle n,\lceil hn^{2/3}\rceil\rangle} - T_{\langle 0,a'\rangle,\langle n,\lceil hn^{2/3}\rceil\rangle}|
\end{aligned}$$

By symmetry, now it suffices to bound

$$\mathbb{P}\left[\max_{\substack{|b|,|b'|<hn^{2/3}\\ |b-b'|<\theta n^{2/3}}} |T_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,b\rangle} - T_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,b'\rangle}| > \frac{1}{4}(t\theta^{1/2-0.01}n^{1/3} + Ch\theta n^{1/3})\right].$$

For this we split $\{\langle 0, b\rangle : |b| < hn^{2/3}\}$ into segments of length $n^{2/3}$, and apply Lemma 2.38 to each of them to get the desired bound. □

We next prove Lemma 2.37. Again, using Theorem 1.17 we reduce the problem to Airy_2 processes, and then by applying Theorem 1.15 we could just prove the result

for Brownian motions.

Proof of Lemma 2.37. Denote $\mathcal{P}_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ as the process given by

$$\mathcal{P}_{\alpha,\beta}(x) := \alpha^{1/3}\mathcal{P}_1(\alpha^{-2/3}x) + (1-\alpha)^{1/3}\mathcal{P}_1'((1-\alpha)^{-2/3}(x - 2^{-2/3}\beta)),$$

where $\mathcal{P}_1'$ is an independent copy of $\mathcal{P}_1$. By Theorem 1.17, it suffices to bound the probability of the following event: there exist $-2^{1/3}h < x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < 2^{1/3}h$, with $x_2 - x_1, x_3 - x_2, x_4 - x_3, x_5 - x_4, x_6 - x_5 > 2^{-2/3}\theta$, such that $x_1 = \operatorname{argmax}_{[-2^{4/3}h, 2^{4/3}h]} \mathcal{P}_{\alpha,\beta}$, and

$$\mathcal{P}_{\alpha,\beta}(x_1) < \mathcal{P}_{\alpha,\beta}(x_i) + 2^{-4/3}t\theta^{1/2}, \quad \forall i = 2,3,4,5,6.$$

One could replace $\mathcal{P}_{\alpha,\beta}$ by a (two-sided) Brownian motion with diffusivity $2\alpha^2 + 2(1-\alpha)^2$, and study the probability of the same event. By Lemma 2.39 below this probability is bounded by Ct^5 for C depending on h. Finally, we apply Theorem 1.15 and get the desired bound. □

We finally bound the events on Brownian motion.

Lemma 2.39. *There exists a constant $C > 0$, such that for any $t, \theta > 0$, the following event holds with probability at most Ct^5. For $W : [-2,2] \to \mathbb{R}$ being a two sided Brownian motion, there are $-1 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < 1$, with $x_2 - x_1, x_3 - x_2, x_4 - x_3, x_5 - x_4, x_6 - x_5 > \theta$, such that $x_1 = \operatorname{argmax}_{[-2,2]} W$, and*

$$W(x_1) < W(x_i) + t\theta^{1/2}, \quad \forall i = 2,3,4,5,6.$$

Proof. Fix $T_1 \in [-1,1]$, and let $\mathcal{E}$ be the event where $W(T_1) = \max_{[-2,2]} W$. For $i = 2,3,4,5,6$, let $T_i = \min\{x \geq T_{i-1} + \theta : W(x) \geq W(x_1) - t\theta^{1/2}\}$. It suffices to show that $\mathbb{P}[T_6 \leq 1 \mid \mathcal{E}] < Ct^5$ for some constant $C > 0$. For $i = 2,3,4,5,6$, conditioned

on $\mathcal{E}$ and the event $T_{i-1} \le 1$, and given the values of T_{i-1} and $W(T_{i-1}) - W(T_1)$, the process $x \mapsto W(T_{i-1} + x) - W(T_1)$ on $[0, 2 - T_{i-1}]$ has the same law of W', which is a Brownian motion on $[0, 2 - T_{i-1}]$ starting from $W'(0) = W(T_{i-1}) - W(T_1)$ and conditioned below zero (for $i = 2$ this degenerates to a Brownian meander). Using reflection principle we have that $\mathbb{P}[\max_{[\theta, 2-T_{i-1}]} W' \ge -t\theta^{1/2}] < C't$ for some $C' > 0$. So we have that $\mathbb{P}[T_i \le 1 \mid \mathcal{E}, T_{i-1} \le 1] < C't$. Since $T_i \le 1$ is implied by $T_{i-1} \le 1$ for each i, we have $\mathbb{P}[T_6 \le 1 \mid \mathcal{E}] < (C't)^5$, which implies the conclusion. □

2.7.2 Disjoint paths

In this subsection we prove Lemma 2.36. The idea is to show that for a path restricted to be close to another (deterministic) path for a while, its weight is unlikely to be small (compared to that of a geodesic with the same end points). We then use the FKG inequality to move from a deterministic path to a geodesic.

Lemma 2.40. *For sufficiently small $c_0 > 0$, there is $c_1 > 0$, such that for large enough l and any $r \in \mathbb{Z}$, we have*

$$\mathbb{E}\left[\max_{0 \le a, b \le c_0 l^{2/3}} T_{\langle 0,a\rangle, \langle l, r+b\rangle}\right] < 4l - c_1 l^{1/3}.$$

Proof. Take $u = \langle -\lfloor c_0^{3/2} l \rfloor, 0\rangle$ and $v = \langle l + \lfloor c_0^{3/2} l \rfloor, r'\rangle$, where r' is the number in $\lfloor c_0 l^{2/3} \rfloor \mathbb{Z}$ with $r \le r' \le r + c_0 l^{2/3}$. Note that

$$\mathbb{E}\left[\max_{0 \le a, b \le c_0 l^{2/3}} T_{\langle 0,a\rangle, \langle l, r+b\rangle}\right] \le \mathbb{E}T_{u,v} - \mathbb{E}\left[\min_{0 \le a \le c_0 l^{2/3}} T_{u, \langle -1, a\rangle}\right] - \mathbb{E}\left[\min_{0 \le b \le c_0 l^{2/3}} T_{\langle l+1, r+b\rangle, v}\right].$$

By Proposition 2.2, we have

$$\mathbb{E}\left[\min_{0 \le a \le c_0 l^{2/3}} T_{u, \langle -1, a\rangle}\right],\ \mathbb{E}\left[\min_{0 \le b \le c_0 l^{2/3}} T_{\langle l+1, r+b\rangle, v}\right] \ge 4c_0^{3/2} l - C c_0^{1/2} l^{1/3}$$

for some constant $C > 0$. We also claim that for l sufficiently large.

$$\mathbb{E}T_{u,v} \leq 4(l + 2c_0^{3/2}l) - C'l^{1/3} \tag{2.11}$$

for some $C' > 0$. When $l^{-2/3}|r| > C''$ for some C'' depending on C', (2.11) is by (2.3). When $l^{-2/3}|r| \leq C''$, for each l there are at most $3C''/c_0$ possible numbers r' can take. For each of them, by Theorem 1.17 the corresponding $T_{u,v}$ after rescaling converges (as $l \to \infty$) to one point of the Airy_2 process, whose law is given by the GUE Tracy-Widom distribution. Thus (2.11) follows since the GUE Tracy-Widom distribution has negative expectation. By choosing c_0 sufficiently small we complete the proof. □

For the next lemma, as above we denote $T^*_{u,v} = T_{u,v} - \omega(v)$ for any vertices $u \leq v \in \mathbb{Z}^2$.

Lemma 2.41. *For $l, M \in \mathbb{N}$, and any $r_0, \ldots, r_M \in \mathbb{Z}$ such that $|r_0 - r_M| < M^{5/6}l^{2/3}$, we have*

$$\mathbb{P}\left[\max_{0\leq a_0,\ldots,a_M\leq cl^{2/3}} \sum_{i=0}^{M-1} T^*_{\langle il, r_i+a_i\rangle, \langle (i+1)l, r_{i+1}+a_{i+1}\rangle} \geq 4Ml - cMl^{1/3}\right] < Ce^{-cM}$$

for some absolute constants $c, C > 0$, when l is large enough.

Proof. In this proof we let $C, c > 0$ be large and small enough constants, and their values can change from line to line.

We denote $S_i = \max_{0\leq a_i, a_{i+1}\leq c_0 l^{2/3}} T^*_{\langle il, r_i+a_i\rangle, \langle (i+1)l, r_{i+1}+a_{i+1}\rangle}$ for each $0 \leq i \leq M-1$. By Lemma 2.40, we have $\mathbb{E}[S_i] < 4l - c_1 l^{1/3}$, where $c_0, c_1 > 0$ are some constants.

Next we apply Proposition 2.2. When $|r_i - r_{i+1}| \leq 0.9l$ we could directly apply it; and when $|r_i - r_{i+1}| > 0.9l$, the slope condition may not be satisfied, thus we use the fact that $T^*_{\langle il, r_i+a_i\rangle, \langle (i+1)l, r_{i+1}+a_{i+1}\rangle} < T^*_{\langle il, r_i+a_i\rangle, \langle (i+1)l+\lfloor 0.1l\rfloor, r_{i+1}+a_{i+1}\rangle}$, and upper bound the later using Proposition 2.2. We conclude that $\mathbb{P}[S_i > 4l + xl^{1/3}] < Ce^{-cx}$, for any

$x > 0$.

Note that S_i for each i are independent. Thus by a Bernstein type bound on sum of independent variables with exponential tails, we have

$$\mathbb{P}\left[\max_{0\le a_0,\dots,a_M\le c_0 l^{2/3}} \sum_{i=0}^{M-1} T^*_{\langle il, r_i+a_i\rangle,\langle (i+1)l, r_{i+1}+a_{i+1}\rangle} \ge 4Ml - \frac{c_1}{2}Ml^{1/3}\right] < Ce^{-cM}.$$

Then the conclusion follows. □

Proof of Lemma 2.36. Take any directed path Γ from u to v, such that for the intersections of $\Gamma_{u,v}$ with $\mathbb{L}_m$ and $\mathbb{L}_{m+Ml}$, denoted as u', v', we have $|ad(u') - ad(v')| < 2M^{5/6}l^{2/3}$. Denote $\mathcal{D}_\Gamma$ as the following event: there exists a directed path γ from $\mathbb{L}_m$ to $\mathbb{L}_{m+Ml}$, such that

- γ is disjoint from Γ.
- The weight of γ (i.e. $T(\gamma)$) is at least $4Ml - c_0 Ml^{1/3}$.
- For each $i = 0, 1, \dots, M$, $\|\Gamma \cap \mathbb{L}_{m+il} - \gamma \cap \mathbb{L}_{m+il}\|_1 < 2c_0 l^{2/3}$.

Now we consider the event $\Gamma_{u,v} = \Gamma$. Under this event we have $\mathcal{D}^{\mathbf{0},\langle n,b\rangle}_{M,l,m} = \mathcal{D}_\Gamma$. Also, $\Gamma_{u,v} = \Gamma$ is a negative event of the field on $\mathbb{Z}^2 \setminus \Gamma$, while $\mathcal{D}_\Gamma$ is determined by the field on $\mathbb{Z}^2 \setminus \Gamma$, and is a positive event of the field on $\mathbb{Z}^2 \setminus \Gamma$. By the FKG inequality we have

$$\mathbb{P}[\mathcal{D}^{\mathbf{0},\langle n,b\rangle}_{M,l,m} \mid \Gamma_{u,v} = \Gamma] = \mathbb{P}[\mathcal{D}_\Gamma \mid \Gamma_{u,v} = \Gamma] \le \mathbb{P}[\mathcal{D}_\Gamma].$$

By Lemma 2.41, when $c_0 < c$ we have $\mathbb{P}[\mathcal{D}_\Gamma] < Ce^{-cM}$, for some constants $C, c > 0$. By averaging over all Γ we get the conclusion. □

2.8 Convergence of one point distribution

In this section we prove Theorem 1.3 and 1.4. The general idea is to show that the law of the value of a specific vertex in the geodesic is close to that of nearby vertices along

the geodesic; and this is achieved by a coalescing argument. Then we use Proposition 2.33 to argue that certain time average is close to the stationary one.

We start by deducing the following estimate on coalescence of geodesics, which is the main step towards proving Theorem 1.3. It directly follows from Proposition 2.4 and Lemma 2.6.

Lemma 2.42. *For each $\rho \in (0,1)$, there is a constant $C > 0$, such that for any $r \in \mathbb{N}$, and $k > 2$, we have $\mathbb{P}[\Gamma^\rho_{\mathbf{0}} \cap \mathbb{L}_{\lfloor rk \rfloor} \neq \gamma \cap \mathbb{L}_{\lfloor rk \rfloor}] < C \log(k) k^{-2/3}$, and $\mathbb{P}[\Gamma^\rho_{\mathbf{0}} \cap \mathbb{L}_{\lfloor rk \rfloor} \neq \gamma \cap \mathbb{L}_{\lfloor rk \rfloor}] < C \log(k) k^{-2/3}$, where $\gamma = \Gamma^\rho_{\langle \lfloor r/2 \rfloor, 0 \rangle_\rho}$ or $\gamma = \Gamma^\rho_{\langle \lfloor r/2 \rfloor, 0 \rangle_\rho + (1,0)}$.*

Proof. Denote the intersections of $\Gamma^\rho_{\mathbf{0}}$ and γ with $\mathbb{L}_r$ as $\langle r, b_r \rangle_\rho$ and $\langle r, b'_r \rangle_\rho$. By Lemma 2.6 and Proposition 2.4, there is a constant $C_0 > 0$ such that

$$\mathbb{P}[|b_r|, |b'_r| \leq C_0 \log(k) r^{2/3}] > 1 - C_0 k^{-1},$$

and

$$\mathbb{P}[\Gamma^\rho_{\langle r, -\lfloor C_0 \log(k) r^{2/3} \rfloor - 1 \rangle_\rho} \cap \mathbb{L}_{\lfloor rk \rfloor} \neq \Gamma^\rho_{\langle r, \lfloor C_0 \log(k) r^{2/3} \rfloor + 1 \rangle_\rho} \cap \mathbb{L}_{\lfloor rk \rfloor}] < C_0^2 \log(k) (k-1)^{-2/3}.$$

Thus the conclusion follows by monotonicity of geodesics. □

Proof of Theorem 1.3. Take any $s \in \mathbb{N}$ and any measurable function $f : \mathbb{R}^{[\![-s,s]\!]} \to [0,1]$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2}$, we shall show that $\lim_{i \to \infty} \mathbb{E} f(\omega\{\Gamma^\rho_{\mathbf{0}}[i]\}) = \nu^\rho(f)$.

For $i, r \in \mathbb{N}$ and $k > 2$ with $i > 2rk$, we consider $\Gamma^\rho_{\mathbf{0}}$ with $\gamma = \Gamma^\rho_{\langle \lfloor r/2 \rfloor, 0 \rangle_\rho + (1,0)}$ (when r is even) or $\gamma = \Gamma^\rho_{\langle \lfloor r/2 \rfloor, 0 \rangle_\rho + (1,0)}$ (when r is odd). By Lemma 2.42, we have $\omega\{\Gamma^\rho_{\mathbf{0}}[i]\} = \omega\{\gamma[i-r]\}$, with probability at least $1 - C \log(k) k^{-2/3}$. Since $\omega\{\gamma[i-r]\}$ has the same law as $\omega\{\Gamma^\rho_{\mathbf{0}}[i-r]\}$, we must have that

$$|\mathbb{E} f(\omega\{\Gamma^\rho_{\mathbf{0}}[i]\}) - \mathbb{E} f(\omega\{\Gamma^\rho_{\mathbf{0}}[i-r]\})| \leq C \log(k) k^{-2/3}.$$

By averaging over r for $0 \le r \le i/4k$, we have

$$|\mathbb{E}f(\omega\{\Gamma^\rho_{\mathbf{0}}[i]\}) - \mathbb{E}\mu_{\Gamma^\rho_{\mathbf{0}}[i-\lfloor i/4k\rfloor],\Gamma^\rho_{\mathbf{0}}[i]}(f)| \le C\log(k)k^{-2/3}.$$

By Lemma 2.6, and Proposition 2.33, for any fixed $k > 0$, we have $\mu_{\Gamma^\rho_{\mathbf{0}}[i-\lfloor i/4k\rfloor],\Gamma^\rho_{\mathbf{0}}[i]}(f) \to \nu^\rho(f)$ in probability as $i \to \infty$. Thus we have that $\limsup_{i\to\infty} |\mathbb{E}f(\omega\{\Gamma^\rho_{\mathbf{0}}[i]\}) - \nu^\rho(f)| \le C\log(k)k^{-2/3}$. Since k can be arbitrarily large, the conclusion holds. □

The proof of Theorem 1.4 is similar, while for finite geodesics we use Lemma 2.5 and Proposition 2.3 instead of Lemma 2.6 and Proposition 2.4.

Lemma 2.43. *For each $\rho \in (0,1)$, there is a constant $C > 0$, such that for any $r, n \in \mathbb{N}$ and $k > 2$, with $n \ge 2rk$, we have $\mathbb{P}[\Gamma_{\mathbf{0},\mathbf{n}^\rho} \cap \mathbb{L}_{\lfloor rk\rfloor} \ne \gamma \cap \mathbb{L}_{\lfloor rk\rfloor}] < C\log(k)k^{-2/3}$, and $\mathbb{P}[\Gamma_{\mathbf{0},\mathbf{n}^\rho} \cap \mathbb{L}_{n-\lfloor rk\rfloor} \ne \gamma \cap \mathbb{L}_{n-\lfloor rk\rfloor}] < C\log(k)k^{-2/3}$, where $\gamma = \Gamma_{\langle\lfloor r/2\rfloor,0\rangle_\rho+(1,0),\mathbf{n}^\rho+\langle\lfloor r/2\rfloor,0\rangle_\rho+(1,0)}$ or $\gamma = \Gamma_{\langle\lfloor r/2\rfloor,0\rangle_\rho,\mathbf{n}^\rho+\langle\lfloor r/2\rfloor,0\rangle_\rho}$.*

Proof. Since $n \ge 2rk$, we just show $\mathbb{P}[\Gamma_{\mathbf{0},\mathbf{n}^\rho} \cap \mathbb{L}_{\lfloor rk\rfloor} \ne \gamma \cap \mathbb{L}_{\lfloor rk\rfloor}] < Ck^{-2/3}$, and by symmetry the other inequality would follow.

Denote the intersections of $\Gamma_{\mathbf{0},\mathbf{n}^\rho}$ and γ with $\mathbb{L}_r$ as $\langle r, b_-\rangle_\rho$ and $\langle r, b'_-\rangle_\rho$, respectively; and the intersections of $\Gamma_{\mathbf{0},\mathbf{n}^\rho}$ and γ with $\mathbb{L}_{n-r}$ as $\langle n-r, b_+\rangle_\rho$ and $\langle n-r, b'_+\rangle_\rho$, respectively. There is a constant $C_0 > 0$, such that

$$\mathbb{P}[|b_-|, |b'_-| \le C_0\log(k)r^{2/3}],\ \mathbb{P}[|b_+|, |b'_+| \le C_0\log(k)r^{2/3}] > 1 - C_0k^{-1}$$

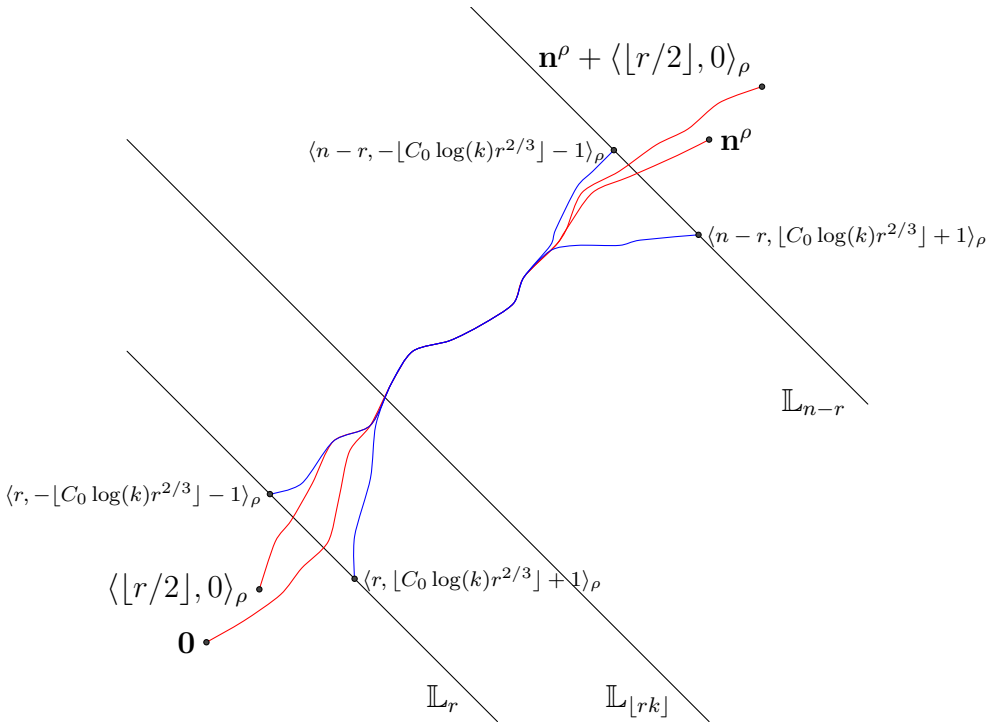


Figure 2.8: An illustration of the proof of Lemma 2.43. The geodesics $\Gamma_{\mathbf{0},\mathbf{n}^\rho}$ and $\Gamma_{\langle \lfloor r/2 \rfloor,0\rangle_\rho,\mathbf{n}^\rho+\langle \lfloor r/2 \rfloor,0\rangle_\rho}$ are sandwiched between $\Gamma_{\langle r,-\lfloor C_0 \log(k) r^{2/3} \rfloor-1\rangle_\rho,\langle n-r,-\lfloor C_0 \log(k) r^{2/3} \rfloor-1\rangle_\rho}$ and $\Gamma_{\langle r,\lfloor C_0 \log(k) r^{2/3} \rfloor+1\rangle_\rho,\langle n-r,\lfloor C_0 \log(k) r^{2/3} \rfloor+1\rangle_\rho}$.

by Lemma 2.5; and

$$
\begin{aligned}
&\mathbb{P}[\Gamma_{\langle r,-\lfloor C_0 \log(k) r^{2/3} \rfloor-1\rangle_\rho,\langle n-r,-\lfloor C_0 \log(k) r^{2/3} \rfloor-1\rangle_\rho} \cap \mathbb{L}_{\lfloor rk \rfloor} \\
&\quad \neq \Gamma_{\langle r,\lfloor C_0 \log(k) r^{2/3} \rfloor+1\rangle_\rho,\langle n-r,\lfloor C_0 \log(k) r^{2/3} \rfloor+1\rangle_\rho} \cap \mathbb{L}_{\lfloor rk \rfloor}] \\
\leq &\mathbb{P}[\Gamma_{\langle r,-\lfloor C_0 \log(k) r^{2/3} \rfloor-1\rangle_\rho,\langle n-r,-\lfloor C_0 \log(k) r^{2/3} \rfloor-1\rangle_\rho} \cap \mathbb{L}_{\lfloor rk \rfloor} \\
&\quad \neq \Gamma_{\langle r,-\lfloor C_0 \log(k) r^{2/3} \rfloor-1\rangle_\rho,\langle n-r,\lfloor C_0 \log(k) r^{2/3} \rfloor+1\rangle_\rho} \cap \mathbb{L}_{\lfloor rk \rfloor}] \\
&\quad + \mathbb{P}[\Gamma_{\langle r,-\lfloor C_0 \log(k) r^{2/3} \rfloor-1\rangle_\rho,\langle n-r,\lfloor C_0 \log(k) r^{2/3} \rfloor+1\rangle_\rho} \cap \mathbb{L}_{\lfloor rk \rfloor} \\
&\quad \neq \Gamma_{\langle r,\lfloor C_0 \log(k) r^{2/3} \rfloor+1\rangle_\rho,\langle n-r,\lfloor C_0 \log(k) r^{2/3} \rfloor+1\rangle_\rho} \cap \mathbb{L}_{\lfloor rk \rfloor}] \\
< &C_0^2 \log(k)(k-1)^{-2/3},
\end{aligned}
$$

where the last inequality is by Proposition 2.3. Then the conclusion follows by monotonicity of geodesics (see Figure 2.8). □

Proof of Theorem 1.4. Take any $s \in \mathbb{N}$ and any measurable function $f : \mathbb{R}^{[\![-s,s]\!]} \to$

$[0,1]$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2}$, now we shall show that $\lim_{n\to\infty} \mathbb{E}f(\omega\{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor]\}) = \nu^\rho(f)$.

Without loss of generality we assume that $\alpha \leq 1$. For $n, r \in \mathbb{N}$ and $k > 2$ with $\alpha n > 2rk$, we consider $\gamma = \Gamma_{\langle \lfloor r/2\rfloor,0\rangle_\rho, \mathbf{n}^\rho + \langle \lfloor r/2\rfloor,0\rangle_\rho}$ (when r is even) or $\gamma = \Gamma_{\langle \lfloor r/2\rfloor,0\rangle_\rho + (1,0), \mathbf{n}^\rho + \langle \lfloor r/2\rfloor,0\rangle_\rho + (1,0)}$ (when r is odd). By Lemma 2.43, we have

$$\mathbb{P}[\omega\{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor]\} = \omega\{\gamma[\lfloor \alpha n\rfloor - r]\}] \geq 1 - C\log(k)k^{-2/3}.$$

Since $\omega\{\gamma[\lfloor \alpha n\rfloor - r]\}$ has the same law as $\omega\{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor - r]\}$, we must have that

$$|\mathbb{E}f(\omega\{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor]\}) - \mathbb{E}f(\omega\{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor - r]\})| \leq C\log(k)k^{-2/3}.$$

By averaging over r for $0 \leq r \leq \alpha n/4k$, we have

$$|\mathbb{E}f(\omega\{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor]\}) - \mathbb{E}\mu_{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor - \lfloor \alpha n/4k\rfloor], \Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor]}(f)| \leq C\log(k)k^{-2/3}.$$

By Lemma 2.5 and Proposition 2.33, for fixed k we have $\mu_{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor - \lfloor \alpha n/4k\rfloor], \Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor]}(f) \to \nu^\rho(f)$ in probability as $n \to \infty$. Thus we have that $\limsup_{i\to\infty} |\mathbb{E}f(\omega\{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n\rfloor]\}) - \nu^\rho(f)| \leq C\log(k)k^{-2/3}$. Then the conclusion follows since k is arbitrarily taken. □

2.9 Exponential concentration via counting argument

Using a covering argument, we can prove the following exponential concentration of the empirical distribution, for both finite or semi-infinite geodesics.

Proposition 2.44. *For each $\rho \in (0,1)$ $s \in \mathbb{Z}_{\geq 0}$, and any bounded function $f : \mathbb{R}^{[\![-s,s]\!]^2} \to \mathbb{R}$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2}$, and any $\epsilon > 0$, we have*

$$\mathbb{P}[|\mu^\rho_{\mathbf{0};r}(f) - \nu^\rho(f)| > \epsilon] < Ce^{-cr},$$

for r large enough, and $C, c > 0$ depending on ρ, s, f, ϵ.

Proposition 2.45. *Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of integers such that $\lim_{n\to\infty} n^{-2/3}|b_n| < \infty$. Then for any $s \in \mathbb{Z}_{\geq 0}$, any bounded function $f : \mathbb{R}^{[\![-s,s]\!]^2} \to \mathbb{R}$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2}$, and any $\epsilon > 0$, we have*

$$\mathbb{P}[|\mu_{\mathbf{0},\langle n,b_n\rangle_\rho}(f) - \nu^\rho(f)| > \epsilon] < Ce^{-cn},$$

for n large enough, and $C, c > 0$ depending on ρ, s, f, ϵ.

From Proposition 2.44 we can deduce Theorem 1.2

Proof of Theorem 1.2. By Proposition 2.44, for any bounded $f : \mathbb{R}^{[\![-s,s]\!]^2} \to \mathbb{R}$ and $\epsilon > 0$, we have that $\sum_{r\in\mathbb{N}} \mathbb{P}[|\mu^\rho_{\mathbf{0};r}(f) - \nu^\rho(f)| > \epsilon] < \infty$; so almost surely, there exists some (random) r_0 such that $|\mu^\rho_{\mathbf{0};r}(f) - \nu^\rho(f)| \leq \epsilon$ for any $r > r_0$. Thus we have that $\mu^\rho_{\mathbf{0};r}(f) \to \nu^\rho(f)$ almost surely. The conclusion follows by taking all $s \in \mathbb{N}$, and f over all characteristic functions $\mathbb{1}[\prod_{i\in[\![-s,s]\!]^2}(-\infty, x_i]]$, for $\{x_i\}_{i\in[\![-s,s]\!]^2} \in \mathbb{Q}^{[\![-s,s]\!]^2}$. □

To prove these exponential concentration bounds (Proposition 2.44 and 2.45), we cover the geodesics with short finite ones, and use Proposition 2.33.

For the rest of this section we fix $\rho \in (0, 1)$. We take $m \in \mathbb{N}$, and assume that $m^{2/3} \in \mathbb{Z}$. For each $i, j \in \mathbb{Z}$ we denote $L_{i,j}$ as the segment joining $\langle im, (2j+1)m^{2/3}\rangle_\rho$ and $\langle im, (2j+1)m^{2/3}\rangle_\rho$. For each sequence $j_0, j_1, \ldots, j_k$, we let $P_{j_0,\ldots,j_k}$ be the collection of paths from L_{0,j_0} to L_{k,j_k}, intersecting each L_{i,j_i}, $0 \leq i \leq k$. For any $k \in \mathbb{N}$ and $D > 0$, we also denote $P_{k,D}$ as the union of all $P_{j_0,j_1,\ldots,j_k}$ such that $j_0 = 0$ and $\sum_{i=1}^k (j_i - j_{i-1})^2 > Dk$.

Lemma 2.46. *There exists $c_0 > 0$, such that when m, k, D are large enough,*

$$\mathbb{P}\left[\exists\gamma \in P_{k,D}, T(\gamma) > \frac{2km}{(1-\rho)^2+\rho^2} - (b_+ - b_-)(\rho^{-1} - (1-\rho)^{-1}) - c_0 Dk m^{1/3}\right] < e^{-c_0 k},$$

where $b_-, b_+ \in \mathbb{Z}$ such that $\langle 0, b_- \rangle, \langle km, b_+ \rangle_\rho$ are the intersections of γ with $\mathbb{L}_0, \mathbb{L}_{km}$, respectively.

Proof. First, by Theorem 2.2 and (2.3), and fundamental computations, there exist $C_1, c_1 > 0$ such that for m large enough and any $j \in \mathbb{Z}$, $x > 0$,

$$\mathbb{E}\left[\max_{\substack{\langle 0,b\rangle_\rho \in L_{0,0},\\ \langle m,b'\rangle_\rho \in L_{1,j}}} T_{\langle 0,b\rangle_\rho, \langle m,b'\rangle_\rho} + (b'-b)(\rho^{-1}-(1-\rho)^{-1})\right] < \frac{2m}{(1-\rho)^2+\rho^2} + (C_1 - c_1 j^2)m^{1/3},$$

$$\mathbb{P}\left[\max_{\substack{\langle 0,b\rangle_\rho \in L_{0,0},\\ \langle m,b'\rangle_\rho \in L_{1,j}}} T_{\langle 0,b\rangle_\rho, \langle m,b'\rangle_\rho} + (b'-b)(\rho^{-1}-(1-\rho)^{-1}) > \frac{2m}{(1-\rho)^2+\rho^2} + (x - c_1 j^2)m^{1/3}\right] < C_1 e^{-c_1 x}.$$

Note that

$$\max_{\gamma \in P_{j_0, j_1, \ldots, j_k}} T(\gamma) \le \sum_{i=1}^{k-1} \max_{\substack{u \in L_{i-1, j_{i-1}},\\ v \in L_{i,j_i}}} T^*_{u,v} + \max_{\substack{u \in L_{k-1, j_{k-1}},\\ v \in L_{k,j_k}}} T_{u,v}$$

Here $T^*_{u,v} = T_{u,v} - \omega(v)$ for any $u \le v \in \mathbb{Z}^2$. Then by a Bernstein type estimate for independent random variables with exponential tails, we have

$$\mathbb{P}\left[\max_{\gamma \in P_{j_0, j_1, \ldots, j_k}} T(\gamma) + (b_+ - b_-)(\rho^{-1} - (1-\rho)^{-1}) > \frac{2mk}{(1-\rho)^2+\rho^2} - \frac{c_1}{2} D k m^{1/3}\right] < C_2 e^{-c_2\left(Dk + \sum_{i=1}^k (j_i - j_{i-1})^2\right)},$$

for any $D > 1$ and $j_0 = 0$, $\sum_{i=1}^k (j_i - j_{i-1})^2 > Dk$, where $C_2, c_2 > 0$ are constants, $\langle 0, b_- \rangle, \langle km, b_+ \rangle_\rho$ are the intersections of γ with $\mathbb{L}_0, \mathbb{L}_{km}$. Summing over all such sequences $j_0, j_1, \ldots, j_k$, the right hand side is bounded by

$$C_2 e^{-c_2 Dk} \left(\sum_{j \in \mathbb{Z}} e^{-c_2 j^2}\right)^k.$$

By taking D large so that $e^{c_2 D/2} > \sum_{j\in\mathbb{Z}} e^{-c_2 j^2}$, we get the conclusion. □

Proof of Proposition 2.44. For any $u \le v \in \mathbb{Z}^2$, denote

$$\mu^*_{u,v} := \frac{1}{|\Gamma_{u,v}| - 1} \sum_{w\in\Gamma_{u,v}, w\neq v} \delta_{\omega\{w\}};$$

i.e. it is the empirical distribution along $\Gamma_{u,v}$, excluding the last vertex v. Without loss of generality we also assume that $0 \le f \le 1$.

Take D large enough, and then m large, as required by Lemma 2.46; and also m is large enough such that

$$\mathbb{P}\left[\max_{|a|,|b|<\epsilon^{-2}m^{2/3}} |\mu^*_{\langle 0,a\rangle_\rho,\langle m,b\rangle_\rho}(f) - \nu^\rho(f)| > \epsilon^2\right] < \varepsilon,$$

by Proposition 2.33. Here ε is a small number depending on D, ϵ and to be determined. Take a sequence $j_0, \dots, j_k$, such that $j_0 = 0$ and $\sum_{i=1}^k (j_i - j_{i-1})^2 \le Dk$. We let $I' \subset \{1, \dots, k\}$ be the collection of indices such that $|j_i - j_{i-1}| < \epsilon^{-2}/2 - 1$ for each $i \in I'$. Then $|I'| > (1 - \epsilon/2)k$, when ϵ is small enough (depending on D). Next we let $I \subset I'$ such that for each $i \in I$,

$$\max_{u\in L_{i-1,j_{i-1}}, v\in L_{i,j_i}} |\mu^*_{u,v}(f) - \nu^\rho(f)| \le \epsilon^2.$$

For each $i \in I'$ we have $i \in I$ with probability at least $1 - \varepsilon$, and this is independent for each i. Then by Chernoff bound and taking ε small enough (depending on D, ϵ), we can make $\mathbb{P}[|I'| - |I| > \epsilon^2 k] < (D+1)^{-2k}$. Let γ be the path consisting of the first $2km + 1$ vertices of $\Gamma^\rho_{\mathbf{0}}$; i.e. γ is the part of $\Gamma^\rho_{\mathbf{0}}$ on and between $\mathbb{L}_0$ and $\mathbb{L}_{km}$. Given that $\gamma \in P_{j_0,\dots,j_k}$, and $|I'| - |I| \le \epsilon^2 k$, for any $r \in \mathbb{N}$ with $km \le r < (k+1)m$ we must have that $|\mu^\rho_{\mathbf{0};r}(f) - \nu^\rho(f)| \le \epsilon/2 + \epsilon^2 + 1/(k+1)$. So when $k > \epsilon^{-2}$ we have

$$\mathbb{P}\left[\gamma \in P_{j_0,\dots,j_k}, |\mu^\rho_{\mathbf{0};r}(f) - \nu^\rho(f)| > \epsilon\right] < (D+1)^{-2k}.$$

Thus by summing over all sequences $j_0, \dots, j_k$ with $j_0 = 0$, $\sum_{i=1}^{k}(j_i - j_{i-1})^2 \le Dk$, we have

$$\mathbb{P}\left[\gamma \notin P_{k,D}, |\mu^{\rho}_{\mathbf{0};r}(f) - \nu^{\rho}(f)| > \epsilon\right] < \binom{Dk+k-1}{k-1}(D+1)^{-2k} < e^{-ck}$$

for some $c > 0$ depending on D.

Now it remains to bound $\mathbb{P}[\gamma \in P_{k,D}]$. By Lemma 2.46, we have

$$\begin{aligned}&\mathbb{P}[\gamma \in P_{k,D}]\\ &\quad< e^{-c_0 k} + \mathbb{P}\left[T(\gamma) \le \frac{2km}{(1-\rho)^2+\rho^2} - b_+(\rho^{-1} - (1-\rho)^{-1}) - c_0 Dkm^{1/3}\right], \quad (2.12)\end{aligned}$$

where $\langle km, b_+\rangle_\rho$ is the intersection of $\Gamma^{\rho}_{\mathbf{0}}$ with $\mathbb{L}_{km}$. When the event in the right hand side of (2.12) happens, we must have that (at least) one of the following happens:

- $|b_+| > km^{2/3}$,
- $\max_{|b|\le km^{2/3}} B^{\rho}(\langle km, b\rangle_\rho, \langle km, 0\rangle_\rho) - b(\rho^{-1} - (1-\rho)^{-1}) \ge c_0 Dkm^{1/3}/3$,
- $T^{*}_{\mathbf{0},\langle km,0\rangle_\rho} \le \frac{2km}{(1-\rho)^2+\rho^2} - c_0 Dkm^{1/3}/2$,

where $T^{*}_{\mathbf{0},\langle km,0\rangle_\rho} = T_{\mathbf{0},\langle km,0\rangle_\rho} - \omega(\langle km, 0\rangle_\rho)$. To see this, we assume the contrary, i.e. none of the above three events happen. If we let the first intersection of $\Gamma^{\rho}_{\mathbf{0}}$ with $\Gamma^{\rho}_{\langle km,0\rangle_\rho}$ be w, then we must have

$$\begin{aligned}T(\gamma) > T_{\mathbf{0},w} - T_{\langle km,b_+\rangle_\rho,w} &\ge T^{*}_{\mathbf{0},\langle km,0\rangle_\rho} + T_{\langle km,0\rangle_\rho,w} - T_{\langle km,b_+\rangle_\rho,w}\\ &> \frac{2km}{(1-\rho)^2+\rho^2} - b_+(\rho^{-1} - (1-\rho)^{-1}) - 5c_0 Dkm^{1/3}/6,\end{aligned}$$

which contradicts with the event in the right hand side of (2.12). Finally, we claim that we can bound the probability of each of the three events by $e^{-c'k}$, for $c' > 0$ and k large enough, depending on m, D. For the first event the bound is by Lemma 2.6.

For the second event, note that $b \mapsto B^\rho(\langle km, b\rangle_\rho, \langle km, 0\rangle_\rho) - b(\rho^{-1} - (1-\rho)^{-1})$ is a (two-sided) centered random walk; for the third event, use Theorem 2.1. Thus the conclusion follows. □

Proof of Proposition 2.45. The first half of this proof follows the same way as the proof of Proposition 2.44, and we conclude that the following is true when D is large enough and then m is large enough. Suppose that $km \le n < (k+1)m$ for some $k \in \mathbb{N}$, $k > \epsilon^{-2}$. Let γ be the path from $\mathbb{L}_0$ to $\mathbb{L}_{km}$, consisting of the first $2km+1$ vertices of $\Gamma_{\mathbf{0},\langle n, b_n\rangle_\rho}$. Then we have

$$\mathbb{P}\left[\gamma \notin P_{k,D}, |\mu_{\mathbf{0},\langle n,b_n\rangle_\rho}(f) - \nu^\rho(f)| > \epsilon\right] < \binom{Dk+k-1}{k-1}(D+1)^{-2k} < e^{-ck} \quad (2.13)$$

for some $c > 0$ depending on D. It remains to bound $\mathbb{P}[\gamma \in P_{k,D}]$. By Lemma 2.46, we have

$$\mathbb{P}[\gamma \in P_{k,D}] < e^{-c_0 k} + \mathbb{P}\left[T(\gamma) \le \frac{2km}{(1-\rho)^2+\rho^2} - b_+(\rho^{-1} - (1-\rho)^{-1}) - c_0 Dkm^{1/3}\right],$$

where $\langle km, b_+\rangle_\rho$ is the intersection of $\Gamma_{\mathbf{0},\langle n,b_n\rangle_\rho}$ with $\mathbb{L}_{km}$. When the event in the right hand side of (2.13) happens, we must have that (at least) one of the following happens:

- $\max_{b\in\mathbb{Z}} T_{\langle km,b\rangle_\rho,\langle n,b_n\rangle_\rho} - (b-b_n)(\rho^{-1}-(1-\rho)^{-1}) \ge c_0 Dkm^{1/3}/3$,
- $T^*_{\mathbf{0},\langle km,b_n\rangle_\rho} \le \frac{2km}{(1-\rho)^2+\rho^2} - b_n(\rho^{-1}-(1-\rho)^{-1}) - c_0 Dkm^{1/3}/2$,

where $T^*_{\mathbf{0},\mathbf{r}^\rho} = T_{\mathbf{0},\mathbf{r}^\rho} - \omega(\mathbf{r}^\rho)$. To see this, we assume the contrary, i.e. none of the above events happen. Then we must have

$$\begin{aligned} T(\gamma) > T_{\mathbf{0},\langle n,b_n\rangle_\rho} - T_{\langle km,b_+\rangle_\rho,\langle n,b_n\rangle_\rho} &\ge T^*_{\mathbf{0},\langle km,b_n\rangle_\rho} - T_{\langle km,b_+\rangle_\rho,\langle n,b_n\rangle_\rho} \\ &> \frac{2km}{(1-\rho)^2+\rho^2} - b_+(\rho^{-1}-(1-\rho)^{-1}) - 5c_0Dkm^{1/3}/6, \end{aligned}$$

which contradicts with the event in the right hand side of (2.13). Finally, we claim that we can bound the probability of each of the two events by $e^{-c'k}$, for $c' > 0$ and k large enough, depending on m, D. For the first event, note that $n - km < m$, then $|b - b_n| < 2m$, and the bound can be obtained by taking a union bound over all directed paths from $\mathbb{L}_{km}$ to $\langle n, b_n \rangle_\rho$. For the second event, apply Theorem 2.1. Thus the conclusion follows. □

Chapter 3

Fractal geometry of the space-time difference profile in the directed landscape

3.1 A roadmap of the proofs

This chapter is devoted to proving the results in Section 1.3, which mainly concerns the spatial-temporal behavior of the difference profile $\mathcal{D}$. As mentioned above, such study involves several new ideas, which we describe in more details now.

We first setup some conventions throughout this chapter. We use $\mathscr{L}$ to denote the Lebesgue measure on $\mathbb{R}$. We would need to consider the space of all non-empty compact subsets of $\mathbb{R}^2$ which equipped with the Hausdorff metric, becomes a complete metric space, with the induced topology referred to as the Hausdorff topology. We will also freely use the notation argmax even when there isn't a unique maximum (but there must exist at least one) using it to denote an arbitrary one.

We next recall the definitions of the Hausdorff dimension and Hausdorff measure of a set.

Definition 3.1. For any $d \geq 0$ and metric space X, the d-dimensional Hausdorff measure of X is defined as

$$\lim_{\delta \searrow 0} \inf \left\{ \sum_i \operatorname{diam}(U_i)^d : \{U_i\} \text{ is a countable cover of } X \text{ with } 0 < \operatorname{diam}(U_i) < \delta \right\}.$$

The Hausdorff dimension of X is

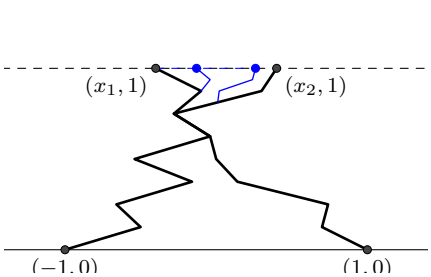
$$\inf\{d > 0 : \text{ the } d\text{-dimensional Hausdorff measure of } X \text{ is zero } \}.$$

We start by stating two general strategies to obtain upper and lower bounds for the Hausdorff dimension for X being a subset of $\mathbb{R}^n$ (for some $n \in \mathbb{N}$).

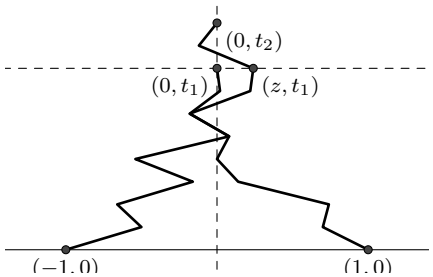
- Upper bound: Show that for any $U \subset \mathbb{R}^n$, $\mathbb{P}[X \cap U \neq \emptyset] < \operatorname{diam}(U)^{n-d}$ (potentially with some extra logarithmic factors). Then for any compact $S \subset \mathbb{R}^n$, on dividing S into roughly δ^{-n} subsets, each with diameter $\leq \delta$, X is covered by the union of roughly δ^{-d} of them. Thus for any $\epsilon > 0$, the $d + \epsilon$-dimensional Hausdorff measure of X is zero. So the Hausdorff dimension of X is at most d.

- Lower bound: This is usually the harder direction and involves constructing a measure m whose support is contained in X, such that $m(X) > 0$, and for any measurable $U \subset X$ the Hölder estimate $m(U) < \operatorname{diam}(U)^d$ (potentially with extra sub-polynomial factors) holds. Then for any countable cover $\{U_i\}$ of X, one must have $\sum_i \operatorname{diam}(U_i)^d > m(X)$, implying that the Hausdorff dimension of X is at least d.

Before diving into our arguments, for contrast, let us first briefly sketch the nature of the arguments appearing in the analysis of $\mathcal{D}(\cdot, t)$ in [10] (from here on we will fix $t = 1$) and why they fail in the case of $\mathcal{D}(x, \cdot)$ (we will fix $x = 0$). A key starting observation in [10] is the following (also see Figure 3.1a): for two points $x_1 < x_2$, $\mathcal{D}(x, 1)$ is constant on $[x_1, x_2]$ if the geodesics $\pi_{(-1,0;x_1,1)}$ $\pi_{(1,0;x_2,1)}$ coalesce.



(a) If the geodesics $\pi_{(-1,0;x_1,1)}$ and $\pi_{(1,0;x_2,1)}$ coalesce, any geodesic from $(-1, 0)$ or $(1, 0)$ to any $(x, 1)$ with $x \in [x_1, x_2]$ must stay with $\pi_{(-1,0;x_1,1)}$ or $\pi_{(1,0;x_2,1)}$ up to the end of their intersection. This implies that $\mathcal{D}(x, 1)$ is constant on $[x_1, x_2]$.



(b) If all geodesics from $(-1, 0)$ and $(1, 0)$ to (x, t_1) coalescence, for x in an order $(t_2-t_1)^{2/3}$ neighborhood of 0, the process $\mathcal{D}(0, \cdot)$ is (likely to be) constant on $[t_1, t_2]$.

Figure 3.1: Illustrations for the arguments in [10] (for upper bounding the Hausdorff dimension), and the same strategy in the temporal direction.

Thus to prove an upper bound on the Hausdorff dimension of $\overline{\mathsf{NC}}_1$, [10] uses as input the following estimate from [64]: For any $x_1 < x_2$, $y_1 < y_2$, and $s < t$, if we let $\mathrm{Disj}^{[s,t]}_{(x_1,x_2)\to(y_1,y_2)}$ be the event that $\pi_{(x_1,s;y_1,t)}$ and $\pi_{(x_2,s;y_2,t)}$ are unique and disjoint, then

Theorem 3.2. *There is a constant C, such that for any $\epsilon > 0$, we have*

$$\mathbb{P}[\mathrm{Disj}^{[0,1]}_{(-1,1)\to(-\epsilon,\epsilon)}] < C\epsilon^{1/2}e^{C|\log(\epsilon)|^{5/6}}.$$

Thus, for any small ϵ, ignoring sub-polynomial factors, the probability that $\overline{\mathsf{NC}}_1$ intersects any interval of size ϵ is at most $\epsilon^{1/2}$, which then leads to an upper bound of $1/2$ on the Hausdorff dimension, according to the above general strategy.

For the lower bound, the argument relies on Hölder regularity of $\mathcal{L}$. Namely that the Parabolic Airy_2 process is locally Hölder $1/2-$ regular (in fact it is globally parabolic and locally Brownian, see e.g., [36, 63, 30]) and so is $\mathcal{D}(\cdot, 1)$ being the difference of two Airy_2 processes rooted at ± 1. By interpreting $-\mathcal{D}(\cdot, 1)$ as the distribution function of a measure and the above general strategy, we get a Hausdorff

dimension lower bound of 1/2.

It is instructive to apply now the same strategy to analyze the process $\mathcal{D}(0,\cdot)$ instead (also see Figure 3.1b). For two points $t_1 < t_2$, consider the geodesics $\pi_{(-1,0;0,t_1)}$ and $\pi_{(1,0;0,t_2)}$. Let the latter intersect the line $t = t_1$ at location z, then by a similar argument as before, one can essentially conclude (the precise conclusion is somewhat different which we will not spell out here) that $\mathcal{D}(0,\cdot)$ is constant on $[t_1, t_2]$ unless the geodesics $\pi_{(-1,0;0,t_1)}$ and $\pi_{(1,0;z,t_1)}$ are disjoint. Now, it is known, and goes back to the work of Johansson [72] (see also [15, 65, 43]), that the geodesics in LPP and hence $\mathcal{L}$ are 2/3− Hölder regular, which then implies that $z \approx |t_2 - t_1|^{2/3}$. This in conjunction with the above estimate on existence of disjoint geodesics imply that the probability that $\widehat{\mathsf{NC}}$ intersects $[t_1, t_2]$ is at most, up to sub-polynomial factors, $|t_2 - t_1|^{1/3}$ which translates into a Hausdorff dimension upper bound of 2/3 for $\widehat{\mathsf{NC}}$.

On the other hand, applying the same reasoning as before, we now seek to use the fact that $\mathcal{L}$ is 1/3− Hölder continuous in the temporal direction, implying at least as much regularity for $\mathcal{D}(0,\cdot)$. This then yields, again by the above general strategy, a lower bound of 1/3 for the Hausdorff dimension which unlike the spatial case, falls short of matching the upper bound.

The reason for the disparity is that the process $\mathcal{D}(0,\cdot)$ is a mean zero process and hence not monotone. In the spatial case, when applying the above general strategy, the process $\mathcal{D}(\cdot,1)$ was itself inducing a measure owing to monotonicity, whereas the process $\mathcal{D}(0,\cdot)$ induces a signed measure instead. Thus the argument which essentially works with the "absolute value" of the measure, or the "un-signed" measure fails to capture certain cancellations exhibited by $\mathcal{D}(0,\cdot)$, which we expect leads to a square-root fluctuation effect boosting the lower bound from 1/3 to 2/3.

Thus to obtain a sharp lower bound we will seek to construct new measures capturing the appropriate cancellation. This involves many new ideas and is quite technical. We will review the key steps in the sequel but first let us digress a bit allowing us to

introduce a few new objects which will play central roles in our arguments.

Geodesic local-time: As is the case in Brownian motion, the zero set of the semi-infinite geodesic $\pi_{\mathbf{0}}$ denoted by Z, will turn out to be a particularly fundamental object. As indicated in Section 1.2, we will construct a random measure induced by the function L supported on Z which we will term the geodesic local time, and show that it is 1/3 Hölder-regular using which we will show that Z has Hausdorff dimension 1/3 (as stated in Theorem 1.11). This will follow from the following estimate, whose proof will be briefly reviewed at the end of this section:

Lemma 3.3. *There are universal constants $C, c > 0$ such that the following is true. For any closed interval $I \subset \mathbb{R}$ and $J \subset \mathbb{R}_{\geq 0}$, we have*

$$\mathbb{P}[\mathscr{L}(\{t \in J : \pi_{\mathbf{0}}(t) \in I\}) > M\mathscr{L}(I)\mathscr{L}(J)^{1/3}] < Ce^{-cM}$$

for any $M > 0$.

Competition interfaces: The next object that we need to consider pertains to level sets. As is perhaps not surprising from the fact that we are investigating the behavior of the difference profile $\mathcal{D}$, it would be particularly convenient to consider level sets of the form

$$\vartheta_\ell(t) = \inf\{x \in \mathbb{R} : \mathcal{D}(x, t) \leq \ell\}. \tag{3.1}$$

It turns out however that a remarkable identity holds if we were to take $\ell = 0$ and consider the difference profile not rooted at ± 1, but with Brownian initial data instead. Namely, let $\mathcal{B}$ be a standard two sided Brownian motion. For any $x \in \mathbb{R}, t >$

0, we let

$$\mathcal{L}^L(x,t) = \sup_{y\leq 0} \mathcal{L}(y,0;x,t) + \mathcal{B}(y),$$
$$\mathcal{L}^R(x,t) = \sup_{y\geq 0} \mathcal{L}(y,0;x,t) + \mathcal{B}(y),$$

be the passage times induced by the Brownian initial data from the negative and positive parts of the $x-$axis respectively. One can show that the function $x \mapsto \mathcal{L}^L(x,t) - \mathcal{L}^R(x,t)$ is non-increasing in x (see the beginning of Section 3.4). We then let

$$\vartheta^{\mathcal{B}}(t) = \inf\{x \in \mathbb{R} : \mathcal{L}^L(x,t) - \mathcal{L}^R(x,t) \leq 0\}. \tag{3.2}$$

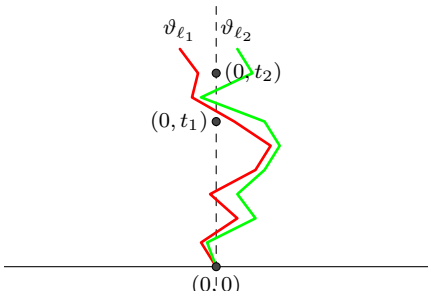
One then has the remarkable duality result:

Proposition 3.4. *The processes $\vartheta^{\mathcal{B}}$ and $\pi_{\mathbf{0}}$ are equal in distribution.*

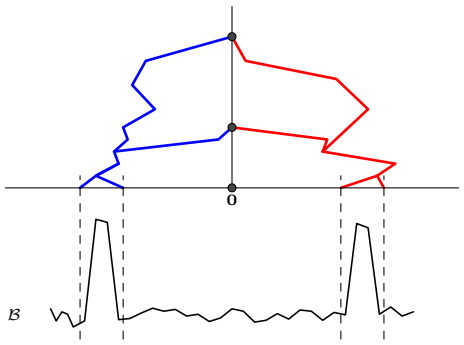
In the discrete setting (of exponential LPP), this goes back to [50, 7, 49], and we get the above directed landscape version using a limit transition from [44]. It is also developed in [90] independently, using a similar procedure.

