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Parametrizing Rings of Rank 6

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PARAMETRIZING RINGS OF RANK 6

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Abstract

We give various methods of parametrizing sextic rings (rings of rank 6) seeking to provide limited generalizations of Bhargava's Higher Composition Laws.

Most of the parametrization methods take form of constructing sextic rings from certain algebraic forms. For $A \in \mathbb{Z}^3 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^5$ a $3 \times 3 \times 5$ form, we construct a sextic ring $\mathcal{R}_A$ whose multiplication coefficients are given as polynomials in entries of A . The ring $\mathcal{R}_A$ is constructed from hypercohomology of a minimal free resolution of six points in $\mathbb{P}^2$ determined by certain equations in A , and it can be realized as subring of the ring of global functions on those six points. Similarly, we construct a sextic ring from an element of $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^5$ from hypercohomology of a minimal free resolution of six points in $\mathbb{P}^1 \times \mathbb{P}^1$ determined by certain equations.

The parametrization method in the Chapter 5 takes opposite approach – it constructs a Segre cubic threefold from a sextic ring with nonzero discriminant. This gives a one-to-one correspondence between primitive sextic nondegenerate rings and Segre cubics with integral coefficients with gcd of its coefficients 1 up to $\mathrm{GL}_5(\mathbb{Z})$ -equivalence. This method allows us to give a geometric interpretation for the Higher Composition Law for quintic rings. For a form $A \in \mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5$ which parametrizes a pair (R, S) of a quintic ring R and its sextic resolvent ring S in the quintic Higher Composition Law, one can construct a Segre cubic from A whose corresponding sextic ring becomes a subring of $S \otimes \mathbb{Q}$.

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Chapter 1

Introduction

In this thesis, we will discuss several methods of parametrizing sextic (rank 6) rings, given in the form of maps between sextic rings and algebraic forms. For a representation of algebraic group (G, V) defined over $\mathbb{Z}$ (where G is group scheme defined over $\mathbb{Z}$ and V is representation of G defined over $\mathbb{Z}$) our parametrization methods in Chapters 3, 4, and 6 take the form of maps $V(\mathbb{Z}) \rightarrow \{\text{sextic rings}\}$, constructing a sextic ring from an integral form (which is considered as an element of $V(\mathbb{Z})$). The parametrization method in Chapter 5 gives a map in the opposite direction from sextic rings to forms: it attaches an integral cubic form in 5 variables to a given sextic ring R .

A *ring of rank n* ($n \in \mathbb{Z}_{>0}$) is defined as a ring with 1 whose $\mathbb{Z}$ -module structure is isomorphic to $\mathbb{Z}^n$ as $\mathbb{Z}$ -module. Any rank n ring comes with a unique invariant called the *discriminant* (defined as the determinant of the matrix of the symmetric bilinear form $(x, y) \mapsto \text{tr}_{R/\mathbb{Z}}(xy)$), and we call a ring *nondegenerate* if its discriminant is nonzero. Nondegenerate rings of rank n can be considered as generalization for the ring of integers of a degree n number field and its orders. More generally, one can define a *rank n algebra over a ring R* as an R -algebra with 1 that is locally free of rank n . Over a scheme S , a *rank n algebra over S* is defined as an $\mathcal{O}_S$ -algebra that is

a locally free $\mathcal{O}_S$ -module of rank n ([49]).

For a ring R with rank n having $\mathbb{Z}$ -basis $1 = \alpha_0, \alpha_1, \dots, \alpha_{n-1}$, its *multiplication coefficients* with respect to the basis denotes the integral coefficients which appears in representing $\alpha_i \alpha_j$ as a linear combination of basis elements, that is, numbers $c_{ij}^k \in \mathbb{Z}$ satisfying

$$\alpha_i \alpha_j = \sum_{k=0}^{n-1} c_{ij}^k \alpha_k.$$

While a ring can be specified by listing all of its multiplication coefficients, it is generally really hard to classify rings of rank n only using multiplication coefficients.

Parametrizing the rings of rank n can be roughly said to be any process of expressing a ring with numbers, that is at least simpler than giving the description of all multiplication coefficients. For example, the so-called Levi–Delone–Faddeev correspondence ([35], [19], [29]) establishes the following one-to-one correspondence between cubic (rank 3) rings and integral binary cubic forms $(\text{Sym}^3 \mathbb{Z}^2)^* = \{ax^3 + bx^2y + cxy^2 + dy^3 : a, b, c, d \in \mathbb{Z}\}$. This correspondence $f \mapsto R_f$ ($f = ax^3 + bx^2y + cxy^2 + dy^3$) is given explicitly as follows:

$$\begin{aligned} R_f &= \mathbb{Z} + \mathbb{Z}\omega + \mathbb{Z}\theta, \\ \omega\theta &= -ad, \quad \omega^2 = -ac + b\omega - a\theta, \quad \theta^2 = -bd + d\omega - c\theta \end{aligned}$$

Here $R = R_f$ is constructed as a free $\mathbb{Z}$ -module with basis $(1, \omega, \theta)$, and the above relations expressing $\omega\theta, \omega^2, \theta^2$ as a linear combination of basis elements suffice to determine all multiplications on R_f . If we consider the standard action of $g \in \text{GL}_2(\mathbb{Z})$ on the cubic form f , its effects on R_f can be described as follows. Denote $\bar{\omega}, \bar{\theta}$ as the images of ω, θ respectively under the map $R \rightarrow R/\mathbb{Z}$. Then g acts on a vector $(\bar{\omega}, \bar{\theta})$ as the standard action on $\mathbb{Z}^2$ times determinant of g . As all $\mathbb{Z}$ -bases of $R/\mathbb{Z}$ are $\text{GL}_2(\mathbb{Z})$ -equivalent, considering equivalence classes with respect to the $\text{GL}_2(\mathbb{Z})$ -action induces a one-to-one correspondence between cubic rings and equivalence class

of binary cubics. In a series of works ([2], [3]) named as the “Higher Composition Laws”, Bhargava established the parametrization of quartic rings (rank $n = 4$ rings) and quintic rings (rank $n = 5$ rings) as one-to-one correspondences given in the following form

$$\{\text{rank } n \text{ rings}\} \leftrightarrow V(\mathbb{Z})/G(\mathbb{Z})$$

where (G, V) is a representation of an algebraic group defined over $\mathbb{Z}$, and $V(\mathbb{Z})/G(\mathbb{Z})$ denotes the set of equivalence classes of integral orbits. The representations used are given as follows.

$$(V(\mathbb{Z}), G(\mathbb{Z})) = (\mathbb{Z}^2 \otimes (\text{Sym}^2 \mathbb{Z}^3)^*, \text{SL}_2(\mathbb{Z}) \times \text{GL}_3(\mathbb{Z})) \text{ for quartic rings}$$

$$(V(\mathbb{Z}), G(\mathbb{Z})) = (\mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5, \text{SL}_4(\mathbb{Z}) \times \text{GL}_5(\mathbb{Z})) \text{ for quintic rings.}$$

The formulas for multiplication coefficients of the corresponding ring are given as explicit polynomials in the entries of $V(\mathbb{Z})$, and the action of $G(\mathbb{Z})$ acts as integral coordinate changes for basis of the ring. As this action keeps the ring same, a $G(\mathbb{Z})$ -equivalence class of elements in $V(\mathbb{Z})$ represents an isomorphism class of a rank n ring.

These parametrization methods for rings of rank 3, 4, 5 led to a solution to the problem of counting rings and number fields. The problem of finding asymptotic growth for the number of number fields having bounded discriminant, that is, finding the asymptotic growth of the number

$$N_n(A, B) = \#\{\text{number field } F \text{ of degree } n$$

$$\text{with its discriminant satisfying } A < \text{Disc}(F) < B\},$$

could be tackled using these parametrizations of the rings of rank n . Davenport and Heilbronn [17] used the above Levi–Delone–Faddeev correspondence of the cubic

rings to find an asymptotic for $N_3(0, X)$ and $N_3(-X, 0)$, by characterizing cubic forms giving cubic maximal orders and counting the corresponding lattice points in a certain fundamental region for the action of $\mathrm{GL}_2(\mathbb{Z})$ on $(\mathrm{Sym}^3 \mathbb{Z}^2)^*$. This approach was refined and made more convenient to use in [5] such that one doesn't need to construct equations for explicit fundamental regions, and was applied to $N_4(A, B)$ and $N_5(A, B)$ in [6] and [7] respectively, each based on the parametrization models in [2] and [3] respectively.

The exact asymptotic for the quantities $N_n(0, X)$ and $N_n(-X, 0)$ is not known yet for any $n \geq 6$, as there is no one-to-one correspondence known between rank n rings and integral forms which is powerful enough to reduce the problem to lattice point counting. To either tackle this counting problem, or provide generalizations for the Bhargava's Higher Composition Laws, there have been several attempts to construct ring parametrization models for rank ≥ 6 . Instead of finding the strict one-to-one correspondence of the following form as given in Bhargava's model

$$\{\text{rank } n \text{ rings}\} \leftrightarrow V(\mathbb{Z})/G(\mathbb{Z}),$$

one may still attempt the partial correspondence having the form of

$$\{\text{rank } n \text{ rings with additional structure}\} \leftrightarrow V(\mathbb{Z})/G(\mathbb{Z}).$$

This correspondence does not need to be one-to-one, and it can take the form of either the maps from rings to forms ($\rightarrow$) or the maps from forms to rings ($\leftarrow$). One example is the construction of a rank n ring from a binary n -ic form $f = f_0 x^n + f_1 x^{n-1} y + \cdots + f_n y^n$, which appeared in [39] and was studied in more detail in [47]. The ring R_f constructed from a form f is given as a rank n ring whose multiplication coefficients are given as polynomials in f_i (explicit formula can be found in [47]). Here the representation used here is $G(\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z})$ and $V(\mathbb{Z}) = (\mathrm{Sym}^n \mathbb{Z}^2)^*$, and the

standard action of $G(\mathbb{Z})$ on $V(\mathbb{Z})$ acts on R_f as a linear action on $\mathbb{Z}$ -basis elements of R_f whose determinant is 1 (thus preserving the equivalence class of the ring R_f). For $n = 3$ it is identical to the Levi–Delone–Faddeev correspondence, but for $n > 3$ the correspondence does not represent all rank n rings: there are rank n rings R that are not isomorphic to R_f .

In this thesis, we suggest some new models for parametrizing sextic (rank 6) rings, by combining several ideas from various works which addresses the generalization of Bhargava’s ring parametrization models. Most of parametrization models in this thesis (Chapter 3, 4, 6) can be described as above form of maps $V(\mathbb{Z})/G(\mathbb{Z}) \rightarrow \{\text{sextic rings}\}$ for different set of algebraic representation (G, V) which are integral (defined over $\mathbb{Z}$). Of course it might be possible to find parametrization models which do not have this form. For example, one might consider a certain map from rings to forms, and characterize which forms come from rank n rings. While parametrizations in this form is also considered briefly (Chapter 5), generally we would like to focus on maps from integral forms to sextic rings. We hope that this work can, in particular, be applied in finding out asymptotic growth of sextic rings in author’s future works.

1.1 Paradigms of ring parametrizations

How can we find such correspondences between sextic rings and algebraic forms? Before addressing this question and describing the results of this thesis, we would like to discuss some common features which can be found in previous works on the ring parametrizations. One of those common features is to relate rank n rings to certain n points in some projective space, and this idea appears repeatedly in many parametrization models including the Bhargava’s Higher Composition Laws. Once geometry of those n points is identified, a construction using hypercohomology of sheaves in Wood’s works (appearing in [47], [49], [48] etc.) can be applied to the most

of those ring parametrization models.

Those two ideas (n points in projective space and the hypercohomology construction) also become central in many of the parametrization models for sextic rings in this thesis. We will first describe how those two ideas appear in previous works on the ring parametrizations. Then we will move onto our parametrization models for sextic rings, and describe how the most of the those models (Chapter 3, 4, 6) are structured around them.

(1) Points in projective space

One of the occuring principle is that rank n rings can be related to some n geometric points in projective space. For example, the ring parametrization models for rank n rings in the Bhargava's Higher Composition Laws all determine equations for n geometric points in $\mathbb{P}^{n-2}$.

- Binary cubic forms $((\text{Sym}^3 \mathbb{Z}^2)^*)$ parametrizing rank 3 rings ([19], [29]) : a generic binary cubic form $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ cuts out three points in $\mathbb{P}^1$ via the equation $f(x, y) = 0$.
- Pairs of ternary quadratic forms $(\mathbb{Z}^2 \otimes (\text{Sym}^2 \mathbb{Z}^3)^*)$ parametrizing rank 4 rings ([2]): two generic quadratic forms $a(x, y, z), b(x, y, z)$ determine four points in $\mathbb{P}^2$ as their intersections $a(x, y, z) = b(x, y, z) = 0$.
- 4-tuples of 5×5 alternating bilinear forms $(\mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5)$ parametrizing rank 5 rings ([3]): for a generic such 4-tuple (A_1, A_2, A_3, A_4) ($A_i \in \wedge^2 \mathbb{Z}^5$), the five Pfaffians of the five 4×4 principal minors of alternating matrix $x_1 A_1 + \dots + x_4 A_4$ cuts out 5 points in $\mathbb{P}^3$. This result will be discussed more in the Chapter 6 of this thesis.

While this heuristic of relating n points in $\mathbb{P}^{n-2}$ to rank n rings was first stated explicitly in the Higher Composition Laws ([3]), this idea can be linked to several other works related to ring parametrizations. The above mentioned algebraic forms

determining equations for n points in $\mathbb{P}^{n-2}$ appear in a preceding work of Wright and Yukie ([52]), which parametrizes nondegenerate algebras of rank $n = 3, 4, 5$ over field k of characteristic $\neq 2, 3, 5$ using those same algebraic forms. If we consider those forms as suitable algebraic representations (G, V) (e.g. $G = \mathrm{GL}_2 \times \mathrm{SL}_3$, $V = k^2 \otimes \mathrm{Sym}^2 k^3$ for $n = 4$), then a generic form $A \in V(k)$ determines n geometric points in $\mathbb{P}^{n-2}$ via equations described above. The rank n algebra over k corresponding to a form A is given as the global function ring of the k -scheme formed by the union of those n points.

In general, a nondegenerate algebra R of rank n over a field k is equivalent to a degree n étale cover $\mathrm{Spec} R \rightarrow \mathrm{Spec} k$. Casnati's work [16] considers a degree n Gorenstein cover of schemes $\rho : X \rightarrow Y$ over field k , and shows that such ρ can be always factored into a closed embedding $X \rightarrow \mathbb{P}_Y^{n-2}$ and the natural map $\mathbb{P}_Y^{n-2} \rightarrow Y$. When $Y = k$, the closed embedding $X \rightarrow \mathbb{P}_Y^{n-2}$ corresponds to some n geometric points in $\mathbb{P}_k^{n-2}$.

In [16], the structure of the minimal free resolution of the cover $X \rightarrow \mathbb{P}_Y^{n-2}$ is also uniquely determined. In this context, [46] further considered the structure of a minimal free resolution of n geometric points - especially the natural DG-algebra structure, which could be used in constructing the global function ring. It also described S_n -equivariance properties of the DG-algebra structure which allow ones to describe the action of the Galois group on the global function ring of those n points.

While those examples so far all consider n points in $\mathbb{P}^{n-2}$, it is also possible to construct a rank n algebra over field k by considering n points in $\mathbb{P}^m$ for $m < n - 2$. One simple example is a binary form $f = f(x, y)$ of degree n , which cuts out n distinct geometric points in $\mathbb{P}^1$ when discriminant of f is nonzero. In this case, the algebra associated to f simply becomes $k[x]/f(x, 1)$, provided that $f(1, 0) \neq 0$. (If $f(1, 0) = 0$, then one can choose any binary form which is $\mathrm{SL}_2(\mathbb{Z})$ -equivalent to f and has nonzero x^n -coefficient.) This algebra coincides with the global function ring

of the k -scheme formed by union of those n points. If we consider binary n -ic forms (degree n binary forms) over $\mathbb{Z}$ and try to construct a rank n ring associated to it, the ring $\mathbb{Z}[x]/f(x, 1)$ does not become ring of finite rank in general, so one needs another approach when we want to construct an integral ring.

(2) Hypercohomology construction

Procedure for constructing an integral ring out of an algebraic form can be made rather explicit, provided that an appropriate syzygy-like complex is given. The method of constructing rank n ring via hypercohomology of a complex of sheaves was established by Wood's works ([47], [49], [48]) and was applied to several ring parametrization models. In this method, one first considers a complex of sheaves on $\mathbb{P}^m$ which resembles minimal free resolution of n geometric points, then constructs a rank n ring from hypercohomology module of the complex. For example, for the case of a binary n -ic form f and associated rank n ring, we consider the following sequence of sheaves on $\mathbb{P}_{\mathbb{Z}}^1$:

$$\mathcal{O}(-n) \xrightarrow{\times f} \mathcal{O}, \quad \mathcal{O} = \mathcal{O}_{\mathbb{P}^1}.$$

If $f \neq 0$, this provides a minimal free resolution for the sheaf $\mathcal{O}/f$, which becomes structure sheaf for the vanishing scheme $V(f) \subset \mathbb{P}^1$ of f . One can show that its hypercohomology module is free $\mathbb{Z}$ -module of rank n . Also the hypercohomology module can be naturally embedded into the global function ring of $V(f)$, and can be given a ring structure inherited from the multiplication structure of the global function ring.

In general, we consider a situation where a system of polynomial equations is made out of our given algebraic form, and this system determines n points in $\mathbb{P}^m$ for generic case. (For the above example, the equation was $f(x, y) = 0$ in $\mathbb{P}^1$ made out of a binary n -ic form f , and it determines n solutions for generic f .) Let $\mathcal{I}$ be the ideal sheaf for the system of equations, and consider a complex $\mathcal{C}$ consisting of locally free sheaves on $\mathbb{P}^m$ which becomes a minimal free resolution of $\mathcal{O}/\mathcal{I}$ for generic case.

For $\mathcal{C}$ satisfying some mild conditions, it turns out that the hypercohomology module $H^0 R\Gamma(\mathcal{C})$ becomes a free $\mathbb{Z}$ -module of rank n and also can be given a natural ring structure.

Wood applies this hypercohomology construction to parametrization models including the ring parametrization using binary n -ic forms and the quartic ring parametrization. In [47], the hypercohomology construction applied to binary n -ic forms gives the same rank n ring as the one given in [39]. Similarly, in [49], this construction is applied to the Koszul complex associated to two ternary quadrics to produce the exact quartic ring constructed out of those quadrics in the quartic Higher Composition Law of [2].

Most of the sextic ring parametrization models in this thesis follow those two ideas of considering n points in some projective space and computing the rank n ring via hypercohomology of certain complex of sheaves. All parametrization models in the Chapter 3, 4, 6, each coming together with a algebraic representation (G, V) defined over $\mathbb{Z}$, will all be discussed in the following order.

- Points in projective spaces: for each parametrization method with given representation (V, G) , we will first give a system of equations determined by an element A of V whose solution becomes six points in certain projective space for generic case.
- Hypercohomology construction: we will first consider an ideal sheaf $\mathcal{I}$ for those six points in the ambient projective space, and build a suitable complex of sheaves which behaves like minimal free resolution of $\mathcal{O}/\mathcal{I}$ for generic value of A . The desired rank 6 ring is constructed out of the hypercohomology module of the complex.

In constructing the rank 6 ring $\mathcal{R}(A)$ in above order, it will follow that all multiplication coefficients of $\mathcal{R}(A)$ are given as integral polynomials in coordinates of $A \in V$,

and the discriminant of the sextic ring will become a polynomial in A which becomes relative invariant with respect to the action of G .

We might also consider additional invariants of orbits in $V(\mathbb{Q})/G(\mathbb{Q})$, which correspond to a list of elements in sextic resolvent algebra of $\mathcal{R}(A) \otimes \mathbb{Q}$ - a sextic algebra constructed in a Galois-theoretic way using properties of outer automorphisms of S_6 . (The exact definition of sextic resolvent algebra will be discussed in the Section 3.4.) One can construct such elements in many parametrization models of sextic rings, including the ones appearing in Chapter 3, 4 and the sextic ring parametrization using binary sextic forms (special case of the parametrization using binary n -ic forms). This is due to that geometric invariants of six points in various of projective spaces (including $\mathbb{P}^1$, $\mathbb{P}^2$, and $\mathbb{P}^1 \times \mathbb{P}^1$) all can be constructed using properties of outer automorphisms of S_6 , ([33]) and properties of those invariants will be utilized extensively in constructing elements in the sextic resolvent algebra.

1.2 Structure of this paper

Each chapter of this thesis is devoted to a parametrization model for sextic rings or sextic $\mathbb{Q}$ -algebras, with two exceptions of Chapter 2 and Chapter 7. Chapter 2 functions as a ‘preliminary part’ of the whole thesis, by describing the hypercohomology construction technique which will be used extensively duringt the constructions of sextic ring. Chapter 7 deals with a different topic, discussing several concepts regarding $\mathbb{Z}$ -schemes related to rank n nondegenerate rings. While the main results of each chapters (except Chapter 2) are indepenent to each other, there are some common parts appearing in different chapters (such as descriptions for the Segre cubic or the sextic resolvent algebras). To avoid redundancy, those common parts will be described only once at their earliest appearances.

Chapter 2 generalizes the hypercohomology construction which appeared in Wood’s

works including [47] and [49], and provides statements which can be applied to a variety of parametrization models including the ones appearing in this thesis. After briefly reviewing how hypercohomology appears in the cubic and quartic ring parametrizations as described in [49], we will describe a class of complexes of sheaves on projective spaces whose hypercohomology module can be given a structure of rank n ring. Then we will illustrate how to explicitly compute the hypercohomology ring and discuss how this ring relates to the n geometric points determined by the equations of the complex. Results in this chapter will be used repeatedly in subsequent chapters of this thesis.

Chapter 3 is about sextic rings associated to $3 \times 3 \times 5$ forms: we have $V(\mathbb{Z}) = \mathbb{Z}^3 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^5$ acted by $G(\mathbb{Z}) = \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_5(\mathbb{Z})$. For a $3 \times 3 \times 5$ form A , we describe a system of equations which determine six geometric points in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^4$ for generic A . Denote those six points $\mathfrak{X}_A$.

By using the descriptions for the complete geometry of $3 \times 3 \times 5$ forms in [38], one can show that a generic orbit in $V(\mathbb{C})/G(\mathbb{C})$ corresponds to an PGL_2 -equivalence class of six points in $\mathbb{P}^2$ not on a conic, via the correspondence mapping A to the projection of $\mathfrak{X}_A$ onto $\mathbb{P}^2$. We will build two complexes consisting of sheaves on $\mathbb{P}^2$ and $\mathbb{P}^4$, each becoming minimal free resolution of projections of $\mathfrak{X}_A$ onto $\mathbb{P}^2$ and $\mathbb{P}^4$ respectively for generic $A \in V(\mathbb{Q})$, and show that their hypercohomology rings are isomorphic sextic rings. For $A \in V(\mathbb{Q})$, the set of invariants for the corresponding six points in $\mathbb{P}^2$ correspond to an element of the sextic resolvent algebra of the sextic ring constructed, and this element can be used to further classify orbits of $V(\mathbb{Q})/G(\mathbb{Q})$.

Chapter 4 is about sextic rings associated to $2 \times 2 \times 2 \times 5$ forms: $V(\mathbb{Z}) = \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^5$ acted by $G(\mathbb{Z}) = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{GL}_5(\mathbb{Z})$. A $2 \times 2 \times 2 \times 5$ form A describes six points $\mathfrak{X}_A$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^4$ for generic A . Similar to the $3 \times 3 \times 5$ case, One can construct two complexes each giving equations for projections of $\mathfrak{X}_A$ onto $\mathbb{P}^4$ and $\mathbb{P}^1 \times \mathbb{P}^1$ respectively for generic case, and their hypercohomology rings turn out to be

isomorphic sextic rings. By using properties of this sextic ring, we can further show that show that a generic orbit of $V(\mathbb{C})/G(\mathbb{C})$ corresponds to $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ -orbit of six points in $\mathbb{P}^1 \times \mathbb{P}^1$ not on single bidegree $(2, 1)$ or $(1, 2)$ -form. The invariants for six points in $\mathbb{P}^1 \times \mathbb{P}^1$ correspond to a pair of elements in the sextic resolvent algebra of the sextic ring constructed.

Chapter 5 is about sextic $\mathbb{Q}$ -algebras and Segre cubic threefolds, and it takes a little different approach from other parametrization models. A Segre cubic threefold is degree 3 cubic of $\mathbb{P}^4$ which has 10 nodes, and such cubics are all PGL_5 -equivalent. (See [21]) It is also known that all automorphisms of Segre cubic can be realized as PGL_5 -action on $\mathbb{P}^4$ and they form automorphism group isomorphic to S_6 . For a Segre cubic, we will construct six geometric points which are invariant under its automorphism group, and show that these points determine a sextic nondegenerate algebra. Conversely, one can construct a Segre cubic from a nondegenerate ring, providing its inverse correspondence. This shows that a sextic nondegenerate $\mathbb{Q}$ -algebra corresponds to a $\mathrm{PGL}_5(\mathbb{Q})$ -equivalence class of Segre cubics with rational coefficients, though the explicit construction of the $\mathbb{Q}$ -algebra corresponding to Segre cubic to $\mathbb{Q}$ -algebra is not yet known.

Chapter 6 provides geometric interpretations for the Bhargava's parametrization model for quintic rings in [3], and attempts to reproduce the construction using the hypercohomology construction. The form (V, G) used in the quintic ring parametrization is given as $V(\mathbb{Z}) = \mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5$ and $G(\mathbb{Z}) = \mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_4(\mathbb{Z})$, and the 5 geometric points in $\mathbb{P}^3$ corresponding to the quintic ring R was already described. But for the sextic resolvent ring S - constructed together with the quintic ring as part of the complete correspondence $V(\mathbb{Z})/G(\mathbb{Z}) \leftrightarrow (R, S)$ (see [3] for the exact definition of the sextic resolvent ring), the 6 geometric points corresponding to S were not discussed so far. We will construct special 6 geometric points whose global function ring over $\mathbb{Q}$ gives the sextic resolvent algebra $S \otimes \mathbb{Q}$.

