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Fine-scale properties of random functions

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Fine-scale properties of random functions

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

We study the monochromatic ensemble of random functions in the generality of a compact Riemannian manifold of any dimension. We prove equidistribution of local integrals at scales within a logarithmic factor of the optimal wave scale. On the two-dimensional sphere, we prove a limit theorem for the distribution of these integrals. We also study nodal domains, giving explicit (but embarrassing) lower bounds for the Nazarov-Sodin constant in dimension 2 and 3 and an estimate of the high-dimensional behaviour.

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Chapter 1

Introduction

A good example of the random functions we have in mind is the random spherical harmonic of high degree. This is defined by taking any orthonormal basis of spherical harmonics ϕ_j of degree m and forming the sum

$$\phi = \sum c_j \phi_j$$

where the coefficients c_j are independent Gaussians of mean zero and equal variance. One could consider other distributions for the coefficients, but we will focus on the Gaussian case.

One can generalize beyond spherical geometry. On any compact Riemannian manifold, take orthonormal eigenfunctions of the Laplacian with frequency in a short window around T. The parameter T plays the role of the degree m for spherical harmonics. It is necessary to take a window in order to have a growing number of basis functions ϕ_j. Then one forms a sum with Gaussian coefficients as above. If the window is short compared to T, then ϕ is a stand-in for a "random eigenfunction" with eigenvalue T^2. The problem with literally taking a random eigenfunction is that when an eigenvalue has multiplicity 1, the random function would simply be a deterministic function multiplied by a random scalar.

A spherical harmonic of degree m has a wavelength of order $1/m$. Likewise, in other geometries with a short window of frequencies near T, the wave scale is $1/T$. By "fine-scale" properties of these random functions, we mean ones that involve the behaviour at distance close to $1/T$. For example, we will see that nodal domains typically have this diameter. Another quantity that we study in this thesis is the integral of ϕ^2 over a ball of radius r. If rT diverges but only very slowly, say like a power of $\log(T)$, then we regard this as "fine-scale". There is a fundamental barrier to probing these functions past the wave scale, analogous to the Planck scale in quantum mechanics. At a scale r such that $rT \to \infty$, one hopes to see a large sample containing

many wavelengths of the function. If rT is bounded, then one risks catching the function at an arbitrary point within its natural cycle instead of seeing many cycles.

There are several reasons to study random spherical harmonics, and so much the more for their generalizations to other geometries. They provide a random model for non-random objects, including some of interest in number theory, via Berry's Random Wave Model. Second, there are many difficult questions to do with Laplace eigenfunctions, and the random harmonic sometimes suggests extremal behaviour or gives an interesting perspective. Third, there is intrinsic probabilistic interest in the properties of these random functions. Sometimes they are relevant for models of noise in the real world, such as the Cosmic Microwave Background or errors in medical imaging (see [73] and Chapter 5 of [2]). Let us first describe some of this motivation in more detail and then outline the new theorems proved in this thesis.

1.1 The Random Wave Model

Berry's Random Wave Model [11] uses a monochromatic random wave as a stand-in to make predictions about non-random eigenfunctions. This is expected to be a good approximation for chaotic systems. For example, the model applies to Laplace eigenfunctions on a manifold of negative curvature. It is not so easy to make a precise statement capturing the idea that eigenfunctions of a chaotic system somehow behave like monochromatic random waves. At what scale is the behaviour similar? Which properties can reliably be predicted from the random model? How many exceptions should be allowed? Recently, Abert, Bergeron, and Le Masson have advanced a conjecture that phrases the Random Wave Model in terms of Benjamini-Schramm convergence [1]. Ingremeau has given a similar formulation and proved tightness of the measures that arise in his approach [38].

Much like a single Gaussian random variable is specified by just its mean and variance, a Gaussian random field $F(x)$ is specified by its mean at each point x together with the correlations between pairs of values:

$$\mathbb{E}\left[\big(F(x)-\mathbb{E}[F(x)]\big)\big(F(y)-\mathbb{E}[F(y)]\big)\right].$$

If x and y vary only over a finite set, then this is essentially the covariance matrix of a random vector. We usually assume that $\mathbb{E}[F(x)] = 0$ identically, so that F is a so-called *centered* field. Then it is the two-point correlations that determine F. The *monochromatic* Gaussian random field earns its name because this function is synthesized from just a single frequency:

$$\mathbb{E}\left[F(x)F(y)\right] = \frac{1}{|S^{n-1}|}\int_{|\xi|=1} e^{i\xi\cdot(x-y)}d\xi. \tag{1.1.1}$$

When $n = 2$, a random function with this covariance structure can be expressed in polar coordinates as

$$F(x) = F(r,\theta) = \mathrm{Re}\left(\sum_{n\in\mathbb{Z}} c_n J_{|n|}(r)e^{in\theta}\right)$$

where the coefficients c_n are independent Gaussians of mean 0 and constant variance.

The random wave model applies at the *wave scale*, meaning at distances on the order of $1/\sqrt{\lambda}$ for Laplace eigenfunctions with eigenvalue λ. On a manifold of negative sectional curvatures, Berry's insight is that the eigenfunctions are well described by the random wave model as $\lambda \to \infty$. Thus they converge to $F(x)$ in a statistical sense. Two parts to this conjectured limit should be distinguished from each other: First, that the limiting field is Gaussian and, second, that its correlations are given by the monochromatic ensemble.

1.2 Quantum unique ergodicity

The eigenvalue equation $(\Delta + \lambda)\phi = 0$ imparts a quantum interpretation to spherical harmonics: $|\phi|^2$ is the probability density of a single quantum particle on a sphere. Does the particle have favourite regions of the sphere? What are the possible limit measures of $|\phi_\lambda|^2$ as $\lambda \to \infty$ along a subsequence? In the case of quantum unique ergodicity, the only possible limit is the volume measure on the underlying manifold M. By QUE for a Riemannian manifold M, we mean that for any fixed measurable subset A of M,

$$\int_A \phi_\lambda^2 d\,\mathrm{vol} \to \mathrm{vol}(A)$$

for any sequence of Laplace eigenfunctions ϕ_λ with growing eigenvalue $\lambda \to \infty$. More generally, one can lift the eigenfunctions to measures on the phase space S^*M, and quantum unique ergodicity is the property that Liouville measure is the only possible limit of these measures as $\lambda \to \infty$.

Whether this property holds depends on M and can be very difficult to establish. But it does not hold on the sphere. Consider S^2. A ball of small radius r has volume close to πr^2, as in the Euclidean case. However, for every degree, there is an explicit harmonic that assigns measure roughly $2r/\pi$ to such a ball, which is much larger than its volume. There is also a conceptual explanation for the failure of ergodicity on the sphere: The geodesic flow is periodic [18]. Thus QUE is known to be false on $M = S^2$, because of the zonal spherical harmonics for example, but Rudnick and Sarnak conjecture that it is true on any compact negatively curved surface [59]. The quantum ergodicity theorem proved by Shnirelman [64], [65], Colin de Verdiére [18], and Zelditch [74] shows that negative curvature implies convergence along a full subsequence of eigenfunctions, or equivalently on average over the eigenfunctions, but there may be many other subsequential limits besides the uniform measure. The stronger property of QUE has been shown for examples of arithmetic origin in work of

Lindenstrauss [49], [50], and Bourgain-Lindenstrauss [14], Jakobson [41], Holowinsky [36], Holowinsky-Soundararajan [35]. Outside these examples, work of Anantharaman [3], Anantharaman-Nonnenmacher [4], Anantharaman-Silberman [5], and Dyatlov-Jin [21] places significant constraints on the measures that arise as quantum limits, but it remains unknown whether the uniform measure is the only possibility.

From this point of view, it is of interest to randomize and see whether one at least has uniform distribution with high probability. VanderKam [69] showed that one does have equidistribution for random spherical harmonics on the sphere, where QUE is known to fail. A more refined question is whether there is equidistribution even if the test set A shrinks as the frequency grows. This scenario has been studied recently in papers of Han [29] (assuming high multiplicity), Han-Tacy [30] (with a spectral window instead of high multiplicity), Granville-Wigman [27] (on an arithmetic torus guaranteeing high multiplicity), Lester-Rudnick [46] (on higher-dimensional tori), Humphries [37] (for non-random functions on arithmetic surfaces, with the averaging being done over the sphere center instead). In particular, Theorem 4.4 from [30] estimates the probability that there is some point with a given deviation, much like our Theorem 1.4.1 but in a slightly different context where $\int_M \phi^2$ is conditioned to be exactly 1 instead of fluctuating near 1. For this theorem, Han and Tacy take $r = T^{-p}$ with p close to $1/2$, whereas we take r equal to T^{-1} up to a logarithmic power. Instead of their shrinking deviation $r^n \lambda^{-\delta}$, we take a fixed $\varepsilon > 0$. Thus the deviations we study are easier, but we are closer to the wave scale $1/T$.

1.3 Nodal domains

The nodal set of f is defined by $\{x \in M \ ; \ f(x) = 0\}$. The connected components of $\{f(x) \neq 0\}$ are called *nodal domains*. In the two-dimensional case, the connected components of $\{f(x) = 0\}$ are called *nodal lines*. According to Courant's nodal domain

theorem, the m'th eigenfunction has at most m nodal domains. As a caveat, note that the nodal set could form a grid, in which case a single nodal line with many singularities bounds all of the nodal domains. There is no lower bound on the number of nodal domains – that is, better-than-constant as the eigenvalue grows – as shown by explicit examples on the torus and sphere discovered by Stern [66]. Stern's results appeared in her thesis and were referred to in Courant-Hilbert but were not easily accessible. Lewy rediscovered her theorems on spherical harmonics roughly fifty years later [48]. See also Bérard and Helffer's discussion of spherical harmonics with few nodal domains [10].

These examples show that the number of nodal domains need not increase as the eigenvalue grows. However, Nazarov-Sodin proved that a random harmonic will have a number of nodal domains on the same order of magnitude as Courant's upper bound, with high probability. To state their results, we denote by $N(f)$ the number of connected components of the zero set $f^{-1}\{0\}$.

Theorem (Nazarov-Sodin) There is a positive constant $a > 0$ such that

$$\lim_{n\to\infty} \frac{\mathbb{E}N(f)}{n^2} = a$$

and, moreover, $N(f)$ exhibits exponential concentration in the sense that for any $\epsilon > 0$, there are positive constants $c(\epsilon)$ and $C(\epsilon)$ such that

$$\mathbf{P}\left\{\left|\frac{N(f)}{n^2} - a\right| > \epsilon\right\} \leq C(\epsilon)e^{-c(\epsilon)n}.$$

This is a difficult theorem because the nodal count $N(f)$ is a global feature: Distant points on the sphere could be part of the same component of the zero set, and correlations at both long and short distances are relevant. This theorem together with considerations of the lengths of the nodal lines (which, in total, amount to at most a constant multiple of n, by deterministic facts about harmonics) shows that most of

the nodal domains of a typical harmonic are small, with diameter comparable to $1/n$. It is not possible to improve this theorem by obtaining a better-than-exponential rate of concentration. Nazarov and Sodin perturb the zonal spherical harmonic (which has only n nodal lines) to give an exponential lower bound on the probability of $N(f)$ being smaller than expected: To any $\delta > 0$, no matter how small, there is a corresponding $C(\delta)$ such that

$$\mathbb{P}\{N(f) < \delta n^2\} \geq e^{-C(\delta)n}.$$

Even though exponential concentration is the correct order here, there is still room for improvement in the exponential rate $c(\epsilon)$. Nazarov and Sodin optimize various parameters appearing in their proof and find that a constant multiple of ϵ^{15} is admissible. Presumably the true rate is faster.

Rozenshein has proved exponential concentration in the setting of a higher-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ [58]. On the torus, the Laplace eigenfunctions are spanned by complex exponentials $e^{2\pi i x\cdot k}$ with $k \in \mathbb{Z}^d$ instead of spherical harmonics. The corresponding eigenvalue is $4\pi^2 L^2$ where $L^2 = k_1^2 + \cdots k_d^2$ is a sum of d integer squares. Let $\mathcal{H}_L$ be the space of eigenfunctions of eigenvalue $4\pi^2 L^2$. Write N_L for the number of connected components of the nodal set of a random element of $\mathcal{H}_L$, and m_L for the median of N_L/L^d.

Theorem (Rozenshein) For any $\epsilon > 0$, there are positive constants $C(\epsilon)$ and $c(\epsilon)$ such that for L a sum of d squares,

$$\mathbb{P}\left\{\left|\frac{N_L}{L^d} - m_L\right|\right\} \leq C(\epsilon)e^{-c(\epsilon)\dim\mathcal{H}_L}.$$

The number of ways of writing a given L^2 as a sum of d squares involves some subtle arithmetic. For $d \geq 5$, the exponential concentration can still be proved with expectation instead of median. For smaller $d = 3$ or 4, one can prove exponential

concentration under some assumptions about L: The prime 2 should divide L at most to a bounded multiplicity, no matter how large L grows. For $d = 2$, there are additional conditions but the theorem still holds for $L \to \infty$ along a density-1 subsequence.

Note that on the sphere S^2, the eigenvalues are $n(n+1)$ with multiplicity $2n+1$, so $\dim \mathcal{H}_L$ on the torus is comparable to n on the sphere. Rozenshein shows that one can take $c(\epsilon) \approx \epsilon^{(d+2)^2-1}$ for the torus, which is consistent with Nazarov and Sodin's ϵ^{15} on the sphere with $d = 2$.

Another direction is to ask more detailed questions about the zero set, whether on S^2 or another space. For example, each nodal domain is a two-dimensional region with some number of holes. What are the statistics of the number of nodal domains with different connectivities? Sarnak and Wigman have proved that there is a universal limiting distribution answering this question [62] and, with Canzani, even more intricate questions about the nesting of nodal domains inside one another [63], [17]. They prove that this is a probability distribution, so that no mass has escaped in the limit, and that each atom has a non-zero mass. Barnett and Jin have conducted a numerical simulation sampling this distribution for a range of connectivities. It would be of great interest to estimate the tails of this distribution. Here we quote the first few values, taken from the announcement [62] by Sarnak and Wigman.

Holes	Limiting proportion
0	0.91171
1	0.05143
2	0.01322
3	0.00628
4	0.00364

Despite the exponential concentration of $N(f)/n^2$ around its mean a, we know very little about its variance. Exponential concentration gives some upper bound on the fluctuations in $N(f)$, but doesn't help us directly bound the variance from above

without some control of $c(\epsilon)$ and $C(\epsilon)$. More important, it seems to be a difficult problem to prove lower bounds on the variance of $N(f)$. Bogomolny and Schmit, based on a percolation model [13], predict that the variance of $N(f)$ grows quadratically with n, as does the mean.

1.4 Outline of this thesis

The new results described in this thesis are related to nodal domains and to equidistribution of random functions. In Chapter 4, we give an explicit version of the barrier argument of Nazarov-Sodin in order to provide a lower bound for the constant in their theorem on the number of nodal lines. (Theorem: $a \geq$ 0.00000000000000000000000 000000000000000000000000000000000000000000000000000000000000001) This lower bound seems to be much smaller than the true value: Numerical simulations by Nastasescu [51] suggest that a is only slightly less than 0.06. The lower bound from the barrier method involves Gaussian tail probabilities and hence leads to a very small numerical value. We also discuss how the barrier lower bound and other bounds for the Nazarov-Sodin constant behave in higher dimensions.

Even though QUE may fail for certain exceptional sequences of spherical harmonics, VanderKam [69] shows that it does hold with probability tending to 1 for ϕ_λ in a randomly chosen orthonormal basis. In Chapter 3, we study a slightly different model for random spherical harmonics. Namely, instead of generating an entire basis at once, we sample from the monochromatic ensemble. We also study this ensemble on an arbitrary compact manifold in Chapter 2.

An important feature of our work is that the target set or "observable" A is not fixed. This aspect where A shrinks as the eigenvalue grows has not been considered until recent papers such as Han [29], Han-Tacy [30], Granville-Wigman [27], Lester-Rudnick [46], Humphries [37]. In particular, Han's Corollary 5 from [29] allows one to

take a ball shrinking at the rate $r = m^{-1/2}$. Our Theorem 1.4.1 accelerates this to, for example, $r = m^{-1}\log(m)^2$.

The choice of variance $1/(2m+1)$ guarantees that if we integrate over a geodesic ball $B_r(z)$,

$$\mathbb{E}\left[\int_{B_r(z)} \phi^2\right] = \frac{\mathrm{vol}(B_r)}{4\pi} = \sin^2(r/2).$$

In expectation, the random measure $\phi^2 d\mathrm{vol}$ thus weights the ball $B_r(z)$ by its volume fraction. For an individual ϕ, there is some deviation from the expected value, and this is our interest. Notice that the expected value is independent of the center z, as it must be since the ensemble is invariant under rotation of S^2.

On the sphere, we consider the random variables

$$X_z = \frac{1}{\mathrm{vol}(B_r)} \int_{B_r(z)} \phi^2,$$

normalized so that $\mathbb{E}\left[X_z\right] = \frac{1}{4\pi}$ is of order 1 for all $r > 0$ and $m \geq 1$. This corresponds to Gaussian coefficients of variance $1/(2m+1)$. The *discrepancy* is

$$D(r,m) = \sup_z |X_z - \mathbb{E}[X_z]| = \sup_z \left| X_z - \frac{1}{4\pi} \right|.$$

Theorem 1.4.1. *If $r \to 0$ and $m \to \infty$ in such a way that*

$$\frac{rm}{\log m} \to \infty,$$

then for any fixed $\epsilon > 0$,

$$\mathbb{P}\{D(r,m) > \epsilon\} \to 0.$$

In fact, the proof we give shows that

$$\mathbb{P}\{D(r,m) > \epsilon\} \leq C(\epsilon) m^2 e^{-c(\epsilon) rm}. \tag{1.4.1}$$

for some positive constants $c(\epsilon)$ and $C(\epsilon)$, with $c(\epsilon)$ on the order of ϵ^2. The hypothesis that $rm/\log m \to \infty$ guarantees that the factor m^2 can be absorbed, no matter how small a value ϵ is given. Thus the discrepancy $D(r, m)$ converges to 0 in probability as long as $rm \to \infty$ asymptotically faster than $\log m$. This means the random measure $\phi^2 d\text{vol}$ is approximately uniform at a scale $r \approx 1/m$, larger than the wave scale $1/m$ by only a slowly growing function. This is a quantum mechanical effect: There is enough mass but it is not being distributed evenly because the Planck scale sets a fundamental limit.

There is a heuristic justification of Theorem 1.4.1 worth keeping in mind during the proof. To accurately sample a polynomial of degree m requires a grid spacing of order $1/m$, and hence roughly m^2 points on S^2. With high probability, the maximum of N independent Gaussians of unit variance is of order $\sqrt{\log N}$. Taking $N \asymp m^2$ and approximating the supremum by a maximum over N points, we thus expect

$$\sup_z |X_z - \mathbb{E}[X_z]]| = \sqrt{\text{var}} \sup_z \left| \frac{X_z - \mathbb{E}}{\sqrt{\text{var}}} \right| \approx \sqrt{\text{var}} \sqrt{\log m}.$$

One of our key estimates is that the variance is of order $1/(rm)$. So the discrepancy should be small when

$$\frac{\log m}{rm} \to 0.$$

More generally, take a compact Riemannian manifold M. The Laplace eigenfunctions $\phi_j : M \to \mathbb{R}$ satisfy

$$\Delta \phi_j + t_j^2 \phi_j = 0.$$

and form an orthonormal basis for $L^2(M)$ with respect to the volume form of g.

The *monochromatic ensemble* takes the form

$$\phi(x) = \sum_{T - \eta(T) \le t_j < T} c_j \phi_j(x) \tag{1.4.2}$$

where the coefficients c_j are independent, identically distributed Gaussian random variables of mean 0. The parameter T is large and corresponds to the degree m from the spherical case.

Consider a ball $B = B_r(z)$ with center $z \in M$ whose radius $r > 0$ is allowed to vary with T. We can normalize so that $\int_B \phi^2$, in expectation, is close to $\mathrm{vol}(B)$. This corresponds to Gaussian coefficients c_j of variance proportional to $1/N$, where N is the number of eigenvalues in the window.

Theorem 1.4.2. *In dimension $n \geq 3$, if $rT/\log(T) \to \infty$ and the spectral window obeys $\eta(T)/\log(T) \to \infty$ and $\eta(T) \lesssim T^{1/2}$, then for any $\varepsilon > 0$,*

$$\mathbb{P}\left\{\sup_z \left| \frac{1}{\mathrm{vol}(B_r(z))} \int_{B_r(z)} |\phi|^2 - \mathbb{E}\left[\frac{1}{\mathrm{vol}(B_r(z))} \int_{B_r(z)} |\phi|^2\right] \right| \geq \varepsilon \right\} \to 0.$$

Assuming further that $rT/\log(T)^2 \to \infty$, the same conclusion holds also in dimension 2.

We have written $|\phi|^2$ instead of ϕ^2 even though the basis functions ϕ_j are real-valued and the coefficients c_j are also real. This is because all of our considerations apply also to complex-valued functions $\phi : M \to \mathbb{C}$, with the basis functions taking complex values and the coefficients c_j having independent Gaussians as their real and imaginary part. We will concentrate on the real-valued case. This is partly for ease of notation, allowing us to write ϕ^2 instead of $|\phi|^2$ or $\phi_j\phi_k$ instead of $\phi_j\overline{\phi_k}$. Also, complex-valued eigenfunctions may equidistribute at a finer scale than their real counterparts. For example, a pure phase e^{iTx} is an eigenfunction on the torus whose associated measure is uniform at all scales since $|e^{ix}|^2 dx = dx$, whereas the real part $\cos(Tx)$ is not uniform below the scale $1/T$.

The same trigonometric example of $\cos(Tx)$ shows that one cannot expect equidistribution at the scale $1/T$, so that the rate $rT/\log(T) \to \infty$ is optimal as far as the exponent on T goes. The other novel feature to be emphasized in Theorem 1.4.2 is that

we take a supremum over z. Thus, with high probability, ϕ is uniformly distributed everywhere at once, and at almost the optimal scale. We imagine something like a quantum coupon collector. In the coupon collector problem, coupons are taken at random and one asks for the expected number of trials until the collector has at least one coupon of each type. More stringently, how many trials are needed in order to have a high chance of sampling each type of coupon close to its proportional number of times? Theorem 1.4.2 is along similar lines, but one asks what regime of r and T leads to a high chance of the random measure $\phi^2 d\,\mathrm{vol}$ assigning roughly the right mass to every ball.

Let us elaborate on this elementary example, because our main theorem makes similar use of the union bound and Chernoff bound. Suppose n balls are independently thrown at random into k boxes (or, equivalently, n trials are taken to collect coupons of k types). On average, each box receives an equal share n/k of the boxes. What is the probability that some box receives more or less than expected by at least $\varepsilon n/k$? One could find all such outcomes combinatorially, but the union bound gives an easy estimate from above. Let b_i be the number of balls in box i. The union bound is

$$\mathbb{P}\left(\exists\ i\ \left|b_i - \frac{n}{k}\right| \geq \varepsilon\frac{n}{k}\right) \leq k\mathbb{P}\left(\left|b_i - \frac{n}{k}\right| > \varepsilon\frac{n}{k}\right)$$

which sacrifices a factor of k in exchange for reducing the problem to a calculation with a single box. Let $p = 1/k$ be the probability of any single ball landing in box i. Write the number b_i as a sum of terms Y_j, each indicating whether one of the balls lands in box i. Then we can calculate the moment generating function of b_i by independence: For any s,

$$\mathbb{E}\left[e^{sb_i}\right] = \prod_{j=1}^{n} \mathbb{E}\left[e^{sY_j}\right] = (pe^s + 1 - p)^n$$

since e^{sY_j} is e^s with probablity p or else e^0 in the complementary event that $Y_j = 0$.

The Chernoff bound on the upper tail probability is

$$\mathbb{P}(b_i \geq (1+\varepsilon)np) \leq \mathbb{E}\left[e^{sb_i}\right] \exp(-snp(1+\varepsilon)).$$

The parameter s is at our disposal. Choosing $s = 0$ would give the trivial bound and one hopes to get an improved bound from a better s. Guided by calculus and the explicit form of $\mathbb{E}[\exp(sb_i)]$, we take $s = \log(1+\varepsilon)$ to obtain

$$\begin{aligned}
\mathbb{P}(b_i \geq (1+\varepsilon)np) &\leq (1+p(e^s-1))^n \exp(-snp(1+\varepsilon)) \\
&= \exp(n\log(1+p(e^s-1)) - snp(1+\varepsilon)) \\
&= \exp\left(-np(1+\varepsilon)\log(1+\varepsilon) + n\log(1+\varepsilon p)\right)
\end{aligned}$$

We are interested in the case that $p \to 0$ as $n \to \infty$, so that the exponent is

$$-np(1+\varepsilon)\log(1+\varepsilon) + n\log(1+\varepsilon p) \sim -np(1+\varepsilon)\log(1+\varepsilon) + np\varepsilon.$$

The result of the Chernoff bound is

$$\mathbb{P}(b_i \geq (1+\varepsilon)np) \leq \exp(-npc(\varepsilon))$$

where $c(\varepsilon)$ is a positive constant, for any given ε. Similar considerations apply to the lower tail where $b_i \leq (1-\varepsilon)np$. Taking $p = 1/k$ and combining these tail bounds with the union bound, we have

$$\mathbb{P}\left(\exists\ i\ \left|b_i - \frac{n}{k}\right| \geq \varepsilon\frac{n}{k}\right) \leq k\exp(-c(\varepsilon)n/k).$$

This shows that, as long as

$$\frac{n}{k\log k} \to \infty,$$

there is only a vanishingly small probability of some box having an ε-deviation from the expected number of balls, no matter how small the given $\varepsilon > 0$. If n were only of size k, it would be just barely possible to have a ball in every box, and not likely to occur by random chance. The argument just given shows that if n is only slightly larger than k, up to this logarithmic factor, then random chance will not only put a ball in every box, but do so close to the expected number of times. In Chapters 2 and 3, we will apply the same "union plus Chernoff" approach to the local integrals $\int_B \phi^2$ with varying center and random ϕ.

Chapter 2

Statistics of local integrals

By a *local integral*, we mean

$$\int_B |\phi|^2$$

where ϕ is a function on some space and B is a subset of that space (typically a small one if we are to have a truly "local" integral). The statistics of these integrals depend on the random ensemble from which ϕ is drawn. We have a specific ensemble in mind – namely the monochromatic ensemble from the introduction – but first state a formula that gives correlations between such integrals in some generality.

Lemma 2.0.3. *Suppose c_j are independent random variables with first and third moments 0, variance σ^2, and fourth moment $3\sigma^4$. Suppose $\phi_j : M \to \mathbb{C}$ are functions on some measure space M, and $\phi = \sum_j c_j \phi_j$ is the corresponding random function. Then for any measurable subsets $B \subseteq M$, $B' \subseteq M$,*

$$\operatorname{cov}\left[\int_B |\phi|^2, \int_{B'} |\phi|^2\right] = 2\sigma^4 \int_B \int_{B'} K(x,x')^2 dx dx' \tag{2.0.1}$$

where $K(x,x') = \sum_j \phi_j(x)\overline{\phi_j(x')}$.