Now, how does this help us? Suppose for the moment that the same were true for the level sets $\vartheta_\ell(\cdot)$ for all ℓ, in place of $\vartheta^{\mathcal{B}}(\cdot)$. Recalling that we only seek to prove a lower bound of 2/3 for the Hausdorff dimension of $\widehat{\mathsf{NC}}$, we will now construct a measure supported on $\widehat{\mathsf{NC}}$ which will be 2/3 Hölder-regular. At a very high level, this is achieved by taking the average of the local time for each of the level sets. The construction proceeds as in the case of the geodesic local time. Now using the above assumed comparison between level sets and the geodesic, it follows that each of the individual level set local times are 1/3 Hölder regular using Theorem 1.11. Given this, we obtain the desired 2/3 Hölder regularity, using Hölder regularity of $\mathcal{L}$.

To see this, consider an interval $[t_1, t_2]$. Now each ϑ_ℓ induces a 1/3 Hölder -regular measure κ_ℓ. To define this we start by defining the corresponding occupation measures



(a) If both ϑ_{ℓ_1} and ϑ_{ℓ_2} intersect the interval $\{0\} \times [t_1, t_2]$, likely $|\ell_2 - \ell_1|$ is in the order of at most $|t_2 - t_1|^{1/3}$.



(b) An illustration of the event where the Brownian motion $\mathcal{B}$ has a very spiky behavior with spikes near ± 1, and geodesic coalescence.

Figure 3.2

analogous to (1.4). For each $h > 0$, we denote by $\lambda_{\ell,h}$ the measure on $\mathbb{R}$, such that

$$\lambda_{\ell,h}(A) = \int_0^h \mathbb{1}[\vartheta_\ell(t) \in A]dt, \tag{3.3}$$

for any measurable $A \subset \mathbb{R}$. Given this, κ_ℓ on an interval (g, h) is then defined to be the density of $\lambda_{\ell,h} - \lambda_{\ell,g}$ at 0, that is,

$$\kappa_\ell((g,h)) = \lim_{w \searrow 0} (2w)^{-1}(\lambda_{\ell,h}([-w,w]) - \lambda_{\ell,g}([-w,w])). \tag{3.4}$$

(The actual definition proceeds a bit differently owing to certain measure theoretic considerations which we will ignore in this discussion). Define now the measure κ by setting

$$\kappa(A) = \int \kappa_\ell(A)d\ell. \tag{3.5}$$

This is the measure that we use to lower bound the Hausdorff dimension of $\widehat{\mathsf{NC}}$ by showing that κ is supported in $\widehat{\mathsf{NC}}$ and further for any interval $[t_1, t_2]$, $\kappa([t_1, t_2]) \approx |t_2 - t_1|^{2/3}$. Very briefly, the latter holds because by definition $\kappa([t_1, t_2]) = \int \kappa_\ell([t_1, t_2])d\ell$. Now by the claimed 1/3 Hölder regularity of the geodesic local time and the assumed

comparison between κ_ℓ and the same, the integrand is at most $|t_2 - t_1|^{1/3}$. Now by the $1/3$ Hölder regularity of the landscape in the temporal direction (a consequence of $1/3$ being the universal exponent governing weight fluctuations in the KPZ universality class), for ϑ_{ℓ_1} and ϑ_{ℓ_2} to both intersect the interval $\{0\} \times [t_1, t_2]$, we must have, again ignoring sub-polynomial factors, $|\ell_2 - \ell_1| \leq |t_2 - t_1|^{1/3}$ (see Figure 3.2a), which yields the desired estimate $\kappa([t_1, t_2]) = \int \kappa_\ell([t_1, t_2]) = O(|t_2 - t_1|^{2/3})$ since the integrand is non-zero only on a interval of size at most $|t_2 - t_1|^{1/3}$.

Note that the above reasoning is contingent on the assumption that the measures κ_ℓ are Hölder $1/3$ regular which was a consequence of the claim that the level curves ϑ_ℓ are "similar" to the geodesic $\pi_{\mathbf{0}}$. We rely on Proposition 3.4 to prove a result of this kind which suffices for our purpose. At a very high level, the arguments in this part is based on the observation that if the Brownian motion $\mathcal{B}$ has a very spiky behavior with spikes near ± 1 respectively as in Figure 3.2b then this resembles the original situation of considering the difference profile rooted at ± 1 respectively. In particular, in presence of such spikes, owing to geodesic coalescence, $\vartheta^{\mathcal{B}}$ is the same as ϑ_ℓ for some random ℓ, where the law of the latter dominates a multiple of Lebsegue measure owing to the fluctuation of the difference in the height of the spikes at ± 1, which can be obtained by resampling suitable parts of $\mathcal{B}$.

Theorem 1.9 is proved in the same way as Theorem 1.10 and is in fact somewhat easier technically since instead of working with the "density" measures in (3.4), we work with the occupation measures in (3.3) instead and the "occupation" counterpart of κ by taking their integral over different values of ℓ. The same set of arguments then concludes the proof. In this case the measure we work with is simply ζ, such that for any $a < b$ and $0 \leq g < h$, we let

$$\zeta([a, b] \times (g, h)) = \int_g^h [\mathcal{D}(a, t) - \mathcal{D}(b, t)]dt = \int_\ell [\lambda_{\ell,h}([a, b]) - \lambda_{\ell,g}([a, b])]d\ell. \tag{3.6}$$

Now when $b-a = h-g \approx \epsilon$, using the same argument as before one can conclude that $\zeta([a,b] \times (g,h)) \approx \epsilon^{5/3}$ leading to the desired lower bound on the dimension.

The remainder of the discussion is devoted to providing a brief overview of the ideas going into the other novel aspects of the chapter. This includes the construction of the semi-infinite geodesic $\pi_{\mathbf{0}}$ as well as the proof of the duality relation with the competition interface (3.2), and finally the construction of the geodesic local time and the counterpart for the level curves (3.1). In [90] the first two ingredients have been developed independently using somewhat related ideas. The semi-infinite geodesics and competition interfaces are also constructed in the recent works [96, 29], in the directed landscape or the pre-limiting model of Brownian LPP.

Construction of $\pi_{\mathbf{0}}$: The construction relies on geodesic coalescence. To construct a semi-infinite geodesic we show that the finite geodesics $\pi_{(p;z_n,h_n)}$, for any sequences $\{z_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ satisfying that $z_n \to \infty$ as $n \to \infty$ and $|z_n - h_n r| < h_n^{2/3}$ for each $n \in \mathbb{N}$, owing to geodesic coalescence, all share the same initial segment. Further, the length of the initial segment itself goes off to infinity as $n \to \infty$. $\pi_{\mathbf{0}}$ is then defined to be the path made up of such initial segments.

While the construction is intuitive, it does involve some amount of work to show that it works. Namely it has the desired direction. We further show that there is a unique such semi-infinite geodesic.

We will not comment much on how the above is carried out except that we rely on a construction of the Busemann function in the directed landscape setting (again relying on geodesic coalescence). We then show that (for fixed time) the Busemann function is a Brownian motion and that $\pi_{\mathbf{0}}(t)$, is the argmax of the sum of the Busemann function and an independent copy of an appropriately scale Parabolic Airy_2 process. By the known Brownianity of both the functions and their independence, it follows that $\pi_{\mathbf{0}}(t)$ is unique, at least for rational t and hence for all t by continuity.

Further, it follows by estimates of Brownian motion as well as the Parabolic Airy_2 process, that $\pi_{\mathbf{0}}(t) \approx O(t^{2/3})$ and thereby implying that $\pi_{\mathbf{0}}$ indeed has the desired direction.

Proof of duality: The duality is known to hold in the discrete model of Exponential LPP. To prove the directed landscape version, we rely on the recent convergence results proved in [44], which in particular implies that both the semi-infinite geodesic and the competition interfaces in the discrete setting converge to their counterparts $\pi_{\mathbf{0}}$ and $\vartheta^{\mathcal{B}}$ which is enough to conclude the proof. To be completely precise, however, the results in [44] only can be used to deduce convergence of certain "compactified" versions of the above objects. To make use of this, we work with certain truncations of $\pi_{\mathbf{0}}$ and $\vartheta^{\mathcal{B}}$, and their discrete versions, show that with high probability the truncated versions agree with the untruncated ones, the discrete ones converge to the continuous counterparts and hence the discrete duality transfers to the desired continuous duality statement.

Construction of the geodesic local time: Akin to the construction of the Brownian local time, this proceeds by first constructing the occupation measure for $\pi_{\mathbf{0}}$ as defined in (1.4) at height h, by setting for any measurable $A \subset \mathbb{R}$,

$$\nu_h(A) = \int_0^h \mathbb{1}[\pi_{\mathbf{0}}(t) \in A]dt. \tag{3.7}$$

All that remains is to show that ν_h has a density at 0, (see the Proposition 1.12), i.e., almost surely, the limit $\lim_{w \searrow 0}(2w)^{-1}\nu_h([-w, w])$ exists and is finite. While the finiteness is a consequence of the estimate in Lemma 3.3, we briefly explain the idea behind the proof of the existence of a density. The starting point is the observation that, by Lebesgue's theorem for the differentiability of monotone functions, for any

measure on $\mathbb{R}$ that is locally finite, almost every point is a point of density. However, a priori, this set of density points (which is a random set) could avoid 0. We now want to show that almost surely that is not the case. Now this would be immediate if the occupation measure ν_h was translation invariant in law, i.e., for any real number x, $\nu_h(\cdot) \stackrel{d}{=} \nu_h(\cdot + x)$, since any translation invariant set must contain a given point, in this case 0, with probability one. The proof now proceeds by constructing ν'_h, a translation invariant "proxy" for ν_h, namely considering the occupation measure of a collection of semi-infinite geodesics started from a Poisson process of intensity one. The latter measure does admit a density at 0 almost surely given a sample of the Poisson process and the landscape $\mathcal{L}$. Now owing to geodesic coalescence and that there is a positive probability that the Poisson process consists of a point quite close to the origin and no other points in the vicinity, it follows that the occupation measure of the two measures agree, (at least away from 0), i.e.,

$$\nu'_h - \nu'_g = \nu_h - \nu_g$$

for any $0 < g < h$. This is enough to finish the proof once we show that ν_g is negligible if g is small enough.

We finish with a brief discussion on the proof of Lemma 3.3. A slightly more general statement in the discrete setting of Exponential LPP had already appeared previously in [92]. For completeness we include the arguments adapted to the setting of the Directed Landscape and describe some of the ideas here.

To begin, a simple argument shows that $\mathscr{L}(I)\mathscr{L}(J)^{1/3}$ is the correct order of the random variable of interest, namely $\mathscr{L}(\{t \in J : \pi_{\mathbf{0}}(t) \in I\})$. This is because, by KPZ transversal fluctuation considerations, for any height t, $\pi_{\mathbf{0}}(t)$ is distributed roughly uniformly on an interval $t^{2/3}$ and hence, if $J = [a, b]$,

$$\mathbb{E}[\mathscr{L}(\{t \in J : \pi_{\mathbf{0}}(t) \in I\})] \approx \int_a^b \frac{\mathscr{L}(I)}{t^{2/3}} \lesssim \mathscr{L}(I)\mathscr{L}(J)^{1/3}$$

The proof of the lemma now relies on the above expectation bound for "all" I and J and dominating $\mathscr{L}(\{t \in J : \pi_{\mathbf{0}}(t) \in I\}$ by a sum of independent random variables and appealing to standard concentration of measure results. However the geodesic does not admit a decomposition into independent segments. To get around this, one uses the observation that a similar expectation bound as above holds when one considers the maximum occupation time, say among all geodesics with endpoints outside $I \times J$. Note that such a class of geodesics in particular includes segments of the geodesic $\pi_{\mathbf{0}}$ crossing such a box but crucially such a quantity across different boxes is independent.

Thus the occupation time in a box can be dominated by an independent sum of occupation times in smaller boxes with the same width, which by scale invariance, corresponds to the original box with a larger width. At this point we use induction, assuming the desired exponential tail for larger widths (which is straightforward to establish for a large enough width) and standard concentration bounds to conclude the same for a smaller width, going down in dyadic scales.

3.2 Properties of the directed landscape

In this section we record some properties of the directed landscape, that we will be using throughout this chapter.

3.2.1 Basic symmetries and estimates

There are several symmetries of the directed landscape.

Lemma 3.5 ([42, Lemma 10.2], [44, Proposition 1.23])**.** *As continuous functions on*

$\mathbb{R}^4_\uparrow$, $\mathcal{L}$ *is equal in distribution as*

$$(x,s;y,t) \mapsto \mathcal{L}(-y,-t;-x,-s), \quad (x,s;y,t) \mapsto \mathcal{L}(-x,s,-y,t);$$

and

$$(x,s;y,t) \mapsto \mathcal{L}(x+cs+z,s+r;y+ct+z,t+r)+(t-s)^{-1}((x-y-c(t-s))^2-(x-y)^2)$$

for any $c,z,r \in \mathbb{R}$*; and*

$$(x,s;y,t) \mapsto w\mathcal{L}(w^{-2}x,w^{-3}s;w^{-2}y,w^{-3}t);$$

for any $w>0$*. We shall refer to these invariances as flip invariance, skew-shift invariance (or translation invariance when* $c=0$*), and scaling invariance, respectively.*

A basic property of the directed landscape is the following quadrangle inequality, which is equivalent to Lemma 1.8. This and versions of it in different settings has been proved and widely used in many previous studies [42, 45, 44, 10].

Lemma 3.6 ([42, Lemma 9.1])**.** *For any* $s<t$*,* $x_1<x_2$*,* $y_1<y_2$*, we have*

$$\mathcal{L}(x_1,s;y_1,t)+\mathcal{L}(x_2,s;y_2,t) \geq \mathcal{L}(x_1,s;y_2,t)+\mathcal{L}(x_2,s;y_1,t).$$

We next state the following basic regularity estimates on $\mathcal{L}$. Recall that $\|\cdot\|$ denotes the usual l^2 norm.

Lemma 3.7 ([42, Corollary 10.7])**.** *There is a random number* R *such that the following is true. First, for any* $M>0$ *we have* $\mathbb{P}[R>M]<Ce^{-cM^{3/2}}$ *for some universal constants* $c,C>0$*. Second, for any* $u=(x,s;y,t) \in \mathbb{R}^4_\uparrow$*, we have*

$$\left|\mathcal{L}(x,s;y,t)+\frac{(x-y)^2}{t-s}\right| \leq R(t-s)^{1/3}\log^{4/3}(2(\|u\|+2)^{3/2}/(t-s))\log^{2/3}(\|u\|+2).$$

We remark that the logarithmic factors come from the exponential tails of $\mathcal{L}(u)$ for any fixed $u \in \mathbb{R}^4_\uparrow$, continuity estimates of $\mathcal{L}$, and a union bound and one cannot hope to obtain a uniform estimate without such factors due to the translation invariance and space-time decorrelation of $\mathcal{L}$. We will also use the following continuity estimate of $\mathcal{L}$.

Lemma 3.8 ([42, Proposition 1.6])**.** *Let K be a compact subset of $\mathbb{R}^4_\uparrow$. There is a random number R depending only on K, such that the following is true. First, for any $M > 0$ we have $\mathbb{P}[R > M] < Ce^{-cM^{3/2}}$ for some universal constants $c, C > 0$. Second, for any $(x, s; y, t), (x', s'; y', t') \in K$, we have*

$$\left|\left(\mathcal{L}(x,s;y,t) + \frac{(x-y)^2}{t-s}\right) - \left(\mathcal{L}(x',s';y',t') + \frac{(x'-y')^2}{t'-s'}\right)\right| \leq R\left(\tau^{1/3}\log^{2/3}(1+\tau^{-1}) + \xi^{1/2}\log^{1/2}(1+\xi^{-1})\right),$$

where $\tau = |s-s'| \vee |t-t'|$ and $\tau = |x-x'| \vee |y-y'|$.

Like in the exponential LPP setting, (using these regularity estimates above) in the directed landscape we get the following estimates on transversal fluctuation of geodesics, which is uniform in the endpoints.

Lemma 3.9. *There is a random number R such that the following is true. First, for any $M > 0$ we have $\mathbb{P}[R > M] < Ce^{-cM^{9/4}\log(M)^{-4}}$ for some constants $c, C > 0$. Second, for any $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$, any geodesic π_u, and $(s+t)/2 \leq r < t$, we have*

$$\left|\pi_u(r) - \frac{x(t-r) + y(r-s)}{t-s}\right| < R(t-r)^{2/3}\log^3(1 + \|u\|/(t-r)). \tag{3.8}$$

Similar bound holds when $s < r < (s+t)/2$ by symmetry.

Estimates similar to this have appeared in the literature, see e.g. [42, Proposition 12.3], and [56, Theorem 1.4] for a uniform in endpoints estimate (in the Brownian

LPP setting). The proof of Lemma 3.9 does not contain particularly new ideas, and relies on computations using Lemma 3.7 and Lemma 3.8. For completeness, we give the details in Appendix A.

The next subsection records several estimates concerning the geometry of geodesics in $\mathcal{L}$.

3.2.2 Intersections of geodesics

We begin with an estimate on the number of intersections that geodesics can have with a line.

Lemma 3.10. *For any $\alpha \in (0,1)$, there are constants $c, C > 0$ such that the following is true. For any $M > \ell^3$, we have $\mathbb{P}[|\{\pi_{(x,0;y,1)}(\alpha) : |x|, |y| \le \ell, \pi_{(x,0;y,1)} \text{is unique}\}| > M] < Ce^{-cM^{1/384}}$.*

This can be deduced from a similar result in the exponential LPP setting, proved in [13]. The condition "$\pi_{(x,0;y,1)}$ is unique" is assumed due to technical reasons, and is used in passing the geodesics in exponential LPP to the limit. In fact, while adapting the arguments of [13] to the directed landscape setting, we can prove a stronger version of Lemma 3.10 by removing this extra condition, we choose to not pursue this for the simplicity of the proof. The current formulation would suffice for applications in this thesis.

Proposition 3.11 ([13, Proposition 3.10])**.** *Let A_n and B_n be line segment along the lines $\mathbb{L}_0$ and $\mathbb{L}_n$, with midpoints $(xn^{2/3}, -xn^{2/3})$ and (n,n), respectively, and each has length $2n^{2/3}$. Let $F_{n,m} = \left(\bigcup_{u\in A_n, v\in B_n} \Gamma_{u,v}\right) \cap \mathbb{L}_m$, for $0 < m < n$. For any $\psi \in (0,1)$, $\alpha \in (0,1/2)$, there is a constant $c > 0$, such that when $|x| < \psi n^{1/3}$ and $\alpha n < m < (1-\alpha)n$, we have*

$$\mathbb{P}[|F_{n,m}| > M] < e^{-cM^{1/128}}$$

for any sufficiently large $M < n^{0.01}$ and sufficiently large n.

We remark that while the statement of [13, Proposition 3.10] is for $\alpha = 1/3$, the proof therein goes through essentially verbatim for a general $\alpha \in (0, 1)$.

Using this and the limit transition from Theorem 1.14, we can prove Lemma 3.10.

Proof of Lemma 3.10. Below we let $C, c > 0$ be constants depending on α, whose values may change from line to line.

Let $A_{n,\ell}$ and $B_{n,\ell}$ be the line segments along the lines $\mathbb{L}_0$ and $\mathbb{L}_n$, with midpoints $\mathbf{0}$ and (n, n), respectively, and length $2^{5/3}\ell n^{2/3}$. Let $F_{n,m,\ell} = \left(\bigcup_{u\in A_{n,\ell}, v\in B_{n,\ell}} \Gamma_{u,v}\right) \cap \mathbb{L}_m$, for $0 < m < n$. Then by splitting $A_{n,\ell}$ and $B_{n,\ell}$ into segments of length $2n^{2/3}$, and applying Proposition 3.11 to each pair, we get the following result. For any $M > 0$, and any n large enough (depending on ℓ and M), we have

$$\mathbb{P}[|F_{n,\lfloor \alpha n\rfloor,\ell}| > \ell^2 M] < C\ell^2 e^{-cM^{1/128}}.$$

Thus, when $M > \ell^3$, we have

$$\mathbb{P}[|F_{n,\lfloor \alpha n\rfloor,\ell}| > M] < Ce^{-cM^{1/384}}.$$

Finally, consider the coupling from Theorem 1.14. Suppose that in the directed landscape, we have $|\{\pi_{(x,0;y,1)}(\alpha) : |x|, |y| \le \ell, \pi_{(x,0;y,1)}\text{is unique}\}| > M$ for some $M > \ell^3$. Then for any n large enough, one would have $|F_{n,\lfloor \alpha n\rfloor,\ell}| > M$, since any unique geodesic $\pi_{(x,0;y,1)}$, is the Hausdorff limit of a sequence of pre-limiting geodesics. Thus the conclusion follows. □

3.2.3 Ordering of geodesics

We recall the notion of leftmost geodesics from [42]: for any $(x, s; y, t) \in \mathbb{R}^4_\uparrow$, a geodesic π from (x, s) to (y, t) is called the leftmost geodesic, if for any other geodesic τ from

(x, s) to (y, t), there is $\tau(t') \geq \pi(t')$ for any $t' \in [s, t]$. By [42, Lemma 13.2], almost surely, for any $(x, s; y, t) \in \mathbb{R}^4_\uparrow$ there is a leftmost geodesic from (x, s) to (y, t). We also have the following ordering property.

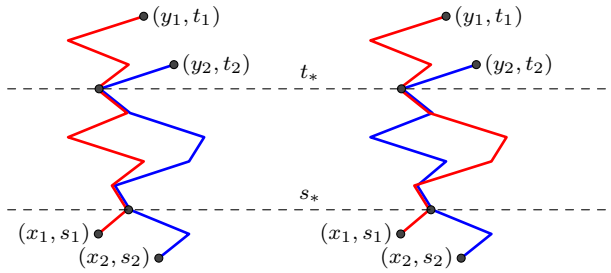


Figure 3.3: An illustration of the proof of Lemma 3.12: the left figure depicts $\pi_{(x_1,s_1;y_1,t_1)}$ and $\pi_{(x_2,s_2;y_2,t_2)}$, and the right figure depicts π'_1 (the red path) and π'_2 (the blue path), which are alternative candidates for the geodesics from (x_1, s_1) to (y_1, t_1) and from (x_2, s_2) to (y_2, t_2), respectively.

Lemma 3.12. *Almost surely the following is true. For any* $(x_1, s_1; y_1, t_1), (x_2, s_2; y_2, t_2) \in \mathbb{R}^4_\uparrow$ *with* $s_1 \vee s_2 \leq t_1 \wedge t_2$, *the set* $O = \{t' \in \mathbb{R} : \pi_{(x_1,s_1;y_1,t_1)}(t') = \pi_{(x_2,s_2;y_2,t_2)}(t')\}$ *is either empty or a closed interval, if one of the following two conditions hold:*

1. *both* $\pi_{(x_1,s_1;y_1,t_1)}$ *and* $\pi_{(x_2,s_2;y_2,t_2)}$ *are the leftmost geodesics;*
2. *at least one of* $\pi_{(x_1,s_1;y_1,t_1)}$ *and* $\pi_{(x_2,s_2;y_2,t_2)}$ *is the unique geodesic (between its endpoints).*

Proof. We assume that $O \neq \emptyset$. The denote $s_* = \inf O$ and $t_* = \sup O$. As the geodesics are continuous, we have that $s_*, t_* \in O$. Now we define $\pi'_1 : [s_1, t_1] \to \mathbb{R}$ and $\pi'_2 : [s_2, t_2] \to \mathbb{R}$ as follows: for any $t' \in [s_1, t_1] \setminus [s_*, t_*]$, we let $\pi'_1(t') = \pi_{(x_1,s_1;y_1,t_1)}(t')$, and for any $t' \in [s_2, t_2] \setminus [s_*, t_*]$ we let $\pi'_2(t') = \pi_{(x_2,s_2;y_2,t_2)}(t')$. For any $t' \in (s_*, t_*)$ we let $\pi'_1(t') = \pi_{(x_2,s_2;y_2,t_2)}(t')$ and $\pi'_2(t') = \pi_{(x_1,s_1;y_1,t_1)}(t')$. See Figure 3.3 for an illustration. Then it is straightforward to check that

$$\|\pi'_1\|_{\mathcal{L}} + \|\pi'_2\|_{\mathcal{L}} = \|\pi_{(x_1,s_1;y_1,t_1)}\|_{\mathcal{L}} + \|\pi_{(x_2,s_2;y_2,t_2)}\|_{\mathcal{L}} = \mathcal{L}(x_1, s_1; y_1, t_1) + \mathcal{L}(x_2, s_2; y_2, t_2).$$

This implies that π_1' and π_2' are also geodesics.

In the case where both $\pi_{(x_1,s_1;y_1,t_1)}$ and $\pi_{(x_2,s_2;y_2,t_2)}$ are the leftmost geodesics, for any $t' \in (s_*, t_*)$ we must have that $\pi_1'(t') \geq \pi_{(x_1,s_1;y_1,t_1)}(t')$, which implies that $\pi_{(x_2,s_2;y_2,t_2)}(t') \geq \pi_{(x_1,s_1;y_1,t_1)}(t')$. By a symmetric argument we have that $\pi_{(x_2,s_2;y_2,t_2)}(t') \leq \pi_{(x_1,s_1;y_1,t_1)}(t')$, so we must have that $t' \in O$, and $O = [s_*, t_*]$.

We next consider the case where $\pi_{(x_1,s_1;y_1,t_1)}$ is the unique geodesic. Then by uniqueness we must have $\pi_1' = \pi_{(x_1,s_1;y_1,t_1)}$, so we conclude that $\pi_{(x_2,s_2;y_2,t_2)}(t') = \pi_{(x_1,s_1;y_1,t_1)}(t')$ for any $t' \in (s_*, t_*)$, and $O = [s_*, t_*]$. By symmetry the case where $\pi_{(x_2,s_2;y_2,t_2)}$ is the unique geodesic also follows. □

Disjointness of geodesics

Recall that for any $x_1 < x_2$, $y_1 < y_2$, and $s < t$, $\mathrm{Disj}^{[s,t]}_{(x_1,x_2)\to(y_1,y_2)}$ is the event where $\pi_{(x_1,s;y_1,t)}$ and $\pi_{(x_2,s;y_2,t)}$ are unique and disjoint. This disjointness would be equivalent to strict inequality of the quadrangle inequality (Lemma 3.6).

Lemma 3.13. *For any $x_1 < x_2$, $y_1 < y_2$, and $s < t$, the event $\mathrm{Disj}^{[s,t]}_{(x_1,x_2)\to(y_1,y_2)}$ is equivalent to that the geodesics $\pi_{(x_1,s;y_1,t)}$ and $\pi_{(x_2,s;y_2,t)}$ are both unique, and*

$$\mathcal{L}(x_1, s; y_1, t) + \mathcal{L}(x_2, s; y_2, t) > \mathcal{L}(x_1, s; y_2, t) + \mathcal{L}(x_2, s; y_1, t). \tag{3.9}$$

Statements similar to this have also appeared in the proof of [17, Proposition 4.1].

Proof. Whenever $\pi_{(x_1,s;y_1,t)}$ and $\pi_{(x_2,s;y_2,t)}$ are not disjoint, we can take x_*, t_* with

$$\pi_{(x_1,s;y_1,t)}(t_*) = \pi_{(x_2,s;y_2,t)}(t_*) = x_*,$$

and construct π_1' from (x_1, s) to (y_2, t) and π_2' from (x_2, s) to (y_1, t), by switching the paths after t_*. Then $\|\pi_1'\|_{\mathcal{L}} + \|\pi_2'\|_{\mathcal{L}} = \mathcal{L}(x_1, s; y_1, t) + \mathcal{L}(x_2, s; y_2, t)$, and (3.9) cannot hold.

Now we assume that (3.9) does not hold; then by the quadrangle inequality (Lemma 3.6), we must have that the left hand side equals the right hand side. We then consider any geodesic $\pi_{(x_1,s;y_2,t)}$ and $\pi_{(x_2,s;y_1,t)}$. Take x_*, t_* with

$$\pi_{(x_1,s;y_2,t)}(t_*) = \pi_{(x_1,s;y_2,t)}(t_*) = x_*,$$

then we construct π_1' from (x_1, s) to (y_1, t) and π_2' from (x_2, s) to (y_2, t), by switching the paths after t_*. Now we have

$$\|\pi_1'\|_{\mathcal{L}} + \|\pi_2'\|_{\mathcal{L}} = \mathcal{L}(x_1, s; y_2, t) + \mathcal{L}(x_1, s; y_2, t) = \mathcal{L}(x_1, s; y_1, t) + \mathcal{L}(x_2, s; y_2, t),$$

so π_1' and π_2' must be geodesics. By uniqueness we have $\pi_1' = \pi_{(x_1,s;y_1,t)}$ and $\pi_2' = \pi_{(x_2,s;y_2,t)}$, and they are not disjoint. □

We recall the following estimate on the disjoint probability. This is a limiting version of [64, Theorem 1.1], and is slightly more general than Theorem 3.2.

Theorem 3.14. *There is a constant C depending on g, h, such that for any $g \le t \le h$, and $\epsilon > 0$, $l > 1$ we have*

$$\mathbb{P}[\mathrm{Disj}^{[0,t]}_{(-l,l)\to(-\epsilon,\epsilon)}] < Cl\epsilon^{1/2}e^{C|\log(\epsilon)|^{5/6}}.$$

Proof. By Lemma 3.13, $\mathrm{Disj}^{[0,t]}_{(-l,l)\to(-\epsilon,\epsilon)}$ implies that

$$\mathcal{L}(-1, 0; -\epsilon, t) - \mathcal{L}(-1, 0; \epsilon, t) > \mathcal{L}(1, 0; -\epsilon, t) - \mathcal{L}(1, 0; \epsilon, t);$$

thus for some $k \in [\![0, \lceil l \rceil]\!]$, we have

$$\begin{aligned}\mathcal{L}(-\lceil l\rceil + 2k, 0; -\epsilon, t) - \mathcal{L}(-\lceil l\rceil + 2k, 0; \epsilon, t)\\ > \mathcal{L}(-\lceil l\rceil + 2k + 2, 0; -\epsilon, t) - \mathcal{L}(-\lceil l\rceil + 2k + 2, 0; \epsilon, t).\end{aligned}$$

Assuming (the probability 1 event) that these geodesics are unique, this implies

$$\bigcup_{k=0}^{\lceil l \rceil} \mathrm{Disj}^{[0,t]}_{(-\lceil l \rceil + 2k, -\lceil l \rceil + 2k + 2) \to (-\epsilon, \epsilon)},$$

by Lemma 3.13. Then by the skew-shift invariance of the directed landscape, we have

$$\mathbb{P}[\mathrm{Disj}^{[0,t]}_{(-l,l)\to(-\epsilon,\epsilon)}] \leq \lceil l \rceil \mathbb{P}[\mathrm{Disj}^{[0,t]}_{(-1,1)\to(-\epsilon,\epsilon)}].$$

We now claim that

$$\mathbb{P}[\mathrm{Disj}^{[0,t]}_{(-1,1)\to(-\epsilon,\epsilon)}] < C\epsilon^{1/2} e^{C|\log(\epsilon)|^{5/6}}.$$

for some constant C depending on g, h. Such a disjointness probability estimate is proved in the setting of Brownian LPP, as [64, Theorem 1.1]; and we pass it to the directed landscape limit using [42, Theorem 1.8], which states that (under an appropriate coupling) any unique geodesic in the directed landscape is the limit (under uniform convergence) of the corresponding ones in Brownian LPP. The above two inequalities imply the conclusion. □

3.3 Busemann function and semi-infinite geodesics

In this section we construct semi-infinite geodesics in the directed landscape. As stated in Section 3.1, we shall first construct the Busemann function, and use it to define the semi-infinite geodesics. We will further deduce various useful properties of these constructed objects.

3.3.1 Construction of Busemann function

For each $l > 0$, let

$$U_l = [-2l^{2/3}, 2l^{2/3}] \times [l, 2l]. \tag{3.10}$$

For any compact set $B \subset \mathbb{R}^2$, we denote as $\mathcal{C}_{B,l}$ the following coalescence event: there exists (x_*, t_*) such that for any $p \in B$ and $q \in U_l$, any geodesic $\pi_{(p;q)}$ contains t_* in its range, and $\pi_{(p;q)}(t_*) = x_*$. The following shows that this is rather likely.

Lemma 3.15. *For any compact $B \subset \mathbb{R}^2$, there is a constant $C > 0$, such that $\mathbb{P}[\mathcal{C}_{B,l}] > 1 - Cl^{-1/3}e^{C|\log(l)|^{5/6}}$ for any $l > 0$.*

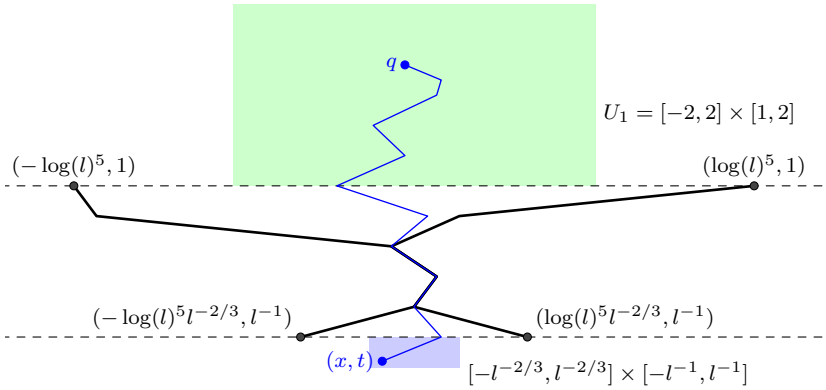


Figure 3.4: An illustration of the proof of Lemma 3.15.

The above is a general version of statements that have already appeared in the literature before (see e.g., [14, 12, 102] which works in the exponential LPP setting). We follow a similar proof relying on the bound given by Theorem 3.14 (which accounts for the extra factor of $e^{C|\log(l)|^{5/6}}$), combined with transversal fluctuation estimates and the ordering of geodesics.

Proof. By scale invariance, we can assume that $B = [-1,1] \times [-1,1]$ without loss of generality. We also assume that l is large enough (since otherwise the conclusion holds obviously by taking C large enough). Let the event $\mathcal{C}'_l$ be the scaled version of $\mathcal{C}_{B,l}$, i.e., there exists (x_*, t_*) such that $l^{-1} \le t_* \le 1$, and $\pi_{(x,t;q)}(x_*) = t_*$ for any $|x| \le l^{-2/3}$, $|t| \le l^{-1}$, and $q \in U_1$, and any geodesic $\pi_{(x,t;q)}$. Hence $\mathbb{P}[\mathcal{C}_{B,l}] = \mathbb{P}[\mathcal{C}'_l]$,

Below we let $C, c > 0$ denote large and small constants, whose values may change from line to line. Let R be the random varialbe given by Lemma 3.9. We now consider the following events.

1. Let $\mathcal{E}_1$ be the event that $R > \log(l)$.

2. Let $\mathcal{E}_2$ be the event $\mathrm{Disj}^{[l^{-1},1]}_{(-\log(l)^5 l^{-2/3}, \log(l)^5 l^{-2/3}) \to (-\log(l)^5, \log(l)^5)}$; recall that this is the event where the geodesics $\pi_{(-\log(l)^5 l^{-2/3}, l^{-1}; -\log(l)^5, 1)}$ and $\pi_{(\log(l)^5 l^{-2/3}, l^{-1}; \log(l)^5, 1)}$ are unique and disjoint (while the coordinates are somewhat technical looking, we hope Figure 3.4 would convey the basic setup).

We next show that (up to a probability zero event) $\mathcal{E}_1^c \cap \mathcal{E}_2^c \subset \mathcal{C}_l'$ (see Figure 3.4 for an illustration). Using Lemma 3.9, for any $|x| \le l^{-2/3}$, $|t| \le l^{-1}$, and $q \in U_1$, we have $|\pi_{(x,t;q)}(1)| \le \log(l)^5$ and $|\pi_{(x,t;q)}(l^{-1})| \le \log(l)^5 l^{-2/3}$ under $\mathcal{E}_1^c$. Now on $\mathcal{E}_2^c$ and hence on $\mathcal{E}_1^c \cap \mathcal{E}_2^c$, we can find (x_*, t_*) such that

$$\pi_{(-\log(l)^5 l^{-2/3}, l^{-1}; -\log(l)^5, 1)}(t_*) = \pi_{(\log(l)^5 l^{-2/3}, l^{-1}; \log(l)^5, 1)}(t_*) = x_*.$$

We assume (the probability 1 event) that the geodesics $\pi_{(-\log(l)^5 l^{-2/3}, l^{-1}; -\log(l)^5, 1)}$ and $\pi_{(\log(l)^5 l^{-2/3}, l^{-1}; \log(l)^5, 1)}$ are unique. By the ordering of finite geodesics (Lemma 3.12), for any $|x| \le l^{-2/3}$, $|t| \le l^{-1}$, and $q \in U_1$, one also has $\pi_{(x,t;q)}(t_*) = x_*$.

By Lemma 3.9, we have $\mathbb{P}[\mathcal{E}_1] < Ce^{-c(\log(l))^{9/4} \log(\log(l))^{-4}}$, by Theorem 3.14 we have

$$\mathbb{P}[\mathcal{E}_2] < C \log(l)^{15/2} l^{-1/3} e^{C|\log(l)|^{5/6}}.$$

Adding these bounds together yields the conclusion. □

We then immediately get the following convergence of difference of passage times.

Corollary 3.16. *For any compact $B \subset \mathbb{R}^2$ and any $r \in \mathbb{R}$, almost surely there is a (random) number h_0 such that the following is true. For any $p, q \in B$, $h \ge h_0$, and $z \in \mathbb{R}$ with $|z - hr| < h^{2/3}$, we have that $\mathcal{L}(p; z, h) - \mathcal{L}(q; z, h) = \mathcal{L}(p; h_0 r, h_0) - \mathcal{L}(q; h_0 r, h_0)$.*

Proof. By skew-shift invariance of the directed landscape (Lemma 3.5), we can assume that $r = 0$ without loss of generality. By Lemma 3.15 and the Borel-Cantelli lemma,

there is a random $k_0 \in \mathbb{N}$, such that $\mathcal{C}_{B,2^k}$ holds for any $k \geq k_0$. Then for any $p, q \in B$, $k \geq k_0$ and $p' \in U_{2^k}$, we have $\mathcal{L}(p;0,2^k) - \mathcal{L}(q;0,2^k) = \mathcal{L}(p;p') - \mathcal{L}(q;p')$; in particular $\mathcal{L}(p;0,2^k) - \mathcal{L}(q;0,2^k) = \mathcal{L}(p;0,2^{k+1}) - \mathcal{L}(q;0,2^{k+1})$. This implies that for any $h \geq 2^{k_0}$ and $|z| < h^{2/3}$, we have $\mathcal{L}(p;0,2^{k_0}) - \mathcal{L}(q;0,2^{k_0}) = \mathcal{L}(p;z,h) - \mathcal{L}(q;z,h)$, since $(z,h) \in \cup_{k \geq k_0} U_{2^k}$. Thus the conclusion follows by taking $h_0 = 2^{k_0}$. □

As in the Exponential LPP setting, the above coalescence results allow us to define the Busemann function in the directed landscape.

Definition 3.17. We define the Busemann function as

$$\mathcal{M}^r(x,s;y,t) := \lim_{n \to \infty} \mathcal{L}(x,s;z_n,h_n) - \mathcal{L}(y,t;z_n,h_n)$$

for any $(x,s;y,t) \in \mathbb{R}^4$ and $r \in \mathbb{R}$, where $\{z_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ are any sequences such that $z_n \to \infty$ as $n \to \infty$, and $|z_n - h_n r| < h_n^{2/3}$ for each $n \in \mathbb{N}$. By Corollary 3.16 we see that for any fixed $r \in \mathbb{R}$, almost surely this limit exists for any $(x,s;y,t) \in \mathbb{R}^4$, and is independent of the choice of the sequences. Also, $\mathcal{M}^r$ is almost surely continuous as a function on $\mathbb{R}^4$. For simplicity of notations, we would often write $\mathcal{M}$ for $\mathcal{M}^0$.

The next lemma asserts Brownianity of the Busemann functions.

Lemma 3.18. *For any $r, t \in \mathbb{R}$, the process of $x \mapsto \mathcal{M}^r(x,t;0,t) - 2xr$ is a Brownian motion.*

Proof. By skew-shift invariance (Lemma 3.5) it suffices to assume $r = t = 0$. For any $h > 0$, by scaling invariance, the process $x \mapsto \mathcal{L}(x,0;0,h) - \mathcal{L}(\mathbf{0};0,h)$ has the same law as $x \mapsto h^{-1/3}\mathcal{L}(xh^{-2/3},0;0,1) - \mathcal{L}(\mathbf{0};0,1)$, which is the same as $x \mapsto h^{-1/3}(\mathcal{P}_1(h^{-2/3}x) - \mathcal{P}_1(0))$, where $\mathcal{P}_1$ is the parabolic Airy$_2$ process. By [80, Theorem 4.14], as $h \to \infty$ it converges to Brownian motion, in finite dimensional distributions. Also by Corollary 3.16, $x \mapsto \mathcal{L}(x,0;0,h) - \mathcal{L}(\mathbf{0};0,h)$ converges to $x \mapsto \mathcal{M}(x,t;0,t)$

uniformly in compact sets, so $x \mapsto \mathcal{M}(x, 0; \mathbf{0})$ has the same finite dimensional distributions as a Brownian motion. Since $\mathcal{M}$ is continuous, the conclusion follows. □

Lemma 3.19. *For any $r_1 < r_2$, the following is true almost surely. For any $x_1 < x_2$ and $t \in \mathbb{R}$, there is*

$$\mathcal{M}^{r_1}(x_1, t; x_2, t) \geq \mathcal{M}^{r_2}(x_1, t; x_2, t). \tag{3.11}$$

Proof. It suffices to prove this for fixed (thus all rational) $x_1 < x_2$ and $t \in \mathbb{R}$ since then by continuity of $\mathcal{M}^{r_1}$ and $\mathcal{M}^{r_2}$ we can get (3.11) to hold for all $x_1 < x_2$ and $t \in \mathbb{R}$ simultaneously.

From the construction, we can find some rational $h_0 > t$, such that

$$\mathcal{M}^{r_1}(x_1, t; x_2, t) = \mathcal{L}(x_1, t; h_0 r_1, h_0) - \mathcal{L}(x_2, t; h_0 r_1, h_0),$$

and

$$\mathcal{M}^{r_2}(x_1, t; x_2, t) = \mathcal{L}(x_1, t; h_0 r_2, h_0) - \mathcal{L}(x_2, t; h_0 r_2, h_0).$$

Then by the quadrangle inequality (Lemma 3.6), the conclusion follows. □

We now move on to the formal construction of semi-infinite geodesics.

3.3.2 Construction of semi-infinite geodesics

Lemma 3.20. *For any $r \in \mathbb{R}$, almost surely the following is true. For any $p = (x, s) \in \mathbb{R}^2$, there exists a continuous path $\pi_p^r : [s, \infty) \to \mathbb{R}$ satisfying the following condition.*

Consider any compact $B \subset \mathbb{R}^2$, and any sequences $\{z_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ satisfying $z_n \to \infty$ as $n \to \infty$ and $|z_n - h_n r| < h_n^{2/3}$ for each $n \in \mathbb{N}$, and $t \in \mathbb{R}$. Then there is some $n_0 \in \mathbb{N}$ such that $\pi_{(p;z_n,h_n)}(t') = \pi_p^r(t')$ for any $n \geq n_0$, any $p = (x, s) \in B$ with $s \leq t$, and any $s \leq t' \leq t$. Here $\pi_{(p;z_n,h_n)}$ is taken as the leftmost geodesic.

Proof. The finite geodesics in this proof are taken to be the leftmost one. We also just prove for the case where $r = 0$, since the argument for general r follows from the skew-shift invariance.

By Lemma 3.15 and using the Borel-Cantelli lemma, almost surely, for each $m \in \mathbb{N}$ there is a random $k_m \in \mathbb{N}$, such that the coalescence event $\mathcal{C}_{[-m,m]\times[-m,m],2^k}$ (recall the definition from the beginning of Section 3.3.1) holds for any $k \geq k_m$. Then for any $p = (x, s) \in [-m, m] \times [-m, m]$, $k \geq k_m$, $s \leq t \leq m$, and $p' \in U_{2^k}$ (defined in (3.10)), we have $\pi_{(p;0,2^k)}(t) = \pi_{(p;p')}(t)$ by Lemma 3.12; in particular $\pi_{(p;0,2^k)}(m) = \pi_{(p;0,2^{k+1})}(m)$. This implies that for any $h \geq 2^{k_m}$ and $|z| < h^{2/3}$, and $s \leq t \leq m$, we have $\pi_{(p;0,2^{k_m})}(t) = \pi_{(p;z,h)}(t)$, since $(z, h) \in \cup_{k\geq k_m} U_{2^k}$.

For any $p = (x, s) \in [-m, m] \times [-m, m]$, we just define $\pi^0_p(t) = \pi_{(p;0,2^{k_m})}(t)$ for any $s \leq t \leq m$. Since $\pi_{(p;0,2^{k_m})}(t) = \pi_{(p;0,2^{k_m+1})}(t)$ (for any $s \leq t \leq m$) we would have that this definition is consistent for different m. Now for a general compact set $B \subset \mathbb{R}^2$ and $t \in \mathbb{R}$, the lemma follows by taking m large enough so that $B \subset [-m, m] \times [-m, m]$ and $m > t$. □

As already indicated in the introduction, given $r \in \mathbb{R}$, unless otherwise noted, for any $p \in \mathbb{R}^2$, we will use π^r_p to denote the path constructed in Lemma 3.20 and $\pi_p = \pi^0_p$ for simplicity of notations.

Just like the composition law (1.1) satisfied by finite geodesics, the number $\pi^r_p(t)$ would be the maximizer of the sum of two profiles: the passage time from p, and the Busemann function in the r-direction.

Lemma 3.21. *For any fixed $r, t \in \mathbb{R}$, almost surely the following is true. For any $p = (x, s) \in \mathbb{R}^2$ with $s < t$, we have*

$$\mathcal{L}(p; \pi^r_p(t), t) + \mathcal{M}^r(\pi^r_p(t), t; 0, t) = \max_x \mathcal{L}(p; x, t) + \mathcal{M}^r(x, t; 0, t).$$

Proof. By Corollary 3.16 and Lemma 3.20, for any $n \in \mathbb{N}$, we can find h_n large

enough such that $\mathcal{M}^r(x,t;0,t) = \mathcal{L}(x,t;h_n r, h_n) - \mathcal{L}(0,t;h_n r, h_n)$ for any $|x| \le n$, and $\pi_p^r(t) = \pi_{(p;h_n r,h_n)}(t)$ for any $p \in [-n,n] \times [-n,t)$.

Then for any $p = (x,s) \in \mathbb{R}^2$ with $s < t$, and any $n \in \mathbb{N}$ with $n > 2|\pi_p^r(t)|$ and $p \in [-n,n] \times [-n,n]$, by the composition law (1.1) we have that

$$\mathcal{L}(p;\pi_p^r(t),t) + \mathcal{M}^r(\pi_p^r(t),t;0,t) = \max_{|x|\le n} \mathcal{L}(p;x,t) + \mathcal{M}^r(x,t;0,t).$$

By sending $n \to \infty$ the conclusion follows. □

We next verify that the constructed paths π_p^r are semi-infinite geodesics.

It is straight forward to see that the paths are locally geodesics. Fix any $r \in \mathbb{R}$. By Lemma 3.20, for each $p = (x,s) \in \mathbb{R}^2$, and any $s \le t_1 < t_2$, the restriction of π_p^r to $[t_1,t_2]$ is a subpath of $\pi_{(p;hr,h)}$ for some large enough h; so it is also a geodesic.

It remains to show that π_p^r have the desired asymptotic directions. For that, we need the following transversal fluctuation estimate on semi-infinite geodesics. It will also be used in several other places in the rest of this chapter, and can be viewed as a limiting version of Lemma 2.10. While we do not prove this, up to the constants, this is also the best upper bound one can hope for, as a matching lower bound can be obtained by analyzing the location of the argmax of the parabolic Airy_2 process plus a Brownian motion, and using the Brownian Gibbs property of the Airy line ensemble (the latter is an invariance property which is crucially used in Appendix C).

Lemma 3.22. *There exist constants $C, c > 0$ such that the following is true. For any $r,t \in \mathbb{R}$ and $p = (x,s) \in \mathbb{R}^2$ satisfying $s < t$, we have $\mathbb{P}[|\pi_p^r(t) - (x + r(t-s))| > w(t-s)^{2/3}] < Ce^{-cw^3}$.*

Proof. In this proof, we let $C, c > 0$ be universal constants, whose value can change from line to line. By translation and skew-shift invariance of the directed landscape (thus of semi-infinite geodesics), we just need to prove this for $p = \mathbf{0}$, $r = 0$, and

$t = 1$. By Lemma 3.21, the event $|\pi_{\mathbf{0}}(1)| > w$ implies that

$$\max_{|y|>w} \mathcal{L}(\mathbf{0}; y, 1) + \mathcal{M}(y, 1; 0, 1) \geq \mathcal{L}(\mathbf{0}; 0, 1).$$

Denote this event as $\mathcal{E}$. For any $n \in \mathbb{Z}$ we let $\mathcal{E}_n$ be the event where

$$\max_{y\in[n,n+1]} \mathcal{L}(\mathbf{0}; y, 1) + \mathcal{M}^r(y, 1; 0, 1) \geq \mathcal{L}(\mathbf{0}; 0, 1).$$

Then we have $\mathbb{P}[\mathcal{E}] \leq \sum_{n\geq w} \mathbb{P}[\mathcal{E}_n] + \sum_{n\leq w-1} \mathbb{P}[\mathcal{E}_n]$.

Next, note that $\mathcal{E}_n$ implies one of the following three events:

1. $\max_{y\in[n,n+1]} \mathcal{L}(\mathbf{0}; y, 1) > -2n^2/3$.
2. $\mathcal{L}(\mathbf{0}; 0, 1) < -n^2/3$.
3. $\max_{y\in[n,n+1]} \mathcal{M}^r(y, 1; 0, 1) > n^2/3$.

For the first event, by skew-shift invariance of the directed landscape, it has the same probability as the follow event:

$$\max_{y\in[0,1]} \mathcal{L}(\mathbf{0}; y, 1) - (y + n)^2 + y^2 > -2n^2/3.$$

By Lemma 3.7, its probability is bounded by Ce^{-cn^3}. Also, the probabilities of the second and third event above are also bounded by Ce^{-cn^3}, using Lemma 3.7 and estimates on Brownian motions, respectively. Thus we have $\mathbb{P}[\mathcal{E}_n] < Ce^{-cn^3}$. By summing over $n \geq w$ and $n \leq w - 1$, the conclusion follows. □

We also need the following result which is on the overlaps between the constructed paths, and essentially says that the paths are "ordered". It is a consequence of the construction and Lemma 3.12. It will be improved later (Lemma 3.28) for the case of paths in the same direction.

Lemma 3.23. *For any $r_1, r_2 \in \mathbb{R}$, almost surely the following is true. For any $p_2 = (x_1, s_1), p_2 = (x_2, s_2) \in \mathbb{R}^2$, let $O = \{t \geq s_1 \vee s_2 : \pi_{p_1}^{r_1}(t) = \pi_{p_2}^{r_2}(t)\}$ be the set of "overlap times". If $r_1 \neq r_2$, then O is either empty or a closed interval. If $r_1 = r_2$, then O is either empty, or $O = [t_*, \infty)$ for some $t_* \geq s_1 \vee s_2$.*

Proof. We first consider the case where $r_1 \neq r_2$. In this case we must have that O is contained in a compact set, since the asymptotic directions of $\pi_{p_1}^{r_1}$ and $\pi_{p_2}^{r_2}$ are different. For any $t > s_1 \vee s_2$, by Lemma 3.20 there is some (random) h large enough (depending on p_1, p_2, r_1, r_2 and t) such that $\pi_{p_1}^{r_1}(t') = \pi_{(p_1;hr_1,h)}(t')$ for any $s_1 \leq t' \leq t$, and $\pi_{p_2}^{r_2}(t') = \pi_{(p_2;hr_2,h)}(t')$ for any $s_2 \leq t' \leq t$. Thus by Lemma 3.12, the set $O \cap (-\infty, t]$ is either empty, or a closed interval. By sending $t \to \infty$ we get the conclusion.

We next consider the case where $r_1 = r_2$. Again we take any $t > s_1 \vee s_2$, and by Lemma 3.20 there is some (random) h large enough (depending on p_1, p_2, $r_1 = r_2$ and t) such that $\pi_{p_1}^{r_1}(t') = \pi_{(p_1;hr_1,h)}(t')$ for any $s_1 \leq t' \leq t$, and $\pi_{p_2}^{r_2}(t') = \pi_{(p_2;hr_1,h)}(t')$ for any $s_2 \leq t' \leq t$. By Lemma 3.12, the set $\{s_1 \vee s_2 \leq t' \leq h : \pi_{(p_1;hr_1,h)}(t') = \pi_{(p_2;hr_1,h)}(t')\}$ is a closed interval with right endpoint being h; so the set $O \cap (-\infty, t]$ is either empty, or a closed interval with right endpoint being t. Again by sending $t \to \infty$ we get the conclusion. □

Now we finish proving that the constructed paths are semi-infinite geodesics.