Chapter 7 discusses several notions regarding $\mathbb{Z}$ -schemes associated to parametrizations of rank n rings. While we considered n geometric points associated to rank n rings in ring parametrization model, the inverse correspondence - constructing a rank n ring out of n geometric points - can be rather confusing. We will look at several scheme-theoretic notions involving n Galois conjugate points, such as scheme-theoretic closure of n geometric points and maps $\text{Spec} R \rightarrow \mathbb{P}^m$, and give examples which illustrates their inconsistency with existing ring parametrization models.

Chapter 2

Hypercohomology ring for resolution of n points

2.1 Introduction

In this chapter, we describe a procedure of constructing a rank n ring using hypercohomology of a complex of sheaves on some projective space, which was first introduced by Wood in [47], [49] etc. and was applied to the parametrization models of rank 3 and 4 rings. We will provide a generalized account of the hypercohomology construction procedure so that it can be used to construct rank n rings in more various situations. In particular, results of this chapter will be used extensively in sextic ring parametrization models appearing in the later chapters of this thesis (Chapter 3, 4, 6).

In the parametrizations of rings of rank $n = 3, 4, 5$ given in [19], [2] and [3] respectively, we mentioned that those models for rank n rings can be related to n geometric points in $\mathbb{P}^{n-2}$ for each $n = 3, 4, 5$. For an algebraic form which corresponds to a rank n ring in those parametrization models, one can construct a certain system of equations which determines n points in $\mathbb{P}^{n-2}$ for generic case.

One way to associate those n points to a rank n ring is to consider the ring of global functions of the subscheme cut by those equations. For example, for a binary cubic form $f = ax^3 + bx^2y + cxy^2 + dy^3$ whose discriminant is nonzero, we consider a subscheme $X = V(f) \subset \mathbb{P}_{\mathbb{Z}}^1$ formed by an equation $f(x, y) = 0$. One can explicitly compute its global function ring $\Gamma(X)$ as cubic ring of rank 3 (in the Section 3.2.2) and this construction $f \mapsto \Gamma(V(f))$ from a cubic form to a cubic ring coincides with the cubic ring parametrization of [19]. The construction of rank n rings associated to binary n -ic forms in [39], etc. can be also explained using the ring of global functions.

Wood in series of paper ([49], [47], [50] etc.) established the method of using hypercohomology of a certain complex instead of the global function ring. For a scheme $X \subset \mathbb{P}^k$ whose ideal sheaf is $\mathcal{I}$, its global function ring $\Gamma(X)$ can be interpreted as $H^0(\mathcal{O}/\mathcal{I})$ (where $\mathcal{O}$ is the structure sheaf for $\mathbb{P}^k$). If one can construct a minimal free resolution of $\mathcal{O}/\mathcal{I}$ as a complex $\mathcal{C}$ of sheaves on $\mathbb{P}^k$, then $H^0(\mathcal{O}/\mathcal{I})$ coincides with the zeroth hypercohomology $H^0R\Gamma(\mathcal{C})$ of $\mathcal{C}$. The main idea of the hypercohomology construction is to consider hypercohomology of a well-chosen complex $\mathcal{C}$ which becomes minimal free resolution of $\mathcal{O}/\mathcal{I}$ for generic case.

One of the advantages of considering hypercohomology $H^0R\Gamma(\mathcal{C})$ instead of the global function ring is that it behaves more consistently. In the context of ring parametrization, our algebraic form A is given as an element in the representation of algebraic groups (G, V) , and the equations are given as polynomials in entries of A (more exactly, polynomials in linear functionals of V). If maps of our complex $\mathcal{C}$ can be described in polynomials in A , then its hypercohomology ring has multiplication coefficients which can be described in polynomials in A . One can show that $H^0R\Gamma(\mathcal{C})$ can be given a rank n ring structure for any values of A , including the cases where the subscheme becomes degenerate and the global function ring is no longer a rank n ring. For the example of a binary cubic form $f = ax^3 + bx^2y + cxy^2 + dy^3$ corresponding to a cubic ring, the hypercohomology construction can give cubic rings for all f , even

including the case $f = 0$ where considering the global function ring $\Gamma(V(f))$ does not work.

This consistency of hypercohomology construction even allows us to extend it over general base scheme S and $\mathcal{O}_S$ -modules. In this setting, we consider a locally free $\mathcal{O}_S$ -module V and relative projective space $\mathbb{P}(V)$ over S instead of $\mathbb{P}_{\mathbb{Z}}^k$.

Our algebraic form is usually considered as a global section of $F(V)$ for some representation-theoretic functor F - in the example of binary cubic forms we have $F = \text{Sym}^3 \otimes \wedge^2$ for locally free $\mathcal{O}_S$ -module V of rank 2. We consider a complex of sheaves $\mathcal{C}$ on $\mathbb{P}(V)$ whose maps can be described as polynomials in this algebraic form, and the zeroth hypercohomology $H^0 R\pi_*(\mathcal{C})$ of structure morphism $\pi : \mathbb{P}(V) \rightarrow S$ can be given a $\mathcal{O}_S$ -algebra structure. In this way, Wood could generalize Bhargava's cubic and quartic ring constructions in the context of $\mathcal{O}_S$ -algebras over general scheme S .

This chapter seeks to provide an account for this hypercohomology construction appearing in [49] and [47] etc. in more unified way, with providing some additional general results. Results in this chapter can be applied to more various complexes which resembles resolutions of n points, and those will be used in constructing rank 6 rings in many sextic ring parametrization models coming in the later parts of this thesis.

In Section 2.2 we will first discuss motivating examples of ring parametrizations of rank 3 and 4, and introduce how those ring parametrizations can be interpreted and generalized to arbitrary base in the context of hypercohomology construction.

In Section 2.3 we discuss a certain class of complexes of sheaves on $\mathbb{P}^k$ which has shapes of resolution of n points, and show that hypercohomology of such complexes become isomorphic to $\mathbb{Z}^n$ as $\mathbb{Z}$ -module. (Theorem 2.3.3) We will suggest an explicit expression of hypercohomology elements using the Čech cohomology for standard cover for $\mathbb{P}^k$.

In Section 2.4 we discuss a natural ring structure on hypercohomology induced

from differential graded algebra (DG-algebra) structure on the complex. We will suggest an explicit method of calculating multiplicative structure on the hypercohomology module, which works consistently with the Čech cohomology expression of hypercohomology elements. This method will be applied to the two examples of Section 2.2 (the cubic and quartic ring parametrizations), to illustrate how one can calculate the hypercohomology ring in practice.

In Section 2.5 we compare the hypercohomology ring $H^0 R\Gamma(\mathcal{C})$ with the global function ring of scheme $V(\mathcal{I})$ when $\mathcal{C}$ ‘almost’ becomes minimal free resolution of $\mathcal{O}/\mathcal{I}$, in particular when $\mathcal{I}$ cuts out n points for generic case. It follows that the hypercohomology ring is isomorphic to global functions provided $\mathcal{C}$ is exact. For certain complexes giving free resolution of n points in $\mathbb{P}^{n-2}$ generically, we show the nondegeneracy of hypercohomology ring can provide a limited inverse to this statement (Theorem 5.3). Results of this section will be applied to author’s later works on ring parametrizations using $2 \times 2 \times 2 \times 5$ forms and $3 \times 3 \times 5$ forms.

We will consider every equations and schemes over $\mathbb{Z}$ or $\mathbb{Q}$, except that in Section 2.6 we discuss the generalization to hypercohomology of complexes of sheaves on projective spaces over arbitrary base scheme S . We will also discuss how can one generalize some of our results to hypercohomology over spaces other than $\mathbb{P}^k$ (notably $\mathbb{P}^k \times \mathbb{P}^l$).

2.2 Motivating Examples

Before we discuss the general case, we review the case of the cubic and quartic ring parametrizations, each first discussed in [19] and [2] respectively and both formulated via the hypercohomology construction in [49] later.

2.2.1 Binary cubic forms and cubic rings

For the rest of this section, let $f = ax^3 + bx^2y + cxy^2 + dy^3$ be a binary cubic form with integral coefficients, $a, b, c, d \in \mathbb{Z}$. The so-called Levi–Delone–Faddeev correspondence (appeared in [19] and formalized in [29]) constructs a cubic ring $R(f)$ defined as a free $\mathbb{Z}$ -module with basis $1, \omega, \theta$ whose multiplications are given by the following explicit formulas:

$$\omega\theta = -ad, \quad \omega^2 = -ac + b\omega - a\theta, \quad \theta^2 = -bd - d\omega + c\theta$$

One can check that an action of $g \in \mathrm{GL}_2(\mathbb{Z})$ on f transforms the basis ω, θ by $\det(g)$ times the standard action $(\omega', \theta')^t = g(\omega, \theta)^t$. Also, one can construct an inverse mapping of $f \mapsto R(f)$: for a cubic ring R , a binary cubic form $R/\mathbb{Z} \simeq \mathbb{Z}^2$ to $\wedge^3 R \simeq \mathbb{Z}$ defined as $x \mapsto 1 \wedge x \wedge x^2$ becomes one inverse. It follows that there is a one-to-one correspondence between cubic rings and $\mathrm{GL}_2(\mathbb{Z})$ -equivalence classes of binary cubic forms $\mathrm{Sym}^3(\mathbb{Z}^2)$.

One geometric interpretation of the ring $R(f)$ can be given as follows.

Proposition 2.2.1. *Let f be a nonzero cubic form. The cubic ring $R(f)$ constructed by Levi–Delone–Faddeev correspondence is isomorphic to the ring of global functions of $V(f) \subset \mathbb{P}_{\mathbb{Z}}^1$.*

A global function for a scheme X with an affine cover $\{U_i\}$ can be represented as a set of functions on each U_i which coincides on intersections $U_i \cap U_j$. For $\mathbb{P}_{\mathbb{Z}}^1 = \mathrm{Proj} \mathbb{Z}[x, y]$, we consider two distinguished open sets $D(x) \simeq \mathrm{Spec} \mathbb{Z}[y/x]$ and $D(y) \simeq \mathrm{Spec} \mathbb{Z}[x/y]$ providing an affine cover. These also provide affine cover for $V(f) \subset \mathbb{P}_{\mathbb{Z}}^1$ such that

$$V(f) \cap D(x) \simeq \mathrm{Spec} \mathbb{Z}[y/x]/f(y/x), \quad V(f) \cap D(y) \simeq \mathrm{Spec} \mathbb{Z}[x/y]/f(x/y).$$

Here $f(y/x) = f(x, y)/x^3$ and $f(x/y) = f(x, y)/y^3$. In this perspective, a global

functions of $V(f)$ corresponds to polynomials $h_1(y/x) \in \mathbb{Z}[y/x]/f$ and $h_2(x/y) \in \mathbb{Z}[x/y]/f$ such that $h_1 - h_2 \equiv 0$ on $\mathbb{Z}[x/y, y/x]/f$. To find such (h_1, h_2) , consider the following expression.

$$\begin{aligned}\frac{f}{x^2y} &= a\frac{x}{y} - \left(-b - c\frac{y}{x} - d\left(\frac{y}{x}\right)^2\right) \\ \frac{f}{xy^2} &= \left(a\left(\frac{x}{y}\right)^2 + b\frac{x}{y} + c\right) - \left(-d\frac{y}{x}\right)\end{aligned}$$

In both lines, the right hand sides consist of differences between a polynomial in x/y and a polynomial in y/x , each corresponding to functions on $D(y)$ and $D(x)$. And the left hand sides suggest that those differences are equivalent to 0 modulo f . Thus interpreting the right hand sides as $h_1(x/y) - h_2(y/x)$ gives desired pairs (h_1, h_2) representing global functions on $V(f)$.

One motivation of thinking $f/(x^2y)$ and $f/(xy^2)$ can be given as follows. As $h_1(x/y) - h_2(y/x)$ should be a multiple of f in $\mathbb{Z}[x/y, y/x]$, it should be of form $f \cdot g(x/y)$ for some Laurent polynomial g . If we split g by monomials, one can express $h_1 - h_2$ as a $\mathbb{Z}$ -linear combination of expressions of form $f \cdot x^k y^{-k}$ ($k \in \mathbb{Z}$). All such expressions $f \cdot x^k y^{-k-3}$ ($k \in \mathbb{Z}$) split into polynomial in x/y and polynomial in y/x . For all k except $k = -1, -2$ one of the split polynomial is zero and the split is trivial modulo f , and choices of $k = -1, -2$ give above nontrivial functions. One can check that those two functions and 1 (1 on both $D(x)$ and $D(y)$) form a $\mathbb{Z}$ -basis for global functions, and satisfy the same multiplication rules as those of $R(f)$ under the identification

$$\omega = a\frac{x}{y} = -b - c\frac{y}{x} - d\left(\frac{y}{x}\right)^2, \quad \theta = a\left(\frac{x}{y}\right)^2 + b\frac{x}{y} + c = -d\frac{y}{x}.$$

In the hypercohomology perspective, the above argument translates as follows. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$ be the structure sheaf of $\mathbb{P}_{\mathbb{Z}}^1$. For $f \neq 0$, the ideal sheaf $\mathcal{I}$ for $V(f)$

consists of multiples of f and satisfies following exact sequence.

$$0 \rightarrow \mathcal{O}(-3) \xrightarrow{f} \mathcal{O} \rightarrow \mathcal{O}/\mathcal{I} \rightarrow 0$$

The global function ring $\Gamma(V(f))$ is isomorphic to $H^0(\mathbb{P}_{\mathbb{Z}}^1, \mathcal{O}/\mathcal{I})$.

The idea of hypercohomology construction is to consider the resolution $\mathcal{O}(-3) \rightarrow \mathcal{O}$ instead of $\mathcal{O}/\mathcal{I}$ and consider hypercohomology of the complex. If $\mathcal{O}(-3) \rightarrow \mathcal{O}$ is exact and therefore becomes a resolution of $\mathcal{O}/\mathcal{I}$, then one can show the following isomorphism of $\mathbb{Z}$ -modules

$$H^0(\mathcal{O}/\mathcal{I}) \simeq H^0 R\Gamma(\mathcal{O}(-3) \xrightarrow{f} \mathcal{O}).$$

One can also induce an algebra structure on right hand side deduced from DG-algebra structure on the resolution, which turns out to be isomorphic to the natural algebra structure on the global function ring. It turns out that this ring constructed from hypercohomology behaves more nicely than the global function ring, in the sense that it yields a well-defined cubic algebra even when $\mathcal{O}(-3) \rightarrow \mathcal{O}$ is not exact, including singular cases like $f = 0$.

This perspective was suggested by Deligne in the letter [18], in the broader context of parametrizing cubic $\mathcal{O}_S$ -algebra over general scheme S . In this context, every $\mathbb{Z}$ -modules and sheaves on $\mathbb{P}^1$ are replaced by $\mathcal{O}_S$ -modules and $\mathbb{P}(V)$ for V locally free rank 2 $\mathcal{O}_S$ -module. For a twisted cubic form f on V (considered as element of $\text{Sym}^3(V) \otimes \wedge^2(V^*)$), one can consider the similar sequence $\mathcal{O}(-3) \rightarrow \mathcal{O}$ and consider its hypercohomology of the pushforward $H^0 R\pi_*$ of the structure morphism $\pi : \mathbb{P}(V) \rightarrow S$. This provides the correspondence between general cubic algebras over any base scheme S and cubic forms over S .

Finally, we note that our construction of global functions using $f/(x^2y)$ and

$f/(x^2y)$ can be explained through the following diagram:

$$\begin{array}{ccc} & \Gamma(\mathcal{O}_x \oplus \mathcal{O}_y) & \\ & \downarrow & \\ \Gamma(\mathcal{O}_{xy}(-3)) & \longrightarrow & \Gamma(\mathcal{O}_{xy}) \end{array}$$

Here the vertical complex comes from the Čech complex for cover $\mathbb{P}_{\mathbb{Z}}^1 = D(x) \cup D(y)$ (note that $\mathcal{O}_x = \mathcal{O}|_{D(x)}$)

$$0 \rightarrow \mathcal{O}_x \oplus \mathcal{O}_y \rightarrow \mathcal{O}_{xy} \rightarrow 0$$

and the horizontal complex is our resolution $\mathcal{O}(-3) \rightarrow \mathcal{O}$. The elements $1/(x^2y), 1/(xy^2)$ are elements of $\Gamma(\mathcal{O}_{xy})(-3)$ which represent elements of $H^1(\mathcal{O}(-3))$. (If we calculate $H^1(\mathcal{O}(-k))$ using the Čech cohomology with cover $D(x) \cup D(y)$, $H^1(\mathcal{O}(-k))$ can be identified with set of total degree $-k$ elements in $\mathbb{Z}[x, y, x^{-1}, y^{-1}]$ modulo $(\mathbb{Z}[x, y, x^{-1}] + \mathbb{Z}[x, y, y^{-1}])$, thus it has $\mathbb{Z}$ basis $x^{-1}y^{-k+1}, \dots, x^{-k+1}y^{-1}$.) The process of splitting $f/(x^2y)$ into polynomials in x/y and y/x corresponds to ‘lifting along diagonal’ - applying rightward map and taking any inverse image of the downward map. This can be visualized as follows:

$$\begin{array}{ccc} & a\frac{x}{y} - \left(-b - c\frac{y}{x} - d\left(\frac{y}{x}\right)^2\right) & \\ & \downarrow & \\ \frac{1}{x^2y} & \longrightarrow & \frac{f}{x^2y} \end{array}$$

We will later establish that this lifting process actually gives hypercohomology module for various complexes resembling resolutions of n points.

2.2.2 Pairs of ternary quadrics and quartic rings

The parametrization of quartic (rank 4) rings using pairs of ternary quadratic forms was established by Bhargava in [2], and was later generalized by Wood in [49] to arbitrary base scheme using the hypercohomology construction.

This quartic ring parametrization corresponds a pair of ternary quadric forms $f = \sum_{1 \leq i \leq j \leq 3} a_{ij} x_i x_j$ and $g = \sum_{1 \leq i \leq j \leq 3} b_{ij} x_i x_j$ ($a_{ij}, b_{ij} \in \mathbb{Z}$) to a quartic ring, whose multiplication structure is described in the following formulas. Let $1 = \alpha_0, \alpha_1, \alpha_2, \alpha_3$ be the $\mathbb{Z}$ -basis of the corresponding quartic ring. For variables $\lambda_{ij}^{kl} = a_{ij} b_{kl} - a_{kl} b_{ij}$, multiplication coefficients of the with respect to this basis are determined by the following formulas

$$\begin{aligned} c_{ij}^k &= \text{sgn}(ijk) \cdot \lambda_{ii}^{jj}, \\ c_{ii}^j &= \text{sgn}(ijk) \cdot \lambda_{ik}^{ii}, \\ c_{ij}^i - c_{kj}^k &= \text{sgn}(ijk) \cdot \lambda_{jj}^{ik}, \\ c_{ii}^i - c_{ij}^j - c_{ik}^k &= \text{sgn}(ijk) \cdot \lambda_{ij}^{ik} \end{aligned}$$

where (i, j, k) is a permutation of $(1, 2, 3)$ and $\text{sgn}(ijk)$ denotes the signature of the permutation (i, j, k) . To put it more exactly, all integers c_{ij}^k satisfying above are equivalent up to the effect of change of shifting α_i by integers $\alpha_i \mapsto \alpha_i + C_i$, thus giving the same ring with basis of $R/\mathbb{Z}$ specified. (In [2], those C_i s are made explicit through certain ‘standardization’ making some c_{ij}^i zero.) The remaining coefficients c_{ij}^0 are determined uniquely for such values of c_{ij}^k given. In this correspondence, the coefficients a_{ij} and b_{ij} of quadrics f and g can be interpreted as coordinates of certain quadratic ‘resolvent map’ from our quartic ring to some cubic ring. This resolvent map can be used to construct the inverse correspondence - recovering the pairs (f, g) which gives a specific quartic ring.

In geometric perspective, the above quartic ring can be linked to the ring of global functions of $V(f, g) \subset \mathbb{P}_{\mathbb{Z}}^2$. If f and g do not share common components, then their intersection over $\overline{\mathbb{Q}}$ becomes distinct 4 geometric points, and we can express the resolution for those points as the following Koszul complex (here $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2}$ is structure

sheaf)

$$\mathcal{K}_{f,g} : \mathcal{O}(-4) \xrightarrow{(-g,f)^t} \mathcal{O}(-2)^2 \xrightarrow{(f,g)} \mathcal{O}.$$

It can be shown that the hypercohomology of $\mathcal{K}_{f,g}$ can be given a ring structure whose multiplication is induced from DG-algebra structure of $\mathcal{K}_{f,g}$, and this ring is isomorphic to the quartic ring constructed above. This hypercohomology ring can be defined for any integral forms f, g (even including $f = g = 0$ case) although it will no longer be isomorphic to the global function ring $\Gamma(V(f, g))$. It is also possible to generalize the whole process over arbitrary base scheme S to obtain complete parametrization for quartic $\mathcal{O}_S$ -algebra as in [49].

The explicit elements of the hypercohomology ring of $\mathcal{K}_{f,g}$ corresponding to global functions of $V(f, g)$ can be described using the following diagram.

$$\begin{array}{ccc} & & \Gamma(\mathcal{O}_{x_1} \oplus \mathcal{O}_{x_2} \oplus \mathcal{O}_{x_3}) \\ & & \downarrow \\ \Gamma((\mathcal{O}_{x_1x_2} \oplus \mathcal{O}_{x_1x_3} \oplus \mathcal{O}_{x_2x_3})(-2)) & \longrightarrow & \Gamma(\mathcal{O}_{x_1x_2} \oplus \mathcal{O}_{x_1x_3} \oplus \mathcal{O}_{x_2x_3}) \\ & & \downarrow \\ \Gamma(\mathcal{O}_{x_1x_2x_3}(-4)) & \longrightarrow & \Gamma(\mathcal{O}_{x_1x_2x_3}(-2)) \end{array}$$

Similar to the diagram appearing in the cubic ring parametrization case, the vertical maps all come from the Čech complex for cover $\{D(x_1), D(x_2), D(x_3)\}$ of $\mathbb{P}_{\mathbb{Z}}^2$ and horizontal maps come from $\mathcal{K}_{f,g}$. We start from $1/(x_1x_2x_3x_i) \in \Gamma(\mathcal{O}_{x_1x_2x_3}(-4))$ and repeat the lifting process of applying rightward map and finding its preimage under the downward map. The final element in $\Gamma(\mathcal{O}_{x_1} \oplus \mathcal{O}_{x_2} \oplus \mathcal{O}_{x_3})$ has form of

$$\left(h_1\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right), h_2\left(\frac{x_1}{x_2}, \frac{x_3}{x_2}\right), h_3\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right) \right) \in \Gamma(\mathcal{O}_{x_1}) \oplus \Gamma(\mathcal{O}_{x_2}) \oplus \Gamma(\mathcal{O}_{x_3})$$

such that differences $h_i - h_j$ are spanned by f and g : the image of (h_1, h_2, h_3) under the downward map is $(h_1 - h_2, h_1 - h_3, h_2 - h_3)$ and it is in the image of the rightward map induced from $\mathcal{O}(-2)^2 \rightarrow \mathcal{O}$ of $\mathcal{K}_{f,g}$. Thus (h_1, h_2, h_3) can be considered as a global

function of $V(f, g)$, where each h_i describes the function restricted to $D(x_i) \cap V(f, g)$. We will show this process actually gives hypercohomology of $\mathcal{K}_{f,g}$, and revisit this example again.

2.3 Module structure of hypercohomology

In this section, we consider a class of complexes of sheaves on $\mathbb{P}^k$ which resemble free resolutions of n points, and compute hypercohomology modules of such complexes. This class is described in the statement of Theorem 2.3.3, and it includes many complexes appearing in various ring parametrization models including the cubic and quartic ring parametrizations of Section 2.2. Proving that the hypercohomology of a complex in this class has $\mathbb{Z}$ -module structure isomorphic to $\mathbb{Z}^n$ only requires basic properties of hypercohomology (including hypercohomology spectral sequence), which we will first briefly review.

We also give an explicit description of hypercohomology elements using a list of Laurent polynomials in projective coordinates $x_1, \dots, x_{k+1}$ of $\mathbb{P}^k$. We note that elements of Čech cohomology of $\mathcal{O}(t)$ ($t \in \mathbb{Z}$) using a cover $D(x_1), \dots, D(x_{k+1})$ of $\mathbb{P}^k$ can be expressed in Laurent polynomials of $x_1, \dots, x_{k+1}$. By considering the *Čech complex* - a complex we will define soon - which can be used to calculate Čech cohomology in $\mathbb{P}^k$, we can adapt the method of calculating Čech cohomology to calculate hypercohomology generally. For a complex $M_\bullet$ whose entries consist of direct sums of $\mathcal{O}(t)$, we will show its hypercohomology $H^0 R\Gamma(M_\bullet)$ is isomorphic to the total complex of the tensor product of the Čech complex and $M_\bullet$, and describe its elements in Laurent polynomials.

2.3.1 Preliminaries: basic properties of hypercohomology

Descriptions for homological algebra and hypercohomology given here will mainly follow that of [45]. We will use $H^i R F$ as notation for the i th hyper-derived functor of F , adopting the notation for the total derived functor in derived category. (This is to make our notation consistent with the Wood's works.)