Proof. We compute the covariance $\mathbb{E}[\int_B |\phi|^2 \int_{B'} |\phi|^2] - \mathbb{E}[\int_B |\phi|^2]\mathbb{E}[\int_{B'} |\phi|^2]$ by expanding $|\phi|^2$ and using linearity of expectation to exchange $\mathbb{E}$ with the sums and integrals. For the expectation of the product, we have

$$\mathbb{E}\left[\int_B \phi^2 \int_{B'} \phi^2\right] = \int_B \int_{B'} \sum_i \sum_j \sum_k \sum_l \phi_i(x)\overline{\phi_j(x)}\phi_k(x')\overline{\phi_l(x')}\mathbb{E}[c_i c_j c_k c_l] dx dx'.$$

Since the coefficients are independent and have mean 0, the expectation $\mathbb{E}[c_i c_j c_k c_l]$ is $3\sigma^4$ if all indices i, j, k, and l are equal, σ^4 if they are equal in pairs, and 0 in all other cases. In light of the different cases $i = j \neq k = l$, $i = k \neq j = l$, or $i = l \neq j = k$, it

follows that

$$\mathbb{E}\left[\int_B |\phi|^2 \int_{B'} |\phi|^2\right]$$
$$= \sigma^4 \left(3\sum_i |\phi_i(x)|^2|\phi_i(x')|^2 + \sum_{i\neq k} |\phi_i(x)|^2|\phi_k(x')|^2 + 2\sum_{i\neq j} \phi_i(x)\overline{\phi_i(x')}\phi_j(x')\overline{\phi_j(x')}\right)$$

The factor of 3 means that the first term exactly supplies the missing diagonal terms $i = k$, $i = j$, and $i = l$ (which we have merged with $i = k$, the two cases giving the same contribution) in the three other sums. The completed sums then factor, so that

$$\mathbb{E}\left[|\phi(x)|^2|\phi(x')|^2\right] = \sigma^4 \left(\sum_i |\phi_i(x)|^2 \sum_k |\phi_k(x')|^2 + 2\sum_i \phi_i(x)\overline{\phi_i(x')} \sum_j \phi_j(x)\overline{\phi_j(x')}\right)$$
$$= \sigma^4 \left(K(x,x)K(x',x') + 2K(x,x')^2\right)$$

For the product of the expectations, we have

$$\mathbb{E}\left[\int_B |\phi|^2\right] = \sum_i \sum_j \mathbb{E}[c_i c_j] \int_B \phi_i \overline{\phi_j} = \sigma^2 \int_B K(x,x)dx$$

by independence of the coefficients. Thus subtraction gives

$$\operatorname{cov}\left[\int_B |\phi|^2, \int_{B'} |\phi|^2\right] = \mathbb{E}\left[\int_B |\phi|^2 \int_{B'} |\phi|^2\right] - \mathbb{E}\left[\int_B |\phi|^2\right] \mathbb{E}\left[\int_{B'} |\phi|^2\right]$$
$$= \sigma^4 \int_B \int_{B'} \left(K(x,x)K(x',x') + 2K(x,x')^2\right) - \sigma^4 \int_B \int_{B'} K(x,x)K(x',x')dxdx'$$
$$= 2\sigma^4 \int_B \int_{B'} K(x,x')^2 dxdx'$$

which is (2.0.1). □

The motivating example for this lemma is that $\mathbb{E}[\mathfrak{z}^4] = 3\mathbb{E}[\mathfrak{z}^2]^2$ if $\mathfrak{z}$ is a Gaussian of mean 0. The variance formula (2.0.1) holds equally well for coefficients following any probability distribution whose first, second, and fourth moments share this property.

In the Gaussian case, we can calculate even further. Let us give another proof of Lemma 3.3.1, or at least the case $B = B'$ thereof, that also applies to higher moments.

2.1 Moments of a quadratic form in Gaussians

Our local integrals $\int \phi^2$ are given by a quadratic form in Gaussian random variables:

$$\frac{1}{\mathrm{vol}(B)} \int_B |\phi|^2 = \sum_j \sum_k c_j c_k \frac{1}{\mathrm{vol}(B)} \int_B \phi_j \overline{\phi_k} = \mathfrak{z}^T A \mathfrak{z}$$

where $\mathfrak{z}$ is a vector of Gaussians $z_j = c_j/\sigma$ of variance 1, and the matrix A has entries

$$A_{jk} = \frac{\sigma^2}{\mathrm{vol}(B)} \int_B \phi_j \overline{\phi_k}$$

If we diagonalize A so that $A = U^T \Lambda U$, with U an orthogonal matrix, then the vector $y = U\mathfrak{z}$ in eigencoordinates will also follow the law $N(0, I_N)$. Then we have

$$\mathfrak{z}^T A \mathfrak{z} = \sum_j \lambda_j y_j^2.$$

In particular, since the Gaussians y_j have variance 1, the expected value of a quadratic form in Gaussians is the trace:

$$\mathbb{E}[\mathfrak{z}^T A \mathfrak{z}] = \lambda_1 + \ldots + \lambda_M = \mathrm{tr}(A).$$

We can also find the higher moments. From diagonalizing the form as above and evaluating a Gaussian integral, it follows that the moment generating function of $\mathfrak{z}^T A \mathfrak{z}$ is

$$g(s) = \mathbb{E}\left[e^{s\mathfrak{z}^T A \mathfrak{z}}\right] = \prod_{j=1}^{M} (1 - 2s\lambda_j)^{-1/2} \tag{2.1.1}$$

Indeed, by independence, we can factor $g(s)$:

$$g(s) = \mathbb{E}\left[e^{\sum_j s\lambda_j y_j^2}\right] = \prod_j \mathbb{E}\left[e^{s\lambda_j y_j^2}\right].$$

Each term is a Gaussian integral:

$$\mathbb{E}\left[e^{s\lambda y^2}\right] = \int_{-\infty}^{\infty} e^{s\lambda y^2} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} = (1-2\lambda s)^{-1/2}.$$

We can find the moments by differentiating $g(s)$ at $s = 0$:

$$g^{(k)}(0) = \mathbb{E}\left[(\mathfrak{z}^T A \mathfrak{z})^k\right].$$

Our interest is more in the central moments, which are gotten from

$$g_0(s) = \mathbb{E}\left[e^{s\left(\mathfrak{z}^T A \mathfrak{z} - \mathbb{E}[\mathfrak{z}^T A \mathfrak{z}]\right)}\right]$$

via

$$g_0^{(k)}(0) = \mathbb{E}\left[\left(\mathfrak{z}^T A \mathfrak{z} - \mathbb{E}[\mathfrak{z}^T A \mathfrak{z}]\right)^k\right]$$

Note that $g(0) = g_0(0) = 1$. Logarithmic differentiation of (2.6.1) shows that

$$g'(s) = g(s) \sum_{j=1}^{M} \frac{\lambda_j}{1-2s\lambda_j}$$
$$g_0'(s) = g_0(s) \sum_{j=1}^{M} \lambda_j \left(\frac{1}{1-2s\lambda_j} - 1\right)$$

so an important role is played by the function

$$h(s) = \sum_{j=1}^{M} \frac{\lambda_j}{1-2s\lambda_j} = \operatorname{tr} A(I-2sA)^{-1}$$

and the first moment is simply $\operatorname{tr} A$, as we have seen before. Repeated use of the

product rule shows that

$$g^{(k+1)}(s) = \sum_{l=0}^{k} \binom{k}{l} g^{(k-l)}(s) h^{(l)}(s).$$

The derivatives of h are elementary:

$$h^{(l)}(s) = \sum_{j=1}^{M} \frac{\lambda_j (2\lambda_j)^l l!}{(1 - 2s\lambda_j)^{l+1}}.$$

Evaluating at $s = 0$ gives a recursive formula for the moment of order $k+1$ in terms of the previous k moments:

$$g^{(k+1)}(0) = \sum_{l=0}^{k} \binom{k}{l} 2^l l! \operatorname{tr} A^{l+1} g^{(k-l)}(0)$$

For example, taking $k = 1$ gives another way to get at the variance:

$$\begin{aligned} g^{(2)}(0) &= \operatorname{tr}(A) g^{(1)}(0) + 2 \operatorname{tr}(A^2) g^{(0)}(0) = \operatorname{tr}(A)^2 + 2 \operatorname{tr}(A^2) \\ \operatorname{var}[\mathfrak{Z}^T A \mathfrak{Z}] &= g^{(2)}(0) - g^{(1)}(0)^2 = 2 \operatorname{tr}(A^2) \end{aligned}$$

To compare this with Lemma 2.0.3, we must relate the matrix A and the kernel K. Recall that

$$K(x, x') = \sum_j \phi_j(x) \overline{\phi_j(x')}$$

The kernel determines the traces of powers of A. Indeed, since the (j,k)-entry of A is

$$A_{jk} = \frac{1}{\operatorname{vol}(B)} \int_B \phi_j \overline{\phi_k},$$

the entries of A^p are

$$A^{(p)}_{jk} = \frac{1}{\operatorname{vol}(B)^p} \sum_{k_1} \cdots \sum_{k_{p-1}} \int_B \phi_j \overline{\phi_{k_1}} \int_B \phi_{k_1} \overline{\phi_{k_2}} \cdots \int_B \phi_{k_{p-2}} \overline{\phi_{k_{p-1}}} \int_B \phi_{k_{p-1}} \overline{\phi_k}.$$

When we sum the diagonal entries, we get

$$\begin{aligned}\operatorname{tr}(A^p) &= \sum_j A_{jj}^{(p)} \\ &= \operatorname{vol}(B)^{-p} \sum_j \sum_{k_1} \cdots \sum_{k_{p-1}} \int_B \phi_j \overline{\phi_{k_1}} \int_B \phi_{k_1} \overline{\phi_{k_2}} \cdots \int_B \phi_{k_{p-2}} \overline{\phi_{k_{p-1}}} \int_B \phi_{k_{p-1}} \overline{\phi_j}\end{aligned}$$

As in the proof of Lemma 2.0.3, we express this as a multiple integral:

$$\begin{aligned}\operatorname{tr}(A^p) = \operatorname{vol}(B)^{-p} &\int_B dx_1 \cdots \int_B dx_p \\ &\sum_j \sum_{k_1} \cdots \sum_{k_{p-1}} \phi_j(x_1)\overline{\phi_{k_1}(x_1)}\phi_{k_1}(x_2)\overline{\phi_{k_2}(x_2)} \ldots \phi_{k_{p-2}}(x_{p-1})\overline{\phi_{k_{p-1}}(x_{p-1})}\phi_{k_{p-1}}(x_p)\overline{\phi_j(x_p)}\end{aligned}$$

The integrand factors:

$$\begin{aligned}&\sum_j \sum_{k_1} \cdots \sum_{k_{p-1}} \phi_j(x_1)\overline{\phi_{k_1}(x_1)}\phi_{k_1}(x_2)\overline{\phi_{k_2}(x_2)} \ldots \phi_{k_{p-2}}(x_{p-1})\overline{\phi_{k_{p-1}}(x_{p-1})}\phi_{k_{p-1}}(x_p)\overline{\phi_j(x_p)} \\ &= \sum_j \phi_j(x_1)\overline{\phi_j(x_p)} \sum_{k_1} \overline{\phi_{k_1}(x_1)}\phi_{k_1}(x_2) \cdots \sum_{k_{p-1}} \overline{\phi_{k_{p-1}}(x_{p-1})}\phi_{k_{p-1}}(x_p) \\ &= K(x_1, x_p)K(x_2, x_1) \cdots K(x_p, x_{p-1})\end{aligned}$$

We summarize this as follows:

Lemma 2.1.1. *If A is the matrix with entries*

$$A_{jk} = \frac{1}{\operatorname{vol}(B)} \int_B \phi_j \overline{\phi_k}$$

and K is the kernel given by

$$K(x, x') = \sum_j \phi_j(x)\overline{\phi_j(x')}$$

then

$$\operatorname{tr}(A^p) = \frac{1}{\operatorname{vol}(B)^p} \int_B \cdots \int_B \prod_{j=1}^{p} K(x_j, x_{j-1})\ dx_1 \dots dx_p$$

with the indices interpreted cyclically so that x_0 means x_p.

In particular, with $p = 2$, we have

$$\operatorname{tr}(A^2) = \frac{1}{\operatorname{vol}(B)^2} \int_B dx_1 \int_B dx_2 |K(x_1, x_2)|^2$$

Thus $2\operatorname{tr}(A^2)$, the variance obtained from this method, agrees with Lemma 2.0.3.

When we continue to $p = 3$, we find that

$$g^{(3)}(0) = 8\operatorname{tr}(A^3) + 6\operatorname{tr}(A^2)\operatorname{tr}(A) + \operatorname{tr}(A)^3.$$

Passing to the central moment, most of the terms cancel out:

$$\mathbb{E}\left[\left(\mathfrak{z}^T A \mathfrak{z} - \mathbb{E}[\mathfrak{z}^T A \mathfrak{z}]\right)^3\right] = g^{(3)}(0) - 3g^{(1)}(0)g^{(2)}(0) + 2g^{(1)}(0)^3 = 8\operatorname{tr}(A^3).$$

The central moments $g_0^{(k)}(0)$ are given by the recursion

$$g_0^{(k+1)}(0) = \sum_{l=1}^{k} \binom{k}{l} 2^l l! \operatorname{tr}(A^{l+1}) g_0^{(k-l)}(0)$$

by the same process as the non-central moments except with the term $l = 0$ omitted so that the average is 0. This shows that the fourth central moment is

$$g_0^{(4)}(0) = 2^3 3! \operatorname{tr}(A^4) + 6\operatorname{tr}(A^2) g_0^{(2)}(0) = 48\operatorname{tr}(A^4) + 12\operatorname{tr}(A^2)^2.$$

The fifth one is

$$g_0^{(5)}(0) = 2^4 4! \operatorname{tr}(A^5) + 160\operatorname{tr}(A^3)\operatorname{tr}(A^2)$$

and the sixth is

$$g_0^{(6)}(0) = 2^5 5! \operatorname{tr}(A^6) + 1440 \operatorname{tr}(A^4)\operatorname{tr}(A^2) + 640 \operatorname{tr}(A^3)^2 + 120 \operatorname{tr}(A^2)^3.$$

We normalize by the standard deviation:

$$\sigma = \sqrt{\mathbb{E}\left[(\mathfrak{z}^T A \mathfrak{z} - \mathbb{E}[\mathfrak{z}^T A \mathfrak{z}])^2\right]} = \sqrt{2 \operatorname{tr}(A^2)}$$
$$Z = \frac{\mathfrak{z}^T A \mathfrak{z} - \mathbb{E}[\mathfrak{z}^T A \mathfrak{z}]}{\sigma}$$

Then we have

$$\mathbb{E}[Z^3] = \sqrt{8} \frac{\operatorname{tr}(A^3)}{\operatorname{tr}(A^2)^{3/2}}$$
$$\mathbb{E}[Z^4] = 3 + 12 \frac{\operatorname{tr}(A^4)}{\operatorname{tr}(A^2)^2}$$

Note also that the last term in $g_0^{(6)}(0)$, namely $120 \operatorname{tr}(A^2)^3$, yields 15 when divided by σ^6. The numbers 3 and 15 are the fourth and sixth moments of a standard Gaussian, while the odd moments vanish. This suggests that Z follows the normal distribution to some approximation. For the two-dimensional sphere, we will confirm this in chapter 3.

There is a combinatorial way to approach the higher moments, along the lines of the calculation in lemma 2.0.3. To compute $\mathbb{E}[X^p]$, first raise X to a power

$$\begin{aligned}\left(\int_B \phi^2\right)^p &= \int \dots \int \phi(x_1)^2 \cdots \phi(x_p)^2 \\ &= \int \dots \int \sum_{j_1} \sum_{k_1} \cdots \sum_{j_p} \sum_{k_p} c_{j_1} c_{k_1} \dots c_{j_p} c_{k_p} \phi_{j_1}(x_1)\phi_{k_1}(x_1) \cdots \phi_{j_p}(x_p)\phi_{k_p}(x_p).\end{aligned}$$

When we take expectation, since the odd moments $\mathbb{E}[c_j^{2a+1}]$ are 0 and the coefficients

are independent, many terms drop out:

$$\mathbb{E}[c_{j_1} c_{k_1} \dots c_{j_p} c_{k_p}] = 0$$

unless the indices can be matched up into non-zero combinations such as $\mathbb{E}[c_a^2]^3 \mathbb{E}[c_b^4] \dots$. All the nonzero terms correspond to partitions of $2p$ into even parts. For example, $p = 2$ gives the variance, and we saw that for a nonzero contribution, the indices must either be equal in pairs or else all equal. These terms come from the partitions $4 = 4$ and $4 = 2 + 2$. For higher p, it remains possible to enumerate the partitions of $2p$ and weight them according to the Gaussian moments that appear. This is similar to the theorem of Isserlis [40], rediscovered by Wick [72], which expresses higher Gaussian moments $\mathbb{E}[X_1 X_2 \dots]$ given the covariance matrix of moments $\mathbb{E}[X_i X_j]$.

2.2 The monochromatic ensemble

Consider a compact manifold M together with a Riemannian metric g. The corresponding Laplace operator is given in coordinates by

$$-\Delta_g \phi = \frac{1}{\sqrt{\det(g)}} \sum_{i=1}^{n} \partial_i \left(\sum_{j=1}^{n} g^{ij} \sqrt{\det(g)} \partial_j \phi \right) \tag{2.2.1}$$

where g^{ij} are the entries of the inverse of the metric g as a matrix at each point, and ∂_i is the derivative with respect to the coordinate x^i. By compactness, the spectrum of the Laplacian is a discrete sequence of eigenvalues $0 = t_0^2 \leq t_1^2 \leq t_2^2 \leq \dots \to \infty$, possibly with multiplicity. The corresponding eigenfunctions $\phi_j : M \to \mathbb{C}$ satisfy

$$\Delta \phi_j + t_j^2 \phi_j = 0.$$

These eigenfunctions form an orthonormal basis for $L^2(M)$, the L^2 space with respect to integration against the volume form of g. Thus one can expand functions in terms of the Laplace eigenfunctions, and a natural model for a random function on M is to randomize the coefficients in such an expansion. The *monochromatic ensemble* takes the specific form

$$\phi(x) = \sum_{T-\eta(T)\leq t_j<T} c_j \phi_j(x) \tag{2.2.2}$$

where the coefficients c_j are independent, identically distributed Gaussian random variables of mean 0. The parameter T is large. The window $\eta(T)$ may also be large but such that $\eta(T)/T \to 0$. Then all of the frequencies t_j in the sum are approximately T, hence the name "monochromatic".

In terms of the general notation in Lemma 2.0.3 above, the space M is a compact manifold, the basis functions ϕ_j are Laplace eigenfunctons with eigenvalues in a short interval, and the kernel is

$$K(x,x') = \sum_{T-\eta(T)<t_j<T} \phi_j(x)\phi_j(x').$$

We consider random variables

$$X_z = \frac{1}{\mathrm{vol}(B_r(z))} \int_{B_r(z)} \phi^2 d\,\mathrm{vol} \tag{2.2.3}$$

as z varies over the surface. In order to use (2.0.1) to show that the variance of X_z vanishes in the limit, we need more information about the kernel $K(x,x')$. This is given to us by semiclassical analysis. We have

$$\mathbb{E}[\phi(x)\overline{\phi(x')}] = \sum_j \sum_k \phi_j(x)\overline{\phi_k(x')}\mathbb{E}[c_j c_k] = \sigma^2 K(x,x'). \tag{2.2.4}$$

A natural normalization is to require

$$\mathbb{E}\left[\frac{1}{\mathrm{vol}(M)}\int_M |\phi|^2\right] = 1.$$

To arrange this, the variance of the coefficients must be

$$\sigma^2 = \frac{\mathrm{vol}(M)}{\int_M K(x,x)dx} = \frac{\mathrm{vol}(M)}{\sum \int_M |\phi_j|^2}$$

The basis functions are orthonormal in $L^2(M)$, so the denominator is just the number of eigenvalues in the interval, say N:

$$\sum_j \int_M \phi_j^2 = \#\{j \ ; \ T - \eta(T) \leq t_j \leq T\} = N.$$

Thus we choose the variance of the coefficients to be

$$\sigma^2 = \mathrm{var}[c] = \frac{\mathrm{vol}(M)}{N} \asymp N^{-1}.$$

For other sets $B \subseteq M$, we then have

$$\mathbb{E}\left[\int_B |\phi|^2\right] = \sigma^2 \int_B K(x,x)dx = \mathrm{vol}(B)\frac{\int_B K(x,x)dx/\,\mathrm{vol}(B)}{\int_M K(x,x)dx/\,\mathrm{vol}(M)}$$

In the homogeneous case, $K(x,x)$ is independent of x and the expectation is simply $\mathrm{vol}(B)$. In general, it is never very far from $\mathrm{vol}(B)$, as we will see from Weyl's law:

$$\sigma^2 \int_B K(x,x)dx = \mathrm{vol}(B)\sigma^2\left(\frac{N}{\mathrm{vol}(M)} + O(T^{n-1})\right) = \mathrm{vol}(B)\left(1 + O\big(\eta^{-1}\big)\right)$$

2.3 Input from semiclassics

To estimate the variance and other moments, we need to know the size of $K(x, x')$. Here is the basic estimate:

Claim 1. *On a compact manifold of dimension n, with spectral kernel*

$$K(x, x') = \sum_{T-\eta<t_j\leq T} \phi_j(x)\overline{\phi_j(x')}$$

defined over a window $\eta(T) \to \infty$ growing arbitrarily slowly and such that $\eta(T) \lesssim T^{1/2}$, we have

$$K(x, x') \lesssim T^{n-1}\eta(T)$$

for all x, x' and an improved bound for well-separated pairs:

$$K(x, x') \lesssim T^{n-1}\eta\left((Td(x, x'))^{-(n-1)/2} + \eta^{-1}\right) \tag{2.3.1}$$

improving on the trivial bound once $d(x, x') > 1/T$.

The basis for claim 1 is Hörmander's Theorem 4.4 from [36], which implies

$$\sum_{t_j\leq T} \phi_j(x)\phi_j(y) = \frac{T^n}{(2\pi)^n} \int_{|\xi|_g<1} e^{i\langle \exp_y^{-1}(x),\xi\rangle_g} \frac{d\xi}{\sqrt{|g_y|}} + O\left(T^{n-1}\right) \tag{2.3.2}$$

where the error term is uniform over pairs (x, y) with $d(x, y) < r_0/T$ for any r_0. This in turn is based on Lax's parametrix for the wave equation, constructed in [45]. This implies Weyl's law in the form

$$K(x, x') = c\left(T^n - (T - \eta(T))^n\right) \frac{J_{n/2-1}(Td(x, x'))}{(Td(x, x'))^{n/2-1}} + O\left(T^{n-1}\right)$$

To subtract the sum up to $T - \eta(T)$ from the sum up to T, we assume that $\eta(T) \lesssim T^{1/2}$. This allows us to absorb the higher terms in $(T - \eta(T))^n$ into the error

$O(T^{n-1})$ already present:

$$T^n - (T-\eta)^n = nT^{n-1}\eta + O\left(T^{n-2}\eta^2\right) = nT^{n-1}\eta + O\left(T^{n-1}\right)$$

$$K(x,y) = nT^{n-1}\eta(T)\int_{|\xi|_g<1} e^{i\langle \exp_y^{-1}(x),\xi\rangle_g}\frac{d\xi}{\sqrt{|g_y|}} + O\left(T^{n-1}\right)$$

$$K(x,y) = cT^{n-1}\eta(T)\frac{J_{n/2-1}(Td(x,x'))}{(Td(x,x'))^{n/2-1}} + O\left(T^{n-1}\right)$$

This gives a trivial bound

$$K(x,y) \lesssim T^{n-1}\eta(T)$$

which is useful for nearby pairs (x,y) but can be improved by incorporating cancellation in the integral for larger separations. From $J_\nu(u) \lesssim u^{-1/2}$, we see that

$$\begin{aligned} K(x,y) &\lesssim T^{n-1}\eta(T)(Td(x,y))^{-n/2+1-1/2} + T^{n-1} \\ &\lesssim T^{n-1}\eta(T)\left((Td(x,y))^{-(n-1)/2} + \eta^{-1}\right) \end{aligned}$$

2.4 Upper bound on the variance

Lemma 2.4.1. *For any point z in a Riemannian manifold of dimension n, if the random variable X_z is defined in (2.2.3) as above with radius obeying $rT \to \infty$ and spectral window $\eta \le T^{1/2}$, then*

$$\operatorname{var}[X_z] \lesssim \left((rT)^{-(n-1)/2} + \eta^{-1}\right)^2 \tag{2.4.1}$$

In particular, if $\eta^{-1}(rT)^{(n-1)/2}$ is bounded, then the variance is bouned by $(rT)^{-(n-1)}$. For windows shorter than $(rT)^{(n-1)/2}$, the bound becomes η^{-2}. The same estimates apply to $\operatorname{tr}(A^2)$ or $\sum \lambda_j^2$ because the variance is simply $2\operatorname{tr}(A^2)$.

Proof. The variance is given by equation (2.0.1) in terms of

$$\int_{B_r(z)} \int_{B_r(z)} K(x,x')^2 dx dx'.$$

By the triangle inequality, $d(x,x') \leq d(x,z) + d(z,x') < 2r$. Since the integrand is nonnegative, we can bound the inner integral by

$$\int_{B_r(z)} K(x,x')^2 dx \leq \int_{B_{2r}(x')} K(x,x')^2 dx.$$

To integrate over a ball, it is natural to introduce polar coordinates with respect to the center. The radial coordinate $\rho = d(x,x')$ ranges from 0 to $2r$. The volume form is given approximately by

$$d\,\mathrm{vol}(x) = (1 + O(\rho^2))\rho^{n-1} d\rho d\omega. \tag{2.4.2}$$

Indeed, the volume form is obtained from the metric g by $\sqrt{\det(g)}$ and we have the expansion

$$\sqrt{\det(g)} = 1 - \frac{1}{6}\mathrm{Ric}_{kl}(x')x^k x^l + O(|x|^3) = 1 + O(\rho^{-2}).$$

We integrate the estimate

$$K(x,x') \lesssim T^{n-1}\eta\left((T\rho)^{-(n-1)/2} + \eta^{-1}\right).$$

This diverges as $\rho \to 0$, since we would be better off using the trivial bound for

$\rho < 1/T$, but the singularity is integrable. We obtain

$$\begin{aligned}\int_{B_r(z)} K(x,x')^2 dx' &\lesssim (T^{n-1}\eta)^2 \int_0^{2r} \left((T\rho)^{-(n-1)/2} + \eta^{-1}\right)^2 \rho^{n-1} d\rho \\ &\lesssim T^{2n-2}\eta^2 r^n \left((rT)^{-(n-1)} + \eta^{-1}(rT)^{-(n-1)/2} + \eta^{-2}\right) \\ &\lesssim T^{2n-2}\eta^2 r^n \left((rT)^{-(n-1)/2} + \eta^{-1}\right)^2\end{aligned}$$

Integrating over x and noting that $\mathrm{vol}(B_r) \asymp r^n$, we obtain

$$\int_B \int_B K(x,x')^2 dx' dx \lesssim \mathrm{vol}(B)^2 \left(T^{n-1}\eta\right)^2 \left((rT)^{-(n-1)/2} + \eta^{-1}\right)^2$$

Recall that we have normalized to have Gaussian coefficients of variance proportional to $T^{n-1}\eta$. Thus this factor will cancel, leaving

$$\mathrm{var}\left[\frac{1}{\mathrm{vol}(B)} \int_B |\phi|^2\right] = \frac{\sigma^4}{\mathrm{vol}(B)^2} \int_B \int_B K^2 \lesssim \left((rT)^{-(n-1)/2} + \eta^{-1}\right)^2$$

as claimed in Lemma 2.4.1. □

Note that replacing K with its maximum gives

$$\int_B \int_B K(x,x')^2 dx' dx \lesssim \mathrm{vol}(B)^2 \left(T^{n-1}\eta\right)^2.$$

This trivial bound would only show the variance is bounded, whereas the calculation above shows that the variance vanishes as long as $rT \to \infty$ and $\eta \to \infty$.