Lemma 3.24. *For any fixed $r \in \mathbb{R}$, almost surely the constructed path π_p^r is a semi-infinite geodesic, for each $p \in \mathbb{R}^2$.*

Proof. Using skew-shift invariance, it suffice to prove for the case where $r = 0$.

We start with a fixed $p \in \mathbb{R}^2$. Above we have stated that any segment of π_p is a geodesic, and it remains to show that it has the desired asymptotic direction. By Lemma 3.22, and the Borel-Cantelli lemma, almost surely there exists some random $k_0 > 0$, such that for any $k > k_0$, $k \in \mathbb{N}$, we have $|\pi_p(2^k)| < 2^{2k/3+0.1k}$. For any $k > k_0$,

since π_p between $(\pi_p(2^k), 2^k)$ and $(\pi_p(2^{k+1}), 2^{k+1})$ is a geodesic, using Lemma 3.9 we can bound $\sup_{t\in[2^k,2^{k+1}]} |\pi_p(t)|$. Thus we conclude that $\lim_{t\to\infty} t^{-1}\pi_p(t) = 0$.

Now we have that almost surely, for any $p \in \mathbb{Q}^2$, we have $\lim_{t\to\infty} t^{-1}\pi_p(t) = 0$. Assuming this event, and consider some $p = (x, s) \notin \mathbb{Q}^2$. Then we can find some $t, x_-, x_+ \in \mathbb{Q}$, such that $t > s$ and $x_- < \pi_p(s) < x_+$. By Lemma 3.23, we must have that $\pi_{(x_-,t)}(t') \le \pi_p(t') \le \pi_{(x_+,t)}(t')$ for any $t' \ge t$. Since $\lim_{t\to\infty} t^{-1}\pi_{(x_-,t)}(t) = \lim_{t\to\infty} t^{-1}\pi_{(x_+,t)}(t) = 0$, we must have $\lim_{t\to\infty} t^{-1}\pi_p(t) = 0$. □

3.3.3 Properties of semi-infinite geodesics

The first property is about the overlap of semi-infinite geodesics for starting points around a fixed point. For any continuous path $\pi_n : [s_n, t_n] \to \mathbb{R}$, $n \in \mathbb{N}$, and $\pi : [s, t] \to \mathbb{R}$, we say that $\pi_n \to \pi$ *in the overlap topology*, if for all large enough n, $O_n = \{t' \in [s, t] \cap [s_n, t_n] : \pi_n(t') = \pi(t')\}$ is an interval, and the endpoints of O_n converge to s and t. We quote the following result on convergence in the overlap topology for finite geodesics.

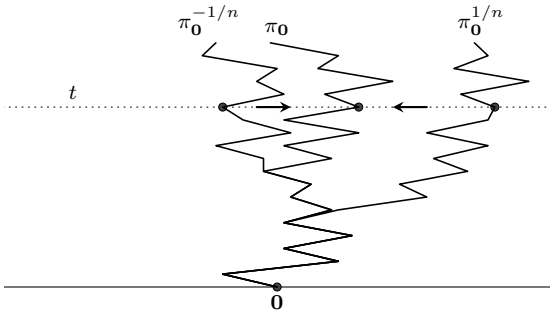
Lemma 3.25 ([43, Lemma 3.1, Lemma 3.3])**.** *Almost surely the following is true. Let $(p_n; q_n) \to (p; q) \in \mathbb{R}^4_\uparrow$, and let π_n be any sequence of geodesics from p_n to q_n. Suppose that there is a unique geodesic π from p to q. Then $\pi_n \to \pi$ in the overlap topology.*

From this, we can get overlap of semi-infinite geodesics.

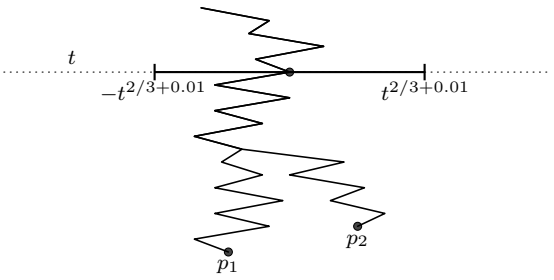
Lemma 3.26. *For any $r \in \mathbb{R}$, $p = (x, s) \in \mathbb{R}$ and $t > s$, almost surely there is an open neighborhood $\hat{B}$ of p, such that $\pi^r_p(t') = \pi^r_{p'}(t')$ for any $p' \in \hat{B}$ and $t' \ge t$.*

Proof. Take a compact set B containing an open neighborhood of p. We assume (the probability 1 event) that for any $h \in \mathbb{N}$, $h > s$, there is a unique geodesic $\pi_{(p;hr,h)}$ from p to (hr, h).

By Lemma 3.20, we can find some large enough (random) $h \in \mathbb{N}$ with $h > t$, such that $\pi_{(p';hr,h)}(t) = \pi^r_{p'}(t)$ for any $p' = (x', s') \in B$ with $s' \le t$. By Lemma 3.25,



(a) Lemma 3.27: both $\pi_{\mathbf{0}}^{-1/n}(t)$ and $\pi_{\mathbf{0}}^{1/n}(t)$ converge to $\pi_{\mathbf{0}}(t)$.



(b) Lemma 3.28: under $\mathcal{E}_1^c$ where $|\pi_{p_1}(t)| \vee |\pi_{p_2}(t)| \le t^{2/3+0.01}$ and $\mathcal{E}_2^c$ where $x \mapsto \mathcal{L}(p_1; x, t) - \mathcal{L}(p_2; x, t)$ is constant in $[-t^{2/3+0.01}, t^{2/3+0.01}]$, π_{p_1} and π_{p_2} would coalesce before t.

Figure 3.5: Illustrations for the proofs of Lemma 3.27 and 3.28.

there is an open neighborhood $\hat{B}$ of p, such that $\hat{B} \subset B$, and for any $p' \in \hat{B}$ there is $\pi_{(p';hr,h)}(t) = \pi_{(p;hr,h)}(t)$. We then have that $\pi_p^r(t) = \pi_{p'}^r(t)$ for any $p' \in \hat{B}$. By Lemma 3.23 we have $\pi_p^r(t') = \pi_{p'}^r(t')$ for any $p' \in \hat{B}$ and $t' \ge t$. □

If we fix both the starting point and the direction, then almost surely the semi-infinite geodesic is unique.

Lemma 3.27. *For fixed $r \in \mathbb{R}$ and $p \in \mathbb{R}^2$, almost surely there is a unique semi-infinite geodesic from p in the r direction.*

The proof uses the ordering of semi-infinite geodesics to sandwich any semi-infinite geodesic between two other semi-infinite geodesics with the same starting point but slightly different directions. We will also use the maximizer formulation of semi-infinite geodesics (Lemma 3.21), and monotonicity of Busemann functions (Lemma 3.19).

Proof. It suffices to prove uniqueness. By skew-shift invariance of the directed landscape (Lemma 3.5), without loss of generality we can assume that $r = 0$ and $p = \mathbf{0}$. We will work on the almost sure event that for any $m, n \in \mathbb{N}$, there is a unique geodesic from $\mathbf{0}$ to $(m/n, m)$, and a unique geodesic from $\mathbf{0}$ to $(-m/n, m)$.

Let $\pi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a semi-infinite geodesic in the 0 direction. Take any fixed $t > 0$, and $n, n' \in \mathbb{N}$ with $n < n'$. By the asymptotic directions and Lemma 3.20, we can find some $m \in \mathbb{N}$ such that $\pi_{\mathbf{0}}^{r}(t) = \pi_{(\mathbf{0};mr,m)}(t)$, for each $r \in \{-1/n, 1/n, -1/n', 1/n'\}$, and $|\pi(m)| < m/n'$. Thus by Lemma 3.12 we have

$$\pi_{\mathbf{0}}^{-1/n}(t) \leq \pi_{\mathbf{0}}^{-1/n'}(t) \leq \pi(t) \leq \pi_{\mathbf{0}}^{1/n'}(t) \leq \pi_{\mathbf{0}}^{1/n}(t).$$

Then the limits $x_- := \lim_{n\to\infty} \pi_{\mathbf{0}}^{-1/n}(t)$ and $x_+ := \lim_{n\to\infty} \pi_{\mathbf{0}}^{1/n}(t)$ exist, and $x_- \leq \pi(t) \leq x_+$. We would next show that $x_- = x_+$, and then $\pi(t)$ must equal $x_- = x_+$. See Figure 3.5a for an illustration.

Before proceeding further, we explain how to prove that $x_- = x_+$. Recall (from Lemma 3.21) that the function $\mathcal{L}(\mathbf{0}; \cdot, t) + \mathcal{M}^{-1/n}(\cdot, t; 0, t)$ takes its maximum value at $\pi_{\mathbf{0}}^{-1/n}(t)$. Then we prove uniform convergence of $\mathcal{M}^{-1/n}(\cdot, t; 0, t)$ to $\mathcal{M}(\cdot, t; 0, t)$, in a compact interval. By passing to the $n \to \infty$ limit, we get that x_- is the argmax of the function $\mathcal{L}(\mathbf{0}; \cdot, t) + \mathcal{M}(\cdot, t; 0, t)$; and similarly this is also true for x_+. By uniqueness of the argmax we conclude that $x_- = x_+$.

By Lemma 3.21, for any $y \in \mathbb{R}$ we have

$$\mathcal{L}(\mathbf{0}; \pi_{\mathbf{0}}^{-1/n}(t), t) + \mathcal{M}^{-1/n}(\pi_{\mathbf{0}}^{-1/n}(t), t; x_-, t) \geq \mathcal{L}(\mathbf{0}; y, t) + \mathcal{M}^{-1/n}(y, t; x_-, t). \quad (3.12)$$

We would next send $n \to \infty$ for this equation. By continuity of $\mathcal{L}$ and $\lim_{n\to\infty} \pi_{\mathbf{0}}^{-1/n}(t) = x_-$, we have that

$$\lim_{n\to\infty} \mathcal{L}(\mathbf{0}; \pi_{\mathbf{0}}^{-1/n}(t), t) = \mathcal{L}(\mathbf{0}; x_-, t).$$

By Lemma 3.19, we have that

$$\mathcal{M}(\pi_{\mathbf{0}}^{-1/n}(t), t; x_-, t) \leq \mathcal{M}^{-1/n}(\pi_{\mathbf{0}}^{-1/n}(t), t; x_-, t) \leq \mathcal{M}^{-1}(\pi_{\mathbf{0}}^{-1/n}(t), t; x_-, t).$$

By continuity of $\mathcal{M}$ and $\mathcal{M}^{-1}$, we have that as $n \to \infty$, $\mathcal{M}(\pi_{\mathbf{0}}^{-1/n}(t), t; x_-, t) \to 0$

and $\mathcal{M}^{-1}(\pi_{\mathbf{0}}^{-1/n}(t), t; x_-, t) \to 0$; so

$$\lim_{n\to\infty} \mathcal{M}^{-1/n}(\pi_{\mathbf{0}}^{-1/n}(t), t; x_-, t) = 0.$$

For $\mathcal{M}^{-1/n}(y, t; x_-, t)$, by Lemma 3.19 it is non-decreasing as $n \to \infty$, and we always have $\mathcal{M}^{-1/n}(y, t; x_-, t) \le \mathcal{M}(y, t; x_-, t)$. We claim that almost surely, for any $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathcal{M}^{-1/n}(y, t; x_-, t) = \mathcal{M}(y, t; x_-, t). \tag{3.13}$$

It suffices to show that for any $N \in \mathbb{N}$, almost surely this convergence holds for all $y \in [-N, N]$, conditional on that $|x_-| < N$. We note that when $|x_-|, |y| < N$, by Lemma 3.19 there is

$$\begin{aligned}
&\mathcal{M}^{-1/n}(-N, t; N, t) - \mathcal{M}^{-1/n}(y, t; x_-, t)\\
=&\mathcal{M}^{-1/n}(-N, t; y, t) + \mathcal{M}^{-1/n}(x_-, t; N, t)\\
\ge&\mathcal{M}(-N, t; y, t) + \mathcal{M}(x_-, t; N, t)\\
=&\mathcal{M}(-N, t; N, t) - \mathcal{M}(y, t; x_-, t),
\end{aligned}$$

i.e.,

$$\mathcal{M}^{-1/n}(y, t; x_-, t) - \mathcal{M}(y, t; x_-, t) \le \mathcal{M}^{-1/n}(-N, t; N, t) - \mathcal{M}(-N, t; N, t).$$

The right hand side is almost surely positive, while the expectation equals $4N/n$; it is also non-increasing as $n \to \infty$. Thus by sending $n \to \infty$ we have that

$$\mathcal{M}^{-1/n}(-N, t; N, t) - \mathcal{M}(-N, t; N, t) \to 0$$

almost surely. Hence, conditional on $|x_-| < N$, we must have that

$$\limsup_{n\to\infty} \mathcal{M}^{-1/n}(y,t;x_-,t) \le \mathcal{M}(y,t;x_-,t).$$

Similarly we have $\liminf_{n\to\infty} \mathcal{M}^{-1/n}(y,t;x_-,t) \ge \mathcal{M}(y,t;x_-,t)$. So conditional on $|x_-| < N$, (3.13) holds for all $y \in [-N,N]$. Then (3.13) almost surely holds for all $y \in \mathbb{R}$.

Then by sending $n \to \infty$ for (3.12), we conclude that almost surely, $\mathcal{L}(\mathbf{0};x_-,t) \ge \mathcal{L}(\mathbf{0};y,t) + \mathcal{M}(y,t;x_-,t)$ for any $y \in \mathbb{R}$; by symmetry we also have $\mathcal{L}(\mathbf{0};x_+,t) \ge \mathcal{L}(\mathbf{0};y,t) + \mathcal{M}(y,t;x_+,t)$ for any $y \in \mathbb{R}$. Thus we have that the function $x \mapsto \mathcal{L}(\mathbf{0};x,t) + \mathcal{M}(x,t;0,t)$ achieves its maximum at both x_- and x_+. However, $x \mapsto \mathcal{L}(\mathbf{0};x,t)$ is a scaled Airy_2 process, so it is absolutely continuous with respect to Brownian motion (by [36] or Theorem 1.15) in any compact interval; also $x \mapsto \mathcal{M}(x,t;0,t)$ is a Brownian motion by Lemma 3.18. These imply that $x \mapsto \mathcal{L}(\mathbf{0};x,t) + \mathcal{M}(x,t;0,t)$ is also absolutely continuous with respect to Brownian motion in any compact interval. As Brownian motion almost surely has a unique maximum in any compact interval, we conclude that $x_- = x_+$ almost surely.

Thus the above implies that for a given $t > 0$, almost surely $\pi(t)$ is uniquely determined, which then by considering all rational $t > 0$ along with continuity of π, implies the uniqueness of π. □

Finally, we state an improvement of Lemma 3.23 for semi-infinite geodesics in a fixed direction; namely, for fixed two points, the semi-infinite geodesics to this direction coalesce almost surely.

Lemma 3.28. *For any fixed $r \in \mathbb{R}$ and $p_1 = (x_1,s_1), p_2 = (x_2,s_2) \in \mathbb{R}^2$, almost surely the following is true. Let $O = \{t \ge s_1 \vee s_2 : \pi^r_{p_1}(t) = \pi^r_{p_2}(t)\}$, then $O = [t_*,\infty)$ for some $t_* \ge s_1 \vee s_2$.*

Proof. By Lemma 3.5, without loss of generality we will only consider the case $r = 0$.

It suffices to show that $\mathbb{P}[\pi_{p_1}(t) = \pi_{p_2}(t)] \to 1$ as $t \to \infty$; then we have that almost surely $O \neq \emptyset$, and by Lemma 3.23 the conclusion follows.

For fixed $t > s_1 \vee s_2$, we consider the following events.

1. Let $\mathcal{E}_1$ be the event where $|\pi_{p_1}(t)| \vee |\pi_{p_2}(t)| > t^{2/3+0.01}$.

2. Let $\mathcal{E}_2$ be the event where the function $x \mapsto \mathcal{L}(p_1; x, t) - \mathcal{L}(p_2; x, t)$ is not constant for $x \in [-t^{2/3+0.01}, t^{2/3+0.01}]$.

By Lemma 3.21, the functions $x \mapsto \mathcal{L}(p_1; x, t) + \mathcal{M}(x, t; 0, t)$ and $x \mapsto \mathcal{L}(p_2; x, t) + \mathcal{M}(x, t; 0, t)$ take their maximum values at $\pi_{p_1}(t)$ and $\pi_{p_2}(t)$, respectively. Under $\mathcal{E}_2^c$, these two functions differ by a constant in the interval $[-t^{2/3+0.01}, t^{2/3+0.01}]$. As argued in the proof of Lemma 3.27, these two functions are absolutely continuous with respect to a Brownian motion in any compact interval, so almost surely they have a unique argmax in the interval $[-t^{2/3+0.01}, t^{2/3+0.01}]$. Thus (up to a probability zero event) $\mathcal{E}_1^c \cap \mathcal{E}_2^c$ implies that $\pi_{p_1}(t) = \pi_{p_2}(t)$. See Figure 3.5b for an illustration.

As $t \to \infty$, by Lemma 3.22 we have that $\mathbb{P}[\mathcal{E}_1] \to 0$. We net show that $\mathbb{P}[\mathcal{E}_2] \to 0$. For each $i \in \mathbb{Z}$, $|i| < \lceil t^{0.01} \rceil$, let $\mathcal{E}_{2,i}$ be the event where $x \mapsto \mathcal{L}(p_1; x, t) - \mathcal{L}(p_2; x, t)$ is not constant for $x \in [it^{2/3}, (i+1)t^{2/3}]$. Then $\mathcal{E}_2 \subset \cup_{|i|<\lceil t^{0.01} \rceil} \mathcal{E}_{2,i}$, and we need to bound each $\mathbb{P}[\mathcal{E}_{2,i}]$. Using scaling and skew-shift invariance of the directed landscape, we get that $\mathbb{P}[\mathcal{E}_{2,i}] = \mathbb{P}[\mathcal{E}'_{2,i}]$, with $\mathcal{E}'_{2,i}$ being the event where

$$x \mapsto \mathcal{L}(t^{-1/6}x_1 - it^{-1/2}s_1, t^{-1/4}s_1; x, t^{3/4}) - \mathcal{L}(t^{-1/6}x_2 - it^{-1/2}s_2, t^{-1/4}s_2; x, t^{3/4})$$

is not constant for $x \in [0, t^{1/2}]$. Let $B = [-1, 1] \times [-1, 1]$. Then for t large enough, we would have $(t^{-1/6}x_1 - it^{-1/2}s_1, t^{-1/4}s_1)$, $(t^{-1/6}x_2 - it^{-1/2}s_2, t^{-1/4}s_2) \in B$ for any $|i| < \lceil t^{0.01} \rceil$. So by Lemma 3.15, we conclude that there is a constant $C > 0$ such that

$$\mathbb{P}[\mathcal{E}'_{2,i}] < Ct^{-1/4}e^{C|\log(t)|^{5/6}},$$

for each $|i| < \lceil t^{0.01} \rceil$. Thus we have

$$\mathbb{P}[\mathcal{E}_2] \leq \sum_{|i|<\lceil t^{0.01}\rceil} \mathbb{P}[\mathcal{E}'_{2,i}] < Ct^{-1/4} e^{C|\log(t)|^{5/6}} (2\lceil t^{0.01} \rceil + 1)$$

when t is large enough. We then have that $\mathbb{P}[\mathcal{E}_2] \to 0$ as $t \to \infty$, and hence $\mathbb{P}[\pi_{p_1}(t) = \pi_{p_2}(t)] \to 1$ as $t \to \infty$, and the conclusion follows. □

Remark 3.29. We choose to state Lemma 3.28 in the current form as it will be convenient for a later application. In fact, via the ordering of geodesics and bounds on transversal fluctuations, one can prove that almost surely, the statement holds simultaneously for any starting points in $\mathbb{R}^2$.

3.4 Duality in the limit

In this section we prepare the necessary framework of competition interfaces in the directed landscape.

We will start by revisiting equality of law of the competition interface starting from the stationary initial condition and the semi-infinite geodesic in the prelimiting model of exponential LPP, and use Theorem 1.14 from [44] to pass to the limit and conclude the same for the directed landscape.

In Section 1.4.4, the competition interface is defined using $\{\omega^\vee(p)\}_{p\in\mathbb{Z}^2}$ and Busemann function G^ρ. For the ease of notations, in this section we will work with the competition interface defined from $\{\omega(p)\}_{p\in\mathbb{Z}^2}$ instead. We will also mainly work on the $(1,1)$-direction, and denote $G(p) = G^{1/2}(p)$ for any $p \in \mathbb{Z}^2$. Remarkably, the following statements hold:

1. For any $p = (x, y) \in \mathbb{Z}^2$ and $n > y$, if we take $m_* = \operatorname{argmax}_{m\geq x} T_{p,(m,n)} + G(m, n+1)$, then $\Gamma_p = \Gamma_{p,(m_*,n)} \cup \Gamma_{(m_*,n+1)}$, and $G(p) = T_{p,(m_*,n)} + G((m_*, n+1))$.

2. For any down-right path $\mathcal{U} = \{u_k\}_{k\in\mathbb{Z}}$, the random variables $G(u_k) - G(u_{k-1})$

are independent. The law of $G(u_k) - G(u_{k-1})$ is $\mathrm{Exp}(1/2)$ if $u_k = u_{k-1} - (0,1)$, and is $-\mathrm{Exp}(1/2)$ if $u_k = u_{k-1} + (1,0)$. They are also independent of $\omega(p)$ for all $p \in \mathcal{U}_-$, where $\mathcal{U}_-$ is the lower part of $\mathbb{Z}^2 \setminus \mathcal{U}$.

Now, for each $p \in \mathbb{Z}^2$, let $\Gamma_p^\vee$ be the semi-infinite geodesic in the $(-1,-1)$ direction. We shall always assume the almost sure event where $\Gamma_p^\vee$ exists and is unique for every $p \in \mathbb{Z}^2$. We let $G^\vee$ be the $(-1,-1)$ direction Busemann function; i.e. for any $p \in \mathbb{Z}^2$ we let $G^\vee(p) = T_{\mathbf{c},p} - T_{\mathbf{c},\mathbf{0}}$, where $\mathbf{c} \in \mathbb{Z}^2$ is the coalescing point of $\Gamma_p^\vee$ and $\Gamma_{\mathbf{0}}^\vee$. By symmetry, $G^\vee$ has the same distribution and hence similar properties as G. Specifically, we have the following statements.

1. For any $p = (a,b) \in \mathbb{Z}^2$ and $n < b$, if we take $m_* = \mathrm{argmax}_{m \le a} T_{(m,n),p} + G^\vee(m, n-1)$, then $\Gamma_p^\vee = \Gamma_{(m_*,n),p} \cup \Gamma_{(m_*,n-1)}^\vee$, and $G^\vee(p) = T_{(m_*,n),p} + G^\vee(m_*, n+1)$.

2. For any down-right path $\mathcal{U} = \{p_k\}_{k \in \mathbb{Z}}$, the random variables $G^\vee(p_k) - G^\vee(p_{k-1})$ are independent. The law of $G^\vee(p_k) - G^\vee(p_{k-1})$ is $\mathrm{Exp}(1/2)$ if $p_k = p_{k-1} - (0,1)$, and is $-\mathrm{Exp}(1/2)$ if $p_k = p_{k-1} + (1,0)$. They are also independent of $\omega(p)$ for all $p \in \mathcal{U}_+$, where $\mathcal{U}_+$ is the upper part of $\mathbb{Z}^2 \setminus \mathcal{U}$,

As in Section 1.4.4, we can define the competition interface using the up-right growing interface, but now using $G^\vee$ instead of G. For simplicity of notations, in this section we will misuse the notations C_1, C_2, and Δ to denote the two clusters and competition interface defined from $G^\vee$ instead.

The clusters can also be equivalently defined as follows: C_1 consists of $p \in \mathbb{Z}^2_{\ge 0} \setminus \{\mathbf{0}\}$ such that $\Gamma_p^\vee \cap \{(x,0) : x \in \mathbb{N}\} \neq \emptyset$, and C_2 consists of $p \in \mathbb{Z}^2_{\ge 0} \setminus \{\mathbf{0}\}$ such that $\Gamma_p^\vee \cap \{(0,x) : x \in \mathbb{N}\} \neq \emptyset$. Equivalently, let $\mathbb{N} \times \{0\} \subset C_1$, and for any $p \in \mathbb{Z}_{\ge 0} \times \mathbb{N}$, if we let $m_* = \mathrm{argmax}_{m \in \mathbb{Z}} G^\vee(m,0) + T_{(m,1),p}$, then $p \in C_1$ if and only if $m_* > 0$. Also recall that Δ consist of all $p \in \mathbb{Z}^2_{\ge 0}$, such that $p + (1,0) \in C_1$, and $p + (0,1) \in C_2$. See the left panel in Figure 3.7 below for an illustration.

Now Δ and $\Gamma_{\mathbf{0}}$ are not the same almost surely, but from the discussions in Section 1.4.4 one can see that they have the same law.

Lemma 3.30 ([49, Lemma 4.3])**.** *Δ and $\Gamma_{\mathbf{0}}$ are equal in distribution.*

We next setup the duality in the directed landscape. Let $\mathcal{B}$ be a standard two sided Brownian motion. Recall from Section 3.1 that

$$\mathcal{L}^L(x,t) = \sup_{y\le 0} \mathcal{L}(y,0;x,t) + \mathcal{B}(y), \tag{3.14}$$

$$\mathcal{L}^R(x,t) = \sup_{y\ge 0} \mathcal{L}(y,0;x,t) + \mathcal{B}(y), \tag{3.15}$$

for any $x \in \mathbb{R}, t > 0$. Using the quadrangle inequality (Lemma 3.6), we can show that the function $x \mapsto \mathcal{L}^L(x,t) - \mathcal{L}^R(x,t)$ is non-increasing in x. Indeed, for any $x_1 < x_2$ and $y_1 \le 0 \le y_2$, by Lemma 3.6 we have

$$\begin{aligned}(\mathcal{L}(y_1,0;x_1,t) + \mathcal{B}(y_1)) + (\mathcal{L}(y_2,0;x_2,t) + \mathcal{B}(y_2)) \\ \ge (\mathcal{L}(y_1,0;x_2,t) + \mathcal{B}(y_1)) + (\mathcal{L}(y_2,0;x_1,t) + \mathcal{B}(y_2)).\end{aligned}$$

By taking the supremum over $y_1 \le 0 \le y_2$ in the right hand side, we have

$$\mathcal{L}^L(x_1,t) + \mathcal{L}^R(x_2,t) \ge \mathcal{L}^L(x_2,t) + \mathcal{L}^R(x_1,t).$$

We then let

$$\vartheta^{\mathcal{B}}(t) = \inf\{x \in \mathbb{R} : \mathcal{L}^L(x,t) - \mathcal{L}^R(x,t) \le 0\},$$

as given in (3.2).

The goal of this section is to establish Propositon 3.4, i.e., $\vartheta^{\mathcal{B}}$ and $\pi_{\mathbf{0}}$ are equal in distribution. Relying on Lemma 3.30, the Proposition 3.4 follows immediately from the following convergence in distribution. Recall the maps R_n and Graph from Definition 1.13.

Proposition 3.31. *For any* $0 < h$, $\mathsf{R}_n(\Gamma_{\mathbf{0}}) \cap (\mathbb{R} \times [0,h]) \to \mathrm{Graph}(\pi_{\mathbf{0}}) \cap (\mathbb{R} \times [0,h])$ *in distribution in the Hausdorff topology, as* $n \to \infty$.

Proposition 3.32. *For any* $0 < g < h$, $\mathsf{R}_n(\Delta) \cap (\mathbb{R} \times [g,h]) \to \mathrm{Graph}(\vartheta^{\mathcal{B}}) \cap (\mathbb{R} \times [g,h])$ *in distribution in the Hausdorff topology, as* $n \to \infty$.

In Proposition 3.32 we need $g > 0$ due to using Theorem 1.14, where a compact set needs to be taken for uniform convergence.

It remains to prove Proposition 3.31 and 3.32, in the following two subsections respectively.

3.4.1 Weak convergence of semi-infinite geodesics

Let us reiterate from Section 3.1, the general strategy to prove Proposition 3.31. For simplicity and by scaling invariance, it suffices to prove the result for $h = 1$.

Our proof would rely on the convergence of the exponential LPP to the directed landscape. However, a technical issue arises since the convergence stated in Theorem 1.14 is only for compact sets. Thus to use the latter, we will define a "truncated" version of the geodesic $\Gamma_{\mathbf{0}}$, parameterized by some $H > 0$. Theorem 1.14 then allows us to deduce a scaling convergence to a "truncated" version of $\pi_{\mathbf{0}}$. The final step would be to show that the truncated versions agree with the original versions of the geodesics with probabilities increasing to 1 as $H \to \infty$. For the pre-limiting semi-infinite geodesics, this needs to be done uniformly in n. This last step can also be understood as proving tightness for the pre-limiting semi-infinite geodesics, i.e., to give a uniform bound on the transversal fluctuations.

We first truncate the semi-infinite geodesic $\pi_{\mathbf{0}}$ in $\mathcal{L}$, and the idea is to force $\pi_{\mathbf{0}}(1)$ to stay in a compact interval $[-H, H]$. Recall the Busemann function $\mathcal{M}$ from Definition 3.17. For each $H > 0$, we define the *$H-$geodesic in* $\mathcal{L}$ as follows. We take $x_* = \mathrm{argmax}_{x \in [-H,H]} \mathcal{L}(\mathbf{0}; x, 1) + \mathcal{M}(x, 1; 0, 1)$, and let $\pi_{\mathbf{0},H} = \pi_{(\mathbf{0}; x_*, 1)}$.

Lemma 3.33. *Almost surely the H-geodesic $\pi_{\mathbf{0},H}$ is unique.*

This is by the uniqueness of the argmax of the sum of two weight profiles, at each rational time. Similar arguments have been used to prove uniqueness of finite geodesics in [42].

Proof. By Theorem 1.16 and Lemma 3.18, for any $0 < t < 1$ the function

$$y \mapsto \sup_{x \in [-H,H]} \mathcal{L}(y,t;x,1) + \mathcal{M}(x,1;0,1)$$

is absolutely continuous with respect to a Brownian motion, on any compact set. Moreover, on account of being a scaled parabolic Airy_2 process, the function $y \mapsto \mathcal{L}(\mathbf{0};y,t)$ is also absolutely continuous with respect to a Brownian motion on any compact set. Thus, almost surely, the function

$$y \mapsto \mathcal{L}(\mathbf{0};y,t) + \sup_{x \in [-H,H]} \mathcal{L}(y,t;x,1) + \mathcal{M}(x,1;0,1)$$

has a unique maximum in any compact interval. Since this function also has parabolic decay as $|y| \to \infty$ (by Lemma 3.7), it has a unique global maximum, which we denote as $y_{t,*}$. Then we must have $\pi_{\mathbf{0},H}(t) = y_{t,*}$ and hence almost surely the uniqueness of $\pi_{\mathbf{0},H}(t)$ holds for all rational $0 < t < 1$. By continuity of $\pi_{\mathbf{0},H}$ the conclusion follows. □

We next carry out the counterpart truncation for the pre-limiting semi-infinite geodesics. Similarly as above, we define the *exponential LPP (n,H)-geodesic* $\Gamma_{\mathbf{0},H,n}$ as follows. Recalling the Busemann function G, let

$$m_*[n] = \text{argmax}_{m \in \mathbb{Z}, |m-n| \le 2^{5/3} H n^{2/3}} T_{\mathbf{0},(m,n)} + G((m,n+1)),$$

which is almost surely unique (due to continuity of the distribution functions). Let

$\Gamma_{\mathbf{0},H,n} = \Gamma_{\mathbf{0},(m_*[n],n)}$. Noting that the constant $2^{5/3}$ is a result of the scaling function in Section 1.4.1, we have the following convergence result.

Lemma 3.34. *For any $H > 0$, we have $\mathsf{R}_n(\Gamma_{\mathbf{0},H,n}) \to \mathrm{Graph}(\pi_{\mathbf{0},H})$ in distribution in the Hausdorff topology.*

Proof. We take the coupling from Theorem 1.14 of all $\mathcal{K}_n$ and $\mathcal{L}$, on the set $\{(x,s;y,t) \in \mathbb{R}^4 : s < t \le 1\} \subset \mathbb{R}^4_{\uparrow}$. We also consider the function $\mathcal{M}_n : \mathbb{R} \to \mathbb{R}$, where

$$\mathcal{M}_n(x) = 2^{-4/3}n^{-1/3}(G((\lfloor n+2^{5/3}n^{2/3}x\rfloor, n+1)) - G((n,n+1)) + 2(\lfloor n+2^{5/3}n^{2/3}x\rfloor - n)).$$

This is in the same scale as and converges to the function $x \mapsto \mathcal{M}(x,1;0,1)$. As stated at the beginning of this section, along any down-right path, G is a random walk with i.i.d. steps. Thus $\mathcal{M}_n$ is a (rescaled) random walk with i.i.d. steps, and (using Skorokhod's representation theorem) we couple $\mathcal{M}_n$ with $x \mapsto \mathcal{M}(x,1;0,1)$ such that $\mathcal{M}_n$ converges to the latter almost surely uniformly in each compact set. We show that $\mathsf{R}_n(\Gamma_{\mathbf{0},H,n}) \to \mathrm{Graph}(\pi_{\mathbf{0},H})$ almost surely under this coupling.

We work under the probability one event where $\pi_{\mathbf{0},H}$ is unique. In particular, this implies that $\pi_{\mathbf{0},H}(1)$ is the unique maximum of the function $x \mapsto \mathcal{L}(\mathbf{0};x,1) + \mathcal{M}(x,1;0,1)$ in $[-H,H]$, and $\pi_{\mathbf{0},H}$ is the unique geodesic from $\mathbf{0}$ to $(\pi_{\mathbf{0},H}(1),1)$.

Using previous notations, $\mathsf{R}_n((m_*[n],n))$ is a maximum of the function $x \mapsto \mathcal{K}_n(\mathbf{0};x,1) + \mathcal{M}_n(x)$ $\mathsf{R}_n((m_*[n],n))$ in $[-H,H]$. Since $x \mapsto \mathcal{K}_n(\mathbf{0};x,1) + \mathcal{M}_n(x)$ uniformly converges to $x \mapsto \mathcal{L}(\mathbf{0};x,1) + \mathcal{M}(x,1;0,1)$ in $[-H,H]$, we have that $\mathsf{R}_n((m_*[n],n)) \to \pi_{\mathbf{0},H}(1)$. Since $\Gamma_{\mathbf{0},H,n}$ is the geodesic from $\mathbf{0}$ to $(m_*[n],n)$, $\mathsf{R}_n(\Gamma_{\mathbf{0},H,n})$ is a maximal geodesic set of $\mathcal{K}_n$ from $\mathbf{0}$ to $\mathsf{R}_n((m_*[n],n))$. Thus we conclude that any subsequential limit of $\mathsf{R}_n(\Gamma_{\mathbf{0},H,n})$ would be $\mathrm{Graph}(\pi)$ for some $\mathcal{L}$ geodesic π from $\mathbf{0}$ to $(\pi_{\mathbf{0},H}(1),1)$ (by Theorem 1.14), thus we must have $\pi = \pi_{\mathbf{0},H}$, and the conclusion follows. □

We next prove tightness, which says that in the pre-limiting setting, the truncated

semi-infinite geodesics are the same as the original semi-infinite geodesics with high probability. This is achieved via estimates on the transversal fluctuations of semi-infinite geodesics, given by Lemma 2.10.

Lemma 3.35. *There are constants $c, C > 0$, such that for any $H > 0$ and any n large enough (depending on H), we have $\mathbb{P}[\mathsf{R}_n(\Gamma_{\mathbf{0},H,n}) \neq \mathsf{R}_n(\Gamma_{\mathbf{0}}) \cap (\mathbb{R} \times [0,1])] < Ce^{-cH}$.*

Proof. Let $m_* \in \mathbb{Z}$ such that $(m_*, n), (m_*, n+1) \in \Gamma_{\mathbf{0}}$. Then for any $m \in \mathbb{Z}$, $m \neq m_*$, we must have

$$T_{\mathbf{0},(m_*,n)} + G((m_*, n+1)) > T_{\mathbf{0},(m,n)} + G((m, n+1))$$

since otherwise we would have $T_{\mathbf{0},(m_*,n)} + T_{(m_*,n+1),\mathbf{c}} \leq T_{\mathbf{0},(m,n)} + T_{(m,n+1),\mathbf{c}}$ where $\mathbf{c}$ is any point in $\Gamma_{(m_*,n+1)} \cap \Gamma_{(m,n+1)}$. This implies that the path obtained by concatenating $\Gamma_{\mathbf{0},(m,n)}$ and $\Gamma_{(m,n+1),\mathbf{c}}$ has a total weight no less than the total weight of the part of $\Gamma_{\mathbf{0}}$ up to $\mathbf{c}$. However, from the definition of $\Gamma_{\mathbf{0}}$ we have that the part of $\Gamma_{\mathbf{0}}$ up to $\mathbf{c}$ is the geodesic $\Gamma_{\mathbf{0},\mathbf{c}}$. Thus we get a contradiction with the assumption that there is a unique geodesic between $\mathbf{0}$ and $\mathbf{c}$.

Then the event where $\mathsf{R}_n(\Gamma_{\mathbf{0},H,n}) \neq \mathsf{R}_n(\Gamma_{\mathbf{0}}) \cap (\mathbb{R} \times [0,1])$ is equivalent to the event $|m_* - n| > 2^{5/3} H n^{2/3}$. By Lemma 2.10 its probability is upper bounded by Ce^{-cH} for some constants $c, C > 0$, so the conclusion follows. □

The relation between $\pi_{\mathbf{0}}$ and $\pi_{\mathbf{0},H}$ is more obvious, since we have that $\pi_{\mathbf{0},H}$ is the same as $\pi_{\mathbf{0}}$ on $[0,1]$, when $H > |\pi_{\mathbf{0}}(1)|$. Now we can finish the proof of Proposition 3.31.

Proof of Proposition 3.31. We just prove for the case where $h = 1$, and the case of general $h > 0$ follows similar arguments. For any continuous function f on the space of all compact subsets of $\mathbb{R}^2$ with $0 \leq f \leq 1$, by Lemma 3.34 we have $\mathbb{E}[f(\mathsf{R}_n(\Gamma_{\mathbf{0},H,n}))] \to$

$\mathbb{E}[f(\mathrm{Graph}(\pi_{\mathbf{0},H}))]$ as $n \to \infty$ for any $H > 0$. Then we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left|\mathbb{E}[f(\mathsf{R}_n(\Gamma_{\mathbf{0}}) \cap (\mathbb{R}\times[0,1]))] - \mathbb{E}[f(\mathrm{Graph}(\pi_{\mathbf{0}}) \cap (\mathbb{R}\times[0,1]))]\right| \\
&\le \limsup_{n\to\infty} \mathbb{P}[\mathsf{R}_n(\Gamma_{\mathbf{0},H,n}) \neq \mathsf{R}_n(\Gamma_{\mathbf{0}}) \cap (\mathbb{R}\times[0,1])] + \mathbb{P}[\mathrm{Graph}(\pi_{\mathbf{0},H}) \neq \mathrm{Graph}(\pi_{\mathbf{0}}) \cap (\mathbb{R}\times[0,1])] \\
&\le Ce^{-cH} + \mathbb{P}[\mathrm{Graph}(\pi_{\mathbf{0},H}) \neq \mathrm{Graph}(\pi_{\mathbf{0}}) \cap (\mathbb{R}\times[0,1])],
\end{aligned}$$

where the last inequality is by Lemma 3.35. As $H \to \infty$, we have $Ce^{-cH} \to 0$ and $\mathbb{P}[\mathrm{Graph}(\pi_{\mathbf{0},H}) \neq \mathrm{Graph}(\pi_{\mathbf{0}}) \cap (\mathbb{R}\times[0,1])] \le \mathbb{P}[|\pi_{\mathbf{0}}(1)| < H] \to 0$, by Lemma 3.22. So as $H \to \infty$ the last line decays to zero. Thus we have $\mathbb{E}[f(\mathsf{R}_n(\Gamma_{\mathbf{0}}) \cap (\mathbb{R}\times[0,1]))] \to \mathbb{E}[f(\mathrm{Graph}(\pi_{\mathbf{0}}) \cap (\mathbb{R}\times[0,1]))]$, and the conclusion follows. □

We next carry out the counterpart convergence arguments for the competition interface.

3.4.2 Weak convergence of the competition interface

For Proposition 3.32, we again just prove it for $h = 1$, without loss of generality. The proof follows a similar general framework as the proof of Proposition 3.31, since one encounters the same technical difficulty that the convergence stated in Theorem 1.14 only applies to compact sets. We similarly define a "truncated" version of Δ, and use Theorem 1.14 to deduce that its scaling limit is a "truncated" version of $\vartheta^{\mathcal{B}}$. As before, the last step entails showing that the truncated versions of Δ are the same as the original version with high probability. However, compared to the geodesics setting in the last subsection, now it is not immediate the truncated versions of $\vartheta^{\mathcal{B}}$ are close (in total variation distance) to $\vartheta^{\mathcal{B}}$, and this requires some further argument.

We first truncate $\vartheta^{\mathcal{B}}$. The idea is to only use the boundary condition $\mathcal{B}$ only in a compact interval. For each $H > 0$, we define the *H-competition interface in $\mathcal{L}$* as

follows. For any $x \in \mathbb{R}, t > 0$, we let

$$\mathcal{L}_H^L(x,t) = \sup_{-H \le y \le 0} \mathcal{L}(y,0;x,t) + \mathcal{B}(y),$$

and

$$\mathcal{L}_H^R(x,t) = \sup_{0 \le y \le H} \mathcal{L}(y,0;x,t) + \mathcal{B}(y).$$

By arguments similar to the proof of the quadrangle inequality (Lemma 3.6), or the arguments before Propositoin 3.31, we have that the function $x \mapsto \mathcal{L}_H^L(x,t) - \mathcal{L}_H^R(x,t)$ is non-increasing in x. So we can let

$$\vartheta_H^{\mathcal{B}}(t) = \inf\{x \in \mathbb{R} : \mathcal{L}_H^L(x,t) - \mathcal{L}_H^R(x,t) \le 0\}.$$

We can show that actually, almost surely, for each $t > 0$ there is exactly one $x \in \mathbb{R}$ with $\mathcal{L}_H^L(x,t) = \mathcal{L}_H^R(x,t)$.

Lemma 3.36. *For any $H > 0$, almost surely the following is true. For any $t > 0$, $|\{x \in \mathbb{R} : \mathcal{L}_H^L(x,t) - \mathcal{L}_H^R(x,t) = 0\}| = 1$. Thus via continuity of $\mathcal{L}$ and $\mathcal{B}$, $\vartheta_H^{\mathcal{B}}$ must be continuous.*

The proof proceeds as follows: first we show that $\mathcal{L}_H^L(x,t) \neq \mathcal{L}_H^R(x,t)$, for any x and $t > 0$, when they are both rational. Then if there is some $t > 0$ and $x_1 < x_2$ such that $\mathcal{L}_H^L(x_1,t) - \mathcal{L}_H^R(x_1,t) = \mathcal{L}_H^L(x_2,t) - \mathcal{L}_H^R(x_2,t) = 0$, the difference $\mathcal{L}_H^L - \mathcal{L}_H^R$ would be zero in the segment connecting (x_1,t) and (x_2,t). We then show that one can find a rational point (x_*,t_*) where $\mathcal{L}_H^L - \mathcal{L}_H^R$ equals zero, which is already ruled out by the previous line. Finally, using continuity of $\mathcal{L}_H^L - \mathcal{L}_H^R$, and that (for any $t > 0$) $\mathcal{L}_H^L(x,t) - \mathcal{L}_H^R(x,t)$ tends to ∞ (resp. $-\infty$) as $x \to -\infty$ (resp. $x \to \infty$), we conclude that there must exist some x, and hence exactly one such, with $\mathcal{L}_H^L(x,t) = \mathcal{L}_H^R(x,t)$.

Proof. We first show that for any fixed $x \in \mathbb{R}, t > 0$, almost surely we have $\mathcal{L}_H^L(x,t) \neq \mathcal{L}_H^R(x,t)$. Indeed, this would imply that the function $y \mapsto \mathcal{L}(y,0;x,t) + \mathcal{B}(y)$ has

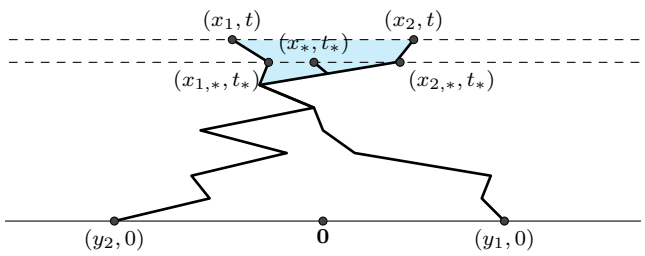


Figure 3.6: An illustration of the arguments for $|\{x \in \mathbb{R} : \mathcal{L}_H^L(x,t) - \mathcal{L}_H^R(x,t) = 0\}| \leq 1$ in the proof of Lemma 3.36: suppose that $\mathcal{L}_H^L(x_1,t) - \mathcal{L}_H^R(x_1,t) = \mathcal{L}_H^L(x_2,t) - \mathcal{L}_H^R(x_2,t) = 0$, the difference $\mathcal{L}_H^L - \mathcal{L}_H^R$ would be zero in the whole blue region, which must include a rational point (x_*, t_*).

the same maximum in $[-H, 0]$ and $[0, H]$, and almost surely this cannot happen since $y \mapsto \mathcal{L}(y, 0; x_*, t) + \mathcal{B}(y)$ in $[-H, H]$ is absolutely continuous with respect to a Brownian motion, by [36] or Theorem 1.15. Thus, almost surely $\mathcal{L}_H^L(x,t) \neq \mathcal{L}_H^R(x,t)$, for any rational x and $t > 0$.

We next show that $|\{x \in \mathbb{R} : \mathcal{L}_H^L(x,t) - \mathcal{L}_H^R(x,t) = 0\}| \leq 1$ for any (not necessarily rational) $t > 0$. For this, we use continuity of $\mathcal{L}$ and geodesics, and basic inequalities of the directed landscape.

We argue by contradiction (see Figure 3.6 for an illustration). Suppose that there is some $t > 0$ and $x_1 < x_2$ such that $\mathcal{L}_H^L(x_1,t) - \mathcal{L}_H^R(x_1,t) = \mathcal{L}_H^L(x_2,t) - \mathcal{L}_H^R(x_2,t) = 0$. Take any $y_1 = \operatorname{argmax}_{y \in [0,H]} \mathcal{L}(y, 0; x_1, t) + \mathcal{B}(y)$. Take any geodesic $\pi_{(y_1,0;x_1,t)}$. Similarly, take any $y_2 = \operatorname{argmax}_{y \in [-H,0]} \mathcal{L}(y, 0; x_2, t) + \mathcal{B}(y)$, and any geodesic $\pi_{(y_2,0;x_2,t)}$. Then we can find a rational $0 < t_* < t$ such that $x_{1,*} < x_{2,*}$, where $x_{1,*} = \pi_{(y_1,0;x_1,t)}(t_*)$

and $x_{2,*} = \pi_{(y_2,0;x_2,t)}(t_*)$. We then have that

$$\begin{aligned}
\mathcal{L}_H^L(x_{1,*}, t_*) &= \sup_{-H \le y \le 0} \mathcal{L}(y, 0; x_{1,*}, t_*) + \mathcal{B}(y) \\
&\le \sup_{-H \le y \le 0} \mathcal{L}(y, 0; x_1, t_*) - \mathcal{L}(x_{1,*}, t_*; x_1, t) + \mathcal{B}(y) \\
&= \mathcal{L}_H^L(x_1, t) - \mathcal{L}(x_{1,*}, t_*; x_1, t) \\
&= \mathcal{L}_H^R(x_1, t) - \mathcal{L}(x_{1,*}, t_*; x_1, t) \\
&= \mathcal{L}_H^R(x_{1,*}, t_*),
\end{aligned}$$

where the inequality is by the reverse triangle inequality of the directed landscape, or the composition law (1.1). Thus we have $\mathcal{L}_H^L(x_{1,*}, t_*) - \mathcal{L}_H^R(x_{1,*}, t_*) \le 0$. Similarly we have $\mathcal{L}_H^L(x_{2,*}, t_*) - \mathcal{L}_H^R(x_{2,*}, t_*) \ge 0$.

As above mentioned, by arguments similar to Lemma 3.6, or those at the beginning of this section, we have that the function $x \mapsto \mathcal{L}_H^L(x, t_*) - \mathcal{L}_H^R(x, t_*)$ is non-increasing. Now we take any rational x_* such that $x_{1,*} \le x_* \le x_{2,*}$. Then we have

$$\begin{aligned}
0 \le \mathcal{L}_H^L(x_{2,*}, t_*) - \mathcal{L}_H^R(x_{2,*}, t_*) \le \mathcal{L}_H^L(x_*, t_*) - \mathcal{L}_H^R(x_*, t_*) \\
\le \mathcal{L}_H^L(x_{1,*}, t_*) - \mathcal{L}_H^R(x_{1,*}, t_*) \le 0.
\end{aligned}$$

This implies that $\mathcal{L}_H^L(x_*, t_*) = \mathcal{L}_H^R(x_*, t_*)$. As we've shown that this almost surely cannot happen, we get a contradiction. We thus conclude that $|\{x \in \mathbb{R} : \mathcal{L}_H^L(x, t) - \mathcal{L}_H^R(x, t) = 0\}| \le 1$ for any $t > 0$.

Finally, by Lemma 3.7 and growth estimates of $\mathcal{B}$, we have that for any $t > 0$, $\mathcal{L}_H^L(x, t)$ has parabolic decay as $x \to \infty$. Also, for $x \le 0$ we have

$$\mathcal{L}_H^L(x, t) \ge \mathcal{B}(x) + \mathcal{L}(x, 0; x, t),$$

so by Lemma 3.7 and growth estimates of $\mathcal{B}$ we have that $\mathcal{L}_H^L(x, t) \ge -|x|^{1/2} \log^3(|x|)$

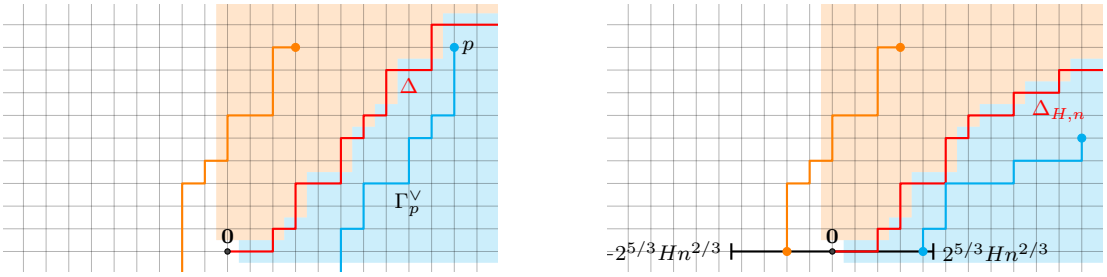


Figure 3.7: A comparison between the competition interfaces Δ and $\Delta_{H,n}$: for Δ we consider semi-infinite geodesics in the $(-1,-1)$-direction; while for $\Delta_{H,n}$, we consider the optimal paths in the $(-1,-1)$-direction passing through $[-2^{5/3}Hn^{2/3}, 2^{5/3}Hn^{2/3}]\times\{0\}$.

for all x small enough. Similalry, $\mathcal{L}_H^R(x,t)$ has parabolic decay as $x \to -\infty$, and $\mathcal{L}_H^R(x,t) \geq -|x|^{1/2}\log^3(|x|)$ for all x large enough. Thus we conclude that

$$\lim_{x\to\infty} \mathcal{L}_H^L(x,t) - \mathcal{L}_H^R(x,t) = -\infty, \quad \lim_{x\to-\infty} \mathcal{L}_H^L(x,t) - \mathcal{L}_H^R(x,t) = \infty.$$

With continuity of $\mathcal{L}_H^L - \mathcal{L}_H^R$ we get that $|\{x \in \mathbb{R} : \mathcal{L}_H^L(x,t) - \mathcal{L}_H^R(x,t) = 0\}| \geq 1$, so the conclusion follows. □

We also define the *exponential LPP* (n,H)*-competition interface* $\Delta_{H,n}$ as follows. Recall that $G^\vee$ is the $(-1,-1)$ direction Busemann function in exponential LPP, defined earlier in this section. Analogous to the un-truncated definitions, we split $\mathbb{Z}^2_{\geq 0}\setminus\{\mathbf{0}\}$ into two parts $C_{H,n,1}$ and $C_{H,n,2}$, as follows. Let $\mathbb{N}\times\{0\} \subset C_{H,n,1}$; and for any $p \in \mathbb{Z}_{\geq 0}\times\mathbb{N}$, if we let

$$m_* = \text{argmax}_{m\in\mathbb{Z},|m|\leq 2^{5/3}Hn^{2/3}} G^\vee(m,0) + T_{(m,1),p}$$

(which is almost surely unique), then $p \in C_{H,n,1}$ if and only if $m_* > 0$. We again note that the constant $2^{5/3}$ is a result of the scaling in Section 1.4.1. Let $\Delta_{H,n}$ be the boundary of $C_{H,n,1}$ and $C_{H,n,2}$: for any $p \in \mathbb{Z}^2_{\geq 0}$, we let $p \in \Delta_{H,n}$ if and only if $p+(1,0) \in C_{H,n,1}$, and $p+(0,1) \in C_{H,n,2}$. See the right panel in Figure 3.7 for an illustration.

We next prove convergence of these (n, H)-competition interfaces to the truncated competition interface in the directed landscape.