Hypercohomology is a natural extension of sheaf cohomology to complexes of sheaves. It is defined as *right hyper-derived functor* - a natural extension of derived functor to the complex of objects - of the global section functor $\Gamma(X, -)$. For general base scheme S with structure map $\pi : X \rightarrow S$, hypercohomology is given as right hyper-derived functor $H^i R \pi_*$ of the direct image sheaf functor π_* .

Similar to the construction of the right derived functor $R^i F$ of a left-exact functor F - finding injective resolution I and taking the i th cohomology of $F(I)$, the right hyper-derived functor is constructed from a *Cartan-Eilenberg resolution* of cochain complex. A (right) Cartan-Eilenberg resolution of a complex $A_\bullet$ is a double complex of $I_{\bullet\bullet}$ consisting of injective objects with an augmentation $A_\bullet \rightarrow I_{\bullet 0}$ such that vertical maps on coboundaries and cohomologies all become injective resolutions. It is known that a Cartan-Eilenberg resolution always exists in categories having enough injectives (including categories of R -modules and $\mathcal{O}_S$ -sheaves). Then hyper-derived functor $H^i R F(A_\bullet)$ of a left-exact functor F can be defined as cohomology of total complex $H^i \text{Tot}(F(I_{\bullet\bullet}))$, and it does not depend on the choice of Cartan-Eilenberg resolution $I_{\bullet\bullet}$.

We list a few properties of hypercohomology, which all follow from general properties of hyper-derived functors. See ([45], Section 5.7) for more information.

Proposition 2.3.1. *1. If $0 \rightarrow A_\bullet \rightarrow B_\bullet \rightarrow C_\bullet \rightarrow 0$ is exact, then we have the following long-exact sequence*

$$\cdots \rightarrow H^i R \Gamma(A_\bullet) \rightarrow H^i R \Gamma(B_\bullet) \rightarrow H^i R \Gamma(C_\bullet) \rightarrow H^{i+1} R \Gamma(A_\bullet) \rightarrow \cdots$$

2. If $A_\bullet$ consists of single object A at degree zero, then $H^i RF(A_\bullet) = H^i(X, A)$.
3. If $A_\bullet$ is exact, then $H^i RF(A_\bullet) = 0$.
4. (Dimension shifting) $H^i R\Gamma(A_{\bullet+j}) = H^{i+j} R\Gamma(A_\bullet)$.
5. If $f : A_\bullet \rightarrow B_\bullet$ is a quasi-isomorphism (becomes isomorphism on cohomology), then it induces isomorphisms $H^i R\Gamma(A_\bullet) \rightarrow H^i R\Gamma(B_\bullet)$ for all i .

Proposition 2.3.2 (Hypercohomology spectral sequence). *There is a convergent hypercohomology spectral sequence converging to $H^i \Gamma(M_\bullet)$ with following E^2 term*

$$E_2^{ij} = H^i(X, H^j(M_\bullet)).$$

If $M_\bullet$ is bounded below, the following hypercohomology spectral sequence also converges.

$$E_1^{ij} = H^i(X, M_j), \quad E_2^{ij} = H^i(H^j(X, M_\bullet))$$

Here $H^j(X, M_\bullet)$ denotes the complex whose k th term is $H^j(X, M_k)$.

Both spectral sequences are obtained from double complex $\Gamma(I_{\bullet\bullet})$ for a Cartan-Eilenberg resolution $I_{\bullet\bullet}$ of A .

The theory of derived categories allow us to consider hypercohomology as derived functor on *derived categories*. Roughly, derived category is defined as a certain quotient of category of chain complex (rigorously saying, localizations by quasi-isomorphisms) such that quasi-isomorphism becomes equivalence between its objects. Instead of giving the full explicit construction of the derived category, we will only note the following universal property of derived category: any functor from the categories of chain complexes which maps quasi-isomorphisms to isomorphisms can be uniquely factored through the derived category. In particular, hyper-derived functors can be defined over derived category. The notion of *total derived functor* allows us to

extend any additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$ to its total derived functor $D(\mathcal{A}) \rightarrow D(\mathcal{B})$, and hyper-derived functor turns out to be cohomology of this total derived functor. ([45], 10.5.7) We will only need this notion of derived category when we construct algebra structure on hypercohomology.

2.3.2 Hypercohomology for complexes with well-posed Tschirnhäuser modules

We give our first main theorem of this chapter, which states that hypercohomology of a complex satisfying certain conditions is isomorphic to $\mathbb{Z}^n$ as $\mathbb{Z}$ -module.

Theorem 2.3.3. *Let $M_\bullet$ be a complex of sheaves on $\mathbb{P}^k$ having form of*

$$M_\bullet : 0 \rightarrow M_{-k} \rightarrow M_{-(k-1)} \rightarrow \cdots \rightarrow M_0 \rightarrow 0$$

satisfying following conditions.

- *We have $M_0 = \mathcal{O}$, where $\mathcal{O}$ is structure sheaf of $\mathbb{P}^k$.*
- *Each M_i is direct sum of $\mathcal{O}(-t)$ ($t \in \mathbb{Z}_{>0}$),*
- *The cohomology of the sequence*

$$0 \rightarrow H^k(M_{-k}) \rightarrow H^k(M_{-k+1}) \rightarrow \cdots \rightarrow H^k(M_0) \rightarrow 0$$

vanishes at all places except the first leftmost place at $H^k(M_{-k})$. Cohomology module T at the first leftmost place is isomorphic to $\mathbb{Z}^{n-1}$.

Then we have an exact sequence of $\mathbb{Z}$ -modules

$$0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(M_\bullet) \rightarrow T \rightarrow 0.$$

Also $H^l R\Gamma(M_\bullet) = 0$ for $l > 0$.

Proof. First note that $R^j\Gamma(\mathcal{O}(-t)) = H^j(\mathcal{O}(-t))$ vanishes except for the two cases of $j = t = 0$ and $t \geq r + 2, j = k$. Thus the cohomology $R^j\Gamma(M_i)$ also vanishes except when $j = 0$ or $j = k$.

As length of $M_\bullet$ is bounded and all $R^j\Gamma(M_i)$ vanishes over degree $j > r$, we can use the convergence of hypercohomology spectral sequence with E_1^{ji} term given by

$$R^j\Gamma(M_i) = H^j(M_i) \Rightarrow H^{i+j}R\Gamma(M_\bullet).$$

Terms $R^j\Gamma(M_i) = H^j(M_i)$ of the E_1 page all vanish except $j = i = 0$ or $j = k$, so this page looks like follows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & H^0(\mathcal{O}) \\ 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\ \vdots & & \vdots & & \ddots & & \vdots \end{array}$$

$$H^k(M_{-k}) \longrightarrow H^k(M_{-k+1}) \longrightarrow \cdots \longrightarrow H^k(M_0)$$

From the assumption, cohomologies for all maps of the E_1 page becomes all zero except at the $H^0(\mathcal{O})$ place and the $H^k(M_{-k})$ place, each isomorphic to $\mathbb{Z}$ and T respectively. So all terms in the E_2 page becomes zero except at two places $E_2^{0,0} = \mathbb{Z}$ and $E_2^{-k,k} = T$, thus the spectral sequence is stablized at $E_2 = E_\infty$.

Now we want to figure out how does the filtration of this spectral sequence turn out to be, which should give information on $H^l R\Gamma(M_\bullet) \simeq H^l(E_\infty)$. For $l \neq 0$, all terms at E_∞^{ji} for $i + j = l$ vanishes, so $H^l(E_\infty) = 0$ for this case. For $l = 0$, then the

filtration of the spectral sequence becomes

$$E_{\infty}^{0,0} \xrightarrow{E_{\infty}^{-k,k}} H^0(E_{\infty})$$

and it turns out to be exactly

$$\mathbb{Z} \xrightarrow{T} H^0 R\pi_*(M_{\bullet})$$

which is equivalent to the following exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(M_{\bullet}) \rightarrow T \rightarrow 0.$$

□

This theorem can be applied to the two examples in the previous section - the parametrizations of cubic and quartic rings. For the cubic ring parametrization using a binary cubic form f , our complex of sheaves on $\mathbb{P}^1$ is given as

$$\mathcal{O}(-3) \xrightarrow{f} \mathcal{O} \quad (\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}, k = 1).$$

As $H^1(\mathcal{O}(-3)) \simeq \mathbb{Z}^2$ and $H^1(\mathcal{O}) \simeq 0$ as $\mathbb{Z}$ -module, this complex satisfies all assumptions of the Theorem 2.3.3. Applying the theorem gives the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(M_{\bullet}) \rightarrow \mathbb{Z}^2 \rightarrow 0$$

showing that hypercohomology is isomorphic to $\mathbb{Z}^3$ as $\mathbb{Z}$ -module. Similarly, for the quartic ring parametrization using a pair of ternary quadratic forms, the Koszul complex on $\mathbb{P}^2$ made by two quadrics f, g

$$\mathcal{K}_{f,g} : \mathcal{O}(-4) \xrightarrow{(-g,f)^t} \mathcal{O}(-2)^2 \xrightarrow{(f,g)} \mathcal{O}$$

also satisfies those assumptions, as $H^2(\mathcal{O}(-4)) \simeq \mathbb{Z}^3$ and $H^2(\mathcal{O}(-2)) = H^2(\mathcal{O}) = 0$. In Wood's work [49], its proofs of the properties that hypercohomology of those complexes are isomorphic to $\mathbb{Z}^n$ ($n = 3, 4$ for the cubic and quartic ring cases respectively) uses long exact sequences repeatedly. Our proof of the Theorem 2.3.3 using hypercohomology spectral sequence can be considered as a generalization of those proofs.

The exact sequence of the form

$$0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(M_\bullet) \rightarrow T \rightarrow 0, \quad T = \ker(H^k(M_{-k}) \rightarrow H^k(M_{-k+1}))$$

can be linked to the notion of Tschirnhausen modules in algebraic geometry. For a cover $\rho : X \rightarrow Y$ of schemes of degree n , its Tschirnhausen module is defined as any locally free $\mathcal{O}_Y$ -module isomorphic to cokernel of $\mathcal{O}_Y \rightarrow \rho_* \mathcal{O}_X$. ([36], [16]) For the case when $Y = \operatorname{Spec} k$ for field k (so that $\mathcal{O}_Y \simeq k$ and $\rho_* \mathcal{O}_X \simeq \Gamma(X)$), its Tsuchirnhausen module T of $X \rightarrow \operatorname{Spec} k$ satisfies an exact sequence

$$0 \rightarrow k \rightarrow \Gamma(X) \rightarrow T \rightarrow 0.$$

Based on this similarity of it and the exact sequence of the Theorem 2.3.3, we would like to name our class of complexes in the Theorem 2.3.3 as follows.

Definition 2.3.4. *Call the conditions of the Theorem 2.3.3 as Well-posed Tschirnhausen Module Conditions (WTM conditions).*

The class of complexes of sheaves on $\mathbb{P}^k$ satisfying WTM conditions includes many complexes which are given as minimal free resolutions of some n points in $\mathbb{P}^k$. For example, a minimal free resolution of n points in $\mathbb{P}^{n-2}$ in general position (no $n-1$ points on a same hyperplane) over field satisfies WTM conditions, as its shape is always of the following form

$$\mathcal{O}(-n) \rightarrow \mathcal{O}(-n+2)^{b_{n,n-3}} \rightarrow \cdots \rightarrow \mathcal{O}(-2)^{b_{n,1}} \rightarrow \mathcal{O}$$

for

$$b_{n,i} = n \binom{n-2}{i} - \binom{n}{i+1}.$$

(See [46]) Complexes appearing in the examples of the cubic and quartic ring parametrizations all fall in this category. In the sextic parametrization models appearing in the Chapter 3 and 4 of this thesis, we will consider complexes having above shape for $n = 6$ when constructing our sextic ring.

For cases $k \neq n - 2$, there are many minimal free resolutions of n points in $\mathbb{P}^k$ satisfying WTM conditions. A minimal free resolution of six points in $\mathbb{P}^2$ which are not on a conic has the following shape

$$\mathcal{O}(-4)^3 \rightarrow \mathcal{O}(-3)^4 \rightarrow \mathcal{O}$$

and one can check that it satisfies the WTM conditions: as $H^2(\mathcal{O}(-4)) \simeq \mathbb{Z}^3$ and $H^2(\mathcal{O}(-3)) \simeq \mathbb{Z}$, its Tschirnhausen module can be given as kernel of a linear map $\mathbb{Z}^9 \rightarrow \mathbb{Z}^4$. If six points are on a conic and are all distinct, those can be represented as a complete intersection of a conic and a cubic in $\mathbb{P}^2$ which has minimal free resolution of form

$$\mathcal{O}(-5) \rightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-3) \rightarrow \mathcal{O}.$$

This complex also satisfies the WTM conditions.

2.3.3 Calculating explicit cohomology elements using Čech complex

Let $x_1, x_2, \dots, x_{k+1}$ be projective coordinates of $\mathbb{P}^k$. For a complex $M_\bullet$ of sheaves on $\mathbb{P}^k$ whose entries are direct sums of $\mathcal{O}(t)$, we will describe a procedure of calculating its hypercohomology, which is analogous to the calculation of Čech cohomology for a covering $\{D(x_1), \dots, D(x_{k+1})\}$ of $\mathbb{P}^k$. For subsets $I = \{i_1, \dots, i_l\}$ of $\{1, \dots, k+1\}$,

denote $x_I = x_{i_1}x_{i_2} \cdots x_{i_l}$. Denote $\mathcal{O}_I = \mathcal{O}_{x_I}$ as restriction of $\mathcal{O}$ to the distinguished open set $D(x_I) \subset \mathbb{P}^k$.

Definition 2.3.5. We define Čech complex of sheaves as a complex having following form.

$$C_\bullet(\mathbb{P}^k) : 0 \rightarrow \bigoplus_{|I|=1} \mathcal{O}_{x_I} \rightarrow \bigoplus_{|I|=2} \mathcal{O}_{x_I} \rightarrow \cdots \bigoplus_{|I|=r} \mathcal{O}_{x_I} \rightarrow 0$$

The maps $\mathcal{O}_{x_I} \rightarrow \mathcal{O}_{x_J}$ are given in the following way. If $J = I \cup \{j\}$ and j is k th element of then it is $(-1)^{k-1}$ times the restriction map, and it is zero otherwise.

For any sheaf M which is direct sum of $\mathcal{O}(t)$, we note that

$$\Gamma(\mathbb{P}(V), M \otimes \mathcal{O}_{x_I}) \simeq \Gamma(D(x_I), M)$$

so if we tensor the Čech complex with $\otimes M$ and take global section functor Γ , it becomes

$$0 \rightarrow \bigoplus_{|I|=1} \Gamma(D(x_I), M) \rightarrow \bigoplus_{|I|=2} \Gamma(D(x_I), M) \rightarrow \cdots \bigoplus_{|I|=r} \Gamma(D(x_I), M) \rightarrow 0$$

and complex of this cohomology is exactly the Čech cohomology of M with respect to the covering $\{D(x_1), D(x_2), \dots, D(x_{k+1})\}$ of $\mathbb{P}^k$. Therefore we have:

Proposition 2.3.6. If M is direct sum of $\mathcal{O}(t)$, then cohomology of $M \otimes C_\bullet(\mathbb{P}^k)$ is isomorphic to $H^i(M)$.

We show that one can apply the similar procedure for hypercohomology too - tensoring with Čech complex, applying global section, and considering its cohomology - except that we consider total cohomology of the double complex.

Theorem 2.3.7. Suppose that $M_\bullet$ is a complex of sheaves whose summands are direct sums of $\mathcal{O}(t)$ for $t \in \mathbb{Z}$. Then its i th hypercohomology $H^i R\Gamma(M_\bullet)$ is isomorphic to the i th total cohomology of the double complex $\Gamma(M_\bullet \otimes C_\bullet(\mathbb{P}^k))$.

Proof. We consider the spectral sequence coming from the double complex $\Gamma(M_\bullet \otimes C_\bullet(\mathbb{P}^k))$ obtained from filtration by columns, with d_0 starting from downward map. By the above observation for Čech complex, its ${}^I E_1^{ij}$ terms become $H^j(M_i)$ and maps ${}^I d_1$ become complex maps of $M_\bullet$. Thus this page coincides with E_1 page of the hypercohomology spectral sequence $H^j(M_i)$, so by the mapping lemma ([45], 5.2.4) those two spectral sequences become identical after the first page. This shows that we can calculate $H^0 R\Gamma(M_\bullet)$ as total cohomology of the $\Gamma(M_\bullet \otimes C_\bullet(\mathbb{P}^k))$, regardless of whether $M_\bullet \otimes C_\bullet(\mathbb{P}^k)$ becomes a right Cartan-Eilenberg resolution for $M_\bullet$. $\square$

During the rest of this section, denote $C_\bullet(\mathbb{P}^k)$ by $C_\bullet$ for simplicity. For $M_\bullet$ satisfying the WTM conditions, we can combine Theorem 2.3.3 and Theorem 2.3.7 to obtain explicit elements in the total cohomology $H^i \text{Tot}(\Gamma(M_\bullet \otimes C_\bullet))$. We first note that for any $I \subset [k+1]$ we have

$$\Gamma(\mathcal{O}_{x_I}(t)) \simeq (\text{degree } t \text{ elements in } \mathbb{Z}[x_1, \dots, x_{k+1}, x_I^{-1}])$$

and calculation of Čech cohomology allows us to identify $H^k(\mathcal{O}(t))$ with degree t elements in $\mathbb{Z}[x_1, \dots, x_{k+1}, x_1^{-1}, \dots, x_{k+1}^{-1}]$ modulo the image of $\Gamma(C_{k-1} \otimes \mathcal{O}(t)) \rightarrow \Gamma(C_k \otimes \mathcal{O}(t))$. So we can pick unique representative of $H^k(\mathcal{O}(t))$ in the following set.

$$H^k(\mathcal{O}(t)) \simeq (\text{degree } t \text{ elements in } \frac{1}{x_1 x_2 \cdots x_{k+1}} \mathbb{Z}[x_1^{-1}, \dots, x_{k+1}^{-1}])$$

Suppose that we are given an element α of $T = \ker(H^k(M_{-k}) \rightarrow H^k(M_{-(k-1)}))$. We give the following algorithm which constructs a preimage of α under the map $H^0 R\Gamma(M_\bullet) \rightarrow T$ appearing in the following exact sequence of the Theorem 2.3.3

$$0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(M_\bullet) \rightarrow T \rightarrow 0.$$

1. Let $\alpha^{(k)}$ be the representative of $\alpha \in T$ in $\Gamma(M_{-k} \otimes C_{-k})$.

2. Repeat the following for $i = k - 1, k - 2, \dots, 0$: construct $\alpha^{(i)} \in \Gamma(M_{-i} \otimes C_{-i})$ satisfying

$$d_{\rightarrow}(\alpha^{(i+1)}) + d_{\downarrow}(\alpha^{(i)}) = 0.$$

Here $d_{\rightarrow}, d_{\downarrow}$ are the complex maps of $\Gamma(M_{\bullet} \otimes C_{\bullet})$

$$d_{\rightarrow} : \Gamma(M_i \otimes C_j) \rightarrow \Gamma(M_{i+1} \otimes C_j)$$

$$d_{\downarrow} : \Gamma(M_i \otimes C_j) \rightarrow \Gamma(M_i \otimes C_{j+1})$$

Those come from maps of $M_{\bullet}$ and $C_{\bullet}$ respectively, with additional factor of $(-1)^i$ for $d_{\downarrow}$.

3. Consider $\tilde{\alpha} = \sum_k \alpha^{(k)}$ as an element of the total complex $\text{Tot}(\Gamma(M_{\bullet} \otimes C_{\bullet}))$. Then $d(\tilde{\alpha}) = 0$.

4. The cohomology element of $H^0 \text{Tot}(\Gamma(M_{\bullet} \otimes C_{\bullet}))$ represented by $\tilde{\alpha}$ is one of preimages of $\alpha \in T$ in $H^0 R\Gamma(M_{\bullet}) \rightarrow T$.

Finding preimages of $d_{\downarrow}$ can be done using matrix operations, as maps of the Čech complex simply act linearly on monomials.

Proposition 2.3.8. *The algorithm described above produces a preimage of $d_{\downarrow}$.*

Proof. We need a more explicit description for converging entries of the spectral sequence, and will use some notations and accounts from [44] for it. For a bounded double complex with downward differentials $d_{\downarrow}$ and rightward differentials $d_{\rightarrow}$, we consider a spectral sequence ${}^I E_{\bullet}^{\bullet}$ obtained from vertical filtration. For ${}^I E_{\bullet}$, in the explicit convergence formula for total complex

$$0 \dots \xrightarrow{E_{\infty}^{p+1, q-1}} F^{p+1} H^{p+q} \xrightarrow{E_{\infty}^{p, q}} F^p H^{p+q} \xrightarrow{E_{\infty}^{p-1, q+1}} \dots H^{p+q=n},$$

terms $F^p H^{p+q}$ are given as 'closed (p, q) -strips'

$$F^p H^{p+q} = \{(x^{p-k, q+k})_{k \geq 0} \in \prod_k E_0^{p-k, q+k} : \\ d_{\downarrow} x^{p, q} = 0, d_{\rightarrow} x^{p-k, q+k} = d_{\downarrow} x^{p-k-1, q+k+1}\}.$$

We would like to consider this convergence formula for H^0 of $\Gamma(M_{\bullet} \otimes C_{\bullet})$, which looks like as follows. In Theorem 2.3.7 we already established that its spectral sequence coincides with the hypercohomology spectral sequence after E_1 step.

$$\begin{array}{ccccccc} \bigoplus \Gamma(M_{-k} \otimes \mathcal{O}_{x_i}) & \longrightarrow & \cdots & \longrightarrow & \Gamma(\bigoplus M_{-1} \otimes \mathcal{O}_{x_i}) & \longrightarrow & \bigoplus \Gamma(\mathcal{O}_{x_i}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bigoplus \Gamma(M_{-k} \otimes \mathcal{O}_{x_i x_j}) & \longrightarrow & \cdots & \longrightarrow & \bigoplus \Gamma(M_{-1} \otimes \mathcal{O}_{x_i x_j}) & \longrightarrow & \bigoplus \Gamma(\mathcal{O}_{x_i x_j}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & \longrightarrow & \ddots & \longrightarrow & \vdots & \longrightarrow & \vdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Gamma(M_{-k} \otimes \mathcal{O}_{x_1 \cdots x_{k+1}}) & \longrightarrow & \cdots & \longrightarrow & \Gamma(M_{-1} \otimes \mathcal{O}_{x_1 \cdots x_{k+1}}) & \longrightarrow & \Gamma(\mathcal{O}_{x_1 \cdots x_{k+1}}) \end{array}$$

We give their 'coordinates' for conveniences.

$$\begin{aligned} &(-k, 0), \dots, (-2, 0), (-1, 0), (0, 0), \\ &(-k, 1), \dots, (-2, 1), (-1, 1), (0, 1), \\ &\dots, \dots, \dots, \dots, \dots, \\ &(-k, k), \dots, (-2, k), (-1, k), (0, k). \end{aligned}$$

As we have $E_{\infty}^{-i, i} = 0$ except $i = 0$ and $i = k$, the convergence formula for H^0 becomes

$$0 = F^1 H^0 \xrightarrow{E_{\infty}^{0,0}} F^0 H^0 = F^{-1} H^0 = \dots = F^{-k+1} H^0 \xrightarrow{E_{\infty}^{-k,k}} F^{-k} H^0 = H^0.$$

We already noted that $E_\infty^{0,0} \simeq \mathbb{Z}$ and $E_\infty^{-k,k} \simeq T$, so this reduces to the sequence of Theorem 2.3.3

$$\mathbb{Z} \xrightarrow{T} H^0 R\Gamma(M_\bullet).$$

The explicit description of the above quotient

$$H^0 = F^{-k}H^0 / F^{-k+1}H^0 \simeq E_\infty^{-k,k} \simeq T$$

operates on element $\tilde{\alpha} \in F^{-k}H^0$ as follows. If we represent

$$\tilde{\alpha} = \sum_t \alpha^{(t)}, \quad \alpha^{(t)} \in \Gamma(M_{-t} \otimes C_t),$$

then the above map mods out $\alpha^{(k-1)} + \dots + \alpha^{(0)}$ part in $F^{-k+1}H^0$ and maps to $\alpha^{(0)} \in T$.

One can note that the above algorithm calculates a preimage of this map when $\alpha = \alpha^{(0)}$ is given, and the Theorem 2.3.3 guarantees the existence of one preimage. Kernel of this quotient map $F^0H^0 \simeq \mathbb{Z}$ is by definition the kernel of $C_0 \rightarrow C_1$, which can be identified with $\Gamma(\mathcal{O}) \simeq \mathbb{Z}$, and injects into $\Gamma(C_0) \simeq \oplus \Gamma(\mathcal{O}_{x_i})$ via $1 \mapsto (1, 1, \dots, 1)$ for $1 \in \Gamma(\mathcal{O}_{x_i})$. $\square$

This algorithm is actually identical to the procedure for calculating global functions in [49], although it was not explicitly noted that it actually produces hypercohomology for affine base case regardless of exactness of $M_\bullet$. In the next section, we will discuss application of this algorithm to examples we mentioned in Section 2.2, together with natural algebra structure given on hypercohomology.

2.4 Algebra structure of hypercohomology

If a complex of sheaves is given a differential graded algebra (DG-algebra) structure, then its hypercohomology module can be given a induced multiplicative structure. Many complexes appearing in the ring parametrizations come together with DG-algebra structure given, and they satisfy WTM conditions of the Theorem 2.3.3. This allows us to construct a multiplication on hypercohomology module isomorphic to $\mathbb{Z}^n$, and we will show that this becomes a well-defined ring of rank n .