2.5 Union bound

We assume that $rT \to \infty$ and $\eta \to \infty$ so that, for each fixed center z, $\mathrm{var}[X_z] \to 0$. Then Chebyshev guarantees that $\mathbb{P}\{|X_z - \mathbb{E}[X_z]| > \varepsilon\} \lesssim \varepsilon^{-2}(rT)^{-1} \to 0$ for any given $\varepsilon > 0$, so that each random variable X_z is at least somewhat concentrated. What

is the largest deviation over all $z \in M$? Our goal is to show that on any compact manifold of dimension n,

$$\mathbb{P}\{\sup_{z\in M} |X_z - \mathbb{E}[X_z]| > \varepsilon\} \to 0 \tag{2.5.1}$$

We write the random variable of interest as

$$X_z = \frac{1}{\mathrm{vol}(B_r(z))} \int_{B_r(z)} |\phi|^2. \tag{2.5.2}$$

It has expectation $\mathbb{E}[X_z] = 1 + O(\eta^{-1})$ of order 1. The key point is that for a monochromatic wave ϕ of frequency T, the modulus of continuity at scale $1/T$ is under control. This allows one to replace the supremum over all $z \in M$ by a maximum over roughly T^n sample points, where $n = \dim(M)$. The union bound is that for a finite number of points

$$\mathbb{P}\{|X_z - \mathbb{E}X_z| > \varepsilon \text{ for some } z\} \leq (\text{number of points}) \max_z \mathbb{P}\{|X_z - \mathbb{E}X_z| > \varepsilon\}. \tag{2.5.3}$$

For our application, the number of points is proportional to T^n. There will be only a $o(1)$ probability of there being *some* point z at which a deviation of ε occurs, provided the probability of a deviation at any *single* point z is $o(T^{-n})$. Thus the union bound reduces the problem to a calculation at a single point. This calculation can be done by a Chernoff bound.

Passing to the grid brings with it another error: Conceivably the integrals around all the gridpoints are within ε of their average, but nevertheless the integral around some point off the grid differs considerably. The probability of such an "off-grid" error is very small, because ϕ is unlikely to oscillate so quickly at scale $1/T$.

To be more precise, suppose there is a point z such that

$$|X_z - \mathbb{E}[X_z]| > \varepsilon.$$

Take a grid of points z_j such that every point of M is within $1/T$ of a gridpoint. The number of gridpoints is thus of order T^n. We have

$$\varepsilon < |X_z - X_{z_j}| + |X_{z_j} - \mathbb{E}[X_{z_j}]| + |\mathbb{E}[X_{z_j}] - \mathbb{E}[X_z]|$$

Thus one of the three terms must be greater than $\varepsilon/3$. The difference of expected values is non-random and small: Both are $1 + O(\eta^{-1})$, so their difference is $O(\eta^{-1})$. Eventually, this will not be greater than $\varepsilon/3$ since we assume $\eta(T) \to \infty$. Alternatively, note that

$$\begin{aligned} |\mathbb{E}[X_{z_j} - \mathbb{E}[X_z]| &= \sigma^2 \left| \frac{1}{\mathrm{vol}(B_r(z))} \int_{B_r(z)} K(x,x)dx - \frac{1}{\mathrm{vol}(B_r(z_j)} \int_{B_r(z_j)} K(x,x)dx \right| \\ &\lesssim \frac{\mathrm{vol}(B_r(z) \Delta B_r(z))}{\mathrm{vol}(B_r)} \end{aligned}$$

To bound the volume of the symmetric difference, note that

Claim 2. *If $B_r(z)$ and $B_r(z')$ are balls of radius $r \to 0$ centered at points z, z' separated by less than r in a Riemannian manifold of dimension n,*

$$\mathrm{vol}(B_r(z) \Delta B_r(z')) \lesssim r^{n-1} d(z, z').$$

Indeed, for small radii r, we can compare to Euclidean balls or simply to a Euclidean box with $n-1$ sidelengths of order r and a remaining side of order s. The bound

$r^{n-1}s$ holds for larger separations as well, but becomes worse than the bound

$$\mathrm{vol}(B \Delta B') \lesssim \mathrm{vol}(B) + \mathrm{vol}(B') \lesssim r^n.$$

With a separation of less than $1/T$ between z and z_j, we therefore have

$$|\mathbb{E}[X_{z_j}] - \mathbb{E}[X_z]| \lesssim \frac{r^{n-1}T^{-1}}{r^n} = \frac{1}{rT}.$$

Assuming $rT \to \infty$, this term will be less than $\varepsilon/3$. Thus the difference of expected values will eventually be less than $\varepsilon/3$ whether we assume $\eta \to \infty$ or $rT \to \infty$ (and later, we will assume that both of them diverge faster than logarithmically). In the case of an ε-difference of $\int_B |\phi|^2$ from its mean, it is one of the other two terms that must be greater than $\varepsilon/3$ (and in fact, almost greater than $\varepsilon/2$ once rT and η are large enough).

Suppose it is the integrals around z versus z' that differ by more than $\varepsilon/3$. We have

$$\left| \int_B |\phi|^2 - \int_{B'} |\phi|^2 \right| \lesssim \int_{B \Delta B'} |\phi|^2 \lesssim \mathrm{vol}(B \Delta B') \|\phi\|_\infty^2$$

Since $d(z, z_j) < 1/T$, the same volume bound as above gives

$$\frac{\varepsilon}{3} \lesssim r^{-n} \left(r^{n-1} T^{-1} \|\phi\|_\infty^2 \right)$$

That is,

$$\|\phi\|_\infty \gtrsim \sqrt{\varepsilon r T}$$

To control the probability of ϕ having such a large maximum, we use another union bound. Again, take a grid of roughly T^n points. Either there is a gridpoint w_j at which $|\phi(w_j)| \geq C\sqrt{\varepsilon r T}$ or else there are two points separated by only $1/T$ at which the values of ϕ differ by at least $C\sqrt{\varepsilon r T}$. The latter is ruled out because $1/T$ is

the wave scale for ϕ: Whereas the values $\phi(w)$ are Gaussian with unit variance, its derivatives are Gaussian with variance T^2, so this case would require a Gaussian to be more than $\sqrt{\varepsilon r T}$ standard deviations above its mean. This occurs with probability less than $\exp(-c\varepsilon r T)$. Likewise, having $|\phi(w_j)| \geq C\sqrt{\varepsilon r T}$ requires a Gaussian to be more than $\sqrt{\varepsilon r T}$ standard deviaions above its mean. From the union bound,

$$\mathbb{P}(\|\phi\|_\infty \geq c\sqrt{\varepsilon r T}) \lesssim T^n \exp(-c'\varepsilon r T)$$

which is negligible as long as $rT/\log(T) \to \infty$. Thus we can move to the final case: The probability that an integral around any single point shows a deviation of more than $\varepsilon/3$.

2.6 Chernoff bound

Each variable X_z is a quadratic form in the coefficients c_j. Writing $B = B_r(z)$, we have

$$X_z = \frac{1}{\mathrm{vol}(B)} \int_B |\phi|^2 = \sum_j \sum_k c_j c_k \frac{1}{\mathrm{vol}(B)} \int_B \phi_j \overline{\phi_k}.$$

We scale by the variance to write $c_j = \sigma \mathfrak{z}_j$, where $\mathfrak{z}$ is a standard Gaussian of mean 0 and variance 1. Thus

$$X_z = \mathfrak{z}^T A \mathfrak{z}$$

where the matrix A has entries

$$A_{jk} = \frac{\sigma^2}{\mathrm{vol}(B)} \int_B \phi_j \overline{\phi_k}.$$

Note that this matrix depends on z, as well as r and T, but we have suppressed this in the notation. Since A is a symmetric matrix, or Hermitian if we prefer to start from complex-valued eigenfunctions ϕ_j, we may diagonalize to write $A = U^T D U$ where U

is orthogonal (or unitary) and D is diagonal with entries, say, λ_j. In eigencoordinates, the random variable X_z becomes

$$X_z = \mathfrak{z}^T A\mathfrak{z} = (U\mathfrak{z})^T D(U\mathfrak{z}) = \sum_j \lambda_j y_j^2$$

where $y = U\mathfrak{z}$ is again a standard Gaussian vector (or complex Gaussian). Note that, since $X_z \geq 0$, all of the eigenvalues λ_j are nonnegative. Evaluating a Gaussian integral, it follows that the moment generating function of $\mathfrak{z}^T A\mathfrak{z}$ is

$$g(s) = \mathbb{E}\left[e^{s\mathfrak{z}^T A\mathfrak{z}}\right] = \prod_{j=1}^{N}(1-2s\lambda_j)^{-1/2} \tag{2.6.1}$$

where λ_j are the eigenvalues of A. This is defined as long as $1 - 2s\lambda_j > 0$ for all j, so s must be small enough. Specifically, $g(s)$ is defined for $s < 1/(2\lambda_{\max})$, where $\lambda_{\max}$ is the largest eigenvalue of A. Estimates for $g(s)$ allow us to execute a Chernoff bound on the tail probability. For any $s > 0$, $X > \mathbb{E}[X] + \varepsilon$ if and only if $e^{sX} > e^{s\mathbb{E}[X]+s\varepsilon}$, so by Markov's inequality

$$\mathbb{P}\{X > \mathbb{E}[X] + \varepsilon\} \leq g(s)e^{-s\mathbb{E}[X]-s\varepsilon} = \exp(-s\varepsilon - s\mathbb{E}[X] + \log g(s))$$

In the case at hand, where $X = \mathfrak{z}^T A\mathfrak{z}$, we have

$$-s\varepsilon - s\mathbb{E}[X] + \log g(s) = -s\varepsilon - s\mathbb{E}[X] + \frac{1}{2}\sum_j -\log(1-2s\lambda_j).$$

Expanding the logarithm in a power series (provided $2s\lambda_{\max} < 1$), we have

$$\frac{1}{2}\sum_j -\log(1-2s\lambda_j) = \sum_{m=1}^{\infty}\frac{1}{2m}\sum_j (2s\lambda_j)^m.$$

The term $m = 1$ contributes $s \sum_j \lambda_j = s\mathbb{E}[X]$. This cancels the expected value above so that

$$\begin{aligned} -s\varepsilon - s\mathbb{E}[X] + \log g(s) &= -s\varepsilon + \sum_{m\geq 2} \frac{1}{2m} \sum_j (2s\lambda_j)^m \\ &= -s\varepsilon + s^2 \sum_j \lambda_j^2 + \sum_{m\geq 3} \frac{1}{2m} \sum_j (2s\lambda_j)^m. \end{aligned}$$

If such an s is allowed, then we can minimize the sum of the first two terms by choosing

$$s_? = \frac{\varepsilon}{2\sum \lambda_j^2}$$

However, it is not clear whether $2s_?\lambda_{\max} < 1$, that is, whether $g(s_?)$ is defined. We would need to know that $\lambda_{\max}/\sum_j \lambda_j^2 < 1/\varepsilon$. In the case of S^2, we will show that $\lambda_{\max}$ and $\sum \lambda_j^2$ are of the same order of magnitude, so that $s_?$ is a valid choice once ε is small enough. Here, we choose a different s to guarantee that $2s\lambda_{\max} < 1$, namely

$$s = c \left(\sum_j \lambda_j^2 \right)^{-1/2}$$

where $c < 1/2$. Note that $\lambda_{\max} \leq \sqrt{\sum \lambda_j^2}$, so that this is a valid choice of s.

Claim: There is a constant A such that

$$\log g(s) - s\mathbb{E}[X] \leq As^2 \sum_j \lambda_j^2$$

Indeed, this follows from Taylor's theorem. For a twice differentiable function f, we have

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)dt$$

Applied to the function $f(x) = -\log(1-x)$, this gives

$$-\log(1-x) = x + \int_0^x \frac{1}{(1-t)^2}(x-t)dt.$$

In particular, for $x \leq a$ we have

$$-\log(1-x) - x \leq x^2(1-a)^{-2}$$

so we may take $A = (1-a)^{-2}$ to have a bound valid for all x up to a. We take $x = 2s\lambda_j$ where $s = c(\sum \lambda_j^2)^{-1/2}$ with $0 < c < 1/2$. These values of x are at most

$$x = 2s\lambda_j \leq 2c\frac{\lambda_{\max}}{(\sum \lambda_j^2)^{1/2}} \leq 2c.$$

Taylor's theorem then gives

$$-\log(1-2s\lambda_j) - 2s\lambda_j \leq (1-2c)^{-2}4s^2\lambda_j^2 = \frac{4c^2}{(1-2c)^2}\lambda_j^2 / \sum_i \lambda_i^2$$

Summing over j and dividing by 2, we get

$$-g(s) - s\sum_j \lambda_j \leq \frac{2c^2}{(1-2c)^2}$$

Hence, noting again that $\sum_j \lambda_j = \mathbb{E}[X]$, we have proved the claim.

With this estimate in hand, we can bound the tail probability as follows:

$$\mathbb{P}\{X > \mathbb{E}[X] + \varepsilon\} \leq e^{2c^2/(1-2c)^2} \exp\left(-c\varepsilon\big(\sum_j \lambda_j^2\big)^{-1/2}\right)$$

The lower tail, where $X < \mathbb{E}[X] - \varepsilon$, is slightly different but can be treated by the same method. We have $X < \mathbb{E}[X] - \varepsilon$ if and only if $-X > \mathbb{E}[-X] + \varepsilon$, so we can apply the argument above with $-X$ in place of X. Instead of $g(s)$, the relevant

function for the Chernoff bound is

$$g_-(s) = \mathbb{E}\left[e^{-sX}\right] = \prod_j (1 + 2s\lambda_j)^{-1/2}.$$

This function $g_-(s)$ is now defined for all $s \geq 0$ whereas $g(s)$ is defined only for sufficiently small s. The Chernoff bound is

$$\mathbb{P}\{-X > \mathbb{E}[-X] + \varepsilon\} \leq g_-(s) e^{s\mathbb{E}[X]} e^{-s\varepsilon}.$$

We have $-\log(1+x) \leq -x + x^2/2$ for all $x \geq 0$, so that

$$\log g_-(s) + s\mathbb{E}[X] \leq \frac{1}{4}\sum_j (2s\lambda_j)^2 \leq c^2$$

where we choose $s = c\big(\sum \lambda_j^2\big)^{-1/2}$ as above. This shows that the lower tail probability obeys the same bound as the upper tail probability, namely

$$\mathbb{P}\{-X > \mathbb{E}[-X] + \varepsilon\} \leq e^{c^2} \exp\left(-c\varepsilon\big(\sum \lambda_j^2\big)^{-1/2}\right).$$

In fact, we can show it obeys an even better bound because $g_-(s)$ is defined for all s and so we could simply choose $s = s_?$. However, since this does not help with the upper tail, we simply state the bound this way so that it applies to both:

Lemma 2.6.1. *For any $\varepsilon > 0$, there is a positive $c(\varepsilon) > 0$ such that*

$$\mathbb{P}(|X - \mathbb{E}[X]| > \varepsilon) \lesssim \exp\left(-c(\varepsilon)\big(\sum \lambda_j^2\big)^{-1/2}\right).$$

To absorb a factor T^n from the union bound over a grid of spacing $1/T$, we need

$$T^n \exp\left(-c(\varepsilon)\big(\sum_j \lambda_j^2\big)^{-1/2}\right) \to 0$$

no matter how small the given ε. Thus it is enough to show that

$$\frac{1}{\log T}\left(\sum_j \lambda_j^2\right)^{-1/2} \to \infty. \tag{2.6.2}$$

This follows from Lemma 2.4.1. Indeed, the numbers λ_j are the eigenvalues of the matrix A, so that

$$\sum_j \lambda_j^2 = \mathrm{tr}(A^2).$$

On the other hand,

$$\mathrm{tr}(A^2) = \frac{1}{2}\,\mathrm{var}[X] \lesssim \left((rT)^{-(n-1)/2} + \eta^{-1}\right)^2.$$

Therefore

$$\left(\sum_j \lambda_j^2\right)^{-1/2} \gtrsim \left((rT)^{-(n-1)/2} + \eta^{-1}\right)^{-1}.$$

The condition (2.6.2) is equivalent to

$$\log(T)\left((rT)^{-(n-1)/2} + \eta^{-1}\right) \to 0.$$

Thus both $\log(T)(rT)^{-(n-1)/2}$ and $\log(T)\eta^{-1}$ should vanish. If $n = 2$, we assume that $rT/\log(T)^2 \to \infty$ in order to arrange the first of these. For $n \geq 3$, note that we have already assumed that $\log(T)(rT)^{-1} \to 0$ in order to control the probability of an "off-grid" deviation. This implies that $\log(T)(rT)^{-(n-1)/2} \to 0$. In any dimension, we make the extra assumption that $\log(T)\eta^{-1} \to 0$, or equivalently, $\eta/\log(T) \to \infty$, as in the hypotheses of Theorem 1.4.2.

2.7 How bad is the union bound?

The union bound might not look like a very accurate approximation. The two sides of

$$\mathbb{P}\left\{\bigcup_j E_j\right\} \leq \sum_j \mathbb{P}\{E_j\}$$

may be very far apart if there is significant overlap between the events E_j. If one has mutually independent events E_j, each of small probability, then the union bound is not far off. By independence,

$$\mathbb{P}\{\cup_j E_j\} = 1 - \prod_j (1 - \mathbb{P}(E_j)) \approx \sum_j \mathbb{P}(E_j)$$

because, when one distributes the multiplication, we assume the probabilities are small enough that products of two or more terms can be neglected. Thus the seemingly wasteful union bound is fairly accurate for independent events of low probability. For any subset S of M, if the points of S are well-separated, then the covariance formula in Lemma 2.0.3 shows that the local masses X_z are only weakly correlated. Although there is a considerable difference between mutual independence and pairwise weak correlations, this is an indication that the union bound might be close to the truth in this case.

2.8 How about the Chernoff bound?

Let us test the accuracy of the Chernoff bound in a simple example, namely where all of the coefficients λ_j are the same, say $\lambda_j = 1$. In this case, the quadratic form in

Gaussians is not only diagonal, but radial:

$$X = \sum_{j=1}^{D} \lambda_j y_j^2 = |y|^2.$$

This is known as a χ^2 random variable with D degrees of freedom. Its average is D. The moment generating function is

$$\mathbb{E}\left[e^{sX}\right] = (1-2s)^{-D/2}.$$

Since the mean is growing, it is natural to write the deviations multiplicatively. That is, to consider the tail event $X > (1+\varepsilon)\mathbb{E}[X]$, there being a very high chance of an additive deviation $X > \mathbb{E}[X] + \varepsilon$ when D is large. The Chernoff bound is

$$\mathbb{P}\{X > (1+\varepsilon)D\} \leq (1-2s)^{-D/2} e^{-s(1+\varepsilon)D}.$$

By calculus, the optimal value of s solves $1 - 2s = 1/\varepsilon$, that is,

$$s = \frac{\varepsilon}{1+\varepsilon}\frac{1}{2}.$$

The Chernoff bound therefore gives

$$\mathbb{P}\{X > (1+\varepsilon)D\} \leq \exp((\log(1+\varepsilon) - \varepsilon)D) \approx \exp(-\varepsilon^2 D/2).$$

On the other hand, an explicit calculation is possible. Since y is a standard

Gaussian vector in D dimensions, we have

$$\begin{aligned}\mathbb{P}\left\{|y|^2 > (1+\varepsilon)D\right\} &= \int_{\sqrt{(1+\varepsilon)D}}^{\infty} \left(\frac{e^{-r^2/2}}{\sqrt{2\pi}}\right)^D r^{D-1} dr \int_{S^{D-1}} d\omega \\ &= \int_{(1+\varepsilon)D}^{\infty} x^{D/2} e^{-x/2} \frac{dx}{x} 2^{-D/2}/\Gamma(D/2) \\ &= \frac{\int_{\frac{(1+\varepsilon)D}{2}}^{\infty} u^{D/2} e^{-u} \frac{du}{u}}{\int_0^{\infty} u^{D/2} e^{-u} \frac{du}{u}}.\end{aligned}$$

Note the change of variables $x = r^2$, $u = x/2$. In both the numerator and the denominator, the integrand is

$$\exp\left(-u + \left(\frac{D}{2} - 1\right) \log u\right).$$

This takes its largest value when $u = D/2 - 1$, which is not included in the numerator's range when $\varepsilon > 0$, so we expect the numerator to be much smaller than the denominator. To see how much smaller, we use the Laplace method. Expanding around the unique critical point $D/2 - 1$ gives

$$\begin{aligned}f(u) &= -u + (D/2-1)\log u \\ &= (D/2-1)\left(\log(D/2-1) - 1\right) + \frac{-1}{D-2}\left(u - (D/2-1)\right)^2 + \dots\end{aligned}$$

which leads to the standard Stirling asymptotic for the denominator $\Gamma(D/2)$. The full range $0 < u < \infty$ means that $w = (u - (D/2-1))/\sqrt{D/2-1}$ covers the bulk of the Gaussian $e^{-w^2/2}$, which integrates to the $\sqrt{2\pi}$ in Stirling's formula. In the numerator, on the other hand, $u > (1+\varepsilon)D/2$ and w is relegated to the tails of the bell curve:

$$w = \frac{u - (D/2-1)}{\sqrt{D/2-1}} > \frac{\varepsilon(D-1)/2 + 1 + \varepsilon/2}{\sqrt{D/2-1}} \approx \varepsilon\sqrt{D/2}.$$

In consequence,

$$\begin{aligned}
&\mathbb{P}\{|y|^2 > (1+\varepsilon)D\} \\
&= \frac{(1+o(1))(D/2-1)^{D/2-1}e^{-(D/2-1)}\sqrt{D/2-1}}{(1+o(1))(D/2-1)^{D/2-1}e^{-(D/2-1)}\sqrt{D/2-1}} \frac{1}{\sqrt{2\pi}} \int_{\varepsilon\sqrt{D/2}}^{\infty} e^{-w^2/2} dw \\
&= (1+o(1))e^{-\varepsilon^2(D-1)/4} \frac{1}{\varepsilon\sqrt{\pi(D-1)}},
\end{aligned}$$

using the estimate $\mathbb{P}\{z > t\} \sim e^{-t^2/2}/t$ for a standard Gaussian. So the actual exponential rate is $\varepsilon^2/2$, rather than the rate $\varepsilon - \log(1+\varepsilon)$ provided by the Chernoff bound. Expanding the logarithm shows that these agree to a first approximation but the latter is smaller (i.e. a slower rate) by $\varepsilon^3/3$ for small ε. This raises hopes that the Chernoff bound is well suited to our purposes.

Chapter 3

The two-dimensional sphere

On the two-dimensional sphere, we can sharpen the results from chapter 3 by removing the spectral window needed in general. The Laplace eigenvalues on S^2 are $m(m+1)$ with multiplicity $2m+1$ for each $m \geq 0$. Thus instead of taking a window, we simply randomize over the large number of harmonics of degree m. The monochromatic ensemble thus consists of *random spherical harmonics* $\phi : S^2 \to \mathbb{R}$ given by

$$\phi = \sum c_j \phi_j$$

where the $2m+1$ functions ϕ_j form an orthonormal basis for degree m spherical harmonics on S^2. The coefficients are independent Gaussians of mean 0 and variance $1/(2m+1)$. The degree m plays the role of the frequency parameter T, since $T = \sqrt{m(m+1)}$.

This explicit form will allow us to say more in the case of S^2 than for other manifolds. In particular, the two-point function is a Legendre polynomial

$$K(x,y) = \sum_j \phi_j(x)\phi_j(y) = \frac{2m+1}{4\pi} P_m(\cos\theta) \tag{3.0.1}$$

where θ is the spherical distance between x and y. Instead of estimating $K(x,y)$ along the lines of Weyl's Law, we can use more precise asymptotic expressions for $P_m(\cos\theta)$, especially Hilb's formula.

The choice of variance $1/(2m+1)$ guarantees that if we integrate over a geodesic ball $B_r(z)$,

$$\mathbb{E}\left[\int_{B_r(z)} \phi^2\right] = \frac{\mathrm{vol}(B_r)}{4\pi} = \sin^2(r/2). \tag{3.0.2}$$

Indeed,

$$\mathbb{E}\left[\int_{B_r(z)} \phi^2\right] = \int_{B_r(z)} \mathbb{E}[\phi^2] = \int_{B_r(z)} \sum \phi_j(x)^2 \mathbb{E}[c_j^2] dx = \int_{B_r(z)} \frac{2m+1}{4\pi} \frac{1}{2m+1}$$

by linearity of expectation, expanding the square, and the fact that, for any orthonor-

mal basis of harmonics ϕ_j,

$$\sum_j \phi_j(x)^2 = \frac{2m+1}{4\pi},$$

(see Fact 3.0.3 below). Thus the expectation is the volume fraction, as claimed. Notice that the expected value in Equation (3.0.2) is independent of the center z, as it must be since the ensemble is invariant under rotation of S^2. This is a slight but convenient simplification compared to the general case, where the expected value changes from point to point.

To normalize, consider the random variables

$$X_z = \frac{1}{\mathrm{vol}(B_r)} \int_{B_r(z)} \phi^2$$

so that $\mathbb{E}[X_z] = \frac{1}{4\pi}$ is of order 1 for all $r > 0$ and $m \geq 1$. As in chapter 2, expanding the square in ϕ^2 shows that X_z is a quadratic form in Gaussian random variables:

$$X_z = \sum_j \sum_k c_j c_k \frac{1}{\mathrm{vol}(B_r(z))} \int_{B_r(z)} \phi_j \phi_k \tag{3.0.3}$$

The main feature of the sphere compared to other surfaces is that this quadratic form can be diagonalized explicitly. Indeed, fix the point z and work in spherical coordinates with respect to this origin, with θ being the distance from z and α being an azimuthal angle around the axis. Separation of variables leads to a basis of functions of the form $P(\cos\theta)T(\alpha)$ where P is essentially a derivative of the Legendre polynomial and T is a trigonometric funcion $e^{\pm ij\alpha}$. Any two such functions are orthogonal over $B_r(z)$ for every radius r: The angular factors $T(\alpha)$ are already orthogonal, and the integral over $0 \leq \theta \leq r$ then plays no role. Taking a basis of such functions, we then have

$$X_z = \sum_j c_j^2 \frac{1}{\mathrm{vol}(B_r(z))} \int_{B_r(z)} \phi_j^2. \tag{3.0.4}$$

We have the following semicircle law for the coefficients of this quadratic form.

Proposition 3.0.1. *Fix a point $z \in S^2$ and consider $X = X_z$. If we choose for our basis functions ϕ_j the standard ultraspherical polynomials rotated so that z is at the North pole $(0,0,1)$, then*

$$X = \sum \lambda_\nu \mathfrak{z}_\nu^2$$

where the $\mathfrak{z}_\nu$ are independent standard Gaussians for $0 \leq \nu \leq 2m$ and the coefficients λ_ν satisfy

$$\lambda_{2k} = \lambda_{2k+1} = \frac{1}{2\pi^2}\sqrt{1-(k/(rm))^2}\frac{1}{rm}\left(1 + O_\eta\left(\frac{k^{2/3+\eta}}{rm}\right)\right). \tag{3.0.5}$$

for any $\eta > 0$ and a ratio $0 \leq k/(rm) < 1$ bounded away from 1. For $k \sim rm$,

$$\lambda_k \ll_\eta (rm)^{-4/3+\eta}. \tag{3.0.6}$$

For k so large that $k + k^p > rm$, where $p > 1/3$, we have

$$\lambda_k \ll_p \frac{\exp(-ck^{(3p-1)/2})}{(rm)^2} \tag{3.0.7}$$

for a constant $c > 0$.

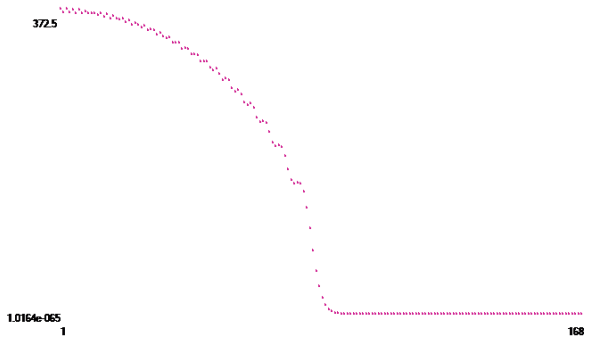


Figure 3.0.1: Dropoff of Bessel integrals $\int_0^{rm} uJ_k(u)^2du \div \int_0^{\pi m} uJ_k(u)^2du$ for $0 \leq k \leq 2rm$ with $m = 10000$, $r = \log(m)^2$. Made with pari-gp

This semicircle dropoff is illustrated in Figure 3. Perhaps one should expect some

GOE behaviour because A is given by integrals $\int_B \phi_j \phi_k$ and there is a large orthogonal group of symmetries acting by change of basis on the functions ϕ_j. However, our proof relies on explicit calculations with a specific basis.