Lemma 3.37. *For any $H > 0$ and $0 < g < 1$, we have $\mathsf{R}_n(\Delta_{H,n}) \cap ([-H, H] \times [g, 1]) \to \mathrm{Graph}(\vartheta_H^{\mathcal{B}}) \cap ([-H, H] \times [g, 1])$ in distribution in the Hausdorff topology.*

We note that this where the requirement of $g > 0$ in Proposition 3.32 comes from.

In the proof we work under the coupling from Theorem 1.14, and show almost sure convergence. Note that (compared to Lemma 3.34), here we only consider the sets $\mathsf{R}_n(\Delta_{H,n})$ and $\mathrm{Graph}(\vartheta_H^{\mathcal{B}})$ in $[-H, H] \times [0, 1]$ rather than $\mathbb{R} \times [0, 1]$. This can be viewed as a further truncation, and is also due to the compactness limitation of Theorem 1.14.

Proof. We take the coupling from Theorem 1.14 of all $\mathcal{K}_n$ and $\mathcal{L}$ on the set $\{(x, s; y, t) \in \mathbb{R}^4 : 0 < s < t\} \subset \mathbb{R}^4_{\uparrow}$. We also consider the function $\mathcal{B}_n : \mathbb{R} \to \mathbb{R}$, where

$$\mathcal{B}_n(x) = 2^{-4/3} n^{-1/3} (G^{\vee}((\lceil 2^{5/3} n^{2/3} x \rceil, 0)) - 2\lceil 2^{5/3} n^{2/3} x \rceil).$$

This is a discrete version of the Brownian motion $\mathcal{B}$. Using the properties of $G^{\vee}$ stated at the beginning of this section, $\mathcal{B}_n$ is a rescaled random walk with i.i.d. steps, and is independent of $\mathcal{K}_n$ on $\{(x, s; y, t) \in \mathbb{R}^4 : 0 < s < t\}$. We can couple $\mathcal{B}_n$ with $\mathcal{B}$ such that $\mathcal{B}_n \to \mathcal{B}$ almost surely, uniformly on each compact set, by Skorokhod's representation theorem.

We show that under this coupling, almost surely, we have $\mathsf{R}_n(\Delta_{H,n}) \cap ([-H, H] \times [g, 1]) \to \mathrm{Graph}(\vartheta_H^{\mathcal{B}}) \cap ([-H, H] \times [g, 1])$.

For each $x \in \mathbb{R}$ and $t > 0$, we let

$$\mathcal{L}^L_{H,n}(x, t) = \sup_{-H \le y \le 0} \lim_{s \searrow 0} \mathcal{K}_n(y, s; x, t) + \mathcal{B}_n(y),$$

and

$$\mathcal{L}^R_{H,n}(x,t) = \sup_{0\le y\le H} \lim_{s\searrow 0} \mathcal{K}_n(y,s;x,t) + \mathcal{B}_n(y).$$

We use the limit $\lim_{s\searrow 0}\mathcal{K}_n(y,s;x,t)$ (instead of $\mathcal{K}_n(y,0;x,t)$) due to the rounding setup in Section 1.4.1: under the rouding setup, the limit depends on the values of $\mathcal{K}_n$ on $\{(x,s;y,t)\in\mathbb{R}^4 : 0<s<t\}$, thus is independent of $\mathcal{B}_n$.

From the convergence of $\mathcal{K}_n$ to $\mathcal{L}$ (Theorem 1.14) and the convergence of $\mathcal{B}_n$ to $\mathcal{B}$, we have $\mathcal{L}^L_{H,n}\to\mathcal{L}^L_H$ and $\mathcal{L}^R_{H,n}\to\mathcal{L}^R_H$, both uniformly in the set $[-2H,2H]\times[g/2,2]$. From the above constructions and the definitons of $C_{H,n,1}$ and $C_{H,n,1}$, we have that

$$C_{H,n,1} = \{p\in\mathbb{Z}^2_{\ge 0}\setminus\{\mathbf{0}\} : \mathcal{L}^L_{H,n}(\mathsf{R}_n(p)) - \mathcal{L}^R_{H,n}(\mathsf{R}_n(p)) < 0\},$$

and

$$C_{H,n,2} = \{p\in\mathbb{Z}^2_{\ge 0}\setminus\{\mathbf{0}\} : \mathcal{L}^L_{H,n}(\mathsf{R}_n(p)) - \mathcal{L}^R_{H,n}(\mathsf{R}_n(p)) > 0\}.$$

Further, from the definition of Graph, for any $(x,t)\in(-H,H)\times(g,1)$, we have that $\vartheta^{\mathcal{B}}_H(t)=x$ if $(x,t)\in\mathrm{Graph}(\vartheta^{\mathcal{B}}_H)$. Take any $\delta>0$ small enough (depending on g, H, and (x,t)). By Lemma 3.36 we have $\mathcal{L}^L_H(x-\delta/2,t)>\mathcal{L}^R_H(x-\delta/2,t)$ and $\mathcal{L}^L_H(x+\delta/2,t)<\mathcal{L}^R_H(x+\delta/2,t)$. By continuity of $\mathcal{L}$, thus of $\mathcal{L}^L_H$ and $\mathcal{L}^R_H$, and uniform convergence, we have that for any n large enough, we can find $(x_{L,n},t_n)\in\mathsf{R}_n(C_{H,n,2})$ and $(x_{R,n},t_n)\in\mathsf{R}_n(C_{H,n,1})$, in the $\delta/3$ neighborhood of $(x-\delta/2,t)$ and $(x+\delta/2,t)$ respectively. Now consider the δ neighborhood of (x,t): its pre-image under R_n must contain a lattice path from $\mathsf{R}_n^{-1}((x_{L,n},t_n))$ and $\mathsf{R}_n^{-1}((x_{L,n},t_n))$. Since $\Delta_{H,n}$ is the boundary of $C_{H,n,1}$ and $C_{H,n,2}$, the lattice path must intersect $\Delta_{H,n}$, so there must be a vertex of $\mathsf{R}_n(\Delta_{H,n})$ in the δ neighborhood of (x,t). Thus we conclude that (x,t) is in any subsequential limit of $\mathsf{R}_n(\Delta_{H,n})\cap([-H,H]\times[g,1])$.

If $(x,t)\notin\mathrm{Graph}(\vartheta^{\mathcal{B}}_H)$, by Lemma 3.36, we have $\mathcal{L}^L_H(x,t)\neq\mathcal{L}^R_H(x,t)$ and without loss of generality we assume that $\mathcal{L}^L_H(x,t)<\mathcal{L}^R_H(x,t)$. Then by continuity of $\mathcal{L}^L_H$ and $\mathcal{L}^R_H$, and uniform convergence, there is a neighborhood of (x,t), that is disjoint from

$\mathsf{R}_n(C_{H,n,1})$ for any n large enough. Thus (x,t) cannot be in any subsequential limit of $\mathsf{R}_n(\Delta_{H,n}) \cap ([-H,H] \times [g,1])$. This completes the proof. □

We next prove that the competition interfaces are with high probability the same as the truncated versions. We start with the pre-limiting ones, and the arguements are via estimates on transversal fluctuations of semi-infinite geodesics in the $(-1,-1)$ direction.

Lemma 3.38. *There are constants $c, C > 0$, such that for any $H > 0$ and any n large enough (depending on H), we have $\mathbb{P}[\mathsf{R}_n(\Delta_{H,n}) \cap ([-H,H] \times [0,1]) \neq \mathsf{R}_n(\Delta) \cap (\mathbb{R} \times [0,1])] < Ce^{-cH}$.*

Proof. Let $p_1 = (n - \lfloor n^{2/3}H \rfloor, n)$ and $p_2 = (n + \lfloor n^{2/3}H \rfloor, n)$, and U_1 be the parallelogram whose vertices are $(-2^{5/3}n^{2/3}H, 0)$, $\mathbf{0}$, and $(n - 2^{5/3}n^{2/3}H, n)$, (n,n), and $U_2 = U_1 + (2^{5/3}n^{2/3}H, 0)$. See Figure 3.8 for an illustration of these objects.

We now consider the events where $\Gamma^\vee_{p_1} \cap (\mathbb{Z} \times [\![0,n]\!]) \subset U_1$ and $\Gamma^\vee_{p_2} \cap (\mathbb{Z} \times [\![0,n]\!]) \subset U_2$. By Lemma 2.10 these happen with probability at least $1 - Ce^{-cH}$, for some constants $c, C > 0$.

It now suffices to show that these events imply that $\mathsf{R}_n(\Delta_{H,n}) \cap ([-H,H] \times [0,1]) = \mathsf{R}_n(\Delta) \cap (\mathbb{R} \times [0,1])$. Assuming that the events occur, let $m_{1,*}, m_{2,*} \in \mathbb{Z}$ such that $(m_{1,*}, 0), (m_{1,*}, 1) \in \Gamma^\vee_{p_1}$, and $(m_{2,*}, 0), (m_{2,*}, 1) \in \Gamma^\vee_{p_2}$. Then we have

$$m_{1,*} = \operatorname{argmax}_{m \in \mathbb{Z}} G^\vee((m,0)) + T_{(m,1),p_1}, \quad m_{2,*} = \operatorname{argmax}_{m \in \mathbb{Z}} G^\vee((m,0)) + T_{(m,1),p_2}.$$

Then we would have that $-2^{5/3}n^{2/3}H \leq m_{1,*} \leq 0$ and $0 < m_{2,*} \leq 2^{5/3}n^{2/3}H$. We now take $p = (x,y) \in \mathbb{Z}_{\geq 0} \times [\![0,n]\!] \setminus \{\mathbf{0}\}$ that is between $\Gamma^\vee_{p_1}$ and $\Gamma^\vee_{p_2}$; i.e. there are $x_1 \leq x \leq x_2$ such that $(x_1, y) \in \Gamma^\vee_{p_1}$ and $(x_2, y) \in \Gamma^\vee_{p_2}$.

Let $m_* \in \mathbb{Z}$ such that $(m_*, 0), (m_*, 1) \in \Gamma^\vee_p$. Due to the tree structure of the $(-1,-1)$-direction semi-infinite geodesics, we must have that $m_{1,*} \leq m_* \leq m_{2,*}$, and thus $m_* \in [-2^{5/3}Hn^{2/3}, 2^{5/3}Hn^{2/3}]$. This implies that $p \in C_{H,n,1}$ if and only if $p \in C_1$,

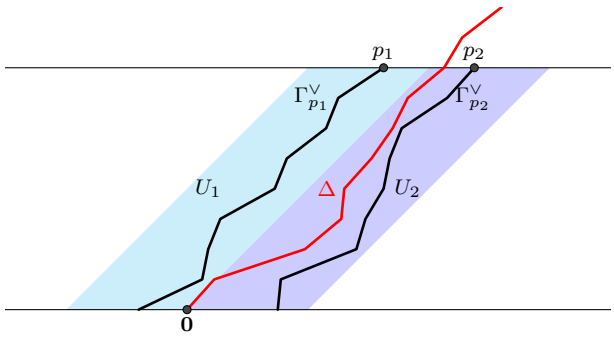


Figure 3.8: An illustration of the proof of Lemma 3.38: if $\Gamma^\vee_{p_1}$ and $\Gamma^\vee_{p_2}$ in $\mathbb{Z} \times [\![0, n]\!]$ are contained in the parallelograms U_1 and U_2, respectively, the competition interface Δ (in $\mathbb{Z} \times [\![0, n]\!]$) should be between them and equal $\Delta_{H,n}$.

and that $p \in C_{H,n,2}$ if and only if $p \in C_2$. Since $\Gamma^\vee_{p_1} \cap \mathbb{Z}^2_{\geq 0} \setminus \{\mathbf{0}\} \subset C_1 \cap U_1$ and $\Gamma^\vee_{p_2} \cap \mathbb{Z}^2_{\geq 0} \setminus \{\mathbf{0}\} \subset C_2 \cap U_2$, we conclude that $\Delta_{H,n} = \Delta \cap (\mathbb{Z} \times [\![0, n]\!]) \subset U_1 \cup U_2$, which implies that $\mathsf{R}_n(\Delta_{H,n}) \cap ([-H, H] \times [0, 1]) = \mathsf{R}_n(\Delta) \cap (\mathbb{R} \times [0, 1])$. □

As we have pointed out, unlike for semi-infinite geodesics, it is not immediate that $\vartheta^{\mathcal{B}}_H$ becomes $\vartheta^{\mathcal{B}}$ when H is large enough. We prove this result now.

Lemma 3.39. *Almost surely there is a (random) $H_0 > 0$, such that* $\mathrm{Graph}(\vartheta^{\mathcal{B}}_H) \cap ([-H, H] \times [0, 1]) = \mathrm{Graph}(\vartheta^{\mathcal{B}}) \cap (\mathbb{R} \times [0, 1])$ *for any $H > H_0$.*

Proof. We let C denote a large constant in this proof, whose value may change from line to line. Let R_1 be given by Lemma 3.7, and let $R_2 = \sup_x |\mathcal{B}(x)| - |x|$. Let $R = R_1 \vee R_2 \vee C$. For any $x, y \in \mathbb{R}$ and $0 < t \leq 1$, suppose that $|y| > 10|x| \vee 10$, then we have

$$\begin{aligned}
&\mathcal{L}(0, 0; x, t) - \mathcal{L}(y, 0; x, t) - \mathcal{B}(y) \\
>& -\frac{x^2}{t} + \frac{(x-y)^2}{t} - |y| - R \\
&- 2Rt^{1/3} \log^{4/3}(2(|x| + |y| + 3)^{3/2}/t) \log^{2/3}(|x| + |y| + 3) \\
\geq& \frac{y^2 - 2xy}{t} - |y| - R - 2R \log^2(2(|x| + |y| + 3)^{3/2}) \\
>& \frac{y^2}{2} - 5R \log^2(10(|y| + 1)).
\end{aligned}$$

Since the last line is positive if y is large enough, this implies that we can find some (random) $H_1 > 0$, such that for any $H > 10H_1$, $|x| < H_1$ and $0 < t \le 1$, we have $\mathcal{L}^L_H(x,t) = \mathcal{L}^L(x,t)$ and $\mathcal{L}^R_H(x,t) = \mathcal{L}^R(x,t)$. We also have that for any $x, y \in \mathbb{R}$ and $0 < t \le 1$,

$$\begin{aligned}
&\mathcal{L}(x,0;x,t) + \mathcal{B}(x) - \mathcal{L}(y,0;x,t) - \mathcal{B}(y)\\
>&\frac{(x-y)^2}{t} - |x| - |y| - 2R\\
&- 2Rt^{1/3}\log^{4/3}(2(2|x| + |y| + 3)^{3/2}/t)\log^{2/3}(2|x| + |y| + 3)\\
\ge&\frac{(x-y)^2}{t} - |x| - |y| - 2R - 2R\log^2(2(2|x| + |y| + 3)^{3/2})\\
>&\frac{(x-y)^2}{t} - |x| - |y| - 5R\log^2(10(|x| + |y| + 1)).
\end{aligned}$$

This implies that we can find some (random) $H_2 > 0$, such that $\mathcal{L}^L(x,t) < \mathcal{L}^R(x,t)$ for any $x > H_2$, $0 < t \le 1$; and $\mathcal{L}^L(x,t) > \mathcal{L}^R(x,t)$ for any $x < -H_2$, $0 < t \le 1$. By taking $H_0 = 10H_1 \vee H_2$ the conclusion follows. □

Now we can deduce Proposition 3.32, and its proof is similar to the proof of Proposition 3.31.

Proof of Proposition 3.32. Again we just prove for the case where $h = 1$, and the case of general $h > 0$ follows similar arguments. For any continuous function f on the space of all compact subsets of $\mathbb{R}^2$ with $0 \le f \le 1$, by Lemma 3.37 we have

$$\mathbb{E}[f(\mathsf{R}_n(\Delta_{H,n}) \cap ([-H,H] \times [g,1]))] \to \mathbb{E}[f(\mathrm{Graph}(\vartheta^{\mathcal{B}}_H) \cap ([-H,H] \times [g,1]))]$$

as $n \to \infty$ for any $H > 0$. Then by Lemma 3.38 we have

$$\begin{aligned}
&\limsup_{n\to\infty} |\mathbb{E}[f(\mathsf{R}_n(\Delta) \cap (\mathbb{R} \times [g,1]))] - \mathbb{E}[f(\mathrm{Graph}(\vartheta^{\mathcal{B}}) \cap (\mathbb{R} \times [g,1]))]|\\
&\qquad \le Ce^{-cH} + \mathbb{P}[\mathrm{Graph}(\vartheta^{\mathcal{B}}_H) \cap ([-H,H] \times [g,1]) \ne \mathrm{Graph}(\vartheta^{\mathcal{B}}) \cap (\mathbb{R} \times [g,1])].
\end{aligned}$$

As $H \to \infty$ the right hand side decays to zero, by Lemma 3.39. Thus we have $\mathbb{E}[f(\mathsf{R}_n(\Delta) \cap (\mathbb{R} \times [g,1]))] \to \mathbb{E}[f(\mathrm{Graph}(\vartheta^{\mathcal{B}}) \cap (\mathbb{R} \times [g,1]))]$, and the conclusion follows. □

With the above, somewhat long, sequence of preparatory results, we can finally delve in to the proof of Theorem 1.9.

3.5 Hausdorff dimension of NC: Proof of Theorem 1.9

Our first observation is that for the difference profile $\mathcal{D}$, being constant in a 2D neighborhood is equivalent to being constant in the space direction. Recall the sets $\overline{\mathsf{NC}}_t$ and NC from the statement of Theorem 1.9.

Lemma 3.40. *Almost surely, we have* $\mathsf{NC} \cap (\mathbb{R} \times \{t\}) = \overline{\mathsf{NC}}_t$ *for any* $t > 0$.

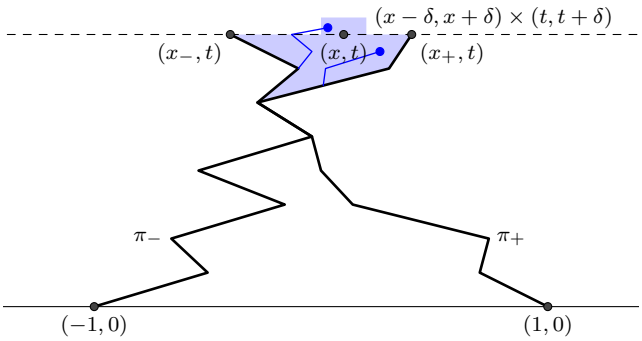


Figure 3.9: An illustration of the proof of Lemma 3.40: we prove that $\mathcal{D}$ is flat in the blue region.

The idea is to show that, if $\mathcal{D}$ is constant on a segment $(x_-, x_+) \times \{t\}$, it would also be constant in a 2D open neighborhood of it, using monotonicity of $\mathcal{D}$ and transversal fluctuation estimates of geodesics.

Proof. The inclusion of $\overline{\mathsf{NC}}_t \subset \mathsf{NC} \cap (\mathbb{R} \times \{t\})$ is obvious, and we prove the other inclusion now. Take any $(x,t) \in \mathbb{H}$ such that $(x,t) \notin \overline{\mathsf{NC}}_t$, i.e., $\mathcal{D}(\cdot,t)$ is constant in a neighborhood of x. Denote $X = \mathcal{D}(x,t)$. We can find $x_- < x < x_+$ such that

$$\mathcal{D}(x_-,t) = \mathcal{D}(x_+,t) = X.$$

We will show that the difference profile is constant in a 2D neighborhood of (x,t). Although the proof below mainly consists of equations, there is a geometric heuristic behind this, as illustrated by Figure 3.9. From the equality, we would expect that the geodesics $\pi_{(-1,0;x_-,t)}$ and $\pi_{(1,0;x_+,t)}$ coalesce, assuming their uniqueness (by Lemma 3.13). This then implies that for all p and p' in a neighborhood of (x,t), $\pi_{(-1,0;p)}$ and $\pi_{(1,0;p')}$ coalesce, due to transversal fluctuation estimates and the ordering of the geodesics, and this would imply that the difference profile is constant in the neighborhood of (x,t), whose generic point is, say, (z,s).

We argue separately for (z,s) with $s < t$ (case (1)) or $s > t$ (case (2)).

Case 1. We let π_- and π_+ be any geodesics from $(1,0)$ to (x_-,t), and $(-1,0)$ to (x_+,t) respectively. Take any $0 < s < t$ with $\pi_-(s) < \pi_+(s)$. We then have

$$\mathcal{D}(\pi_-(s),s) = \mathcal{L}(-1,0;\pi_-(s),s) - \mathcal{L}(1,0;x_-,t) + \mathcal{L}(\pi_-(s),s;x_-,t) \le \mathcal{D}(x_-,t) =: X$$

and

$$\mathcal{D}(\pi_+(s),s) = \mathcal{L}(-1,0;x_+,t) - \mathcal{L}(1,0;\pi_+(s),s) - \mathcal{L}(\pi_+(s),s;x_+,t) \ge \mathcal{D}(x_+,t) = X.$$

By monotonicity of $\mathcal{D}(\cdot,s)$, for any $z \in (\pi_-(s),\pi_+(s))$ we have

$$\mathcal{D}(x_-(s),s) \ge \mathcal{D}(z,s) \ge \mathcal{D}(x_+(s),s),$$

so $\mathcal{D}(z,s) = X$. Since the geodesics π_- and π_+ are continuous, we conclude that there exists some $\delta > 0$, such that for any $t-\delta < s \leq t$ and $|z-x| < \delta$, we have $\mathcal{D}(z,s) = X$.

Case 2: Take any $s \geq t$ and $z \in \mathbb{R}$. Consider any geodesics π'_- from $(-1,0)$ to (z,s), and π'_+ from $(1,0)$ to (z,s). We claim that if $\pi'_-(t), \pi'_+(t) \in [x_-, x_+]$, we must have $\mathcal{D}(z,s) = X$. Indeed, assuming that $\pi'_-(t), \pi'_+(t) \in [x_-, x_+]$, we have

$$\mathcal{D}(z,s) \leq \mathcal{D}(\pi'_-(t), t) \leq \mathcal{D}(x_-, t) = X,$$

where the first inequality is by $\mathcal{L}(-1,0;z,s) = \mathcal{L}(-1,0;\pi'_-(t),t) + \mathcal{L}(\pi'_-(t),t;z,s)$ and $\mathcal{L}(1,0;z,s) \geq \mathcal{L}(1,0;\pi'_-(t),t) + \mathcal{L}(\pi'_-(t),t;z,s)$, and the second inequality is by Lemma 3.6. Similarly we have

$$\mathcal{D}(z,s) \geq \mathcal{D}(\pi'_+(t), t) \geq \mathcal{D}(x_+, t) = X,$$

so the claim follows.

By Lemma 3.9, (almost surely) we can find some $\delta > 0$, such that for any $t \leq s \leq t + \delta$ and $|z - x| < \delta$, and any geodesic π' from $(-1,0)$ to (z,s) or from $(1,0)$ to (z,s), there is $\pi'(t) \in [x_-, x_+]$. Thus by the above claim we have $\mathcal{D}(z,s) = X$ (for each $(z,s) \in [t, t+\delta] \times (x-\delta, x+\delta)$) and hence the proof is complete. □

We now are in a position to embark on proving Theorem 1.9. The upper bound is straightforward, and the arguments are similar to those in [10]. The lower bound is significantly more delicate, and we now briefly reiterate our strategy from Section 3.1.

We define a measure ζ on $\mathbb{H}$, such that for any $a < b$ and $0 \leq g < h$,

$$\zeta([a,b] \times (g,h)) = \int_g^h [\mathcal{D}(a,t) - \mathcal{D}(b,t)]dt. \tag{3.16}$$

We first observe that the support of ζ is contained in NC, since for any $(x,t) \in \mathbb{H}\setminus\mathsf{NC}$, there is an open neighborhood of (x,t) with zero measure under ζ. We further prove that ζ is not degenerate, i.e., $\zeta(\mathbb{H}) > 0$. In fact, by Lemma 3.7, as $a \to \infty$ we have

$$\mathcal{D}(-a,t) - \mathcal{D}(a,t) \to \infty$$

uniformly for $t \in [g,h]$, for any $0 < g < h$. Thus we conclude that

$$\zeta(\mathbb{H}) \geq \lim_{a\to\infty} \zeta([-a,a]\times[g,h]) = \infty. \tag{3.17}$$

Given the above no-degeneracy of the measure, to show that NC has Hausdorff dimension at least $5/3$, it now suffices to prove the following "2D Hölder" bound.

Proposition 3.41. *For any $0 < g < h$ and $N, \epsilon > 0$, almost surely*

$$\sup_{A\subset[-N,N]\times[g,h]} \frac{\zeta(A)}{\operatorname{diam}(A)^{5/3-\epsilon}} < \infty.$$

As indicated in Section 3.1, to prove such bound on the measure ζ, we will obtain a "level set decomposition" of it.

3.5.1 Level set decomposition

We consider "level functions" of the difference profile, as given in (3.1); i.e., for each $\ell \in \mathbb{R}$ and $t \in \mathbb{R}_+$, we take

$$\vartheta_\ell(t) = \inf\{x \in \mathbb{R} : \mathcal{D}(x,t) \leq \ell\}.$$

Then by continuity of $\mathcal{L}$ we have $\mathcal{D}(\vartheta_\ell(t),t) = \ell$. Actually, for all but countably many ℓ, the infimum in the above definition is not needed.

Lemma 3.42. *For each $\ell \in \mathbb{R}$, except for countably many of them, there is a unique $x \in \mathbb{R}$ with $\mathcal{D}(x,t) = \ell$ for every $t \in \mathbb{R}_+$.*

Proof. Suppose that for some $\ell \in \mathbb{R}$ there are $t \in \mathbb{R}_+$ and $x_1 < x_2$, such that $\mathcal{D}(x_1,t) = \mathcal{D}(x_2,t) = \ell$. By monotonicity of $\mathcal{D}(\cdot,t)$, we have $\mathcal{D}(x,t) = \ell$ for any $x_1 \le x \le x_2$. Then by Lemma 3.40, one can find some rational x' and $t' > 0$ with $\mathcal{D}(x',t') = \ell$. Thus the conclusion follows. □

Another basic property of ϑ_ℓ is its continuity.

Lemma 3.43. *For each $\ell \in \mathbb{R}$, ϑ_ℓ is a continuous function on $\mathbb{R}_+$.*

As a standard exercise in measure theory, using Lemma 3.43 and the fact that $\ell \mapsto \vartheta_\ell(t)$ is right continuous for each t, one can show that $(\ell,t) \mapsto \vartheta_\ell(t)$ is a measurable function on $\mathbb{H}$. Now, for each $\ell \in \mathbb{R}$, we define the occupation measure of $\vartheta_\ell(\cdot)$ to be denoted ζ_ℓ on $\mathbb{H}$, such that for any $0 \le g < h$ and $a < b$, we have

$$\zeta_\ell([a,b] \times (g,h)) = \int_g^h \mathbb{1}[\vartheta_\ell(t) \in [a,b]]dt. \tag{3.18}$$

Recalling (3.16), observing that $\mathcal{D}(a,t) - \mathcal{D}(b,t) = \int \mathbb{1}[\vartheta_\ell(t) \in [a,b]]d\ell$, using Fubini's theorem, we have that

$$\begin{aligned}\zeta([a,b] \times (g,h)) &= \int_g^h \int \mathbb{1}[\vartheta_\ell(t) \in [a,b]]d\ell dt \\ &= \int \int_g^h \mathbb{1}[\vartheta_\ell(t) \in [a,b]]dtd\ell = \int \zeta_\ell([a,b] \times (g,h))d\ell.\end{aligned}$$

Thus (by standard measurable theory arguments)

$$\zeta(A) = \int \zeta_\ell(A)d\ell \tag{3.19}$$

for any measurable $A \subset \mathbb{H}$. Then to bound $\zeta(A)$, we shall bound an average of $\zeta_\ell(A)$ over ℓ.

Before proceeding further we prove the continuity of each ϑ_ℓ.

Proof of Lemma 3.43. We argue by contradiction. Suppose that ϑ_ℓ is not continuous at some $t > 0$. Denote $x = \vartheta_\ell(t)$. Then there is a sequence $\{t_k\}_{k=1}^{\infty}$, such that $t_k \to t$ while $\vartheta_\ell(t_k)$ does not converge to x, as $k \to \infty$. By Lemma 3.7, $|\vartheta_\ell(t_k)|$ are bounded as $k \to \infty$, so by taking a subsequence we can assume that $\lim_{k\to\infty} \vartheta_\ell(t_k) = y$ for some $y \neq x$. Since $\mathcal{D}(\vartheta_\ell(t_k), t_k) = \ell$ for each k, by continuity of $\mathcal{L}$ we have $\mathcal{D}(y, t) = \ell$. Thus we have $y > x$. We take some $x < z < y$. By Lemma 3.40, there is some $\delta > 0$, such that for any $s \in (t - \delta, t + \delta)$, $\mathcal{D}(z, s) = \ell$. In particular, we can find some large enough k, such that $t_k \in (t - \delta, t + \delta)$, and $z < \vartheta_\ell(t_k)$, whereas $\mathcal{D}(z, t_k) = \ell$. This contradicts the definition of $\vartheta_\ell(t_k)$. □

3.5.2 Comparison to competition interface

To analyze the measures ζ_ℓ, we study the level functions ϑ_ℓ, by connecting them with the competition interface starting from the stationary initial condition.

We start by recalling certain objects. Namely, recall that we let $\mathcal{B}$ be a standard two sided Brownian motion, $\mathcal{L}^L(x, t)$ and $\mathcal{L}^R(x, t)$ be the left and right weight profiles from $\mathcal{B}$, and that $\vartheta^{\mathcal{B}}$ denotes the competition interface (from (3.2)). Recall also the equality in law statement in Proposition 3.4. We define $\zeta^{\mathcal{B}}$ as the occupation measure for $\vartheta^{\mathcal{B}}$ on $\mathbb{H}$, i.e., for any $a < b$ and $0 \leq g < h$,

$$\zeta^{\mathcal{B}}([a, b] \times (g, h)) = \int_g^h \mathbb{1}[\vartheta^{\mathcal{B}}(t) \in [a, b]]dt. \tag{3.20}$$

As indicated in Section 3.1, the main idea is to show that, on an event with positive probability, $\vartheta^{\mathcal{B}}$ equals ϑ_ℓ for some random ℓ; thus bounds on $\zeta^{\mathcal{B}}$ would imply bounds on ζ_ℓ averaged over ℓ. The former would be obtained by comparison with the semi-infinite geodesic $\pi_{\mathbf{0}}$.

The following is a crucial estimate on the occupation measure $\zeta^{\mathcal{B}}$ (and is equivalent to Lemma 3.3).

Lemma 3.44. *There are universal constants $C, c > 0$ such that the following is true. For any closed interval $I \subset \mathbb{R}$ and $J \subset \mathbb{R}_{\geq 0}$, we have*

$$\mathbb{P}[\zeta^{\mathcal{B}}(I \times J) > M\mathscr{L}(I)\mathscr{L}(J)^{1/3}] < Ce^{-cM}$$

for any $M > 0$.

However, it turns out that for later applications, we will need a stronger version (see Lemma 3.54) of the above. As already mentioned, the proofs are adaptations of arguments appearing previously in [92]. For completeness we include the proofs in the Appendix B.

In the remainder of this section we finish the proof of Theorem 1.9. In Section 3.5.3 we set up the events where $\vartheta^{\mathcal{B}}$ equals ϑ_ℓ for some random ℓ. We prove Proposition 3.41 in Section 3.5.4, using Lemma 3.44, and the relations between $\vartheta^{\mathcal{B}}$ and the level functions. In Section 3.5.5 we prove the upper bound on the Hausdorff dimension.

3.5.3 Spiky boundary behavior

We will first establish how one can relate $\vartheta^{\mathcal{B}}$, and ϑ_ℓ for different ℓ. For each $0 < g < h$, $N > 0$, and $\delta > 0$, let $U_{g,h,N}$ denote the box $[-N, N] \times [g/2, 2h]$. We consider the event $\text{Good}_{g,h,N,\delta}$, where the following natural conditions (with somewhat technical looking quantifiers which the reader could without hesitation pay no attention to on first read) hold.

1. Coalescence of geodesics to $U_{g,h,N}$: there exist some $t_-, t_+ > 0$, $x_-, x_+ \in \mathbb{R}$, such that for any $x \in (-1-\delta, -1+\delta)$ and $q \in U_{g,h,N}$, there is $\pi_{(x,0;q)}(t_-) = x_-$; and for any $x \in (1-\delta, 1+\delta)$ and $q \in U_{g,h,N}$, there is $\pi_{(x,0;q)}(t_+) = x_+$. See the top half of Figure 3.10 for an illustration of the event.

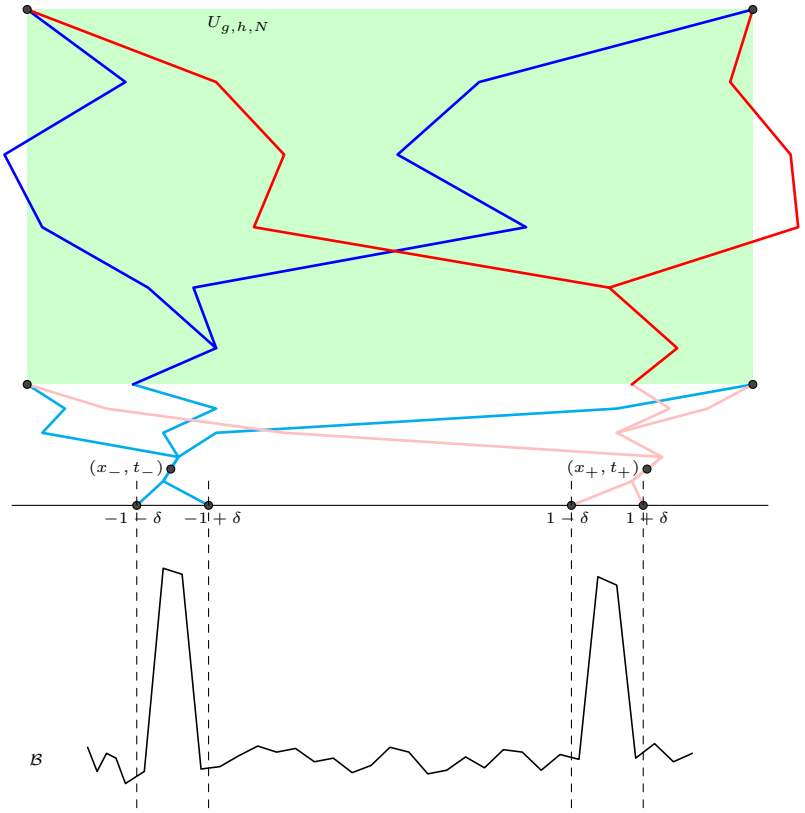


Figure 3.10: The top half of this figure illustrates the event $\mathrm{Good}_{g,h,N,\delta}$, and the bottom half of this figure illustrates the event where $\mathcal{B}$ has spikes around -1 and 1. These two events imply that for any p in the box $U_{g,h,N}$, the optimal paths would land near -1 or 1.

2. Regularity of the directed landscape: $R < \delta^{-0.01}$, where R is the maximum of the random variables given by Lemma 3.7, Lemma 3.8 for $K = ([-4N, 4N] \times [-1, g/6]) \times ([-4N, 4N] \times [g/3, 2h])$, and Lemma 3.9.

The next result says that the above event holds with high probability as δ tends to 0.

Lemma 3.45. *For any $0 < g < h$, $N > 0$, we have $\lim_{\delta \searrow 0} \mathbb{P}[\mathrm{Good}_{g,h,N,\delta}] = 1$.*

Proof. We denote the two events as $\mathcal{E}_1, \mathcal{E}_2$ respectively.

By Lemmas 3.7, 3.8, and 3.9, we have $\mathbb{P}[\mathcal{E}_2] \to 1$ as $\delta \searrow 0$. Thus it remains to analyze $\mathcal{E}_1$. Assuming that $\mathcal{E}_2$ holds, by Lemma 3.9, there exists some $Q \in \mathbb{N}$, determined by g, h, δ, N (which are the parameters appearing in the definition of the

good event), such that

$$\pi_{(-1-\delta,0;Q,2h)}(t) > N,\ \pi_{(1+\delta,0;-Q,2h)}(t) < -N$$

for any $g/2 \leq t \leq 2h$. By Theorem 3.14, almost surely there is some random $\delta' > 0$, such that $\pi_{(-1-\delta',0;-Q,2h)}$ intersects $\pi_{(-1+\delta',0;Q,2h)}$, and $\pi_{(1-\delta',0;-Q,2h)}$ intersects $\pi_{(1+\delta',0;Q,2h)}$, and all of these geodesics are unique. Then we just take t_-, x_- such that $\pi_{(-1-\delta',0;-Q,2h)}(t_-) = \pi_{(-1+\delta',0;Q,2h)}(t_-) = x_-$, and t_+, x_+ such that $\pi_{(1-\delta',0;-Q,2h)}(t_+) = \pi_{(1+\delta',0;Q,2h)}(t_+) = x_+$. Thus by the ordering of geodesics (Lemma 3.12), we have that for any $x \in (-1-\delta', -1+\delta')$ and $q \in U_{g,h,N}$, there is $\pi_{(x,0;q)}(t_-) = x_-$; and for any $x \in (1-\delta', 1+\delta')$ and $q \in U_{g,h,N}$, there is $\pi_{(x,0;q)}(t_+) = x_+$. This implies that $\mathbb{P}[\mathcal{E}_2 \setminus \mathcal{E}_1] \to 0$ as $\delta \searrow 0$. Thus the conclusion follows. □

We next consider the key event $\mathrm{ConstDiff}_{g,h,N}$ which allows us to relate the difference profile rooted at $(\pm 1, 0)$ to the one corresponding to Brownian initial data. This is defined as follows: there exists a (random) number α, such that for any $p \in U_{g,h,N}$, there is

$$\alpha = \mathcal{D}(p) - (\mathcal{L}^L(p) - \mathcal{L}^R(p)),$$

i.e., the two difference profiles are a shift of each other when restricted to $U_{g,h,N}$. As an useful consequence, we have that under $\mathrm{ConstDiff}_{g,h,N}$,

$$\{(\vartheta_\alpha(t), t) : t \geq 0\} \cap U_{g,h,N} = \{(\vartheta^{\mathcal{B}}(t), t) : t \geq 0\} \cap U_{g,h,N}.$$

We next come to the crucial result which allows us to go between the level sets of $\mathcal{D}$ and $\vartheta^{\mathcal{B}}$.

Lemma 3.46. *For any $0 < g < h$, $N > 0$, and any $\delta > 0$, $H > 0$, there is some $\delta' > 0$, such that for any interval $I \subset [-H, H]$, and any sampling of $\mathcal{L}$ such that*

$\text{Good}_{g,h,N,\delta}$ *holds, we have*

$$\mathbb{P}[\text{ConstDiff}_{g,h,N}, \alpha \in I \mid \mathcal{L}] > \delta' \mathscr{L}(I).$$

where $\mathscr{L}(\cdot)$ *denotes the Lebesgue measure.*

Re-iterating from Section 3.1, the proof of this lemma is to force $\mathcal{B}$ to have spikes around -1 and 1. Thus for any p in a box $U_{g,h,N}$, the optimal path to the left half boundary (i.e. the path corresponding to $\mathcal{L}^L(p)$) would land near -1; and the optimal path to the right half boundary (i.e. the path corresponding to $\mathcal{L}^R(p)$) would land near 1, respectively. See Figure 3.10 for an illustration. Further, we will show that on the coalescence event $\text{Good}_{g,h,N,\delta}$, the event $\text{ConstDiff}_{g,h,N}$ would hold. Finally, by a simple argument involving resampling $\mathcal{B}$, we will also establish that the random difference α is independent of the events $\text{Good}_{g,h,N,\delta}$ and $\mathcal{B}$ being spiky and has a smooth distribution.

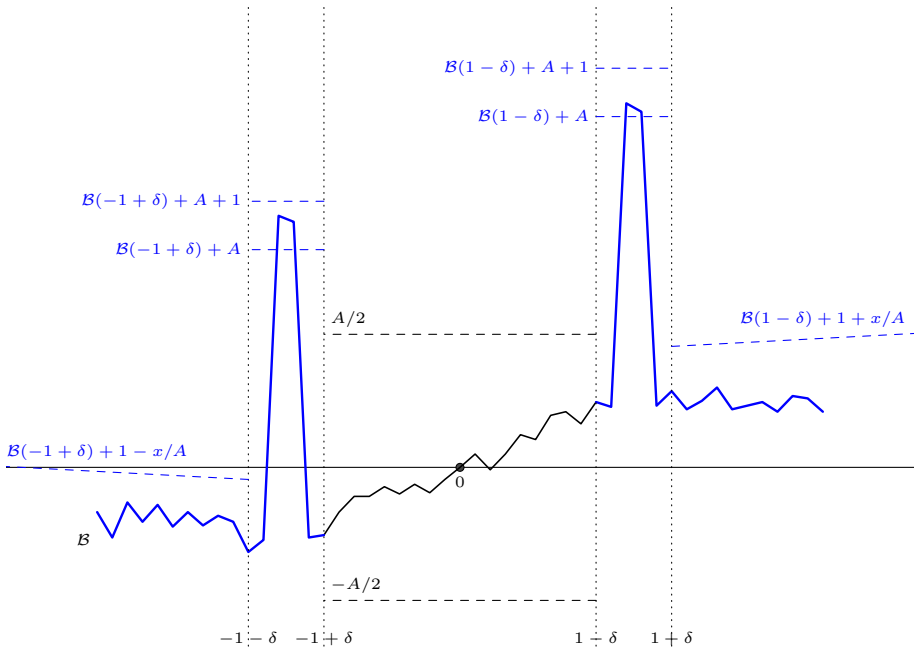


Figure 3.11: An illustration of the events in the proof of Lemma 3.46: the blue curves are the processes $\mathcal{B}(1-\delta+x)-\mathcal{B}(1-\delta)$ and $\mathcal{B}(-1+\delta-x)-\mathcal{B}(-1+\delta)$ for $x>0$, which determine the event $\mathcal{E}$. Conditional on these two processes, $\mathcal{B}$ within $[-1-\delta,-1+\delta]$ is still a two-sided Brownian motion, and determines $\mathcal{E}'$.

Proof. Since $\mathcal{L}$ is conditioned on, the only randomness is from the Brownian motion $\mathcal{B}$. The general idea is to make $\mathcal{B}$ "spiky" in the intervals $[-1-\delta, -1+\delta]$ and $[1-\delta, 1+\delta]$, such that for any $p \in U_{g,h,N}$, we have

$$\mathcal{L}^L(p) = \sup_{|y+1|\le\delta} \mathcal{L}(y, 0; p) + \mathcal{B}(y), \quad \mathcal{L}^R(p) = \sup_{|y-1|\le\delta} \mathcal{L}(y, 0; p) + \mathcal{B}(y). \tag{3.21}$$

Assuming $\text{Good}_{g,h,N,\delta}$, we have that (3.21) implies $\text{ConstDiff}_{g,h,N}$. Indeed, by the coalecence condition of the event $\text{Good}_{g,h,N,\delta}$ (as illustrated by Figure 3.10, and recall the points (x_-, t_-) and (x_+, t_+) there), we have for any $p_1, p_2 \in U_{g,h,N}$,

$$\mathcal{L}^L(p_1) - \mathcal{L}^L(p_2) = \mathcal{L}(x_-, t_-; p_1) - \mathcal{L}(x_-, t_-; p_2) = \mathcal{L}(-1, 0; p_1) - \mathcal{L}(-1, 0; p_2),$$
$$\mathcal{L}^R(p_1) - \mathcal{L}^R(p_2) = \mathcal{L}(x_+, t_+; p_1) - \mathcal{L}(x_+, t_+; p_2) = \mathcal{L}(1, 0; p_1) - \mathcal{L}(1, 0; p_2).$$

These imply that $\mathcal{D}(p_1) - (\mathcal{L}^L(p_1) - \mathcal{L}^R(p_1)) = \mathcal{D}(p_2) - (\mathcal{L}^L(p_2) - \mathcal{L}^R(p_2))$, so $\text{ConstDiff}_{g,h,N}$ holds. The value of α (from the definition of the event $\text{ConstDiff}_{g,h,N}$), can be altered by "adjusting the difference in the heights of the spikes", i.e., by adding different constants to $\mathcal{B}$ in $[-1-\delta, -1+\delta]$ and $[1-\delta, 1+\delta]$. This is done by decomposing the process $\mathcal{B}$ into three independent parts: the processes $\mathcal{B}(1-\delta+x) - \mathcal{B}(1-\delta)$ and $\mathcal{B}(-1+\delta-x) - \mathcal{B}(-1+\delta)$ for $x > 0$, and $\mathcal{B}$ in $[-1+\delta, 1-\delta]$, and modifying the last part while fixing the first two. See Figure 3.11 for an illustration of some of the events we define next.

Take A to be a large enough number depending on g, h, δ, N (which are parameters in the definition of the good event) and H. We consider the event $\mathcal{E}$, given by the following conditions:

- $\sup_{x<-1-\delta} \mathcal{B}(x) - \mathcal{B}(-1+\delta) + x/A < 1$,
- $\sup_{x>1+\delta} \mathcal{B}(x) - \mathcal{B}(1-\delta) - x/A < 1$,
- $\sup_{x\in(-1-\delta,-1+\delta)} \mathcal{B}(x) - \mathcal{B}(-1+\delta) \in [A, A+1]$,

- $\sup_{x\in(1-\delta,1+\delta)} \mathcal{B}(x) - \mathcal{B}(1-\delta) \in [A, A+1]$.

In words, the first two conditions stipulate a bound for the processes $\mathcal{B}(1-\delta+x) - \mathcal{B}(1-\delta)$ and $\mathcal{B}(-1+\delta-x) - \mathcal{B}(-1+\delta)$for $x > 2\delta$ that force them to be essentially bounded by 1 for an interval of size A (a large number) while beyond that, not growing faster than the linear function $|x|/A$. Note that the last part can be ensured in fact by any function that grows faster than $\sqrt{x}$.

The third and fourth conditions say that the processes $\mathcal{B}(1-\delta+x) - \mathcal{B}(1-\delta)$ and $\mathcal{B}(-1+\delta-x) - \mathcal{B}(-1+\delta)$ attain the value A in $[0, 2\delta]$. Thus together, the above ensures the spiky behavior we desire.

As indicated above, we also want to adjust the difference in the spike heights and to be able to do that we define for any $x \in \mathbb{R}, t > 0$, the following proxies for $\mathcal{L}^L(x,t)$ and $\mathcal{L}^R(x,t)$, by restricting the range of y and shifting by the value $\mathcal{B}$ at the endpoint. Namely,

$$\hat{\mathcal{L}}^L(x,t) = \sup_{y\le -1+\delta} \mathcal{L}(y,0;x,t) + \mathcal{B}(y) - \mathcal{B}(-1+\delta),$$

and

$$\hat{\mathcal{L}}^R(x,t) = \sup_{y\ge 1-\delta} \mathcal{L}(y,0;x,t) + \mathcal{B}(y) - \mathcal{B}(1-\delta).$$

Observe that $\hat{\mathcal{L}}^L$ and $\hat{\mathcal{L}}^R$ are independent of $\mathcal{B}$ in $[-1+\delta, 1-\delta]$ and we will use the randomness in the latter to alter the heights of the spikes. Recalling the parameter g appearing in the definition of $\mathrm{Good}_{g,h,N,\delta}$, let $p_0 = (0,g)$, and define α' via

$$\alpha' = \mathcal{D}(p_0) - (\hat{\mathcal{L}}^L(p_0) - \hat{\mathcal{L}}^R(p_0)).$$

Recalling that I is the interval in the statement of Lemma 3.46, we next consider the event $\mathcal{E}'$, where

- $\mathcal{B}(-1+\delta) - \mathcal{B}(1-\delta) \in \alpha' - I$;
- $\sup_{|x|\le 1-\delta} |\mathcal{B}(x)| < A/2$.

Note that for a given α', the event $\mathcal{E}'$ is independent of $\mathcal{E}$. Using basic properties of Brownian motions, we can find some $\delta' > 0$, depending on all parameters, such that $\mathbb{P}[\mathcal{E}] > \sqrt{\delta'}$; and conditional on $\mathcal{E}$, and any $\mathcal{L}$ such that $\mathrm{Good}_{g,h,N,\delta}$ holds, we have $\mathbb{P}[\mathcal{E}' \mid \mathcal{E}, \mathcal{L}] > \sqrt{\delta'}\mathscr{L}(I)$. Here for the second part, we use that $\mathbb{P}[\mathcal{E}' \mid \mathcal{E}, \mathcal{L}, \alpha'] = \mathbb{P}[\mathcal{E}' \mid \mathcal{L}, \alpha'] = \mathbb{P}[\mathcal{E}' \mid \mathrm{Good}_{g,h,N,\delta}, \alpha']$, and that under the event $\mathrm{Good}_{g,h,N,\delta}$, $\alpha' - I \subset [-A/2, A/2]$ by taking A large enough.

Thus we have

$$\mathbb{P}[\mathcal{E} \cap \mathcal{E}' \mid \mathcal{L}] = \mathbb{P}[\mathcal{E}' \mid \mathcal{E}, \mathcal{L}]\mathbb{P}[\mathcal{E}] > \delta'\mathscr{L}(I),$$

for any $\mathcal{L}$ such that $\mathrm{Good}_{g,h,N,\delta}$ holds.

It remains to show that $\mathcal{E} \cap \mathcal{E}' \cap \mathrm{Good}_{g,h,N,\delta} \subset \mathrm{ConstDiff}_{g,h,N} \cap \{\alpha \in I\}$. Under the event $\mathcal{E} \cap \mathcal{E}' \cap \mathrm{Good}_{g,h,N,\delta}$, since A is taken large enough, we have that (3.21) holds for any $p \in U_{g,h,N}$. Indeed, for this, we just use the event $\mathcal{E} \cap \mathcal{E}'$ to lower bound $\sup_{|y+1|\le\delta} \mathcal{B}(y)$ and $\sup_{|y-1|\le\delta} \mathcal{B}(y)$, and upper bound $\mathcal{B}(y)$ for $y \notin [-1-\delta, -1+\delta] \cup [1-\delta, 1+\delta]$; and we use $R < \delta^{-0.01}$ (where R is the random variable given by Lemma 3.7) from the event $\mathrm{Good}_{g,h,N,\delta}$ to bound $\mathcal{L}(y, 0; p)$ for any $y \in \mathbb{R}$. Then (3.21) implies $\mathrm{ConstDiff}_{g,h,N}$, as argued at the beginning of this proof. Under $\mathrm{ConstDiff}_{g,h,N}$, the event $\mathcal{E}'$ implies that $\alpha' - \alpha = \mathcal{B}(-1+\delta) - \mathcal{B}(1-\delta)$, so $\alpha \in I$. Thus we conclude that $\mathcal{E} \cap \mathcal{E}' \cap \mathrm{Good}_{g,h,N,\delta} \subset \mathrm{ConstDiff}_{g,h,N} \cap \{\alpha \in I\}$. □

3.5.4 Proof of the lower bound

Proposition 3.41 is an immediate consequence of the following result.

Lemma 3.47. *For any $0 < g < h$ and $N, \epsilon > 0$, there is a random number Q such that the following is true. For any $k \in \mathbb{N}$ and $i \in [\![-2^k, 2^k - 1]\!]$, $j \in [\![0, 2^k - 1]\!]$ we have*

$$\zeta([i2^{-k}N, (i+1)2^{-k}N] \times [j2^{-k}(h-g) + g, (j+1)2^{-k}(h-g) + g]) < Q2^{-(5/3-\epsilon)k}.$$

Let us iterate from Section 3.1 the source of the exponent 5/3 in the right hand side. By the regularity condition of the directed landscape under the event $\text{Good}_{g,h,N,\delta}$, the measure ζ_ℓ on a set of size $2^{-k} \times 2^{-k}$ is nonzero only for ℓ in an interval of length (in the order of) $2^{-k/3}$. Thus to bound ζ on a set of size $2^{-k} \times 2^{-k}$, we just bound an integral of ζ_ℓ (on the set) for ℓ in an interval of length $2^{-k/3}$, and take the maximum over all such intervals. By Lemma 3.46, (under $\text{Good}_{g,h,N,\delta}$) such a quantity is bounded by $2^{-k/3}$ times $\zeta^{\mathcal{B}}$ (on the set of size $2^{-k} \times 2^{-k}$). Using Lemma 3.44, we can bound $\zeta^{\mathcal{B}}$ (on a set of size $2^{-k} \times 2^{-k}$) by the order of $2^{-4k/3}$, with exponential tails. These combined would give the power of 5/3.

Proof. We let $C, c > 0$ denote large and small constants depending on g, h, N, ϵ, and their values may change from line to line. All other constants and parameters in this proof would also depend on g, h, N, ϵ.

For any $\delta > 0$, it suffices to prove the result under the event $\text{Good}_{g,h,N,\delta}$. To control ζ on a dyadic box, for any ℓ, we let Q_ℓ be the supremum of

$$2^{(4/3-\epsilon/2)k}\zeta_\ell([i2^{-k}N, (i+1)2^{-k}N] \times [j2^{-k}(h-g)+g, (j+1)2^{-k}(h-g)+g]),$$

over all $k \in \mathbb{N}$, $i \in [\![-2^k, 2^k - 1]\!]$, $j \in [\![0, 2^k - 1]\!]$. We note that, as we expect that ζ_ℓ would assign mass of the order of $2^{-k/3} \times 2^{-k} = 2^{-4k/3}$ to any dyadic box of size $2^{-k} \times 2^{-k}$, we multiply by the factor of $2^{(4/3-\epsilon/2)k}$ to make Q_ℓ to be of an order slightly smaller than 1. Now for any $H > 0$ large enough (depending on δ), for any $\varepsilon > 0$, we denote

$$J_\varepsilon = \sup_{A \subset [-H,H], \mathscr{L}(A) < \delta^{-1}\varepsilon^{1/3}\log^{2/3}(1+\varepsilon^{-1})} \int_A Q_\ell d\ell,$$

and

$$J = \sup_{\varepsilon \in (0,1)} \frac{J_\varepsilon}{\varepsilon^{1/3}\log(1+\varepsilon^{-1})}. \tag{3.22}$$

The definition of J_ε is motivated by the following (above mentioned) heuristic: by

continuity estimates of $\mathcal{L}$ (ensured by the second condition of $\mathrm{Good}_{g,h,N,\delta}$), for any $k \in \mathbb{N}, i \in [\![-2^k, 2^k-1]\!], j \in [\![0, 2^k-1]\!]$, the measure $\zeta_\ell([i2^{-k}N, (i+1)2^{-k}N] \times [j2^{-k}(h-g)+g, (j+1)2^{-k}(h-g)+g])$ is non-zero only for ℓ in an interval of length roughly in the order of $2^{-k/3}$.