While the induced multiplicative structure on hypercomology is given from the natural properties of the derived category and total derived functors, we show that it can be described in rather explicitly way when our complex satisfies WTM conditions. In the last section, we explicitly constructed hypercomology elements of $H^0 R\Gamma(M_\bullet)$ as total cohomology elements of $\Gamma(M_\bullet \otimes C_\bullet(\mathbb{P}^k))$, which could be expressed as lists of Laurent polynomials with coefficients in $M_\bullet$. We give a procedure of explicitly calculating multiplication between those total cohomology elements, by utilizing a suitable DG-algebra structure on the Čech complex $C_\bullet(\mathbb{P}^k)$.

2.4.1 Preliminaries: DG-algebra structure for double complex

We review the construction of total tensor product of two chain complexes. ([45], 2.7.1) For complexes $M_\bullet$ and $N_\bullet$, their tensor product double complex $M_\bullet \otimes N_\bullet$ has $M_i \otimes N_j$ as (i, j) -entry, with horizontal differential $d \otimes 1$ and vertical differential $(-1)^i$ times of $1 \otimes d$. The total complex of this double complex $\text{Tot}(M_\bullet \otimes N_\bullet)$ is called the *total tensor product*. Taking the total tensor product of chain complexes are associative, that is, we have a following natural isomorphism

$$\text{Tot}(\text{Tot}(M_\bullet \otimes N_\bullet) \otimes L_\bullet) \simeq \text{Tot}(M_\bullet \otimes \text{Tot}(N_\bullet \otimes L_\bullet)).$$

We also have a natural isomorphism regarding commutativity of the total tensor product, which involves sign changes of $(-1)^{ij}$.

$$\text{Tot}(M_\bullet \otimes N_\bullet) \simeq \text{Tot}(N_\bullet \otimes M_\bullet), \quad M_i \otimes N_j \xrightarrow{(-1)^{ij}} N_j \otimes M_i$$

Meanwhile, differential graded algebra (DG-algebra) structure on a complex is defined as collections of multiplications $M_i \otimes M_j \rightarrow M_{i+j}$ satisfying following conditions

$$d(\alpha\beta) = (d\alpha)\beta + (-1)^i \alpha(d\beta), \quad \alpha \in M_i, \beta \in M_j.$$

A DG-algebra is called associative if it is associative as algebra, and commutative if it satisfies the anticommutativity rule

$$\alpha\beta = (-1)^{ij} \beta\alpha \quad (\alpha \in M_i, \beta \in M_j).$$

One can observe that a DG-algebra structure is equivalent a the map of complexes $\text{Tot}(M_\bullet \otimes M_\bullet) \rightarrow M_\bullet$.

Suppose that $M_\bullet$ and $N_\bullet$ have DG-algebra structures, each of those equivalent to complex maps $\text{Tot}(M_\bullet \otimes M_\bullet) \rightarrow M_\bullet$ and $\text{Tot}(N_\bullet \otimes N_\bullet) \rightarrow N_\bullet$. Then we can give a natural map

$$\begin{aligned} & \text{Tot}(\text{Tot}(M_\bullet \otimes N_\bullet) \otimes \text{Tot}(M_\bullet \otimes N_\bullet)) \\ & \simeq \text{Tot}(\text{Tot}(M_\bullet \otimes \text{Tot}(N_\bullet \otimes M_\bullet)) \otimes N_\bullet) \quad (\text{associativity}) \\ & \simeq \text{Tot}(\text{Tot}(M_\bullet \otimes \text{Tot}(M_\bullet \otimes N_\bullet)) \otimes N_\bullet) \quad (\text{commutativity}) \\ & \simeq \text{Tot}(\text{Tot}(M_\bullet \otimes M_\bullet) \otimes \text{Tot}(M_\bullet \otimes M_\bullet)) \quad (\text{associativity}) \\ & \rightarrow \text{Tot}(M_\bullet \otimes N_\bullet) \end{aligned}$$

which becomes valid map of complexes. The multiplicative structure coming from this map can be written explicitly in the following way. For $m_1, m_2 \in M_\bullet$ and $n_1, n_2 \in N_\bullet$ homogeneous elements, we have

$$(m_1 \otimes n_1) \cdot (m_2 \otimes n_2) = (-1)^{\deg(m_2)\deg(n_1)}(m_1 \cdot m_2) \otimes (n_1 \cdot n_2).$$

It is even possible to use following abusive notation

$$(m_1 n_1)(m_2 n_2) = m_1 n_1 m_2 n_2 = (-1)^{\deg(m_2)\deg(n_1)} m_1 m_2 n_1 n_2$$

with the rule $nm = (-1)^{\deg(m)\deg(n)}mn$. We can call this *tensor product of DG-algebras*, and one can check it satisfies the universal property of the tensor product. Product of two associative/commutative DG-algebras is associative/commutative respectively.

Proposition 2.4.1. *Multiplication of DG-algebra restricts to give graded algebra structure on graded module $\oplus H^i(M_\bullet)$ (with $H^i(M_\bullet)$ being its i th graded component). If DG-algebra is associative/commutative, then this graded algebra is also associative/commutative respectively.*

Proof. If α, β are homogeneous elements, $d\alpha = d\beta = 0$ implies $d(\alpha\beta) = 0$. So the product of two boundary elements is boundary element. If $\alpha = d\gamma$ and $d\beta = 0$, then $\alpha \cdot \beta = d(\gamma \cdot \beta)$. So the product of cycle and boundary is cycle. This shows the multiplication can be restricted to cohomology

$$H^i(M_\bullet) \times H^j(M_\bullet) \rightarrow H^{i+j}(M_\bullet).$$

□

2.4.2 DG-algebra structure on hypercohomology

In ring parametrizations, our complex $M_\bullet$ is often assumed to have a DG-algebra structure which is associative and commutative. Complexes appearing in the examples of the Section 2.2 all fit into category of Koszul complexes, which comes with a natural DG-algebra structure. A minimal free resolution of n points in $\mathbb{P}^{n-2}$ in general position can always be given canonical DG-algebra structure (See [46]). In the later chapters of this thesis, other cases of minimal free resolutions for n points in $\mathbb{P}^k$ having DG-algebra structure will be discussed.

If $M_\bullet$ is given a DG-algebra structure, then one can naturally give induced algebra structure on hypercohomology $H^0 R\Gamma(M_\bullet)$. The construction uses the theory of derived functors, and can be found in ([49], Section 9). Before discussing this natural construction, we want to find an explicit description of multiplicative structure on $H^0 R\Gamma(M_\bullet)$ which is consistent with DG-algebra structure on $M_\bullet$. We remind the Theorem 2.3.7 that if entries of $M_\bullet$ are direct sums of $\mathcal{O}(t)$ for $t \in \mathbb{Z}$, then we have

$$H^i R\Gamma(M_\bullet) \simeq H^i(\text{Tot}(\Gamma(M_\bullet \otimes C_\bullet))).$$

We will construct an explicit multiplicative structure on the righthand side, and show that it coincides with the natural construction.

Suppose that there exists a DG-algebra structure on the Čech complex $C_\bullet = C_\bullet(\mathbb{P}^k)$ such that the following map

$$\mathcal{O} \rightarrow \bigoplus_i \mathcal{O}_i \rightarrow C_\bullet(\mathbb{P}^k), \quad 1 \mapsto (1, 1, \dots, 1)$$

is unit for the DG-algebra, that is, the two maps

$$\begin{aligned}\mathrm{Tot}(\mathcal{O} \otimes C_\bullet) &\rightarrow \mathrm{Tot}(C_\bullet \otimes C_\bullet) \rightarrow C_\bullet, \\ \mathrm{Tot}(C_\bullet \otimes \mathcal{O}) &\rightarrow \mathrm{Tot}(C_\bullet \otimes C_\bullet) \rightarrow C_\bullet.\end{aligned}$$

are isomorphisms. Then we can construct a DG-algebra structure on $\mathrm{Tot}(M_\bullet \otimes C_\bullet)$ through the tensor product of DG-algebras. As $M_i \otimes C_j$ s are direct sums of sheaves of the form $M_i \otimes \mathcal{O}_{x_I}$, which are coherent sheaves supported on affine subset, they are flat for any i and j . Thus taking Γ comutes with multiplication map of $\mathrm{Tot}(M_\bullet \otimes C_\bullet)$, meaning that one can give DG-algebra structure on $\mathrm{Tot}\Gamma(M_\bullet \otimes C_\bullet)$ too. From this DG-algebra structure one can give algebra structure on H^0 of this complex, the desired multiplication on $H^0\mathrm{Tot}\Gamma(M_\bullet \otimes N_\bullet)$.

Unfortunately, one can find such DG-algebra structure on the Čech complex only when the base ring is $\mathbb{Q}$. Thus this construction can't be used in defining multiplication universally. One can use the natural definition when defining multiplication, and over base $\mathbb{Q}$ (we have shown that hypercohomology for $M_\bullet$ commutes with base change) natural definition coincides with explicit construction. This shows that the explicit construction still can be formally used in explicit calculation of homology, and should yield integral results regardless of choice of multiplication on Čech cohomology.

Theorem 2.4.2. *Let $M_\bullet$ be a complex of sheaves whose summands are direct sums of $\mathcal{O}(t)$. Suppose that the Čech complex $C_\bullet(\mathbb{P}^k)$ is given a DG-algebra structure such that the natural map $\mathcal{O} \rightarrow \bigoplus_i \mathcal{O}_i \rightarrow C_\bullet(\mathbb{P}^k)$ becomes its unit. The induced multiplicative structure on $H^0\mathrm{Tot}\Gamma(M_\bullet \otimes C_\bullet(\mathbb{P}^k))$ coincides with the natural multiplication induced on hypercohomology $H^0R\Gamma(M_\bullet)$.*

Proof. Throughout this proof, write $C_\bullet(\mathbb{P}^k)$ as $C_\bullet$ for simplicity.

We will first review the natural construction given in ([49], Section 9). Let $\mathcal{C}(\mathcal{O} - \mathbf{mod})$, $\mathcal{D}(\mathcal{O} - \mathbf{mod})$ be the category of chain complexes of $\mathcal{O}_{\mathbb{P}^k}$ -modules and its

derived category respectively, and q be the ‘localization’ functor $q : \mathcal{C}(\mathcal{O}\text{--}\mathbf{mod}) \rightarrow \mathcal{D}(\mathcal{O}\text{--}\mathbf{mod})$. A DG-algebra structure on $M_\bullet$ corresponds to a map of complexes $\mathrm{Tot}(M_\bullet \otimes M_\bullet) \rightarrow M_\bullet$, which allows us to construct a morphism

$$qM_\bullet \otimes^L qM_\bullet \rightarrow q\mathrm{Tot}(M_\bullet \otimes M_\bullet) \rightarrow qM_\bullet$$

where $qM_\bullet \otimes^L qM_\bullet$ denotes the total tensor product in category of derived functors ([45], 10.6) - the superscript $\otimes^L$ means it is formed as total left derived functor of tensor product. The first map is obtained from the universal property of total tensor product, and the second map is application of q to $\mathrm{Tot}(M_\bullet \otimes M_\bullet) \rightarrow M_\bullet$. When the total right-derived functor $R\Gamma$ is applied to this multiplicative structure on $qM_\bullet$, one can obtain a map

$$R\Gamma(qM_\bullet) \otimes R\Gamma(qM_\bullet) \rightarrow R\Gamma(qM_\bullet \otimes^L qM_\bullet)$$

by composing two maps obtained from the projection formula ([45], 10.8.1) and counit of $R\Gamma$ (note Γ is pushforward for the structure map $\mathbb{P}^k \rightarrow \mathbb{Z}$ thus becomes right adjoint). By applying H^0 on both sides, one can give multiplicative structure on $H^0 R\Gamma(qM_\bullet)$, which coincides with the ordinary hypercohomology $H^0 R\Gamma(M_\bullet)$. (Here $H^0 R\Gamma$ is the ordinary hypercohomology functor in $\mathcal{C}(\mathcal{O}\text{--}\mathbf{mod})$).

We would like to compare this natural construction with the our construction described just above, and show that those two coincide indeed. Our construction firstly considers the multiplicative structure on $\mathrm{Tot}(M_\bullet \otimes C_\bullet)$ naturally coming from $M_\bullet$ and $C_\bullet$, applies Γ which commutes with tensor product, and induces multiplicative structure on H^0 of $\mathrm{Tot}(\Gamma(M_\bullet \otimes C_\bullet))$. We consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}(M_\bullet \otimes M_\bullet) & \longrightarrow & M_\bullet \\ \downarrow & & \downarrow \\ \mathrm{Tot}(\mathrm{Tot}(M_\bullet \otimes C_\bullet) \otimes \mathrm{Tot}(M_\bullet \otimes C_\bullet)) & \longrightarrow & \mathrm{Tot}(M_\bullet \otimes C_\bullet) \end{array}$$

whose vertical maps are induced from the unit $\mathcal{O} \rightarrow C_\bullet$. By universal property of q , we can distribute q to tensor product as above, and consider the following diagram.

$$\begin{array}{ccc} qM_\bullet \otimes^L qM_\bullet & \longrightarrow & qM_\bullet \\ \downarrow & & \downarrow \\ q\mathrm{Tot}(M_\bullet \otimes C_\bullet) \otimes^L q\mathrm{Tot}(M_\bullet \otimes C_\bullet) & \longrightarrow & q\mathrm{Tot}(M_\bullet \otimes C_\bullet) \end{array}$$

When we apply the total right-derived functor $R\Gamma$ to this diagram and compose the horizontal maps with the natural maps of the following form (the bottom map is obtained by replacing $M_\bullet$ with $\mathrm{Tot}(M_\bullet \otimes C_\bullet)$)

$$R\Gamma(qM_\bullet) \otimes R\Gamma(qM_\bullet) \rightarrow R\Gamma(qM_\bullet \otimes qM_\bullet),$$

we can induce a multiplicative structure on $R\Gamma q\mathrm{Tot}(M_\bullet \otimes C_\bullet)$, which commutes with multiplicative structure on $R\Gamma qM_\bullet$. Meanwhile, we note that all entries $M_i \otimes C_j$ of $M_\bullet \otimes C_\bullet$ are all Γ -acyclic, so action of $R\Gamma$ on $q\mathrm{Tot}(M_\bullet \otimes C_\bullet)$ is simply componentwise: we have

$$R\Gamma q\mathrm{Tot}(M_\bullet \otimes C_\bullet) \simeq q\mathrm{Tot}(\Gamma(M_\bullet \otimes C_\bullet)).$$

If we take its zeroth cohomology, we can obtain the following isomorphism using the universal property of q .

$$H^0 R\Gamma q\mathrm{Tot}(M_\bullet \otimes C_\bullet) \simeq H^0 \mathrm{Tot}(\Gamma(M_\bullet \otimes C_\bullet)).$$

Thus we have naturally constructed a multiplicative structure on $H^0 \mathrm{Tot}(\Gamma(M_\bullet \otimes C_\bullet))$ which commutes with multiplicative structure on $H^0 R\Gamma(qM_\bullet) \simeq H^0 R\Gamma(M_\bullet)$. (The $R\Gamma$ on left side denotes total right-derived functor, and $H^0 R\Gamma$ on right side denotes ordinary hypercohomology.)

If we explicitly track this multiplication on $H^0 \mathrm{Tot}(\Gamma(M_\bullet \otimes C_\bullet))$ by considering all the maps described above, it turns out that the resulting multiplication coincides

with the multiplication induced by componentwise multiplication in $\text{Tot}(\Gamma(M_\bullet \otimes C_\bullet))$. As entries $M_i \otimes C_j$ of $M_\bullet \otimes C_\bullet$ are direct sums of sheaves of form $M_i \otimes \mathcal{O}_{x_I}$, those are Γ -acyclic. Action of a right-derived functor on acyclic complexes is given componentwise and counit of $R\Gamma$ on acyclic complexes becomes isomorphism, allowing us to check that multiplication on $H^0\text{Tot}(\Gamma(M_\bullet \otimes C_\bullet))$ becomes componentwise multiplication in $\text{Tot}(\Gamma(M_\bullet \otimes C_\bullet))$. We already established that maps between $H^0R\Gamma(M_\bullet)$ and $H^0\text{Tot}(\Gamma(M_\bullet \otimes C_\bullet))$ are isomorphic, so this proves the theorem. $\square$

2.4.3 DG-algebra structure for Čech complex

In this section, we will describe a DG-algebra structure on $C_\bullet(\mathbb{P}^k)$ such that the natural map $\mathcal{O} \rightarrow C_\bullet(\mathbb{P}^k)$ becomes unit.

Remark. It turns out that there exists infinitely many DG-algebra structures possible, and all those choices do not matter when calculating multiplication in hypercohomology (due to previous theorem). The DG-algebra structure constructed in this section is symmetric with respect to the permutation of $k+1$ variables, which allows simpler description. Meanwhile, it is impossible to construct such DG-algebra with integral coefficient, and we need to assume that everything here are defined over $\mathbb{Q}$.

For $I \subseteq [k+1] = \{1, \dots, k+1\}$, $I \neq \emptyset$, define c_I as element 1 of $\mathcal{O}_I$, and denote our whole complex $C_\bullet$ as $\bigoplus_{I \neq \emptyset} \mathcal{O}_I c_I$. It suffices to define multiplications between components $\mathcal{O}_I c_I$ and $\mathcal{O}_J c_J$.

We introduce some additional notations for convenience. For $I = \{i_1, \dots, i_l\}$ with $i_1 < i_2 < \dots < i_l$ and a permutation $\sigma \in S_l$, define

$$c(i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(l)}) = \text{sgn}(\sigma) c_I.$$

Also define $c(j_1 \cdots j_l) = 0$ if there are duplicate elements among j_t . One can easily check the following.

Proposition 2.4.3. *For a list $(i_1, i_2, \dots, i_l) \in [k+1]^l$ and $\sigma \in S_l$, we have*

$$c(i_{\sigma(1)}i_{\sigma(2)} \cdots i_{\sigma(l)}) = \text{sgn}(\sigma)c(i_1i_2 \cdots i_l).$$

Also, in this notation, one can write differentials of the Čech complexes in the following convenient way. For any list $I = (i_1, \dots, i_l) \in [k+1]^l$, denote $It = (i_1, \dots, i_l, t)$ ($t \in [k+1]$) as concatenation of I and t .

Proposition 2.4.4. *We have*

$$d(c(I)) = \sum_{t=1}^{k+1} c(It)$$

where d is the complex map for $C_\bullet$.

Proof. If there are duplicates among elements of I , then both sides are zero. If elements of I are all distinct, it suffices to assume that elements of I are sorted, as permuting I changes both sides by signature of the permutation.

Let t be the e th smallest element among $I \cup t$, such that $i_1 < \cdots < i_{l-e+1} < t < i_{l-e} < \cdots < i_l$. Then

$$c(It) = c(i_1i_2 \cdots i_l t) = (-1)^e c(i_1 \cdots i_{l-e+1} t i_{l-e} \cdots i_l) = (-1)^{e-1} c_{I \cup t}.$$

Meanwhile, the map of Čech complex from $\mathcal{O}_I$ to $\mathcal{O}_{I \cup \{t\}}$ is $(-1)^{e-1}$ times the identity. So the description $d(c(I)) = \sum_{t=1}^{k+1} c(It)$ coincides with map of Čech complex. $\square$

For example, when $k+1 = 5$, we have

$$\begin{aligned} d(c_{\{1,3\}}) &= d(c(13)) = c(131) + c(132) + c(133) + c(134) + c(135) \\ &= c(132) + c(134) + c(135) \\ &= -c_{\{1,2,3\}} + c_{\{1,3,4\}} + c_{\{1,3,5\}}. \end{aligned}$$

Now we define our multiplication as follows. For I, J be any sequences in whose length is a, b respectively, and $t \in [k+1]$, let

$$c(It) \cdot c(Jt) = (-1)^{ab} \frac{a!b!}{(a+b+1)!} c(IJt).$$

And define $c(I) \cdot c(J) = 0$ if the number of common elements of I and J are not exactly 1. These conditions, provided that those are consistent, define multiplication on variables c_I in unique way. We show that this construction of multiplicative structure on $C_\bullet$ is indeed the DG-algebra structure we desired. Before stating the explicit construction, we introduce a notation. For two disjoint subsets I, J , define

$$\text{ind}(I, J) = (-1)^{\#\{(i,j) \in I \times J : j > i\}}.$$

This number can be interpreted as signature of the permutation which permutes the concatenated list of I and J (by considering them as sequences which are already ordered in ascending order) to list corresponding to $I \cup J$. This statement is simply equivalent to saying

$$c(IJ) = \text{ind}(I, J) c_{I \cup J} \quad \text{if } I, J \text{ are both ordered.}$$

Proposition 2.4.5. *For $I, J \subset [k+1]$, $I, J \neq \emptyset$, define the multiplication on basis elements of $C_\bullet = \bigoplus_{I \neq \emptyset} \mathcal{O}_I c_I$ as*

$$c_I \cdot c_J = \begin{cases} (-1)^{(|I|-1)(|J|-1)} \frac{(|I|-1)! (|J|-1)!}{(|I|+|J|-1)!} n(I, J) c_{I \cup J} & \text{if } |I \cap J| = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $n(I, J)$ is defined as follows: if $I \cap J = \{l\}$ then $n(I, J) = \text{ind}(I \setminus l, J \setminus l)$. The

multiplication $\mathcal{O}_I c_I \times \mathcal{O}_J c_J \rightarrow \mathcal{O}_{I \cup J} c_{I \cup J}$ is defined as

$$ac_I \cdot bc_J = (\tilde{a}\tilde{b})(c_I \cdot c_J) \in \mathcal{O}_{I \cup J} c_{I \cup J}$$

where $a \in \mathcal{O}_I$ and $b \in \mathcal{O}_J$ are mapped to $\tilde{a} \in \mathcal{O}_{I \cup J}$ and $\tilde{b} \in \mathcal{O}_{I \cup J}$ via the restriction map. This multiplication rule extended over all $C_\bullet$ via distributivity gives a DG-algebra structure on $C_\bullet$ which has the natural map $\mathcal{O} \rightarrow \bigoplus_i \mathcal{O}_i \rightarrow C_\bullet$ as its unit.

Proof. Step 1) First we show that the multiplication map preserves grades on $C_\bullet$. The component $\mathcal{O}_I c_I$ is a summand of $C_{|I|-1}$, so it has grade $|I| - 1$. If the multiplication $c_I \cdot c_J$ is nontrivial, then $|I \cap J| = 1$, and it should follow that $(|I| - 1) + (|J| - 1) = (|I \cup J| - 1)$. This implies that grade is preserved in the multiplication of form $c_I \cdot c_J = k c_{I \cup J}$ ($k \in \mathbb{Q}$).

Step 2) We show that above mentioned rule

$$c(It) \cdot c(Jt) = (-1)^{ab} \frac{a!b!}{(a+b+1)!} c(IJt)$$

holds for any sequences I and J with lengths a, b respectively. It suffices to show when I, J and t all consists of all distinct elements, because both sides are zero otherwise. If it is the case, we can consider permutations on I and J which sorts them, and those actions will change signs of both sides in the same way. So we can only consider I and J which are ordered in ascending order. In that case, we have

$$c(It) = \text{ind}(I, t) c_{I \cup t}, c(Jt) = \text{ind}(J, t) c_{J \cup t}$$

and

$$c(IJt) = \text{ind}(I, J) c(Kt) = \text{ind}(I, J) \text{ind}(I \cup J, t) c_{I \cup J \cup t}$$

where K is the ordered sequence of $I \cup J$. By calculating the multiplication, we obtain

$$\begin{aligned} c(It) \cdot c(Jt) &= \text{ind}(I, t) \text{ind}(J, t) c_{I \cup t} \cdot c_{J \cup t} \\ &= (-1)^{ab} \frac{a!b!}{(a+b+1)!} \text{ind}(I, t) \text{ind}(J, t) \text{ind}(I, J) c_{I \cup J \cup t} \end{aligned}$$

so it suffices to show

$$\text{ind}(I, t) \text{ind}(J, t) = \text{ind}(I \cup J, t).$$

This equation can be easily deduced from the definition $\text{ind}(I, t) = \prod_{i \in I, i < t} (-1)$.

Step 3) We show that the condition for DG-algebra should hold for any homogeneous α, β , that is, the following holds.

$$d(\alpha \cdot \beta) = (d\alpha) \cdot \beta + (-1)^{\deg(\alpha)} \alpha \cdot (d\beta)$$

It suffices to verify this statement for $(\alpha, \beta) = (c(I), c(J))$, and the whole statement follows from distributivity. That is, for any sequences I and J whose length is a and b respectively, we want to prove

$$d(c(I) \cdot c(J)) = d(c(I)) \cdot c(J) + (-1)^{a-1} c(I) \cdot d(c(J)).$$

We can assume that both I and J consists of distinct entries, otherwise all multiplications above are zero.

Case 1) $|I \cap J| \geq 2$

We have $c(I) \cdot c(J) = 0$. Also $d(c(I)) \cdot c(J) = \sum_t c(It) \cdot c(J) = 0$ and similarly $c(I) \cdot d(c(J)) = 0$, so it holds.