As a consequence, we can choose a better parameter s in the Chernoff bound compared to the one we settled for in the general case. The resulting bound on the tails of X_z involves rm in the exponent instead of only $\sqrt{rm}$:

Lemma 3.0.2. *For any $\varepsilon > 0$ and any fixed $z \in S^2$, there are positive $C(\varepsilon) > 0$ and $c(\varepsilon) > 0$ such that*

$$\mathbb{P}\left\{\left|X_z - \frac{1}{4\pi}\right| > \varepsilon\right\} \leq C(\varepsilon) e^{-c(\varepsilon) rm}.$$

The constant $c(\varepsilon)$ in the exponent can be taken proportional to ε^2.

Assuming that rm is asymptotically larger than $\log m$, this exponential decay in rm is enough to absorb any power of m sacrificed in tribute to the union bound. Thus instead of having to take r large enough relative to the spectral window, we can conclude equidistribution at any scales r with $rm/\log m \to \infty$ arbitrarily slowly.

We will also show that the local integrals obey a central limit theorem as $rm \to \infty$. Fix any point $z \in S^2$ and standardize $X = X_z$ to have mean 0 and variance 1:

$$Z = \frac{X - \mathbb{E}[X]}{\sqrt{\mathrm{var}[X]}}.$$

We prove that as $rm \to \infty$, Z converges in distribution to the "bell curve" $N(0,1)$. This follows from pointwise convergence of the characteristic functions

$$\mathbb{E}\left[e^{itZ}\right] \to e^{-t^2/2}$$

for each fixed $t \in \mathbb{R}$. Our first motivation for this result was as a way to prove the tail

bound. We might expect that

$$\mathbb{P}(X - \mathbb{E}[X] > \varepsilon) = \mathbb{P}(Z > \varepsilon \operatorname{var}[X]^{-1/2}) \lesssim \exp(-\varepsilon^2 \operatorname{var}[X]^{-1})$$

because Z is approximately Gaussian. At this point, we knew that $\operatorname{var}[X]^{-1}$ is of order rm, so this seemed to imply exponential decay in rm, as required. However, the pointwise convergence applies only with *fixed* t, whereas this would require t to grow at the rate $\operatorname{var}[X]^{-1/2}$. For this reason, we turned to the Chernoff bound instead to complete the application to shrinking scale equidistribution. Nevertheless, there is intrinsic interest in finding the limiting distribution of the local integrals. The proof is more difficult than the tail bound in the sense that it uses higher moments of X_z beyond the variance.

Some facts from analysis

For ease of reference, here are some of the tools we use below.

Fact 3.0.3. ***(Addition formula for spherical harmonics)*** *For any orthonormal basis of spherical harmonics ϕ_j of degree m, and for any points x and y on S^2,*

$$\sum_j \phi_j(x)\phi_j(y) = \frac{2m+1}{4\pi} P_m(x \cdot y). \tag{3.0.8}$$

Here, P_m is the Legendre polynomial of degree m normalized so that $P_m(1) = 1$. In particular, $|P_m| \le 1$.

Fact 3.0.4. ***(Bernstein's inequality)*** *The Legendre polynomial P_m satisfies*

$$P_m(\cos\theta)^2 \le \frac{2}{\pi} \frac{1}{m \sin\theta} \tag{3.0.9}$$

for all $\theta > 0$.

Fact 3.0.5. ***(Basis of ultraspherical harmonics)*** *Fix any point $z \in S^2$ as origin.*

There is an orthonormal basis of spherical harmonics of degree m that are orthogonal not only over S^2 but also over any spherical cap $B_r(z)$ centered at z.

In fact, the standard basis of "Y_l^m"s has this property. Let the distance θ and the longitude α be spherical coordinates with respect to the point z. Then the $2m+1$ functions

$$\phi_{j,T} = \frac{P_m^j(\cos\theta)T(j\alpha)}{\left(\int_0^{2\pi}\int_0^{\pi} P_m^j(\cos\theta)^2 T(j\alpha)^2 \sin\theta d\theta d\alpha\right)^{1/2}} \tag{3.0.10}$$

form an orthonormal basis for spherical harmonics of degree m. The indices j and T run over $j = 0, 1, \ldots, m$ and $T \in \{\sin, \cos\}$, excluding the case where $j = 0$ and $T = \sin$, which gives 0. These basis functions are orthogonal over any spherical cap $B_r(z)$ around z, no matter how small the radius r, because the functions $T(j\alpha)$ are orthogonal over the circle $0 \le \alpha \le 2\pi$. The polynomials P_m^j are given by

$$P_m^j(\cos\theta) = \frac{j!}{(2j)!}\frac{(m+j)!}{m!}(\sin\theta)^j P_{m-j}^{(j,j)}(\cos\theta).$$

in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $n = m - j$ and $\alpha = \beta = j$. We follow Szegő's treatment in section 4.7 of [67]. When $j = 0$, we have the Legendre polynomial of degree m. As j increases, $P_m^j(x)$ vanishes to higher and higher order at $x = 1$. This endpoint $x = 1$ corresponds to the point z on the sphere when we take $x = \cos\theta$, θ being the distance to z.

Fact 3.0.6. ***(Hilb asymptotics)***

$$P_m^j(\cos\theta) = h_{j,m}\left(\sqrt{\frac{\theta}{\sin\theta}}J_j((m+1/2)\theta) + O\left(\frac{(m-j)!}{m!}m^j\left(\frac{\sin\theta}{2}\right)^j \theta^{1/2}(m-k)^{-3/2}\right)\right) \tag{3.0.11}$$

where

$$h_{j,m} = \frac{j!2^j}{(2j)!}\frac{(m+j)!}{(m-j)!}(m+1/2)^{-j}.$$

The factor $h_{j,m}$ disappears when we normalize in L^2 and thus plays no role.

Equation (3.0.11) is a special case of Szegő's asymptotic (formula (8.21.17) in [67]) for Jacobi polynomials $P_n^{(\alpha,\beta)}$. For $\alpha > -1$ and any real β, with $N = n + (\alpha+\beta+1)/2$, we have the estimate

$$\left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta} P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\Gamma(n+\alpha+1)}{n!}\sqrt{\frac{\theta}{\sin\theta}}\frac{J_\alpha(N\theta)}{N^\alpha} + \varepsilon(n,\theta).$$

The error satisfies

$$\varepsilon(n,\theta) = \begin{cases} \theta^{1/2}O(n^{-3/2}) \text{ if } c/n \le \theta \le \pi^- < \pi \\ \theta^{\alpha+2}O(n^\alpha) \text{ if } 0 < \theta \le c/n \end{cases}$$

for any fixed π^- less than π and any $c > 0$, the implicit O constants being subject to the choice of these parameters. In particular, $\varepsilon(n,\theta) \lesssim \theta^{1/2}n^{-3/2}$ holds for all θ. In the special case where $\alpha = \beta = j$ and $n = m - j$, Szegő's asymptotic implies Fact 3.0.6. The case $\alpha = \beta = 0$ is Hilb's formula for Legendre polynomials, namely

$$P_m(\cos\theta) = \sqrt{\frac{\theta}{\sin\theta}}J_0((m+1/2)\theta) + O\left(\frac{1}{m^{3/2}}\right)$$

For k smaller than, say, $m/3$, we have $(1-k/m)^{-3/2} \le 2$. For k much smaller than $\sqrt{m}$, the factor $(m-k)!(m+1/2)^k/m!$ is also bounded. In that case, a consequence of equation (3.0.11) is that (for k much smaller than $\sqrt{m}$)

$$\begin{aligned}\frac{\int_0^r P_m^k(\cos\theta)^2\sin\theta d\theta}{\int_0^\pi P_m^k(\cos\theta)^2\sin\theta d\theta} &= \frac{\int_0^r \theta J_k((m+1/2)\theta)^2 d\theta + O\left(2^{-k}(m-k)^{-3/2}r^k/k\right)}{\int_0^\pi \theta J_k((m+1/2)\theta)^2 d\theta + O\left(2^{-k}(m-k)^{-3/2}k^{-1/2}\right)} \\ &= \frac{\int_0^r \theta J_k((m+1/2)\theta)^2 d\theta}{\int_0^\pi \theta J_k((m+1/2)\theta)^2 d\theta}\left(1 + O\left(m^{-1/2}2^{-k}k^{-1/2}\right)\right) \\ &= \frac{\int_0^{rm} xJ_k(x)^2 dx}{\int_0^{\pi m} xJ_k(x)^2 dx}\left(1 + O\left(m^{-1/2}2^{-k}\right)\right).\end{aligned}$$

Thus Hilb's formula naturally leads to the following integrals.

Fact 3.0.7. ***(Some integrals involving Bessel functions)***

$$\int_0^t x J_k(x)^2 dx = \frac{t^2}{2}\left(J_k(t)^2 - J_{k-1}(t)J_{k+1}(t)\right) \tag{3.0.12}$$

$$\int_0^t u J_0(u)^2 du = \frac{1}{2}t^2\left(J_0(t)^2 + J_1(t)^2\right). \tag{3.0.13}$$

This is formula 5.54 in [26]. It can be checked by differentiating both sides and using the recurrence relation between J_k, J_k', and $J_{k\pm1}$. The second is formula (10.22.29) in the Digital Library of Mathematical Functions [55], and can be construed as the $k = 0$ case of (3.0.12) with $J_{-1} = -J_1$. We don't use (3.0.12) in the proof, but we did use it to compute the integrals for Figure 3.

Fact 3.0.8. ***(Asymptotics of J Bessel functions)*** *For $k > x > 0$, we have*

$$J_k(x) = (2\pi)^{-1/2}k^{-1/2}(1-u^2)^{-1/4}e^{k(\sqrt{1-u^2}-\sinh^{-1}(u^{-1}))}\left(1 + O\left(\frac{1}{\sqrt{x^2-k^2}}\right)\right) \tag{3.0.14}$$

where $u = x/n$ is strictly between 0 and 1. For $x > k$, write $x = k\sec\beta$ with $0 < \beta < \pi/2$. Then

$$J_k(k\sec\beta) = \sqrt{\frac{2}{\pi k\tan\beta}}\left(\cos(k(\tan\beta - \beta) - \pi/4) + O\left(\frac{1}{k\tan\beta}\right)\right) \tag{3.0.15}$$

noting that $k\tan\beta = \sqrt{x^2-k^2}$. When k and x are too close, that is, $|x-k| < Ck^{1/3}$, these approximations become inaccurate and we use the upper bound

$$J_k(x) \ll k^{-1/3} \tag{3.0.16}$$

although it is possible to be much more precise.

The first of these is formula 7.13.2 (14) in volume 2 of the Bateman Manuscript

Project [22], page 87. Note that

$$\frac{d}{du}\left(\sqrt{1-u^2} - \sinh^{-1}(u^{-1})\right) = \frac{u^{-1}-u}{(1-u^2)^{1/2}} > 0$$

so the quantity in the exponent increases with u from its limit $-\infty$ as $u \to 0$ to its value $-\log(1+\sqrt{2})$ at $u = 1$. The Bessel function $J_n(x)$ is exponentially small for small x and oscillates with a decaying amplitude $\sqrt{2/(\pi x)}$ for large x. See formula 8.41(4) on p.244 of [71] for equation (3.0.15). In between, there is a transition range of length $Cn^{1/3}$ centered at $x = n$. In this region, $J_n(x)$ achieves a maximum value of order $n^{-1/3}$ and also reaches its first positive zero. This maximum of order $n^{-1/3}$ is considerably larger than the amplitude $n^{-1/2}$ for x beyond the transition range, and can be regarded as a "boost" from the Airy function. The result, stated as 8.2(1) on p.231 of [71], is

$$J_n(n) = \frac{\Gamma(1/3)}{2^{2/3}3^{1/6}\pi} n^{-1/3} + O(n^{-2/3}).$$

In this regime, where $|x-n|$ is of order $n^{1/3}$ or smaller, Watson established an asymptotic for $J_n(x)$ stated as formulas (1) and (2) on p.249 of [71] depending on which of x and n is the larger. Olver gives an asymptotic expansion for $J_n(n+\tau n^{1/3})$ in [56].

As a corollary of the behaviour of $J_\nu(t)$ for large t, we have

Fact 3.0.9. ***(Bessel version of*** $\sin^2 + \cos^2 = 1$***)*** *As* $t \to \infty$,

$$J_\nu(t)^2 + J_{\nu+1}(t)^2 \sim \frac{2}{\pi t}\left(1 + O_\nu\left(\frac{1}{t}\right)\right).$$

We are imprecise about the dependence of the error term on ν because we only use it with $\nu = 0$ in connection with Equation (3.0.13).

Fact 3.0.10. *If* $f(y)$ *is real-valued and continuously differentiable for* $a < y < b$ *with*

$f'(y)$ positive and monotone, and $\inf f' > 0$*, then*

$$\int_a^b e^{if(y)} dy \lesssim \frac{1}{\inf f'} \tag{3.0.17}$$

This is shown using integration by parts on p.124 of [68].

3.1 Ultraspherical basis and proof of the semicircle law

We fix $z \in S^2$ and use the basis from Fact 3.0.5. The key advantage of this basis is that the off-diagonal entries of the matrix A in $X = \mathfrak{z}^T A \mathfrak{z}$ all vanish. Thus

$$X = \sum_{k=1}^{2m+1} \lambda_k \mathfrak{z}_k^2 \tag{3.1.1}$$

where each random variable $\mathfrak{z}_k$ is a standard Gaussian and there are no cross terms. The coefficients λ_k are, for $1 \le j \le m$,

$$\begin{aligned}
\lambda_1 &= \frac{1}{(2m+1)\operatorname{vol}(B_r)} \int_0^r P_m^0(\cos\theta)^2 \sin\theta d\theta \div \int_0^\pi P_m(\cos\theta)^2 \sin\theta d\theta \\
\lambda_{2j} = \lambda_{2j+1} &= \frac{1}{(2m+1)\operatorname{vol}(B_r)} \int_0^r P_m^j(\cos\theta)^2 \sin\theta d\theta \div \int_0^\pi P_m^j(\cos\theta)^2 \sin\theta d\theta.
\end{aligned}$$

Our opening move is Hilb's formula:

$$\begin{aligned}
\lambda_k &= \frac{1}{(2m+1)\operatorname{vol}(B_r)} \frac{\int_0^r P_m^k(\cos\theta)^2 \sin\theta d\theta}{\int_0^\pi P_m^k(\cos\theta)^2 \sin\theta d\theta} \\
&= \frac{1}{(2m+1)\operatorname{vol}(B_r)} \frac{\int_0^{rm} x J_k(x)^2 dx}{\int_0^{\pi m} x J_k(x)^2 dx} \left(1 + O\left(\frac{1}{2^k m^{1/2}}\right)\right)
\end{aligned}$$

To appraise the coefficients λ_k with k growing, we approximate the integral

$\int_0^t xJ_k(x)^2dx$ using Fact 3.0.8. Consider an initial range $x < k - k^p$, an intermediate range where $k - k^p < x < k + k^p$, and a final range where $k + k^p < x < t$. To begin, $0 < p < 1$. In the initial range $x < k$ so we change variables to $x = k \text{ sech } \alpha$ and use equation (3.0.14). The lower limit $x = 0$ corresponds to $\alpha \to \infty$ while the upper limit $x = k - k^p$ corresponds to $\alpha = \alpha_0 = \cosh^{-1}(k/(k - k^p)) \sim \sqrt{2}k^{(p-1)/2}$. This gives

$$\int_0^{k-k^p} xJ_k(x)^2dx \ll k \exp(2k(\tanh \alpha_0 - \alpha_0)) < \exp(-ck^{(3p-1)/2}), \tag{3.1.2}$$

for some $c > 0$, since $\tanh \alpha - \alpha \sim -\alpha^3/3$ for small α. The constant c is positive and could be taken close to $2/3$. Thus (3.1.2) shows that the initial range can be neglected as long as we choose $p > 1/3$. Over the transition range, we have

$$\int_{k-k^p}^{k+k^p} xJ_k(x)^2dx \ll k^p k(k^{-1/3})^2 \ll k^{1/3+p}. \tag{3.1.3}$$

For large $x = k \sec \beta$, we have

$$\begin{aligned} xJ_k(x)^2 &= k \sec \beta \frac{2}{\pi k \tan \beta} \left(\cos^2(k(\tan \beta - \beta) - \pi/4) + O\left(\frac{1}{k \tan \beta}\right)\right) \\ &= \frac{1}{\pi \sin \beta} \left(1 + \sin(2k(\tan \beta - \beta)) + O\left(\frac{1}{k \tan \beta}\right)\right). \end{aligned}$$

The change of measure $dx = k \sec \beta \tan \beta d\beta = d\beta k \sin \beta / \cos^2(\beta)$ cancels the $\sin \beta$ in the denominator above. Thus on the final stretch of the integration,

$$\int_{k+k^p}^t xJ_k(x)^2dx = k \int_{\sec^{-1}(1+k^{p-1})}^{\sec^{-1}(t/k)} \sec^2(\beta) \left(1 + \sin(2k(\tan \beta - \beta)) + O\left(\frac{1}{k \tan \beta}\right)\right) \frac{d\beta}{\pi}$$

The lower limit of integration, $\sec^{-1}(1 + k^{p-1})$, is roughly 0. The O term contributes

$O(\log k + \log(t^2 - k^2))$ when integrated by a change of variables $u = \tan\beta$:

$$\begin{aligned}\int_{\sec^{-1}(1+k^{p-1})}^{\sec^{-1}(t/k)} \frac{1}{\tan\beta}\sec^2(\beta)d\beta &= \log\tan\beta\Big]_{\sec^{-1}(1+k^{p-1})}^{k=\sec^{-1}(t/k)} \\ &= \log\sqrt{t^2/k^2-1} - \log\sqrt{2k^{p-1}+k^{2(p-1)}} \\ &= \frac{1}{2}\log(t^2-k^2) - \frac{1}{2}(p\log k + \log 2 + \log(1+k^{p-1}/2)).\end{aligned}$$

This can be regarded as an error term as long as $t^2 - k^2$ is large. The term $\sec^2(\beta)\sin(2k(\tan\beta - \beta))$ oscillates enough to be of lower order when integrated. Indeed, change variables to $y = \tan\beta$, $dy = \sec^2(\beta)d\beta$ so that the integral is

$$k\int_{\sqrt{2k^{p-1}+k^{2(p-1)}}}^{\sqrt{(t/k)^2-1}} \sin(2k(y - \arctan y))dy = k\,\mathrm{Im}\left[\int_a^b e^{if(y)}dy\right]$$

where $f(y) = 2k(y - \arctan y)$, $b = \sqrt{(t/k)^2-1}$, and $a = \sqrt{2k^{p-1}+k^{2(p-1)}}$. We have

$$f'(y) = 2k\frac{y^2}{1+y^2}$$

which is positive and increasing, with a minimum value of $f'(a) \asymp k^p$ on the interval of integration. It follows from Fact 3.0.10 that

$$\int_a^b e^{if(y)}dy \lesssim \frac{1}{f'(a)} \lesssim k^{-p}$$

and therefore

$$k\int_{\sqrt{2k^{p-1}+k^{2(p-1)}}}^{\sqrt{(t/k)^2-1}} \sin(2k(y - \arctan y))dy = O(k^{1-p}).$$

The main term is therefore

$$\begin{aligned}\frac{k}{\pi}\int \sec^2(\beta)d\beta &= \frac{k}{\pi}\tan\beta\Big]_{\sec^{-1}(1+k^{p-1})}^{\beta=\sec^{-1}(t/k)}\\ &= \frac{1}{\pi}\sqrt{t^2-k^2}+O(k\sqrt{(1+k^{p-1})^2-1})\\ &= \frac{1}{\pi}\sqrt{t^2-k^2}+O(k^{(p+1)/2})\end{aligned}$$

In order for this to be larger than our estimates for the initial range, we take $3p-1>0$. For the intermediate range to be smaller than the main term, we take $1/3+p<1$. Thus any exponent $1/3<p<2/3$ is allowed. For definiteness, we can take $p=1/2$, although a value closer to $1/3$ would be more natural from the point of view of the transition for J_k. Combining the three ranges shows that for $k<t$ (strictly, for $k+k^p<t$)

$$\int_0^t xJ_k(x)^2dx = \frac{1}{\pi}\sqrt{t^2-k^2}+O\big(e^{-ck^{(3p-1)/2}}+k^{(p+1)/2}+\log t+k^{1-p}+\log(t^2-k^2)\big).$$

We would like to take $p=1/3$ to balance the powers of k, but the implicit constant diverges because of the initial range. However, we can choose p slightly larger than $1/3$ to obtain, for any $\eta>0$,

$$\int_0^t xJ_k(x)^2dx = \frac{1}{\pi}\sqrt{t^2-k^2}+O_\eta(k^{2/3+\eta}+\log(t^2-k^2)). \tag{3.1.4}$$

When k is slightly larger, so that $k-k^p>t$, only the initial segment contributes. In this case, the integral is dominated by $\exp(-ck^{(3p-1)/2})$ and is therefore negligible. If $k-k^p<t<k+k^p$ so that the transition region contributes, the integral is still at most $O(k^{1/3+p})$.

The coefficients at hand are given by a ratio of these integrals with $t=rm$ relative

to $t = \pi m$. In the latter case, t is always substantially larger than k and we get

$$\int_0^{\pi m} x J_k(x)^2 dx = m + O_\eta(k^{2/3+\eta}).$$

The ratio is

$$\frac{\int_0^{rm} x J_k(x)^2 dx}{\int_0^{\pi m} x J_k(x)^2} = \frac{r}{\pi}\sqrt{1-(k/(rm))^2} + O_\eta\left(\frac{k^{2/3+\eta}}{m}\right).$$

When we incorporate the error from Hilb's formula, we get

$$\lambda_k = \frac{1}{(2m+1)\,\mathrm{vol}(B_r)}\left(\frac{r}{\pi}\sqrt{1-\left(\frac{k}{rm}\right)^2} + O_\eta\left(\frac{k^{2/3+\eta}}{m} + \frac{rm}{2^k m^{3/2}}\right)\right). \tag{3.1.5}$$

That is, since each appears for two different basis functions (sin versus cos)

$$\lambda_{2k} = \lambda_{2k+1} = \frac{1}{2\pi^2}\sqrt{1-(k/(rm))^2}\frac{1}{rm}\left(1 + O_\eta\left(\frac{k^{2/3+\eta}}{rm}\right)\right). \tag{3.1.6}$$

This explains the elliptical shape in Figure 3. Also, to leading order, the coefficients just for $k < rm$ are enough to match the expected value of X. Indeed,

$$\mathbb{E}\left[\sum_{j<2rm} \mathfrak{z}_j^2 \lambda_j\right] \sim 2\frac{1}{2\pi^2}\int_0^1 \sqrt{1-u^2}du = \frac{1}{4\pi} = \mathbb{E}[X] \tag{3.1.7}$$

up to an error of $O((rm)^{-1/3})$. We also have, with D the nearest integer to rm,

$$\mathrm{var}\left[\sum_{k=1}^{D} \mathfrak{z}_k^2 \lambda_k\right] = \sum_k 2\lambda_k^2 = \frac{\pi^{-4}}{D}\left(\int_0^1 1-u^2 du + O(1/(rm))\right) = \frac{2}{3\pi^4}\frac{1}{D} + O(D^{-2}) \tag{3.1.8}$$

which is another way to see that the variance is of order $1/(rm)$, and even to find the constant of proportionality. Higher moments can likewise be expressed in terms of sums of powers of λ_k, and then estimated by integrals of $(1-u^2)^{M/2}$.

3.2 Proof of the central limit theorem

For $s \geq 0$ small enough that $1 - 2s\lambda_j > 0$ for all j, we had

$$\mathbb{E}\left[e^{sX}\right] = \prod_j (1 - 2s\lambda_j)^{-1/2} \tag{3.2.1}$$

To compare Z with a standard Gaussian, we use its characteristic function. For a Gaussian G of mean 0 and variance 1, we have

$$\mathbb{E}[e^{itG}] = e^{-t^2/2}.$$

For Z, we have

$$\mathbb{E}[e^{itZ}] = e^{-i\mathbb{E}[X]t/\sigma}\mathbb{E}[iXt/\sigma] \tag{3.2.2}$$

Expanding $\log(1 - 2s\lambda_j)$ in a power series gives

$$\log \mathbb{E}\left[e^{sX}\right] = \sum_j \log(1 - 2s\lambda_j)^{-1/2} = s\sum_j \lambda_j + s^2 \sum_j \lambda_j^2 + \sum_{p=3}^{\infty} \frac{(2s)^p}{p} \sum_j \lambda_j^p. \tag{3.2.3}$$

Write, provisionally, $s = it/\sigma$. For this to be a valid choice, we will have to verify that, for all j,

$$2|t|\lambda_j/\sigma < 1. \tag{3.2.4}$$

Note that $\mathbb{E}[X] = \sum_j \lambda_j$. Therefore the first term $s\sum_j \lambda_j$ will cancel when we subtract $\mathbb{E}[X]$, leaving

$$\log \mathbb{E}\left[e^{itZ}\right] = -t^2 \frac{\sum_j \lambda_j^2}{\sigma^2} + \sum_{p=3}^{\infty} \frac{(2it)^p}{p} \frac{\sum_j \lambda_j^p}{\sigma^p}. \tag{3.2.5}$$

Since the variance σ^2 is exactly $2\sum \lambda_j^2$, the first term is $-t^2/2 = \log \mathbb{E}[e^{itG}]$. To show that Z is approximately Gaussian, the key step is to show that for $p \geq 3$

$$\frac{\sum \lambda_j^p}{\sigma^p} \to 0. \tag{3.2.6}$$

To verify (3.2.6), we can use the semicircle law from Proposition 3.0.1 to estimate $\sum \lambda_j^p$. There are three cases in the semicircle law, which we think of as "bulk", "edge", and "tail". First, we bound the contributions from j near rm (the edge) or larger (the tail). We have

$$\sum_{j \approx rm} \lambda_j^p \lesssim (rm)^{1/3} (rm)^{p(-4/3+\eta)}$$

For larger j, we have

$$\sum_{j + j^{1/3+\delta} > rm} \lambda_j^p \lesssim \frac{1}{(rm)^2} \sum_{k > rm} \exp(-ck^{3\delta/2})$$

The tail sum is very small. For example, comparing to an integral gives

$$\sum_{x > T} e^{-x^\varepsilon} \lesssim \int_T^\infty e^{-x^\varepsilon} dx \lesssim \frac{1}{\varepsilon} e^{-T^\varepsilon} T^{1-\varepsilon}$$

with $\varepsilon = 3\delta/2$, $x \approx c^{2/(3\delta)} k$ so that $x^\varepsilon \approx ck^{3\delta/2}$, and $T = c^{1/\varepsilon} rm$. Meanwhile, the "bulk" contribution is

$$\begin{aligned} \sum_{j < rm - (rm)^{1/3+\delta}} \lambda_j^p &= 2 \sum_{k=1}^{rm} \left(\frac{1}{2\pi^2} \sqrt{1 - (k/(rm))^2} \frac{1}{rm} \left(1 + O_\eta \left(\frac{k^{2/3+\eta}}{rm} \right) \right) \right)^p \\ &= \frac{2}{(2\pi^2)^p} \frac{1}{rm} \sum_{k=1}^{rm} \left(1 - \left(\frac{k}{rm} \right)^2 \right)^{p/2} \frac{rm}{(rm)^p} \left(1 + O_\eta \left(\frac{pk^{2/3+\eta}}{rm} \right) \right) \end{aligned}$$

For the main term, we have a Riemann sum approximation

$$\frac{1}{rm} \sum_{k=1}^{rm} (1 - (k/(rm))^2)^{p/2} \sim \int_0^1 (1 - u^2)^{p/2} du.$$

For the error term, we have

$$\frac{2p}{(2\pi^2)^p (rm)^{p+1}} \sum_{k=1}^{rm} \left(1 - \left(\frac{k}{rm}\right)^2\right)^{p/2} k^{2/3+\eta} \lesssim \frac{p}{(2\pi^2)^p} \frac{(rm)^{2/3+\eta}}{(rm)^p} \int_0^1 (1-u^2)^{p/2} u^{2/3+\eta} du.$$

Thus the bulk contribution is

$$\sum_{j \in \text{bulk}} \lambda_j^p = \frac{2}{(2\pi^2)^p} (rm)^{-p+1} \left(\int_0^1 (1-u^2)^{p/2} du + O_{p,\eta}\big((rm)^{-1/3+\eta}\big) \right). \tag{3.2.7}$$

Compare this to the edge contribution, bounded by $(rm)^{(-4/3+\eta)p+1/3}$, and the tail contribution, which obeys the even stronger bound $\exp(-c(rm)^{3\delta/2})$. For any chosen $\eta < 1/3$, these are negligible compared to the main term of order $(rm)^{-p+1}$ and even to the error term of order $(rm)^{-p+1-1/3+\eta}$. Thus we have, for the sum over all three ranges,

$$\sum_j \lambda_j^p = \frac{2}{(2\pi^2)^p} (rm)^{-p+1} \left(\int_0^1 (1-u^2)^{p/2} du + O_{p,\eta}\big((rm)^{-1/3+\eta}\big) \right)$$

with an implicit constant proportional to $p \int_0^1 (1-u^2)^{p/2} u^{2/3+\eta} du$. In particular,

$$\sigma^2 = 2 \sum_j \lambda_j^2 \sim \frac{1}{\pi^4} \frac{1}{rm}. \tag{3.2.8}$$

From this, we obtain

$$\frac{\sum \lambda_j^p}{\sigma^p} \lesssim \frac{(rm)^{-p+1}}{(rm)^{-p/2}} = (rm)^{-p/2+1}, \tag{3.2.9}$$

which is negligible as $rm \to \infty$ provided $p \geq 3$. Even when we sum over all $p \geq 3$, we obtain a geometric series:

$$\sum_{p=3}^{\infty} \frac{\sum \lambda_j^p}{\sigma^p} \lesssim rm \sum_{p=3}^{\infty} \sqrt{rm}^{-p} \lesssim (rm)^{-1/2}.$$

A final detail remains: We took $s = it/\sigma$, which we now justify by using the

semicircle law to show that the power series for $\log(1-2s\lambda_j)$ does converge at that point. From the semicircle law, the largest λ_j are of order $1/(rm)$, whereas σ is of order $1/\sqrt{rm}$. It follows that, for any given $t \in \mathbb{R}$, once rm is sufficiently large we do have

$$2|t|\frac{\lambda_j}{\sigma} < 1$$

for all j. The series will converge once rm is so large that

$$\frac{1}{\sqrt{rm}} < \frac{1}{2|t|}. \tag{3.2.10}$$

Combining these estimates, we have

$$\log \mathbb{E}[e^{itZ}] = \frac{-t^2}{2} + \sum_{p=3}^{\infty} \left(\frac{\sum_j \lambda_j^p}{\sigma^p} \right) (2it)^p = -\frac{t^2}{2} + O\left((rm)^{-1/2}|t|^3\right)$$

and hence

$$\mathbb{E}\left[e^{itZ}\right] = \left(1 + O\big((rm)^{-1/2}|t|^3\big)\right) e^{-t^2/2}.$$

For any fixed t, this converges to $e^{-t^2/2}$ as $rm \to \infty$, which completes our mission.