This number J is essentially (up to a constant factor) the desired Q (in the statement of this lemma). This will be shown in two steps: (1) we bound the upper tail of $\mathbb{1}[\mathrm{Good}_{g,h,N,\delta}]J$; (2) we bound ζ on small rectangles, using J under the event $\mathrm{Good}_{g,h,N,\delta}$.

Step 1. We carry out the first step next relying on the following input about the competition interface with stationary initial condition. Let the counterpart of Q_ℓ for the stationary initial condition be defined as

$$Q^{\mathcal{B}} = \sup_{k\in\mathbb{N}, i\in[\![-2^k,2^k-1]\!], j\in[\![0,2^k-1]\!]} 2^{(4/3-\epsilon/2)k} \times \zeta^{\mathcal{B}}([i2^{-k}N, (i+1)2^{-k}N] \times [j2^{-k}(h-g)+g, (j+1)2^{-k}(h-g)+g]).$$

By applying Lemma 3.44 to the boxes $[i2^{-k}N, (i+1)2^{-k}N] \times [j2^{-k}(h-g)+g, (j+1)2^{-k}(h-g)+g]$, and a union bound over all k, i, j, we have

$$\mathbb{P}[Q^{\mathcal{B}} > M] < C_1 e^{-c_1 M},$$

for any $M > 0$, for some constants $c_1, C_1 > 0$. Thus by Lemma 3.46, we have

$$\mathbb{E}\left[\mathbb{1}[\mathrm{Good}_{g,h,N,\delta}] \int_{-H}^{H} \mathbb{1}[Q_\ell > M] d\ell\right] \leq \delta'^{-1} \mathbb{P}[Q^{\mathcal{B}} > M] < \delta'^{-1} C_1 e^{-c_1 M}. \tag{3.23}$$

where δ' appears in the statement of Lemma 3.46. We also assume that $\delta' \leq \delta$, since otherwise we can repalce δ' by $\delta' \wedge \delta$.

The next few lines are technical and can be a bit hard to parse, but are used to

spell out the algebra needed to translate this expectation bound (3.23) into bounds on the tail of $\mathbb{1}[\mathrm{Good}_{g,h,N,\delta}]J$. For any $\varepsilon, M > 0$, let

$$J_{\varepsilon,M} = \sup_{A\subset[-H,H],\mathscr{L}(A)<\delta^{-1}\varepsilon^{1/3}\log^{2/3}(1+\varepsilon^{-1})} \int_A \mathbb{1}[Q_\ell > M]d\ell.$$

By simply bounding the integrand by 1, we have

$$\mathbb{1}[\mathrm{Good}_{g,h,N,\delta}]J_{\varepsilon,M} \leq \delta^{-1}\varepsilon^{1/3}\log^{2/3}(1+\varepsilon^{-1}). \tag{3.24}$$

On the other hand, since $J_{\varepsilon,M} \leq \int_{-H}^{H} \mathbb{1}[Q_\ell > M]d\ell$, by (3.23) we have

$$\mathbb{E}[\mathbb{1}[\mathrm{Good}_{g,h,N,\delta}]J_{\varepsilon,M}] \leq \delta'^{-1}C_1 e^{-c_1 M}. \tag{3.25}$$

We take

$$E_\varepsilon = \begin{cases} \sup_M c_1^{-1}\log(C_1^{-1}\delta' J_{\varepsilon,M}) + M/2 & \text{if } \mathrm{Good}_{g,h,N,\delta} \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

which then rewritten yields

$$\mathbb{1}[\mathrm{Good}_{g,h,N,\delta}]J_{\varepsilon,M} \leq \delta'^{-1}C_1 e^{-c_1 M/2 + c_1 E_\varepsilon}, \ \ \forall M > 0. \tag{3.26}$$

We next bound the upper tail of E_ε (which in particular implies that almost surely it is finite). Assume that $\mathrm{Good}_{g,h,N,\delta}$ holds, then from (3.24) and the fact that $\delta' < \delta$, we have $c_1^{-1}\log(C_1^{-1}\delta' J_{\varepsilon,M}) + M/2 < C$ for any $M < 1$. Also, by (3.25), and Markov's inequality, for any $i \in \mathbb{N}$ and $M > 0$, we have

$$\mathbb{P}[\{c_1^{-1}\log(C_1^{-1}\delta' J_{\varepsilon,i}) + (i+1)/2 > M\} \cap \mathrm{Good}_{g,h,N,\delta}] < e^{-c_1((i-1)/2+M)}.$$

Thus by a union bound over $i \in \mathbb{N}$, and noting that $J_{\varepsilon,M}$ is non-increasing in M, we have $\mathbb{P}[E_\varepsilon > M] < Ce^{-c_1 M}$ for any $M > 0$.

Denote by M_ε the smallest non-negative number such that

$$\delta'^{-1} C_1 e^{-c_1 M_\varepsilon/2 + c_1 E_\varepsilon} \le \delta^{-1} \varepsilon^{1/3} \log^{2/3}(1+\varepsilon^{-1}).$$

Then we have

$$\begin{aligned}
\mathbb{1}[\mathrm{Good}_{g,h,N,\delta}] J_\varepsilon &\le \int_0^\infty \mathbb{1}[\mathrm{Good}_{g,h,N,\delta}] J_{\varepsilon,M} dM \\
&\le \delta^{-1}\varepsilon^{1/3}\log^{2/3}(1+\varepsilon^{-1}) M_\varepsilon + \int_{M_\varepsilon}^\infty \mathbb{1}[\mathrm{Good}_{g,h,N,\delta}] J_{\varepsilon,M} dM \\
&\le \delta^{-1}\varepsilon^{1/3}\log^{2/3}(1+\varepsilon^{-1}) M_\varepsilon + \int_{M_\varepsilon}^\infty \delta'^{-1} C_1 e^{-c_1 M/2 + c_1 E_\varepsilon} dM \\
&= \delta^{-1}\varepsilon^{1/3}\log^{2/3}(1+\varepsilon^{-1}) M_\varepsilon + 2c_1^{-1}\delta'^{-1} C_1 e^{-c_1 M_\varepsilon/2 + c_1 E_\varepsilon} \\
&\le (\delta^{-1} M_\varepsilon + 2c_1^{-1}\delta^{-1})\varepsilon^{1/3}\log^{2/3}(1+\varepsilon^{-1}).
\end{aligned}$$

Here the first inequality is by the definition of J_ε and $J_{\varepsilon,M}$, the second inequality is by splitting the integral at M_ε and (3.24), the third inequality is by (3.26), and the last inequality is by the definition of M_ε. From the tail of E_ε and the relation between E_ε and M_ε, we conclude that for any $M > 0$,

$$\mathbb{P}[\mathbb{1}[\mathrm{Good}_{g,h,N,\delta}] J_\varepsilon > M\varepsilon^{1/3}\log^{2/3}(1+\varepsilon^{-1})] < C'_\delta e^{-c'_\delta M},$$

where C'_δ, c'_δ are constants depending on δ. Recall J from (3.22). By taking the union bound over an exponentially decreasing sequence of ε, and noting that J_ε is monotone in ε (for ε small enough), we conclude that $\mathbb{P}[\mathbb{1}[\mathrm{Good}_{g,h,N,\delta}] J > M] < C_\delta e^{-c_\delta M}$, for any $M > 0$, where C_δ, c_δ are constants depending on δ.

Step 2. We now implement the second step. Under the event $\mathrm{Good}_{g,h,N,\delta}$, using its second condition and Lemma 3.7, we can bound $|\mathcal{D}|$ in $[-N, N] \times [g, h]$. Then as H

is taken to be large enough depending on δ, we have $\zeta_\ell([-N,N]\times[g,h])=0$ for any $|\ell|>H$. Thus for any measurable $A\subset[-N,N]\times[g,h]$, we have $\zeta(A)=\int_{-H}^{H}\zeta_\ell(A)d\ell$. Hence, for any $k\in\mathbb{N}$ large enough (depending on δ) and $i\in[\![-2^k,2^k-1]\!]$, $j\in[\![0,2^k-1]\!]$ we have

$$\begin{aligned}
&\zeta([i2^{-k}N,(i+1)2^{-k}N]\times[j2^{-k}(h-g)+g,(j+1)2^{-k}(h-g)+g])\\
=&\int_{-H}^{H}\zeta_\ell([i2^{-k}N,(i+1)2^{-k}N]\times[j2^{-k}(h-g)+g,(j+1)2^{-k}(h-g)+g])d\ell\\
<&2^{-(4/3-\epsilon/2)k}2^{-k/3}\log(1+2^k)J\\
<&CJ2^{-(5/3-\epsilon)k}.
\end{aligned}$$

Here the first inequality is by the definition of J, and the following reasoning: under the second condition of $\mathrm{Good}_{g,h,N,\delta}$ and using Lemma 3.8, for any ℓ_1,ℓ_2, if $\zeta_{\ell_1}([i2^{-k}N,(i+1)2^{-k}N]\times[j2^{-k}(h-g)+g,(j+1)2^{-k}(h-g)+g])$ and $\zeta_{\ell_2}([i2^{-k}N,(i+1)2^{-k}N]\times[j2^{-k}(h-g)+g,(j+1)2^{-k}(h-g)+g])$ are both nonzero, we must have

$$|\ell_1-\ell_2|<C\delta^{-0.01}2^{-k/3}\log^{2/3}(1+2^k)<\delta^{-1}2^{-k/3}\log^{2/3}(1+2^k),$$

where the second inequality is by taking δ small enough. Finally, by taking $Q=CJ$ the conclusion follows. □

This lemma immediately implies Proposition 3.41. Recall from Section 3.1 the general strategy to prove lower bound for Hausdorff dimension, Proposition 3.41 and (3.17) imply that almost surely the Hausdorff dimension of NC is at least 5/3.

3.5.5 Proof of the upper bound

We next upper bound the Hausdorff dimension of the set NC. This is a more straightforward argument involving splitting $\mathbb{H}$ into boxes, and computing the expected number of boxes intersecting NC.

Proposition 3.48. *For any $0 < g < h$, $N > 0$, and any $d > 5/3$, almost surely the d-dimensional Hausdorff measure of the set $\mathsf{NC} \cap ([-N, N] \times [g, h])$ is zero.*

The following is the key estimate relying on non-coalescence of geodesics similar to the argument in [10].

Proposition 3.49. *For any $0 < g < h$ and $N > 0$, there is a constant C, such that for any $g \leq t < t + \epsilon \leq h$ and $-N \leq x < x + \epsilon \leq N$, we have*

$$\mathbb{P}[\mathsf{NC} \cap ((x, x + \epsilon) \times (t, t + \epsilon)) \neq \emptyset] < C\epsilon^{1/3} e^{C|\log(\epsilon)|^{5/6}}.$$

Before jumping into the proof we take a moment to explain the form of the above bound. For any geodesic from $(-1, 0)$ or $(1, 0)$ to $(x, x + \epsilon) \times (t, t + \epsilon)$, the fluctuation of its intersection with the line $\mathbb{R} \times \{t\}$ would be of the order of $\epsilon^{2/3}$. Thus the event $\mathsf{NC} \cap ((x, x + \epsilon) \times (t, t + \epsilon)) \neq \emptyset$ would roughly imply disjointness of geodesics from $(-1, 0)$ and $(1, 0)$ to $(x - \epsilon^{2/3}, t)$ and $(x + \epsilon^{2/3}, t)$, respectively. Theorem 3.14 implies that the probability is of the order of $\epsilon^{1/3}$, with the extra sub-polynomial factor.

Assuming Proposition 3.49, the proof of Proposition 3.48 is quick.

Proof of Proposition 3.48. Take any large enough $n \in \mathbb{N}$ and let

$$\mathscr{S}_n = \{i, j \in \mathbb{Z} : ([i/n, (i + 1)/n] \times [j/n, (j + 1)/n]) \cap \mathsf{NC} \cap ([-N, N] \times [g, h]) \neq \emptyset\}.$$

By Proposition 3.49, we have that $\mathbb{E}[|\mathscr{S}_n|] < Cn^{5/3} e^{C(\log(n))^{5/6}}$, where C is a constant depending only on g, h, N. Taking $0 < \alpha < d - 5/3$, we then have that

$$\mathbb{P}[n^{-d}|\mathscr{S}_n| > n^{-\alpha}] < Cn^{5/3-d+\alpha} e^{C(\log(n))^{5/6}}.$$

This implies by the Borel-Cantelli lemma, that almost surely, we can find a random

k_0, such that for any integer $k > k_0$,

$$2^{-dk}|\mathscr{S}_{2^k}| \leq 2^{-\alpha k}.$$

Since the right hand side converges to 0 as $k \to \infty$, and that $\mathsf{NC} \cap ([-N, N] \times [g, h]) \subset \cup_{i \in \mathscr{S}_n} [i/n, (i+1)/n] \times [j/n, (j+1)/n]$, the conclusion follows. □

We now finish the proof of Proposition 3.49. Recall that for any $x_1 < x_2$, $y_1 < y_2$, and $s < t$, $\mathrm{Disj}^{[s,t]}_{(x_1,x_2)\to(y_1,y_2)}$ denotes the event where $\pi_{(x_1,s),(y_1,t)}$ and $\pi_{(x_2,s),(y_2,t)}$ are unique and disjoint.

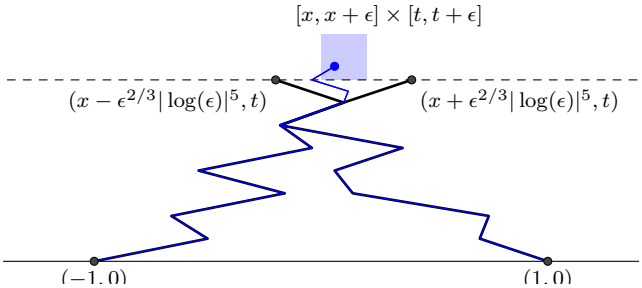


Figure 3.12: An illustration of the proof of Proposition 3.49: the events $\mathcal{E}_1^c$ and $\mathcal{E}_2^c$.

Proof of Proposition 3.49. We can assume that ϵ is small enough since otherwise the inequality is obvious by taking the constant large enough.

We consider the following two events.

1. Let $\mathcal{E}_1$ be the event $\mathrm{Disj}^{[0,t]}_{(-1,1)\to(x-\epsilon^{2/3}|\log(\epsilon)|^5, x+\epsilon^{2/3}|\log(\epsilon)|^5)}$.

2. Let $\mathcal{E}_2$ be the event that there is some $y \in [x, x+\epsilon]$ and $s \in [t, t+\epsilon]$, such that

$$|\pi_{(-1,0;y,s)}(t) - x| \vee |\pi_{(1,0;y,s)}(t) - x| > \epsilon^{2/3}|\log(\epsilon)|^5.$$

Under $\mathcal{E}_1^c \cap \mathcal{E}_2^c$, we have that (almost surely) the function $(y, s) \mapsto \mathcal{L}(-1, 0; y, s) - \mathcal{L}(1, 0; y, s)$ is constant on $[x, x+\epsilon] \times [t, t+\epsilon]$ (see Figure 3.12 for an illustration).

Indeed, by $\mathcal{E}_1^c$ we can find some $t_* \in (0,t)$ and x_*, such that

$$\pi_{(-1,0;x-\epsilon^{2/3}|\log(\epsilon)|^5,t)}(t_*) = \pi_{(1,0;x+\epsilon^{2/3}|\log(\epsilon)|^5,t)}(t_*) = x_*.$$

We also assume (the probability 1 event) that the geodesics $\pi_{(-1,0;x-\epsilon^{2/3}|\log(\epsilon)|^5,t)}$ and $\pi_{(1,0;x+\epsilon^{2/3}|\log(\epsilon)|^5,t)}$ are unique. Then by $\mathcal{E}_2^c$ and the ordering of geodesics (Lemma 3.12), we must have that

$$\pi_{(1,0;y,s)}(t_*) = \pi_{(-1,0;y,s)}(t_*) = x_*,$$

for any $(y,s) \in [x,x+\epsilon] \times [t,t+\epsilon]$. We then have that

$$\mathcal{L}(-1,0;y,s) - \mathcal{L}(1,0;y,s) = \mathcal{L}(-1,0;x_*,t_*) - \mathcal{L}(1,0;x_*,t_*),$$

which implies that the left hand side is a constant (in $[x,x+\epsilon] \times [t,t+\epsilon]$). Thus we have that the event $\mathsf{NC} \cap ((x,x+\epsilon) \times (t,t+\epsilon)) \neq \emptyset$ (almost surely) implies $\mathcal{E}_1 \cup \mathcal{E}_2$.

We next bound the probabilities of these events. Below we let $C, c > 0$ denote large and small constants depending on g, h, N whose values may change from line to line. By Theorem 3.14 we have that $\mathbb{P}[\mathcal{E}_1] < C\epsilon^{1/3} e^{C|\log(\epsilon)|^{5/6}}$. For $\mathbb{P}[\mathcal{E}_2]$, we take R to be the random varialbe given by Lemma 3.9. Then $\mathbb{P}[R > |\log(\epsilon)|] < Ce^{-c|\log(\epsilon)|^{9/4}/\log(|\log(\epsilon)|)^4}$, and $R \leq |\log(\epsilon)|$ implies $\mathcal{E}_2^c$. So we have

$$\mathbb{P}[\mathcal{E}_2] < Ce^{-c|\log(\epsilon)|^{9/4}/\log(|\log(\epsilon)|)^4}.$$

Thus $\mathbb{P}[\mathcal{E}_1 \cup \mathcal{E}_2] < C\epsilon^{1/3} e^{C|\log(\epsilon)|^{5/6}}$, and the conclusion follows. □

Having completed our analysis of the two dimensional process $\mathcal{D}$, we now shift our focus on one dimensional objects for which, as outlined in Section 3.1, the arguments are more delicate. We start with the geodesic local time.

3.6 The geodesic local time: Proof of Theorem 1.11

Recall from the discussion following the statement of the theorem that we construct a non-decreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$, which we call the geodesic local time of $\pi_{\mathbf{0}}$, in analogue with the well known Brownian local time. This L induces a (random) measure μ on $\mathbb{R}_{\geq 0}$, supported on the set Z defined in (1.3). We show that almost surely, L is $1/3 - \epsilon$ Hölder for any $\epsilon > 0$, and that it is almost surely non-degenerate, i.e., $L > 0$ on $\mathbb{R}_+$. This implies that the set Z has Hausdorff dimension at least $1/3$. The upper bound of the Hausdorff dimension is easier, by splitting the line $\{(0,t) : t \geq 0\}$ into segments, and bounding the expected number of segments that intersect with $\pi_{\mathbf{0}}$.

In Section 3.6.1 we construct the geodesic local time L. The Hölder property of L is proved in Section 3.6.2. In Section 3.6.3 we show that L is almost surely non-zero. Finally we prove the Hausdorff dimension upper bound in Section 3.6.4.

3.6.1 Construction: occupation measure and the limit

We recall from (1.4) the occupation measure ν_h for the geodesic $\pi_{\mathbf{0}}$ at height h, where for any measurable $A \subset \mathbb{R}$

$$\nu_h(A) = \int_0^h \mathbb{1}[\pi_{\mathbf{0}}(t) \in A] dt. \tag{3.27}$$

The geodesic local time L will be constructed as in the classical case of Brownian motion by showing the existence of the density of ν_h at 0. For this, we prove Proposition 1.12, which asserts that almost surely the limit $\lim_{w \searrow 0} (2w)^{-1} \nu_h([-w, w])$ exists and is finite. In fact, in the forthcoming work [59], we will show that properly normalized counting measure on the zero set of the geodesic in the discrete prelimiting model of Exponential LPP converges to L, similar to how it is classically known that the random walk local time converges to its Brownian counterpart.

Reiterating the outline from Section 3.1, note that by Lebesgue's theorem for the differentiability of monotone functions, for any (locally finite) measure on $\mathbb{R}$, its density exists almost everywhere. However, for our purposes we would need it to exist at 0. To ensure this we will construct the occupation measure ν_h' of semi-infinite geodesics starting from random points. The latter has the advantage of being translation invariant which is enough to imply that almost surely its density exists at a fixed point. We will then use coalescence of geodesics to compare ν_h and ν_h', and deduce the same for ν_h.

We start by recording the following bound on the number of intersections of semi-infinite geodesics with a line segment in the space direction.

Lemma 3.50. *For any $t > 0$, and interval $A \subset \mathbb{R}$, we have*

$$\mathbb{E}[|\{\pi_{(x,0)}(t) : x \in \mathbb{R}, \pi_{(x,0)} \text{ is unique}\} \cap A|] < C\mathscr{L}(A)t^{-2/3},$$

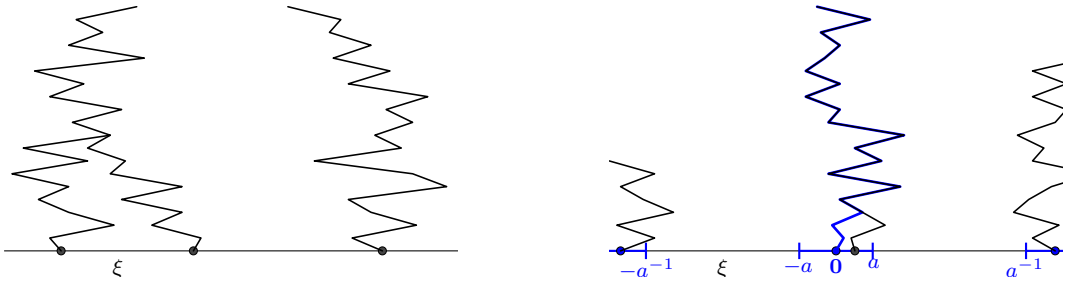
where C is a universal constant.

We start by briefly explaining the occurrence of the factor of $t^{-2/3}$: by the KPZ scaling (of the directed landscape $\mathcal{L}$), we expect $\pi_{(x,0)}$ for x varying in an interval of length (of the order of) $t^{2/3}$ to coalesce before height t. Thus there would be constant many points in $\{\pi_{(x,0)}(t) : x \in \mathbb{R}, \pi_{(x,0)} \text{ is unique}\}$, in an interval of length $t^{2/3}$.

Next we prove Proposition 1.12 assuming Lemma 3.50.

Proof of Proposition 1.12. The proof goes in two steps: in the first step we prove that the density of $\nu_h - \nu_g$ exists at 0 for any $0 < g < h$, using a translation invariant argument; in the second step we bound $\limsup_{w \searrow 0}(2w)^{-1}\nu_g([-w, w])$, and show that it decays to zero as $g \searrow 0$.

Step 1. We start with the first step, which is illustrated by Figure 3.13. Let $\xi \subset \mathbb{R}$ be a Poisson point process with rate 1. We consider the measure ν_h' such that for any



(a) Semi-infinite geodesics starting from $\xi \times \{0\}$.

(b) $\pi_{\mathbf{0}}$ and semi-infinite geodesics starting from $\xi \times \{0\}$, conditional on $\mathcal{E}_a$.

Figure 3.13: An illustration of the proof of Proposition 1.12.

measurable $A \subset \mathbb{R}$,

$$\nu'_h(A) = \int_0^h \sum_{x \in \xi} \mathbb{1}[\pi_{(x,0)}(t) \in A] dt.$$

Recall that for any (locally finite) measure, the density exists and is finite Lebesgue almost everywhere, by Lebesgue's theorem for the differentiability of monotone functions. We next show that ν'_h is locally finite. Note that under the probability one event where $\pi_{(x,0)}$ is unique for any $x \in \xi$, we have $\nu'_h(A) \leq \int_0^h |\{\pi_{(x,0)}(t) : x \in \mathbb{R}, \pi_{(x,0)} \text{ is unique}\} \cap A| dt$. So by Lemma 3.50 we have that almost surely $\nu'_h(A) < \infty$ for any compact A. Then if we let E be the set where such limit $\lim_{w \searrow 0} (2w)^{-1} \nu'_h([x-w, x+w])$ does not exist or is infinite, we would have that E is Lebesgue measurable[1] with $\mathscr{L}(E) = 0$. By translation invariance of the law of ν'_h and Fubini's theorem, we have $\mathbb{P}[0 \in E] = \mathbb{E}[\mathscr{L}(E \cap [0,1])] = 0$. Thus we have that almost surely the limit $\lim_{w \searrow 0} (2w)^{-1} \nu'_h([-w, w])$ exists and is finite.

We next compare ν' and ν. Take $0 < g < h$. For $0 < a < 1$, consider the following event

$$\mathcal{E}_a := \{|\xi \cap [-a, a]| = |\xi \cap [-a^{-1}, a^{-1}]| = 1\}.$$

The idea being that on $\mathcal{E}_a$, the presence of only a single point near the origin would

[1]The fact that E is measurable can be argued as follows: for each $w > 0$, the function $x \mapsto (2w)^{-1} \nu'_h([x-w, x+w])$ is continuous except for countably many points, thus it is measurable. Then E is measurable, since it is the set where a sequence of measurable functions does not converge or converges to infinity.

allow us to carry out the desired comparison. Since $\mathbb{P}[\mathcal{E}_a] > 0$, conditional on $\mathcal{E}_a$ we still have that almost surely the limit $\lim_{w \searrow 0}(2w)^{-1}(\nu_h'([-w,w]) - \nu_g'([-w,w]))$ exists and is finite.

We note that, almost surely, as $x \to 0$, the geodesic $\pi_{(x,0)} \to \pi_{\mathbf{0}}$ in the overlap topology (by Lemma 3.26). Also, by Lemma 3.22 and the Borel-Cantelli lemma, we have that almost surely $|\pi_{(x,0)}(h)| \to \infty$ as $|x| \to \infty$. Then by Lemma 3.9, and ordering of finite geodesics, we further have that almost surely $\min_{t \in [0,h]} |\pi_{(x,0)}(t)| \to \infty$ as $|x| \to \infty$. Hence, given $0 < g < h$, almost surely for all small enough a, on $\mathcal{E}_a$,

$$\nu_h([-w,w]) - \nu_g([-w,w] = \nu_h'([-w,w]) - \nu_g'([-w,w].$$

Thus we have that the limit $\lim_{w \searrow 0}(2w)^{-1}(\nu_h([-w,w]) - \nu_g([-w,w]))$ exists and is finite (almost surely for the directed landscape).

Step 2. We next carry out the second step, i.e., to bound $\limsup_{w \searrow 0}(2w)^{-1}\nu_g([-w,w])$. The idea is to bound ν_g by the occupation measure of the union of $\pi_{(x,0)}$ *for all* $x \in \mathbb{R}$, and to apply Lemma 3.50. We begin by denoting by ν_g'' the measure such that for any Borel measurable set A,

$$\nu_g''(A) = \int_0^g |\{\pi_{(x,0)}(t) : x \in \mathbb{R}, \pi_{(x,0)} \text{ is unique}\} \cap A| dt.$$

From this definition, almost surely $\nu_g(A) \leq \nu_g''(A)$ for any compact A. Also, for compact A, by Lemma 3.50 we have $\mathbb{E}[\nu_g''(A)] < \infty$, so almost surely we have $\nu_g''(A) < \infty$. Since the density of ν_g'' exists Lebesgue almost everywhere, by translation invariance of the law of ν_g'', we have that almost surely $\lim_{w \searrow 0}(2w)^{-1}\nu_g''([-w,w])$ exists. Assuming this limit exists, we have

$$\limsup_{w \searrow 0}(2w)^{-1}\nu_g([-w,w]) \leq \lim_{w \searrow 0}(2w)^{-1}\nu_g''([-w,w]).$$

We next bound the upper tail of this limit. Let $E_z \subset \mathbb{R}$ consist of all x, where $\lim_{w \searrow 0}(2w)^{-1}\nu_g''([x-w, x+w])$ exists and is at least z. Then E_z is measurable, by the same arguments as proving that E is measurable (see the footnote above). Since the law of ν_g'' is translation invariant, we have $\mathbb{P}[0 \in E_z] = (2l)^{-1}\mathbb{E}[\mathscr{L}(E_z \cap [-l, l])]$ for any $l > 0$. We also have[2]

$$\nu_g''([-l,l]) \geq \int_{E_z \cap [-l,l]} \lim_{w \searrow 0}(2w)^{-1}\nu_g''([x-w, x+w])dx \geq z\mathscr{L}(E_z \cap [-l,l]),$$

so

$$\mathbb{P}[0 \in E_z] \leq (2l)^{-1}z^{-1}\mathbb{E}[\nu_g''([-l,l])] < Cg^{1/3}z^{-1}$$

for some constant $C > 0$. Thus we conclude that

$$\mathbb{P}[\limsup_{w \searrow 0}(2w)^{-1}\nu_g([-w,w]) > z] < Cg^{1/3}z^{-1}.$$

By the Borel-Cantelli lemma, this implies that

$$\lim_{g \searrow 0}\limsup_{w \searrow 0}(2w)^{-1}\nu_g([-w,w]) = 0$$

almost surely. Thus we get Proposition 1.12. □

It remains to prove Lemma 3.50.

Proof of Lemma 3.50. Recall that to justify the factor of $t^{-2/3}$, we would need to prove coalescence of semi-infinite geodesics, and for that we rely on Lemma 3.10.

By scaling invariance it suffices to prove this result for $t = 1$; and by translation invariance it suffices to consider a fixed interval $A = [-1, 1]$; i.e. we just need to show that $\mathbb{E}[|\{\pi_{(x,0)}(1) : x \in \mathbb{R}, \pi_{(x,0)} \text{ is unique}\} \cap [-1,1]|]$ is finite. As throughout

[2]The first inequality is by the following argument: for any $w > 0$, we have $\nu_g''([-l-w, l+w]) \geq \int_{-l}^{l}(2w)^{-1}\nu_g''([x-w, x+w])dx$. Then the inequality follows by sending $w \to 0$ and using Fatou's lemma for the functions $x \mapsto (2w)^{-1}\nu_g''([x-w, x+w])$.

this thesis, below we use $C, c > 0$ to denote large and small constants whose values may change from line to line.

Take $m \in \mathbb{N}$ and we consider $\pi_{(-m,0)}$ and $\pi_{(m,0)}$. By passing Lemma 2.10 to the limit using Proposition 3.31, we have that

$$\mathbb{P}[\sup_{s\in[0,2]} |\pi_{(-m,0)}(s) + m| > m/2] = \mathbb{P}[\sup_{s\in[0,2]} |\pi_{(m,0)}(s) - m| > m/2] < Ce^{-cm^3}. \quad (3.28)$$

By the ordering of semi-infinite geodesics (Lemma 3.23), when $\sup_{s\in[0,2]} |\pi_{(-m,0)}(s) + m| \le m/2$ and $\sup_{s\in[0,2]} |\pi_{(m,0)}(s) - m| \le m/2$, we must have that

$$\sup_{s\in[0,2],x\in[-m,m]} |\pi_{(x,0)}(s)| \le 3m/2,$$

and hence

$$\begin{aligned} &|\{\pi_{(x,0)}(1) : x \in \mathbb{R}, \pi_{(x,0)} \text{ is unique}\} \cap [-1,1]| \\ &\qquad \le |\{\pi_{(x,0),(y,2)}(1) : |x|, |y| \le 3m/2, \pi_{(x,0),(y,2)} \text{ is unique}\} \cap [-1,1]|. \end{aligned}$$

Then by (3.28) and Lemma 3.10, we have that $|\{\pi_{(x,0)}(1) : x \in \mathbb{R}, \pi_{(x,0)} \text{ is unique}\} \cap [-1,1]|$ has a stretched exponential tail, so the conclusion holds. □

We record the following useful immediate corollary.

Corollary 3.51. *For any $t > 0$ and interval $A \subset \mathbb{R}$, we have $\mathbb{P}[\pi_{\mathbf{0}}(t) \in A] < C\mathscr{L}(A)t^{-2/3}$, where C is a universal constant.*

Such an estimate in the pre-limit has appeared as [8, Theorem 2] where the proof also uses translation invariance of the models.

We finish this subsection by recording the following absolute continuity property of the occupation measure ν_h, which will be used later.

Proposition 3.52. *Almost surely the measure ν_h is absolutely continuous to the Lebesgue measure for any $h > 0$.*

The proof is similar to the one showing absolute continuity for the Brownian occupation measure, in studying Brownian local time (see e.g., [81, Theorem 3.25]).

Proof. Since ν_h is monotone in h, it suffices to prove this for fixed $h > 0$. By an equivalent definition of absolute continuity, we need to show that almost surely, for ν_h-almost every $x \in \mathbb{R}$, we have

$$\liminf_{w \searrow 0} \frac{\nu_h([x-w, x+w])}{w} < \infty.$$

We take arbitrary $H > 0$, and by Fatou's lemma we have

$$\begin{aligned}
&\mathbb{E} \int_{-H}^{H} \liminf_{w \searrow 0} \frac{\nu_h([x-w, x+w])}{w} d\nu_h(x) \\
\leq &\liminf_{w \searrow 0} w^{-1} \mathbb{E} \int_{-H}^{H} \nu_h([x-w, x+w]) d\nu_h(x) \\
= &\liminf_{w \searrow 0} w^{-1} \int_0^h \int_0^h \mathbb{P}[|\pi_{\mathbf{0}}(t)| < H, |\pi_{\mathbf{0}}(t) - \pi_{\mathbf{0}}(s)| < w] dt ds.
\end{aligned}$$

For each $i \in \mathbb{Z}$, we have

$$\int_0^h \int_0^h \mathbb{P}[\pi_{\mathbf{0}}(t), \pi_{\mathbf{0}}(s) \in [(i-1)w, (i+1)w]] dt ds = \mathbb{E}[\nu_h([(i-1)w, (i+1)w])^2] \leq Cw^2,$$

where $C > 0$ is a constant depending on h, H, and the inequality is by Lemma 3.44. Then we have

$$\begin{aligned}
&\int_0^h \int_0^h \mathbb{P}[|\pi_{\mathbf{0}}(t)| < H, |\pi_{\mathbf{0}}(t) - \pi_{\mathbf{0}}(s)| < w] dt ds \\
&\qquad \leq \sum_{i \in \mathbb{Z}: w(|i|-1) \leq H} \int_0^h \int_0^h \mathbb{P}[\pi_{\mathbf{0}}(t), \pi_{\mathbf{0}}(s) \in [(i-1)w, (i+1)w]] dt ds \\
&\qquad \leq (2\lfloor H/w \rfloor + 3) Cw^2 < 3CHw
\end{aligned}$$

when w is small enough. Thus we conclude that ν_h is absolutely continuous with respect to Lebesgue measure. □

3.6.2 1/3 Hölder regularity of the geodesic local time

By Proposition 1.12, almost surely we can define $L(h) = \lim_{w \searrow 0}(2w)^{-1}\nu_h([-w, w])$ for any non-negative rational h. The following Hölder estimate of L in particular implies continuity and will allow us to extend L to $\mathbb{R}_{\geq 0}$, and conclude that L is almost surely $\frac{1}{3} - \epsilon$ Hölder for any $\epsilon > 0$.

Proposition 3.53. *There exist constants $c, C > 0$ such that the following is true. For any $g, h \in \mathbb{Q}$, $0 \leq g < h$, and $M > 0$, we have*

$$\mathbb{P}[L(h) - L(g) > M(h-g)^{1/3}] < Ce^{-cM}.$$

In fact, we will prove a more general estimate. Namely, for some intervals I, J, we will obtain a uniform bound on the occupation measure in the set $I \times J$, for any geodesic whose endpoints are at the boundary of $I \times J$. It would imply both Proposition 3.53 and Lemma 3.44, and have many other applications in the remainder of this chapter.

For any $u = (p; q) = (x, s; y, t) \in \mathbb{R}^4_\uparrow$, recall that π_u denotes any geodesic from $p = (x, s)$ to $q = (y, t)$. For any interval I we define

$$W[u, I] = \int_s^t \mathbb{1}[\pi_u(r) \in I, \pi_u \text{ is unique}]dr.$$

In words, $W[u, I]$ is the total amount of time π_u spends in I, if π_u is the unique geodesic; and $W[u, I] = 0$ otherwise.

For any intervals I, J, we denote

$$\Phi_{I,J} = \{(p; q) \in \mathbb{R}^4_\uparrow : p, q \in \partial(I \times J)\}, \tag{3.29}$$

where $\partial(I \times J)$ denotes the boundary of $I \times J$, i.e., $\partial(I \times J) = (\{\inf I, \sup I\} \times J) \cup (I \times \{\inf J, \sup J\})$, and

$$W_{I,J} = \sup_{u \in \Phi_{I,J}} W[u, I].$$

We note that $W_{I,J}$ is also an upper bound for the occupation measure in $I \times J$, for any unique geodesic π_u, where $u = (p; q) = (x, s; y, t) \in \mathbb{R}^4_\uparrow$ with $p, q \notin I \times J$. Here this occupation measure is defined as

$$\int_{[s,t] \cap J} \mathbb{1}[\pi_u(r) \in I] dr.$$

Indeed, if the graph of π_u intersects $I \times J$, the above expression equals $W[(p'; q'), I]$, for p', q' being the first and last intersection of the graph of π_u with $I \times J$. Since π_u is unique, $\pi_{(p';q')}$ must also be unique, so $W[(p'; q'), I] \leq W_{I,J}$. Similarly, for any $p \notin I \times J$, if π_p is unique, its occupation measure in $I \times J$ is also upper bounded by $W_{I,J}$. We now state the following uniform estimate.

Lemma 3.54. *There are universal constants $C, c > 0$, such that for any $w, M > 0$ we have*

$$\mathbb{P}[W_{[-w,w],[0,1]} > Mw] < Ce^{-cM}.$$

Its proof follows induction arguments that are essentially the same as those in [92, Section 3], where a version for exponential LPP is proved. We provide the proof adapted to the directed landscape setting in Appendix B.

3.6.3 Non-degeneracy of the geodesic local time

The goal of this section is to show that the local time L is strictly positive on $\mathbb{R}_+$.

Proposition 3.55. *Almost surely, $L(h) > 0$ for any $h > 0$.*

Before proving the above, we quickly prove the lower bound of the Hausdorff dimension.

Proof of Theorem 1.11: lower bound. By Proposition 3.53 L is continuous, and we can define a measure μ on $\mathbb{R}_{\geq 0}$, via $\mu([g,h]) = L(h) - L(g)$ for any $0 \leq g < h$. We next claim that μ is supported on Z (although we believe that the support is exactly equal to Z, the proof of this unfortunately still eludes us). Indeed, as $\pi_{\mathbf{0}}$ is continuous, for any $t \in \mathbb{R}_+ \setminus Z$ we can find some $\delta > 0$ with $|\pi_{\mathbf{0}}(s)| > \delta$ for any $|s - t| \leq \delta$. Thus $\nu_{t+\delta}([-\delta,\delta]) - \nu_{t-\delta}([-\delta,\delta]) = 0$, and hence $L(t+\delta) = L(t-\delta)$, leading to $\mu([t-\delta, t+\delta]) = 0$.

Take any $h, \epsilon > 0$. By Proposition 3.55 we have (almost surely) $\mu([0,h]) > 0$. By Proposition 3.53, (almost surely) there is a constant $C > 0$ such that $L([a,b]) < C(b-a)^{1/3-\epsilon}$ for any $0 \leq a < b \leq h$. Hence by the general strategy from Section 3.1, the Hausdorff dimension of $Z \cap [0,h]$ is at least $1/3 - \epsilon$. Since h, ϵ are arbitrary, the Hausdorff dimension of Z is at least $1/3$. □

The proof of Proposition 3.55 is via the monotonicity and scaling invariance of the function L, and the following 0-1 law of the tail sigma algebra of the directed landscape.

For any $r \in \mathbb{N}$, we let $\mathcal{F}_{\geq r}$ be the sigma algebra generated by $\mathcal{L}(x,s;y,t)$ for $r \leq s < t$, and let $\mathcal{F}_\infty = \cap_{r>0} \mathcal{F}_{\geq r}$.

Lemma 3.56. *Any $\mathcal{F}_\infty$-measurable event has probability 0 or 1.*

Proof. This follows from the definition of $\mathcal{F}_\infty$. For any $(x,s;y,t) \in \mathbb{R}^4_\uparrow$, and $r > t$, $\mathcal{L}(x,s;y,t)$ is independent of $\mathcal{F}_{\geq r}$. This implies that $\mathcal{L}$ is independent of $\mathcal{F}_\infty$. So any $\mathcal{F}_\infty$-measurable event is independent of itself, thus has probability 0 or 1. □

Proof of Proposition 3.55. By scaling invariance and monotonicity of the function L, it suffices to show that $\mathbb{P}[L(1) > 0] = 1$. We first show that $\mathbb{P}[L(1) > 0] > 0$. For this it suffices to show that $\mathbb{E}[L(1)] > 0$. We note that by Lemma 3.54, the random variables $(2w)^{-1}\nu_1([-w,w])$ for all small enough w are uniformly integrable. This

implies that $\mathbb{E}[(2w)^{-1}\nu_1([-w,w])] \to \mathbb{E}[L(1)]$ as $w \searrow 0$; so it suffices to lower bound $\mathbb{E}[(2w)^{-1}\nu_1([-w,w])]$.

By Lemma 3.57 below and scaling invariance, we have that $\mathbb{P}[\pi_{\mathbf{0}}(t) \in [-w,w]] > cw$ for any $0 \leq t \leq 1$; and by integrating over t we get $\mathbb{E}[(2w)^{-1}\nu_1([-w,w])] > c/2$ for any $0 < w < 1$. We thus conclude that $\mathbb{E}[L(1)] > 0$ and hence $\mathbb{P}[L(1) > 0] > 0$.

Now suppose that $\mathbb{P}[L(1) > 0] = \alpha$ for some $0 < \alpha < 1$. We then claim that conditional on $L(1) > 0$, we have $\lim_{h\to\infty} L(h) - L(g) > 0$ for any $g > 0$. Indeed, we take

$$g_* = \inf\{g > 0 : L(h) = L(g), \forall h > g\} \cup \{\infty\}.$$

By scaling invariance of L, the probability $\mathbb{P}[g_* > g]$ is same for any $g > 0$; thus we have $\mathbb{P}[g < g_* \leq h] = 0$ for any $0 < g < h$, so $\mathbb{P}[0 < g_* < \infty] = 0$. Then conditional on $L(1) > 0$ we have that $g_* = \infty$ almost surely. Also by scaling invariance we have that $\mathbb{P}[L(h) > 0] = \alpha$ for any $h > 0$, so by monotonicity of L we have that conditional on $L(1) > 0$, almost surely $L(h) > 0$ for any $h > 0$, and $g_* = \infty$. Then we conclude that the events $L(1) > 0$ and $g_* = \infty$ are almost surely equivalent; i.e., the probability of exactly one of them happening equals zero.

We will now show $g_* = \infty$ happens almost surely. Towards this for any $r \in \mathbb{N}$ and $h > 0$, we denote ν_h^r as the occupation measure for $\pi_{(0,r)}$ at height $h + r$; i.e., for any measurable $A \subset \mathbb{R}$, we let

$$\nu_h^r(A) = \int_0^h \mathbb{1}[\pi_{(0,r)}(r+t) \in A]dt,$$

and we let $L^r(h) = \lim_{w\searrow 0}(2w)^{-1}\nu_h^r([-w,w])$. Almost surely this limit exists and is finite, by Proposition 1.12 and translation invariance. Then we can define $g_*^r = \inf\{g > 0 : L^r(h) = L^r(g), \forall h > g\} \cup \{\infty\}$. By the coalescence of semi-infinite geodesics (Lemma 3.28), we have that $g_* = \infty$ is almost surely equivalent to $g_*^r = \infty$, thus to $L^r(1) > 0$, for any $r \in \mathbb{N}$. Thus we can consider the $\mathcal{F}_\infty$ measurable event

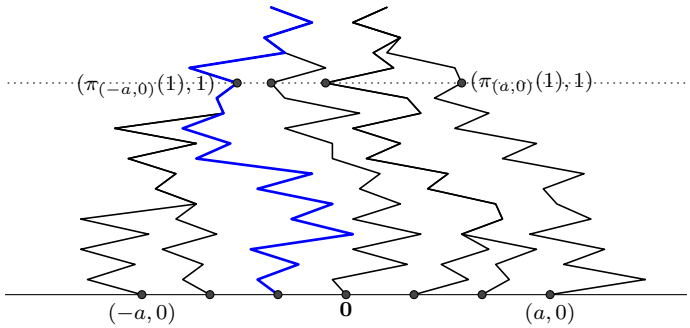


Figure 3.14: An illustration for the proof of Lemma 3.57: consider $\pi_{(ai/n,0)}(1)$ for each $i \in [\![-n, n]\!]$; assuming $\pi_{(-a,0)}(1) > -a$ and $\pi_{(a,0)}(1) < a$, there must exist an i with $|\pi_{(ai/n,0)}(1) - ai/n| \leq a/n$ (such semi-infinite geodesic $\pi_{(ai/n,0)}$ is colored blue). Then $\mathbb{P}[\pi_{(-a,0)}(1) > -a, \pi_{(a,0)}(1) < a] > 0$ implies that $\mathbb{P}[\pi_{\mathbf{0}}(1) \in [-a/n, a/n]]$ is at least in the order of $1/n$.

$|\{r \in \mathbb{N} : L^r(1) > 0\}| = \infty$. By the arguments above, this is almost surely equivalent to $L(1) > 0$. These imply that $\mathbb{P}[|\{r \in \mathbb{N} : L^r(1) > 0\}| = \infty] = \alpha$. By Lemma 3.56, this contradicts with the fact that $0 < \alpha < 1$, so we must have that $\mathbb{P}[L(1) > 0] = 1$. □

It remains to prove the following technical result.

Lemma 3.57. *For any $0 < w < 1$ we have $\mathbb{P}[\pi_{\mathbf{0}}(1) \in [-w, w]] > cw$, where $c > 0$ is a universal constant.*

The matching upper bound for this probability is already given by Corollary 3.51. A similar lower bound in the pre-limit is given by [8, Theorem 2] and our arguments are similar.

Unlike the upper bound, such a lower bound does not directly follow from translation invariance. In addition, we need to use the ordering of geodesics, and some additional geometric arguments.

Proof. We claim that there is some $a \in [1, 2]$ such that

$$\mathbb{P}[\pi_{(-a,0)}(1) > -a, \pi_{(a,0)}(1) < a] > 0. \tag{3.30}$$

We note that here $[1,2]$ is chosen arbitrarily and can be replaced by any closed interval in $\mathbb{R}_+$. We show this by contradiction. Using Corollary 3.51, and translation and scaling invariance of $\mathcal{L}$, we have that almost surely $\pi_{(x,0)}(t) \neq x$ for any $x, t \in \mathbb{Q}$. If the claim is not true, then by translation and scaling invariance, almost surely the following is true: for any $x, y, t \in \mathbb{Q}$, with $t > 0$ and $y - x \in [2,4]$, we have $(\pi_{(x,0)}(t) - x)(\pi_{(y,0)}(t) - y) > 0$. By using this repeatedly, we have that almost surely, $\pi_{\mathbf{0}}(t)(\pi_{(x,0)}(t) - x) > 0$ for any $x, t \in \mathbb{Q}$, $t > 0$. Consider $\pi_{\mathbf{0}}$ and $\pi_{(1,0)}$: there is some (random) $t_0 > 0$ such that $\pi_{\mathbf{0}}(t) = \pi_{(1,0)}(t)$ for any $t > t_0$. Then either $\liminf_{t\to\infty} \pi_{\mathbf{0}}(t) > 1$ or $\limsup_{t\to\infty} \pi_{(1,0)}(t) < 0$. By flip invariance, these two events have equal probability, so we have

$$0 < \mathbb{P}[\liminf_{t\to\infty} \pi_{\mathbf{0}}(t) > 1] < 1. \tag{3.31}$$

However, by coalescing of semi-infinite geodesics, the event $\liminf_{t\to\infty} \pi_{\mathbf{0}}(t) > 1$ is (up to a probability zero event) equivalent to the event $\liminf_{t\to\infty} \pi_{(0,r)}(t) > 1$, for any $r \in \mathbb{R}$; thus it is (up to a probability zero event) equivalent to an event that is $\mathcal{F}_\infty$ measurable. So (3.31) contradicts Lemma 3.56, and we can find an $a \in [1,2]$ such that (3.30) holds.

Now we fix such an $a \in [1,2]$. It suffices to prove this lemma for $w = \frac{a}{n}$, $n \in \mathbb{N}$. We claim that under $\pi_{(-a,0)}(1) > -a, \pi_{(a,0)}(1) < a$, there exists some $i \in [\![-n, n]\!]$, such that $|\pi_{(ai/n,0)}(1) - ai/n| \leq a/n$ (see Figure 3.14). Indeed, otherwise we denote

$$I_- = \{i \in [\![-n, n]\!] : \pi_{(ai/n,0)}(1) < a(i-1)/n\}$$

and

$$I_+ = \{i \in [\![-n, n]\!] : \pi_{(ai/n,0)}(1) > a(i+1)/n\}.$$

We have $-n \in I_+$ and $n \in I_-$, and $[\![-n, n]\!] \subset I_- \cup I_+$. Then we can find some

$i \in [\![-n, n-1]\!]$ such that $i \in I_+$ and $i+1 \in I_-$. These would lead to a contradiction with the ordering of semi-infinite geodesics (Lemma 3.23). Then we conclude that $|\pi_{(ai/n,0)}(1) - ai/n| \leq a/n$ for some $i \in [\![-n, n]\!]$; thus by translation invariance we have

$$\mathbb{P}[\pi_{\mathbf{0}}(1) \in [-a/n, a/n]] > (2n+1)^{-1}\mathbb{P}[\pi_{(-a,0)}(1) > -a, \pi_{(a,0)}(1) < a],$$

so our conclusion follows. □

3.6.4 Hausdorff dimension of the zero set: upper bound

To finish the proof of Theorem 1.11, it remains to prove the upper bound for the Hausdorff dimension of the set $Z = \{t \geq 0 : \pi_{\mathbf{0}}(t) = 0\}$. This is immediately implied by the following result.

Proposition 3.58. *For any $0 < g < h$, and any $d > 1/3$, almost surely the d-dimensional Hausdorff measure of the set $Z \cap [g, h]$ is zero.*

Recall the formal definition of Hausdorff dimension in Definition 3.1. As indicated in Section 3.1, we would split $[g, h]$ into segments, and bound the expected number of segments that intersect with Z. The key estimate is as follows.

Proposition 3.59. *There is a constant C depending on g, h, such that for any $g \leq t < t + \epsilon \leq h$, we have*

$$\mathbb{P}[Z \cap (t, t+\epsilon) \neq \emptyset] < C\epsilon^{2/3}|\log(\epsilon)|^5.$$

There are two mechanisms for $Z \cap (t, t+\epsilon) \neq \emptyset$: either $\pi_{\mathbf{0}}(t)$ is in a neighborhood of 0 of size in the order of $\epsilon^{2/3}$, or $\pi_{\mathbf{0}}$ has a transversal fluctuation in $(t, t+\epsilon)$ larger than its typical order (of $\epsilon^{2/3}$). We shall bound the probabilities of both events. The proof is similar in spirit to the proof of Proposition 3.49.

Proof. We can assume that ϵ is small enough since otherwise the inequality is obvious by taking the constant large enough.

We consider the following two events, corresponding to the above two mechanisms.

- Let $\mathcal{E}_1$ be the event where $|\pi_{\mathbf{0}}(t)| \leq \epsilon^{2/3}|\log(\epsilon)|^5$.
- Let $\mathcal{E}_2$ be the event where there is some $s \in [t, t+\epsilon]$, such that

$$|\pi_{(\mathbf{0};0,s)}(t)| > \epsilon^{2/3}|\log(\epsilon)|^5.$$

Then we have that the event $Z \cap (t, t+\epsilon) \neq \emptyset$ implies $\mathcal{E}_1 \cup \mathcal{E}_2$. Indeed, suppose that there is some $s \in Z \cap (t, t+\epsilon)$, then $\pi_{(\mathbf{0};0,s)}$ is a part of $\pi_{\mathbf{0}}$; i.e., for any $s' \in [0, s]$ we have $\pi_{(\mathbf{0};0,s)}(s') = \pi_{\mathbf{0}}(s')$. In particular, we would have either $|\pi_{\mathbf{0}}(t)| \leq \epsilon^{2/3}|\log(\epsilon)|^5$, or $\pi_{(\mathbf{0};0,s)}(t)| > \epsilon^{2/3}|\log(\epsilon)|^5$, implying that either $\mathcal{E}_1$ or $\mathcal{E}_2$ occurs.

It now suffices to bound the probabilities of these events by the order of $\epsilon^{2/3}|\log(\epsilon)|^5$. The bound on $\mathbb{P}[\mathcal{E}_1]$ is by Corollary 3.51. For $\mathbb{P}[\mathcal{E}_2]$, we us Lemma 3.9, as bounding $\mathbb{P}[\mathcal{E}_2]$ in the proof of Proposition 3.49. We omit the details here. □

Proof of Proposition 3.58. For any large enough $n \in \mathbb{N}$, let

$$\mathscr{Z}_n = \{i \in \mathbb{Z} : [i/n, (i+1)/n] \cap Z \cap [g, h] \neq \emptyset\}.$$

By Proposition 3.59, $\mathbb{E}[|\mathscr{Z}_n|] < Cn^{2/3}(\log(n))^5$, where C is a constant depending only on g, h. For $0 < \alpha < d - 1/3$, we then have

$$\mathbb{P}[n^{-d}|\mathscr{Z}_n| > n^{-\alpha}] < Cn^{1/3-d+\alpha}(\log(n))^5.$$

This implies that almost surely, we can find a random k_0, such that for any $k > k_0$,

$$2^{-dk}|\mathscr{Z}_{2^k}| \leq 2^{-\alpha k}.$$

Since the right hand side $\to 0$ as $k \to \infty$, and $Z \cap [g,h] \subset \cup_{i \in \mathscr{Z}_n}[i/n,(i+1)/n]$, the conclusion follows. $\square$

We have now spelled out all the key ingredients and ideas for us to be able to plunge in to the proof of Theorem 1.10.