Case 2) $I \cap J = \emptyset$

The left side is zero. Let $J = (j_1 j_2 \cdots j_b)$ and $\hat{J}_e = (j_1 \cdots \hat{j}_e \cdots j_b)$ be the list with j_e

omitted. Then we have

$$\begin{aligned}
d(c(I)) \cdot c(J) &= \sum_t c(It) \cdot c(J) = \sum_{t \in J} c(It) \cdot c(J) \\
&= \sum_{e=1}^b (-1)^{b-e} c(Ij_e) c(\hat{J}_e j_e) \\
&= \sum_{e=1}^b \frac{(-1)^{a(b-1)} a! (b-1)!}{(a+b)!} (-1)^{b-e} c(I \hat{J}_e j_e) \\
&= \sum_{e=1}^b \frac{(-1)^{a(b-1)} a! (b-1)!}{(a+b)!} c(IJ) = \frac{(-1)^{ab-a} a! b!}{(a+b)!} c(IJ)
\end{aligned}$$

and similarly for $I = (i_1 i_2 \cdots i_a)$ and $\hat{I}_f = (i_1 \cdots \hat{i}_f \cdots i_a)$ be the list with i_f omitted,

$$\begin{aligned}
c(I) \cdot d(c(J)) &= \sum_t c(I) \cdot c(Jt) = \sum_{t \in I} c(I) \cdot c(Jt) \\
&= \sum_{f=1}^a (-1)^{a-f} c(\hat{I}_f i_f) c(J i_f) \\
&= \sum_{f=1}^a \frac{(-1)^{(a-1)b} (a-1)! b!}{(a+b)!} (-1)^{a-f} c(\hat{I}_f J i_f) \\
&= \sum_{f=1}^a \frac{(-1)^{(a-1)b} (a-1)! b!}{(a+b)!} (-1)^{a+b-f} c(\hat{I}_f i_f J) \\
&= \sum_{f=1}^a \frac{(-1)^{(a-1)b} (a-1)! b!}{(a+b)!} (-1)^b c(IJ) = \frac{(-1)^{ab} a! b!}{(a+b)!} c(IJ)
\end{aligned}$$

Thus we have

$$(RHS) = ((-1)^{ab-a} + (-1)^{a-1} (-1)^{ab}) \frac{a! b!}{(a+b)!} c(IJ) = 0.$$

Case 3) $|I \cap J| = 1$

Both sides change sign in identical way when I and J are permuted, so we can assume

I, J has form of $I = I_0 t$ and $J = J_0 t$. Then

$$c(I) \cdot c(J) = c(I_0 t) c(J_0 t) = (-1)^{(a-1)(b-1)} \frac{(a-1)!(b-1)!}{(a+b-1)!} c(I_0 J_0 t)$$

so

$$d(c(I) \cdot c(J)) = (-1)^{(a-1)(b-1)} \sum_l \frac{(a-1)!(b-1)!}{(a+b-1)!} c(I_0 J_0 t l).$$

For the right hand side, we have

$$\begin{aligned} d(c(I)) \cdot c(J) &= \sum_l c(I_0 t l) c(J_0 t) = - \sum_l c(I_0 l t) c(J_0 t) \\ &= \sum_l \frac{(-1)^{a(b-1)+1} a! (b-1)!}{(a+b)!} c(I_0 l J_0 t) \\ &= \sum_l \frac{(-1)^{a(b-1)+b+1} a! (b-1)!}{(a+b)!} c(I_0 J_0 t l) \end{aligned}$$

and

$$\begin{aligned} (-1)^{a-1} c(I) \cdot d(c(J)) &= (-1)^{a-1} \sum_l c(I_0 t) c(J_0 t l) = (-1)^a \sum_l c(I_0 t) c(J_0 l t) \\ &= \sum_l \frac{(-1)^{(a-1)b+a} (a-1)! b!}{(a+b)!} c(I_0 J_0 l t) \\ &= \sum_l \frac{(-1)^{(a-1)b+a-1} (a-1)! b!}{(a+b)!} c(I_0 J_0 t l) \end{aligned}$$

so

$$\begin{aligned} &d(c(I)) \cdot c(J) + (-1)^{a-1} c(I) \cdot d(c(J)) \\ &= \sum_l \frac{(-1)^{a(b-1)+b+1} a! (b-1)!}{(a+b)!} c(I_0 J_0 t l) + \sum_l \frac{(-1)^{(a-1)b+a-1} (a-1)! b!}{(a+b)!} c(I_0 J_0 t l) \\ &= (-1)^{(a-1)(b-1)} \sum_l \frac{(a-1)!(b-1)!}{(a+b-1)!} c(I_0 J_0 t l) = c(I) \cdot c(J). \end{aligned}$$

Step 4) Finally, we show that the map

$$\mathcal{O} \rightarrow \bigoplus_i \mathcal{O}_i \rightarrow C_\bullet$$

becomes a unit in this DG-algebra. The image of 1 under this map is given as $\sum_i c_i$, and by noting that

$$c_i \cdot c_I = c_I \cdot c_i = \begin{cases} \frac{1}{|I|} c_I & \text{if } i \in I \\ 0 & \text{otherwise} \end{cases}$$

we can see

$$\left(\sum_i c_i \right) \cdot c_I = c_I \cdot \left(\sum_i c_i \right) = c_I$$

□

Remark. One can also see that the DG-algebra defined above is commutative, as one can show

$$c_I \cdot c_J = (-1)^{(|I|-1)(|J|-1)} c_J \cdot c_I$$

from observing the following (when $I \cap J = \{l\}$)

$$\text{ind}(I \setminus l, J \setminus l) \text{ind}(I \setminus l, J \setminus l) = (-1)^{(|I|-1)(|J|-1)}.$$

It is not associative though: we have $c_1 \cdot c_1 = c_1$ and $c_1 \cdot c_{1,2} = c_{1,2}/2$, so $(c_1 \cdot c_1) \cdot c_{1,2} \neq c_1 \cdot (c_1 \cdot c_{1,2})$.

2.4.4 Example: binary cubic forms and cubic rings

We revisit the example of parametrization of cubic rings by binary cubic forms, and interpret its construction using hypercohomology. Let $f = ax_1^3 + bx_1^2x_2 + cx_1x_2^2 + dx_2^3$

be our cubic form. We first consider the complex

$$M_{\bullet} : \mathcal{O}(-3) \rightarrow \mathcal{O}$$

where its map is given as multiplication by f . Denote m_1 and m_0 as basis elements 1 in M_1 and M_0 respectively. Then the map of $M_{\bullet}$ is basically given as $d(m_1) = fm_0$. It becomes DG-algebra under the following rules

$$m_0 \cdot m_1 = m_1 \cdot m_0 = m_1, \quad m_0 \cdot m_0 = m_0.$$

This complex becomes a free resolution of $\mathcal{O}/(f)$ for generic f , and satisfies WTM conditions. We note that the cohomology $H^1(M_1)$ has $\mathbb{Z}$ -basis given as $m_1/(x_1^2x_2)$ and $m_2/(x_1x_2^2)$.

Meanwhile, maps of the Čech complex in $\mathbb{P}^1$ are given as

$$dc_1 = c_{12}, \quad dc_2 = -c_{12}, \quad dc_{12} = 0$$

and the algebra structure we constructed in Section 2.4.3 is given by the rules (and all other multiplications zero)

$$c_1 \cdot c_1 = c_1, \quad c_2 \cdot c_2 = c_2, \quad c_1 \cdot c_{12} = c_2 \cdot c_{12} = \frac{1}{2}c_{12}.$$

We consider the double complex $M_{\bullet} \otimes C_{\bullet}(\mathbb{P}^1)$ which has form of

$$\begin{array}{ccc} M_1 \otimes (\mathcal{O}_1 \oplus \mathcal{O}_2) & \longrightarrow & M_0 \otimes (\mathcal{O}_1 \oplus \mathcal{O}_2) \\ \downarrow & & \downarrow \\ M_1 \otimes \mathcal{O}_{12} & \longrightarrow & M_0 \otimes \mathcal{O}_{12} \end{array}$$

Its maps $d_{\rightarrow}$ and $d_{\downarrow}$ correspond to maps of $M_{\bullet}$ and $C_{\bullet}$ respectively with some sign change that $d_{\downarrow} : M_p \otimes C_q \rightarrow M_p \otimes C_{q+1}$ is $(-1)^p$ times the map $C_q \rightarrow C_{q+1}$. We have

shown that $\Gamma(H^0\text{Tot}(M_\bullet \otimes C_\bullet(\mathbb{P}^1)))$ is identified with hypercohomology $H^0\Gamma(M_\bullet)$, with multiplication structure given as in the Theorem 2.4.2.

Denote φ_1, φ_2 as preimages of the cohomology elements $m_1/(x_1^2x_2)$ and $m_2/(x_1x_2^2)$ of $H^1(M_1)$ under the map $H^0\Gamma(M_\bullet) \rightarrow H^1(M_1)$ respectively, and consider them as elements in $H^0\text{Tot}\Gamma(M_\bullet \otimes C_\bullet)$. We can write them as

$$\varphi_i = \varphi_i^{(1)} + \varphi_i^{(0)}, \quad \varphi_i^{(t)} \in \Gamma(M_t \otimes C_t)$$

such that

$$d(\varphi_i) = d_{\rightarrow}(\varphi_i^{(1)}) + d_{\downarrow}(\varphi_i^{(0)}) = 0.$$

We start with letting

$$\varphi_1^{(1)} = \frac{m_1c_{12}}{x_1^2x_2}, \quad \varphi_2^{(1)} = \frac{m_1c_{12}}{x_1x_2^2}.$$

Then

$$\begin{aligned} d_{\rightarrow}\varphi_1^{(1)} &= \frac{(ax_1^3 + bx_1^2x_2 + cx_1x_2^2 + dx_2^3)m_0c_{12}}{x_1^2x_2} \\ &= \left(b + c\frac{x_2}{x_1} + d\frac{x_2^2}{x_1^2}\right)m_0c_{12} + a\frac{x_1}{x_2}m_0c_{12} \end{aligned}$$

thus by picking

$$\varphi_1^{(0)} = -\left(b + c\frac{x_2}{x_1} + d\frac{x_2^2}{x_1^2}\right)m_0c_1 + a\frac{x_1}{x_2}m_0c_2$$

one can see that $d_{\rightarrow}\varphi_1^{(1)} + d_{\downarrow}\varphi_1^{(0)} = 0$. This choice of $\varphi_1^{(0)}$ can vary by constant times of $m_0c_1 + m_0c_2$, and we will see that our particular choice yields the identical multiplication table to that of the Levi–Delone–Faddeev correspondence. Let those

choice of $\varphi_1^{(0)}, \varphi_2^{(0)}$ be

$$\begin{aligned}\varphi_1 &= \frac{1}{x_1^2 x_2} m_1 c_{12} - \left(b + c \frac{x_2}{x_1} + d \frac{x_2^2}{x_1^2} \right) m_0 c_1 + a \frac{x_1}{x_2} m_0 c_2 \\ \varphi_2 &= \frac{1}{x_1 x_2^2} m_1 c_{12} - d \frac{x_2}{x_1} m_0 c_1 + \left(c + b \frac{x_1}{x_2} + a \frac{x_1^2}{x_2^2} \right) m_0 c_2\end{aligned}$$

together with identity element given as

$$\varphi_0 = m_0 c_1 + m_0 c_2.$$

We show that the $\mathbb{Z}$ -basis $(\varphi_0, -\varphi_1, -\varphi_2)$ of $H^0 R\Gamma(M_\bullet)$ has same multiplication table as $(1, \omega, \theta)$ of Levi–Delone–Faddeev correspondence. That is, we want to verify the following multiplications formulas. (We already established that φ_0 becomes unit.)

$$\varphi_1 \varphi_2 = -ad\varphi_0$$

$$\varphi_1^2 = -ac\varphi_0 - b\varphi_1 + a\varphi_2$$

$$\varphi_2^2 = -bd\varphi_0 - d\varphi_1 + c\varphi_2.$$

The multiplication on $\text{Tot}(M_\bullet \otimes C_\bullet)$ follows the same multiplication rules of $M_\bullet$ and $C_\bullet$, together with sign changes involved in $\text{Tot}(C_\bullet \otimes M_\bullet) \rightarrow \text{Tot}(M_\bullet \otimes C_\bullet)$: this simply implies $mc = (-1)^{\deg(m)\deg(c)} cm$ for homogeneous elements $m \in M_\bullet$ and $c \in C_\bullet$. For example, we have

$$(m_1 c_{12})(m_0 c_1) = (-1)^{1 \cdot 0} m_1 (m_0 c_{12} c_1) = \frac{1}{2} m_1 c_{12}.$$

With that in mind, we can explicitly calculate above multiplications. For example,

$$\begin{aligned}
\varphi_1^2 &= -\frac{1}{x_1^2 x_2} \left(b + c \frac{x_2}{x_1} + d \frac{x_2^2}{x_1^2} \right) (m_1 c_{12} m_0 c_1 + m_0 c_1 m_1 c_{12}) \\
&\quad + \frac{2}{x_1^2 x_2} \frac{a x_1}{x_2} (m_1 c_{12} m_0 c_2 + m_0 c_2 m_1 c_{12}) \\
&\quad + \left(b + c \frac{x_2}{x_1} + d \frac{x_2^2}{x_1^2} \right)^2 m_0 c_1 m_0 c_1 + \left(a \frac{x_1}{x_2} \right)^2 m_0 c_2 m_0 c_2 \\
&= \left(-\frac{b}{x_1^2 x_2} + \frac{a}{x_1 x_2^2} - \frac{c}{x_1^3} - \frac{d x_2}{x_1^4} \right) m_1 c_{12} \\
&\quad + \left(b + c \frac{x_2}{x_1} + d \frac{x_2^2}{x_1^2} \right)^2 m_0 c_1 + \left(a \frac{x_1}{x_2} \right)^2 m_0 c_2 \\
&= -\frac{b}{x_1^2 x_2} m_1 c_{12} + b \left(b + c \frac{x_2}{x_1} + d \frac{x_2^2}{x_1^2} \right) m_0 c_1 - b a \frac{x_1}{x_2} m_0 c_2 \\
&\quad + \frac{a}{x_1 x_2^2} m_1 c_{12} - a d \frac{x_2}{x_1} m_0 c_1 + a \left(c + b \frac{x_1}{x_2} + a \frac{x_1^2}{x_2^2} \right) m_0 c_2 \\
&\quad - a c m_0 c_1 - a c m_0 c_2 \\
&\quad + \left(-\frac{c}{x_1^3} - \frac{d x_2}{x_1^4} \right) m_1 c_{12} + (a x_1^3 + b x_1^2 x_2 + c x_1 x_2^2 + d x_2^3) \left(\frac{c}{x_1^3} + \frac{d x_2}{x_1^4} \right) m_0 c_1 \\
&= -b \varphi_1 + a \varphi_2 - a c \varphi_0 + d_{\downarrow} \left(\left(\frac{c}{x_1^3} + \frac{d x_2}{x_1^4} \right) m_1 c_1 \right) + d_{\rightarrow} \left(\left(\frac{c}{x_1^3} + \frac{d x_2}{x_1^4} \right) m_1 c_1 \right) \\
&= -b \varphi_1 + a \varphi_2 - a c \varphi_0 + d_{tot} \left(\left(\frac{c}{x_1^3} + \frac{d x_2}{x_1^4} \right) m_1 c_1 \right)
\end{aligned}$$

and image of d_{tot} is zero in total cohomology. One can similarly calculate

$$\begin{aligned}
\varphi_2^2 &= -\frac{1}{x_1 x_2^2} \frac{d x_2}{x_1} m_1 c_{12} + \frac{1}{x_1 x_2^2} \left(c + b \frac{x_1}{x_2} + a \frac{x_1^2}{x_2^2} \right) m_1 c_{12} \\
&\quad + \left(d \frac{x_2}{x_1} \right)^2 m_0 c_1 + \left(c + b \frac{x_1}{x_2} + a \frac{x_1^2}{x_2^2} \right)^2 m_0 c_2 \\
&= \left(-\frac{d}{x_1^2 x_2} + \frac{c}{x_1 x_2^2} + \frac{b}{x_2^3} + \frac{a x_1}{x_2^4} \right) m_1 c_{12} + \left(d \frac{x_2}{x_1} \right)^2 m_0 c_1 + \left(c + b \frac{x_1}{x_2} + a \frac{x_1^2}{x_2^2} \right)^2 m_0 c_2 \\
&= -d \varphi_1 + c \varphi_2 - b d \varphi_0 + d_{tot} \left(\left(\frac{b}{x_2^3} + \frac{a x_1}{x_2^4} \right) m_1 c_2 \right)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_1\varphi_2 &= -\frac{1}{x_1^2x_2}m_1c_{12}\frac{dx_2}{x_1}m_0c_1 + \frac{1}{x_1^2x_2}m_1c_{12}\left(c + b\frac{x_1}{x_2} + a\frac{x_1^2}{x_2^2}\right)m_0c_2 \\
&\quad - \frac{1}{x_1x_2^2}m_1c_{12}\left(b + c\frac{x_2}{x_1} + d\frac{x_2^2}{x_1^2}\right)m_0c_1 + \frac{1}{x_1x_2^2}m_1c_{12}\frac{ax_1}{x_2}m_0c_2 \\
&\quad + \left(b + c\frac{x_2}{x_1} + d\frac{x_2^2}{x_1^2}\right)\frac{dx_2}{x_1}m_0c_1 + \frac{ax_1}{x_2}\left(c + b\frac{x_1}{x_2} + a\frac{x_1^2}{x_2^2}\right)m_0c_2 \\
&= -ad(m_0c_1 + m_1c_2) + \left(-\frac{d}{x_1^3} + \frac{a}{x_2^3}\right)m_1c_{12} + \frac{d}{x_1^3}fm_0c_1 + \frac{a}{x_2^3}fm_0c_2 \\
&= -ad\varphi_0 + d_{tot}\left(\frac{d}{x_1^3}m_1c_1 + \frac{a}{x_2^3}m_1c_2\right).
\end{aligned}$$

2.4.5 Example: pairs of ternary quadrics and quartic rings

We also reexamine the example of the quartic ring parametrization. Let $f = \sum_{1 \leq i \leq j \leq 3} a_{ij}x_ix_j$, $g = \sum_{1 \leq i \leq j \leq 3} b_{ij}x_ix_j$ be two ternary quadrics. Then the multiplication coefficients c_{ij}^k (defined as integers satisfying $\alpha_i\alpha_j = \sum_{k=0}^3 c_{ij}^k\alpha_k$) satisfy the following relations

$$\begin{aligned}
c_{ij}^k &= \text{sgn}(ijk) \cdot \lambda_{ii}^{jj}, \\
c_{ii}^j &= \text{sgn}(ijk) \cdot \lambda_{ik}^{ii}, \\
c_{ij}^i - c_{kj}^k &= \text{sgn}(ijk) \cdot \lambda_{jj}^{ik}, \\
c_{ii}^i - c_{ij}^j - c_{ik}^k &= \text{sgn}(ijk) \cdot \lambda_{ij}^{ik}
\end{aligned}$$

for any permutations (i, j, k) of $(1, 2, 3)$, where $\lambda_{ij}^{kl} = a_{ij}b_{kl} - a_{kl}b_{ij}$ and $\text{sgn}(ijk)$ denotes the signature of the permutation (i, j, k) . We will illustrate how this ring becomes isomorphic to the hypercohomology ring of the Koszul complex

$$M_\bullet = \mathcal{K}_{f,g} : \mathcal{O}(-4) \rightarrow \mathcal{O}(-2)^2 \rightarrow \mathcal{O}.$$

Denote $m_{f \wedge g}, m_f, m_g, m_0$ as the unit element of $\mathcal{O}(-4)$, two copies of $\mathcal{O}(-2)$, and $\mathcal{O}$ respectively, such that the maps of $M_\bullet$ are given as follows.

$$d(m_f) = fm_0, \quad d(m_g) = gm_0, \quad d(m_{f \wedge g}) = fm_g - gm_f$$

The natural DG-algebra structure on $M_\bullet$ is given as follows: we have $m_f \cdot m_g = m_{f \wedge g}$, m_0 behaves as unit, and all other multiplications are zero. We will additionally use the notation $m_{g \wedge f} = m_g \cdot m_f = -m_{f \wedge g}$.

We describe the Čech complex for $\mathbb{P}_{\mathbb{Z}}^2$ with the algebra structure defined in Section 2.4.3. Its maps can be described as

$$\begin{aligned} dc_1 &= c_{12} + c_{13}, & dc_2 &= c_{21} + c_{23}, & dc_3 &= c_{31} + c_{32} \\ dc_{12} &= c_{123}, & dc_{23} &= c_{231}, & dc_{31} &= c_{312}. \end{aligned}$$

Here we use the notation of Section 2.4.3 in the sense that $c_{ij} = c(ij)$: for example we have $c_{21} = -c_{12}$ and $c_{231} = -c_{132} = c_{123}$. Its nontrivial multiplications consist of permutations of the following formulas:

$$c_1^2 = c_1, \quad c_1 \cdot c_{12} = \frac{1}{2}c_{12}, \quad c_1 \cdot c_{123} = \frac{1}{3}c_{123}, \quad c_{13} \cdot c_{23} = \frac{1}{6}c_{123}.$$

Similar to the cubic ring example, we identify hypercohomology with the total cohomology of $\Gamma(M_\bullet \otimes C_\bullet)$. Denote $\varphi_1, \varphi_2, \varphi_3$ as preimages of the cohomology elements $m_{f \wedge g}/(x_1^2 x_2 x_3), m_{f \wedge g}/(x_1 x_2^2 x_3), m_{f \wedge g}/(x_1 x_2 x_3^2)$ in $H^2(M_2)$ respectively under the map $H^0\Gamma(M_\bullet) \rightarrow H^2(M_2)$, and consider them as elements in $H^0\Gamma\text{Tot}(M_\bullet \otimes C_\bullet)$. We can write them as

$$\varphi_i = \varphi_i^{(2)} + \varphi_i^{(1)} + \varphi_i^{(0)}, \quad \varphi_i^{(t)} \in \Gamma(M_t \otimes C_t)$$

such that

$$d_{\rightarrow}(\varphi_i^{(2)}) + d_{\downarrow}(\varphi_i^{(1)}) = 0, \quad d_{\rightarrow}(\varphi_i^{(1)}) + d_{\downarrow}(\varphi_i^{(0)}) = 0.$$

For $\varphi_1^{(2)} = m_{f \wedge g} c_{123} / (x_1^2 x_2 x_3)$, one choice of $\varphi_1^{(1)}$ and $\varphi_1^{(0)}$ can be found explicitly using above two rules. We start from

$$\begin{aligned} d_{\rightarrow}(\varphi_1^{(2)}) &= \frac{f m_g - g m_f}{x_1^2 x_2 x_3} c_{123} \\ &= \left(\frac{a_{11}}{x_2 x_3} + \left(\frac{a_{12}}{x_1 x_3} + \frac{x_2 a_{22}}{x_1^2 x_3} \right) + \left(\frac{a_{23}}{x_1^2} + \frac{a_{13}}{x_1 x_2} + \frac{x_3 a_{33}}{x_1^2 x_2} \right) \right) m_g c_{123} \\ &\quad - \left(\frac{b_{11}}{x_2 x_3} + \left(\frac{b_{12}}{x_1 x_3} + \frac{x_2 b_{22}}{x_1^2 x_3} \right) + \left(\frac{b_{23}}{x_1^2} + \frac{b_{13}}{x_1 x_2} + \frac{x_3 b_{33}}{x_1^2 x_2} \right) \right) m_f c_{123} \end{aligned}$$

One of its preimage can be taken as follows: (keep in mind that $d_{\downarrow}$ changes sign compared to the complex map of $C_{\bullet}$ at degree -1)

$$\begin{aligned} \varphi_1^{(1)} &= \frac{a_{11}}{x_2 x_3} m_g c_{23} - \left(\frac{a_{12}}{x_1 x_3} + \frac{x_2 a_{22}}{x_1^2 x_3} \right) m_g c_{13} + \left(\frac{a_{13}}{x_1 x_2} + \frac{a_{23}}{x_1^2} + \frac{x_3 a_{33}}{x_1^2 x_2} \right) m_g c_{12} \\ &\quad - \frac{b_{11}}{x_2 x_3} m_f c_{23} + \left(\frac{b_{12}}{x_1 x_3} + \frac{x_2 b_{22}}{x_1^2 x_3} \right) m_f c_{13} - \left(\frac{b_{13}}{x_1 x_2} + \frac{b_{23}}{x_1^2} + \frac{x_3 b_{33}}{x_1^2 x_2} \right) m_f c_{12}. \end{aligned}$$

As $b_{pq}f - a_{pq}g = \sum_{1 \leq i \leq j \leq 3} \lambda_{ij}^{pq} x_i x_j$, we have

$$\begin{aligned} d_{\rightarrow}(\varphi_1^{(1)}) &= \sum_{1 \leq i \leq j \leq 3} \left(\frac{\lambda_{11}^{ij} x_i x_j}{x_2 x_3} m_0 c_{23} - \frac{\lambda_{12}^{ij} x_i x_j}{x_1 x_3} m_0 c_{13} - \frac{\lambda_{22}^{ij} x_i x_j x_2}{x_1^2 x_3} m_0 c_{13} \right. \\ &\quad \left. + \frac{\lambda_{13}^{ij} x_i x_j}{x_1 x_2} m_0 c_{12} + \frac{\lambda_{23}^{ij} x_i x_j}{x_1^2} m_0 c_{12} + \frac{\lambda_{33}^{ij} x_i x_j x_3}{x_1^2 x_2} m_0 c_{12} \right) \end{aligned}$$

If we expand everything out then we can find one preimage as

$$\begin{aligned} \varphi_1^{(0)} &= \left(\frac{\lambda_{22}^{23} x_2^2}{x_1^2} + \frac{\lambda_{22}^{33} x_2 x_3}{x_1^2} + \frac{\lambda_{12}^{33} x_3}{x_1} + \frac{\lambda_{22}^{13} x_2}{x_1} + \frac{\lambda_{12}^{23} x_2}{x_1} + \lambda_{12}^{13} \right) m_0 c_1 \\ &\quad + \left(\frac{\lambda_{33}^{11} x_3}{x_2} + \frac{\lambda_{13}^{11} x_1}{x_2} + \lambda_{23}^{11} \right) m_0 c_2 + \left(\frac{\lambda_{12}^{11} x_1}{x_3} + \frac{\lambda_{11}^{22} x_2}{x_3} \right) m_0 c_3. \end{aligned}$$

We can obtain similar expressions for φ_2 and φ_3 by considering cyclic permutations of x_1, x_2, x_3 .