3.3 Bounds for the variance

In the case of S^2, the variance formula from chapter 3 reads

Lemma 3.3.1. *For any point $z \in S^2$,*

$$\mathrm{var}[X_z] = \frac{2}{\mathrm{vol}(S^2)^2} \int_{B_r(z)} \int_{B_r(z)} P_m(x \cdot x')^2 \frac{dx}{\mathrm{vol}(B_r)} \frac{dx'}{\mathrm{vol}(B_r)}, \tag{3.3.1}$$

where P_m is the Legendre polynomial of degree m normalized so that $P_m(1) = 1$. In particular,

$$\mathrm{var}[X_z] \asymp \frac{1}{rm}. \tag{3.3.2}$$

Equation (3.3.1) is an exact formula: It holds regardless of the relative sizes of r and m. But if $rm \to \infty$, then (3.3.2) shows that the variance converges to 0. This is good enough for us to conclude using Chebyshev's inequality that at any point z

$$\mathbb{P}\left\{|X_z - \mathbb{E}[X_z]| > \varepsilon\right\} \leq \frac{\operatorname{var}(X_z)}{\varepsilon^2} \lesssim \frac{1}{\varepsilon^2}\frac{1}{rm} \to 0,$$

as long as $rm \to \infty$. For smaller r, the variance remains of order 1 or even diverges.

Proof. The algebraic part of the argument is the same as in chapter 2. We turn to the proof of Equation (3.3.2) to illustrate how it follows from classical estimates for $P_m(\cos\theta)$. We can give explicit values for constants that were left implicit in the general case. We have Bernstein's inequality

$$P_m(\cos\theta)^2 \leq \frac{2}{\pi}\frac{1}{m\sin\theta} \leq \frac{1}{m\theta}$$

which improves on the trivial bound $P_m(\cos\theta)^2 \leq 1$ once $\theta = d(x, x') > 1/m$. Since $d(x, x')$ ranges all the way up to $2r$, if we assume that $rm \to \infty$, most values of θ appearing in the integral will enjoy a substantially improved bound on $P_m(\cos\theta)$. Fix $x \in B_r(z)$. The points x' lie in a ball $B_{2r}(x)$ around x, by the triangle inequality, and the integral of $P_m^2 \geq 0$ can only increase if we include all $x' \in B_{2r}(x)$ instead of only those in $B_r(z) \cap B_{2r}(x)$. Therefore, using spherical coordinates with respect to x on $B_{2r}(x)$,

$$\begin{aligned}
\int_{B_r(z)}\int_{B_r(z)} P_m(x\cdot x')^2 dx'dx &\leq \int_{B_r(z)}\int_0^{2\pi}\int_0^{2r} P_m(\cos\theta)^2 \sin\theta d\theta d\alpha dx \\
&\leq \int_{B_r(z)} 2\pi \int_0^{2r} \frac{2}{\pi}\frac{1}{m\sin\theta}\sin\theta d\theta dx \\
&= 8\frac{r\operatorname{vol}(B_r)}{m} \\
&\leq 2\pi\frac{\operatorname{vol}(B_r)^2}{rm}
\end{aligned}$$

by Bernstein's inequality (Fact 3.0.4). We also used $\mathrm{vol}(B_r) = 4\pi \sin(r/2)^2$ and $\sin(r/2) \geq r/\pi$ for $r \leq \pi$. Thus, by (3.3.1), $\mathrm{var}[X_z] \leq C/(rm)$ with $C = 1/(4\pi)$.

The upper bound on $\mathrm{var}[X_z]$ holds for any fixed m. To give a lower bound, we assume $rm \to \infty$. Then Hilb's asymptotics for P_m show that this integral really is of order $(rm)^{-1}\mathrm{vol}(B_r)^2$. Let $x \cdot x' = \cos\theta$, so $\theta = d(x, x')$, and let $\xi = d(z, x)$. By the triangle inequality, $B_{r-\xi}(x) \subset B_r(z)$. The integrand is nonnegative, so we have a lower bound

$$\begin{aligned}
&\int_{B_r(z)} \int_{B_r(z)} P_m(x \cdot x')^2 dx dx' \\
&\geq \int_{B_r(z)} \int_{B_{r-\xi}(x)} P_m(x \cdot x')^2 dx' dx \\
&= \int_0^{2\pi} \int_0^r \int_{\alpha=0}^{2\pi} \int_{\theta=0}^{r-\xi} P_m(\cos\theta)^2 \sin\theta d\theta d\alpha' \sin\xi d\xi d\alpha \\
&= (2\pi)^2 \int_0^r \int_0^{r-\xi} \left(\frac{\theta}{\sin\theta} J_0((m+1/2)\theta)^2 + O(m^{-3/2}) \right) \sin\theta d\theta \sin\xi d\xi \\
&= (2\pi)^2 \int_0^r \int_0^{(r-\xi)(m+1/2)} u J_0(u)^2 du (m+1/2)^{-2} \sin\xi d\xi + O\left(m^{-3/2} r^4\right) \\
&= (2\pi)^2 \int_0^r \frac{(r-\xi)^2}{2} \left(J_0^2 + J_1^2\right) ((r-\xi)(m+1/2)) \sin\xi d\xi + O(m^{-3/2}\,\mathrm{vol}(B_r)^2)
\end{aligned}$$

At this point, we restrict the range of integration further to $0 \leq \xi < (1-\delta)r$ so that $(r-\xi)(m+1/2) \geq \delta rm$, which grows without bound by assumption. This allows us to use 3.0.9. The result is that

$$\int_{B_r(z)} \int_{B_r(z)} P_m(x \cdot x')^2 \frac{dx dx'}{\mathrm{vol}(B_r)^2} \geq \frac{(1-\delta)^2}{rm} \left(\frac{1}{2} - \frac{1-\delta}{3} \right) + O((rm)^{-2} + m^{-3/2})$$

Taking $\delta \to 0$, we have that for $rm \to \infty$,

$$\mathrm{var}[X_z] \geq \frac{2}{(4\pi)^2} \frac{1}{6} \frac{1}{rm} \geq \frac{1}{480} \frac{1}{rm} \tag{3.3.3}$$

We have used the Facts 3.0.6, 3.0.7, and 3.0.9. □

There is a factor of 12π between the crude upper and lower bounds above. One could use spherical trigonometry to evaluate the double integral more exactly, but upper and lower bounds of order $1/(rm)$ are all we need. We can also express the variance as $2\sum \lambda_j^2$ and use Proposition 3.0.1 to estimate the coefficients λ_j. See equation (3.1.8).

3.4 Union bound over a grid

This step proceeds very similarly to the general case but we point out certain simplifications. There is no possibility of $\mathbb{E}[X_z]$ and $\mathbb{E}[X_{z'}]$ differing at all, saving us that step from the general case. It is also possible to be more precise about the "off-grid" scenario. Form a (deterministic) grid of points z_j on S^2 such that every point is within δ of one of the gridpoints. If there is a point z such that

$$\left| X_z - \frac{1}{4\pi} \right| > \varepsilon,$$

then we can expect the discrepancy to be high also for a nearby gridpoint. Indeed, if $d(z, z_j) < \delta$, then

$$\varepsilon < \left| X_z - \frac{1}{4\pi} \right| \leq \left| X_{z_j} - \frac{1}{4\pi} \right| + \left| X_z - X_{z_j} \right| \lesssim \left| X_{z_j} - \frac{1}{4\pi} \right| + \frac{\delta}{r} \|\phi\|_\infty^2.$$

The last step follows from comparing integrals over two nearby balls as follows. For two sets B and B', we have

$$\left| \int_B \phi^2 - \int_{B'} \phi^2 \right| \leq \int_{B \Delta B'} \phi^2 \leq \mathrm{vol}(B \Delta B') \|\phi\|_\infty^2$$

For balls $B = B_r(z)$ and $B' = B_r(z')$, the volume of the symmetric difference depends both on r and on the separation $\delta = d(z, z')$ between their centers. We have

$$\mathrm{vol}(B_r(z)\Delta B_r(z')) = O(\delta r)$$

by comparison with Euclidean rectangles, or by a more accurate calculation. Passing to averages, this gives

$$|X_z - X_{z'}| \leq \frac{\mathrm{vol}(B_r(z)\Delta B_r(z'))}{\mathrm{vol}(B_r)}\|\phi\|_\infty^2 \lesssim \frac{\delta}{r}\|\phi\|_\infty^2.$$

So either (writing X_j for X_{z_j}) there is a j such that $|X_j - \frac{1}{4\pi}| > \varepsilon/2$
or $\|\phi\|_\infty^2 \gtrsim r\varepsilon/\delta$.

To handle the off-grid case on an arbitrary manifold, we gave a crude estimate of $\|\phi\|_\infty$. On the sphere, we can quote more precise results. It follows from Théorème 7 in the paper [15] of Burq and Lebeau that $\|\phi\|_\infty$ is, with high probability, on the order of $\sqrt{\log m}$. Canzani and Hanin give another proof of this in [16]. Thus the latter case where $\|\phi\|_\infty^2$ is at least of order $r\varepsilon/\delta$ is very unlikely provided that we have a growing lower bound:

$$\frac{\|\phi\|_\infty^2}{\log m} \gtrsim \frac{r\varepsilon}{\delta \log m} \to \infty$$

as $rm \to \infty$. We can rewrite this in the form

$$\frac{\|\phi\|_\infty^2}{\log m} \gtrsim \frac{rm}{\log m}\frac{\varepsilon}{m\delta}$$

By hypothesis, rm is asymptotically larger than $\log m$. So, for any fixed ε, we can choose δ to be $1/m$. Then the probability of this case occurring will go to 0 as $rm \to \infty$.

For the former case, we have a union bound:

$$\begin{aligned}\mathbb{P}\left\{\exists j \ \left|X_j - \frac{1}{4\pi}\right| > \varepsilon/2\right\} &\leq (\text{number of points})\,\mathbb{P}\left\{\left|X_1 - \frac{1}{4\pi}\right| > \varepsilon/2\right\} \\ &\lesssim \delta^{-2}\mathbb{P}\left\{\left|X_1 - \frac{1}{4\pi}\right| > \varepsilon/2\right\} \\ &\lesssim m^2\mathbb{P}\left\{\left|X_1 - \frac{1}{4\pi}\right| > \varepsilon/2\right\}\end{aligned}$$

With $\delta = 1/m$ as above, we see that the union bound has cost us a factor of m^2, and we would pay an even steeper price of m^d to apply it in d dimensions. To afford it, we appeal to Lemma 3.0.2. Since $rm/\log m \to \infty$, the bound $\exp(-c(\varepsilon)rm)$ is $o(m^{-d})$ for any d.

In fact, Burq and Lebeau show that $\mathbb{P}\{\|\phi\|_\infty > c_0\sqrt{\log m} + r\} \leq Ce^{-cr^2}$ for a specific constant c_0 and positive constants C and c. In our context, this shows that the probability of the latter case is exponentially small with respect to $\varepsilon^2 rm$. Thus it is no worse than the bound from Lemma 3.0.2 that we apply to the former case. The rate of convergence in Theorem 1.4.1 is thus $O(\exp(-c(\varepsilon)rm))$.

3.5 Chernoff bound

On the sphere, we can choose almost the optimal s in the Chernoff bound. In general, it is not clear whether the corresponding choice is valid: It might be too large. We repeat the argument to illustrate what goes right on the sphere. The tail bound Lemma 3.0.2 is a special case of a more general fact about quadratic forms in Gaussians, which we state as

Proposition 3.5.1. *If $\mathfrak{z}_j$ are independent Gaussians of mean 0 and variance 1 for*

$1 \le j \le D$, *and the weights* $\lambda_j \ge 0$ *satisfy*

$$\frac{A^-}{D} \le \sum_{j=1}^{D} \lambda_j^2 \le \frac{A^+}{D} \tag{3.5.1}$$

and

$$\max_{j=1}^{D} \lambda_j \le \frac{M}{D} \tag{3.5.2}$$

then the random variable $X = \sum_j \lambda_j \mathfrak{z}_j^2$ *has exponential concentration as* $D \to \infty$*: For any fixed* $\varepsilon > 0$*, there is a positive rate* $c(\varepsilon) > 0$ *such that*

$$\mathbb{P}\{|X - \mathbb{E}[X]| > \varepsilon\} \le \exp(-c(\varepsilon)D). \tag{3.5.3}$$

For example, if each λ_j is $1/D$, then X is a rescaled χ^2 random variable with D degrees of freedom, which exhibits concentration for large D. The role of the hypotheses is just to allow us to truncate the Taylor expansion of $\log(1 \pm x)$, and assumption (3.5.2) could be relaxed to an upper bound on $\sum \lambda_j^3$. In our application, D will be of order rm.

Proof. The Chernoff bound is

$$\mathbb{P}\{X > \mathbb{E}[X] + \varepsilon\} = \mathbb{P}\left\{e^{sX} > e^{s(\mathbb{E}[X]+\varepsilon)}\right\} \le \frac{\mathbb{E}\left[e^{sX}\right]}{e^{s(\mathbb{E}[X]+\varepsilon)}}$$

where, given $\varepsilon > 0$, the parameter s is chosen to minimize the upper bound. Choosing $s = 0$ would give the trivial bound that probabilities are at most 1. Choosing an s for which $\mathbb{E}[e^{sX}]$ is infinite would be even worse. We write $X = \sum_j \lambda_j \mathfrak{z}_j^2$ for a quadratic form in Gaussian random variables $\mathfrak{z}_j$. In our case, the sum is indexed by $1 \le j \le 2m + 1$ and

$$\lambda_j = \frac{1}{2m+1} \frac{1}{\mathrm{vol}(B_r)} \int_{B_r} \phi_j^2.$$

In general, we take $j \le D$ as our indices and allow the coefficients $\lambda_j = \lambda_j(D)$ to

depend on the number of variables. The moment generating function can be computed explicitly. For $s \geq 0$ small enough that $1 - 2s\lambda_j > 0$ for all j,

$$\mathbb{E}\left[e^{sX}\right] = \prod_j (1 - 2s\lambda_j)^{-1/2} \tag{3.5.4}$$

since, by independence of the variables $\mathfrak{z}_j$, the quantity on the left factors as a product of Gaussian integrals. By differentiation, the optimal s would solve

$$\sum_j \frac{\lambda_j}{1 - 2s\lambda_j} = \mathbb{E}[X] + \varepsilon.$$

Expanding the left in a geometric series gives

$$\sum_{\nu=1}^{\infty} (2s)^{\nu-1} \sum_j \lambda_j^\nu = \mathbb{E}[X] + \varepsilon.$$

Note that the first term $\nu = 1$ in the sum on the left is $\sum_j \lambda_j = \mathbb{E}[X]$, which cancels with the right. We may thus rewrite the equation for the optimal s as

$$\sum_{\nu=2}^{\infty} (2s)^{\nu-1} \sum_j \lambda_j^\nu = \varepsilon. \tag{3.5.5}$$

Any choice of s gives some bound, and it is natural to choose s by truncating this geometric series and solving the resulting equation. Keeping only the first term gives

$$s_1 = \frac{\varepsilon}{2} \frac{1}{\sum_j \lambda_j^2}.$$

One could keep two terms and solve a quadratic equation to get

$$s_2 = \frac{\sum_j \lambda_j^2}{4 \sum_j \lambda_j^3} \left(\sqrt{1 + 4\varepsilon \frac{\sum \lambda_j^3}{(\sum \lambda_j^2)}} - 1 \right)$$

which agrees with s_1 to first order in ε. We will content ourselves with s_1. When we

expand the logarithm, the terms of order ε^1 cancel so that s_1 gives

$$\begin{aligned}
&\mathbb{P}\{X > \mathbb{E}[X] + \varepsilon\} \leq \mathbb{E}[e^{s_1 X}]e^{-s_1(\mathbb{E}[X]+\varepsilon)} \\
&= \prod_j \left(1 - \frac{\varepsilon}{\sum \lambda_j^2}\lambda_j\right)^{-1/2} \exp\left(-\varepsilon\mathbb{E}[X]/\left(2\sum \lambda_j^2\right)\right) \exp\left(-\varepsilon^2/\left(2\sum \lambda_j^2\right)\right) \\
&= \exp\left(-\frac{\varepsilon^2}{2\sum \lambda_j^2} - \varepsilon\frac{\sum \lambda_j}{2\sum \lambda_j^2} - \frac{1}{2}\sum_j \log\left(1 - \varepsilon\frac{\lambda_j}{\sum_k \lambda_k^2}\right)\right) \\
&= \exp\left(-\frac{\varepsilon^2}{4\sum \lambda_j^2} + \sum_{\nu=3}^{\infty} \frac{1}{2\nu}\frac{\sum \lambda_j^\nu}{(\sum \lambda_j^2)^\nu}\varepsilon^\nu\right)
\end{aligned}$$

For $0 \leq x < 1/3$, we have the one-variable calculus exercise

$$-\log(1-x) \leq x + \frac{3}{4}x^2.$$

Indeed, the claim follows for small x from the series expansion for log and the range $x < 1/3$ guarantees that the difference between the right and the left is in fact increasing. If we can take $x = \varepsilon\lambda_j/\sum \lambda_k^2$, which we will see shortly really is less than $1/3$, then this will bound the product:

$$\prod_j \left(1 - \frac{\varepsilon\lambda_j}{\sum \lambda_k^2}\right)^{-1/2} \leq \exp\left(\frac{1}{2}\frac{\varepsilon\mathbb{E}[X]}{\sum \lambda_k^2} + \frac{3}{8}\frac{\varepsilon^2}{\sum \lambda_k^2}\right)$$

The terms that are first-order in ε cancel and the numbers have been rigged so that $3/8 - 1/2 = -1/8 < 0$, which gives a negative coefficient of ε^2. The resulting bound is

$$\mathbb{P}\{X > \mathbb{E}[X] + \varepsilon\} \leq \exp\left(-\frac{\varepsilon^2}{8}\frac{1}{\sum_j \lambda_j^2}\right)$$

Assuming that $\sum \lambda_j^2 \leq A_2/D$, this implies that

$$\mathbb{P}\{X > \mathbb{E}[X] + \varepsilon\} \leq \exp(-c(\varepsilon)D)$$

with $c(\varepsilon) = \varepsilon^2/(8A_2)$ quadratic in ε. Thus the probability of a deviation above the mean is exponentially small in D, as required. We claimed above that for each j, we may assume $\varepsilon\lambda_j / \sum \lambda_k^2 < 1/3$ or, in other words, that $\lambda_{\max} < \frac{1}{3\varepsilon} \sum \lambda_k^2$. One could certainly replace $1/3$ by any $\alpha < 1$ through a more vigorous Taylor expansion. The important point is that $\lambda_{\max}$ and $\sum \lambda_k^2$ have the same order of magnitude as $D \to \infty$, namely $1/D$. For if $\lambda_{\max} \leq M/D$ and $A^-/D \leq \sum \lambda_k^2$, then we will be guaranteed that $\lambda_{\max} < 1/(3\varepsilon) \sum \lambda_k^2$ as long as $\varepsilon < A^-/(3M)$ is sufficiently small (in absolute terms, with no reference to D).

For the lower tail, we rewrite $X < \mathbb{E}[X] - \varepsilon$ as $-X > \mathbb{E}[-X] + \varepsilon$ and apply the argument above to $Y = -X$. The details are slightly different because the moment generating function is now

$$\mathbb{E}[e^{sY}] = \prod_k (1 + 2s\lambda_k)^{-1/2} \tag{3.5.6}$$

with a $1 + 2s\lambda_k$ instead of $1 - 2s\lambda_k$ in each factor. Thus any $s \geq 0$ is allowed and yields the bound

$$\begin{aligned}
\mathbb{P}\{X < \mathbb{E}[X] - \varepsilon\} &= \mathbb{P}\{Y > \mathbb{E}[Y] + \varepsilon\} \\
&= \mathbb{P}\{e^{sY} > e^{s(\mathbb{E}[Y]+\varepsilon)}\} \\
&\leq \mathbb{E}[e^{sY}] \exp(-s(\mathbb{E}[Y] + \varepsilon)) \\
&= \exp(s(\mathbb{E}[X] - \varepsilon)) \prod_k (1 + 2s\lambda_k)^{-1/2} \\
&= \exp\left(-\varepsilon s + s \sum_k \lambda_k + \frac{1}{2} \sum_k - \log(1 + 2s\lambda_k) \right)
\end{aligned}$$

The optimal s would solve

$$\sum_k \frac{\lambda_k}{1 + 2s\lambda_k} = \mathbb{E}[X] - \varepsilon.$$

The first-order choice of s is again

$$s_1 = \frac{\varepsilon}{2} \frac{1}{\sum_k \lambda_k^2}$$

although the second-order choice is different than in the case of the upper tail:

$$s_2^- = \frac{\sum_k \lambda_k^2}{4 \sum_k \lambda_k^3} \left(1 - \sqrt{1 - 4\varepsilon \frac{\sum_k \lambda_k^3}{(\sum_k \lambda_k^2)^2}} \right).$$

Choosing s_1 and using the inequality $-\log(1+x) \leq -x + x^2/2$ for $x \geq 0$ gives an upper bound of

$$\begin{aligned}
&\exp\left(-\varepsilon s + s \sum_k \lambda_k + \frac{1}{2} \sum_k - \log(1 + 2s\lambda_k) \right) \\
&\leq \exp\left(-\frac{\varepsilon^2}{2 \sum_k \lambda_k^2} + \frac{\varepsilon^2}{4(\sum_k \lambda_k^2)} \right) \\
&= \exp\left(-\frac{\varepsilon^2}{4} \left(\sum_k \lambda_k^2 \right)^{-1} \right) \\
&\leq \exp(-c(\varepsilon) D).
\end{aligned}$$

Thus the lower tail is also exponentially unlikely in D, provided only that $\sum \lambda_k^2 \leq A^+ D$. □

Another way to prove Propostion 3.5.1 is to complexify and consider $\mathbb{E}[e^{itX}]$ instead of $\mathbb{E}[e^{sX}]$. Inverting the Fourier transform recovers the density of X. One can shift contours to show that the density is exponentially small away from $\mathbb{E}[X]$, but some care is needed in truncating the integral $\int_{-\infty}^{\infty} e^{-ixt} \mathbb{E}[e^{itX}] dt$ to a finite range $\int_{-T}^{T}$ and shifting the finite segment to an imaginary height $[-T, T] + iH$. The parameters T and H will both be small multiples of D, depending on the constants in the hypotheses, with sizes constrained relative to each other.

The sum $\sum \lambda_j^2$ is nothing but the variance (or half the variance) that we have seen is of order $1/(rm)$. The largest coefficient $\lambda_{\max}$ is also of order $1/(rm)$, from the semicircle law. For ε small enough, we are thus guaranteed that $\varepsilon\lambda_j / \sum \lambda_k^2 < 1/3$ for all j, as promised above. The argument above then applies, showing that the probability is exponentially small in rm. This is enough to overcome any factor m^2 or even a higher power coming from the union bound, as long as rm is asymptotically larger than $\log m$. What we lack at present in the general case is a guarantee that $\lambda_{\max}$ is of the same order of magnitude as $\sum \lambda_j^2$ (or smaller).

Conclusion

We have approximated the supremum

$$\sup_{z \in S^2} \left| \frac{1}{\mathrm{vol}(B_r)} \int_{B_r(z)} \phi^2 - \frac{1}{4\pi} \right|$$

by a maximum over only finitely many points z. To control the error introduced this way, we made a brutish argument based on the union bound. We discuss a more sophisticated tool below, but the union bound is not as crude as it might seem. The exponentially light tail given by Lemma 3.0.2 is at the heart of why Theorem 1.4.1 is true. A helpful analogy is given by k balls thrown at random into n boxes, where one asks for the probability that each box receives close to k/n balls as expected.

Dudley [20] proved a general bound that applies to a separable, subgaussian process X_t indexed by a metric space (T, d). Normalizing so that $\mathbb{E}[X_t] = 0$ for convenience, the subgaussian assumption is that for all $\lambda \geq 0$,

$$\mathbb{E}[e^{\lambda(X_s - X_t)}] \leq e^{\lambda^2 d(s,t)^2/2}.$$

Dudley's conclusion is that

$$\mathbb{E}\left[\sup_{t \in T} X_t\right] \lesssim \int_0^\infty \sqrt{\log N(T, d, \varepsilon)} d\varepsilon,$$

where $N(T, d, \varepsilon)$ is the smallest number of balls of radius ε, in terms of the metric d, needed to cover T. The constant hidden inside $\lesssim$ is absolute and could be taken to be 12. This entropy method was used effectively by Feng and Zeldtich in [24] and by Canzani and Hanin [16]. In applications, the metric d is given by

$$d(s,t) = \sqrt{\mathbb{E}[(X_s - X_t)^2]},$$

and it is not quite a metric because it is possible to have $d(s,t) = 0$ with $s \neq t$. In our context of random spherical harmonics, $T = S^2$ is the sphere and

$$X_z = X_z^{\pm} = \pm \left(\frac{1}{\mathrm{vol}(B_r)} \int_{B_r(z)} \phi^2 - \frac{1}{4\pi} \right).$$

The sign $\pm$ ensures that deviations above and below the mean can both be controlled. Taking χ and χ' in the proof of Lemma 3.3.1 to be the indicator functions of the balls $B_r(z)$ and $B_r(z')$, we can express the (squared) metric $d(z,z')^2$ as

$$\frac{4}{\mathrm{vol}(S^2)^4} \left(\int_{B_r} \int_{B_r} P_m(x \cdot x')^2 \frac{dx dx'}{\mathrm{vol}(B_r)^2} - \int_{B_r(z)} \int_{B_r(z')} P_m(x \cdot x')^2 \frac{dx dx'}{\mathrm{vol}(B_r)^2} \right).$$

By spherical symmetry, the first term $\int_{B_r} \int_{B_r}$ does not depend on the center of the ball B_r while the second term $\int_{B_r(z)} \int_{B_r(z')}$ depends only on the spherical distance between z and z'. We have $d(z,z') = 0$, and indeed the first term exactly equals the second when $z = z'$. The first term is of order $1/(rm)$, as we saw in Lemma 3.3.1. As z and z' become more distant, the second term decreases because of the decay of $P_m(x \cdot x')^2$ given, for example, by Fact 3.0.6. It would be interesting to give another proof of Theorem 1.4.1 by understanding the geometry of S^2 under this metric and, in particular, estimating the covering numbers $N(T, d, \varepsilon)$.