3.7 Hausdorff dimension of $\widehat{\mathsf{NC}}$: Proof of Theorem 1.10

Our general strategy would be similar to that implemented in Section 3.5. However while the upper bound straight forwardly follows from Proposition 3.49, there are several key differences in the lower bounds, which we explain below.

3.7.1 Proof of the upper bound

The proof is similar to that of the proof of Proposition 3.58 with the application of Proposition 3.59 replaced by an application of Proposition 3.49. We provide the details below for completeness at the risk of being slightly repetitive. For any $0 < g < h$, and any $d > 2/3$, we show that almost surely, the d-dimensional Hausdorff measure of the set $\widehat{\mathsf{NC}} \cap [g,h]$ is zero. Take any large enough $n \in \mathbb{N}$, and let

$$\mathscr{S}_n = \{i \in \mathbb{Z} : [i/n,(i+1)/n] \cap S \cap [g,h] \neq \emptyset\}.$$

By Proposition 3.49, we have that $\mathbb{E}[|\mathscr{S}_n|] < Cn^{2/3}e^{C(\log(n))^{5/6}}$, where C is a constant depending only on g,h. Taking $0 < \alpha < d - 2/3$, we then have

$$\mathbb{P}[n^{-d}|\mathscr{S}_n| > n^{-\alpha}] < Cn^{2/3-d+\alpha}e^{C(\log(n))^{5/6}}.$$

This implies that almost surely, we can find a random k_0, such that for any $k > k_0$,

$$2^{-dk}|\mathscr{S}_{2^k}| \leq 2^{-\alpha k}.$$

The proof can then be finished as for Proposition 3.58. □

We next move on to the proof of the lower bound. For this, we need to construct a measure κ on $\mathbb{R}_+$, which is the density of the measure ζ from Section 3.5, as described in Section 3.1 (see (3.5) and the ensuing discussion). We shall call it *the measure induced by the difference profile density*, and show that it is supported on $\widehat{\mathsf{NC}}$. We then prove a Hölder property of κ, and that κ is strictly positive. Recall the level functions ϑ_ℓ defined as

$$\vartheta_\ell(t) = \inf\{x \in \mathbb{R} : \mathcal{D}(x,t) \leq \ell\}.$$

The Hölder property of κ is proved by writing κ as local times of ϑ_ℓ, averaged over $\ell \in \mathbb{R}$. Via results in Section 3.5.3, this is reduced to analyzing the local time of the competition interface $\vartheta^{\mathcal{B}}$, which has the same law as the semi-infinite geodesic $\pi_{\mathbf{0}}$ (Proposition 3.4).

There are two main additional difficulties in the setting of $\widehat{\mathsf{NC}}$ (compared to that of NC in Section 3.5). First, the construction of κ is not straightforward, and we need arguments similar to those in Section 3.6.1. In addition, there is some work needed to show that κ is an average of occupation measures of ϑ_ℓ. We resolve these difficulties in Section 3.7.2. Second, the fact that κ is non-degenerate is not obvious, and requires some work using estimates about the Airy line ensemble. We conduct these analyses in Section 3.7.4. The desired Hölder property of κ is proved in Section 3.7.3.

Recall the notations $U_{g,h,N}$ and $\text{Good}_{g,h,N,\delta}$ from Section 3.5.3. For this section, we shall usually take $N = 1/2$, and use the abbreviations $U_{g,h} = U_{g,h,1/2}$ and $\text{Good}_{g,h,\delta} = \text{Good}_{g,h,1/2,\delta}$, for simplicity of notations.

3.7.2 Construction and average of occupation measures

Recall $\zeta^{\mathcal{B}}$, ζ_ℓ for each $\ell \in \mathbb{R}$, and ζ on $\mathbb{H}$, which are defined in (3.20), (3.18), and (3.16) respectively, and are given by

$$\zeta^{\mathcal{B}}([a,b] \times (g,h)) = \int_g^h \mathbb{1}[\vartheta^{\mathcal{B}}(t) \in [a,b]]dt,$$

and

$$\zeta_\ell([a,b] \times (g,h)) = \int_g^h \mathbb{1}[\vartheta_\ell(t) \in [a,b]]dt,$$

$$\zeta([a,b] \times (g,h)) = \int_g^h [\mathcal{D}(a,t) - \mathcal{D}(b,t)]dt,$$

for any $a < b$ and $0 \le g < h$. In this section, we will mostly consider these measures on rectangles. To ease notations, for each $h \ge 0$, we denote by $\lambda_h^{\mathcal{B}}$ and $\lambda_{\ell,h}$ the measures on $\mathbb{R}$, such that

$$\lambda_h^{\mathcal{B}}(A) = \zeta^{\mathcal{B}}(A \times (0,h]) = \int_0^h \mathbb{1}[\vartheta^{\mathcal{B}}(t) \in A]dt,$$

and

$$\lambda_{\ell,h}(A) = \zeta_\ell(A \times (0,h]) = \int_0^h \mathbb{1}[\vartheta_\ell(t) \in A]dt,$$

for any measurable $A \subset \mathbb{R}$. Thus $\lambda_h^{\mathcal{B}}$ and $\lambda_{\ell,h}$ can be viewed as the occupation measure for $\vartheta^{\mathcal{B}}$ and ϑ_ℓ, respectively, for the time interval $[0,h]$. Also for each $0 \le g < h$, we denote by $\lambda_{(g,h]}$ the measures on $\mathbb{R}$, such that

$$\lambda_{(g,h]}(A) = \zeta(A \times (g,h]) \tag{3.32}$$

for any measurable $A \subset \mathbb{R}$.

Remark 3.60. For $\lambda_{(g,h]}$, we shall usually take $g > 0$, because $\lambda_{(0,h]}$ is infinite. Indeed, by symmetry of the directed landscape, for any $a < b$ and $t > 0$ we have $\mathbb{E}[\mathcal{D}(a,t) - \mathcal{D}(b,t)] = 4(b-a)/t$, and the fluctuation of $\mathcal{D}(a,t) - \mathcal{D}(b,t)$ is $O(t^{1/3})$ (by Lemma 3.7). Thus when $g = 0$, we have $\lambda_{(0,h]}([a,b]) = \int_0^h [\mathcal{D}(a,t) - \mathcal{D}(b,t)]dt = \infty$. On the other hand, when $g > 0$, we have $\mathbb{E}[\lambda_{(g,h]}([a,b])] < \infty$, so almost surely we have $\lambda_{(g,h]}([a,b]) < \infty$.

The relation (3.19) for the set $A \times (g,h]$ can be written as

$$\lambda_{(g,h]}(A) = \int [\lambda_{\ell,h}(A) - \lambda_{\ell,g}(A)]d\ell. \tag{3.33}$$

For the remainder of this section, we will use $\lambda_{(g,h]}$, $\lambda_h^{\mathcal{B}}$, $\lambda_{\ell,h}$, instead of ζ, $\zeta^{\mathcal{B}}$, ζ_ℓ, at the risk of introducing new notation. This is because we will mostly consider sets of the form of $A \times (g,h]$, and $\lambda_h^{\mathcal{B}}$ has the same law as ν_h (defined in Section 3.6.1) by Proposition 3.4.

We next consider the densities of these measures at zero, by defining, for any $0 \le g < h$,

$$K_{g,h} =: \lim_{w \searrow 0} (2w)^{-1} \lambda_{(g,h]}([-w,w]), \tag{3.34}$$

$$K_{g,h}^{\mathcal{B}} := \lim_{w \searrow 0} (2w)^{-1} (\lambda_h^{\mathcal{B}}([-w,w]) - \lambda_g^{\mathcal{B}}([-w,w])),$$

and

$$K_{\ell,g,h} := \lim_{w \searrow 0} (2w)^{-1} (\lambda_{\ell,h}([-w,w]) - \lambda_{\ell,g}([-w,w])).$$

We now show that these limits are (in some sense) well-defined and finite.

For the stationary initial setting, as $\lambda_h^{\mathcal{B}}$ has the same law as ν_h, almost surely $K_{g,h}^{\mathcal{B}}$ is well defined and finite by Proposition 1.12, and $\lambda_h^{\mathcal{B}}$ is absolutely continuous with respect to Lebesgue measure by Proposition 3.52.

We next study the limits in defining $K_{g,h}$ and $K_{\ell,g,h}$.

Lemma 3.61. *For any $0 < g < h$, almost surely $K_{g,h}$ is well defined, i.e., the limit exists and is finite.*

Proof. This proof is similar to the proof of Proposition 1.12. We construct a translation invariant version of the difference profile. Let $\xi \subset \mathbb{R}$ be a Poisson point process with rate 1. For each $0 \leq g < h$, we consider a measure $\lambda'_{(g,h]}$ such that for any $a < b$, we have

$$\lambda'_{(g,h]}([a,b]) = \int_g^h \sum_{x\in\xi} (\mathcal{L}(x-1,0;a\vee(x-1),t) - \mathcal{L}(x+1,0;a\vee(x-1),t)) \\ - (\mathcal{L}(x-1,0;b\wedge(x+1),t) - \mathcal{L}(x+1,0;b\wedge(x+1),t))dt.$$

Basically this is just the sum of $\lambda_{(g,h]}$ centered at each $x \in \xi$ and truncated by $[x-1, x+1]$. The truncation is taken so that at any point y, the measure only depends on passage times of finitely many starting points, and each is on the segment $[y-2, y+2] \times \{0\}$. Crucially however, even with this truncation the law of the measure $\lambda'_{(g,h]}$ is translation invariant.

Via the translation invariance, we have that $\lim_{w\searrow 0}(2w)^{-1}\lambda'_{(g,h]}([-w,w])$ exists and is finite almost surely, when $g > 0$, using the same arguments as in the proof of Proposition 1.12 and the fact that $\lambda_{(g,h]}$ is finite in any compact interval (as mentioned in Remark 3.60).

Next, for small enough δ, we consider the event $\mathcal{E}_\delta$, where $|\xi \cap [-\delta,\delta]| = |\xi \cap [-3,3]| = 1$. Then, given $0 < g < h$ and recalling the event $\mathrm{Good}_{g,h,\delta}$ from Section 3.5.3, we notice that under the latter, we have

$$(\mathcal{L}(x_1-1,0;p_1) - \mathcal{L}(x_1+1,0;p_1)) - (\mathcal{L}(x_1-1,0;p_2) - \mathcal{L}(x_1+1,0;p_2)) \\ = (\mathcal{L}(x_2-1,0;p_1) - \mathcal{L}(x_2+1,0;p_1)) - (\mathcal{L}(x_2-1,0;p_2) - \mathcal{L}(x_2+1,0;p_2))$$

for any $|x_1|, |x_2| < \delta$ and $p_1, p_2 \in U_{g,h} = [-1/2, 1/2] \times [g/2, 2h]$. In words, for any

$|x_1|, |x_2| < \delta$, the difference profiles centered at x_1 and x_2 are the same inside the box $U_{g,h}$, up to adding a (random) constant.

From the above arguments, we have that

$$(2w)^{-1}(\lambda_{(g,h]}([-w,w]) = (2w)^{-1}(\lambda'_{(g,h]}([-w,w])$$

for any $0 < w < 1/2$, assuming the event $\mathcal{E}_\delta \cap \mathrm{Good}_{g,h,\delta}$. By Lemma 3.45 we have that $\mathbb{P}[\mathrm{Good}_{g,h,\delta}] \to 1$ as $\delta \searrow 0$. So by the independence of $\mathcal{E}_\delta$ and $\mathrm{Good}_{g,h,\delta}$, we have $\mathbb{P}[\mathcal{E}_\delta \cap \mathrm{Good}_{g,h,\delta}] = \mathbb{P}[\mathcal{E}_\delta]\mathbb{P}[\mathrm{Good}_{g,h,\delta}] > 0$ when δ is small enough. These imply that conditional on the event $\mathcal{E}_\delta \cap \mathrm{Good}_{g,h,\delta}$, we still have the (almost sure) existence and finiteness of the limit $\lim_{w \searrow 0}(2w)^{-1}(\lambda'_{(g,h]}([-w,w]))$, thus we have the (almost sure) existence and finiteness of the limit $\lim_{w \searrow 0}(2w)^{-1}(\lambda_{(g,h]}([-w,w]))$. Since $\mathbb{P}[\mathrm{Good}_{g,h,\delta}] \to 1$ as $\delta \searrow 0$, and $\mathcal{E}_\delta$ is independent of $\lambda_{(g,h]}$, the existence and finiteness of the limit hold almost surely. □

Lemma 3.62. *For any $0 < g < h$, almost surely, for almost every $\ell \in \mathbb{R}$, $K_{\ell,g,h}$ is well defined (i.e., the limit exists and is finite), and $\lambda_{\ell,h} - \lambda_{\ell,g}$ is absolutely continuous with respect to Lebesgue measure. By* (3.33) *we then conclude that (almost surely) $\lambda_{(g,h]}$ is also absolutely continuous with respect to Lebesgue measure.*

Proof. Take any $\delta, H > 0$. By Proposition 3.4 and Proposition 1.12, there is a probability one event, under which $K^{\mathcal{B}}_{g,h}$ is well-defined and finite; and by Proposition 3.52, there is a probability one event, under which $\lambda^{\mathcal{B}}_h - \lambda^{\mathcal{B}}_g$ is absolutely continuous with respect to Lebesgue measure. On the other hand, by Lemma 3.46, under the event $\mathrm{Good}_{g,h,\delta}$ (which has positive probability by Lemma 3.45 and taking δ small) almost surely we have $\mathbb{P}[K^{\mathcal{B}}_{g,h} = K_{\alpha,g,h}, \lambda^{\mathcal{B}}_h - \lambda^{\mathcal{B}}_g = \lambda_{\alpha,h} - \lambda_{\alpha,g} \mid \mathcal{L}] > 0$, with α being random and has (conditional on $\mathcal{L}$) probability density $\geq \delta'$ on $[-H, H]$. These imply that under the event $\mathrm{Good}_{g,h,\delta}$, almost surely there is a full measure subset of $[-H, H]$, which is random and determined by $\mathcal{L}$, such that for each ℓ in this set, $K_{\ell,g,h}$ is well-

defined and finite, and $\lambda_{\ell,h} - \lambda_{\ell,g}$ is absolutely continuous with respect to Lebesgue measure. Then the conclusion follows by taking $\delta \searrow 0$ and $H \to \infty$, and using Lemma 3.45. □

Remark 3.63. We explain the reason why the above two lemmas require that $g > 0$ (unlike in the case of semi-infinite geodesics). For Lemma 3.61, it is simply because $\lambda_{(g,h]}$ is finite only when $g > 0$, as discussed in Remark 3.60. For Lemma 3.62, one can actually show convergence to $K_{\ell,g,h}$ when $g = 0$; but since that is not used in the remainder of this thesis we omit the details. In fact, the occupation measures for short times can be (roughly) described as follows. For any $(x, t) \in \mathbb{H}$ and $\ell \in \mathbb{R}$ with $|x + \ell t/4| \gg t^{4/3}$, it is unlikely to have $\mathcal{D}(x, t) = \ell$, by Lemma 3.7. This implies that $|\vartheta_\ell(t) + \ell t/4|$ is at most in the order of $t^{4/3}$. From these we expect that $\lambda_{\ell,1}([-w, w])$ is in the order of $|\ell|^{-1}w$, if $\ell \neq 0$; and it is in the order of $w^{3/4}$ if $\ell = 0$.

By Lemma 3.62, we now have that almost surely, $K_{g,h}$ is well-defined and is positive and finite for each rational $0 < g < h$. We also have that almost surely for $\mathcal{L}$, we can find a random full measure set $\Theta \subset \mathbb{R}$, which can be viewed as a function of $\mathcal{L}$, such that for any $\ell \in \Theta$, and each rational $0 < g < h$, $K_{\ell,g,h}$ is well-defined and finite.

Recall the measures κ and κ_ℓ described in Section 3.1. We are now ready to define them. For any rational $0 < h_1 < h_2 < h_3$, we have $K_{h_1,h_2} + K_{h_2,h_3} = K_{h_1,h_3}$. Hence we can define the measure κ on $\mathbb{R}_+$, by letting

$$\kappa((g, h)) = \sup_{g<g'<h'<h,\ g',h' \in \mathbb{Q}} K_{g',h'}, \tag{3.35}$$

for any $0 < g < h$. We take the supremum from interior since we need $\kappa((g, h))$ be right continuous in g and left continuous in h. Using standard measure theory arguments, this suffices to define κ as a measure on $\mathbb{R}_+$.

We now move on to the construction of the decomposition of κ indicated in (3.5),

which is the one dimensional analogue of (3.33). For each $\ell \in \Theta$ and $0 < g < h$, we let

$$\kappa_\ell((g,h)) = \sup_{g<g'<h'<h,\ g',h' \in \mathbb{Q}} K_{\ell,g',h'}, \tag{3.36}$$

and we extend this to define a measure κ_ℓ on $\mathbb{R}_+$.

Remark 3.64. Actually, almost surely $K_{\ell,g,h}$ is continuous in g and h for ℓ in a random set with full measure. This is by Proposition 3.53 and the equality in distribution between $\pi_{\mathbf{0}}$ and $\vartheta^{\mathcal{B}}$ (Lemma 3.4), and taking $\kappa^{\mathcal{B}}$ as κ_ℓ for a random ℓ using Lemma 3.46 (as done in the proof of Lemma 3.62). Thus $\kappa_\ell((g,h)) = K_{\ell,g,h}$ for almost every ℓ and any rational $0 < g < h$.

We next show that κ is the integral of κ_ℓ over ℓ. For the convenience of notations, we let κ_ℓ be an arbitrary measure on $\mathbb{R}_+$ for each $\ell \in \mathbb{R} \setminus \Theta$. Since $\mathbb{R} \setminus \Theta$ has zero Lebesgue measure, the choice of such κ_ℓ does not affect the integral.

Lemma 3.65. *Almost surely, $\kappa(A) = \int \kappa_\ell(A) d\ell$, for any measurable set A that is contained in a compact subset of $\mathbb{R}_+$.*

To prove this, we first prove the same relation for $K_{g,h}$ and $K_{\ell,g,h}$, for fixed g, h.

Lemma 3.66. *For any $0 < g < h$, almost surely we have that $K_{g,h} = \int K_{\ell,g,h} d\ell$.*

We note that this is a "density" and hence a limiting version of (3.33). The proof of this lemma is via a uniform integrability result, when passing from $\lambda_{(g,h]}$ and $\lambda_{\ell,h} - \lambda_{\ell,g}$ to $K_{g,h}$ and $K_{\ell,g,h}$. We would again rely on the connections with $\lambda_h^{\mathcal{B}} - \lambda_g^{\mathcal{B}}$, and with semi-infinite geodesics, and the estimate in Lemma 3.54.

Proof. Fixing $0 < g < h$, for brevity, let

$$f_w(\ell) = (2w)^{-1}(\lambda_{\ell,h}([-w,w]) - \lambda_{\ell,g}([-w,w])),$$

for $w > 0$, and

$$f(\ell) = K_{\ell,g,h}.$$

From Lemma 3.61 and Lemma 3.62, we have that almost surely, $f_w \to f$ almost everywhere, and also $\|f_w\|_{L_1}$ converges as $w \searrow 0$. Now we want to show (almost surely) the limit of $\lim_{w\searrow 0} \|f_w\|_{L_1} = \|f\|_{L_1}$. For this we show for any $H > 0$, there is some (random) positive sequence $w_1, w_2, \cdots \to 0$, such that $f_{w_n} \to f$ in $L_1([-H, H])$.

Note that almost surely, all f_w with $0 < w < 1$ and f are supported on some compact interval. This is because, by Lemma 3.7, for the level fucntion ϑ_ℓ, we have $|\vartheta_\ell(t)| > 1$ for $t \in [g, h]$ when $|\ell|$ is large enough. Thus it suffices to show that $\{f_{w_n}\}_{n=1}^{\infty}$ are uniformly integrable in $[-H, H]$, for any $H > 0$.

Analogous to $f_w(\cdot)$, let

$$f_w^{\mathcal{B}} = (2w)^{-1}(\lambda_h^{\mathcal{B}}([-w, w]) - \lambda_g^{\mathcal{B}}([-w, w])).$$

We then have (by Lemma 3.54),

$$\mathbb{P}[f_w^{\mathcal{B}} > M] < Ce^{-cM}$$

for some $c, C > 0$ that are independent of w (but depend on g, h). To show uniform integrablity, we let $\gamma_{w,m} = \int_{-H}^{H} \mathbb{1}[f_w(\ell) > m] f_w(\ell) d\ell$, for any $m \in \mathbb{N}$. Now, for any $\delta > 0$, let δ' be given by Lemma 3.46. Then,

$$\begin{aligned}\mathbb{E}[\mathbb{1}[\mathrm{Good}_{g,h,\delta}] \liminf_{w\searrow 0} \gamma_{w,m}] &\le \liminf_{w\searrow 0} \mathbb{E}[\mathbb{1}[\mathrm{Good}_{g,h,\delta}]\gamma_{w,m}] \\ &\le 1/\delta' \liminf_{w\searrow 0} \mathbb{E}[\mathbb{1}[f_w^{\mathcal{B}} > m] f_w^{\mathcal{B}}] < Ce^{-cm}/\delta',\end{aligned}$$

where the first inequality is by Fatou's lemma, and $C, c > 0$ are constants independent of w, m. Recall (from Lemma 3.45) that $\lim_{\delta\searrow 0} \mathbb{P}[\mathrm{Good}_{g,h,\delta}] = 1$, and note that $\mathrm{Good}_{g,h,\delta}$ does not depend on w or m. These imply that $\lim_{m\to\infty} \liminf_{w\searrow 0} \gamma_{w,m} = 0$ almost surely.

Via a diagonal argument, almost surely we can take a (random) positive subse-

quence $w_1, w_2, \ldots \to 0$ such that $\lim_{n\to\infty} \gamma_{w_n,m} = \liminf_{w\searrow 0} \gamma_{w,m}$ for each $m \in \mathbb{N}$. Hence

$$\lim_{m\to\infty} \lim_{n\to\infty} \gamma_{w_n,m} = \lim_{m\to\infty} \liminf_{w\searrow 0} \gamma_{w,m} = 0.$$

That is, $\{f_{w_n}\}_{n=1}^{\infty}$ are uniformly integrable in $[-H, H]$. For H large enough, we have all f_w and f are supported on $[-H, H]$ (and the largeness is random). Thus for the subsequence $w_1, \ldots$, we have $f_{w_n} \to f$ in L_1; this implies that $\|f_{w_n}\|_{L_1} \to \|f\|_{L_1}$, and hence $\|f_w\|_{L_1} \to \|f\|_{L_1}$. □

Using the above, the proof of Lemma 3.65 is straightforward.

Proof of Lemma 3.65. For the arguments below, we assume the probability one event that Lemma 3.66 holds for any rational $0 < g < h$.

We first consider the case where A is an interval (g, h) for some rational $0 < g < h$. Take a rational sequence $g_n \searrow g$ and a rational sequence $h_n \nearrow h$. Then $\kappa(A) = \lim_{n\to\infty} K_{g_n,h_n}$ and $\kappa_\ell(A) = \lim_{n\to\infty} K_{\ell,g_n,h_n}$ for any $\ell \in \Theta$. Since $K_{\ell,g_n,h_n} \le K_{\ell,g,h}$ and $K_{\ell,g,h}$ is integrable as a function of ℓ (by Lemma 3.66), we can apply dominated convergence theorem to conclude that

$$\int \kappa_\ell(A) d\ell = \lim_{n\to\infty} \int K_{\ell,g_n,h_n} d\ell.$$

By Lemma 3.66, the right hand side equals $\lim_{n\to\infty} K_{g_n,h_n} = \kappa(A)$, so the conclusion holds in the case where A is a rational interval.

Then we can extend this to any measurable set A contained in a compact set in $\mathbb{R}_+$, using $\pi - \lambda$ theorem. For this, we need to show that for any rational $0 < g < h$, the collection $\{A \subset (g, h) : \kappa(A) = \int \kappa_\ell(A) d\ell\}$ is a Dynkin system: closed under complements is obvious, and closed under countable disjoint union is by dominated convergence theorem (since $\kappa_\ell(A) \le \kappa_\ell((g, h))$ for any $A \subset (g, h)$ and any $\ell \in \Theta$). □

As was indicated in Section 3.1, the following statement highlights the usefulness

of κ.

Lemma 3.67. *Almost surely, the measure κ is supported inside $\widehat{\mathsf{NC}}$.*

Proof. In this proof we assume the event where we can find a measurable set $\Theta' \subset \Theta \subset \mathbb{R}$ with $\mathscr{L}(\mathbb{R} \setminus \Theta') = 0$, such that each $\ell \in \Theta'$ satisfies the following: (1) for each $t \in \mathbb{R}_+$, there is a unique $x \in \mathbb{R}$ with $\mathcal{D}(x,t) = \ell$; (2) $K_{\ell,g,h}$ is well-defined and finite for each rational $0 < g < h$. By Lemma 3.42 and Lemma 3.62, this event happens almost surely.

We now fix some $0 < g < h$, and show that κ restricted on (g,h) is supported inside $\widehat{\mathsf{NC}} \cap (g,h)$.

Denote S to be the closure of $\cup_{\ell \in \Theta'} \{t : g < t < h, \vartheta_\ell(t) = 0\}$. Here the closure is taken to ensure the measurability of S. We claim that $S \subset \widehat{\mathsf{NC}}$. Indeed, for any $t \notin \widehat{\mathsf{NC}}$, we have $\mathcal{D}(0,s) = \ell_*$ for each $s \in A$, where A is a neighborhood of t and $\ell_* \in \mathbb{R}$ is a constant. If $t \in S$, by the definition of S we can find some $s \in A$ and some $\ell' \in \Theta'$, such that $\vartheta_{\ell'}(s) = 0$. Thus we must have that $\ell' = \ell_*$, so $\ell_* \in \Theta'$. This implies that $\vartheta_{\ell_*}(s) = 0$ for each $s \in A$ and $K_{\ell_*,g,h}$ is well-defined and finite. However, from its definition we have $K_{\ell_*,g,h} = \infty$ when $\vartheta_{\ell_*}(s) = 0$ for each $s \in A$. Thus we must have $t \notin S$.

We shall next show that κ restricted to (g,h) is supported inside S. Take any $\ell \in \Theta'$. By Lemma 3.43, the set $\{t : g \le t \le h, \vartheta_\ell(t) = 0\}$ is closed, and for any $g < t < h$ with $\vartheta_\ell(t) \neq 0$, we can find an open neighborhood around $(0,t)$ that is disjoint from the trajectory of ϑ_ℓ. The later implies that for any $g < t < h$ with $\vartheta_\ell(t) \neq 0$, one can find some $\delta > 0$ such that $\kappa_\ell((t-\delta, t+\delta)) = 0$. Thus $\kappa_\ell(\{t : g < t < h, \vartheta_\ell(t) \neq 0\}) = 0$. Since $(g,h) \setminus S \subset \{t : g < t < h, \vartheta_\ell(t) \neq 0\}$ we have $\kappa_\ell((g,h) \setminus S) = 0$. By Lemma 3.65, we have that $\kappa((g,h) \setminus S) = \int \kappa_\ell((g,h) \setminus S) d\ell = 0$. This implies that κ restricted to (g,h) is supported inside S, thus supported inside $\widehat{\mathsf{NC}}$. □

To show that the Hausdorff dimension of $\widehat{\mathsf{NC}}$ is at least $2/3$, it remains to show that κ is $2/3-\epsilon$ Hölder in any interval $[g,h]$ for any $\epsilon>0$, and $\kappa([g,h])>0$ for small enough $g>0$. These two steps are carried out in the next two subsections.

3.7.3 $2/3$ Hölder of the difference profile density

Lemma 3.68. *For any $0<g<h$ and $\epsilon>0$, almost surely there is some random Q, such that $\kappa([a,b])<Q(b-a)^{2/3-\epsilon}$ for any $g\le a<b\le h$.*

The proof follows a similar strategy as the proof of Lemma 3.47. Specifically, it suffices to prove $K_{a,b}<Q(b-a)^{2/3-\epsilon}$ for any $a,b\in\mathbb{Q}$ satisfying $g\le a<b\le h$ and $b-a<1$. By the regularity condition of the directed landscape under the event $\mathrm{Good}_{g,h,\delta}$, $K^{\ell}_{a,b}$ is nonzero only for ℓ in an interval of length (in the order of) $(b-a)^{1/3}$. To bound $K_{a,b}$, we just bound an integral of $K^{\ell}_{a,b}$ over ℓ in an interval of length (in the order of) $(b-a)^{1/3}$, and take the supremum over all such intervals. Using Lemma 3.46, (under $\mathrm{Good}_{g,h,\delta}$) such a supremum of integrals is bound by $(b-a)^{1/3}$ times $K^{\mathcal{B}}_{a,b}$; and by Proposition 3.53, $K^{\mathcal{B}}_{a,b}$ is in the order of $(b-a)^{1/3}$ with exponential tails.

Proof. Recall the almost sure event where we have a set $\Theta\subset\mathbb{R}$ with full measure, such that for every $\ell\in\Theta$, and any $0<a<b$, $a,b\in\mathbb{Q}$, the numbers $K_{a,b}$, $K_{\ell,a,b}$, and $K^{\mathcal{B}}_{a,b}$ are well defined and finite. We assume this event for the rest of this proof.

Fix any small enough $\delta>0$. It suffices to bound $K_{a,b}$ for any rational a,b satisfying $g\le a<b\le h$ and $b-a<1$, under the event $\mathrm{Good}_{g,h,\delta}$. For each ℓ, let

$$Q_\ell=\sup_{g\le a<b\le h,a,b\in\mathbb{Q}}\frac{K_{\ell,a,b}}{(b-a)^{1/3}(1+|\log(b-a)|^2)}.$$

Here the extra factor of $1+|\log(b-a)|^2$ is due to the fact that we take the supremum over a,b, and the power of 2 can be replaced by any number >1. To bound $K_{a,b}$, we would bound the integral of Q_ℓ over ℓ. We take $H>0$, large enough and depending

on g, h, δ. For any $\varepsilon > 0$ we denote

$$J_\varepsilon = \sup_{A \subset [-H,H], \mathscr{L}(A) < \delta^{-1}\varepsilon^{1/3}\log^{2/3}(1+\varepsilon^{-1})} \int_A Q_\ell d\ell,$$

and let

$$J = \sup_{\varepsilon \in (0,1)} \frac{J_\varepsilon}{\varepsilon^{1/3}\log^3(1+\varepsilon^{-1})}.$$

The motivation behind these definitions is that, by regularity estimates of $\mathcal{L}$, $K_{\ell,a,b}$ is non-zero only for ℓ in an interval of length of the order of $(b-a)^{1/3}$.

As in the proof of Lemma 3.47, this random variable J is essentially the desired Q in the statement of the lemma, and the remainder of this proof mainly consists of two steps: (1) we bound the upper tail of $\mathbb{1}[\mathrm{Good}_{g,h,\delta}]J$; (2) we bound $K_{a,b}$ for any $g \le a < b \le h$, $a, b \in \mathbb{Q}$ using J under the event $\mathrm{Good}_{g,h,\delta}$. Unavoidably, the following sequence of arguments, like in the proof of Lemma 3.47, will be somewhat technical.

Step 1. We first bound $\mathbb{1}[\mathrm{Good}_{g,h,\delta}]J$. As in the proof of Lemma 3.47, we take inputs from estimates on the competition interface with stationary initial condition. Define

$$Q^{\mathcal{B}} = \sup_{g \le a < b \le h, a, b \in \mathbb{Q}} \frac{K^{\mathcal{B}}_{a,b}}{(b-a)^{1/3}(1+|\log(b-a)|^2)}.$$

By Lemma 3.46 and taking δ' there, for any $M > 0$ we have

$$\mathbb{E}\left[\mathbb{1}[\mathrm{Good}_{g,h,\delta}] \int_{-H}^{H} \mathbb{1}[Q_\ell > M] d\ell\right] < \delta'^{-1}\mathbb{P}[Q^{\mathcal{B}} > M] < \delta'^{-1} C_1 e^{-c_1 M}, \tag{3.37}$$

where $c_1, C_1 > 0$ are some constants (depending on g, h) and the last inequality is by Proposition 3.53.

Recall the arguments in Step 1 in the proof of Lemma 3.47, where (3.23) is translated into an upper tail bound of $\mathbb{1}[\mathrm{Good}_{g,h,\delta}]J$ there. Using the same algebra, (3.37)

can be translated into the following bound on the upper tail of $\mathbb{1}[\text{Good}_{g,h,\delta}]J$: for any $M > 0$, we have that

$$\mathbb{P}[\mathbb{1}[\text{Good}_{g,h,\delta}]J > M] < C_\delta e^{-c_\delta M},$$

for C_δ, c_δ being constants depending on g, h, δ.

Step 2. Under $\text{Good}_{g,h,\delta}$, using its second condition and Lemma 3.7, we can bound $|\mathcal{D}|$ in $[-1/2, 1/2] \times [g, h]$. Consequently, we have that $K_{\ell,g,h} = 0$ for any $|\ell| > H$, as H is large enough depending on g, h, δ. Then by Lemma 3.66, for any rational a, b with $g \le a < b \le h$, we have $K_{a,b} = \int_{-H}^{H} K_{\ell,a,b} d\ell$. We note that by the second condition of $\text{Good}_{g,h,\delta}$ and using Lemma 3.8, for any ℓ_1, ℓ_2, if both $K_{\ell_1,a,b}$ and $K_{\ell_2,a,b}$ are nonzero, we must have

$$|\ell_1 - \ell_2| \le C\delta^{-0.01}(b-a)^{1/3}\log^{2/3}(1+(b-a)^{-1}) < \delta^{-1}(b-a)^{1/3}\log^{2/3}(1+(b-a)^{-1}),$$

where $C > 0$ is a constant depending on g, h, and the last inequality is by taking δ small enough depending on g, h. Thus by the definitions of J_{b-a} and J, we have (for $b - a < 1$)

$$\begin{aligned} K_{a,b} = \int_{-H}^{H} K_{\ell,a,b} d\ell &\le (b-a)^{1/3}(1+|\log(b-a)|^2) J_{b-a} \\ &< (b-a)^{2/3}(1+|\log(b-a)|^2)\log^3(1+(b-a)^{-1}) J. \end{aligned}$$

By taking Q to be J multiplied by a constant depending on g, h, ϵ, we get the desired bound $K_{a,b} < Q(b-a)^{2/3-\epsilon}$ under $\text{Good}_{g,h,\delta}$. Finally, by taking $\delta \searrow 0$, we get that the desired bound holds almost surely. □

3.7.4 Non-degeneracy of the difference profile density

In this subsection we implement the last step in our strategy to lower bound the Hausdorff dimension of $\widehat{\mathsf{NC}}$, namely showing that κ is strictly positive This along with Lemma 3.67 and Lemma 3.68 concludes the proof of the lower bound of Theorem 1.10, as indicated by the general strategy in Section 3.1.

Proposition 3.69. *Almost surely, there exist* $0 < g < h$ *such that* $\kappa([g,h]) > 0$.

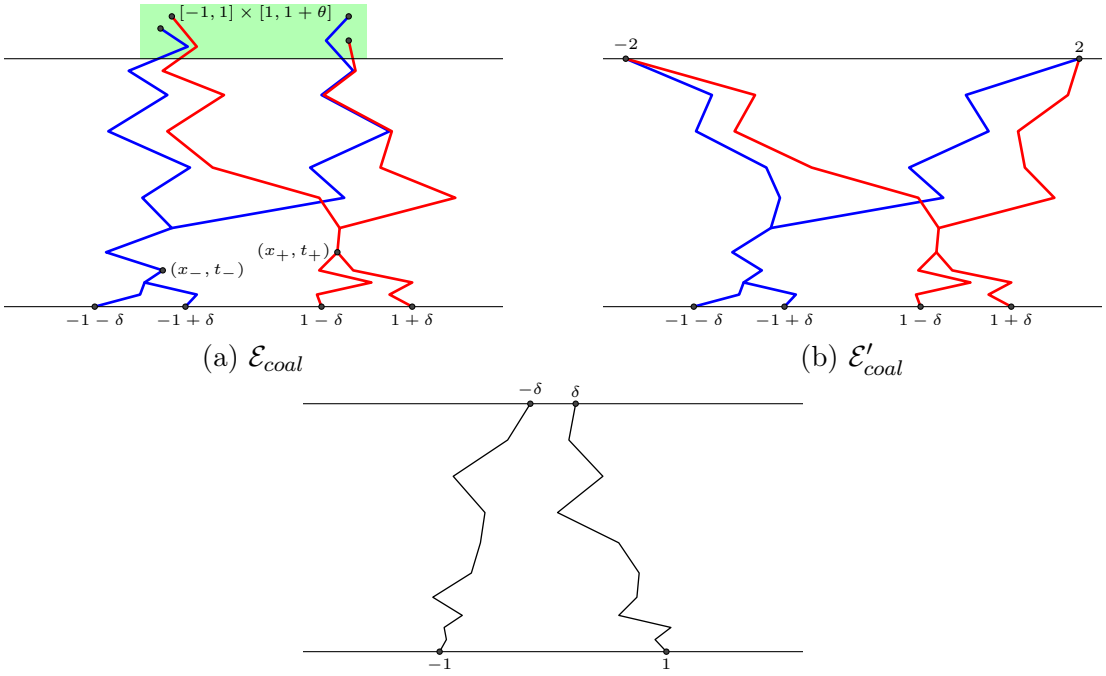
Note that for fixed $0 < g < h$, with positive probability, owing to coalesence of geodesics, the difference profile $\mathcal{D}$ can be flat in the box $[-1,1] \times [g,h]$, thus $\kappa([g,h]) = 0$, with positive probability.

The proof of Proposition 3.69 follows a similar strategy as the proof of Proposition 3.55: we start by showing that for a fixed interval, κ is positive with positive probability; then we use ergodicity to show that κ is almost surely strictly positive. Recall the notation of $K_{g,h}$ for any $0 < g < h$, from (3.34).

Lemma 3.70. *We have* $\mathbb{P}[K_{1,2} > 0] > 0$.

To prove this we need estimates using the Airy line ensemble. We defer the Airy line ensemble arguments to Appendix C, and state the rest of the arguments here.

Proof. Take some small $\theta > 0$ to be determined. It suffices to prove that $\mathbb{E}[K_{1,1+\theta}] > 0$. The general idea is to use translation invariance and coalescence of geodesics, to reduce this to proving that, for some small $\delta > 0$, $\lambda_{(1,1+\theta]}([-\delta,\delta]) > 0$ with positive probability (recall (3.32) for the definition of $\lambda_{(1,1+\theta]}$); and this event is further implied by some coalescence and disjointness of geodesics. Such coalescence and disjointness of geodesics are studied using the Airy line ensemble, and we leave the arguments to Appendix C.



(a) $\mathcal{E}_{coal}$ (b) $\mathcal{E}'_{coal}$

(c) $\mathcal{E}_{disj}$

Figure 3.15: Illustrations of the events $\mathcal{E}_{coal}$, $\mathcal{E}'_{coal}$, and $\mathcal{E}_{disj}$ in the proof of Lemma 3.70: assuming $\mathcal{E}_{disj}$, we have $\int_{-\delta}^{\delta} \overline{K}^s_{1,1+\theta} ds > 0$ for some random $\theta > 0$, where $\overline{K}^s_{1,1+\theta}$ is $K_{1,1+\theta}$ translated by $(s,0)$ in its definition. Then by translation invariance (and assuming $\mathbb{P}[\mathcal{E}_{coal} \cap \mathcal{E}_{disj}] > 0$), we have $\mathbb{E}[K_{1,1+\theta}] > 0$ for small enough θ. To show $\mathbb{P}[\mathcal{E}_{coal} \cap \mathcal{E}_{disj}] > 0$ we reduce $\mathcal{E}_{coal}$ to $\mathcal{E}'_{coal}$, and use tools from the Airy line ensemble to show $\mathbb{P}[\mathcal{E}'_{coal} \cap \mathcal{E}_{disj}] > 0$ (in Appendix C).

We first define translated versions of $K_{1,1+\theta}$. For any $s \in \mathbb{R}$ we let

$$\overline{K}^s_{1,1+\theta} = \lim_{w \searrow 0} (2w)^{-1} \int_1^{1+\theta} [(\mathcal{L}(-1+s,0;-w+s,t) - \mathcal{L}(1+s,0;-w+s,t)) - (\mathcal{L}(-1+s,0;w+s,t) - \mathcal{L}(1+s,0;w+s,t))]dt.$$

By Lemma 3.61 and translation invariance of the directed landscape, for fixed s, θ, almost surely $\overline{K}^s_{1,1+\theta}$ is well-defined and finite, and has the same distribution as $K_{1,1+\theta}$.

We also define

$$\begin{aligned}\hat{K}^s_{1,1+\theta} &= \lim_{w \searrow 0} (2w)^{-1} (\lambda_{(1,1+\theta]}([s-w, s+w])) \\ &= \lim_{w \searrow 0} (2w)^{-1} \int_1^{1+\theta} \big[(\mathcal{L}(-1,0;-w+s,t) - \mathcal{L}(1,0;-w+s,t)) \\ &\qquad - (\mathcal{L}(-1,0;w+s,t) - \mathcal{L}(1,0;w+s,t)) \big] dt.\end{aligned}$$

We note that (by Lebesgue's theorem for the differentiability of monotone functions as before), almost surely this is well-defined for almost every s. The difference between these two definitions is that, for $\hat{K}^s_{1,1+\theta}$ only the upper endpoints of the considered passage times are shifted by $(s,0)$, while everything is shifted by $(s,0)$ for $\overline{K}^s_{1,1+\theta}$.

We now explain the reason behind defining $\overline{K}^s_{1,1+\theta}$ and $\hat{K}^s_{1,1+\theta}$. We will write $\mathbb{E}[K_{1,1+\theta}]$ as an integral of $\mathbb{E}[\overline{K}^s_{1,1+\theta}]$ over s. We will show that $\overline{K}^s_{1,1+\theta} = \hat{K}^s_{1,1+\theta}$ under certain coalescence events, so it suffices to consider the expectation of some integral of $\hat{K}^s_{1,1+\theta}$ over s, which can be written as $\lambda_{(1,1+\theta]}([-\delta, \delta])$.

We first define the coalescence event under which $\overline{K}^s_{1,1+\theta} = \hat{K}^s_{1,1+\theta}$ holds. Let $\delta > 0$ be a small number. Let $\mathcal{E}_{coal}$ be the following event (see Figure 3.15 for an illustration): there exist some $t_-, t_+ > 0$, $x_-, x_+ \in \mathbb{R}$, such that for any $x \in (-1-\delta, -1+\delta)$ and $p \in [-1,1] \times [1, 1+\theta]$, there is $\pi_{(x,0;p)}(t_-) = x_-$; and for any $x \in (1-\delta, 1+\delta)$ and $p \in [-1,1] \times [1, 1+\theta]$, there is $\pi_{(x,0;p)}(t_+) = x_+$. Under $\mathcal{E}_{coal}$, for any $p_1, p_2 \in [-1,1] \times [1, 1+\theta]$ and $|s| < \delta$, we have

$$\mathcal{L}(-1,0;p_1) - \mathcal{L}(-1,0;p_2) = \mathcal{L}(-1+s,0;p_1) - \mathcal{L}(-1+s,0;p_2).$$

This is because, from the coalescence of geodesics as illustrated by Figure 3.15, we have that both sides would equal $\mathcal{L}(x_-, t_-; p_1) - \mathcal{L}(x_-, t_-; p_2)$. Similarly,

$$\mathcal{L}(1,0;p_1) - \mathcal{L}(1,0;p_2) = \mathcal{L}(1+s,0;p_1) - \mathcal{L}(1+s,0;p_2).$$

By taking the difference of the above two equations, letting $p_1 = (-w+s, t)$, $p_2 = (w+s, t)$, integrating over t from 1 to $1+\theta$, dividing both hand sides by $2w$, and sending $w \searrow 0$, we conclude that $\overline{K}^s_{1,1+\theta} = \hat{K}^s_{1,1+\theta}$ under $\mathcal{E}_{coal}$. We then have

$$2\delta\mathbb{E}[K_{1,1+\theta}] = \int_{-\delta}^{\delta} \mathbb{E}[\overline{K}^s_{1,1+\theta}]ds \geq \int_{-\delta}^{\delta} \mathbb{E}[\mathbb{1}[\mathcal{E}_{coal}]\hat{K}^s_{1,1+\theta}]ds = \mathbb{E}[\mathbb{1}[\mathcal{E}_{coal}] \int_{-\delta}^{\delta} \hat{K}^s_{1,1+\theta} ds]. \tag{3.38}$$

It remains to show that (for appropriate choice of δ, θ) the last term is positive. Note that as $\lambda_{(1,1+\theta]}$ is absolutely continuous with respect to Lebesgue measure by Lemma 3.62 (almost surely for fixed θ), we can write

$$\begin{aligned}\int_{-\delta}^{\delta} \hat{K}^s_{1,1+\theta} ds &= \lambda_{(1,1+\theta]}([-\delta, \delta]) \\ &= \int_1^{1+\theta} [(\mathcal{L}(-1,0;-\delta,t) - \mathcal{L}(1,0;-\delta,t)) - (\mathcal{L}(-1,0;\delta,t) - \mathcal{L}(1,0;\delta,t))]dt.\end{aligned}$$

We now find an event where the right hand side is positive. Recall that for any $x_1 < x_2$, $y_1 < y_2$, and $s < t$, $\mathrm{Disj}^{[s,t]}_{(x_1,x_2)\to(y_1,y_2)}$ is the event where $\pi_{(x_1,s;y_1,t)}$ and $\pi_{(x_2,s;y_2,t)}$ are unique and disjoint. We then let $\mathcal{E}_{disj} = \mathrm{Disj}^{[0,1]}_{(-1,1)\to(-\delta,\delta)}$. Then by Lemma 3.13 and spatial monotonicity of $\mathcal{D}$, and the fact that almost surely $\mathcal{L}$ is continuous, we have that $\mathcal{E}_{disj}$ implies $\int_{-\delta}^{\delta} \hat{K}^s_{1,1+\theta} ds > 0$.

It remains to show that $\mathbb{P}[\mathcal{E}_{coal} \cap \mathcal{E}_{disj}] > 0$. The event $\mathcal{E}_{coal}$ involves geodesics whose upper endpoints vary in the box $[-1,1] \times [1, 1+\theta]$, and is not easy to analyze using the Airy line ensemble. For this, we consider

$$\mathcal{E}'_{coal} = (\mathrm{Disj}^{[0,1]}_{(-1-\delta,-1+\delta)\to(-2,2)} \cup \mathrm{Disj}^{[0,1]}_{(1-\delta,1+\delta)\to(-2,2)})^c$$

(see Figure 3.15 for an illustration). The event $\mathcal{E}'_{coal}$ only involves geodesics from time 0 to time 1, and can be written as an event of the Airy line ensemble. From this we can show that $\mathbb{P}[\mathcal{E}'_{coal} \cap \mathcal{E}_{disj}] > 0$. We leave the details to Lemma C.1.

By Lemma 3.9, given $\mathcal{E}'_{coal}$ almost surely we have that $\mathcal{E}_{coal}$ holds for some random

$\theta > 0$. Indeed, by Lemma 3.9 we can find some $\theta > 0$, such that for any $x \in (-1-\delta, -1+\delta) \cup (1-\delta, 1+\delta)$ and $p \in [-1,1] \times [1, 1+\theta]$, we have $|\pi_{(x,0;p)}(1)| \leq 2$. Then by ordering of geodesics, for any t_-, t_+ and $x_-, x_+ \in \mathbb{R}$ such that $\pi_{(-1-\delta,0;-2,1)}(t_-) = \pi_{(-1+\delta,0;2,1)}(t_-) = x_-$ and $\pi_{(1-\delta,0;-2,1)}(t_+) = \pi_{(1+\delta,0;2,1)}(t_+) = x_+$, they satisfy the properties in the definition of $\mathcal{E}_{coal}$ above, so $\mathcal{E}_{coal}$ holds. Thus $\mathbb{P}[\mathcal{E}'_{coal} \cap \mathcal{E}_{disj}] > 0$ implies that for some θ small enough (depending on δ), we have $\mathbb{P}[\mathcal{E}_{coal} \cap \mathcal{E}_{disj}] > 0$. We then have that $\mathbb{E}[\mathbb{1}[\mathcal{E}_{coal}] \int_{-\delta}^{\delta} \hat{K}^s_{1,1+\theta} ds] > 0$, which with (3.38) implies the conclusion. □

We need one more ingredient to finish the proof of Proposition 3.69, namely a 0-1 law. For any $r > 0$, we let $\mathcal{F}_r$ be the sigma algebra generated by $\mathcal{L}(x, s; y, t)$, for $0 \leq s < t \leq r$. Let $\mathcal{F}_{0+} = \cap_{r>0} \mathcal{F}_r$ be the corresponding germ sigma-algebra.

Lemma 3.71. *Any $\mathcal{F}_{0+}$-measurable event has probability 0 or 1.*

Proof. We show that $\mathcal{L}(x, s; y, t)$ is independent of $\mathcal{F}_{0+}$ for any $(x, s; y, t) \in \mathbb{R}^4_{\uparrow}$. If $s > 0$ or $t \leq 0$, this is obvious. If $s = 0$, we have $\mathcal{L}(x, s; y, t) = \lim_{\delta \searrow 0} \mathcal{L}(x, s+\delta; y, t)$, which also implies the statement, since each $\mathcal{L}(x, s+\delta; y, t)$ is independent of $\mathcal{F}_{0+}$. Now for $s < 0 < t$, we have the composition law

$$\mathcal{L}(x, s; y, t) = \sup_z \mathcal{L}(x, s; z, 0) + \mathcal{L}(z, 0; y, t).$$

Since $\mathcal{L}(x, s; z, 0)$ is independent of $\mathcal{F}_{0+}$, and that we have just shown that $\mathcal{L}(z, 0; y, t)$ is independent of $\mathcal{F}_{0+}$, we must have that the left hand side is also independent of $\mathcal{F}_{0+}$. Thus the conclusion follows. □

We can finally put things together to finish the proof of Proposition 3.69.

Proof of Proposition 3.69. For each $m \in \mathbb{N}$, we consider $K_{2^{-m-1},2^{-m}}$ (recall its definition from (3.34)), which we assume to be well-defined and finite. By the quadrangle

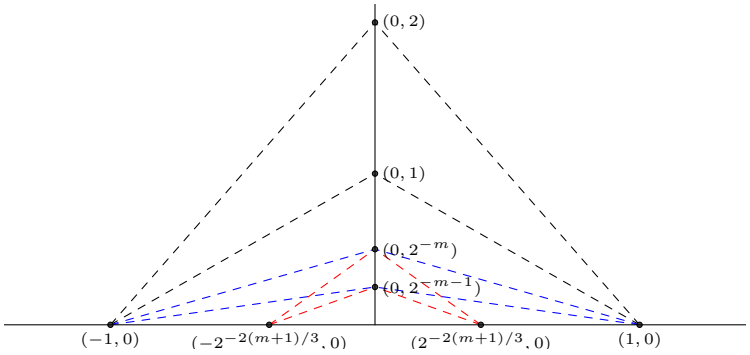


Figure 3.16: An illustration of the proof of Proposition 3.69: the black dashed lines indicate the passage times involved in $K_{1,2}$, and the blue dashed lines indicate the passage times involved in $K_{2^{-m-1},2^{-m}}$. The red dashed lines are rescaled versions of the black ones, which gives a rescaled version of $K_{1,2}$ that lower bounds $K_{2^{-m-1},2^{-m}}$, implying that $\mathbb{P}[K_{2^{-m-1},2^{-m}} > 0] \geq \mathbb{P}[K_{1,2} > 0]$. By the 0-1 law for the germ sigma algebra $\mathcal{F}_{0+}$, almost surely one can find some $m \in \mathbb{N}$ with $K_{2^{-m-1},2^{-m}} > 0$.

inequality (Lemma 3.6), we have

$$\begin{aligned} K_{2^{-m-1},2^{-m}} &= \lim_{w \searrow 0} (2w)^{-1} \int_{2^{-m-1}}^{2^{-m}} \mathcal{D}(-w,t) - \mathcal{D}(w,t) dt \\ &\geq \limsup_{w \searrow 0} (2w)^{-1} \int_{2^{-m-1}}^{2^{-m}} (\mathcal{L}(-2^{-2(m+1)/3}, 0; -w, t) - \mathcal{L}(2^{-2(m+1)/3}, 0; -w, t)) \\ &\qquad - (\mathcal{L}(-2^{-2(m+1)/3}, 0; w, t) - \mathcal{L}(2^{-2(m+1)/3}, 0; w, t)) dt, \end{aligned}$$

where (by scaling invariance) the right hand side has the same distribution as $2^{-2(m+1)/3} K_{1,2}$ (see Figure 3.16). So we must have $\mathbb{P}[K_{2^{-m-1},2^{-m}} > 0] \geq \mathbb{P}[K_{1,2} > 0]$, which implies that

$$\mathbb{P}[\limsup_{m \to \infty} \mathbb{1}[K_{2^{-m-1},2^{-m}} > 0] = 1] > 0$$

by Lemma 3.70. Since $K_{2^{-m-1},2^{-m}}$ is $\mathcal{F}_{2^{-m}}$ measurable, $\limsup_{m \to \infty} \mathbb{1}[K_{2^{-m-1},2^{-m}} > 0]$ is $\mathcal{F}_{0+}$ measurable. Then by Lemma 3.71 we conclude that

$$\mathbb{P}[\limsup_{m \to \infty} \mathbb{1}[K_{2^{-m-1},2^{-m}} > 0] = 1] = 1,$$

which implies the conclusion. □

Appendix A

Transversal fluctuation in the directed landscape

In this appendix we prove Lemma 3.9. We first prove the following weaker version, which can be directly deduced from Lemma 3.7.