While one can verify that multiplications of elements $\varphi_1, \varphi_2, \varphi_3$ satisfy multipli-

cation table of Bhargava's parametrizations by fully calculating their products, we would like to figure out multiplication coefficients with less computations. Suppose that we want to find the multiplication coefficient c_{12}^3 , that is, coefficient of φ_3 in $\varphi_1\varphi_2$.

$$\varphi_1 \cdot \varphi_2 = \sum_{k=0}^3 c_{12}^k \varphi_k$$

We have seen that the map $H^0 R\Gamma(M_\bullet) \rightarrow H^2(M_2)$ of Theorem 2.3.3 maps a total element $\psi = \psi^{(2)} + \psi^{(1)} + \psi^{(0)}$ ($\psi^{(i)} \in \Gamma(M_i \otimes C_i)$) to image of $\psi^{(2)}$ in $\Gamma(M_2 \otimes C_2) \rightarrow H^2(M_2)$, so it maps φ_0 to zero and φ_i to $m_{f \wedge g} c_{123} / (x_1 x_2 x_3 x_i)$. This implies that c_{12}^3 exactly becomes the coefficient of $m_{f \wedge g} c_{123} / (x_1 x_2 x_3^2)$ in $\varphi_1 \cdot \varphi_2$. We track such terms in products of the form $\varphi_1^{(t)} \varphi_2^{(2-t)}$ for $t = 0, 1, 2$, which appears in the calculation of $\varphi_1 \cdot \varphi_2$.

For $t = 0$, in the product of $\varphi_1^{(2)} = m_{f \wedge g} c_{123} / (x_1^2 x_2 x_3)$ and

$$\begin{aligned} \varphi_2^{(0)} = & \left(\frac{\lambda_{33}^{31} x_3^2}{x_2^2} + \frac{\lambda_{33}^{11} x_3 x_1}{x_2^2} + \frac{\lambda_{23}^{11} x_1}{x_2} + \frac{\lambda_{33}^{21} x_3}{x_2} + \frac{\lambda_{23}^{31} x_3}{x_2} + \lambda_{23}^{21} \right) m_0 c_2 \\ & + \left(\frac{\lambda_{11}^{22} x_1}{x_3} + \frac{\lambda_{21}^{22} x_2}{x_3} + \lambda_{31}^{22} \right) m_0 c_3 + \left(\frac{\lambda_{23}^{22} x_2}{x_1} + \frac{\lambda_{22}^{33} x_3}{x_1} \right) m_0 c_1, \end{aligned}$$

the contribution of $\frac{m_{f \wedge g} c_{123}}{x_1 x_2 x_3^2}$ -term is

$$\frac{m_{f \wedge g} c_{123}}{x_1^2 x_2 x_3} \cdot \frac{\lambda_{11}^{22} x_1}{x_3} m_0 c_3 = \frac{\lambda_{11}^{22}}{3} \frac{m_{f \wedge g} c_{123}}{x_1 x_2 x_3^2}.$$

Similarly for $t = 2$, we obtain

$$\frac{\lambda_{11}^{22} x_2}{x_3} m_0 c_3 \cdot \frac{m_{f \wedge g} c_{123}}{x_1 x_2^2 x_3} = \frac{\lambda_{11}^{22}}{3} \frac{m_{f \wedge g} c_{123}}{x_1 x_2 x_3^2}.$$

For $t = 1$, we consider product of

$$\begin{aligned}\varphi_1^{(1)} = & \frac{a_{11}}{x_2x_3}m_gc_{23} - \left(\frac{a_{12}}{x_1x_3} + \frac{x_2a_{22}}{x_1^2x_3}\right)m_gc_{13} + \left(\frac{a_{13}}{x_1x_2} + \frac{a_{23}}{x_1^2} + \frac{x_3a_{33}}{x_1^2x_2}\right)m_gc_{12} \\ & - \frac{b_{11}}{x_2x_3}m_fc_{23} + \left(\frac{b_{12}}{x_1x_3} + \frac{x_2b_{22}}{x_1^2x_3}\right)m_fc_{13} - \left(\frac{b_{13}}{x_1x_2} + \frac{b_{23}}{x_1^2} + \frac{x_3b_{33}}{x_1^2x_2}\right)m_fc_{12}\end{aligned}$$

and ($a_{ij} = a_{ji}$ etc.)

$$\begin{aligned}\varphi_2^{(1)} = & \frac{a_{22}}{x_3x_1}m_gc_{31} - \left(\frac{a_{23}}{x_2x_1} + \frac{x_3a_{33}}{x_2^2x_1}\right)m_gc_{21} + \left(\frac{a_{21}}{x_2x_3} + \frac{a_{31}}{x_2^2} + \frac{x_1a_{11}}{x_2^2x_3}\right)m_gc_{23} \\ & - \frac{b_{22}}{x_3x_1}m_fc_{31} + \left(\frac{b_{23}}{x_2x_1} + \frac{x_3b_{33}}{x_2^2x_1}\right)m_fc_{21} - \left(\frac{b_{21}}{x_2x_3} + \frac{b_{31}}{x_2^2} + \frac{x_1b_{11}}{x_2^2x_3}\right)m_fc_{23},\end{aligned}$$

and the coefficient of $1/(x_1x_2x_3^2)$ comes from

$$\begin{aligned}& \frac{a_{11}m_gc_{23} - b_{11}m_fc_{23}}{x_2x_3} \cdot \frac{a_{22}m_gc_{31} - b_{22}m_fc_{31}}{x_3x_1} \\ & + \frac{-a_{12}m_gc_{13} + b_{12}m_fc_{13}}{x_1x_3} \cdot \frac{a_{12}m_gc_{23} - b_{12}m_fc_{23}}{x_2x_3} \\ & + \frac{x_2(-a_{22}m_gc_{13} + b_{22}m_fc_{13})}{x_1^2x_3} \cdot \frac{x_1(a_{11}m_gc_{23} - b_{11}m_fc_{23})}{x_2^2x_3}.\end{aligned}$$

We calculate above product using

$$c_{23} \cdot c_{31} = c_{13} \cdot c_{23} = \frac{c_{123}}{6}, \quad m_f \cdot m_g = m_{f \wedge g}, \quad m_f^2 = m_g = 0$$

and anticommutativity (we have $m_fc_{23} = -c_{23}m_f$ etc. as all variables are of degree 1) then we can see each terms above contribute $\frac{\lambda_{11}^{22}}{6}, 0, \frac{\lambda_{11}^{22}}{6}$ times $m_{f \wedge g}c_{123}/(x_1x_2x_3^2)$ respectively. By summing up all those contributions for $t = 0, 1, 2$, we can see $c_{12}^3 = \lambda_{11}^{22}$, and we can recover rest of the Bhargava's multiplication table using similar argument.

2.5 Comparing hypercohomology ring and global function rings of n points

While our construction of the hypercohomology ring can be carried out for any complexes with entries consisting of direct sums of $\mathcal{O}(-t)$ ($t \in \mathbb{Z}_{>0}$) satisfying WTM conditions, complexes of our main concern in the context of ring parametrizations are those which become free resolutions for n distinct geometric points.

Suppose that a complex $M_\bullet$ has entries consisting of direct sums of $\mathcal{O}(-t)$ and has its maps given as integral polynomials of variables $a_1, a_2, \dots, a_l$. In the ring parametrization context, those variables are usually given as coordinates of an algebraic form used to construct $M_\bullet$. We can say that $M_\bullet$ has *shape of generic resolution* over $\mathbb{Z}$ if $M_\bullet$ considered as a complex over base of universal algebra $\mathbb{Z}[a_1, \dots, a_l]$ becomes minimal free resolution. If $M_\bullet$ has shape of generic resolution, then there is a Zariski open subset of $\mathbb{Z}[a_1, \dots, a_l]$ such that $M_\bullet$ is exact over $\mathbb{Q}$, because one can consider intersection of all open subsets such that the rank of the each complex map of $M_\bullet$ is set to be maximum possible. It does not hold, however, that the subset of $\mathbb{Z}[a_1, \dots, a_l]$ such that $M_\bullet$ is exact over $\mathbb{Z}$ is Zariski open. For the quartic ring parametrization example using two ternary quadrics f, g , the Koszul complex $\mathcal{K}_{f,g}$ is exact over $\mathbb{Z}$ if and only if f and g have gcd 1 over integers, and the set of such (f, g) is not Zariski open.

In this section we will compare the hypercohomology ring and the global function ring, and show that nondegeneracy of rank n ring $H^0 R\Gamma(M_\bullet)$ is sufficient condition for $M_\bullet$ to be exact over $\mathbb{Q}$ for certain special classes of $M_\bullet$ having shape of generic resolution.

Let $M_\bullet$ be a complex such that $M_0 = \mathcal{O}$, and let $\mathcal{I}$ be image of $M_1 \rightarrow M_0$. One

can consider a complex map $M_\bullet \rightarrow \mathcal{O}/\mathcal{I}$ given as follows:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_2 & \longrightarrow & M_1 & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}/\mathcal{I} \end{array}$$

One can check this map preserves DG-algebra structure, so it induces an algebra map of hypercohomology $H^0 R\Gamma(M_\bullet) \rightarrow \Gamma(\mathcal{O}/\mathcal{I})$. If we identify $H^0 R\Gamma(M_\bullet)$ with total cohomology for $\Gamma(M_\bullet \otimes C_\bullet)$, then this map can be considered as the map induced naturally from

$$\Gamma(M_\bullet \otimes C_\bullet) \rightarrow \Gamma((\mathcal{O}/\mathcal{I}) \otimes C_\bullet)$$

so an element $\varphi \in H^0 R\Gamma(M_\bullet)$, represented as element in total cohomology

$$\varphi = \sum_i \varphi^{(i)}, \varphi_i \in \Gamma(M_i \otimes C_i)$$

is mapped to $\overline{\varphi^{(0)}}$, the cohomology representative of $\varphi^{(0)} \in \Gamma(M_0 \otimes C_0)$ in $\Gamma(\mathcal{O}/\mathcal{I})$. This corresponds to taking the part of φ in $\oplus_i \mathcal{O}_{x_i}$, which can be considered as a explicit expression of global function.

If $M_\bullet$ is exact over $\mathbb{Z}$, then the map $M_\bullet \rightarrow \mathcal{O}/\mathcal{I}$ preserves all cohomology and becomes quasiisomorphism. In this case it induces isomorphism on hypercohomology so $H^0 R\Gamma(M_\bullet) \simeq \Gamma(\mathcal{O}/\mathcal{I})$. We discuss the case when the complex is only exact over $\mathbb{Q}$ ($M_\bullet \otimes \mathbb{Q}$ exact) and show how we can compare global function ring with hypercohomology in this case.

Theorem 2.5.1. *Let $M_\bullet$ be a complex satisfying WTM conditions with $H^0 R\Gamma(M_\bullet) \simeq \mathbb{Z}^n$, and suppose $M_\bullet \otimes \mathbb{Q}$ is exact. Then the map $H^0 R\Gamma(M_\bullet) \rightarrow \Gamma(\mathcal{O}/\mathcal{I})$ is injective map between rank n rings. If $M_\bullet$ is exact over integers, then this map is isomorphism.*

Proof. We have shown that hypercohomology construction commutes with base change when WTM conditions is satisfied. Meanwhile, $M_\bullet \otimes \mathbb{Q}$ is exact, so the map $H^0 R\Gamma(M_\bullet) \rightarrow$

$\Gamma(\mathcal{O}/\mathcal{I})$ is isomorphism when tensored with $\otimes \mathbb{Q}$. This shows that $H^0 R\Gamma(M_\bullet) \rightarrow \Gamma(\mathcal{O}/\mathcal{I})$ is map of $\mathbb{Z}$ -modules of rank n which is injective. $\square$

The index of inclusion $H^0 R\Gamma(M_\bullet) \hookrightarrow \Gamma(\mathcal{O}/\mathcal{I})$ can be explicitly calculated if free resolution of $\mathcal{O}/\mathcal{I}$ over $\mathbb{Z}$ is given. Let $N_\bullet$ be a free resolution of $\mathcal{O}/\mathcal{I}$. Then one can find the map $M_\bullet \rightarrow N_\bullet$. As $\Gamma(\mathcal{O}/\mathcal{I}) \simeq H^0 R\Gamma(N_\bullet)$ and we know that it is rank n ring, we can consider its $\mathbb{Z}$ -basis and construct corresponding elements in total cohomology of $\Gamma(N_\bullet \otimes C_\bullet)$. Meanwhile, the basis element of $H^0 R\Gamma(M_\bullet)$ can be described as total cohomology element of $\Gamma(M_\bullet \otimes C_\bullet)$, so we can consider its image inside $\Gamma(N_\bullet \otimes C_\bullet)$. The index $|\Gamma(\mathcal{O}/\mathcal{I}) : H^0 R\Gamma(M_\bullet)|$ becomes the index of the $\mathbb{Z}$ -module by those elements in $\Gamma(N_\bullet \otimes C_\bullet)$.

Example. We consider the example of quartic ring associated to pair of ternary quadrics f, g . If f, g has no common components and has gcd 1, then $\mathcal{K}_{f,g}$ is exact and hypercohomology becomes identical to global functions. If f, g has integral gcd $d > 1$, then $\mathcal{K}_{f,g}$ is exact over $\mathbb{Q}$ but not over $\mathbb{Z}$. In this case, the free resolution for $\mathcal{O}/\mathcal{I} = \mathcal{O}/(f, g)$ is given as follows: ($f = df_0, g = dg_0$)

$$\mathcal{K}' : \mathcal{O}(-4) \xrightarrow{(-g_0, f_0)^t} \mathcal{O}(-2)^2 \xrightarrow{(f, g)} \mathcal{O}.$$

The map from $\mathcal{K}_{f,g}$ to this complex should be given that $\mathcal{O}(-4) \rightarrow \mathcal{O}(-4)$ is multiplication by d . As both $\mathcal{K}_{f,g}$ and $\mathcal{K}'$ satisfies the WTM conditions, we can also consider the exact sequence

$$\begin{aligned} 0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(\mathcal{K}_{f,g}) \rightarrow H^2(\mathcal{O}(-4)) \rightarrow 0 \\ 0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(\mathcal{K}') \rightarrow H^2(\mathcal{O}(-4)) \rightarrow 0. \end{aligned}$$

By comparing two sequences, we can see the map from $H^0 R\Gamma(\mathcal{K}_{f,g})$ to $H^0 R\Gamma(\mathcal{K}')$ acts as multiplication by d on basis elements except 1. This shows that the index

$|H^0 R\Gamma(\mathcal{K}_{f,g}) : H^0 R\Gamma(\mathcal{K}')|$ is d^3 .

How can we characterize conditions for $M_\bullet$ to be exact? In the context of ring parametrizations, we usually consider a family of complexes having the shape of generic resolution of n points in projective space. If the complex becomes a resolution of n geometric points over $\mathbb{Q}$, then $H^0 R\Gamma(M_\bullet) \otimes \mathbb{Q}$ is isomorphic to global functions on n points, so it should be nondegenerate rank n algebra over $\mathbb{Q}$. We would like to show converse statement that nondegeneracy of $H^0 R\Gamma(M_\bullet)$ implies exactness of $M_\bullet$ for $M_\bullet$ having certain specific shape of generic resolution, including n points in $\mathbb{P}^{n-2}$ in linearly generic position (no $n-1$ points on a hyperplane). The following result is first preliminary step in showing it.

Theorem 2.5.2. *Let $M_\bullet$ satisfy the WTM conditions, and assume that its hypercohomology ring $H^0 R\Gamma(M_\bullet)$ is nondegenerate. Then the map $H^0 R\Gamma(M_\bullet) \rightarrow \Gamma(\mathcal{O}/\mathcal{I})$ is injective.*

Proof.

Lemma. *A rank n ring is nondegenerate if and only if it does not contain nonzero nilpotent element.*

Proof. ($\Leftarrow$) If a ring R is degenerate, then there exists an element $x \in R$ such that $\text{Tr}(xy) = 0$. In particular $\text{Tr}(x^k) = 0$ for all k , so the matrix m_x representing multiplication $r \mapsto rx$ satisfies $\text{Tr}(m_x^k) = 0$. This implies that all eigenvalues of m_x should be zero so m_x should be nilpotent. As $m_x^n = 0$ maps 1 to x^n , $x^n = 0$.

($\Rightarrow$) Suppose x is a nilpotent element. If $z \in R$ is nilpotent element, then $\text{Tr}(z) = \text{Tr}(m_z) = 0$. Considering this for nilpotent element xy leads to that $\text{Tr}(xy) = 0$ for all $y \in R$. □

Suppose that map $H^0 R\Gamma(M_\bullet) \rightarrow \Gamma(\mathcal{O}/\mathcal{I})$ is not injective. When we express hypercohomology elements using the Čech complex, an element $\varphi \in H^0 R\Gamma(M_\bullet)$ which

can be expressed as

$$\varphi = \sum \varphi^{(t)}, \quad \varphi^{(t)} \in \Gamma(M_t \otimes C_t)$$

is mapped to $\varphi^{(0)}$ in $\Gamma(\mathcal{O}/\mathcal{I})$. If φ is mapped to zero in $\Gamma(\mathcal{O}/\mathcal{I})$, then $\varphi^{(0)} \in C_0 = \oplus \mathcal{O}_i$ consists of functions in $\mathcal{O}_i$ which is in $\mathcal{O}_i \otimes \mathcal{I}$. Thus $\varphi^{(0)}$ is in image of $d_{\rightarrow}$ map of $M_1 \rightarrow \mathcal{O}$. Write $\varphi^{(0)} = d_{\rightarrow}(u)$ for $u \in (\Gamma(M_1 \otimes C_0))$, then we can consider $\varphi - (d_{\rightarrow} + d_{\downarrow})(u)$ as representative for φ in $H^0\Gamma(M_{\bullet})$, equivalent to letting $\varphi^{(0)} = 0$. Meanwhile, the multiplication on $\Gamma(M_t \otimes C_t)$ works as

$$(M_t \otimes C_t) \otimes (M_s \otimes C_s) \rightarrow M_{t+s} \otimes C_{t+s}.$$

By considering this degree, $\varphi^{(0)} = 0$ implies that components of φ^k in $M_0 \otimes C_0, \dots, M_{k-1} \otimes M_{k-1}$ are all zero. This shows that $\varphi^n = 0$ so $H^0 R\Gamma(M_{\bullet})$ is nilpotent. $\square$

2.5.1 Free resolutions of n points in $\mathbb{P}^{n-2}$

Theorem 2.5.2 shows that if $H^0 R\Gamma(M_{\bullet})$ is nondegenerate then one can construct rank n nondegenerate ring inside global function ring. Our argument will proceed as follows:

- Use Nullstellensatz to construct n geometric points using the image of rank n ring inside $\Gamma(\mathcal{O}/\mathcal{I})$.
- Show that those n constructed points satisfy certain generic property regarding their positions, by exploiting the explicit expression of global function.
- By comparing $M_{\bullet}$ and free resolution of n points constructed, conclude that $M_{\bullet}$ itself should be free resolution, and $V(\mathcal{I})$ should exactly consist of those n points.

In the rest of this section, we will prove those statements for n points in $\mathbb{P}^{n-2}$, and the generic property of n points associated is that points are in linearly general

position - no $n - 1$ points on a hyperplane. These arguments can be adapted to other cases when the shape for free resolution is completely known. For example, the ring parametrization model of Chapter 3 uses these properties for the case of six points in $\mathbb{P}^2$ not on a conic.

Proposition 2.5.3 (Minimal free resolution of n points in $\mathbb{P}^{n-2}$). *Let X be a set of distinct n geometric points in $\mathbb{P}^{n-2}$ in linearly general position: no $n - 1$ points are on a hyperplane. For $n \geq 4$, the minimal free resolution of X is of following form ($\mathcal{O} = \mathcal{O}_{\mathbb{P}^{n-2}}$)*

$$\mathcal{O}(-n) \rightarrow \mathcal{O}(-n+2)^{\beta_{n,n-2}} \rightarrow \cdots \rightarrow \mathcal{O}(-2)^{\beta_{n,2}} \rightarrow \mathcal{O}$$

and for $n = 3$ it is of form $\mathcal{O}(-3) \rightarrow \mathcal{O}$. Here the betti numbers $\beta_{n,k}$ are given as $\beta_{n,k} = n \binom{n-2}{k} - n \binom{n}{k+1}$.

See ([46], Theorem 138): it is stated that there are several potential proofs possible.

If $M_\bullet$ is minimal free resolution of n points in $\mathbb{P}^{n-2}$, then $H^0 R\Gamma(M_\bullet) \simeq \Gamma(\mathcal{O}/\mathcal{I}) = \Gamma(V(\mathcal{I}))$ is isomorphic to global function ring on n points, so it should be rank n nondegenerate ring. I claim that converse of this statement holds.

Theorem 2.5.4. *Let $M_\bullet$ be a complex of sheaves on $\mathbb{P}^{n-2}$ having Betti numbers of minimal free resolution of n points in $\mathbb{P}^{n-2}$ in linearly general position. If $\mathcal{R} = H^0 R\Gamma(M_\bullet)$ is rank n nondegenerate ring, then for $\mathcal{I}$ the image of $M_1 \rightarrow M_0$, $V(\mathcal{I})$ consists of n points in $\mathbb{P}^{n-2}$ in general position.*

Proof. Denote $\mathcal{I}$ as image of the map $M_1 \rightarrow M_0$ and denote $X = V(\mathcal{I})$. As $\mathcal{R}$ is nondegenerate, we can find n distinct embeddings $\rho_1, \dots, \rho_n : \mathcal{R} \rightarrow \overline{\mathbb{Q}}$. Note that $M_\bullet$ satisfies WTM conditions, and its Tschirnhausen module is isomorphic to $H^{n-2}(\mathcal{O}(-n))$. Let $\varphi_0 = 1 \in \mathcal{R}$, and denote φ_i as element of $\mathcal{R}$ whose image under

the map $\mathcal{R} \rightarrow H^{n-2}(\mathcal{O}(-n))$ in the sequence of Theorem 2.3.3

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{R} \rightarrow H^{n-2}(\mathcal{O}(-n)) \rightarrow 0$$

can be identified with $1/(x_1x_2 \cdots x_{n-1}x_i)$. As $\mathcal{R}$ injects inside $\Gamma(X)$, we can consider φ_i as a global function on X too.

We construct n geometric points on X using $\mathcal{R} \subset \Gamma(X)$. In particular, we want to show that for $j = 1, \dots, n$ there exists a geometric point p_j of X such that φ_i takes value $\rho_j(\varphi_i)$ at p_j . Consider kernel of ρ_j in $\mathcal{R} \otimes \overline{\mathbb{Q}}$. As $\mathcal{R} \otimes \overline{\mathbb{Q}} \subset \Gamma(X \otimes \overline{\mathbb{Q}})$, one can construct a maximal ideal I of $\Gamma(X \otimes \overline{\mathbb{Q}})$ containing kernel of ρ_j . By the weak Nullstellensatz, there exists a point p_j of $V(I)$. Then $\varphi_i - \rho_j(\varphi_i) \in I$ vanishes on p_j , so we have $\varphi_i(p_j) = \rho_j(\varphi_i)$ for all i where $\varphi_i(p_j)$ denotes the evaluation of φ_i at point p_j .

Denote $\varphi_{i,t}$ as a representative of global function φ_i in the open subset $D(x_t)$ of $\mathbb{P}_{\mathbb{Z}}^{n-2}$. If we identify φ_i with total cohomology element of $\Gamma(M_{\bullet} \otimes C_{\bullet})$ and consider representation $\varphi_i = \sum \varphi_i^{(t)}$ for $t \in \Gamma(M_t \otimes C_t)$, then $\varphi_{i,t}$ is the component of $\varphi_i^{(0)} \in \Gamma(C_0) = \oplus_t \Gamma(\mathcal{O}_t)$ in $\Gamma(\mathcal{O}_t)$.

We remind that $\varphi_i^{(t)}$ is constructed via taking map of $M_{\bullet}$ to $\varphi_i^{(t+1)}$ and lift the element with respect to map of Čech complex, for $t = n-1, \dots, 0$. For the case of φ_i , we started from $\varphi_i^{(n-1)} = 1/(x_1x_2 \cdots x_{n-1}x_i)$, and all components of $\varphi_i^{(t)}$ were obtained from multiplying by polynomials and splitting the image into monomials having specified denominators. This shows that all monomials occurring in components of $\varphi_i^{(t)}$ should be polynomial multiples of $1/(x_1x_2 \cdots x_{n-1}x_t)$. But we have $\Gamma(\mathcal{O}_t) \simeq \mathbb{Z}[x_1/x_t, \dots, x_{n-1}/x_t]$, so $\varphi_i^{(t)} \in \Gamma(\mathcal{O}_t)$ is of form

$$\varphi_{i,t} = \frac{(\text{polynomials in } x_1, \dots, x_{n-1})}{x_t^k}$$

for some k . From the condition of it being $1/(x_1x_2 \cdots x_{n-1}x_t)$ -multiple, it follows that

$k = 1$ if $i \neq t$, and we can write $\varphi_{i,t} = l_{i,t}/x_t$ for some linear forms $l_{i,t}$ in variables $x_1, \dots, x_{n-1}$.