On a higher-dimensional sphere S^d in place of S^2, one can still diagonalize the quadratic form. The basis functions are obtained by separation of variables as

$J(\theta)Y(\alpha)$, where θ is the distance to a chosen origin and $\alpha \in S^{d-1}$ is an angular variable ranging over a sphere of dimension one less. The J factors are given by the zonal spherical harmonic and its derivatives, hence in terms of Gegenbauer polynomials instead of Legendre polynomials. The Y factors run over an orthonormal basis of spherical harmonics on S^{d-1}, playing the role of the trigonometric function. Since two such functions $Y(\alpha)$ and $Y'(\alpha)$ are orthogonal, the different functions $J(\theta)Y(\alpha)$ are orthogonal over any disk $\theta < r$.

Chapter 4

A lower bound on the Nazarov-Sodin constant

Consider a random spherical harmonic

$$f = \sum_{k=-n}^{n} \xi_k Y_k$$

where the ξ_k are independent, identically distributed Gaussian random variables of mean 0 and the $Y_k : S^2 \to \mathbb{R}$ are an orthonormal basis of spherical harmonics of degree n. Any non-zero multiple of f has the same zero set, and it is natural to normalize so that the expected value of $\int_{S^2} f^2$ is 1. This corresponds to a variance $\mathbb{E}\xi_k^2 = 1/(2n+1)$ for the coefficients. Nazarov and Sodin [53] prove that, as $n \to \infty$, the number $N(f)$ of connected components of $f^{-1}\{0\}$ obeys

$$\mathbb{E}[N(f)] \sim c_{\mathrm{NS}}(S^2) n^2$$

for some positive constant $c_{\mathrm{NS}}(S^2) > 0$. The method of Nazarov-Sodin can be adapted to higher dimensions and shows that, for spherical harmonics on S^d, the expected number of nodal domains will obey

$$\mathbb{E}[N(f)] \sim c_{\mathrm{NS}}(S^d) n^d.$$

For normalization, note that the random spherical harmonic on S^d is

$$f = \sum_{k=1}^{M} \xi_k Y_k$$

where M is the multiplicity of spherical harmonics of degree n, the functions Y_k are any orthonormal basis of harmonics, and the Gaussian coefficients now have variance $1/M$. We have $M = 2n + 1$ for S^2, and M is of order n^{d-1} in dimension d.

The result of Nazarov and Sodin involves a lower bound on $N(f)$ which would not hold deterministically for all f: Even as $n \to \infty$, there are harmonics with a bounded

number of nodal domains instead of roughly n^d. Thus it is necessary to randomize. The lower bound is proved by populating the sphere with many small disks, in each of which one compares f to a *barrier function*. Each disk contains a nodal domain of the barrier function and, with high probability, the comparison leads to a lower bound on the number of nodal domains of f. The goal of this paper is to see what the barrier method gives explicitly, both in a fixed dimension $d = 2$ or $d = 3$ and in the limit of very high dimension.

It is natural to state the results for the scaled quantity

$$c(d) = \frac{c_{\mathrm{NS}}(S^d)}{\mathrm{vol}(S^d)}$$

and the bounds are so small that it is easier to understand $\log\log \frac{1}{c(d)}$ instead.

Theorem 4.0.2. *As $d \to \infty$,*

$$\frac{4}{3}\log d \leq \log\log \frac{1}{c(d)} \leq d \log \frac{e}{2} + O(\log d)$$

There is a considerable gap, even at this coarse double-logarithmic scale, between the barrier method and the upper bound for $c(d)$ that we outline below. The bound $\log\log 1/c(d) \gtrsim \log d$ is deduced from a deterministic result by taking expectations. Thus it does not take much advantage of randomness, and perhaps $\log\log 1/c(d)$ is closer to the estimate d from the barrier method. Nevertheless, the theoretical maximum provided by Courant's nodal domain theorem is a natural point of comparison.

In specific dimensions, the barrier method gives

Theorem 4.0.3.

$$c(2) \geq 10^{-87}$$

Theorem 4.0.4.

$$c(3) \geq 10^{-1196}$$

In dimension 3, it is difficult to reliably estimate the constant from simulations because there are so few nodal domains. Nevertheless, it would be absurd to suggest 10^{-1196} is anywhere near the true value.

In dimension 2, 10^{-87} is a gross underestimate: The value suggested by simulations is closer to 6×10^{-2}. Nastasescu generated hundreds of harmonics of degree between 30 and 100 and determined the nodal domains of each one [51]. Fitting this data suggests that the Nazarov-Sodin constant for random spherical harmonics on S^2 is

$$0.0598 \pm 0.0003$$

For context, Bogomolny and Schmit had proposed that the constant is

$$\frac{3\sqrt{3} - 5}{\pi} = 0.062437255\ldots$$

on the basis of a percolation model [13]. Nastasescu's work shows that this prediction is too large. She also studied the random Fubini-Study ensemble, where one sums over harmonics of all degrees up to n instead of only those of degree exactly equal to n. The number of components is again of order n^2, but the constant factor is now

$$0.0195 \pm 0.0004.$$

Harnack proved that a real plane curve of genus g has at most $g + 1$ ovals, that is, roughly $n^2/2$ [33]. Comparing this maximum value with the approximate numerical constant $0.02 = 0.04 \times \frac{1}{2}$, one arrives at the attractive slogan that "the random plane curve is 4% Harnack" [67]. Gayet and Welschinger gave lower bounds for the number of components in this and more general related ensembles, as well as the higher Betti numbers ([27], Corollary 0.6). In dimension d, their lower bound is $\exp(-\exp(257d^{3/2}))$, with a top term of order $d^{3/2}$ instead of the d from our theorem

4.0.2. In dimension 2, their lower bound is

$$\exp(-\exp(514\sqrt{2}) + \log(4\pi)) = \exp(-4.9109 \times 10^{315})$$

leaving us still some way from 4% Harnack.

Konrad computed the Nazarov-Sodin constant for random plane waves instead of spherical harmonics [42]. Both ensembles have the same constant because the random plane wave is the scaling limit of the random spherical harmonic. An advantage of working in the plane is that the Fast Fourier Transform allows one to carry the computations farther than is practical with spherical harmonics. Konrad obtained a value

$$0.0589 \pm 1.42 \times 10^{-4}.$$

He also studied the Sinai billiard and the stadium billiard, where the respective constants are roughly 0.0596 and 0.0535. For these calculations, Konrad used a sample of tens of thousands of eigenfunctions. Beliaev and Kereta confirmed the value 0.0589 with an even larger sample [7].

Later, Nazarov and Sodin developed a framework that applies to very general ensembles of random functions f defined over any compact manifold [54]. The nodal set of f is studied by a scaling limit, leading to a random field F on the Euclidean space $\mathbb{R}^d$ of the same dimension as the manifold. Let $N(F, R)$ be the number of nodal domains of F intersecting a box $B = [-R, R]^d$, or equally well a ball $B = B_R(0)$ of radius R or a scaling by R of some other fixed convex body. Any of these variants obeys

$$N(F, R) \sim c_{\mathrm{NS}}(\rho)\,\mathrm{vol}(B)$$

where $c_{\mathrm{NS}}(\rho)$ is a nonnegative constant determined by the law of the random function F. This law is expressed through the spectral measure ρ, which encodes the correlations

between values of F via a Fourier transform:

$$\mathbb{E}[F(x)F(y)] = \int_{\mathbb{R}^d} e^{-2\pi i(x-y)\cdot\xi} d\rho(\xi).$$

In the case of random spherical harmonics, ρ is the uniform measure on the unit sphere $|\xi| = 1$. Likewise, for the monochromatic ensemble on any compact manifold, the scaling limit will be the same. The number of nodal domains of f then scales with the volume of the manifold:

$$c_{\mathrm{NS}}(M) = c_{\mathrm{NS}}(\rho)\,\mathrm{vol}(M)$$

where ρ is the spectral measure of the limiting random field. Nazarov-Sodin show that

$$c_{\mathrm{NS}}(\rho) = \mathbb{E}\left[\frac{1}{\mathrm{vol}(D)}\right]$$

where D is the connected component of $F^{-1}\{0\}$ containing the origin. The positivity of the constant thus implies that D has finite volume with some non-zero probability. The question of whether this probability is 1, or whether there can be an unbounded connected component, is a very interesting one related to percolation. There has been exciting progress establishing percolation of level sets in other ensembles (Rivera-Vanneuville [57], Beliaev-Muirhead [8], Beliaev-Muirhead-Wigman [9], Beffara-Gayet [6]). The results so far do not apply to the monochromatic ensemble because of the sign changes and slower-than-integrable decay of its covariance function $\mathbb{E}[f(x)f(y)]$.

4.1 Barrier method on the sphere

Theorem 4.1.1. *For any $\delta < 1/2$ and any ρ lying between the first and second zeros of $J_{d/2-1}(y)/y^{d/2-1}$, the Nazarov-Sodin constant in dimension d is at least*

$$c(d) := \frac{c_{\mathrm{NS}}(S^d)}{|S^d|} \geq \frac{2\Delta_{\mathbb{R}^d}}{|B_1|}\rho^{-d}(1-2\delta)\mathbb{P}(\mathfrak{z} \geq C_0/c_1)$$

where B_1 is the Euclidean unit ball in $\mathbb{R}^d$ and $|B_1|$ is its volume, $\Delta_{\mathbb{R}^d}$ is the sphere packing density. The quantity $C_0 = C_0(\delta, \rho)$ must be large enough that

$$C_0^2 \geq \frac{1}{\delta}\frac{C}{\mathrm{vol}(S^d)}(\rho+1)^d$$

where

$$C = \frac{1}{2^{d-2}\Gamma(d/2)^2}\frac{|B_1|}{\mathrm{vol}(S^{d-1})}\left(\int_0^1 tJ_{d/2-1}(t)^2dt\right)^{-1}$$

while $c_1 = c_1(\rho)$ is proportional to $|J_{d/2-1}(\rho)/\rho^{d/2-1}$:

$$c_1 = |S^d|^{-1/2}2^{d/2-1}\Gamma(d/2)|J_{d/2-1}(\rho)|\rho^{-(d/2-1)}.$$

In the rest of this section, we explain the barrier method and outline the proof of Theorem 4.1.1, deferring the analysis of C_0 and c_1 to later sections. To prove the existence and positivity of their constant for S^2, Nazarov-Sodin gave essentially the proof below but with δ chosen to be $1/3$ for concreteness, and with $2C_0/c_1$ in place of C_0/c_1. In the notation below, this corresponds to $\varepsilon = 1$ instead of $\varepsilon \to 0$.

Proof. Let $x \in S^d$. Zonal spherical harmonics supply us with a harmonic that changes sign from $\sqrt{M}$ at x to $-\sqrt{M}$ at distance ρ/n from x, where M is the multiplicity of spherical harmonics of degree n. On S^d, M is of order n^{d-1}. This harmonic b_x is called the barrier function at x. It has L^2 norm $||b_x|| = 1$, and we write the sign change as $b_x(x) \geq c_1\sqrt{M}$ while $b_x(y) \leq -c_1\sqrt{M}$ for y a spherical distance ρ/n from

x. Approximating the zonal harmonic by a Bessel function shows that this can be arranged for any ρ as in the statement of the theorem, with a corresponding c_1. For the rest of the proof, ρ and c_1 are positive constants, independent of the degree n.

To compare f to the barrier function, we write $f = \xi_0 b_x + f_x$ where ξ_0 is a random Gaussian of mean 0 and variance $1/M$ and f_x is a random harmonic synthesized from harmonics orthogonal to b_x. We must normalize by $\mathbb{E}||f_x||^2 = 1 - 1/M$ in order to match $\mathbb{E}||f||^2 = 1$. The strategy is that, if f_x is not too large, then f, like b_x, will have a nodal component near x. A convenient way to write the other component is to let

$$f_\pm = \pm\eta_0 b_x + f_x,$$

where η_0 is another Gaussian independent of ξ_0 but identically distributed. These are random spherical harmonics having the same distribution as f and the property that

$$f = \xi_0 b_x + \frac{f_+ + f_-}{2}.$$

Since $f_\pm$ have the same distribution as f, it is enough to bound the maximum of $|f|$ over a small disk. Then $f_\pm$ will be too small to interfere with the barrier function b_x, as long as the coefficient ξ_0 is large enough. We have an estimate on the maximum of f over a ball of radius ρ/n: For any given $\delta > 0$, and any $\rho > 0$, C_0 can be taken sufficiently large that

$$\mathbb{P}\left\{\max_{y\in B(x,\rho/n)} |f| \geq C_0\right\} \leq \delta.$$

This applies equally well for $f_\pm$ in place of f since they have the same distribution.

Now we can apply the nodal trap! As above, write

$$f = \xi_0 b_x + (f_+ + f_-)/2$$

where $f_\pm$ have the same distribution as f, ξ_0 is a Gaussian of mean 0 and variance

$1/M$, and the barrier function b_x obeys $b_x(x) \geq c_1\sqrt{M}$ and $b_x(y) \leq -c_1\sqrt{M}$ for $d(x,y) = \rho/n$. Consider the event Ω_x that $f(x) \geq C_0$ and $f(y) \leq -C_0$ for all $y \in \partial B(x, \rho/n)$. When this happens, the ball $B(x, \rho/n)$ must contain a connected component of $\{f = 0\}$. Suppose that $\xi_0 c_1\sqrt{M} \geq 2C_0$ and $|f_\pm(y)| \leq C_0$ for all y such that $d(x,y) = \rho/n$ and both choices of $\pm$. Then $f(x) \geq C_0$ while $f(y) \leq -C_0$, so Ω_x occurs.

The estimate on the maximum shows that, with probability at least $1 - 2\delta$, we have $|f_\pm| \leq C_0$ simultaneously for both f_+ and f_-. Combining this with the fact that $\mathfrak{z} = \xi_0\sqrt{M}$ is now a standard Gaussian, we have

$$\mathbb{P}(\Omega_x) \geq (1 - 2\delta)\mathbb{P}(\mathfrak{z} \geq 2C_0/c_1).$$

Now take several centers x. Each x leads to a nodal domain with probability at least $\mathbb{P}(\Omega_x)$. If the balls $B(x, \rho/n)$ do not overlap, then all of the nodal lines trapped in this way are distinct. As $n \to \infty$, the radius ρ/n vanishes so the maximum number of points x that can be positioned without overlap is dictated by the Euclidean packing problem with no need for a spherical adjustment. It follows that

$$\#x = (\Delta_{\mathbb{R}^d} + o(1))\frac{\mathrm{vol}(S^d)}{\mathrm{vol}(B(x, \rho/n))} = (1 + o(1))\Delta_{\mathbb{R}^d}\frac{\mathrm{vol}(S^d)}{|B_1|\rho^d}n^d$$

where $|B_1|$ denotes the volume of a Euclidean unit ball in dimension d, namely $\pi^{d/2}/(d/2)!$.

Combining this with the lower bound on $\mathbb{P}(\Omega_x)$, we have

$$\mathbb{E}[N(f)] \geq n^d\rho^{-d}(1 - 2\delta)\mathbb{P}(\mathfrak{z} \geq 2C_0/c_1)\Delta_{\mathbb{R}^d}\frac{\mathrm{vol}(S^d)}{|B_1|}$$

An immediate improvement can be made. The "nodal trap" event Ω_x produces a nodal component near x provided that $\xi_0 c_1\sqrt{M} \geq 2C_0$ and $|f_\pm(y)| \leq C_0$ for all y such

that $d(x,y) = \rho/n$ and both choices of $\pm$. Then $f(x) \geq C_0$ while $f(y) \leq -C_0$, so Ω_x occurs. If we had only $\xi_0 c_1 \sqrt{M} \geq (1+\varepsilon)C_0$, then we would have $f(x) \geq \varepsilon C_0$ while $f(y) \leq -\varepsilon C_0$. This is also enough to produce a nodal component. This allows one to replace $t = 2C_0/c_1$ by the smaller value $t = (1+\varepsilon)C_0/c_1$.

Another improvement: We produced nodal domains by finding sign changes where f is positive at x and negative at distance ρ/n from x, but of course sign changes from negative to positive also yield nodal domains and with equal probability. This gives an overall factor of 2. Taking $n \to \infty$,

$$c(d) \leftarrow \frac{1}{\mathrm{vol}(S^d)} \frac{\mathbb{E}[N(f)]}{n^d} \geq \frac{2\Delta_d}{|B_1|} \rho^{-d} (1-2\delta) \mathbb{P}(\mathfrak{z} \geq (1+\varepsilon)C_0/c_1).$$

Taking the limit as $\varepsilon \to 0$, we can simply take $t = C_0/c_1$ as stated in the theorem. $\square$

In fact, we will see that the packing factor Δ_d can also be removed by looking for nodal domains in more flexible regions instead of spheres. We have adapted this improvement from the work of Ingremeau-Rivera on the two-dimensional constant [39]. In high dimensions, Δ_d is extremely mysterious, but the lower bound $\Delta_d \geq 2^{-d}$ makes it relatively mild compared to the quantities of order d^{-d} in the rest of the bound (ρ will be of order d).

4.2 Mean value inequality in higher dimensions

Claim 2.2 from Nazarov and Sodin [53] is that for a spherical harmonic f of degree n,

$$f(x)^2 \leq Cn^2 \int_{\mathcal{D}(x,1/n)} f^2$$

so that the maximum of f is at most C times the average of f over a spherical cap. This is used to show that $\mathbb{P}(\max|f| \geq C_0) \leq \delta$ for a sufficiently large C_0, as we indicate below. A numerical lower bound on the Nazarov-Sodin constant will require

an explicit value of C, so we give a proof of the mean value inequality and also consider the higher-dimensional case.

Fix the point $x \in S^d$. We will expand the given harmonic f with respect to a particular basis of harmonics:

$$f = \sum_{j=1}^{M} c_j f_j.$$

This special basis is adapted to x in the sense that the f_j are orthogonal not only over S^d, but also over any ball $B(x, r)$ centered at x. Such a basis exists by separation of variables: The functions f_j have the form $b_x^{(j)}(\theta)Y(\alpha)$ where $b_x^{(j)}$ is a derivative of the barrier function, θ is the distance to x, and $Y(\alpha)$ is a harmonic on a lower-dimensional sphere representing the angular variable. One could also understand such a basis more conceptually in terms of the subgroup of rotations fixing x and how the representation of $SO(d+1)$ on spherical harmonics restricts to this copy of $SO(d)$. By either means, the basis enjoys orthogonality over $B(x, 1/n)$, so

$$\int_{B(x,1/n)} f^2 = \sum c_j^2 \int_{B(x,1/n)} f_j^2.$$

By positivity, we have

$$\sum c_j^2 \int_{B(x,1/n)} f_j^2 \geq c_1^2 \int_{B(x,1/n)} f_1^2$$

Multiplying by

$$\text{“}C\text{”} = \frac{f_1(x)^2}{n^d \int_{B(x,1/n)} f_1^2}$$

we find

$$\text{“}C\text{”} \int_{B(x,1/n)} f^2 \geq n^{-d} c_1^2 f_1(x)^2$$

We have

$$f(x) = c_1 f_1(x).$$

because the basis functions f_j all vanish at x except for f_1. Indeed, the others are orthogonal to f_1, and the reproducing kernel property then forces their value to be 0. Thus

$$\text{“}C\text{”} \int_{B(x,1/n)} f^2 \geq n^{-d} f(x)^2.$$

In other words, if the mean value inequality holds for f_1, it then follows for all f, and with the same constant. It remains only to investigate whether

$$\text{“}C\text{”} = \frac{f_1(x)^2}{n^d \int_{B(x,1/n)} f_1^2}$$

can really be bounded above by a constant.

To see this, note that the zonal spherical harmonic is given by $\omega(x,y) = MP_n^d(x \cdot y)/\operatorname{vol}(S^d)$, and the reproducing kernel property implies that the norm is

$$\|\omega\|_2^2 = \int \omega(x,y)\omega(x,y)dy = \omega(x,x) = M/\operatorname{vol}(S^d).$$

Therefore approximating the zonal harmonic by a Bessel function gives

$$f_1(y) = \frac{\omega(x,y)}{\|\omega_x\|_2} = M^{1/2} \operatorname{vol}(S^d)^{-1/2} P(x \cdot y) \approx M^{1/2} \frac{J_{d/2-1}(n\theta)}{(n\theta)^{d/2-1}}$$

We hide the constant of proportionality because both sides $f_1(x)^2$ and $\int f_1(x)^2$ scale the same way. Integrating f_1 gives

$$\text{“}C\text{”} = \left(\lim_{t\to 0} \frac{J_{d/2-1}(t)}{t^{d/2-1}}\right)^2 \left(n^d \int_0^{1/n} \left(\frac{J_{d/2-1}(n\theta)}{(n\theta)^{d/2-1}}\right)^2 \sin(\theta)^{d-1} d\theta \operatorname{vol}(S^{d-1})\right)^{-1} + o(1)$$

Note that the factor $\operatorname{vol}(S^{d-1})$ comes from integrating an angular variable over S^{d-1} in polar coordinates. It is not the volume of the underlying space S^d. From the power series for J_α, we have

$$\lim_{t\to 0} \frac{J_{d/2-1}(t)}{t^{d/2-1}} = \frac{1}{2^{d/2-1}\Gamma(d/2)}.$$

In the integral, let $t = n\theta$ with change of measure $d\theta = dt/n$ and use the small-angle approximation $n^{d-1}\sin(\theta)^{d-1} = \big(t + O(t^3/n^2)\big)^{d-1} \sim t^{d-1}$. This gives

$$\text{“}C\text{”} = \frac{1}{2^{d-2}\Gamma(d/2)^2\,\mathrm{vol}(S^{d-1})}\left(\int_0^1 J_{d/2-1}(t)^2 t dt\right)^{-1} + o(1)$$

which is indeed bounded, independent of n.

It is natural to restate the inequality in terms of averages:

$$f(x)^2 \le \text{“}C\text{”}\ \mathrm{vol}(B(x,1/n))n^d \frac{1}{\mathrm{vol}(B(x,1/n))}\int_{B(x,1/n)} f^2.$$

With the extra volume factor, the constant becomes

$$C = \frac{1}{2^{d-2}\Gamma(d/2)^2}\frac{|B_1|}{\mathrm{vol}(S^{d-1})}\left(\int_0^1 t J_{d/2-1}(t)^2 dt\right)^{-1}$$

It is this volume-adjusted constant that figures in the rest of the barrier argument. For the two-dimensional sphere S^2, we have“C”$= 0.408523$ and a volume-adjusted constant

$$\text{“}C\text{”}\ \mathrm{vol}(B_1) = \frac{\pi}{\mathrm{vol}(S^1)}\left(\int_0^1 J_0(t)^2 t dt\right)^{-1} = 1.2834\ldots$$

so that the maximum of f^2 is at the most about 28% larger than its average, for a harmonic f of high degree. For S^3, we would have a constant $0.2918393\ldots$, or 1.222 if we adjust for volume. Thus the volume-adjusted constant improves slightly.