Lemma A.1. *For any $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$, any geodesic π_u, and $(s+t)/2 \le r < t$, we have*

$$\begin{aligned}&\left|\pi_u(r) - \frac{x(t-r) + y(r-s)}{t-s}\right| \\ &< C(R{+}1)^{1/2} \log(R{+}2)^2 (t{-}s)^{1/6} (t{-}r)^{1/2} \log^{2/3}(2(2\|u\|{+}2)^{3/2}/(t{-}r)) \log^{1/3}(2\|u\|{+}2).\end{aligned} \tag{A.1}$$

where R is the random number from Lemma 3.7, and $C > 0$ is a universal constant. Similar result holds when $s < r < (s+t)/2$ by symmetry.

Proof. Denote the right hand side of (A.1) by D. Let $z = \pi_u(r)$, and $z' = z - \frac{x(t-r)+y(r-s)}{t-s}$. We argue by contradiction: we show that if $|z'| \ge D$, we must have

$$\mathcal{L}(x, s; y, t) - (\mathcal{L}(x, s; z, r) + \mathcal{L}(z, r; y, t)) > 0, \tag{A.2}$$

which contradicts with the fact that π_u is a geodesic. Indeed, by Lemma 3.7, we have

$$\begin{aligned}
&\mathcal{L}(x,s;y,t)\\
&\qquad\geq -\frac{(x-y)^2}{t-s} - R(t-s)^{1/3}\log^{4/3}(2(\|u\|+2)^{3/2}/(t-s))\log^{2/3}(\|u\|+2),\\
&-\mathcal{L}(x,s;z,r)\\
&\geq \frac{(x-z)^2}{r-s} - R(r-s)^{1/3}\log^{4/3}(2(2\|u\|+|z'|+2)^{3/2}/(r-s))\log^{2/3}(2\|u\|+|z'|+2),\\
&-\mathcal{L}(z,r;y,t)\\
&\geq \frac{(z-y)^2}{t-r} - R(t-r)^{1/3}\log^{4/3}(2(2\|u\|+|z'|+2)^{3/2}/(t-r))\log^{2/3}(2\|u\|+|z'|+2).
\end{aligned} \tag{A.3}$$

Here we use that $\|(x,s;z,r)\|, \|(z,r;y,t)\| \leq 2\|u\| + |z'|$. By adding up these three inequalities, we have that the left hand side of (A.2) is lower bounded by

$$\frac{|z'|^2(t-s)}{(r-s)(t-r)} - 3R(t-s)^{1/3}\log^{4/3}(2(2\|u\|+|z'|+2)^{3/2}/(t-r))\log^{2/3}(2\|u\|+|z'|+2). \tag{A.4}$$

One can check that when C is large enough, for any $i \in \mathbb{Z}_{\geq 0}$ we have

$$\frac{(2^iD)^2(t-s)}{(r-s)(t-r)} > 3R(t-s)^{1/3}\log^{4/3}(2(2\|u\|+2^{i+1}D+2)^{3/2}/(t-r))\log^{2/3}(2\|u\|+2^{i+1}D+2),$$

so (A.4) > 0 when $|z'| \in [2^iD, 2^{i+1}D]$. Thus (A.4) > 0 whenever $|z'| \geq D$, and the conclusion follows. □

To upgrade this to Lemma 3.9, we need more refined estimates in a compact subset of $\mathbb{R}^4_\uparrow$. These can be obtained using Lemma 3.8.

Lemma A.2. *Let K be a compact subset of $\mathbb{R}^4_\uparrow$, and R be the maximum of the random numbers from Lemma 3.7 and Lemma 3.8 for K. For any $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$, any geodesic π_u, and $(s+t)/2 \le r < t$, such that $(x, s; \pi_u(r), r), u \in K$, we must have*

$$\left|\pi_u(r) - \frac{x(t-r) + y(r-s)}{t-s}\right| < C(R+1)^{2/3}\log(R+2)^2(t-r)^{2/3}\log^2(C(1+(t-r)^{-1})), \tag{A.5}$$

where $C > 0$ is a constant depending only on K. Similar result holds when $s < r < (s+t)/2$ by symmetry.

Proof. This proof is similar to the proof of the previous corollary. We now denote the right hand side of (A.5) by D, and define z, z' as before. We still show that if $|z'| \ge D$, the inequality (A.2) holds.

Note that (A.3) still holds. By Lemma 3.8, we have

$$\begin{aligned}\left(\mathcal{L}(x,s;y,t) + \frac{(x-y)^2}{t-s}\right) - \left(\mathcal{L}(x,s;z,r) + \frac{(x-z)^2}{r-s}\right)\\ \ge -R\left((t-r)^{1/3}\log^{2/3}(1+(t-r)^{-1}) + |y-z|^{1/2}\log^{1/2}(1+|y-z|^{-1})\right)\end{aligned}$$

Adding this to (A.3), we get that the left hand side of (A.2) is lower bounded by

$$\begin{aligned}\frac{|z'|^2(t-s)}{(r-s)(t-r)} - R(t-r)^{1/3}\log^{4/3}(2(2\|u\|+|z'|+2)^{3/2}/(t-r))\log^{2/3}(2\|u\|+|z'|+2)\\ - R\left((t-r)^{1/3}\log^{2/3}(1+(t-r)^{-1}) + |y-z|^{1/2}\log^{1/2}(1+|y-z|^{-1})\right).\end{aligned}$$

Below we let $C > 0$ denote a constant depending on K, and its value can change from line to line. Then we have $\|u\| < C$, and $|y-z| < C|z'| + C|t-r|$, so the above can be lower bounded by

$$\frac{|z'|^2}{t-r} - CR(t-r)^{1/3}\log^2(C(1+|z'|)/(t-r)) - CR(|z'|^{1/2} + (t-r)^{1/3})\log^{2/3}(C(1+(t-r)^{-1})). \tag{A.6}$$

When $|z'| > D$, the derivative of (A.6) with respect to $|z'|$ is > 0; and when $|z'| = D$, (A.6) is positive. Thus we have that (A.6) is positive whenever $|z'| > D$, and the conclusion follows. □

Proof of Lemma 3.9. We let $C, c > 0$ denote large and small universal constants, and the values may change from line to line. Let $K = ([-1,1] \times [0,1]) \times ([-1,1] \times [2,3])$, and

$$K_+ = \{\left(x, s; \frac{x(t-r) + y(r-s)}{t-s}, r\right) : |x|, |y| \le 2, s \in [0,1], t \in [2,3], r \in [(s+t)/2, t]\}.$$

Let R_1 be the random number from Lemma 3.7, and R_2 be the random number from Lemma 3.8 with K_+. For any $M > 2$, if $R_1, R_2 < M$, by Lemma A.1 and Lemma A.2 for K_+, for any $u = (x, s; y, t) \in K$ and $r \in [(s+t)/2, t]$, we have

$$\left|\pi_u(r) - \frac{x(t-r) + y(r-s)}{t-s}\right| < CM^{1/2} \log(M)^2 (t-r)^{1/2} \log^{2/3}(1 + (t-r)^{-1}),$$

and unless the left hand below side is > 1, we also have

$$\left|\pi_u(r) - \frac{x(t-r) + y(r-s)}{t-s}\right| < CM^{2/3} \log(M)^2 (t-r)^{2/3} \log^2((1 + (t-r)^{-1})).$$

Denote the right hand sides of the above two estimates by D_1 and D_2, respectively. Then if $D_1 \le 1$, we can bound $\left|\pi_u(r) - \frac{x(t-r)+y(r-s)}{t-s}\right|$ by D_2; otherwise we can bound it by $D_1^{4/3}$. Thus we always have

$$\left|\pi_u(r) - \frac{x(t-r) + y(r-s)}{t-s}\right| < CM^{2/3} \log(M)^{8/3} (t-r)^{2/3} \log^2((1 + (t-r)^{-1})).$$

These imply that, if we denote

$$R_* = \sup_{u=(x,s;y,t)\in K, r\in[(s+t)/2,t]} \frac{\left|\pi_u(r) - \frac{x(t-r)+y(r-s)}{t-s}\right|}{(t-r)^{2/3} \log^2((1 + (t-r)^{-1}))},$$

then $\mathbb{P}[R_* > M] < Ce^{-cM^{9/4}\log(M)^{-4}}$ for any $M > 0$.

Now for any $k, a, b, d \in \mathbb{Z}$, we consider

$$\begin{aligned} &K[k,a,b,d] \\ &= \{(2^{2k/3}(x+b/10), 2^k(s+a/10); 2^{2k/3}(y+d/10), 2^k(t+a/10)) : (x,s;y,t) \in K\}, \end{aligned}$$

and

$$R[k,a,b,d] = \sup_{u=(x,s;y,t)\in K[k,a,b,d], r\in[(s+t)/2,t]} \frac{\left|\pi_u(r) - \frac{x(t-r)+y(r-s)}{t-s}\right|}{(t-r)^{2/3}\log^2((1+2^k(t-r)^{-1}))}.$$

By skew-shift invariance and scaling invariance of the directed landscape, each number $R[k,a,b,d]$ has the same distribution as R_*. Since $\cup_{k,a,b,d\in\mathbb{Z}} K[k,a,b,d] = (x,s;y,t) = \mathbb{R}^4_\uparrow$, we can take

$$R = C \sup_{k,a,b,d\in\mathbb{Z}} \frac{R[k,a,b,d]}{\log(1+|k|+|a|+|b|+|d|)}.$$

Then (3.8) holds, and by a union bound over all k, a, b, $d \in \mathbb{Z}$ we have the desired bound on the upper tail of R. □

Appendix B

Occupation time in a tube

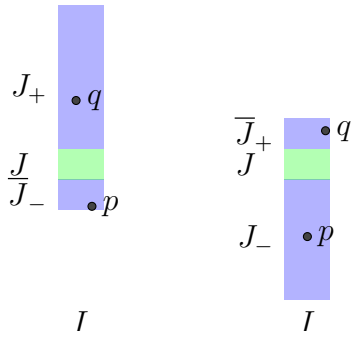


Figure B.1: An illustration of the set $\Phi'_{I,J}$: for $(p;q) = (x,s;y,t) \in \Phi'_{I,J}$, either $s \in \overline{J}_-$ and $t \in J_+$, as illustrated in the left diagram; or $s \in J_-$ and $t \in \overline{J}_+$, as illustrated in the right diagram. The spatial coordinates x, y are always in I.

As promised before, in this appendix we provide the proof of Lemma 3.54. Now, recalling the set $\Phi_{I,J}$ and the random variable $W_{I,J}$ appearing in the discussion preceding the statement of the lemma, it would be simpler to consider the following, similar but different, objects. Namely, for each interval $J = [s,t]$, let $J_- = (-\infty, s]$, $J_+ = [t, \infty)$, and $\overline{J}_- = [2s - t, s]$, $\overline{J}_+ = [t, 2t - s]$. We define

$$\Phi'_{I,J} = \{(p;q) \in \mathbb{R}^4_\uparrow : p \in I \times J_-, q \in I \times \overline{J}_+\} \cup \{(p;q) \in \mathbb{R}^4_\uparrow : p \in I \times \overline{J}_-, q \in I \times J_+\}.$$

See Figure B.1 for an illustration. Then we let

$$W'_{I,J} = \int_J \sup_{u \in \Phi'_{I,J}} \mathbb{1}[\pi_u(r) \in I, \pi_u \text{is unique}] dr.$$

Note that in the definition of $W'_{I,J}$, the supremum over $u \in \Phi'_{I,J}$ is inside the integral over J. This upper bounds the occupation measure in $I \times J$ for any unique geodesic π_u (with $u \in \Phi'_{I,J}$).

We then have the following crucial estimate.

Lemma B.1. *We have $\mathbb{E}[W'_{I,J}] < C_1 \mathscr{L}(I) \mathscr{L}(J)^{1/3}$ for some universal constant C_1.*

An upper bound of the order of $\mathscr{L}(I)\mathscr{L}(J)^{1/3}$ can be explained by a simple scaling argument as follows. First, we expect $\mathbb{E}[W'_{I,J}]$ to be linear in $\mathscr{L}(I)$, at least when $\mathscr{L}(I)$ is small (compared to $\mathscr{L}(J)^{2/3}$). Second, by scaling invariance of $\mathcal{L}$, when $\mathscr{L}(J)$ is multiplied by $h > 0$ and $\mathscr{L}(I)$ is multiplied by $h^{2/3}$, $\mathbb{E}[W'_{I,J}]$ should be multiplied by h.

Compared to $\mathbb{E}[W_{I,J}]$, the expectation $\mathbb{E}[W'_{I,J}]$ is easier to bound: since the time coordinates of the endpoints are outside or at the boundary of J (whereas in the definition of $\Phi_{I,J}$ in (3.29), the endpoints are on the boundary of $I \times J$), we can directly bound $\mathbb{E}[W'_{I,J}]$ using translation invariance of $\mathcal{L}$, and bounds on the number of intersections that geodesics can have with a line (from Lemma 3.10).

This is then used in an inductive proof, where we work on different dyadic scales. The definition of $\Phi'_{I,J}$ enables one to bound the occupation time for a geodesic in different time intervals.

We also remark that, actually, the same upper bound holds for the expectation of the integral $\int_J \sup_{u \in \Phi'_{I,J}, \pi_u \text{ is any geodesic}} \mathbb{1}[\pi_u(r) \in I] dr$; i.e., the uniqueness requirement of the geodesic π_u can be dropped and all possible ones can be considered instead. The uniqueness requirement is inherited from Lemma 3.10, and can actually be removed (see the discussions after Lemma 3.10).

We now prove Lemma 3.54 assuming Lemma B.1, and using an induction argument. A key ingredient is a dyadic decomposition of the time interval $[0,1]$. Further, the exponential tail (in addition to bounding the expectations) in Lemma 3.54 is proved via a concentration estimate, using independence of the directed landscape across different time increments.

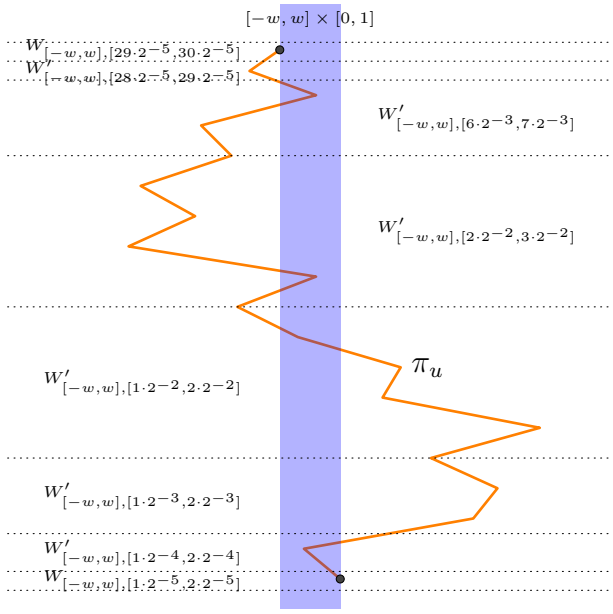


Figure B.2: An illustration of (B.3) in the proof of Lemma 3.54: for $u \in \Phi_{[-w,w],[0,1]}$, we can cover the domain of π_u by intervals J (that are disjoint except for the endpoints), such that for J other than the first and last one, $J \in \Lambda$ and the time of $\pi_u \in [-w,w]$ in J is at most $W'_{[-w,w],J}$; and for J being the first or last one, $J \in \partial\Lambda$ andthe time of $\pi_u \in [-w,w]$ in J is at most $W_{[-w,w],J}$.

Proof of Lemma 3.54. We use induction in w, and prove the following estimates: there exist constants $A, C, c > 0$, such that for any $w > 0$, there is

$$\mathbb{E}[W_{[-w,w],[0,1]}] < A^{2/3+0.01}w, \tag{B.1}$$

and for any $M > 0$,

$$\mathbb{P}[W_{[-w,w],[0,1]} > AMw] < Ce^{-cM}. \tag{B.2}$$

We assume that A, C are large enough and c is small enough, and the precise relation between them is evident from the proof.

The case when $w > A^{-2/3}$ is obvious (for B.2 just take C large enough), so we assume that $w \leq A^{-2/3}$. Now to use an induction argument, we assume that the inequalities hold when w is replaced by any number $\geq 2^{2/3}w$.

We first prove (B.1). Take $g \in \mathbb{N}$ such that $2^g \leq A < 2^{g+1}$. Let Λ consist of all intervals $[a2^{-b}, (a+1)2^{-b}]$ such that $a, b \in \mathbb{Z}$, $0 \leq b \leq g$, and $0 \leq a2^{-b} < (a+1)2^{-b} \leq 1$. Let $\partial\Lambda$ consist of all intervals $[a2^{-g}, (a+1)2^{-g}]$ such that $a \in \mathbb{Z}$, $0 \leq a2^{-g} < (a+1)2^{-g} \leq 1$. We have that

$$W_{[-w,w],[0,1]} \leq \sum_{J\in\Lambda} W'_{[-w,w],J} + 2\max_{J\in\partial\Lambda} W_{[-w,w],J}. \tag{B.3}$$

This is via a dyadic decomposition of π_u for $u \in \Phi_{[-w,w],[0,1]}$, as illustrated by Figure B.2.The precise details of the proof of (B.3) are left for the moment for later.

By Lemma B.1, for each $J = [a2^{-b}, (a+1)2^{-b}] \in \Lambda$, we have $\mathbb{E}[W'_{[-w,w],J}] < 2C_1 w 2^{-b/3}$, thus

$$\sum_{J\in\Lambda} \mathbb{E}[W'_{[-w,w],J}] < \sum_{b=0}^{g}\sum_{a=0}^{2^b-1} 2C_1 w 2^{-b/3} = \sum_{b=0}^{g} 2C_1 w 2^{2b/3} < 10C_1 w 2^{2g/3}.$$

So we have

$$\mathbb{E}[W_{[-w,w],[0,1]}] < 10C_1 w 2^{2g/3} + 2\mathbb{E}[\max_{J\in\partial\Lambda} W_{[-w,w],J}].$$

By induction hypothesis and scale invariance, for each $J \in \partial\Lambda$ we have

$$\mathbb{P}[W_{[-w,w],J} > AMw2^{-g/3}] < Ce^{-cM}.$$

Thus we have

$$\mathbb{E}[\max_{J\in\partial\Lambda} W_{[-w,w],J}] < c^{-1}(\log(2^g) + C)Aw2^{-g/3},$$

and this implies that

$$\mathbb{E}[W_{[-w,w],[0,1]}] < (10C_1 + 4c^{-1}(\log(A) + C))A^{2/3}w < A^{2/3+0.01}w$$

by requiring A being large enough so that $A^{0.01} > (10C_1 + 4c^{-1}(\log(A) + C))$.

We next prove (B.2). Denote $m = \lfloor A^{1/10} \rfloor$. For each $1 \le i \le m$ let $W_i = W_{[-w,w],[(i-1)/m,i/m]}$, and we have $W_{[-w,w],[0,1]} \le \sum_{i=1}^m W_i$. We shall get the exponential tail of $W_{[-w,w],[0,1]}/(Aw)$ by considering the exponential moments of each W_i, using the mutual independence of the latter across i, and applying the induction hypothesis for them.

We write

$$\sum_{i=1}^m W_i \le \sum_{i=1}^m W_i \mathbb{1}[W_i \le Aw/m] + \sum_{i=1}^m W_i \mathbb{1}[W_i > Aw/m] \le Aw + \sum_{i=1}^m W_i \mathbb{1}[W_i > Aw/m].$$

When A is large enough, we have $m \ge 2$, so we can apply the induction hypothesis and use scaling invariance to conclude that

$$\mathbb{E}[W_i] \le A^{2/3+0.01} w m^{-1/3} \qquad \text{and} \qquad \mathbb{P}[W_i > MAwm^{-1/3}] < Ce^{-cM},$$

for any $M > 0$. From these bounds we expect that the sum of W_i should have exponential tail at the order of Aw, and we analyze it as follows. Let $X_i = \frac{m^{1/3}}{Aw} W_i \mathbb{1}[W_i > Aw/m]$, then for any $M > 0$, by the exponential tail we have

$$\mathbb{P}[X_i > M] < Ce^{-cM}.$$

Using Markov's inequality, for any $M > 0$ we also have

$$\mathbb{P}[X_i > M] \le \mathbb{P}[W_i > Aw/m] < \mathbb{E}[W_i]/(Aw/m) \le A^{-1/3+0.01} m^{2/3}.$$

Take $M_* > 0$ such that $Ce^{-cM_*} = A^{-1/3+0.01}m^{2/3}$. Take $a = c/10$. We then have

$$\mathbb{E}[e^{aX_i}] = 1 + \int_0^\infty ae^{aM}\mathbb{P}[X_i > M]dM$$
$$< 1 + \int_0^{M_*} ae^{aM}A^{-1/3+0.01}m^{2/3}dM + \int_{M_*}^\infty ae^{aM}Ce^{-cM}dM < 1 + 2e^{aM_*}A^{-1/3+0.01}m^{2/3},$$

and hence

$$\mathbb{E}[e^{a\sum_{i=1}^m X_i}] < (1 + 2e^{aM_*}A^{-1/3+0.01}m^{2/3})^m < e^{C(A^{-1/3+0.01}m^{2/3})^{1/2}m},$$

where the last inequality follows by using

$$2e^{aM_*}A^{-1/3+0.01}m^{2/3} < C(A^{-1/3+0.01}m^{2/3})^{1/2}m,$$

which is satisfied by taking $C > 10$. Then we have

$$\mathbb{P}[\sum_{i=1}^m W_i > AMw] \le \mathbb{P}[\sum_{i=1}^m X_i > m^{1/3}(M-1)]$$
$$< e^{C(A^{-1/3+0.01}m^{2/3})^{1/2}m - am^{1/3}(M-1)} < Ce^{-cM},$$

where the last inequality is true by taking A large enough (so that $am^{1/3} > 2c$ and $e^{Cm^{-1/3}} < C$), for $M > 2$. For $0 < M \le 2$ we also have (B.2), by taking c small and C large. □

Proof of (B.3). Consider any $u = (x, s; y, t) \in \Phi_{[-w,w],[0,1]}$. The general strategy is to show that $[s, t]$ can be covered the union of (1) two intervals in $\partial\Lambda$ (around the endpoints) and (2) several intervals in Λ that are contained in $[s, t]$. Then we can bound $W[u, [-w, w]]$ by the sum of $W_{[-w,w],J}$ or $W'_{[-w,w],J}$ for J among these intervals.

For each $i \in \mathbb{N}$, we let $a_i^-, a_i^+ \in \mathbb{Z}$ be the numbers with $a_i^- 2^{-i} \le s < (a_i^- + 1)2^{-i}$,

and $(a_i^+ - 1)2^{-i} < t \le a_i^+ 2^{-i}$. Consider collections of intervals:

$$\partial\Lambda_u = \{[a_g^- 2^{-g}, (a_g^- + 1)2^{-g}], [(a_g^+ - 1)2^{-g}, a_g^+ 2^{-g}]\},$$

and

$$\Lambda_u = \{[(a_i^- + 1)2^{-i}, (a_i^- + 2)2^{-i}], [(a_i^+ - 2)2^{-i}, (a_i^+ - 1)2^{-i}] : i \in [\![1, g]\!], a_i^+ - a_i^- > 2\}.$$

We note that $\partial\Lambda_u \subset \partial\Lambda$ and $\Lambda_u \subset \Lambda$. We claim that the union of the intervals in $\partial\Lambda_u \cup \Lambda_u$ covers $[s, t]$. Indeed, if $a_g^+ - a_g^- \le 2$, the union of the intervals in $\partial\Lambda_u$ covers $[s, t]$. Otherwise, let i_* be the smallest integer with $a_{i_*}^+ - a_{i_*}^- > 2$, and we must have $1 < i_* \le g$. We next show that the intervals in $\partial\Lambda_u \cup \Lambda_u$ can be written in a sequence:

$$[a_g^- 2^{-g}, (a_g^- + 1)2^{-g}], [(a_g^- + 1)2^{-g}, (a_g^- + 2)2^{-g}], \cdots, [(a_{i_*}^- + 1)^- 2^{-i_*}, (a_{i_*}^- + 2)2^{-i_*}],$$
$$[(a_{i_*}^+ - 2)2^{-i_*}, (a_{i_*}^+ - 1)2^{-i_*}], \cdots, [(a_g^+ - 2)2^{-g}, (a_g^+ - 1)2^{-g}], [(a_g^+ - 1)2^{-g}, a_g^+ 2^{-g}],$$

such that any two neighboring ones are not disjoint. Then the union of all these intervals is a closed interval.

1. The facts that $[a_g^- 2^{-g}, (a_g^- + 1)2^{-g}] \cap [(a_g^- + 1)2^{-g}, (a_g^- + 2)2^{-g}] \ne \emptyset$ and $[(a_g^+ - 2)2^{-g}, (a_g^+ - 1)2^{-g}] \cap [(a_g^+ - 1)2^{-g}, a_g^+ 2^{-g}] \ne \emptyset$ are obvious.

2. We must have that $a_{i_*}^+ - a_{i_*}^- \le 4$. This is because, otherwise we have $a_{i_*}^+ 2^{-i_*} - a_{i_*}^- 2^{-i_*} > 2^{-i_*+2}$. From the defintions (of a_i^-, a_i^+) above we have

$$a_{i_*-1}^- 2^{-i_*+1} \le a_{i_*}^- 2^{-i_*}, \quad a_{i_*}^+ 2^{-i_*} \le a_{i_*-1}^+ 2^{-i_*+1},$$

so $a_{i_*-1}^+ 2^{-i_*+1} - a_{i_*-1}^- 2^{-i_*+1} > 2^{-i_*+2}$, implying that $a_{i_*-1}^+ - a_{i_*-1}^- > 2$, which contradicts with the choice of i_*. Thus we have $a_{i_*}^+ - a_{i_*}^- = 3$ or 4; and in each case we have that $[(a_{i_*}^- + 1)^- 2^{-i_*}, (a_{i_*}^- + 2)2^{-i_*}]$ and $[(a_{i_*}^+ - 2)2^{-i_*}, (a_{i_*}^+ - 1)2^{-i_*}]$

are not disjoint.

3. From the definition (of a_i^-, a_i^+) above, for any $i \in \mathbb{N}$ we have that

$$a_i^- 2^{-i} \leq a_{i+1}^- 2^{-i-1},\ (a_{i+1}^- + 1)2^{-i-1} \leq (a_i^- + 1)2^{-i},$$

so the intervals $[(a_i^- + 1)^- 2^{-i}, (a_i^- + 2)2^{-i}]$ and $[(a_{i+1}^- + 1)^- 2^{-i-1}, (a_{i+1}^- + 2)2^{-i-1}]$ are not disjoint. Similarly, the intervals $[(a_i^+ - 2)2^{-i}, (a_i^+ - 1)2^{-i}]$ and $[(a_{i+1}^+ - 2)2^{-i-1}, (a_{i+1}^+ - 1)2^{-i-1}]$ are not disjoint.

From the above statements on non-disjointness of intervals in $\partial\Lambda_u \cup \Lambda_u$, we conclude that the union of these intervals is a closed interval. The union also contains s and t, so it must contain $[s, t]$.

Further, because of the way the definitions were set up we crucially now have the following (see Figure B.2). For each $J \in \Lambda_u$, we have

$$\int_J \mathbb{1}[\pi_u(r) \in [-w, w], \pi_u \text{ is unique}]dr \leq W'_{[-w,w],J},$$

and for each $J \in \partial\Lambda_u$, we have

$$\int_{J \cap [s,t]} \mathbb{1}[\pi_u(r) \in [-w, w], \pi_u \text{ is unique}]dr \leq W_{[-w,w],J}.$$

Thus we have

$$W[u, [-w, w]] \leq \sum_{J \in \Lambda} W'_{[-w,w],J} + 2 \max_{J \in \partial\Lambda} W_{[-w,w],J},$$

and that (B.3) holds. □

It remains to prove the bound for $\mathbb{E}[W'_{I,J}]$. We would again do a dyadic decomposition, but now for the locations of the possible starting and ending points of the geodesics.

Proof of Lemma B.1. As $\mathcal{L}$ is translation and scaling invariant we can assume that $J = [0, 1]$, and $I = [-w, w]$ for some $w > 0$.

For each $0 \leq t \leq 1$, we consider

$$A_t = \sup_{u=(p;q):p\in[-w,w]\times[-1,0],q\in[-w,w]\times[1,\infty)} \mathbb{1}[\pi_u(t) \in [-w, w], \pi_u \text{is unique}],$$

and

$$A'_t = \sup_{u=(p;q):p\in[-w,w]\times(-\infty,0],q\in[-w,w]\times[1,2]} \mathbb{1}[\pi_u(t) \in [-w, w], \pi_u \text{is unique}].$$

We then have

$$\mathbb{E}[W'_{[-w,w],[0,1]}] \leq \int_0^1 \mathbb{E}[A_t] + \mathbb{E}[A'_t]dt = 2\int_0^1 \mathbb{E}[A_t]dt \leq 4w^{3/2} + 2\int_{w^{3/2}}^{1-w^{3/2}} \mathbb{E}[A_t]dt,$$

where the equality is by the fact that $\int_0^1 \mathbb{E}[A_t]dt = \int_0^1 \mathbb{E}[A'_t]dt$, due to symmetry. We next show that, there is a universal constant C_2, such that for any $w^{3/2} < t < 1/2$,

$$\mathbb{E}[A_t] < C_2 w t^{-2/3} \tag{B.4}$$

and for $1/2 < t < 1 - w^{3/2}$

$$\mathbb{E}[A_t] < C_2 w (1 - t)^{-2/3}. \tag{B.5}$$

Given the above, integrating over t using (B.4) and (B.5) completes the proof. Thus it remains to prove these two inequalities.

We prove (B.4) first. The general idea is again a dyadic decomposition, but now for the location of the endpoints. Towards this we define the following sets. For each

$a \in \mathbb{N}$ with $a \leq \lceil -\log_2(t) \rceil + 1$, we denote

$$A_{t,a} = \sup_{u=(p;q):p\in[-w,w]\times[-(2^a-1)t,-(2^{a-1}-1)t],q\in[-w,w]\times[1,\infty)} \mathbb{1}[\pi_u(t) \in [-w,w], \pi_u \text{is unique}].$$

Then we have $\mathbb{E}[A_t] \leq \sum_{a=1}^{\lceil -\log_2(t) \rceil + 1} \mathbb{E}[A_{t,a}]$. To bound each $\mathbb{E}[A_{t,a}]$, we use translation invariance. Denote

$$\begin{aligned} B_{t,a} = |[-(2^a t)^{2/3}, (2^a t)^{2/3}] \cap \{\pi_u(t) : u = (p; y, (2^{a-3}+1)t), \\ p \in [-(2^a t)^{2/3}, (2^a t)^{2/3}] \times [-(2^a-1)t, -(2^{a-1}-1)t], \pi_u \text{is unique}\}|. \end{aligned}$$

In words, for $B_{t,a}$, we allow the lower endpoint to be in a wider box (compared to $[-w,w] \times [-(2^a-1)t, -(2^{a-1}-1)t]$ in $A_{t,a}$), and the upper endpoint to vary in the line $\mathbb{R} \times \{(2^{a-3}+1)t\}$. Note that since $(2^{a-3}+1)t \leq 1$, we can bound $A_{t,a}$ by

$$\sup_{u=(p;y,(2^{a-3}+1)t):p\in[-w,w]\times[-(2^a-1)t,-(2^{a-1}-1)t]} \mathbb{1}[\pi_u(t) \in [-w,w], \pi_u \text{is unique}].$$

Also note that $[-(2^a t)^{2/3}, (2^a t)^{2/3}]$ contains $\lfloor (2^a t)^{2/3}/w \rfloor$ non-overlaping translations of $[-w,w]$. Then

$$\mathbb{E}[A_{t,a}] \leq \lfloor (2^a t)^{2/3}/w \rfloor^{-1} \mathbb{E}[B_{t,a}] \leq 2w(2^a t)^{-2/3} \mathbb{E}[B_{t,a}]$$

by translation invariance. We then have

$$\begin{aligned} \mathbb{E}[A_t] \leq \sum_{a=1}^{\lceil -\log_2(t) \rceil + 1} \mathbb{E}[A_{t,a}] &\leq \sum_{a=1}^{\lceil -\log_2(t) \rceil + 1} 2w(2^a t)^{-2/3} \mathbb{E}[B_{t,a}] \\ &= \sum_{a=1}^{\lceil -\log_2(t) \rceil + 1} 2w(2^a t)^{-2/3} \mathbb{E}[B] < 10wt^{-2/3}\mathbb{E}[B], \end{aligned}$$

where

$$B = |[-1,1] \cap \{\pi_u(0) : u = (p; y, 1/8), p \in [-1,1] \times [-1,-1/2], \pi_u \text{is unique}\}|,$$

and by translation and scaling invariance we have each $\mathbb{E}[B_{t,a}] = \mathbb{E}[B]$. Finally, by the following claim we get (B.4).

Claim. $\mathbb{E}[B]$ *is finite.*

Proof of the claim. Take $M > 0$ large enough. Let $\mathcal{E}_M$ denote the event

$$|\{\pi_{(x,-1/4;y,1/8)}(0) : |x|, |y| \le M^{0.01}, \pi_{(x,-1/4;y,1/8)} \text{is unique}\}| \ge M.$$

By Lemma 3.10, and translation and scaling invariance of the directed landscape, we have that $\mathbb{P}[\mathcal{E}_M] \le Ce^{-cM^{1/384}}$ for some universal constants $C, c > 0$.

We next take R from Lemma 3.9. We show that $\{B > M\} \cap \{R \le M^{0.001}\} \subset \mathcal{E}_M$. Indeed, by Lemma 3.9, if $\pi_{(x,s;y,1/8)}(0) \in [-1,1]$ for some $y \in \mathbb{R}$, $x \in [-1,1]$, $s \in [-1,-1/2]$, we must have that $|y| < C(R+1)\log^3(R+1)$, and $|\pi_{(x,s;y,1/8)}(-1/4)| < C(R+1)\log^3(R+1)$, for some universal constant $C > 0$. Thus if $B > M$, we can find $x_1, \cdots, x_M$ and $y_1, \cdots, y_M$, all in the interval $[-C(R+1)\log^3(R+1), C(R+1)\log^3(R+1)]$, such that for each $1 \le i \le M$, $\pi_{(x_i,-1/4;y_i,1/8)}$ is unique; and $\pi_{(x_i,-1/4;y_i,1/8)}(0)$ are mutually different and are all in $[-1,1]$. If (in addition) $R \le M^{0.001}$, we must have that $C(R+1)\log^3(R+1) < M^{0.01}$, so $\mathcal{E}_M$ holds.

Thus we have $\{B > M\} \cap \{R \le M^{0.001}\} \subset \mathcal{E}_M$, which implies that $\mathbb{P}[B > M] < \mathbb{P}[R > M^{0.001}] + \mathbb{P}[\mathcal{E}_M]$. Then by $\mathbb{P}[\mathcal{E}_M] \le Ce^{-cM^{1/384}}$, and the bound on $\mathbb{P}[R > M^{0.001}]$ from Lemma 3.9, the conclusion follows. □

The proof of (B.5) is similar. Denote

$$\hat{A}_t = \sup_{u=(p;q):p\in[-w,w]\times[-1,0],q\in[-w,w]\times[1,2]} \mathbb{1}[\pi_u(t) \in [-w,w], \pi_u \text{is unique}]$$

and

$$\overline{A}_t = \sup_{u=(p;q):p\in[-w,w]\times[-1,0],q\in[-w,w]\times[2,\infty)} \mathbb{1}[\pi_u(t) \in [-w, w], \pi_u \text{is unique}].$$

We the have $\mathbb{E}[A_t] \leq \mathbb{E}[\hat{A}_t] + \mathbb{E}[\overline{A}_t] \leq \mathbb{E}[A_{1-t}] + \mathbb{E}[\overline{A}_t]$. Thus by (B.4) it suffices to bound $\mathbb{E}[\overline{A}_t]$. Via translation invariance, we have $\mathbb{E}[\overline{A}_t] < 2w\mathbb{E}[B]$. Thus by Claim B we get (B.5). □

Appendix C

Estimates using the Airy line ensemble

In this appendix we include the arguments left from the proof of Lemma 3.70. Recall the events $\mathcal{E}'_{coal}$ and $\mathcal{E}_{disj}$ appearing there and that it remained to prove the following result.

Lemma C.1. *We have* $\mathbb{P}[\mathcal{E}'_{coal} \cap \mathcal{E}_{disj}] > 0$.

To prove this we use the connection between the directed landscape and the Airy line ensemble $\mathcal{P} = \{\mathcal{P}_n\}_{n=1}^{\infty}$. As was the key observation in [42] and also later exploited in [45], marginals of the directed landscape can be obtained from last passage percolation across $\mathcal{P}$, and the disjointness of geodesics in the directed landscape can be reduced to disjointness of geodesics across $\mathcal{P}$. What makes it useful to work with $\mathcal{P}$ is a resampling invariance property introduced and termed as the Brownian Gibbs property in [36]. Informally, this implies that fixing $\{\mathcal{P}_n\}_{n=2}^{\infty}$ and $\mathcal{P}_1$ outside some interval; the conditional law of the latter is simply given by a Brownian bridge joining the values at the endpoints of the intervals and further conditioned to avoid $\mathcal{P}_2$. Given this, we will reduce computing probabilities of events about coalescence properties of geodesics to certain Brownian bridge computations.

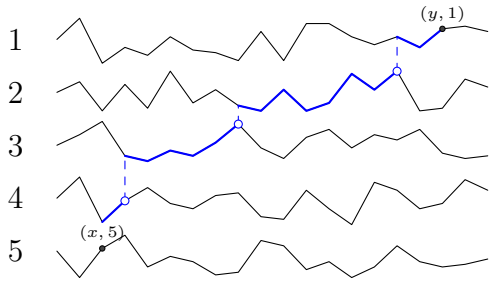


Figure C.1: An illstration of a non-increasing càdlàg function across $\mathcal{P}$ from $(x, 5)$ to $(y, 1)$.

We need a bit of preparation to implement the above strategy. First, as $\mathcal{P}$ has infinitely many lines, the last passage times and geodesics across it are not completely straightforward to define; for this we quote the setup developed in [45], in Section C.1. To study the disjointness of geodesics, we will need to understand last passage percolation across $\mathcal{P}$ whose ending point lives on the second line $\mathcal{P}_2$; we call such optimal paths the "second geodesics" and show their existence using the framework of "disjoint optimizers" from [45], in Section C.2. The proof of Lemma C.1 is finally given in Section C.3.

C.1 Parabolic path across the Airy line ensemble: weight, geodesics, and disjointness

For any $x \leq y$, and any non-increasing càdlàg function $\pi : [x, y] \to \mathbb{N}$, we define the *weight of* π *across* $\mathcal{P}$ as

$$\|\pi\|_{\mathcal{P}} = \sum_{i=\pi(y)}^{\pi(x)} \mathcal{P}_i(y_i) - \mathcal{P}_i(x_i),$$

where $x_i = \inf\{z : \pi(z) \le i\}$ and $y_i = \sup\{z : \pi(z) \ge i\}$. Now, for any $x \le y$ and $n \ge m \in \mathbb{N}$, we define the *last passage value from* (x, n) *to* (y, m) as

$$\mathcal{P}[(x, n) \to (y, m)] = \sup_{\pi} \|\pi\|_{\mathcal{P}},$$

where the supremum is over all non-increasing càdlàg functions π on $[x, y]$, with $\pi(x) \le n$ and $\pi(y) \ge m$ (see Figure C.1). Continuity of $\mathcal{P}$ and compactness of the set of functions ensures that this supremum is always attained.

We let $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}$ be the Airy sheet constructed in [42], which is the fixed time marginal of the directed landscape. Namely, the function $(x, y) \mapsto \mathcal{L}(x, 0; y, 1)$ has the same distribution as $\mathcal{S}$. It can be coupled with $\mathcal{P}$ such that

- $\mathcal{S}(0, y) = \mathcal{P}_1(y)$ for $y \in \mathbb{R}$.
- For $x > 0, z_1 < z_2$, we have

$$\mathcal{S}(x, z_1) - \mathcal{S}(x, z_2) = \lim_{y \to -\infty} \mathcal{P}[(y, \pi(y)) \to (z_1, 1)] - \mathcal{P}[(y, \pi(y)) \to (z_2, 1)], \quad \text{(C.1)}$$

  where $\pi : (-\infty, z_1] \to \mathbb{N}$ is any non-increasing càdlàg function, such that $\lim_{y \to -\infty} \frac{\pi(y)}{2y^2} = x$.

See [42, Definition 1.2] and [45, Definition 2.20]; also see [45, Lemma 2.21] for the second property. Below we always assume the coupling between $\mathcal{S}$ and $\mathcal{P}$.

A *parabolic path* or *path* across $\mathcal{P}$ from $x \ge 0$ to $z \in \mathbb{R}$ is a nonincreasing càdlàg function $\pi : (-\infty, z] \to \mathbb{N}$ such that

$$\lim_{y \to -\infty} \frac{\pi(y)}{2y^2} = x.$$

We then define the weight of π on $\mathcal{P}$ as

$$\|\pi\|_{\mathcal{P}} = \mathcal{S}(x,z) + \lim_{y\to-\infty} \|\pi|_{[y,z]}\|_{\mathcal{P}} - \mathcal{P}[(y,\pi(y)) \to (z,1)]. \tag{C.2}$$

We note that we use the same notation $\|\cdot\|_{\mathcal{P}}$ to denote weight across $\mathcal{P}$ for both a non-increasing càdlàg function in a finite interval, and a parabolic path. The quantity $\|\pi|_{[y,z]}\|_{\mathcal{P}} - \mathcal{P}[(y,\pi(y)) \to (z,1)]$ can be understood as the "loss of weight" of $\pi|_{[y,z]}$, versus the optimal path from $(y,\pi(y))$ to $(z,1)$; whereas the optimal parabolic path from x to z should have weight $\mathcal{S}(x,z)$. The limit exists since $\|\pi|_{[y,z]}\|_{\mathcal{P}} - \mathcal{P}[(y,\pi(y)) \to (z,1)]$ is non-decreasing in y, by [45, Lemma 4.1(i)].

We quote the following results on computing the weight of paths.

Lemma C.2 ([45, Lemma 4.3])**.** *The following statement holds almost surely for $\mathcal{P}$. Let π_1, π_2 be any two parabolic paths across $\mathcal{P}$ from $x \geq 0$ to z_1, z_2 respectively, such that for some $z_0 < z_1 \wedge z_2$, we have $\pi_1(y) = \pi_2(y)$ for any $y \leq z_0$. Then*

$$\|\pi_1\|_{\mathcal{P}} - \|\pi_1|_{[z_0,z_1]}\|_{\mathcal{P}} = \|\pi_2\|_{\mathcal{P}} - \|\pi_2|_{[z_0,z_2]}\|_{\mathcal{P}}$$

The next result states that, for a bounded region in $\mathbb{R}\times\mathbb{N}$, and any parabolic path without a "jump point" inside that region, the weight of the path is measurable with respect to $\mathcal{P}$ outside that region.

Lemma C.3 ([45, Lemma 4.4])**.** *Take any closed interval $I \subset \mathbb{R}$ and $k \in \mathbb{N}$. Let $\mathcal{F}_{I,k}$ be the sigma-algebra generated by all null sets, all $\mathcal{P}_i$ for $i > k$, and $\{\mathcal{P}_i(x) : x \notin I\}$ for $1 \leq i \leq k$. Take any $x \geq 0$, and let Σ_x be the set of parabolic paths π from x to some $z \in \mathbb{R}$ such that either $\pi(y) > k$ for any $y \in (-\infty, z] \cap I$, or else $I \subset (-\infty, z]$ and π is constant on I. Let $F : \Sigma_x \to \mathbb{R}$ be the random function recording path weight in $\mathcal{P}$: $F(\pi) = \|\pi\|_{\mathcal{P}}$. Then F is $\mathcal{F}_{I,k}$-measurable.*

A parabolic path π from $x \geq 0$ to y is a *geodesic* if the weight $\|\pi\|_{\mathcal{P}}$ is finite,

and is maximal among all parabolic paths in $\mathcal{P}$ from x to y; and this is equivalent to $\|\pi\|_{\mathcal{P}} = \mathcal{S}(x, y)$, by [45, Lemma 4.1(ii)]. It is proved in [45, Lemma 4.2] that, almost surely, there exists a geodesic from any $x \geq 0$ to any y; and for any fixed $x \geq 0$ and y, almost surely there is a unique geodesic from x to y. We shall use $\pi\{x, y\}$ to denote any geodesic from x to y.

For paths across the Airy line ensemble, we use the following notion for disjointness. For any paths $\pi_1 : (-\infty, x_1] \to \mathbb{N}$ and $\pi_2 : (-\infty, x_2] \to \mathbb{N}$, we say $\pi_1 < \pi_2$, if $x_1 < x_2$, and for any $x \leq x_1$, we have $\lim_{y \nearrow x} \pi_1(y) < \pi_2(x)$ (we require this instead of $\pi_1(x) < \pi_2(x)$ because π_1 is defined to be càdlàg). We say that π_1 and π_2 are *disjoint*, if $\pi_1 < \pi_2$ or $\pi_2 < \pi_1$; and we say that π_1 *intersects* π_2, if they are not disjoint. Just like in the directed landscape (as stated in Lemma 3.13), the disjointness of geodesics is also equivalent to an inequality for the Airy sheet.

Lemma C.4. *For any $0 \leq x_1 < x_2$, $y_1 < y_2$, assuming that $\pi\{x_1, y_1\}$ and $\pi\{x_2, y_2\}$ are unique, $\pi\{x_1, y_1\}$ and $\pi\{x_2, y_2\}$ being disjoint is equivalent to that*

$$\mathcal{S}(x_1, y_1) + \mathcal{S}(x_2, y_2) > \mathcal{S}(x_1, y_2) + \mathcal{S}(x_2, y_1). \tag{C.3}$$

Proof. We follow a similar idea as in the proof of Lemma 3.13. Whenever $\pi\{x_1, y_1\}$ intersects $\pi\{x_2, y_2\}$, we can find $x_* \leq y_1$ with

$$\lim_{y \nearrow x_*} \pi\{x_1, y_1\}(y) \geq \pi\{x_2, y_2\}(x_*).$$

Again note that we use $\lim_{y \nearrow x_*} \pi\{x_1, y_1\}(y)$ instead of $\pi\{x_1, y_1\}(x_*)$ here, due to that $\pi\{x_1, y_1\}$ is càdlàg. Using this we construct π_1' from x_1 to y_2 and π_2' from x_2 to y_1, by switching the paths $\pi\{x_1, y_1\}$ on $[x_*, y_1]$ and $\pi\{x_2, y_2\}$ on $[x_*, y_2]$. Then by Lemma C.2

$$\|\pi_1'\|_{\mathcal{P}} + \|\pi_2'\|_{\mathcal{P}} = \|\pi\{x_1, y_1\}\|_{\mathcal{P}} + \|\pi\{x_2, y_2\}\|_{\mathcal{P}} = \mathcal{S}(x_1, y_1) + \mathcal{S}(x_2, y_2),$$

where the second equality is by the definition of $\pi\{x_1, y_1\}$ and $\pi\{x_2, y_2\}$. Thus (C.3) cannot hold.

Now we assume that (C.3) does not hold; then by the quadrangle inequality (Lemma 3.6), we must have that $\mathcal{S}(x_1, y_1) + \mathcal{S}(x_2, y_2) = \mathcal{S}(x_1, y_2) + \mathcal{S}(x_2, y_1)$. We then consider any geodesics $\pi\{x_1, y_2\}$ and $\pi\{x_2, y_1\}$. By planarity they should cross, i.e., we can find some $x_* \le y_1$ with

$$\lim_{y \nearrow x_*} \pi\{x_1, y_2\}(y) \le \lim_{y \nearrow x_*} \pi\{x_1, y_2\}(y), \quad \pi\{x_1, y_2\}(x_*) \ge \pi\{x_1, y_2\}(x_*).$$

Then we construct π_1' from x_1 to y_1 and π_2' from x_2 to y_2, by switching the paths $\pi\{x_2, y_1\}$ on $[x_*, y_1]$ and $\pi\{x_1, y_2\}$ on $[x_*, y_2]$. Now by Lemma C.2 we have

$$\begin{aligned}\|\pi_1'\|_{\mathcal{P}} + \|\pi_2'\|_{\mathcal{P}} &= \|\pi\{x_1, y_2\}\|_{\mathcal{P}} + \|\pi\{x_1, y_2\}\|_{\mathcal{P}} \\ &= \mathcal{S}(x_1, y_2) + \mathcal{S}(x_2, y_1) = \mathcal{S}(x_1, y_1) + \mathcal{S}(x_2, y_2),\end{aligned}$$

so π_1' and π_2' must be geodesics. By uniqueness, we have $\pi_1' = \pi\{x_1, y_1\}$ and $\pi_2' = \pi\{x_2, y_2\}$, and they are not disjoint at x_*. □

Using this and Lemma 3.13, disjointness of geodesics in the directed landscape $\mathcal{L}$ can be reduced to disjointness of geodesics in $\mathcal{P}$.

Corollary C.5. *Couple $\mathcal{L}$ with $\mathcal{S}$ such that $\mathcal{S}(x, y) = \mathcal{L}(x, 0; y, 1)$ for any $x, y \in \mathbb{R}$. For any fixed $0 \le x_1 < x_2$, $y_1 < y_2$, the event that $\pi\{x_1, y_1\}$ and $\pi\{x_2, y_2\}$ are disjoint and $\mathrm{Disj}^{[0,1]}_{(x_1,x_2)\to(y_1,y_2)}$ are equivalent (up to an event with probability zero). (Recall that $\mathrm{Disj}^{[0,1]}_{(x_1,x_2)\to(y_1,y_2)}$ is the event where $\pi_{(x_1,0;y_1,1))}$ and $\pi_{(x_2,0;y_2,1)}$ are disjoint.)*

C.2 Disjoint optimizers, ordering, and second geodesics

We also need the notion of disjoint optimizers from [45]. For any $k \in \mathbb{N}$, we let

$$\mathbb{R}^k_{\leq} = \{(x_1, \ldots, x_k) : x_1 \leq \cdots \leq x_k\}.$$

For any $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_k)$ in $\mathbb{R}^k_{\leq}$ with $0 \leq x_1$, a *disjoint k-path* from $\mathbf{x}$ to $\mathbf{y}$ is a collection of parabolic paths $(\pi_1, \ldots, \pi_k)$, such that each π_i is from x_i to y_i, and $\pi_i < \pi_{i+1}$ on $(-\infty, x_i)$, $\pi_i(x_i) \leq \pi_{i+1}(x_i)$, for each $1 \leq i < k$. Note (due to that the paths are càdlàg) this is a weaker condition than π_i and π_{i+1} being disjoint. A *disjoint optimizer* from $\mathbf{x}$ to $\mathbf{y}$ is a disjoint k-path $(\pi_1, \ldots, \pi_k)$, such that $\sum_{i=1}^k \|\pi_i\|_{\mathcal{P}}$ achieves its supremum (over all disjoint k-paths from $\mathbf{x}$ to $\mathbf{y}$). It has been shown, see [45, Proposition 5.8], that almost surely for any $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$ in $\mathbb{R}^k_{\leq}$ with $0 \leq x_1$, there is a disjoint optimizer from $\mathbf{x}$ to $\mathbf{y}$.

For $x \geq 0$ and $y \in \mathbb{R}$, we consider all the parabolic paths $\pi : (-\infty, y] \to \mathbb{N}$ across the Airy line ensemble from x, with $\lim_{z \nearrow y} \pi(z) \geq 2$. Note that at the endpoint we allow $\pi(y) = 1$. We denote any one with the maximum weight as a *second geodesic from x to y*.

Lemma C.6. *Almost surely, for any $x \geq 0$ and y, there is a second geodesic from x to y. In addition, if we take $\pi : (-\infty, y] \to \mathbb{N}$ such that $\pi \equiv 1$, then any second geodesic from x to y together with π would give a disjoint optimizer from $(0, x)$ to (y, y).*

Proof. For the first part, we just take any disjoint optimizer (π_1, π_2) from $(0, x)$ to (y, y), and we claim that π_2 is a second geodesic from x to y. Otherwise, there is a path π_2' from x to y, with $\lim_{z \nearrow y} \pi_2'(z) \geq 2$, and $\|\pi_2'\|_{\mathcal{P}} > \|\pi_2\|_{\mathcal{P}}$. For $\pi : (-\infty, y] \to \mathbb{N}$ with $\pi \equiv 1$, we have $\|\pi\|_{\mathcal{P}} = \mathcal{P}_1(y) = \mathcal{S}(0, y) \geq \|\pi_1\|_{\mathcal{P}}$ (where the inequality is by (C.2)), so $\|\pi\|_{\mathcal{P}} + \|\pi_2'\|_{\mathcal{P}} > \|\pi_1\|_{\mathcal{P}} + \|\pi_2\|_{\mathcal{P}}$, and this contradicts that (π_1, π_2) is a disjoint optimizer.

We next prove the second part. Take any second geodesic π_* from x to y, then $\|\pi_*\|_{\mathcal{P}} = \|\pi_2\|_{\mathcal{P}}$, implying that $\|\pi\|_{\mathcal{P}} + \|\pi_*\|_{\mathcal{P}} \geq \|\pi_1\|_{\mathcal{P}} + \|\pi_2\|_{\mathcal{P}}$. Thus (π, π_*) must also be a disjoint optimizer from $(0, x)$ to (y, y). □

From any $x \geq 0$, to any $y \in \mathbb{R}$, there maybe more than one second geodesic, and below we use $\pi^2\{x, y\} : (-\infty, y] \to \mathbb{N}$ to denote an arbitrary one.