We combine those two properties

- $\varphi_i(p_j) = \rho_j(\varphi_i)$ for all i, j, t
- $\varphi_{i,t} = l_{i,t}/x_t$ for $i \neq t$, $l_{i,t}$ linear forms

to show that points $p_1, \dots, p_n$ are in general position: no $n - 1$ of them on a hyperplane. Suppose on the contrary that $p_1, \dots, p_{n-1}$ are on a hyperplane determined by a linear form l . We consier

$$\tilde{\varphi}_i = \varphi_i - \varphi_i(p_n) \in \Gamma(X) \otimes \overline{\mathbb{Q}}$$

then $\tilde{\varphi}_i$ vanishes on p_n . Let

$$v_i = (\tilde{\varphi}_i(p_1), \tilde{\varphi}_i(p_2), \dots, \tilde{\varphi}_i(p_{n-1})).$$

Then

$$\begin{aligned} (-1)^{n-1} \det(v_1, \dots, v_{n-1}) &= \det \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \tilde{\varphi}_1(p_1) & \tilde{\varphi}_1(p_2) & \dots & \tilde{\varphi}_1(p_{n-1}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{n-1}(p_1) & \varphi_{n-1}(p_2) & \dots & \varphi_{n-1}(p_{n-1}) & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \varphi_1(p_1) & \varphi_1(p_2) & \dots & \varphi_1(p_{n-1}) & \varphi_1(p_n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{n-1}(p_1) & \varphi_{n-1}(p_2) & \dots & \varphi_{n-1}(p_{n-1}) & \varphi_{n-1}(p_n) \end{pmatrix} \\ &= \det (\rho_j(\varphi_{i-1}))_{i,j=1, \dots, n} \quad (\varphi_0 = 1) \\ &= \sqrt{\text{Disc}(\mathcal{R})} \neq 0 \end{aligned}$$

so $v_1, v_2, \dots, v_{n-1}$ are linearly independent.

Consider the following $n - 1$ functions on $D(x_1) \cap X$:

$$\tilde{\varphi}_2, \dots, \varphi_{n-1}, \frac{l}{x_1}.$$

By considering $\varphi_{i,t} = l_{i,t}/x_1$, we know all of those can be represented as (linear form)/ x_1 on X . If those $n - 1$ linear forms corresponding to those functions are linearly dependent, then there exists constants $c_1, \dots, c_{n-1}$ such that

$$c_1 \frac{l}{x_1} + c_2 \tilde{\varphi}_2 + \dots + c_{n-1} \varphi_{n-1} = 0.$$

Meanwhile, if we consider their evaluations at $p_1, \dots, p_{n-1}$, then l vanishes on all those points, so we have

$$(c_2 \tilde{\varphi}_2(p_k) + \dots + c_{n-1} \varphi_{n-1}(p_k))_{k=1, \dots, n-1} = c_2 v_2 + \dots + c_{n-1} v_{n-1} = 0.$$

As $v_1, \dots, v_{n-1}$ are linearly independent, we have $c_2 = c_3 = \dots = c_{n-1} = 0$, and this implies $c_1 = 0$ as $l \neq 0$. This is contradiction, so $n - 1$ linear forms $l, \tilde{\varphi}_2 x_1, \dots, \varphi_{n-1} x_1$ are linearly independent. In particular those span any linear forms, so one can find constants such that

$$1 = \frac{x_1}{x_1} = c_1 \frac{l}{x_1} + c_2 \tilde{\varphi}_2 + \dots + c_{n-1} \varphi_{n-1}.$$

If we consider their evaluations at $p_1, p_2, \dots, p_{n-1}$, we have

$$(1, 1, \dots, 1) = c_2 v_2 + \dots + c_{n-1} p_{n-1}.$$

Meanwhile, we have $v_1, \dots, v_{n-1}$ linearly independent so they form basis of $\overline{\mathbb{Q}}^{n-1}$ so

there exists unique $d_1, \dots, d_{n-1}$ such that

$$(1, 1, \dots, 1) = d_1 v_1 + d_2 v_2 + \dots + d_{n-1} p_{n-1}$$

and the above observations show $d_1 = 0$. Similarly by considering $\tilde{\varphi}_1, \tilde{\varphi}_3, \dots, \varphi_{n-1}, l/x_2$ and showing their linear independence, we obtain $d_2 = 0$. In this way one can show all d_i are zero, which leads to contradiction. Therefore we can conclude that $p_1, \dots, p_{n-1}$ are not on a hyperplane.

It finally remains to show that there are no additional points in X other than $p_1, \dots, p_n$. Suppose there is a point $q \in X$. We can consider a morphism $\mathcal{R} \rightarrow \overline{\mathbb{Q}}$ which is given as evaluation at q . As evaluation of 1 is 1, this morphism is nontrivial so it should be identical to one of ρ_i . This implies $\varphi_j(q) = \rho_i(\varphi_j) = \varphi_j(p_i)$ for all $j = 0, \dots, n-1$.

Suppose $q \neq p_i$. We remind that $\varphi_{j,1}$ is of form $l_{j,1}/x_1$ for some linear form $l_{j,1}$. The dimension of linear forms l such that l/x_1 takes same value at q and p_i is $n-2$. It follows that $n-1$ linear forms x_1 and $l_{2,1}, \dots, l_{n-1,1}$ are linearly dependent, so $\varphi_0, \varphi_2, \dots, \varphi_{n-1}$ are also linearly dependent, contrary to that they form basis of $\mathcal{R}$. Therefore we should have $q = p_i$, and X consists of exactly n points of $p_1, \dots, p_n$. $\square$

We will show that one can actually determine explicit coordinates of those n points in $\mathbb{P}^{n-2}$ if more information on the hypercohomology ring is given. Let R be a nondegenerate ring, and $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ be its $\mathbb{Z}$ -basis. Its *dual basis* $\alpha_0^*, \dots, \alpha_{n-1}^*$ consists of elements $\alpha_i^* \in R \otimes \mathbb{Q}$ satisfying that

$$\mathrm{tr}_{R/\mathbb{Z}}(\alpha_i \alpha_j^*) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

If $\rho_1, \dots, \rho_n : R \rightarrow \mathbb{C}$ are distinct embeddings of R into $\mathbb{C}$, then those dual basis

elements satisfy following two identities

$$\sum_{k=1}^n \rho_k(\alpha_i) \rho_k(\alpha_j^*) = \delta_{ij}, \quad \sum_{k=1}^n \rho_i(\alpha_k) \rho_j(\alpha_k^*) = \delta_{ij}.$$

The second one follows from the first considering that

$$(\rho_i(\alpha_j))_{i,j} (\rho_j(\alpha_i^*))_{i,j} = I \Rightarrow (\rho_j(\alpha_i^*))_{i,j} (\rho_i(\alpha_j))_{i,j} = I.$$

Proposition 2.5.5. *Let $\mathcal{C}$ be a complex of sheaves on $\mathbb{P}^{n-2}$ having form of resolution of n points in $\mathbb{P}^{n-2}$, and suppose $R = H^0 R\Gamma(\mathcal{C})$ is nondegenerate ring. Let $\rho_1, \dots, \rho_n$ be distinct embeddings $R \otimes \mathbb{Q} \rightarrow \mathbb{C}$. Let $\alpha_1, \dots, \alpha_{n-1}$ be elements of R which are preimages of $1/(x_1 \cdots x_{n-1} x_i)$ for $i = 1, \dots, n-1$ respectively under the map $R \rightarrow H^{n-2}(\mathcal{O}(-n))$ given in the Theorem 2.3.3. Then coordinates of the n points determined by $\mathcal{C}$ are given as*

$$p_i = [\rho_i(\alpha_1^*); \rho_i(\alpha_2^*); \cdots; \rho_i(\alpha_{n-1}^*)].$$

Here $\alpha_0^*, \dots, \alpha_{n-1}^* \in R \otimes \mathbb{Q}$ is dual basis of $1 = \alpha_0, \alpha_1, \dots, \alpha_{n-1}$.

Proof. Let $p_1, \dots, p_n$ be points such that evaluation of global functions at p_i is identical to the map $\rho_i : R \otimes \mathbb{Q} \rightarrow \mathbb{C}$. As $p_1, \dots, p_n$ are in linearly general position, there exists some affine coordinates satisfying

$$p_k = [y_{1,k}; y_{2,k}; \cdots; y_{n-1,k}], \quad \sum_{k=1}^n y_{i,k} = 0,$$

and such choice of $y_{i,k}$ is unique up to constant multiplication to all $y_{i,k}$.

For $i, j = 1, \dots, n$, denote $\varphi_{i,j}$ be a global function on $D(x_j)$ corresponding to $1/(x_1 x_2 \cdots x_{n-1} x_i) \in H^{n-2}(\mathcal{O}(-n))$. To put this more exactly, $\varphi_{i,j}$ is the component in $\Gamma(\mathcal{O}_{x_j}) \simeq \mathbb{Z}[x_1, \dots, x_{n-1}, x_j^{-1}]$ of a global function corresponding to $1/(x_1 x_2 \cdots x_{n-1} x_i) \in$

$H^{n-2}(\mathcal{O}(-n))$ via the map

$$H^0 R\Gamma(\mathcal{C}_{t,A}) \rightarrow \Gamma(\mathcal{O}/\mathcal{I}) \subset \bigoplus \Gamma(\mathcal{O}_{x_j}).$$

We know that for $i \neq j$, $\varphi_{i,j}$ is represented as a linear form in $x_1, \dots, x_{n-1}$ divided by x_j , and $\varphi_{i,i}$ is represented as a quadratic form $x_1, \dots, x_{n-1}$ divided by x_i^2 . So $\varphi_{i,j}x_j$ becomes a linear form in $x_1, \dots, x_{n-1}$, either when $i = j$ or $i \neq j$.

Claim. For $i \neq j$, $\varphi_{i,i}x_i - \varphi_{j,j}x_j$ is equivalent to a linear form modulo $\mathcal{I}$, where $\mathcal{I}$ is image of $\mathcal{O}(-2)^{b_{n,2}} \rightarrow \mathcal{O}$.

Proof. We identify α_i with a total cohomology element in $\Gamma(\mathcal{C} \otimes \mathcal{C}_{cech})$, and consider $\varphi_{i,j}$ as component of α_i at $\Gamma(\mathcal{O}_{x_j})$ in $\Gamma(\mathcal{O} \otimes \bigoplus_j \mathcal{O}_{x_j})$. Then $\alpha_i \cdot x_i$ can be identified as a total cohomology element of the following sequence

$$\mathcal{C} \otimes \mathcal{O}(1) : \mathcal{O}(-n+1) \rightarrow \mathcal{O}(-n+3)^{b_{n,n-2}} \rightarrow \dots \rightarrow \mathcal{O}(-1)^{b_{n,2}} \rightarrow \mathcal{O}(1)$$

if we regard x_i as a global section of $\mathcal{O}(1)$. Component of $\alpha_i x_i$ at $\Gamma(\mathcal{O}_{x_j}(1))$ is given as $\varphi_{i,j} \cdot x_j$.

Meanwhile, if we consider the hypercohomology spectral sequence $H^i(M_j) \Rightarrow H^{i+j} R\Gamma(M_\bullet)$ obtained from the above sequence, its E_2 page consisting of terms of the form $H^i(M_j)$ looks like

$$\begin{array}{ccccccc} 0 & & 0 & \cdots & 0 & H^0(\mathcal{O}(1)) & \\ 0 & & 0 & \cdots & 0 & & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ H^{n-2}(\mathcal{O}(-n+1)) & 0 & \cdots & 0 & & & 0 \end{array}$$

so the spectral sequence is stablized at E_2 , and its filtering at degree 0 becomes

equivalent to the following exact sequence

$$0 \rightarrow H^0(\mathcal{O}(1)) \rightarrow H^0 R\Gamma(\mathcal{C} \otimes \mathcal{O}(1)) \rightarrow H^{n-2}(\mathcal{O}(-n+1)) \rightarrow 0.$$

Now consider $\alpha_i x_i - \alpha_j x_j$ in $H^0 R\Gamma(\mathcal{C} \otimes \mathcal{O}(1))$ and its image in $H^{n-2}(\mathcal{O}(-n+1))$ with respect to the above map. If we identify hypercohomology with the total cohomology element $\Gamma(\mathcal{C} \otimes \mathcal{O}(1) \otimes \mathcal{C}_{cech})$, then the above map into $H^{n-2}(\mathcal{O}(-n+1))$ works by simply taking $\mathcal{O}(-n+1) \otimes \mathcal{O}_{x_1 \dots x_{n-1}}$ -component of total cohomology element. As $\mathcal{O}(-n)$ -component of α_i has $1/(x_1 \dots x_{n-1} x_i)$, $\alpha_i x_i - \alpha_j x_j$ is mapped to zero, so it should come from global linear form $H^0(\mathcal{O}(1))$. This means that there exists a linear form l such that $\alpha_i x_i - \alpha_j x_j - l$ is zero in total cohomology. In particular, if we consider the component at $\Gamma(\mathcal{O}_{x_j}(1))$, it implies

$$\varphi_{i,j} x_i - \varphi_{j,j} x_j - l \equiv 0 \pmod{\mathcal{I}}.$$

As we know that $\varphi_{i,j} \equiv \varphi_{i,i} \pmod{\mathcal{I}}$, we have the desired conclusion

$$\varphi_{i,i} x_i - \varphi_{j,j} x_j \equiv l \pmod{\mathcal{I}}.$$

□

We substitute the coordinate of p_k as $x_i = y_{i,k}$, and add by all $k = 1, \dots, n$. By the above assumption that $\sum_k y_{i,k} = 0$, applying this for any linear form gives zero. Additionally noting that

$$\varphi_{i,j}|_{p_k} = \rho_k(\alpha_i)$$

we have ($i \neq j$)

$$0 = \sum_k (\varphi_{i,j} x_j)|_{p_k} = \sum_k \rho_k(\alpha_i) y_{j,k}$$

and

$$0 = \sum_k (\varphi_{i,i}x_i - \varphi_{j,j}x_j)|_{p_k} = \sum_k \rho_k(\alpha_i)y_{i,k} - \sum_k \rho_k(\alpha_j)y_{j,k}.$$

Therefore we have

$$\sum_{k=1}^n \rho_k(\alpha_i)y_{j,k} = C \cdot \delta_{ij}$$

for some C . By comparing it with

$$\sum_{k=1}^n \rho_k(\alpha_i)\rho_k(\alpha_j^*) = \delta_{ij},$$

we conclude that $y_{j,k}$ is C times $\rho_j(\alpha_k^*)$, showing that

$$p_k = [y_{1,k}; y_{2,k}; \cdots; y_{n-1,k}] = [\rho_k(\alpha_1^*); \rho_k(\alpha_2^*); \cdots; \rho_k(\alpha_{n-1}^*)];$$

□

2.6 Generalizations to arbitrary base and other spaces

So far we considered complexes of sheaves on $\mathbb{P}_{\mathbb{Z}}^m$ consisting of direct sums of $\mathcal{O}(-k)$, and considered structure of their hypercohomologies $H^0 R\Gamma(M_{\bullet})$. The first motivation for the hypercohomology construction was to generalize this construction for $\mathcal{O}_S$ -algebras over base scheme S , and it served this purpose successfully in [47] and [49] (precursor works including [36] and [16] can be also said to be in this spirit.) We discuss how some of our above results can be generalized in this generalized setting.

Meanwhile, one might discuss degree n covers coming from bundles on spaces other than $\mathbb{P}^k$: for example, the ring parametrization model in the Chapter 4 will consider some six points in $\mathbb{P}^1 \times \mathbb{P}^1$. One can consider hypercohomology of sheaves on space X for such situations, if proper tools for calculating cohomology on X is provided. We

will discuss how can construct a analogous complex to the Čech complex for a space other than $\mathbb{P}^k$, in particular for $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$.

2.6.1 Hypercohomology over base scheme

Let S be a scheme, and V be a locally rank n $\mathcal{O}_S$ -module. We now consider a complex of sheaves on $\mathbb{P}(V)$, and its hypercohomology $H^i R\pi_*$ for structure morphism $\pi : \mathbb{P}(V) \rightarrow S$. Note that for $S = \text{Spec } \mathbb{Z}$ we have $R\pi_* = \Gamma$.

We consider complex of sheaves having similar form, now over base S . On $\mathbb{P}(V)$, we can still think of sheaves $\mathcal{O}(k) = \mathcal{O}_{\mathbb{P}(V)}(k)$, and it also holds that

$$H^i R\pi_* \mathcal{O}(k) = \begin{cases} \text{Sym}^k V & \text{if } i = 0, k \geq 0 \\ (\text{Sym}^{-k-n} V)^* \otimes (\wedge^n V)^* & \text{if } i = n - 1, k \leq -n \\ 0 & \text{otherwise} \end{cases}$$

By applying the hypercohomology spectral sequence $R^j \pi_*(M_i) \Rightarrow H^{i+j} R^j \Gamma(M_\bullet)$ to complexes of same form in Theorem 2.3.3, one can show the almost identical result.

Theorem 2.6.1 (Theorem 2.3.3 over arbitrary base). *Let $M_\bullet$ be a complex of sheaves on $\mathbb{P}(V)$ having form*

$$M_\bullet : 0 \rightarrow M_{-k} \rightarrow M_{-(k-1)} \rightarrow \cdots \rightarrow M_0 \rightarrow 0$$

satisfying that

- $M_0 = \mathcal{O}$,
- each M_i is direct sum of $\mathcal{O}(-t)$ ($t \in \mathbb{Z}_{>0}$),

- *The cohomology of the sequence*

$$0 \rightarrow R^k \pi_*(M_{-k}) \rightarrow R^k \pi_*(M_{-k+1}) \rightarrow \cdots \rightarrow R^k \pi_*(M_0) \rightarrow 0$$

vanishes at all places except the first place $H^k(M_{-k})$. Cohomology module T at first place becomes isomorphic to $\mathcal{O}_S^{n-1}$.

Then we have an exact sequence of $\mathcal{O}_S$ -modules

$$0 \rightarrow \mathcal{O}_S \rightarrow H^0 R\pi_*(M_\bullet) \rightarrow T \rightarrow 0.$$

Also $H^l R\pi_*(M_\bullet) = 0$ for $l > 0$.

This generalizes the procedures of [47] and [49]. Also, flat cohomology implies cohomology functor commuting with base change, so we have the following result too.

Corollary 2.6.2. *The hypercohomology $H^0 R\pi_*(M_\bullet)$ commutes with base change of S : for $S' \rightarrow S$,*

$$H^0 R(\pi_{S'})_*(M_\bullet \otimes_{\mathcal{O}_{\mathbb{P}(V)}} \mathcal{O}_{\mathbb{P}(V \otimes_S S')}) \simeq (H^0 R\pi_*(M_\bullet)) \otimes_{\mathcal{O}_S} \mathcal{O}'_{S'}.$$

(Here $(\pi_{S'})'_*$ denotes image of π under base change functor $\otimes_S S'$.)

Also a DG-algebra structure on $M_\bullet$ induces a $\mathcal{O}_S$ -algebra structure on $H^0 R\pi_*(M_\bullet)$, and this algebra structure also commutes with corresponding base change. This base change property also suggests that the Čech complex can be used in computing the hypercohomology for nice complexes when the base is affine. Here the Čech complex on $\mathbb{P}_S^k$ can be defined as $C_\bullet(\mathbb{P}_S^k) = C_\bullet(\mathbb{P}_{\mathbb{Z}}^k) \otimes_{\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^k}} \mathcal{O}_{\mathbb{P}_S^k}$.

Corollary 2.6.3 (Theorem 2.3.7 over affine base). *Let S be affine, and suppose that $M_\bullet$ is a sequence of sheaves on $\mathbb{P}_S^k$ whose summands are all direct sums of $\mathcal{O}(t)$.*

Hypercohomology of $H^i R\Gamma(M_\bullet)$ is isomorphic to the i th total cohomology of the double complex $\pi_*(M_\bullet \otimes C_\bullet(\mathbb{P}_S^k))$ of S -modules.

2.6.2 Hypercohomology over other spaces

It is also possible to apply the hypercohomology construction to complexes of sheaves on a space X other than $\mathbb{P}^k$. For a complex of sheaves $M_\bullet$ over a $\mathbb{Z}$ -scheme X , we can consider the following analogue of Theorem 2.3.3 can be established under rather mild condition. The same proof using the hypercohomology spectral sequence $H^j(M_i) \Rightarrow H^{i+j} R\Gamma(M_\bullet)$ can be applied.

Proposition 2.6.4. *Let $M_\bullet$ be a complex of sheaves on X*

$$M_\bullet : 0 \rightarrow M_{-k} \rightarrow \cdots \rightarrow M_0 \rightarrow 0.$$

Suppose that $H^i(M_j)$ vanishes for all (i, j) except $(i, j) = (0, 0)$ and $(i, j) = (k, -k)$ and $(i, j) = (k, -k + 1)$. Then we have the exact sequence of $\mathbb{Z}$ -modules

$$0 \rightarrow H^0(M_0) \rightarrow H^0 R\Gamma(M_\bullet) \rightarrow T \rightarrow 0$$

where T is kernel of $H^k(M_{-k}) \rightarrow H^k(M_{-k+1})$.

We also want to establish an analogue of the explicit construction given in Theorem 2.3.7, which used the Čech complex for $\mathbb{P}^k$. Let $V_1, V_2, \dots, V_k$ be affine subsets covering X , and denote $V_I = \cap_{i \in I} V_i$ for $I \subset \{1, \dots, k\}$. Denote $\mathcal{O}_{V_I}$ as the pushforward of structure sheaf of V_I by the inclusion $V_I \rightarrow X$.

We can define **Čech complex for the cover** $(V_1, \dots, V_k)$ as the complex

$$C_\bullet(X; (V_i)_i) : 0 \rightarrow \bigoplus_{|I|=1} \mathcal{O}_{V_I} \rightarrow \bigoplus_{|I|=2} \mathcal{O}_{V_I} \rightarrow \cdots \bigoplus_{|I|=k} \mathcal{O}_{V_I} \rightarrow 0.$$

The maps $\mathcal{O}_{V_I} \rightarrow \mathcal{O}_{V_J}$ follows the similar rule as Čech complex above. (If $J = I \cup \{j\}$ and j is k th element of then it is $(-1)^{k-1}$ times the restriction map, and zero otherwise.) It is also possible to give a DG-algebra structure on $C_\bullet(X; (V_i)_i)$ which follows the same formula as the one in Proposition 2.3.5.

Theorem 2.6.5. (1) (Generalization of Theorem 2.3.7) Suppose that a complex $C_\bullet$ of sheaves on X satisfies that $H^i\Gamma(M \otimes C_\bullet) \simeq H^i(M)$ for any quasicoherent sheaf M . Then for any complex $M_\bullet$, the hypercohomology $H^i R\Gamma(M_\bullet)$ is isomorphic to $H^i \text{Tot}\Gamma(M_\bullet \otimes C_\bullet)$.

(2) (Generalization of Theorem 2.4.2) Assume that $C_\bullet$ has a DG-algebra structure such that there exists a map $\mathcal{O} \rightarrow C_\bullet$ which becomes unit. Also assume that all entries $M_i \otimes C_j$ of $M_\bullet \otimes C_\bullet$ are Γ -acyclic. Then for $M_\bullet$ having DG-algebra structure, the multiplication structure naturally induced on $H^i R\Gamma(M_\bullet)$ coincides with multiplication structure on $H^0 \text{Tot}\Gamma(M_\bullet \otimes C_\bullet)$ induced by total tensor product.

Proof for both parts are carried out in identical way.

Proposition 2.6.6. The Čech complex $C_\bullet(X; (V_i)_i)$ with DG-algebra structure defined in the Proposition 2.4.5 satisfies all assumptions of above theorem for any $M_\bullet$ consisting of quasicoherent sheaves.