As to the behaviour in high dimensions, we start from the power series

$$J_\nu(z) = z^\nu 2^{-\nu}\sum_{k=0}^{\infty}\frac{(-z^2/4)^k}{k!\Gamma(\nu+k+1)}.$$

With $\nu = d/2 - 1$, we have

$$2^{d-2}\Gamma(d/2)^2 J_{d/2-1}(t)^2 = t^{d-2}\left(\sum_{k=0}^{\infty} \frac{(-t^2/4)^k}{k!}\frac{\Gamma(d/2)}{\Gamma(d/2+k)}\right)^2$$
$$= t^{d-2}\left(1 + O(1/d)\right)$$

Integrating over $0 \le t \le 1$, we obtain

$$\left(\frac{0^\nu}{J_\nu(0)}\right)^2 \int_0^1 t^{d-1}\left(\frac{J_\nu(t)}{t^\nu}\right)^2 dt \sim \int_0^1 t^{1+2\nu} dt = \frac{1}{d}$$

Hence, noting that $\mathrm{vol}(S^{d-1}) = d\,\mathrm{vol}(B_1^d)$, we see that the volume-adjusted constant is

$$C_d = \frac{|B_1|}{\mathrm{vol}(S^{d-1})} d(1 + O(1/d)) = 1 + O\left(\frac{1}{d}\right).$$

4.3 Application to the maximum

For each z within ρ/n of x, the mean value inequality gives

$$f(z)^2 \le C\frac{1}{\mathrm{vol}(B(z,1/n))}\int_{B(z,1/n)} f^2.$$

To give a uniform bound over z, note that $\mathrm{vol}(B(z,1/n)) = \mathrm{vol}(B(x,1/n))$ and, by the triangle inequality, $B(z,1/n) \subseteq B(x,(\rho+1)/n)$. It follows that

$$\max_{B(x,\rho/n)} f^2 \le C\frac{1}{\mathrm{vol}(B(x,1/n))}\int_{B(x,(\rho+1)/n)} f^2$$

Integrating over the sphere gives

$$\int_{S^d} \max_{B(x,\rho n^{-1})} f^2 dx \le C\frac{\mathrm{vol}(B(*, n^{-1}(\rho+1))}{\mathrm{vol}(B(*,1/n)}\int_{S^d} f(y)^2 dy$$

where we have changed the order of integration on the right and noted again that the volume of a spherical cap $B(*, r)$ does not depend on its center. We have also

$$\mathbb{E}\left(\max_{B(x,\rho n^{-1})} f^2\right) = \mathbb{E}\left[\frac{1}{\mathrm{vol}(S^{d-1})}\int_{S^{d-1}} \max_{B(x,\rho n^{-1})} f^2\right]$$

because the expectation is the same for every x. Therefore, using Chebychev's inequality,

$$\begin{aligned} C_0^2 \mathbb{P}\left(\max_{B(x,\rho/n)} |f| \geq C_0\right) &\leq \mathbb{E}\left(\max_{B(x,\rho n^{-1})} f^2\right) \\ &= \frac{1}{\mathrm{vol}(S^d)}\mathbb{E}\int_{S^d} \max_{B(x,\rho n^{-1})} f^2 \\ &\leq C\frac{\mathrm{vol}(B(*, n^{-1}(\rho+1))}{\mathrm{vol}(B(*, 1/n)}\frac{1}{\mathrm{vol}(S^d)}\mathbb{E}\int_{S^d} f(y)^2 dy \end{aligned}$$

By the normalization of f,

$$\mathbb{E}\int_{S^d} f(y)^2 dy = 1.$$

Hence

$$\mathbb{P}\left(\max_{B(x,\rho/n)} |f| \geq C_0\right) \leq C_0^{-2} C\frac{\mathrm{vol}(B(*, n^{-1}(\rho+1))}{\mathrm{vol}(B(*, 1/n)}\frac{1}{\mathrm{vol}(S^d)}.$$

This bound is less than or equal to δ provided that

$$C_0^2 \geq \frac{1}{\delta}\frac{C}{\mathrm{vol}(S^d)}\frac{\mathrm{vol}(B(*, n^{-1}(\rho+1)))}{\mathrm{vol}(B(*, 1/n))}.$$

For large n, the spherical cap has volume proportional to $(n^{-1}(\rho+1))^d$ so the limiting constraint is as stated in Theorem 4.1.1:

$$C_0^2 \geq \frac{1}{\delta}\frac{C(\rho+1)^d}{\mathrm{vol}(S^d)}$$

4.4 More on the barrier function

To determine admissible, let alone optimal, values for ρ and c_1, we need to understand the barrier function in more detail and take care as to normalizations. In order to keep track of the dimension-dependence, we will now use the ordinary "unit-sphere" normalization of S^{d-1} inside $\mathbb{R}^d$ and then convert the results to our preferred notation where d is the dimension of the sphere. We have

$$\mathrm{vol}(S^{d-1}) = d\frac{\pi^{d/2}}{(d/2)!}$$

where $(d/2)!$ is interpreted as $\Gamma(d/2+1)$ for odd d. In particular, $\mathrm{vol}(S^3) = 2\pi^2$. The dimension of the space of spherical harmonics of degree n on S^{d-1} is

$$M = M(d,n) = (2n+d-2)\frac{(n+d-3)!}{n!(d-2)!}$$

and, in particular, the multiplicity on S^3 is $(n+1)^2$. For a fixed d, the behaviour of $M(d,n)$ as $n \to \infty$ is

$$M(d,n) = \frac{2}{(d-2)!}n^{d-2}\left(1 + O_d(1/n)\right).$$

The zonal function is

$$\omega(x,y) = \sum_{j=1}^{M}\phi_j(x)\phi_j(y) = \frac{M}{\mathrm{vol}(S^{d-1})}P_n^d(x\cdot y)$$

where P_n^d is a polynomial normalized by $P_n^d(1) = 1$ and the choice of basis $\{\phi_j\}$ is irrelevant. As long as the basis functions ϕ_j are orthonormal, we have

$$\int_{S^{d-1}}\omega(x,y)f(y)dy = \sum_{j=1}^{M}\phi_j(x)\int_{S^{d-1}}\phi_j(y)f(y)dy = f(x)$$

for any harmonic f of degree M. This is the reproducing kernel property of $\omega(x,y)$. One way to check that we have normalized correctly is that

$$\mathrm{vol}(S^{d-1})\omega(x_0,x_0) = \int_{S^{d-1}} \omega(x,x)dx = \sum_{j=1}^{M} \int_{S^{d-1}} \phi_j(x)^2 dx = M,$$

so we must have $P_n^d(1) = 1$. The barrier function $b_x(y)$ is proportional to $\omega(x,y)$, but we must normalize so that $\int b_x(y)^2 dy = 1$. From the reproducing kernel property, we have

$$\int \omega(x,y)^2 dy = \omega(x,x) = \frac{M}{\mathrm{vol}(S^{d-1})}.$$

Therefore the normalized barrier is

$$b_x(y) = \frac{\omega}{\|\omega\|} = \sqrt{M}\,\mathrm{vol}(S^{d-1})^{-1/2} P_n^d(x \cdot y).$$

The polynomial P_n^d is a well studied special function called the ultraspherical/Gegenbauer polynomial. It is proportional to the Jacobi polynomial

$$P_n^{(\lambda-1/2,\lambda-1/2)}(\cos\theta),$$

where $\lambda = d/2 - 1$ and $x \cdot y = \cos\theta$. This allows us to use Szegő's asymptotic (formula (8.21.17) in [67]) for Jacobi polynomials $P_n^{(\alpha,\beta)}$. For $\alpha > -1$ and any real β, with $N = n + (\alpha+\beta+1)/2$, we have the estimate

$$\left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta} P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\Gamma(n+\alpha+1)}{n!}\sqrt{\frac{\theta}{\sin\theta}}\frac{J_\alpha(N\theta)}{N^\alpha} + \varepsilon(n,\theta).$$

The error satisfies

$$\varepsilon(n,\theta) = \begin{cases} \theta^{1/2} O(n^{-3/2}) \text{ if } c/n \le \theta \le \pi^- < \pi \\ \theta^{\alpha+2} O(n^\alpha) \text{ if } 0 < \theta \le c/n \end{cases}$$

for any fixed π^- less than π and any $c > 0$, the implicit O constants being subject to the choice of these parameters. In particular, $\varepsilon(n, \theta) \lesssim \theta^{1/2} n^{-3/2}$ holds for all θ. For the ultraspherical case, write $x \cdot y = \cos\theta$. We take

$$\alpha = \beta = \lambda - 1/2 = d/2 - 1 - 1/2 = d/2 - 3/2.$$

The resulting N is $N = n + \alpha + 1/2 = n + d/2 - 1$. The trigonometric terms can be combined:

$$\sin(\theta/2)\cos(\theta/2) = \frac{1}{2}\sin\theta.$$

Szegő's asymptotics show that, for an appropriate c depending on the dimension,

$$P_n^d(\cos\theta) \sim c \frac{J_{(d-3)/2}(N\theta)}{(N\theta)^{(d-3)/2}} \left(\frac{\theta}{\sin\theta}\right)^{(d-3)/2}$$

To be sure of the constant factor, note that we have normalized so that

$$P_n^d(1) = 1.$$

As $\theta \to 0$, we have $\theta/\sin(\theta) \sim 1$. The Bessel function can be expanded as follows:

$$\begin{aligned} \frac{J_\alpha(t)}{t^\alpha} &= \sum_{k=0}^{\infty} \frac{1}{2^\alpha \Gamma(\alpha + 1 + k)} \frac{(-t^2/4)^k}{k!} \\ &= \frac{1}{2^\alpha \Gamma(\alpha+1)} \left(1 - \frac{t^2}{4(\alpha+1)} \pm \dots\right) \end{aligned}$$

Taking $\alpha = (d-3)/2$ and comparing with $P_n^d(1) = 1$ implies that

$$1 = P_n^d(1) \sim \frac{c}{2^{(d-3)/2}\Gamma((d-1)/2)}.$$

Thus the constant must be

$$c = 2^{(d-3)/2}\Gamma((d-1)/2).$$

In this way, the zonal spherical function $\omega(x,y) = P_n^d(x \cdot y)M/|S^{d-1}|$ is approximated by

$$\omega(x,y) \sim \frac{M}{|S^{d-1}|} 2^{(d-3)/2}\Gamma((d-1)/2) \frac{J_{(d-3)/2}\big((n+d/2-1)\theta\big)}{\big((n+d/2-1)\theta\big)^{(d-3)/2}} \left(\frac{\theta}{\sin\theta}\right)^{(d-3)/2}$$

and the barrier function by

$$b_x(y) \sim \sqrt{\frac{M}{|S^{d-1}|}} 2^{(d-3)/2}\Gamma((d-1)/2) \frac{J_{(d-3)/2}\big((n+d/2-1)\theta\big)}{\big((n+d/2-1)\theta\big)^{(d-3)/2}} \left(\frac{\theta}{\sin\theta}\right)^{(d-3)/2}.$$

In particular, at distance $\theta = \rho/n$ from x, we have

$$b_x(y) \sim \sqrt{M}|S^{d-1}|^{-1/2} 2^{(d-3)/2}\Gamma((d-1)/2) J_{(d-3)/2}(\rho)\rho^{-(d-3)/2}$$

as $n \to \infty$, whereas the central value is larger:

$$b_x(x) = \sqrt{M}|S^{d-1}|^{-1/2}.$$

To arrange the sign change, we take ρ in between the first and second zeros of the Bessel function $J_{(d-3)/2}$. Then we take

$$c_1 = |S^{d-1}|^{-1/2} \min\left(1, 2^{(d-3)/2}\Gamma((d-1)/2) \frac{|J_{(d-3)/2}(\rho)|}{\rho^{(d-3)/2}}\right)$$

in order to have $b_x(x) \geq c_1\sqrt{M}$ and $b_x(y) \leq -c_1\sqrt{M}$. We will see that it is the second term that is the smaller because ρ is quite large. One option is to choose ρ as the first minimum of the spherical Bessel function, between its first and second zeros, in order

to maximize c_1.

Finally, on S^d instead of S^{d-1}, the relevant Bessel function is $J_{d/2-1}$ and we have

$$c_1 = |S^d|^{-1/2} 2^{d/2-1} \Gamma(d/2) |J_{d/2-1}(\rho)| \rho^{-(d/2-1)}.$$

More on c_1

We have seen that ρ lies between the first two zeros of $J_{d/2-1}$, so it is of order $d/2$. Indeed, the differential equation solved by $J_\nu(x)$ can be written

$$x \frac{d}{dx} \left(x \frac{d}{dx} J_\nu(x) \right) = (\nu^2 - x^2) J_\nu(x),$$

which shows that $xJ'_\nu(x)$ is increasing as long as $x < \nu$ and $J_\nu(x) > 0$. On the other hand, the power series shows that both $J_\nu(x)$ and $J'_\nu(x)$ are positive for small positive x. Together, these show that $J_\nu(x)$ has no zeros for $x < \nu$. In fact, Watson (p.486 [71]) gives the non-asymptotic result that the first positive zero $j_1(\nu)$ of J_ν is in the interval

$$\sqrt{\nu(\nu+2)} < j_1(\nu) < \sqrt{2(\nu+1)(\nu+3)}$$

with the second zero also being of this rough size. Hence ρ is of order $d/2$.

For large d, this leads us quickly to the transition region for $J_\nu(x)$. For large n, the first few zeros j_m of J_n are near n, with corrections of order $n^{1/3}$ dictated by the Airy function. In particular,

$$j_1 = n + \mu_1 n^{1/3} + O(n^{-1/3})$$

for some constant $\mu_1 = 1.855757\ldots$ (see Watson [71] page 521). Since ρ is larger than

$d/2-1$ but very near it, we use Watson's formula in the form

$$\begin{aligned} J_n(n\sec\beta) =& \frac{\tan\beta}{3}\cos\left(n(\tan\beta-\frac{1}{3}\tan(\beta)^3-\beta)\right)\left(J_{-1/3}(\xi)+J_{1/3}(\xi)\right) \\ &+\frac{\tan\beta}{\sqrt{3}}\sin\left(n(\tan\beta-\frac{1}{3}\tan(\beta)^3-\beta)\right)\left(J_{-1/3}(\xi)-J_{1/3}(\xi)\right)+O\left(\frac{1}{n}\right) \end{aligned}$$

where $\xi = n\tan(\beta)^3/3$. See Watson 8.43(5) page 252 [71]. In our range, the argument $x = n\sec\beta$ obeys $x \sim n$ and $|x-n| \asymp n^{1/3}$. Thus

$$\mu_1 n^{1/3} \sim |x-n| = n|\sec\beta - 1| \sim \frac{1}{2}n\beta^2$$

so that β is of order $n^{-1/3}$. The quantity ξ is then of order 1:

$$\xi = \frac{1}{3}n\tan(\beta)^3 \sim \frac{1}{3}n\beta^3 \asymp 1.$$

Expanding $\beta = \arctan(\tan\beta)$ in a power series, we have

$$\tan\beta - \frac{1}{3}\tan(\beta)^3 - \beta \sim \frac{1}{5}\tan(\beta)^5 \sim \frac{1}{5}\beta^5$$

Thus the argument of the trigonometric functions in Watson's formula becomes very small

$$n\left(\tan(\beta)-\frac{1}{3}\tan(\beta)^3-\beta\right) \lesssim n\big(n^{-1/3}\big)^5 \to 0.$$

As a result, only the cosine term survives:

$$\begin{aligned} \cos\left(n\left(\tan\beta-\frac{1}{3}\tan(\beta)^3-\beta\right)\right) &= 1+O\big(n^{-4/3}\big) \\ \sin\left(n\left(\tan\beta-\frac{1}{3}\tan(\beta)^3-\beta\right)\right) &\lesssim n^{-2/3} \end{aligned}$$

Since there is also a factor of $\tan\beta \sim n^{-1/3}$, we can jettison the term $\tan(\beta)\sin(\ldots)$ into the error of $O(1/n)$ already present, along with the error $O(n^{-4/3}\tan\beta)$ from

approximating the cosine by 1. Thus Watson's formula simplifies to

$$J_n(n\sec\beta) = \frac{\tan\beta}{3}\left(J_{-1/3}\left(\frac{1}{3}n\tan(\beta)^3\right) + J_{1/3}\left(\frac{1}{3}n\tan(\beta)^3\right)\right) + O(n^{-1})$$

Note that $\tan\beta \sim \beta \asymp n^{-1/3}$, n being $d/2-1$ for our application, while the term $J_{1/3}+J_{-1/3}$ is of order 1. This shows that $|J_{d/2-1}(\rho)|$ is of order $d^{-1/3}$. This is larger than the size $d^{-1/2}$ that one might have expected using only the asymptotics of $J_\nu(x)$ for large x without taking the transition region into account.

The value of c_1 is thus proportional to

$$\mathrm{vol}(S^d)^{-1/2}2^{d/2-1}\Gamma(d/2)\rho^{-d/2+1}d^{-1/3}$$

with a constant one may vary by tuning $\beta = \arccos(n/\rho)$, namely

$$d^{1/3}\frac{\tan\beta}{3}\left(J_{-1/3}\left(\frac{1}{3}n\tan(\beta)^3\right) + J_{1/3}\left(\frac{1}{3}n\tan(\beta)^3\right)\right)$$

On a first attempt, one would like to take the largest possible c_1 so that the tail probability $\mathbb{P}(\mathfrak{z} \geq C_0/c_1)$ is maximized. There is a tradeoff, since it is not exactly this quantity that figures in the lower bound but rather

$$\rho^{-d}\mathbb{P}(\mathfrak{z} \geq C_0/c_1).$$

This rewards choosing a smaller value of ρ than one would take to maximize the tail probability on its own. Since $\mathbb{P}(\mathfrak{z} > C_0/c_1)$ is so small compared to ρ^{-d}, we will not attempt this optimization in high dimensions. Thus we take ρ to maximize $|J_{d/2-1}(\rho)|$. From

$$\frac{1}{x}\frac{d}{dx}\frac{J_\nu(x)}{x^\nu} = -\frac{J_{\nu+1}(x)}{x^{\nu+1}}.$$

we see that the critical points of $J_\nu(x)/x^\nu$ occur at zeros of $J_{\nu+1}(x)$. For us, $\nu+1 = d/2$,

so we choose ρ to be the first positive zero of $J_{d/2}$.

4.5 Choice of δ

Unlike c_1 and ρ, there is no issue of finding an admissible δ: We can always take $\delta = 1/3$. As $\delta \to 1/2$, the factor $1 - 2\delta$ vanishes because the other harmonics have a high chance of interfering with the barrier. As $\delta \to 0$, the tail factor $\mathbb{P}(\mathfrak{z} > C_0/c_1)$ vanishes because C_0 diverges. In this case, the coefficient of the barrier must be larger and larger to dominate the other harmonics. In principle, there is a tradeoff between these extremes and an optimal choice of δ.

The barrier method gives a lower bound of the form

$$c(d) \gtrsim_d (1 - 2\delta)\mathbb{P}(\mathfrak{z} \geq t(\delta))$$

with an implicit constant determined as above, which we are not trying to optimize, and a tail parameter $t(\delta) = C_0/c_1$ which we would like to balance with the constraints. The bound is monotone decreasing with C_0, so we choose the smallest value permitted, namely

$$C_0^2 = \frac{1}{\delta} \frac{C}{|S^{d-1}|} (\rho + 1)^{d-1}.$$

The quantity to be maximized is $(1 - 2\delta)\mathbb{P}(\mathfrak{z} \geq t(\delta))$, or equivalently, its logarithm. We differentiate with respect to δ to find critical points. Note that, by the fundamental theorem of calculus,

$$\begin{aligned} \frac{d}{d\delta}\mathbb{P}(\mathfrak{z} \geq t(\delta)) &= \frac{d}{d\delta}\left(1 - \int_{-\infty}^{t(\delta)} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}\right) \\ &= -e^{-t(\delta)^2/2} \frac{d}{d\delta} t(\delta) \frac{1}{\sqrt{2\pi}}. \end{aligned}$$

The logarithmic derivative is

$$\frac{d}{d\delta}\log\left((1-2\delta)\mathbb{P}(\mathfrak{z}\geq t(\delta))\right)=\frac{-2}{1-2\delta}-\frac{t'(\delta)e^{-t(\delta)^2/2}/\sqrt{2\pi}}{\mathbb{P}\big(\mathfrak{z}\geq t(\delta)\big)}.$$

Hence the equation for the critical value of δ is

$$\frac{-2}{1-2\delta}-\frac{t'(\delta)e^{-t(\delta)^2/2}/\sqrt{2\pi}}{\mathbb{P}\big(\mathfrak{z}\geq t(\delta)\big)}=0$$

We also have an approximation to the Gaussian tail probability for $t>0$:

$$\mathbb{P}(\mathfrak{z}>t)=\int_t^\infty e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}\geq\frac{1}{\sqrt{2\pi}}e^{-t^2/2}(1/t-1/t^3).$$

For the purposes of choosing δ, since we expect $t(\delta)$ to be large, let us approximate the tail probability by ignoring the $1/t^3$ term. Thus, instead of the critical equation, we solve

$$\frac{-2}{1-2\delta}-t'(\delta)t(\delta)=0.$$

From the estimate of the maximum, we took

$$t(\delta)=\frac{C_0}{c_1}=\frac{1}{\sqrt{\delta}}\left(\frac{C(\rho+1)^d}{|S^d|}\right)^{1/2}\frac{1}{c_1}$$

Hence

$$t'(\delta)t(\delta)=-\frac{1}{2}\delta^{-3/2}\delta^{-1/2}\frac{C(\rho+1)^d}{|S^d|}\frac{1}{c_1^2}.$$

We choose δ by solving

$$\frac{-2}{1-2\delta}+\frac{1}{2c_1^2}\frac{C(\rho+1)^{d-1}}{|S^{d-1}|}\delta^{-2}=0.$$

This is a quadratic equation for δ. After multiplying both sides by $-(1-2\delta)\delta^2/2$, we

have

$$\delta^2 + 2A\delta - A = 0$$

where

$$A = \frac{1}{4c_1^2}\frac{C(\rho+1)^d}{|S^d|} = \delta C_0^2/(4c_1^2).$$

Note that C_0 depends on δ while A does not, and the tail parameter is

$$t^2 = \frac{C_0^2}{c_1^2} = 4\frac{A}{\delta}.$$

The positive solution for δ is

$$\delta = -A + \sqrt{A^2 + A} = A(\sqrt{1 + 1/A} - 1)$$

As $A \to \infty$, $\delta \to 1/2$. As $A \to 0$, $\delta \sim \sqrt{A}$ also vanishes. Of course, A has the definite value given above, which is somewhere in between these two extremes. We will see that it is fairly large. First, let us complete the bound. Using the estimate

$$\mathbb{P}(\mathfrak{z} \geq t) \geq \frac{1}{t}e^{-t^2/2}\frac{1}{\sqrt{2\pi}}(1 - t^{-2})$$

we get

$$\begin{aligned}\frac{c_{\mathrm{NS}}(S^d)}{\mathrm{vol}(S^d)} &\geq \frac{\Delta_{\mathbb{R}^d}}{\mathrm{vol}(B_1 \subset \mathbb{R}^d)}\rho^{-d}(1-2\delta)\mathbb{P}(\mathfrak{z} \geq t(\delta)) \\ &\geq \frac{\Delta_{\mathbb{R}^d}}{\mathrm{vol}(B_1)}\rho^{-d}\frac{1-2\delta}{t(\delta)}\exp(-t(\delta)^2/2)\frac{1 - t(\delta)^{-2}}{\sqrt{2\pi}}\end{aligned}$$

Using the choice above, namely $\delta = \sqrt{A^2 + A} - A$, note that

$$\frac{\delta}{A} = \frac{1}{A + \sqrt{A^2 + A}} \approx \frac{1}{2A}.$$

We have

$$\begin{aligned}\exp(-t(\delta)^2/2) &= \exp(-2A/\delta) \\ &= \exp(-2(A+\sqrt{A^2+A}))\end{aligned}$$

Therefore we can state the lower bound in the form

$$\frac{c_{\mathrm{NS}}(S^d)}{\mathrm{vol}(S^d)} \geq$$
$$\frac{\Delta_{\mathbb{R}^d}\rho^{-d}}{\sqrt{2\pi}\,\mathrm{vol}(B_1)}\frac{\exp\left(-2(A+\sqrt{A^2+A})\right)}{2\sqrt{A+\sqrt{A^2+A}}}(1-2(\sqrt{A^2+A}-A))\left(1-\frac{4}{A+\sqrt{A^2+A}}\right)$$

To see how large A is, note that $C = 1 + O(1/d)$ as $d \to \infty$, while $C_0^2 = C(\rho+1)^d/|S^d|$, and the value above implies

$$\frac{1}{c_1^2} = |S^d|2^{-d+2}\Gamma(d/2)^{-2}|J_{d/2-1}(\rho)|^{-2}\rho^{d/2-1}.$$

Thus

$$\begin{aligned}A &= \frac{1}{4c_1^2}\frac{C(\rho+1)^d}{|S^d|} \\ &= (1+O(1/d))\frac{1}{4}|S^d|2^{-d+2}\Gamma(d/2)^{-2}|J_{d/2-1}(\rho)|^{-2}\rho^{d/2-1}C(\rho+1)^d/|S^d| \\ &\sim e^2 2^{-d}\Gamma(d/2)^{-2}|J_\nu(\rho)|^{-2}\rho^{3d/2-1}.\end{aligned}$$

We have used the fact $\rho \sim d/2$ to write

$$(\rho+1)^d = \rho^d\left(1+\frac{1}{\rho}\right)^d \sim \rho^d e^2.$$

Regardless of our choice of ρ, $|J_\nu(\rho)|^{-2} \asymp d^{2/3}$ is as nothing against $\rho^{3d/2-1}$. Likewise, 2^{-d} and even the mighty $\Gamma(d/2)^{-2}$ are secondary. The parameter A is superexponential in d, growing like $d^{3d/2}$ up to lower-order corrective factors. We restate the lower

bound on the Nazarov-Sodin constant $c(d) = c_{\mathrm{NS}}(S^d)/\operatorname{vol}(S^d)$ as

$$\log\log\frac{1}{c(d)} \leq \log(A) + O(d) = \frac{3}{2}d\log d + O(d)$$

which may be a helpful scale at which to understand such a small number. Taking $\delta = 1/3$ in all dimensions, instead of δ closer and closer to $1/2$ as above, results in a bound of the same $d\log d$ quality but with a larger constant in place of $3/2$ (which then filters through two layers of exponentials to give a much worse lower bound for the actual constant of interest).

Theorem 4.0.2 claims that this $\log d$ can be removed, and we now turn to the proof of this. It follows the same overall barrier strategy, but with an improved estimate of $\mathbb{P}(\max f \geq C_0)$ leading to a smaller C_0 and a better bound. First, let us apply the original barrier method to low dimensions.

4.6 Two and three dimensions

Let us consider the two-dimensional case. We can take $C = 1.2834$ for the constant in the mean value inequality. We have $\rho = 3.83...$ for the location of the minimum of J_0 in between its first and second roots, and $|S^2| = 4\pi$. This gives

$$\begin{aligned} c_1^{-2} &= 4\pi/|J_0(\rho)| = 77.4673\ldots \\ A &= \frac{1}{4}c_1^{-2}C(\rho+1)^2/(4\pi) = 46.1755\ldots \\ \delta &= 0.497321859\ldots \\ c(2) &\geq 4.9\times 10^{-87} \end{aligned}$$

Thus, as claimed, $c(2) \geq 10^{-87}$. This could be improved by numerical optimization over ρ, but not substantially.

For S^3, the Bessel function reduces to pure trigonometry:

$$\frac{J_{1/2}(z)}{z^{1/2}} = \sqrt{\frac{2}{\pi}}\frac{\sin z}{z}.$$

Differentiating for critical points, we find that ρ solves $\tan(\rho) = \rho$. The first positive root of this equation corresponds to a maximum of the Bessel function, so ρ is the second root:

$$\rho = 4.4934\ldots$$

We have the geometric factors $\mathrm{vol}(S^3) = 2\pi^2$ and $\Delta_{\mathbb{R}^3} = \pi/\sqrt{18}$, and $C = 1.222$ is an admissible constant in the mean value inequality. The value of c_1 is

$$c_1 = \frac{\sqrt{2}\Gamma(3/2)}{\sqrt{2\pi^2}}\frac{|J_{1/2}(\rho)|}{\rho^{1/2}} = \frac{-1}{\pi\sqrt{2}}\frac{\sin\rho}{\rho}.$$

From $\tan\rho = \rho$, we obtain

$$c_1^{-2} = 2\pi^2\rho^2/\sin(\rho)^2 = 2\pi^2(1+\rho^2).$$

The key parameter $A = \frac{1}{4}c_1^{-2}C(\rho+1)^d/\mathrm{vol}(S^d)$ becomes

$$A = \frac{1}{4c_1^2}\frac{C(\rho+1)^3}{\mathrm{vol}(S^3)} = \frac{2\pi^2 C(\rho^2+1)}{4}\frac{(\rho+1)^3}{2\pi^2} = 1073.2\ldots$$

leading to a miniscule lower bound $c(3) \geq 10^{-1196}$.