Next we state the following result, which is about the ordering of disjoint optimizers. It can be viewed as a counterpart of Lemma 3.12 or Lemma 3.23 in the disjoint optimizer setting. For any $x_1 < x_2$ and $\pi_1 : (-\infty, x_1] \to \mathbb{N}$ and $\pi_2 : (-\infty, x_2] \to \mathbb{N}$, we define $\pi_1 \vee \pi_2 : (-\infty, x_2] \to \mathbb{N}$ and $\pi_1 \wedge \pi_2 : (-\infty, x_1] \to \mathbb{N}$ via

$$(\pi_1 \vee \pi_2)(x) = \begin{cases} \pi_1(x) \vee \pi_2(x), & \forall x \in (-\infty, x_1], \\ \pi_2(x), & \forall x \in (x_1, x_2], \end{cases}$$

and

$$(\pi_1 \wedge \pi_2)(x) = \pi_1(x) \wedge \pi_2(x), \quad \forall x \in (-\infty, x_1].$$

Lemma C.7. *Take any $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathbb{R}^k_{\leq}$ with $x_1, x'_1 \geq 0$, and $\mathbf{x} \leq \mathbf{x}', \mathbf{y} \leq \mathbf{y}'$ entry-wise. Take any disjoint optimizers $(\pi_1, \ldots, \pi_k)$ from $\mathbf{x}$ to $\mathbf{y}$, and $(\pi'_1, \ldots, \pi'_k)$ from $\mathbf{x}'$ to $\mathbf{y}'$. Then $(\pi_1 \vee \pi'_1, \ldots, \pi_k \vee \pi'_k)$ is a disjoint optimizer from $\mathbf{x}'$ to $\mathbf{y}'$, and $(\pi_1 \wedge \pi'_1, \ldots, \pi_k \wedge \pi'_k)$ is a disjoint optimizer from $\mathbf{x}$ to $\mathbf{y}$.*

For paths in finite intervals (rather than parabolic paths), such result follows obviously from the following facts: the $\wedge$ and $\vee$ of two disjoint optimizers are still disjoint optimizers (with the respective endpoints), and the sum of their weights is the same as the sum of weights of the disjoint optimizers. For parabolic paths, there are extra complexities due to infiniteness of the paths.

Proof. It is straightforward to check that for each $1 \leq i \leq k$, $\pi_i \vee \pi'_i$ is a path from x'_i to y'_i, and $\pi_i \wedge \pi'_i$ is a path from x_i to y_i. For $1 \leq i < k$, we have $\pi_i \wedge \pi'_i < \pi_{i+1} \wedge \pi'_{i+1}$

on $(-\infty, y_i)$, $\pi_i(y_i) \wedge \pi'_i(y_i) \le \pi_{i+1}(y_i) \wedge \pi'_{i+1}(y_i)$; and $\pi_i \vee \pi'_i < \pi_{i+1} \vee \pi'_{i+1}$ on $(-\infty, y'_i)$, $\pi_i(y'_i) \vee \pi'_i(y'_i) \le \pi_{i+1}(y'_i) \vee \pi'_{i+1}(y'_i)$. Thus $(\pi_1 \vee \pi'_1, \ldots, \pi_k \vee \pi'_k)$ is a disjoint k-path from $\mathbf{x}'$ to $\mathbf{y}'$, and $(\pi_1 \wedge \pi'_1, \ldots, \pi_k \wedge \pi'_k)$ is a disjoint k-path from $\mathbf{x}$ to $\mathbf{y}$.

We claim that for each $1 \le i \le k$,

$$\|\pi_i \vee \pi'_i\|_{\mathcal{P}} + \|\pi_i \wedge \pi'_i\|_{\mathcal{P}} = \|\pi_i\|_{\mathcal{P}} + \|\pi'_i\|_{\mathcal{P}}. \tag{C.4}$$

For this, note that for any $y \le y_i$, we have

$$\|(\pi_i \vee \pi'_i)|_{[y,y'_i]}\|_{\mathcal{P}} + \|(\pi_i \wedge \pi'_i)|_{[y,y_i]}\|_{\mathcal{P}} - \|\pi_i|_{[y,y_i]}\|_{\mathcal{P}} - \|\pi'_i|_{[y,y'_i]}\|_{\mathcal{P}} = 0,$$

as the involved paths are finite. By (C.2), to prove (C.4) it now suffices to show that

$$\begin{aligned}\lim_{y\to-\infty} \mathcal{P}[(y, \pi_i(y) \wedge \pi'_i(y)) \to (y_i, 1)] + \mathcal{P}[(y, \pi_i(y) \vee \pi'_i(y)) \to (y'_i, 1)] \\ - \mathcal{P}[(y, \pi_i(y)) \to (y_i, 1)] - \mathcal{P}[(y, \pi'_i(y)) \to (y'_i, 1)] = 0.\end{aligned} \tag{C.5}$$

If there is some y_* such that $\pi_i(y) \le \pi'_i(y)$ for any $y \le y_*$, (C.5) holds obviously. Otherwise, for infinitely many y decreasing to $-\infty$, we have $\pi_i(y) > \pi'_i(y)$. Then since $\lim_{y\to-\infty} \frac{\pi_i(y)}{2y^2} = x_i$, $\lim_{y\to-\infty} \frac{\pi'_i(y)}{2y^2} = x'_i$, and (the assumption that) $x_i \le x'_i$, we must have $x_i = x'_i$. Thus by (C.1) we have

$$\begin{aligned}\lim_{y\to-\infty} \mathcal{P}[(y, \pi_i(y)) \to (y_i, 1)] - \mathcal{P}[(y, \pi_i(y)) \to (y'_i, 1)] \\ = \lim_{y\to-\infty} \mathcal{P}[(y, \pi'_i(y)) \to (y_i, 1)] - \mathcal{P}[(y, \pi'_i(y)) \to (y'_i, 1)].\end{aligned}$$

Note that (before sending $y \to -\infty$) the expression in (C.5) equals either 0 or

$$\mathcal{P}[(y, \pi_i'(y)) \to (y_i, 1)] + \mathcal{P}[(y, \pi_i(y)) \to (y_i', 1)] \\ - \mathcal{P}[(y, \pi_i(y)) \to (y_i, 1)] - \mathcal{P}[(y, \pi_i'(y)) \to (y_i', 1)].$$

Thus (C.5) holds, and so does (C.4). So we have $\sum_{i=1}^k \|\pi_i \vee \pi_i'\|_{\mathcal{P}} + \|\pi_i \wedge \pi_i'\|_{\mathcal{P}} = \sum_{i=1}^k \|\pi_i\|_{\mathcal{P}} + \|\pi_i'\|_{\mathcal{P}}$, and the conclusion follows. □

C.3 Proof of Lemma C.1

We first translate the event $\mathcal{E}_{coal}' \cap \mathcal{E}_{disj}$ into an event on disjointness of geodesics in $\mathcal{P}$. Recall that

$$\mathcal{E}_{disj} = \mathrm{Disj}^{[0,1]}_{(-1,1)\to(-\delta,\delta)} \cap (\mathrm{Disj}^{[0,1]}_{(-1-\delta,-1+\delta)\to(-2,2)})^c \cap (\mathrm{Disj}^{[0,1]}_{(1-\delta,1+\delta)\to(-2,2)})^c,$$

i.e., the geodesics $\pi_{(-1,0;-\delta,1)}$ and $\pi_{(1,0;\delta,1)}$ are disjoint, while, the geodesic pairs given by

$$\{\pi_{(-1-\delta,0;-2,1)}, \pi_{(-1+\delta,0;2,1)}\}, \{\pi_{(1-\delta,0;-2,1)}, \pi_{(1+\delta,0;2,1)}\},$$

coalesce. For easier analysis and to use Corollary C.5, we reflect time and then shift the starting points, since the corollary only allows positive starting points, and the skew-shift invariance property of the directed landscape implies that the disjointness of the geodesics stay invariant as well under such a shift. More precisely, by Lemma 3.5, $\mathcal{E}_{coal}' \cap \mathcal{E}_{disj}$ has the same probability as

$$\mathrm{Disj}^{[0,1]}_{(2-\delta,2+\delta)\to(-1,1)} \cap (\mathrm{Disj}^{[0,1]}_{(0,4)\to(-1-\delta,-1+\delta)})^c \cap (\mathrm{Disj}^{[0,1]}_{(0,4)\to(1-\delta,1+\delta)})^c.$$

Then by Corollary C.5, we have $\mathbb{P}[\mathcal{E}_{coal}' \cap \mathcal{E}_{disj}] = \mathbb{P}[\mathcal{E}_{ALE}]$, where $\mathcal{E}_{ALE}$ is the following event:

1. $\pi\{0,-1-\delta\}$ intersects $\pi\{4,-1+\delta\}$, and $\pi\{0,1-\delta\}$ intersects $\pi\{4,1+\delta\}$.

2. $\pi\{2-\delta,-1\}$ and $\pi\{2+\delta,1\}$ are disjoint.

It remains to show that $\mathcal{E}_{ALE}$ happens with positive probability.

Below we assume that the geodesic $\pi\{x,y\}$ for any rational $x \geq 0$ and y is unique. This event has probability one by [45, Lemma 4.2]. An implication of this assumption is that, for any rational y, $\pi\{0,y\}$ would be the function staying at 1 on $(-\infty,y]$. Recall (from the proof of Lemma 3.70) that $\delta > 0$ is taken to be an arbitrary small number, and below we assume that it is rational.

Step 1: reduce to events on single geodesics. We note that $\pi\{0,-1-\delta\}$ and $\pi\{0,1-\delta\}$ would be constant 1 on the whole domains $(-\infty,-1-\delta]$ and $(-\infty,1-\delta]$ respectively, since these geodesics are assumed to be unique. Then the events $\pi\{0,-1-\delta\}$ intersects $\pi\{4,-1+\delta\}$, and $\pi\{0,1-\delta\}$ intersects $\pi\{4,1+\delta\}$, would be equivalent to $\pi\{4,-1+\delta\}(-1-\delta)=1$ and $\pi\{4,1+\delta\}(1-\delta)=1$; and we denote them as $\mathcal{E}_{ALE,1}$ and $\mathcal{E}_{ALE,2}$ respectively.

For the non-coalescence event, we consider any second geodesics $\pi^2\{2-\delta,-1\}$ and $\pi^2\{2+\delta,0\}$. Then by the asymptotic directions (of $\pi^2\{2-\delta,-1\}$ and $\pi^2\{2+\delta,0\}(y)$), there exists rational $y_0 < -1$, such that $\pi^2\{2-\delta,-1\}(y) < \pi^2\{2+\delta,0\}(y)$ for any $y \leq y_0$. Further we claim that we can take y_0 small enough, such that

$$\|\pi^2\{2+\delta,0\}\|_{\mathcal{P}} - \mathcal{P}_2(0) > \|\pi^2\{2+\delta,y_0\}\|_{\mathcal{P}} - \mathcal{P}_2(y_0), \tag{C.6}$$

which will ensure that $\pi^2\{2+\delta,0\}$ stays below the second line up to y_0, which will allow us to construct events ensuring disjointness of $\pi\{2-\delta,-1\}$ and $\pi\{2+\delta,1\}$. Before describing this, we first show how to establish (C.6) using the following lemma which is essentially the same as the quadrangle inequality.

Lemma C.8. *Almost surely the following is true. The function* $y \mapsto \|\pi^2\{2+\delta, y\}\|_{\mathcal{P}} - \mathcal{P}_2(y)$ *is non-decreasing, and* $\limsup_{y\to-\infty} \|\pi^2\{2+\delta, y\}\|_{\mathcal{P}} - \mathcal{P}_2(y) < \|\pi^2\{2+\delta, 0\}\|_{\mathcal{P}} - \mathcal{P}_2(0)$.

Proof. For any $y_1 < y_2$, and any second geodesic $\pi^2\{2+\delta, y_1\}$, we can construct a path π from $2+\delta$ to y_2, which is the same as $\pi^2\{2+\delta, y_1\}$ on $(-\infty, y_1)$, and equals 1 on $[y_1, y_2]$. We then have that $\|\pi\|_{\mathcal{P}} = \|\pi^2\{2+\delta, y_1\}\|_{\mathcal{P}} - \mathcal{P}_2(y_1) + \mathcal{P}_2(y_2)$. We also have that $\|\pi\|_{\mathcal{P}} \le \|\pi^2\{2+\delta, y_2\}\|_{\mathcal{P}}$, so we have $\|\pi^2\{2+\delta, y_1\}\|_{\mathcal{P}} - \mathcal{P}_2(y_1) \le \|\pi^2\{2+\delta, y_2\}\|_{\mathcal{P}} - \mathcal{P}_2(y_2)$.

For the second part, we assume the contrary; then we have that $\|\pi^2\{2+\delta, y\}\|_{\mathcal{P}} - \mathcal{P}_2(y) = \|\pi^2\{2+\delta, 0\}\|_{\mathcal{P}} - \mathcal{P}_2(0)$ for any $y \le 0$. By Lemma C.2 we have $\|\pi\{2+\delta, 0\}\|_{\mathcal{P}} = \sup_{y\le 0} \|\pi^2\{2+\delta, y\}\|_{\mathcal{P}} + \mathcal{P}[(y, 2) \to (0, 1)]$, which further means

$$\begin{aligned}
\|\pi\{2+\delta, 0\}\|_{\mathcal{P}} &= \sup_{y\le y'\le 0} \|\pi^2\{2+\delta, y\}\|_{\mathcal{P}} - \mathcal{P}_2(y) + \mathcal{P}_2(y') - \mathcal{P}_1(y') + \mathcal{P}_1(0) \\
&= \sup_{y'\le 0} \|\pi^2\{2+\delta, 0\}\|_{\mathcal{P}} - \mathcal{P}_2(0) + \mathcal{P}_2(y') - \mathcal{P}_1(y') + \mathcal{P}_1(0) \qquad \text{(C.7)} \\
&= \|\pi^2\{2+\delta, 0\}\|_{\mathcal{P}} - \mathcal{P}_2(0) + \mathcal{P}_1(0),
\end{aligned}$$

where the last equality is due to the following reason. By the Brownian Gibbs property of $\mathcal{P}$, we have $\mathbb{P}[\mathcal{P}_1(0) - \mathcal{P}_2(0) \le \delta] > 0$ for any $\delta > 0$. Then by stationarity and ergodicity of $\mathcal{P}$ plus a parabola, we have that $\mathbb{P}[\liminf_{y\to\infty} \mathcal{P}_1(y) - \mathcal{P}_2(y) \le \delta] = 1$, so almost surely $\liminf_{y\to\infty} \mathcal{P}_1(y) - \mathcal{P}_2(y) = 0$, and $\sup_{y'\le 0} \mathcal{P}_2(y') - \mathcal{P}_1(y') = 0$.

On the other hand, by the existence of the geodesic $\pi\{2+\delta, 0\}$, there exists some $y_* \le 0$ with $\|\pi\{2+\delta, 0\}\|_{\mathcal{P}} = \|\pi^2\{2+\delta, y_*\}\|_{\mathcal{P}} + \mathcal{P}_1(0) - \mathcal{P}_1(y_*)$, by Lemma C.2. Thus with (C.7) we have

$$\|\pi^2\{2+\delta, 0\}\|_{\mathcal{P}} - \mathcal{P}_2(0) = \|\pi^2\{2+\delta, y_*\}\|_{\mathcal{P}} + -\mathcal{P}_1(y_*).$$

Since we have assumed that $\|\pi^2\{2+\delta, 0\}\|_{\mathcal{P}} - \mathcal{P}_2(0) = \|\pi^2\{2+\delta, y_*\}\|_{\mathcal{P}} - \mathcal{P}_2(y_*)$, we

have $\mathcal{P}_1(y_*) = \mathcal{P}_2(y_*)$. However, by the Brownian Gibbs property of $\mathcal{P}$, we have that $\mathbb{P}[\mathcal{P}_1(y) = \mathcal{P}_2(y),\ \exists y \le 0] = 0$. Thus we conclude that $\mathbb{P}[\|\pi^2\{2+\delta, y\}\|_{\mathcal{P}} - \mathcal{P}_2(y) = \|\pi^2\{2+\delta, 0\}\|_{\mathcal{P}} - \mathcal{P}_2(0),\ \forall y \le 0] = 0$. □

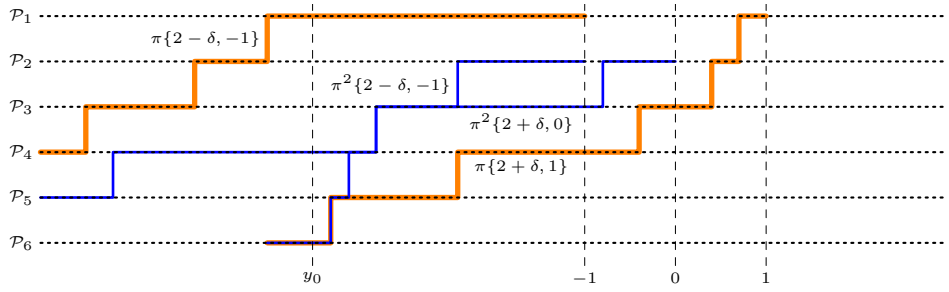


Figure C.2: An illustration of the proof of Lemma C.9: assuming $\pi\{2-\delta,-1\}(y_0) = 1$, $\pi\{2-\delta,-1\}$ is to the left of $\pi^2\{2-\delta,-1\}$; and assuming $\pi\{2+\delta,1\}(0) > 1$, $\pi\{2+\delta,1\}$ is to the right of $\pi^2\{2+\delta,0\}$. By the choice of y_0, $\pi^2\{2-\delta,-1\}$ and $\pi^2\{2+\delta,0\}$ are disjoint in $(-\infty, y_0]$; thus by the ordering of geodesics, $\pi\{2-\delta,-1\}$ and $\pi\{2+\delta,1\}$ are disjoint.

We now construct an event which ensures disjointness of geodesics.

Lemma C.9. *$\pi\{2-\delta,-1\}(y_0) = 1$ and $\pi\{2+\delta,1\}(0) > 1$ imply that $\pi\{2-\delta,-1\}$ and $\pi\{2+\delta,1\}$ are disjoint.*

Proof. The proof is a consequence of ordering properties of geodesics and second geodesics. In particular, we show that (in $(-\infty,-1]$)

$$\pi\{2-\delta,-1\} \le \pi^2\{2-\delta,-1\} \le \pi^2\{2+\delta,0\} \le \pi\{2+\delta,1\}$$

and then establish certain strictness of the inequalities (see Figure C.2). While the inequalities are just a consequence of planarity and uniqueness of the relevant objects, the formal arguments are a bit tedious.

1. Take some rational $y_1 < -1$ with $\pi\{2-\delta,-1\}(y_1) > 1$. Then $(\pi\{0,y_1\}, \pi\{2-\delta,-1\})$ would be a disjoint optimizer from $(0,2-\delta)$ to $(y_1,-1)\}$. We also know that $(\pi\{0,-1\}, \pi^2\{2-\delta,-1\})$ is a disjoint optimizer from $(0,2-\delta)$ to $(-1,-1)$,

by Lemma C.6. Then by Lemma C.7, $(\pi\{0, y_1\}, \pi\{2-\delta, -1\} \wedge \pi^2\{2-\delta, -1\})$ would be another disjoint optimizer from $(0, 2-\delta)$ to $(y_1, -1)$. This implies that $\|\pi\{2-\delta, -1\} \wedge \pi^2\{2-\delta, -1\}\|_{\mathcal{P}} \geq \|\pi\{2-\delta, -1\}\|_{\mathcal{P}} = \mathcal{S}(2-\delta, -1)$, and $\pi\{2-\delta, -1\} \wedge \pi^2\{2-\delta, -1\}$ is also a geodesic from $2-\delta$ to -1, So $\pi\{2-\delta, -1\} = \pi\{2-\delta, -1\} \wedge \pi^2\{2-\delta, -1\}$ by the uniqueness. Thus we have that $\pi\{2-\delta, -1\}(y) \leq \pi^2\{2-\delta, -1\}(y)$ for any $y \in (-\infty, -1]$.

2. When $\pi\{2+\delta, 1\}(0) > 1$, $(\pi\{0, 0\}, \pi\{2+\delta, 1\})$ is a disjoint optimizer from $(0, 2+\delta)$ to $(0, 1)$. By Lemma C.6, $(\pi\{0, 0\}, \pi^2\{2+\delta, 0\})$ is a disjoint optimizer from $(0, 2+\delta)$ to $(0, 0)$. Then by Lemma C.7, $(\pi\{0, 0\}, \pi\{2+\delta, 1\} \vee \pi^2\{2+\delta, 0\})$ is another disjoint optimizer from $(0, 2+\delta)$ to $(0, 1)$, so $\pi\{2+\delta, 1\} \vee \pi^2\{2+\delta, 0\}$ would also be a geodesic from $2+\delta$ to 1. So $\pi\{2+\delta, 1\} = \pi\{2+\delta, 1\} \vee \pi^2\{2+\delta, 0\}$ by the uniqueness, and $\pi\{2+\delta, 1\}(y) \geq \pi^2\{2+\delta, 0\}(y)$ for any $y \in (-\infty, 0]$.

Since $\pi^2\{2-\delta, -1\}$ and $\pi^2\{2+\delta, 0\}$ are disjoint on $(-\infty, y_0]$ by definition of y_0, we have that $\pi\{2-\delta, -1\}$ and $\pi\{2+\delta, 1\}$ are disjoint on $(-\infty, y_0]$. On $[y_0, -1]$ we have $\pi\{2-\delta, -1\} = 1$ and $\pi\{2+\delta, 1\} > 1$ by hypothesis, and so $\pi\{2-\delta, -1\}$ and $\pi\{2+\delta, 1\}$ are also disjoint. □

To use the above, we let $\mathcal{E}_{ALE,3}$ be the event $\pi\{2-\delta, -1\}(y_0) = 1$ and $\mathcal{E}_{ALE,4}$ be the event $\pi\{2+\delta, 1\}(0) > 1$.

From the above lemma we have that

$$\mathcal{E}_{ALE,1} \cap \mathcal{E}_{ALE,2} \cap \mathcal{E}_{ALE,3} \cap \mathcal{E}_{ALE,4} \subset \mathcal{E}_{ALE} = \mathcal{E}'_{coal} \cap \mathcal{E}_{disj}.$$

We will next construct events $\mathcal{E}'_{ALE,1}$, $\mathcal{E}'_{ALE,2}$, $\mathcal{E}'_{ALE,3}$, and $\mathcal{E}'_{ALE,4}$, which imply $\mathcal{E}_{ALE,1}$, $\mathcal{E}_{ALE,2}$, $\mathcal{E}_{ALE,3}$, $\mathcal{E}_{ALE,4}$, respectively. These events are given in terms of inequalities involving $\mathcal{P}_1$, $\mathcal{P}_2$, and weights of certain second geodesics. We will show that, conditional on $\{\mathcal{P}_n\}_{n=2}^{\infty}$, and $\mathcal{P}_1$ outside some compact interval, the intersection

of these events has positive probability. This is by considering $\mathcal{P}_1$ inside that compact interval and using the Brownian Gibbs property of $\mathcal{P}$.

Step 2: reduce to events on $\mathcal{P}_1$ and weights of second geodesics. Let $\mathcal{F}$ be the sigma-algebra generated by $\{\mathcal{P}_n\}_{n=2}^{\infty}$, and $\mathcal{P}_1(x)$ for $x \le y_0 - 1$ and $x \ge 2$. By Lemma C.3, for any rational $x \ge 0$, the random variable $\|\pi^2\{x, y\}\|_{\mathcal{P}}$ for any y is $\{\mathcal{P}_n\}_{n=2}^{\infty}$ measurable; and y_0 is $\{\mathcal{P}_n\}_{n=2}^{\infty}$ measurable. Then these random variables are all $\mathcal{F}$ measurable. By the Brownian Gibbs property of the Airy line ensemble [36], conditional on $\mathcal{F}$, the process $\mathcal{P}_1$ on $[y_0 - 1, 2]$ is a Brownian bridge conditional on avoiding $\mathcal{P}_2$.

It would be useful to remember the following simple relation between geodesics and second geodesics. By Lemma C.2, we have that for any rational $x \ge 0$ and y,

$$\|\pi\{x, y\}\|_{\mathcal{P}} = \sup_{z \le y} \|\pi^2\{x, z\}\|_{\mathcal{P}} + \mathcal{P}_1(y) - \mathcal{P}_1(z), \tag{C.8}$$

and by uniqueness of $\pi\{x, y\}$ this supremum is achieved at a unique $z_* \le y$, such that $\pi\{x, y\}(z) = 1$ for $z \in [z_*, y]$, and $\pi\{x, y\}(z) > 1$ for $z \in (-\infty, z_*)$.

We will make use of a random variable $H > 0$, which is $\mathcal{F}$ measurable, and its exact definition will come up only at the end. Consider the event $\mathcal{E}'_{ALE,1}$:

$$\|\pi^2\{4, -1 - 2\delta\}\|_{\mathcal{P}} - \mathcal{P}_1(-1 - 2\delta) > -H^2 > \sup_{-1-\delta \le x \le -1+\delta} \|\pi^2\{4, x\}\|_{\mathcal{P}} - \mathcal{P}_1(x)$$

as illustrated in Figure C.3 (at $-1 - 2\delta$ and in the interval $[-1 - \delta, -1 + \delta]$). We note that under $\mathcal{E}'_{ALE,1}$, for the function $\|\pi^2\{4, \cdot\}\|_{\mathcal{P}} - \mathcal{P}_1$, its supremum in $[-1 - \delta, -1 + \delta]$ is smaller than its value at $-1 - 2\delta$. By the above relation (C.8) between geodesics and second geodesics, $\mathcal{E}'_{ALE,1}$ implies that the $\pi\{4, -1 + \delta\}$ does not jump to the second line in $[-1 - \delta, -1 + \delta]$, thus $\pi\{4, -1 + \delta\}(-1 - \delta) = 1$ and $\mathcal{E}_{ALE,1}$ holds.

Similarly, let $\mathcal{E}'_{ALE,2}$ be the event where

$$\|\pi^2\{4, 1-2\delta\}\|_{\mathcal{P}} - \mathcal{P}_1(1-2\delta) > -H > \sup_{1-\delta \le x \le 1+\delta} \|\pi^2\{4, x\}\|_{\mathcal{P}} - \mathcal{P}_1(x),$$

and $\mathcal{E}'_{ALE,3}$ be the event where

$$\|\pi^2\{2-\delta, y_0 - 1/2\}\|_{\mathcal{P}} - \mathcal{P}_1(y_0 - 1/2) > -H > \sup_{y_0 \le x \le -1} \|\pi^2\{2-\delta, x\}\|_{\mathcal{P}} - \mathcal{P}_1(x),$$

which are also illustrated in in Figure C.3. We can similarly conclude that $\mathcal{E}'_{ALE,2} \subset \mathcal{E}_{ALE,2}$ and $\mathcal{E}'_{ALE,3} \subset \mathcal{E}_{ALE,3}$.

Finally we let $\mathcal{E}'_{ALE,4}$:

$$\mathcal{P}_1(1/2) - \mathcal{P}_2(1/2) < H^{-1} < \inf_{y_0 \le x \le 0} \mathcal{P}_1(x) - \mathcal{P}_2(x).$$

It would be slightly more complicated to get $\mathcal{E}_{ALE,4}$ from $\mathcal{E}'_{ALE,4}$.

Lemma C.10. *$\mathcal{E}'_{ALE,4} \subset \mathcal{E}_{ALE,4}$ when H is large enough (depending on $\mathcal{F}$).*

The idea behind is that, using (C.6) (in choosing y_0), we can show that the event $\mathcal{E}'_{ALE,4}$ implies $\|\pi^2\{2+\delta, 1/2\}\|_{\mathcal{P}} - \mathcal{P}_1(1/2) > \sup_{x \le 0} \|\pi^2\{2+\delta, x\}\|_{\mathcal{P}} - \mathcal{P}_1(x)$. By the relation (C.8), this further implies that the geodesic $\pi\{2+\delta, 1\}$ jumps to the second line at a point > 0, thus $\mathcal{E}_{ALE,4}$ holds.

We leave the detailed proof of Lemma C.10 to the end, and assume it for now. Conditional on $\mathcal{F}$, the above inequalities for the event $\mathcal{E}'_{ALE,1} \cap \mathcal{E}'_{ALE,2} \cap \mathcal{E}'_{ALE,3} \cap \mathcal{E}'_{ALE,4}$ can be summarized as follows (the inequalities are presented essentially in an order

such that the points/intervals involved move from left to right):

$$
\begin{aligned}
\mathcal{P}_1(y_0 - 1/2) &< \|\pi^2\{2-\delta, y_0 - 1/2\}\|_{\mathcal{P}} + H, && \mathcal{E}'_{ALE,3},\\
\mathcal{P}_1(x) &> \|\pi^2\{2-\delta, x\}\|_{\mathcal{P}} + H, \quad \forall y_0 \le x \le -1, && \mathcal{E}'_{ALE,3},\\
\mathcal{P}_1(x) &> \mathcal{P}_2(x) + H^{-1}, \quad \forall y_0 \le x \le 0, && \mathcal{E}'_{ALE,4},\\
\mathcal{P}_1(-1-2\delta) &< \|\pi^2\{4, -1-2\delta\}\|_{\mathcal{P}} + H^2, && \mathcal{E}'_{ALE,1},\\
\mathcal{P}_1(x) &> \|\pi^2\{4, x\}\|_{\mathcal{P}} + H^2, \quad \forall -1-\delta \le x \le -1+\delta, && \mathcal{E}'_{ALE,1},\\
\mathcal{P}_1(1/2) &< \mathcal{P}_2(1/2) + H^{-1}, && \mathcal{E}'_{ALE,4},\\
\mathcal{P}_1(1-2\delta) &< \|\pi^2\{4, 1-2\delta\}\|_{\mathcal{P}} + H, && \mathcal{E}'_{ALE,2},\\
\mathcal{P}_1(x) &> \|\pi^2\{4, x\}\|_{\mathcal{P}} + H, \quad \forall 1-\delta \le x \le 1+\delta, && \mathcal{E}'_{ALE,2}.
\end{aligned}
\tag{C.9}
$$

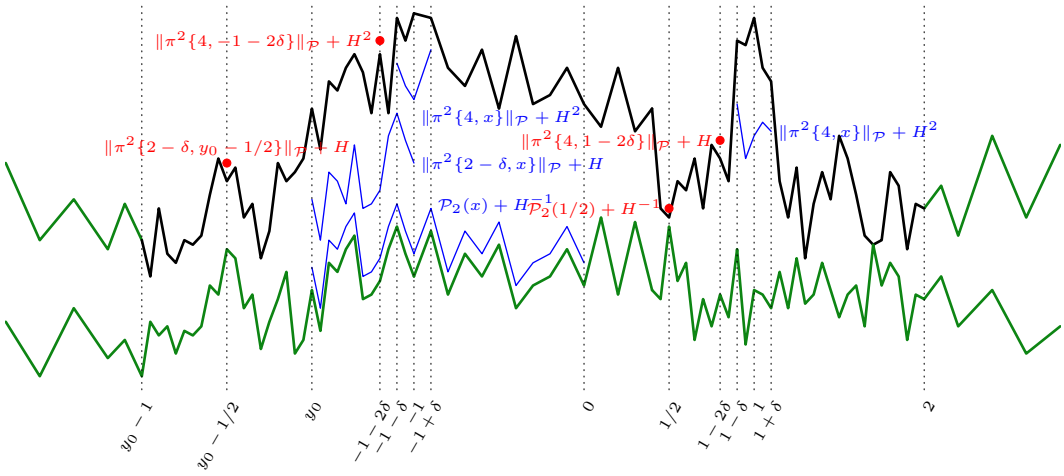


Figure C.3: An illustration of the conditions (C.9) to be satisfied by $\mathcal{P}_1$: the green curves are $\mathcal{P}_2$ and $\mathcal{P}_1$ outside $(y_0 - 1, 2)$, and are $\mathcal{F}$-measurable; the black curve is $\mathcal{P}_1$ in $(y_0 - 1, 2)$, which (give $\mathcal{F}$) is a Brownian bridge conditioned on staying above $\mathcal{P}_2$. The red points are upper bounds and the blue curves are lower bounds.

Note that by (C.8), $\pi^2\{2-\delta, x\}$ and $\pi^2\{4, x\}$ are uniformly bounded for x in any compact interval. Also note that $\mathcal{P}_1$ on $[y_0 - 1, 2]$ is a Brownian bridge conditional on $> \mathcal{P}_2$ (by the Brownian Gibbs property of $\mathcal{P}$ mentioned in Section 1.4.2). Then by looking up the inequalities in (C.9) (see Figure C.3), we can choose H large enough, such that with positive probability, all of them hold simultaneously, and Lemma C.10

also holds. Thus we have

$$\mathbb{P}[\mathcal{E}'_{ALE,1} \cap \mathcal{E}'_{ALE,2} \cap \mathcal{E}'_{ALE,3} \cap \mathcal{E}'_{ALE,4} \mid \mathcal{F}] > 0.$$

By averaging over $\mathcal{F}$, we have that

$$\begin{aligned}\mathbb{P}[\mathcal{E}'_{coal} \cap \mathcal{E}_{disj}] &\geq \mathbb{P}[\mathcal{E}_{ALE,1} \cap \mathcal{E}_{ALE,2} \cap \mathcal{E}_{ALE,3} \cap \mathcal{E}_{ALE,4}] \\ &\geq \mathbb{P}[\mathcal{E}'_{ALE,1} \cap \mathcal{E}'_{ALE,2} \cap \mathcal{E}'_{ALE,3} \cap \mathcal{E}'_{ALE,4}] > 0.\end{aligned}$$

It remains to prove Lemma C.10.

Proof of Lemma C.10. We show $\mathcal{E}'_{ALE,4}$ implies that

$$\|\pi^2\{2+\delta, 1/2\}\|_{\mathcal{P}} - \mathcal{P}_1(1/2) > \sup_{x \leq 0} \|\pi^2\{2+\delta, x\}\|_{\mathcal{P}} - \mathcal{P}_1(x), \tag{C.10}$$

which further implies $\mathcal{E}_{ALE,4}$, by the relation (C.8) between geodesics and second geodesics. Under $\mathcal{E}'_{ALE,4}$, we have that

$$\begin{aligned}&\|\pi^2\{2+\delta, 1/2\}\|_{\mathcal{P}} - \mathcal{P}_1(1/2) \\ >&\|\pi^2\{2+\delta, 1/2\}\|_{\mathcal{P}} - \mathcal{P}_2(1/2) - H^{-1} \\ \geq& \sup_{y_0 \leq x \leq 0} \|\pi^2\{2+\delta, x\}\|_{\mathcal{P}} - \mathcal{P}_2(x) - H^{-1} \\ >& \sup_{y_0 \leq x \leq 0} \|\pi^2\{2+\delta, x\}\|_{\mathcal{P}} - \mathcal{P}_1(x).\end{aligned} \tag{C.11}$$

Here the first and third inequality is by $\mathcal{E}'_{ALE,4}$. The second inequality is essentially a quadrangle inequality. More precisely, it is due to the following reason: for any $y_0 \leq x \leq 0$, any any second geodesic $\pi^2\{2+\delta, x\}$, we can construct a path from $2+\delta$ to $1/2$, which is the same as $\pi^2\{2+\delta, x\}$ on $(-\infty, x)$, and equals 2 on $[x, 1/2]$. By Lemma C.2, the weight of this path is $\|\pi^2\{2+\delta, x\}\|_{\mathcal{P}} - \mathcal{P}_2(x) + \mathcal{P}_2(1/2) \leq \|\pi^2\{2+\delta, 1/2\}\|_{\mathcal{P}}$.

By (C.6), we can take H large enough and $\mathcal{F}$ measurable, such that

$$\|\pi^2\{2+\delta, 0\}\|_{\mathcal{P}} - \mathcal{P}_2(0) - H^{-1} > \|\pi^2\{2+\delta, y_0\}\|_{\mathcal{P}} - \mathcal{P}_2(y_0).$$

and hence by the second inequality above, we have

$$\|\pi^2\{2+\delta, 1/2\}\|_{\mathcal{P}} - \mathcal{P}_1(1/2) > \|\pi^2\{2+\delta, y_0\}\|_{\mathcal{P}} - \mathcal{P}_2(y_0). \tag{C.12}$$

We also have

$$\|\pi^2\{2+\delta, y_0\}\|_{\mathcal{P}} - \mathcal{P}_2(y_0) \geq \sup_{x \leq y_0} \|\pi^2\{2+\delta, x\}\|_{\mathcal{P}} - \mathcal{P}_1(x), \tag{C.13}$$

which is again essentially a quadrangle inequality, and follows similar arguments. Note that (C.10) is implied by (C.11), (C.12), and (C.13), so the conclusion follows. □

Bibliography

[1] David Aldous and Daniel Lanoue. A lecture on the averaging process. *Probab. Surv.*, 9:90–102, 2012.

[2] Philip W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109(5):1492, 1958.

[3] Omer Angel, Alexander Holroyd, and Dan Romik. The oriented swap process. *Ann. Probab.*, 37(5):1970–1998, 2009.

[4] Antonio Auffinger, Michael Damron, and Jack Hanson. *50 years of first-passage percolation*, volume 68 of *University Lecture Series*. American Mathematical Society, 2017.

[5] Jinho Baik, Percy Deift, and Kurt Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12(4):1119–1178, 1999.

[6] Márton Balázs, Ofer Busani, and Timo Seppäläinen. Local stationarity in exponential last-passage percolation. *Probab. Theory Related Fields*, 180(1):113–162, 2021.

[7] Marton Balázs, Eric Cator, and Timo Seppäläinen. Cube root fluctuations for the corner growth model associated to the exclusion process. *Electron. J. Probab.*, 11:1094–1132, 2006.

[8] Riddhipratim Basu and Manan Bhatia. Small deviation estimates and small ball probabilities for geodesics in last passage percolation. arXiv preprint arXiv:2101.01717, 2021.

[9] Riddhipratim Basu and Shirshendu Ganguly. Time correlation exponents in last passage percolation. *In and out of equilibrium 3: Celebrating Vladas Sidoravicius*, pages 101–123, 2018.

[10] Riddhipratim Basu, Shirshendu Ganguly, and Alan Hammond. Fractal geometry of airy_2 processes coupled via the airy sheet. *Ann. Probab*, 49(1):485–505, 2021.

[11] Riddhipratim Basu, Shirshendu Ganguly, Alan Hammond, and Milind Hegde. Interlacing and scaling exponents for the geodesic watermelon in last passage percolation. arXiv preprint arXiv:2006.11448, 2020.

[12] Riddhipratim Basu, Shirshendu Ganguly, and Lingfu Zhang. Temporal correlation in last passage percolation with flat initial condition via Brownian comparison. *Comm. Math. Phys.*, 383(3):1805–1888, 2021.

[13] Riddhipratim Basu, Christopher Hoffman, and Allan Sly. Nonexistence of bigeodesics in integrable models of last passage percolation. arXiv preprint arXiv:1811.04908, 2018.

[14] Riddhipratim Basu, Sourav Sarkar, and Allan Sly. Coalescence of geodesics in exactly solvable models of last passage percolation. *J. Math. Phys.*, 60(9):093301, 2019.

[15] Riddhipratim Basu, Vladas Sidoravicius, and Allan Sly. Last passage percolation with a defect line and the solution of the slow bond problem. arXiv preprint arXiv:1408.3464, 2014.

[16] Erik Bates. Empirical distributions, geodesic lengths, and a variational formula in first-passage percolation. arXiv preprint arXiv:2006.12580, 2020.

[17] Erik Bates, Shirshendu Ganguly, and Alan Hammond. Hausdorff dimensions for shared endpoints of disjoint geodesics in the directed landscape. arXiv preprint arXiv:1912.04164, 2019.

[18] Eli Ben-Naim, Paul L. Krapivsky, and Sidney Redner. Bifurcations and patterns in compromise processes. *Phys. D*, 183(3-4):190–204, 2003.

[19] Elia Bisi, Fabio D. Cunden, Shane Gibbons, and Dan Romik. Sorting networks, staircase young tableaux and last passage percolation. arXiv preprint arXiv:2003.03331, 2020.

[20] Alexei Borodin and Patrik L. Ferrari. Large time asymptotics of growth models on space-like paths I: PushASEP. *Electron. J. Probab.*, 13:1380–1418, 2008.

[21] Alexei Borodin, Patrik L. Ferrari, Michael Prähofer, and Tomohiro Sasamoto. Fluctuation properties of the TASEP with periodic initial configuration. *J. Stat. Phys.*, 129(5):1055–1080, 2007.

[22] Alexei Borodin, Patrik L. Ferrari, and Tomohiro Sasamoto. Large time asymptotics of growth models on space-like paths II: PNG and parallel TASEP. *Comm. Math. Phys.*, 283(2):417–449, 2008.

[23] Alexei Borodin, Vadim Gorin, and Michael Wheeler. Shift-invariance for vertex models and polymers. arXiv preprint arXiv:1912.02957, 2019.

[24] Alexei Borodin and Sandrine Péché. Airy kernel with two sets of parameters in directed percolation and random matrix theory. *J. Stat. Phys.*, 132(2):275–290, 2008.

[25] Jean Bourgain and Carlos E. Kenig. On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math.*, 161(2):389–426, 2005.

[26] Maury Bramson and David Griffeath. Asymptotics for interacting particle systems on $\mathbb{Z}^d$. *Z. Wahrsch. Verw. Gebiete*, 53(2):183–196, 1980.

[27] Alexey Bufetov, Vadim Gorin, and Dan Romik. Absorbing time asymptotics in the oriented swap process. arXiv preprint arXiv:2003.06479, 2020.

[28] Lev Buhovsky, Alexander Logunov, Eugenia Malinnikova, and Mikhail Sodin. A discrete harmonic function bounded on a large portion of $\mathbb{Z}^2$ is constant. arXiv preprint arXiv:1712.07902, 2017.

[29] Ofer Busani, Timo Seppäläinen, and Evan Sorensen. The stationary horizon and semi-infinite geodesics in the directed landscape. arXiv preprint arXiv:2203.13242, 2020.

[30] Jacob Calvert, Alan Hammond, and Milind Hegde. Brownian structure in the KPZ fixed point. arXiv preprint arXiv:1912.00992, 2019.

[31] René Carmona, Abel Klein, and Fabio Martinelli. Anderson localization for Bernoulli and other singular potentials. *Comm. Math. Phys.*, 108(1):41–66, 1987.

[32] Samprit Chatterjee and Eugene Seneta. Towards consensus: Some convergence theorems on repeated averaging. *J. Appl. Probab.*, 14(1):89–97, 1977.

[33] Sourav Chatterjee, Persi Diaconis, Allan Sly, and Lingfu Zhang. A phase transition for repeated averages. *Ann. Probab.*, 50(1):1–17, 2022.

[34] Colin Cooper, Robert Elsässer, Hirotaka Ono, and Tomasz Radzik. Coalescing random walks and voting on connected graphs. *SIAM J. Discrete Math.*, 27(4):1748–1758, 2013.

[35] Ivan Corwin. The Kardar–Parisi–Zhang equation and universality class. *Random Matrices Theory Appl.*, 1(01):1130001, 2012.

[36] Ivan Corwin and Alan Hammond. Brownian gibbs property for airy line ensembles. *Invent. Math.*, 195(2):441–508, 2014.

[37] Ivan Corwin, Alan Hammond, Milind Hegde, and Konstantin Matetski. Exceptional times when the KPZ fixed point violates Johansson's conjecture on maximizer uniqueness. arXiv preprint arXiv:2101.04205, 2021.

[38] Ivan Corwin and Xin Sun. Ergodicity of the Airy line ensemble. *Electron. Commun. Probab.*, 19:1–11, 2014.

[39] David Coupier. Multiple geodesics with the same direction. *Electron. Commun. Probab.*, 16:517–527, 2011.

[40] Duncan Dauvergne. Hidden invariance of last passage percolation and directed polymers. *Ann. Probab.*, 50(1):18–60, 2022.

[41] Duncan Dauvergne. Non-uniqueness times for the maximizer of the KPZ fixed point. arXiv preprint arXiv:2202.01700, 2022.

[42] Duncan Dauvergne, Janosch Ortmann, and Bálint Virág. The directed landscape. arXiv preprint arXiv:1812.00309, 2018.

[43] Duncan Dauvergne, Sourav Sarkar, and Bálint Virág. Three-halves variation of geodesics in the directed landscape. arXiv preprint arXiv:2010.12994, 2020.

[44] Duncan Dauvergne and Bálint Virág. The scaling limit of the longest increasing subsequence. arXiv preprint arXiv:2104.08210, 2021.

[45] Duncan Dauvergne and Lingfu Zhang. Disjoint optimizers and the directed landscape. arXiv preprint arXiv:2102.00954, 2021.

[46] Jian Ding and Charles K. Smart. Localization near the edge for the Anderson Bernoulli model on the two dimensional lattice. *Invent. Math.*, 219(2):467–506, 2020.

[47] Pablo A. Ferrari, L.R.G. Fontes, and Y. Kohayakawa. Invariant measures for a two-species asymmetric process. *J. Stat. Phys.*, 76(5-6):1153–1177, 1994.

[48] Pablo A. Ferrari and James B. Martin. Stationary distributions of multi-type totally asymmetric exclusion processes. *Ann. Probab.*, 35(3):807–832, 2007.

[49] Pablo A. Ferrari, James B. Martin, and Leandro. P. R. Pimentel. A phase transition for competition interfaces. *Ann. Appl. Probab.*, 19(1):281–317, 2009.

[50] Pablo A. Ferrari and Leandro. P. R. Pimentel. Competition interfaces and second class particles. *Ann. Probab*, 33(4):1235–1254, 2005.

[51] Patrik L. Ferrari and Alessandra Occelli. Universality of the GOE tracy-widom distribution for TASEP with arbitrary particle density. *Electron. J. Probab.*, 23:24 pp., 2018.

[52] Patrik L. Ferrari and Herbert Spohn. On time correlations for KPZ growth in one dimension. *SIGMA*, 12(074), 2016.

[53] Jürg. Fröhlich, Fabio Martinelli, Elisabetta Scoppola, and Thomas Spencer. Constructive proof of localization in the Anderson tight binding model. *Comm. Math. Phys.*, 101(1):21–46, 1985.

[54] Jürg Fröhlich and Thomas Spencer. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Comm. Math. Phys.*, 88(2):151–184, 1983.

[55] Pavel Galashin. Symmetries of stochastic colored vertex models. *Ann. Probab.*, 49(5):2175–2219, 2021.

[56] Shirshendu Ganguly and Alan Hammond. The geometry of near ground states in gaussian polymer models. arXiv preprint arXiv:2010.05836, 2020.

[57] Shirshendu Ganguly and Alan Hammond. Stability and chaos in dynamical last passage percolation. arXiv preprint arXiv:2010.05837, 2020.

[58] Shirshendu Ganguly and Milind Hegde. Local and global comparisons of the Airy difference profile to Brownian local time. arXiv preprint arXiv:2103.12029, 2021.

[59] Shirshendu Ganguly and Lingfu Zhang. Convergence of the geodesic local time in last passage percolation. upcoming, 2022.

[60] Shirshendu Ganguly and Lingfu Zhang. Fractal geometry of the space-time difference profile in the directed landscape via construction of geodesic local times. arXiv preprint arXiv:2204.01674, 2022.

[61] Christophe Garban and Jeffrey E. Steif. *Noise sensitivity of Boolean functions and percolation*, volume 5. Cambridge University Press, 2014.

[62] Olle Häggström. A pairwise averaging procedure with application to consensus formation in the Deffuant model. *Acta Appl. Math.*, 119(1):185–201, 2012.

[63] Alan Hammond. Brownian regularity for the Airy line ensemble, and multi-polymer watermelons in Brownian last passage percolation. arXiv preprint arXiv:1609.02971, 2016.

[64] Alan Hammond. Exponents governing the rarity of disjoint polymers in Brownian last passage percolation. *Proc. Lond. Math. Soc. (3)*, 120(3):370–433, 2020.

[65] Alan Hammond and Sourav Sarkar. Modulus of continuity for polymer fluctuations and weight profiles in poissonian last passage percolation. *Electron. J. Probab.*, 25:1–38, 2020.

[66] Jonathan Hermon, Shuangping Li, Dong Yao, and Lingfu Zhang. Mean field behavior during the big bang for coalescing random walk. arXiv preprint arXiv:2105.11585, 2021.

[67] Christopher Hoffman. Coexistence for richardson type competing spatial growth models. *Ann. Appl. Probab.*, 15(1B):739–747, 2005.

[68] Christopher Hoffman. Geodesics in first passage percolation. *Ann. Appl. Probab.*, pages 1944–1969, 2008.

[69] Christopher Hoffman. American Institute of Mathematics problem lists: first passage percolation, 2015. `http://aimpl.org/firstpercolation/` Aug. 2015. Edited by Daniel Ahlberg.

[70] Steven A. Janowsky and Joel L. Lebowitz. Finite-size effects and shock fluctuations in the asymmetric simple-exclusion process. *Phys. Rev. A*, 45(2):618, 1992.

[71] Kurt Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.*, 209(2):437–476, 2000.

[72] Kurt Johansson. Transversal fluctuations for increasing subsequences on the plane. *Probab. Theory Related Fields*, 116(4):445–456, 2000.

[73] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56:889–892, 1986.

[74] Hervé Kunz and Bernard Souillard. Sur le spectre des opérateurs aux différences finies aléatoires. *Comm. Math. Phys.*, 78(2):201–246, 1980.

[75] Nicolas Lanchier. The critical value of the Deffuant model equals one half. *ALEA Lat. Am. J. Probab. Math. Stat*, 9(2):383–402, 2012.

[76] Michel Ledoux and Brain Rider. Small deviations for beta ensembles. *Electron. J. Probab.*, 15:1319–1343, 2010.

[77] Linjun Li and Lingfu Zhang. Anderson-Bernoulli localization on the three-dimensional lattice and discrete unique continuation principle. *Duke Math. J.*, 171(2):327–415, 2022.

[78] Russell Lyons. Factors of IID on trees. *Combin. Probab. Comput.*, 26(2):285–300, 2017.

[79] James B. Martin, Allan Sly, and Lingfu Zhang. Convergence of the environment seen from geodesics in exponential last-passage percolation. arXiv preprint arXiv:2106.05242, 2021.

[80] Konstantin Matetski, Jeremy Quastel, and Daniel Remenik. The KPZ fixed point. arXiv preprint arXiv:1701.00018, 2016.

[81] Peter Mörters and Yuval Peres. *Brownian motion*, volume 30. Cambridge University Press, 2010.

[82] Danny Nam, Allan Sly, and Lingfu Zhang. Ising model on trees and factors of IID. *Comm. Math. Phys.*, 389(2):1009–1046, 2022.

[83] Charles M. Newman. A surface view of first-passage percolation. In *Proceedings of the international congress of mathematicians*, pages 1017–1023. Springer, 1995.

[84] Roberto I. Oliveira. On the coalescence time of reversible random walks. *Trans. Amer. Math. Soc.*, 364(4):2109–2128, 2012.

[85] Roberto I. Oliveira. Mean field conditions for coalescing random walks. *Ann. Probab.*, 41(5):3420–3461, 2013.

[86] Alex Olshevsky and John N Tsitsiklis. Convergence speed in distributed consensus and averaging. *SIAM J. Control Optim.*, 48(1):33–55, 2009.

[87] Leandro. P. R. Pimentel. Duality between coalescence times and exit points in last-passage percolation models. *Ann. Probab.*, 44(5):3187–3206, 2016.

[88] Michael Prähofer and Herbert Spohn. Scale invariance of the PNG droplet and the Airy process. *J. Stat. Phys.*, 108(5):1071–1106, 2002.

[89] Jeremy Quastel and Daniel Remenik. Airy processes and variational problems. In *Topics in percolative and disordered systems*, pages 121–171. Springer, 2014.

[90] Mustazee Rahman and Bálint Virág. Infinite geodesics, competition interfaces and the second class particle in the scaling limit. arXiv preprint arXiv:2112.06849, 2021.

[91] Hermann Rost. Non-equilibrium behaviour of a many particle process: Density profile and local equilibria. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 58(1):41–53, 1981.

[92] Sourav Sarkar, Allan Sly, and Lingfu Zhang. Infinite order phase transition in the slow bond TASEP. arXiv preprint arXiv:2109.04563, 2021.

[93] Sourav Sarkar and Bálint Virág. Brownian absolute continuity of the kpz fixed point with arbitrary initial condition. *Ann. Probab*, 49(4):1718–1737, 2021.

[94] Timo Seppäläinen. Variational formulas, busemann functions, and fluctuation exponents for the corner growth model with exponential weights. arXiv preprint arXiv:1709.05771, 2017.

[95] Timo Seppäläinen. Existence, uniqueness and coalescence of directed planar geodesics: proof via the increment-stationary growth process. *Ann. Inst. Henri Poincaré Probab. Stat.*, 56(3):1775–1791, 2020.

[96] Timo Seppäläinen and Evan Sorensen. Global structure of semi-infinite geodesics and competition interfaces in Brownian last-passage percolation. arXiv preprint arXiv:2112.10729, 2021.

[97] Devavrat Shah. Gossip algorithms. *Found. Trends Networking*, 3(1):1–125, 2009.

[98] Allan Sly and Lingfu Zhang. Stationary distributions for the voter model in $d \geq 3$ are factors of IID. arXiv preprint arXiv:1908.09450, 2019.

[99] Jeffrey E. Steif and Johan Tykesson. Generalized divide and color models. *ALEA, Lat. Am. J. Probab. Math. Stat*, 16:899–955, 2019.

[100] J. van den Berg and Harry Kesten. Asymptotic density in a coalescing random walk model. *Ann. Probab.*, 28(1):303–352, 2000.

[101] J. van den Berg and Harry Kesten. Randomly coalescing random walk in dimension ≥ 3. In *In and out of equilibrium (Mambucaba, 2000)*, volume 51 of *Progr. Probab.*, pages 1–45. Birkhäuser Boston, Boston, MA, 2002.

[102] Lingfu Zhang. Optimal exponent for coalescence of finite geodesics in exponential last passage percolation. *Electron. Commun. Probab.*, 25:1–14, 2020.

[103] Lingfu Zhang. Shift-invariance of the colored TASEP and finishing times of the oriented swap process. arXiv preprint arXiv:2107.06350, 2021.

[104] Nikos Zygouras. Some algebraic structures in the KPZ universality. arXiv preprint arXiv:1812.07204, 2018.