4.7 Estimating the maximum by Dudley's entropy method

Above, we used Chebyshev's inequality

$$\mathbb{P}\left(\max_{B(x,\rho/n)} |f| \geq C_0\right) \leq C_0^{-2}\mathbb{E}\left[\max_{B(x,\rho/n)} f^2\right]$$

together with the estimate

$$\mathbb{E}\left[\max_{B(x,\rho/n)} f^2\right] \leq \frac{C}{|S^d|}\frac{\mathrm{vol}(B(*,(\rho+1)/n))}{\mathrm{vol}(B(*,1/n))} \sim \frac{C}{|S^d|}(\rho+1)^d.$$

Instead of squaring, note that

$$\mathbb{P}\left(\max_{B(x,\rho/n)} |f| \geq C_0\right) \leq 2\mathbb{P}\left(\max_{B(x,\rho/n)} f \geq C_0\right)$$

where the factor of 2 combines both cases $\max f \geq C_0$ and $\min f \leq -C_0$, which have the same probability because f and $-f$ have the same distribution. At each point y, the value $f(y)$ is a Gaussian random variable of mean 0. From Lemma 6.12 in [31], if X_t is a separable Gaussian process indexed by $t \in T$, then the supremum $\sup\{X_t\}_{t\in T}$ is subgaussian with variance proxy

$$\sigma^2 = \sup_{t\in T} \mathrm{var}[X_t].$$

This implies that

$$\mathbb{P}\left(\sup_{t\in T} X_t \geq \mathbb{E}\left[\sup_{t\in T} X_t\right] + a\right) \leq e^{-a^2/(2\sigma^2)}.$$

In our case, the parameter space is $T = B(x, \rho/n)$ and the process is

$$X_t = f(t) = \sum_j c_j \phi_j(t).$$

The variance at any point t is

$$\operatorname{var}[X_t] = \sum_j \phi_j(t)^2 \operatorname{var}[c_j] = \operatorname{var}[c_j]\omega(t,t) = \frac{1}{|S^d|} = \sup_{t\in T} \operatorname{var}[X_t].$$

To estimate the expected value of the supremum, we use Dudley's entropy integral [20] (Corollary 5.25 in [31]):

$$\mathbb{E}\left[\sup_{t\in T} X_t\right] \leq 12 \int_0^\infty \sqrt{\log N(T,d,\varepsilon)}\, d\varepsilon.$$

Here, $N(T,d,\varepsilon)$ is a covering number with respect to a metric on T given in terms of the process by

$$d(s,t) = \sqrt{\mathbb{E}\left[(X_s - X_t)^2\right]}.$$

The integral is written over all positive values of ε, but the integrand vanishes once $N(T,d,\varepsilon) = 1$, that is, once a single ball covers T. For us, by the addition formula, the metric is

$$d(s,t) = \sqrt{\frac{2}{|S^d|}\big(1 - P_n^d(s\cdot t)\big)}.$$

We may call it $d_X(s,t)$ to distinguish it from the spherical metric $\arccos(s\cdot t)$. The parameter space $T = B(x,\rho/n)$ is given by

$$x \cdot t < \cos(\rho/n).$$

If ε is large enough that

$$\sqrt{\frac{2}{|S^d|}\left(1+\left|P_n^d\left(\cos\frac{\rho}{n}\right)\right|\right)}<\varepsilon$$

then a single ball of radius ε centered at $s=x$ covers T. Thus Dudley's integral terminates at this value

$$\begin{aligned}\varepsilon_0 &= \sqrt{\frac{2}{|S^d|}\left(1+\left|P_n^d\big(\cos\frac{\rho}{n}\big)\right|\right)}\\ &\sim \sqrt{\frac{2}{|S^d|}\left(1+2^{d/2-1}\Gamma(d/2)\frac{|J_{d/2-1}(\rho)|}{\rho^{d/2-1}}\right)}\end{aligned}$$

For large d, our choice of $\rho\sim d/2$ makes the term $\rho^{d/2-1}$ very large. For us,

$$\varepsilon_0\sim\sqrt{\frac{2}{|S^d|}}.$$

To bound the covering numbers, let us write $\mathcal{B}$ for balls with respect to the metric d_X and B for balls in the geodesic distance. Each calligraphic ball $\mathcal{B}$ contains a geodesic ball with the same center, the radius of the geodesic ball being smaller by a factor of roughly n. Indeed, the ball $\mathcal{B}(z,\varepsilon)$ is defined by

$$\varepsilon^2>\frac{2}{|S^{d-1}|}\left(1-P_n^d(y\cdot z)\right)$$

which will be satisfied when $y\cdot z$ is close enough to 1, with $\arccos(y\cdot z)$ on the order of $1/n$. By symmetry, how close they must be does not depend on the center z, so we may write

$$\mathcal{B}(z,\varepsilon)\supseteq B(z,f_d\varepsilon/n)$$

where the factor f_d depends only on the dimension. Taking $n\to\infty$ and approximating

the zonal harmonic by a Bessel function, we approximate the metric by

$$\sqrt{\frac{2}{|S^d|}\left(1 - P_n^d\big(\cos\theta\big)\right)} \sim \sqrt{\frac{2}{|S^d|}\left(1 - 2^{d/2-1}\Gamma(d/2)\frac{J_{d/2-1}(n\theta)}{(n\theta)^{d/2-1}}\right)}$$

The limiting condition on f_d is that

$$\frac{2}{|S^d|}\left(1 - 2^{d/2-1}\Gamma(d/2)\frac{J_{d/2-1}(f_d\varepsilon)}{(f_d\varepsilon)^{d/2-1}}\right) \leq \varepsilon^2.$$

This can be written

$$f_d^2 \frac{1 - 2^{d/2-1}\Gamma(d/2)\frac{J_{d/2-1}(f_d\varepsilon)}{(f_d\varepsilon)^{d/2-1}}}{(f_d\varepsilon)^2} \leq \frac{|S^d|}{2}.$$

which would follow from

$$f_d^2 \sup_{y>0} \frac{1 - 2^{d/2-1}\Gamma(d/2)\frac{J_{d/2-1}(y)}{y^{d/2-1}}}{y^2} \leq \frac{|S^d|}{2}.$$

From the power series for $J_{d/2-1}$, we have

$$1 - 2^{d/2-1}\Gamma(d/2)\frac{J_{d/2-1}(y)}{y^{d/2-1}} \sim \frac{y^2}{2d}.$$

This is accurate in the regime $y \to 0$, where the supremum is attained because of the factors y^2 and $y^{d/2-1}$. Thus the containment factor f_d may be taken as

$$f_d = \sqrt{d|S^d|}.$$

Thus, given a covering of $B(x, \rho/n)$ with geodesic balls of size ε/n, the same centers form a covering by calligraphic balls $\mathcal{B}(*, \varepsilon)$. To leading order, the covering numbers for the spherical metric at scale $1/n \to 0$ will agree with their Euclidean counterparts. The number of balls of radius ε needed to cover a Euclidean unit ball in $\mathbb{R}^d$ is between

$1/\varepsilon^d$ and $(3/\varepsilon)^d$ (see, for instance, Lemma 5.13 in [31]). Thus the covering number for a ball of radius ρ by balls of radius $f_d\varepsilon$ is at most $(3\rho/(f_d\varepsilon))^d$. We conclude that, as $n \to \infty$,

$$N(T, d_X, \varepsilon) \leq N(T, d_{S^d}, f_d\varepsilon/n) \leq (1 + o(1)) \left(\frac{3\rho}{f_d\varepsilon}\right)^d$$

This gives an upper bound on Dudley's entropy integral, independent of n up to a $1 + o(1)$ factor:

$$\mathbb{E}\left[\sup_{t\in T} X_t\right] \leq 12 \int_0^\infty \sqrt{\log N(T, d, \varepsilon)} d\varepsilon \leq 12\sqrt{d} \int_0^{\varepsilon_0} \sqrt{\log(\varepsilon^{-1}) + \log(3\rho/f_d)} d\varepsilon$$

Change variables to

$$\begin{aligned} u &= \sqrt{\log\left(\frac{3\rho}{f_d\varepsilon}\right)} \\ \varepsilon &= \frac{3\rho}{f_d} e^{-u^2} \\ d\varepsilon &= -\frac{6\rho}{f_d} u e^{-u^2} du \end{aligned}$$

The integral becomes (without the factor $12\sqrt{d}$)

$$\frac{6\rho}{f_d} \int_{\sqrt{\log \frac{3\rho}{f_d\varepsilon_0}}}^\infty u^2 e^{-u^2} du.$$

This is certainly at most

$$\frac{6\rho}{f_d} \int_0^\infty u^2 e^{-u^2} du = \frac{3\sqrt{\pi}}{2} \frac{\rho}{f_d}.$$

Thus we have a quick bound

$$\mathbb{E}\left[\max_{B(x,\rho/n)} f\right] \leq 18\sqrt{\pi} \frac{\rho\sqrt{d}}{f_d} \leq 18\sqrt{\pi}\rho |S^d|^{-1/2}$$

which can be improved by studying the integral more carefully.

Write the integrand as $\exp(-u^2 + 2\log(u))$, and note that the critical points of $-u^2 + 2\log(u)$ are $u = \pm 1$. Write the lower limit of integration as

$$u_0 = \sqrt{\log\left(\frac{3\rho}{f_d \varepsilon_0}\right)} \sim \sqrt{\log\left(\frac{3}{\sqrt{2}}\frac{\rho}{\sqrt{d}}\right)}$$

Since ρ is chosen to be of order d, $\rho/\sqrt{d} \to \infty$ and we have $u_0 > 1$ for large d. Even for $d = 2$, choosing ρ to be the first minimum of J_0 and using the exact value of ε_0 instead of the approximation $\sqrt{2/|S^d|}$ yields a value $u_0 = 1.2568\ldots$ already greater than 1. Thus the critical points ± 1 do not enter into the integral. Instead, we expand about the lower endpoint:

$$-u^2 + 2\log u = -u_0^2 + 2\log u_0 - 2(u_0 - u_0^{-1})(u - u_0) - 2(1 + u_0^{-2})(u - u_0)^2 + \ldots$$

Thus

$$\int_{u_0}^{\infty} u^2 e^{-u^2} du \sim e^{-u_0^2 + 2\log u_0} \frac{1}{2(u_0 - u_0^{-1})}$$

We have

$$\exp(-u_0^2 + 2\log u_0) = \frac{f_d \varepsilon_0}{3\rho} \log\left(\frac{3\rho}{f_d \varepsilon_0}\right).$$

For large d, u_0 is large and we may neglect u_0^{-1} compared to u_0. Since $f_d = \sqrt{d|S^d|}$ and $\varepsilon_0 \sim \sqrt{2/|S^d|}$, this gives

$$\int_{u_0}^{\infty} u^2 e^{-u^2} du \approx \frac{1}{3\sqrt{2}} \frac{\sqrt{d}}{\rho} \sqrt{\log\left(\frac{3}{\sqrt{2}}\frac{\rho}{\sqrt{d}}\right)}$$

Retrieving the factor $12\sqrt{d}$, we get

$$\mathbb{E}\left[\max_{B(x,\rho/n)} f\right] \leq 12\sqrt{d} \int_{u_0}^{\infty} u^2 e^{-u^2} du \leq (1 + o(1)) 2\sqrt{2} \frac{d}{\rho} \sqrt{\log\left(\frac{3}{\sqrt{2}}\frac{\rho}{\sqrt{d}}\right)}$$

as $d \to \infty$. With $\rho \sim d/2$, this gives

$$\mathbb{E}\left[\max_{B(x,\rho/n)} f\right] \leq (4\sqrt{2} + o(1))\sqrt{\log d}.$$

With variance proxy $\sigma^2 = 1/|S^d|$, we had

$$\mathbb{P}\left(\sup_{t\in T} X_t \geq \mathbb{E}\left[\sup_{t\in T} X_t\right] + a\right) \leq e^{-a^2/(2\sigma^2)}.$$

First, choose a large enough to make this less than $\delta/2$:

$$a = \sqrt{2\sigma^2 \log(2/\delta)} = \sqrt{\frac{2}{|S^d|}\log(2/\delta)}.$$

If we choose

$$C_0 \geq \mathbb{E}\left[\sup_{t\in T} X_t\right] + a = \mathbb{E}\left[\sup_{t\in T} X_t\right] + \sqrt{\frac{2}{|S^d|}}\sqrt{\log(2/\delta)}$$

then the tail probability can be no larger than $\delta/2$. This method guarantees that

$$\mathbb{P}\left(\max_{B(x,\rho/n)} |f| \geq C_0\right) \leq 2\mathbb{P}\left(\max_{B(x,\rho/n)} f \geq C_0\right) \leq \delta,$$

as required, where C_0 grows in proportion to $\sqrt{\log(1/\delta)}$ instead of, as above, $\sqrt{1/\delta}$. However, there is also an additive shift by the expected supremum. By the estimates above, as $d \to \infty$, we may take

$$C_0 = \sqrt{2\log(2/\delta)}|S^d|^{-1/2} + (4\sqrt{2} + o(1))\sqrt{\log d}.$$

Using Stirling's formula, we see that $\sqrt{\log d}$ is negligible compared to $|S^d|^{-1/2}$. There-

fore

$$C_0 = \sqrt{2\log(2/\delta)}|S^d|^{-1/2}(1+o(1)) = \sqrt{2\log(2/\delta)}2^{-1/4}\left(\frac{d}{2\pi e}\right)^{d/4}(1+o(1)).$$

We use the same c_1 as before, hence

$$\begin{aligned} C_0/c_1 &= \frac{\sqrt{2\log(2/\delta)}|S^d|^{-1/2}}{|S^d|^{-1/2}2^{d/2-1}\Gamma(d/2)|J_{d/2-1}(\rho)|\rho^{-(d/2-1)}}(1+o(1)) \\ &= \rho^{d/2-1}\sqrt{\log(2/\delta)}2^{-d/2}\Gamma(d/2)^{-1}|J_{d/2-1}(\rho)|^{-1}(2\sqrt{2}+o(1)). \end{aligned}$$

We have the estimate

$$\begin{aligned} \mathbb{P}(\mathfrak{z} > C_0/c_1) &\sim \frac{1}{\sqrt{2\pi}}\frac{c_1}{C_0}\exp\left(-\frac{1}{2}\left(\frac{C_0}{c_1}\right)^2\right) \\ &= \exp\left(-\rho^{d-1}\log(2/\delta)2^{-d}\Gamma(d/2)^{-2}|J_{d/2-1}(\rho)|^{-2}(4+o(1))\right) \end{aligned}$$

There is also a factor of $1-2\delta$ in the lower bound, leading to

$$\exp\left(\log(1-2\delta) + K\rho^{d-1}\log\frac{\delta}{2}\right)$$

where

$$K = K(d,\rho) = 2^{-d}\Gamma(d/2)^{-2}|J_{d/2-1}(\rho)|^{-2}(4+o(1)).$$

The optimal δ is

$$\delta = \frac{1}{2}\frac{K\rho^{d-1}}{1+K\rho^{d-1}}.$$

Then

$$\log(1-2\delta) + K\rho^{d-1}\log\frac{\delta}{2} \sim -\log(4)K\rho^{d-1} - \log(1+K\rho^{d-1}).$$

Applying Stirling's formula to $\Gamma(d/2)$ gives

$$\Gamma(d/2)^2 \sim \frac{4\pi}{d}\left(\frac{d}{2e}\right)^d.$$

Therefore

$$\begin{aligned} K\rho^{d-1} &= (4+o(1))2^{-d}\Gamma(d/2)^{-2}|J_{d/2-1}(\rho)|^{-2}\rho^{d-1} \\ &= \left(\frac{1}{\pi}+o(1)\right)\left(\frac{e\rho}{d}\right)^d \rho^{-1}|J_{d/2-1}(\rho)|^{-2}. \end{aligned}$$

Since $\rho \sim d/2$, we have $e\rho/d \sim e/2 > 1$, so that $K\rho^{d-1}$ grows exponentially. The barrier method with these parameters gives

$$\begin{aligned} c(d) &\geq \frac{\Delta_{\mathbb{R}^d}}{|B_1|}\rho^{-d}(1-2\delta)\mathbb{P}(\mathfrak{z} > C_0/c_1) \\ &\geq \frac{\Delta_{\mathbb{R}^d}}{|B_1|}\rho^{-d}\exp(-\log(4)K\rho^{d-1} - \log(1+K\rho^{d-1})) \end{aligned}$$

From the estimate on $K\rho^{d-1}$, this can be restated as

$$\log\log\frac{1}{c(d)} \leq d\log\frac{e}{2} + O(\log d).$$

which improves on the dependence $d\log d$ that we achieved using the simpler estimate of the maximum.

However, this improvement does not kick in immediately. For instance, in dimension 2 the entropy-based bound gives a lower bound of only 4×10^{-145}. This is because the additive offset by $\mathbb{E}[\max f]$ leads to a larger C_0 in low dimensions. For instance, even as $\delta \to 1/2$, the estimate $\mathbb{E}[\max] \leq 12\sqrt{d}\int_{u_0}^{\infty} u^2 e^{-u^2}du$ leaves us with

$$C_0 \geq \frac{1}{\sqrt{2\pi}}\sqrt{\log 4} + 2.765 \geq 3.235$$

whereas the simpler estimate allows a smaller value:

$$C_0 \geq \frac{1}{\sqrt{2\pi}}\sqrt{C}(\rho+1) \geq 2.180$$

This makes a tremendous difference because the factor c_1^{-2} increase these numbers tenfold, and then we pass them through the Gaussian tail $\exp(-(C_0/c_1)^2/2)$.

4.8 Method of Ingremeau-Rivera

Ingremeau and Rivera give an improved lower bound on the two-dimensional Nazarov-Sodin constant by combining the barrier method with some significant innovations. They work in the plane instead of the sphere, with the scaling limit F obeying $(\Delta+1)F = 0$ almost surely and a circle of fixed radius r corresponding to a circle of radius r/n on the sphere. In the nodal trap, instead of having the barrier coefficient ξ_0 be large and the other pieces $f_\pm$ be small, they produce a nodal line by arranging that f have no zeros on a circle of radius r. The probability of having zeros on the circle is bounded by using the Kac-Rice formula to calculate the expected number of zeros on the circle. Numerically, this gives better results than the estimate of the maximum. It is not clear how to adapt this step to higher dimensions, where the zeros on the boundary form a hypersurface instead of a discrete configuration. The most attractive feature from our perspective is avoiding the mysterious quantity Δ_d by looking for nodal domains in more flexible regions.

We return to the sphere. Following Ingremeau-Rivera, let G_r be the set of centers x such that f has no zeros on the circle of radius r/n around x. For example, if x is inside some nodal domain and distant by more than r/n from the boundary, then x is such a center. We sum over nodal components c intersecting G_r to get an overestimate

for its volume:

$$\mathrm{vol}(G_r) \leq \sum_c \mathrm{vol}(G_r \cap c) \leq (\#c) \max_c \mathrm{vol}(G_r \cap c).$$

To bound the individual volumes $\mathrm{vol}(G_r \cap c)$, we first bound their diameter and then appeal to the isodiametric inequality to conclude that they have volume no greater than a ball of the same diameter [12].

4.8.1 A proof suggested by Deleporte

To bound the diameter, fix any $x_0 \in G_r$ where r is in between the first and second zeros of the barrier function. We claim that any x in the same nodal domain as x_0 must be within distance r of x_0. Indeed, symmetrize around x_0:

$$g(x) = \int_{\mathrm{Stab}(x_0)} f(gx) dg$$

where dg is Haar measure on the lower-dimensional orthogonal group fixing x_0. The symmetrized function g is also a spherical harmonic of the same degree as f, and by construction it is radial with respect to x_0. There is only one such function, namely the barrier! Thus g must be proportional to the barrier function $\omega(\cdot, x_0)$, say

$$g(x) = cP_n^d(x \cdot x_0)$$

for some constant c. Since $g(x_0) = f(x_0)$, c must have the same sign as $f(x_0)$, which we may assume is positive. Since r is between the first and second zeros of the ultraspherical polynomial (or, for large degrees, the Bessel function $J_{d/2-1}$), $g(x) < 0$ on the shell of radius r/n around x_0. Thus $f(y) < 0$ for some y on this shell. On the other hand, the hypothesis $x_0 \in G_r$ means that f has no zeros on this shell, so it must be that $f(y) < 0$ for *all* such y. Hence f must be negative on this circle of

radius r/n around x_0. For x and x_0 to be in the same nodal component, it must be that $d(x, x_0) < r/n$.

This shows that $\operatorname{diam}(G_r \cap c) < 2r$ for any nodal component c. It follows from the isodiametric inequality that $G_r \cap c$ has volume less than or equal to that of a ball of equal diameter:

$$\operatorname{vol}(G_r \cap c) \leq |B_1| r^d n^{-d}$$

For any spherical harmonic f, random or not, we therefore have

$$\operatorname{vol}(G_r) \leq N(f) |B_1| r^d n^{-d}$$

Now take the expected value of both sides. We have

$$\begin{aligned} \mathbb{E}[\operatorname{vol}(G_r)] &= \int \int_{S^d} \mathbb{I}[x \in G_r] \; d \operatorname{vol}(x) \; d\mathbb{P} \\ &= \int_{S^d} \mathbb{P}\{x \in G_r\} \; d \operatorname{vol}(x) \end{aligned}$$

By symmetry, any point x is equally likely to belong to G_r. For any origin x_0, we have

$$\mathbb{E}[\operatorname{vol}(G_r)] = \operatorname{vol}(S^d) \mathbb{P}(x_0 \in G_r).$$

This gives a lower bound of

$$\mathbb{E}[N(f)] \geq n^d \operatorname{vol}(S^d) |B_1|^{-1} \mathbb{P}(x_0 \in G_r) r^{-d}$$

We need a lower bound for $\mathbb{P}(x_0 \in G_r)$, which proceeds along the same lines as the barrier method. Write

$$f = \xi_0 b_{x_0} + \frac{f_- + f_+}{2}$$

We have

$$|f(y)| \geq |\xi_0||b_{x_0}(y)| - \frac{|f_- + f_+|}{2}$$

so f cannot vanish on a circle of radius r/n around x_0, provided $|\xi_0|$ is large enough compared to $|f_\pm|$. As in the barrier method, we have $|b(y)| = c_1\sqrt{M}$ while $|f_\pm| \leq C_0$ with probability at least $1 - 2\delta$. Therefore

$$\mathbb{P}(x_0 \in G_r) \geq (1 - 2\delta)\mathbb{P}(|\xi_0\sqrt{M}| > C_0/c_1)$$

where $\xi_0\sqrt{M}$ is now a standard Gaussian, say $\mathfrak{z}$. Therefore

$$\frac{\mathbb{E}[N(f)]}{n^d} \geq \mathrm{vol}(S^d)|B_1|^{-1}(1 - 2\delta)\mathbb{P}\left(|\mathfrak{z}| > \frac{C_0}{c_1}\right) r^{-d}.$$

which removes the sphere-packing factor $\Delta_{\mathbb{R}^d}$ from the bound above.

4.9 Upper bound via Courant's nodal domain theorem

The expected value of $N(f)$ is at most its maximum value among all harmonics of the given degree. Courant's theorem is that the N-th eigenfunction has at most N nodal domains. To see what is the eigenvalue of the N-th eigenfunction, we appeal to Weyl's law. An eigenfunction with eigenvalue λ is the N-th eigenfunction where N is the number of eigenvalues less than or equal to λ. On a compact manifold M of dimension d, Weyl's law reads

$$N \sim \lambda^{d/2}\,\mathrm{vol}(M)\frac{\mathrm{vol}(B_1)}{(2\pi)^d}$$

where B_1 is the Euclidean unit ball. Therefore

$$\frac{c_{\mathrm{NS}}(M)}{\mathrm{vol}(M)} \leq \frac{\mathrm{vol}(B_1)}{(2\pi)^d}$$

Using Stirling's formula, we see that this is of order

$$\frac{c_{\mathrm{NS}}(M)}{\mathrm{vol}(M)} \leq \big(1 + o(1)\big) \Big(\frac{e}{2\pi d}\Big)^{d/2} \frac{1}{\sqrt{\pi d}}.$$

In other words

$$\log\log \frac{1}{c(d)} \geq \log(d/2).$$

This is of the same quality $\log d$ as claimed in Theorem 4.0.2, but with a worse constant. In the next section, we improve the constant factor from 1 to 4/3, as claimed. The Courant upper bound is worth calculating explicitly from the point of view of "the random curve is 4% Harnack", that is, to compare the average number of nodal domains to the maximum possible for an eigenfunction.

For the three-dimensional constant, this Courant bound gives

$$c_{\mathrm{NS}}(M^3) \leq \frac{\mathrm{vol}(M)}{6\pi^2}.$$

On the sphere S^3, with volume $2\pi^2$, we get

$$c_{\mathrm{NS}}(S^3) \leq \frac{1}{3}.$$

4.10 The ergodic method of Nazarov-Sodin

In a later paper [54], Nazarov and Sodin generalize their theorem for random spherical harmonics to other ensembles of Gaussian random functions. A random ensemble on a compact manifold can be studied via a *scaling limit*, which yields a random function

on the tangent space. In the example of random spherical harmonics, this scaling limit is the random plane wave. In general, the distribution of the scaling limit is invariant under translations of $\mathbb{R}^d$. Thus its two-point function can be written in the form

$$\mathbb{E}[F(x)F(y)] = \int_{\mathbb{R}^d} e^{2\pi i(x-y)\cdot\lambda} d\rho(\lambda)$$

for some positive measure ρ called the *spectral measure*, where the translation-invariance corresponds to the fact this is a function only of $x - y$. Nazarov and Sodin assume that ρ has finite fourth moment, which guarantees that the resulting random functions F are almost surely in every Hölder space $C^{1+\alpha}(\mathbb{R}^d)$ with $\alpha < 1$. They assume ρ is not supported on a linear hyperplane, which guarantees that the gradient ∇F is non-degenerate. They assume also that ρ has no atoms, which implies that the action of translations is ergodic for the probability measure on functions defined by ρ (theorem of Grenander [28]). Under these three conditions, Nazarov and Sodin prove that there is a constant $\nu \geq 0$ such that the number of connected components of $F^{-1}\{0\}$ contained in a ball of radius R is asymptotic to $\nu \operatorname{vol}(B_R)$, both almost surely and in expectation. If, instead of the unit ball B, we fix any bounded convex set S containing the origin, then the number of components inside the scaled body S_R will be asymptotic to $\nu \operatorname{vol}(S_R)$. Ergodicity guarantees that the limit ν is deterministic. In this generality, it might be that $\nu = 0$. To guarantee that $\nu > 0$, Nazarov and Sodin assume that there is a "barrier-like" function to play the role of the zonal harmonic on the sphere. This function is given as a Fourier transform

$$\widehat{\mu}(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\lambda} d\mu(\lambda)$$

where μ is a finite measure of compact support such that $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\rho)$, Hermitian in the sense that $\mu(-A) = \overline{\mu(A)}$. The assumption is that there are a bounded domain $D \subset \mathbb{R}^d$ and a measure μ such that, for some $u_0 \in D$, $\widehat{\mu}(u_0) > 0$ whereas $\widehat{\mu} < 0$ on

the boundary ∂D. In other words, using only frequencies in the support of ρ, it is possible to synthesize a function with at least one bounded nodal component. Under this assumption, Nazarov-Sodin show that $\nu > 0$. They remark that the constant can be expressed as

$$\nu = \mathbb{E}\left[\frac{1}{\mathrm{vol}(G)}\right]$$

where G is the nodal domain including the origin. Conceivably, this nodal domain could have infinite volume. The theorem of Nazarov-Sodin is that, with positive probability, the volume is finite. Otherwise, ν would be 0. Lower bounds on ν are thus related to the tails of the random variable $\mathrm{vol}(G)$. A softer question, which nonetheless appears very difficult, is whether G is bounded almost surely.

We can also use this formula to give an upper bound on ν in the monochromatic ensemble, where we write $\nu = c(d)$. In this case, the random function F solves the Helmholtz equaton $(\Delta + 1)F = 0$ in $\mathbb{R}^d$. By definition of "nodal set", F vanishes on the boundary of G. Therefore F solves the Dirichlet problem in the domain G, with eigenvalue 1. The Faber-Krahn inequality is that the ball is extremal for the first Dirichlet eigenvalue λ_1 (see [23], [43], [44] for the original articles of Faber and Krahn). Thus, if B is a ball with the same volume as G,

$$1 \geq \lambda_1(G) \geq \lambda_1(B).$$

This is proved by writing the Rayleigh quotient for λ_1 and using rearrangement inequalities. The radius of such a ball must be $(\mathrm{vol}(G)/\mathrm{vol}(B_1))^{1/d}$, where B_1 is the unit ball. On the ball, the first eigenfunction is a Bessel function $J_\alpha(\beta|x|)/(\beta|x|)^\alpha$, where $\alpha = d/2 - 1$ depends only on the dimension while β depends also on the radius of the ball. Namely, since the eigenfunction must vanish on the boundary $|x| = r$, we must have $\beta = j/r$ where j is a root of the Bessel function $J_{d/2-1}$. In order to have the first eigenvalue, we take the first positive root $j_{d/2-1,1}$. Thus Krahn's inequality

can be stated

$$\lambda_1(G) \geq \left(\frac{\mathrm{vol}(B_1)}{\mathrm{vol}(G)}\right)^{2/d} j_{d/2-1,1}$$

Since $\lambda_1(G) \leq 1$, this implies that

$$\frac{1}{\mathrm{vol}(G)} \leq \frac{1}{\mathrm{vol}(B_1)} j_{d/2-1,1}^{-d/2}$$

Taking expectation, the same upper bound holds for the Nazarov-Sodin constant for the d-dimensional monochromatic ensemble:

$$c(d) \leq (d/2)! \left(\frac{1}{j\pi}\right)^{d/2}$$

The first root of J_α occurs near α, with a correction of order $\alpha^{1/3}$, while the ball volume $\mathrm{vol}(B_1)$ is $\pi^{d/2}/(d/2)!$. Thus the bound from Faber-Krahn leads to

$$\log\log \frac{1}{c(d)} \geq \frac{4}{3}\log d + O(1)$$

as claimed in Theorem 4.0.2. This is still of order $\log d$ but improves the constant factor in the Courant bound. Taking $d = 2$, note that the first root of J_0 is $2.4048\ldots$, so this gives an upper bound of $0.13236298\ldots$ In dimension $d = 3$, the Bessel function $J_{1/2}$ is proportional to $\sin(x)/\sqrt{x}$ so that the first root is $j = \pi$. This gives

$$c(3) \leq \frac{1}{4\pi/3}\pi^{-3/2} = 0.04287\ldots$$

One limitation of this method is that equality holds in the Faber-Krahn inequality if and only if the domain is a ball. In dimension 3, the nodal domain containing 0 seems to be very far from a ball. It is more like a supercritical percolation cluster.

Note that Levenshtein's bound for sphere packing [47] is

$$\Delta_{\mathbb{R}^d} \leq \left(\frac{j_{d/2}^{d/2}}{(4\pi)^{d/2}} \operatorname{vol}(B_1) \right)^2$$

which is vaguely reciprocal to this upper bound for the Nazarov-Sodin constant

$$c(d) \leq \left(j_{d/2-1}^{d/2} \operatorname{vol}(B_1) \right)^{-1}.$$

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