Proof. Note that $M \otimes \mathcal{O}_{V_I}$ is isomorphic to restriction of M on the affine subset V_I . As quasicoherent sheaf on affine subset has zero cohomology, $M \otimes \mathcal{O}_{V_I}$ is Γ -acyclic. Also the complex $\Gamma(M \otimes C_\bullet)$ has form

$$\Gamma(M \otimes C_\bullet) : 0 \rightarrow \bigoplus_{|I|=1} \Gamma(V_I, M) \rightarrow \bigoplus_{|I|=2} \Gamma(V_I, M) \rightarrow \cdots \bigoplus_{|I|=k} \Gamma(V_I, M) \rightarrow 0$$

so its i th cohomology becomes isomorphic to the Čech cohomology $H^i(M)$. Properties of the DG-algebra structure on $C_\bullet(X; (V_i)_i)$ can be shown by following the identical proof to the Proposition 2.3.5. $\square$

For the case of space $X_1 \times X_2$, one can use tensor product of two Čech complexes instead. This result will be extensively used in Chapter 4 for the case of $X = \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 2.6.7. *Suppose $C_\bullet^{(1)} = C_\bullet(X_1; (V_i)_i)$ and $C_\bullet^{(2)} = C_\bullet(X_2; (W_i)_i)$ are Čech complexes for open covers (V_i) and (W_i) of X_1 and X_2 respectively. Suppose that $M_\bullet$ on $X_1 \times X_2$ consists of direct sums of form $M \boxtimes N$ for M, N sheaves on X_1, X_2 respectively. Then the complex $\text{Tot}(C_\bullet^{(1)} \boxtimes C_\bullet^{(2)})$ of sheaves on $X_1 \times X_2$ satisfies all assumptions of above theorem for $M_\bullet$.*

Proof. By the above proposition we have

$$H^i(X, M) \simeq H^i \text{Tot} \Gamma(M \otimes C_\bullet^{(1)}), \quad H^i(Y, N) \simeq H^i \text{Tot} \Gamma(N \otimes C_\bullet^{(2)}).$$

According to the Künneth formula

$$H^m(X \times Y, M \boxtimes N) \simeq \bigoplus_{p+q=m} H^p(X, M) \otimes H^q(Y, N),$$

we can deduce

$$\begin{aligned} & \bigoplus_{p+q=m} H^p(X, M) \otimes H^q(Y, N) \\ & \simeq \bigoplus_{p+q=m} H^p \text{Tot} \Gamma(X, M \otimes C_\bullet^{(1)}) \otimes H^q \text{Tot} \Gamma(Y, N \otimes C_\bullet^{(2)}) \\ & \simeq H^m \Gamma(X \times Y, \text{Tot}(M \boxtimes N) \otimes \text{Tot}(C_\bullet^{(1)} \boxtimes C_\bullet^{(2)})) \end{aligned}$$

as outer tensor product commutes with tensor product $((M_1 \otimes M_2) \boxtimes (N_1 \otimes N_2) \simeq (M_1 \boxtimes N_1) \otimes (M_2 \boxtimes N_2))$. Also the DG-algebra structures on $C_\bullet^{(1)}$ and $C_\bullet^{(2)}$ induces multiplicative structure on $\text{Tot}(C_\bullet^{(1)} \boxtimes C_\bullet^{(2)})$, and $\mathcal{O} \rightarrow \text{Tot}(C_\bullet^{(1)} \boxtimes C_\bullet^{(2)})$ becomes unit. To check all entries of $M \otimes \text{Tot}(C_\bullet^{(1)} \boxtimes C_\bullet^{(2)})$ are Γ -acyclic, we note that its entries are direct sums of $M \otimes (\mathcal{O}_V \boxtimes \mathcal{O}_W)$ for open affine subsets $V \subset X$ and $W \subset Y$. As $\mathcal{O}_V \boxtimes \mathcal{O}_W \simeq \mathcal{O}_{V \times W}$, $M \otimes (\mathcal{O}_V \boxtimes \mathcal{O}_W)$ are supported on affine subset $V \times W$ so should

be Γ -acyclic. □

We particularly consider the example of $X = \mathbb{P}^1 \times \mathbb{P}^1$. Let x_1, x_2 and y_1, y_2 be the projective coordinates of first and second $\mathbb{P}^1$ part respectively (so that X is isomorphic to $\text{Proj } \mathbb{Z}[x_1, x_2] \times \text{Proj } \mathbb{Z}[y_1, y_2]$). Denote $\mathcal{O}(m_1, m_2)$ as $\mathcal{O}(m_1) \boxtimes \mathcal{O}(m_2)$. For $m_1, m_2 \geq 0$ or $m_1, m_2 < 0$, its cohomologies can be calculated using the Künneth formula as

$$\begin{aligned} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(m_1, m_2)) &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_2)) \\ &= (\text{degree } (m_1, m_2) \text{ part of } \mathbb{Z}[x_1y_1, x_1y_2, x_2y_1, x_2y_2]), \end{aligned}$$

(degree m_1 in x_1, x_2 and degree m_2 in y_1, y_2)

$$\begin{aligned} H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(m_1, m_2)) \\ &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_1)) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_2)) \oplus H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_2)) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(m_1, m_2)) &= H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_1)) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_2)) \\ &= (\text{degree } (m_1, m_2) \text{ part of } \frac{1}{x_1x_2y_1y_2} \mathbb{Z}[x_1^{-1}, x_2^{-1}, y_1^{-1}, y_2^{-1}]). \end{aligned}$$

As in the above proposition, we can also calculate the cohomology using the the total tensor product of two Čech complexes $C_\bullet(\mathbb{P}^1; (D(x_1), D(x_2)))$ and $C_\bullet(\mathbb{P}^1; (D(y_1), D(y_2)))$,

which is given in the form of

$$\begin{aligned}
& 0 \\
& \rightarrow \mathcal{O}_{D(x_1y_1)} \oplus \mathcal{O}_{D(x_1y_2)} \oplus \mathcal{O}_{D(x_2y_1)} \oplus \mathcal{O}_{D(x_2y_2)} \\
& \rightarrow \mathcal{O}_{D(x_1y_1y_2)} \oplus \mathcal{O}_{D(x_2y_1y_2)} \oplus \mathcal{O}_{D(x_1x_2y_1)} \oplus \mathcal{O}|_{D(x_1x_2y_2)} \\
& \rightarrow \mathcal{O}_{D(x_1x_2y_1y_2)} \\
& \rightarrow 0.
\end{aligned}$$

When we take tensor product with $\mathcal{O}(m_1, m_2)$ and take global section functor, its H^2 can be identified with set of all degree (m_1, m_2) Laurent polynomials in $x_i y_j$ (corresponding to $\Gamma(\mathcal{O}_{D(x_1x_2y_1y_2)}(m_1, m_2))$ modded out by terms having positive degree on any of x_1, x_2, y_1, y_2 . This coincides with the above description for H^2 . We note that the above tensor product is slightly different from the Čech complex obtained from the cover $\{D(x_1y_1), D(x_1y_2), D(x_2y_1), D(x_2y_2)\}$ of $\mathbb{P}^1 \times \mathbb{P}^1$, but one can show those are chain homotopic anyway.

Chapter 3

Sextic rings associated to $3 \times 3 \times 5$ forms

3.1 Introduction

In this chapter, we suggest a method of constructing a sextic ring (an integral ring of rank 6) from a $3 \times 3 \times 5$ integral form $A \in V(Z) = \mathbb{Z}^3 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^5$, where (G, V) is a representation of an algebraic group given as $(G(\mathbb{Z}), V(\mathbb{Z})) = (\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_5(\mathbb{Z}), \mathbb{Z}^3 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^5)$.

Our construction of the sextic ring will provide explicit multiplication coefficients for the sextic ring. For $1 = \alpha_0, \alpha_1, \dots, \alpha_{n-1}$ the $\mathbb{Z}$ -basis of rank n ring R , its *multiplication coefficients* are defined as integers c_{ij}^k satisfying $\alpha_i \alpha_j = \sum_{k=0}^{n-1} c_{ij}^k \alpha_k$. For a form $A \in \mathbb{Z}^3 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^5$, multiplication coefficients $c_{ij}^k(A)$ of our ring $\mathcal{R}(A)$ will be given as polynomials in entries of A which behaves well under the natural action of G on A .

Theorem 3.1.1. *For a form $A \in \mathbb{Z}^3 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^5$, there exists a sextic ring $\mathcal{R}(A)$ with a $\mathbb{Z}$ -basis such that its multiplication coefficients $c_{ij}^k(A)$ are given as integral polynomials of entries of A .*

For an action of $(g_1, g_2, g_3) \in G(\mathbb{Z}) = \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_5(\mathbb{Z})$ on A , $c_{ij}^k(A)$ are invariant under $\mathrm{SL}_3 \times \mathrm{SL}_3$ -action, and $\mathcal{R}(g_3 A)$ is isomorphic to $\mathcal{R}(A)$ in the following way: if $(1 = \alpha_0, \alpha_1, \dots, \alpha_5)$ is basis of $\mathcal{R}(A)$, then $\mathcal{R}(g_3 A)$ is isomorphic to the ring having basis $1, \tilde{\alpha}_1, \dots, \tilde{\alpha}_5$ where $\tilde{\alpha}_i = \det(g_3) \sum_{j=1}^5 (g_3)_{ij} \alpha_j$.

This result suggests that the discriminant of the sextic ring $\mathcal{R}(A)$ is a polynomial in entries of A which becomes a relative invariant with respect to the action of $G = \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_5$. Denote $\mathrm{Disc}(A) = \mathrm{Disc}(\mathcal{R}(A))$ as this degree 60 invariant. We also note that this construction can be generalized over any base ring S : for $A \in V(S) = S^3 \otimes S^3 \otimes S^5$, the same algebraic formulas for c_{ij}^k can be used to define a sextic S -algebra $\mathcal{R}(A)$.

The construction of $\mathcal{R}(A)$ can be linked to six points in $\mathbb{P}^2$ determined by certain system of equations arising from A . For $A \in V(\bar{k})$ over an algebraically closed field $\bar{k}$ of characteristic zero, consider the scheme

$$\mathfrak{X}_A = \left\{ (x, y, t) : \mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \mathbb{P}_t^4 : \sum_{i=1}^5 t_i A_i = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix} \right\}$$

where we consider A as a 5-tuple $(A_1, A_2, A_3, A_4, A_5)$ of 3×3 matrices $A_i \in \mathbb{Z}^3 \otimes \mathbb{Z}^3$ and $x = [x_1; x_2; x_3], y = [y_1; y_2; y_3], t = [t_1; \dots; t_5]$ denote projective coordinates. Let $\mathfrak{X}_{x,A}, \mathfrak{X}_{y,A}, \mathfrak{X}_{t,A}$ be projections to $\mathfrak{X}_A$ onto $\mathbb{P}_x^2, \mathbb{P}_y^2, \mathbb{P}_t^4$ respectively. By considering classical geometry of the Segre embedding and its degree, we can expect that $\mathfrak{X}_A$ consists of six distinct points for generic A . We will show that the condition for $\mathfrak{X}_A$ to become six points is exactly given as $\mathrm{Disc}(A) \neq 0$, and this construction gives an interpretation of $3 \times 3 \times 5$ forms as moduli space for six points in $\mathbb{P}^2$.

Theorem 3.1.2. *Let $\bar{k}$ be an algebraically closed field of characteristic zero. For $A \in V(\bar{k})$, $\mathfrak{X}_A$ consists of distinct six points if and only if $\mathrm{Disc}(A) \neq 0$. For such A , $\mathfrak{X}_{t,A}$ consists of six points in linearly general position (no 5 on a hyperplane), and*

$\mathfrak{X}_{x,A}$ consists of distinct six points of $\mathbb{P}^2$ not on a conic. If we denote $V_{D \neq 0} \subset V$ as open subset of V satisfying $\text{Disc} \neq 0$, there is a one-to-one correspondence

$$V_{D \neq 0}(\bar{k})/G(\bar{k}) \leftrightarrow \left\{ \begin{array}{l} \text{six } \bar{k}\text{-points in } \mathbb{P}^2 \\ \text{not on a conic} \end{array} \right\} / \text{PGL}_3(\bar{k})$$

given by $\mathfrak{X}_{x,A}$.

Theorem 3.1.3. *Let k be a field of characteristic zero. For $A \in V_{D \neq 0}(k)$, $\mathfrak{X}_A$ is isomorphic to $\text{Spec}(\mathcal{R}(A))$.*

Finally, we would like to consider how orbits in $V_{D \neq 0}(\mathbb{Q})/G(\mathbb{Q})$ behave with respect to the construction of $\mathcal{R}(A)$. While I could not find the exact arithmetic object which exactly corresponds to $V_{D \neq 0}(\mathbb{Q})/G(\mathbb{Q})$, I could construct an invariant element s in the sextic resolvent algebra of $\mathcal{R}(A)$ which we will define later in this work, and show that the pair of $(R = \mathcal{R}(A), s \in S)$ can specify corresponding elements of $V_{D \neq 0}(\mathbb{Q})/G(\mathbb{Q})$.

Theorem 3.1.4. *Let R be a nondegenerate sextic $\mathbb{Q}$ -algebra, and S be its sextic resolvent algebra (with respect to outer automorphism of S_6). For $A \in V(\mathbb{Q})$ such that $\mathcal{R}(A) \simeq R$, one can construct $s \in S$ corresponding to A satisfying following two properties.*

(1) For $\psi_1, \psi_2, \dots, \psi_6 : S \rightarrow \mathbb{C}$ be the distinct embeddings, $[\psi_1(s); \dots; \psi_6(s)] \in \mathbb{P}_{\mathbb{C}}^5$ becomes image of Y_A under the map $W : (\mathbb{P}^2)^6 \rightarrow \mathbb{P}^5$ of relative invariants of six points in $\mathbb{P}^2$ (defined in Section 2.3).

(2) We have

$$16\text{tr}_{S/\mathbb{Q}}(s^4) - 4\text{tr}_{S/\mathbb{Q}}(s^2)^2 = \text{Disc}(A).$$

For such A , the mapping $A \mapsto s$ is exactly 2-to-1.

The construction of the sextic ring $\mathcal{R}(A)$ uses the method of the hypercohomology construction, which are outlined in detail in the Chapter 2 of this thesis. As in [47] and

[49], we will consider complexes of sheaves on some projective space with DG-algebra structure, and show that their hypercohomology rings with (its multiplication induced naturally from DG-algebra structure) turn out to be rank 6 ring. We will consider two complexes $\mathcal{C}_{x,A}$ and $\mathcal{C}_{t,A}$, which become minimal free resolution for $\mathfrak{X}_{x,A} \subset \mathbb{P}^2$ and $\mathfrak{X}_{t,A} \subset \mathbb{P}^4$ respectively for generic A . It follows that $H^0 R\Gamma(\mathcal{C}_{x,A})$ and $H^0 R\Gamma(\mathcal{C}_{t,A})$ are isomorphic sextic rings, so we can call both of them $\mathcal{R}(A)$.

Construction of $\mathcal{C}_{x,A}$ and $\mathcal{C}_{t,A}$ relies heavily upon the geometry of $\mathfrak{X}_A$, which can be studied in various perspectives. One possible approach is to consider $\mathfrak{X}_A$ as a determinantal variety. By eliminating variables x, y in $\mathfrak{X}_A$, we obtain

$$\mathfrak{X}_{t,A} = \{t \in \mathbb{P}^4 : \text{rank}(t_1 A_1 + \cdots + t_5 A_5) \leq 1\}$$

which can be interpreted as intersection of determinantal variety consisting of 3×3 matrices which has rank 1 and the hyperplane spanned by $A_1, \dots, A_5$. Lascoux in [34] exhibited minimal free resolution for this degree 6 variety and named it ‘Scandinavian’, and Casnati in [14] generalized this construction to give degree 6 covers of general integral scheme. The construction of $\mathcal{C}_{t,A}$ is also suggested, though it does not give the explicit formula.

Meanwhile, one can consider $3 \times 3 \times 4$ forms instead via the castling transformation, which establishes equivalence between generic open orbits of $3 \times 3 \times 5$ forms and $3 \times 3 \times 4$ forms. The equivalence classes of $3 \times 3 \times 4$ forms over algebraically closed field of characteristic zero were completely characterized by Ng in [38] using the property of certain determinantal varieties. In this process, one can show a set of six points in $\mathbb{P}^2$ not on a conic corresponds to orbit of $3 \times 3 \times 5$ forms over algebraically closed field of $\text{char}(k) \neq 0$.

We can also look for the invariants of six points in $\mathbb{P}^2$ and discuss how those invariants of $\mathfrak{X}_{x,A}$ and $\mathfrak{X}_{y,A}$ behave. The invariants of six points in $\mathbb{P}^2$ were fully discussed

in [23] and [33], and they can be described in polynomials of coordinates whose construction involves properties of an outer automorphism of S_6 . This appearance of outer automorphism indicates that those invariants of $\mathfrak{X}_{x,A}$ should relate to the sextic resolvent algebra of $\mathcal{R}(A)$ for outer automorphism. This perspective also suggests that $\mathfrak{X}_{x,A}$ and $\mathfrak{X}_{y,A}$ are in the Gale transform of each other (see [27] for the definition of Gale transform), and one can view $3 \times 3 \times 5$ forms as moduli space for two sets of six points in $\mathbb{P}^2$ which are in Gale transform.

Combining all of those geometric results to establish above main results on properties of $\mathcal{R}(A)$ will be the second original work of this chapter (with the first being explicit construction of $\mathcal{R}(A)$). We briefly describe the outline of this chapter as follows. In the Section 3.2 we will first review earlier works regarding geometry of $3 \times 3 \times 5$ forms and 6 points in $\mathbb{P}^2$, and formulate results which can be applied directly to $3 \times 3 \times 5$ forms. Those earlier results already suffice to prove some part of the Theorem 3.1.2 - correspondence of certain generic $3 \times 3 \times 5$ orbits and six points on $\mathbb{P}^2$. In the Section 3.3 we construct $\mathcal{R}(A)$ either by providing an explicit description of multiplication coefficients or by using hypercohomology construction. By using the properties of $\mathcal{R}(A)$ and hypercohomology of complex for six points in $\mathbb{P}^4$, we can show the open orbit corresponding to 6 points in $\mathbb{P}^2$ not on a conic exactly becomes $\text{Disc}(A) \neq 0$, completing the proof of Theorem 3.1.2. In the Section 3.4 we will explore what happens when $A \in V(\mathbb{Q})$, and construct an irrational invariant of $V(\mathbb{Q})/G(\mathbb{Q})$ which sits inside the sextic resolvent algebra of $\mathcal{R}(A)$. We will prove the Theorem 3.1.4 that this invariant suffices to provide sufficient information for characterizing orbit of $V(\mathbb{Q})/G(\mathbb{Q})$.

3.2 Geometry of $3 \times 3 \times 5$ forms

We review our notation briefly before we start. Throughout this chapter, denote (G, V) as the representation of $3 \times 3 \times 5$ forms of algebraic group $G = \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_5$. We represent an element $A \in V(\mathbb{Z})$ or $V(k)$ (k : field) as a 5-tuple $(A_1, A_2, A_3, A_4, A_5)$ of 3×3 matrices $A_i \in \mathbb{Z}^3 \otimes \mathbb{Z}^3$ (or $k^3 \otimes k^3$). For such A , define

$$\mathfrak{X}_A = \left\{ (x, y, t) : \mathbb{P}_x^2 \times \mathbb{P}_y^2 \times \mathbb{P}_t^4 : \sum_{i=1}^5 t_i A_i = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix} \right\}$$

and denote $\mathfrak{X}_{x,A}, \mathfrak{X}_{y,A}, \mathfrak{X}_{t,A}$ as its projections to $\mathbb{P}_x^2, \mathbb{P}_y^2, \mathbb{P}_t^4$ respectively. Under the following shorthand notation ($x = (x_1, x_2, x_3)^t$ and $y = (y_1, y_2, y_3)^t$)

$$A(t) = t_1 A_1 + \cdots + t_5 A_5, \quad xy^t = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix},$$

we can write

$$\mathfrak{X}_{x,A} = \{x \in \mathbb{P}^2 : \exists y, t \text{ s.t. } A(t) = xy^t\}$$

$$\mathfrak{X}_{y,A} = \{y \in \mathbb{P}^2 : \exists x, t \text{ s.t. } A(t) = xy^t\}$$

$$\mathfrak{X}_{t,A} = \{t \in \mathbb{P}^4 : \exists x, y \text{ s.t. } A(t) = xy^t\}.$$

All matrices which can be represented as xy^t for some x, y are exactly those having rank 1, and the projective subset of all such matrices $\mathbb{P}^8 \simeq \mathbb{P}(\mathrm{Mat}_{3 \times 3})$ can be interpreted as the image of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^8$. As this Segre embedding is of codimension 4 and degree 6, so intersection of its image and a generic hyperplane $\{A(t) : t \in k^5\}$ of projective dimension 4 becomes a set of six distinct

points. Thus for such case $\mathfrak{X}_{t,A}$ become six points in $\mathbb{P}^4$. As $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_5$ -action on A acts on $\mathfrak{X}_A, \mathfrak{X}_{x,A}, \mathfrak{X}_{y,A}, \mathfrak{X}_{t,A}$ correspondingly as induced projective isomorphism on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^4$ and its parts, properties of $\mathfrak{X}_A$ only depend on $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_5$ -equivalence class of A .

In this section, we would like to explore geometry of $\mathfrak{X}_{t,A}$ for $A \in V(\bar{k})$ where $\bar{k}$ is algebraically closed field of characteristic zero. We will prove the part of the Theorem 3.1.2 which does not involve properties of $\mathcal{R}(A)$, which would be equivalent to the following statement.

Proposition 3.2.1. *Let $\bar{k}$ be an algebraically closed field of characteristic zero, and let $A \in V(\bar{k})$. If $\mathfrak{X}_{x,A}$ consists of distinct six points, then following properties hold.*

- *Six points of $\mathfrak{X}_{t,A} \subset \mathbb{P}^4$ are in linearly general position (no 5 of them on a hyperplane).*
- *Six points of $\mathfrak{X}_{x,A} \subset \mathbb{P}^2$ and $\mathfrak{X}_{y,A} \subset \mathbb{P}^2$ are not on a conic.*

Denote $V_{n.d.}(\bar{k})$ as the subset of such $A \in V(\bar{k})$. Then there is a one-to-one correspondence

$$V_{n.d.}(\bar{k})/G(\bar{k}) \leftrightarrow \left\{ \begin{array}{l} \text{six } \bar{k}\text{-points in } \mathbb{P}^2 \\ \text{not on a conic} \end{array} \right\} / \mathrm{PGL}_3(\bar{k})$$

given by $A \mapsto \mathfrak{X}_{x,A}$.

The proof of the Theorem 3.1.2 will be completed after we construct an invariant $\mathrm{Disc}(A)$ as discriminant of $\mathcal{R}(A)$ and show that $V_{n.d.}$ is actually given as $V_{D \neq 0}$, the set of A such that $\mathrm{Disc}(A) \neq 0$.

For the proof of the Proposition 3.2.1, we will use the complete classification of all $3 \times 3 \times 4$ forms given in [38] up to $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_4$ -equivalence, and utilize a correspondence between certain generic $3 \times 3 \times 4$ forms and $3 \times 3 \times 5$ forms. While

this correspondence does not lead to the complete classification of all $3 \times 3 \times 5$ forms, it should suffice to classify orbits of $3 \times 3 \times 5$ forms in $V_{n.d.}$. After figuring out what $V_{n.d.}$ is, we can show the properties of $\mathfrak{X}_{x,A}$, $\mathfrak{X}_{y,A}$, $\mathfrak{X}_{t,A}$ using an explicit representative of A in $G(\bar{k})$ -orbit.

After its proof we will introduce the invariants of six points in $\mathbb{P}^2$ in [33] and show the correspondence involving these invariants as in Theorem 2.1, and suggest some explicit calculations of forms which will be used in Section 3.3. Throughout this section, all forms are considered over $\bar{k}$, an algebraically closed field of characteristic zero.

3.2.1 Classification of $3 \times 3 \times 4$ forms

We will discuss the Ng's classification results in [38] of $3 \times 3 \times 4$ forms over $\bar{k}$, and use this result to classify all $3 \times 3 \times 5$ forms such that $\mathfrak{X}_A$ becomes six distinct points.

First we note the correspondence between generic orbits of

$$\bar{k}^3 \otimes \bar{k}^3 \otimes \bar{k}^5 / \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_5(\bar{k})$$

and

$$\bar{k}^3 \otimes \bar{k}^3 \otimes \bar{k}^4 / \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_4(\bar{k}).$$

If we denote a $3 \times 3 \times 5$ form and a $3 \times 3 \times 4$ form as tuples of 3×3 matrices $A = (A_1, A_2, A_3, A_4, A_5)$ and $B = (B_1, B_2, B_3, B_4)$ respectively, then the correspondence between them can be described as follows. Consider the trace form $(M, N) \mapsto \mathrm{tr}(M^t N)$ for $M, N \in \mathrm{Mat}_{3 \times 3}(\bar{k})$. Then the forms A and B are in correspondence if and only if $\mathrm{span}(A_1, A_2, A_3, A_4, A_5)$ and $\mathrm{span}(B_1, B_2, B_3, B_4)$ are orthogonal complements to each other with respect to the trace form. As the trace form is invariant under the action of $\mathrm{SL}_3 \times \mathrm{SL}_3$ on $\mathrm{Mat}_{3 \times 3}(\bar{k})$, this correspondence of forms reduce to a correspondence between $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_5$ -equivalence classes of

$3 \times 3 \times 5$ forms and $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_4$ -equivalence classes of $3 \times 3 \times 4$ forms. It can be considered as an analogue for the castling transformation usually defined only on prehomogeneous vector spaces.

Say that a $3 \times 3 \times 4$ form B corresponds to a $3 \times 3 \times 5$ form A in above correspondence. In [38], one considers following determinantal varieties for B . (Notations used here are different from ones in [38]).

$$\mathfrak{X}_{w,B} = \{w \in \mathbb{P}^3 : \det(B_1 w_1 + B_2 w_2 + B_3 w_3 + B_4 w_4) = 0\}$$

$$\mathfrak{X}_{x,B} = \{y \in \mathbb{P}^2 : \mathrm{rank}(\mathrm{span}(x^t B_1, x^t B_2, x^t B_3, x^t B_4)) \leq 2\}$$

$$\mathfrak{X}_{y,B} = \{z \in \mathbb{P}^2 : \mathrm{rank}(\mathrm{span}(B_1 y, B_2 y, B_3 y, B_4 y)) \leq 2\}.$$

If A and B corresponds via the above castling-like correspondence, then we can see

$$\mathfrak{X}_{x,B} = \mathfrak{X}_{x,A}, \quad \mathfrak{X}_{y,B} = \mathfrak{X}_{y,A}$$

from following observations.

$$\begin{aligned} \exists t : xy^t = A(t) &\Leftrightarrow xy^t \in \mathrm{span}(B_1, B_2, B_3, B_4)^\perp \\ &\Leftrightarrow \forall i \, \mathrm{tr}(B_i(xy^t)^t) = 0 \\ &\Leftrightarrow (\forall i \, \mathrm{tr}(B_i y x^t) = x^t B_i y = 0) \end{aligned}$$

Both $\mathfrak{X}_{x,B}$ and $\mathfrak{X}_{x,A}$ are sets of x such that there exists y satisfying those equivalent conditions, so those should be identical.

In [38], a $3 \times 3 \times 4$ form B is called *smooth* if $\mathfrak{X}_{w,B}$ is smooth cubic surface. The set of all smooth forms is Zariski open in the space of all $3 \times 3 \times 4$ forms, so smooth forms can be considered as generic case. Following properties hold for smooth forms.

Proposition 3.2.2 ([38], Proposition 1 and Theorem 2). *Let B be a smooth $3 \times 3 \times 4$*

form.

1. $\mathfrak{X}_{x,B}$ and $\mathfrak{X}_{y,B}$ becomes distinct six points in $\mathbb{P}^2$ in general position: no three of them on a line, and no six of them on a conic.

$\mathfrak{X}_{x,B}$ and $\mathfrak{X}_{y,B}$ are associated as follows. Let M_x be the 3×6 matrix whose column vectors are coordinates of points of $\mathfrak{X}_{x,B}$, and define M_y analogously. Then the three columns of M_y^t form basis of null space of M_x . This relation determines $\mathfrak{X}_{y,B}$ uniquely up to PGL_3 -action. (Generally this transformation is called the Gale transform sending six points in $\mathbb{P}^2$ to six points in $\mathbb{P}^2$.)

2. There is one-to-one correspondence

$$\begin{aligned} & \{\text{smooth forms}\} // \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_4(k) \\ & \Leftrightarrow \mathrm{PGL}_3(k) \backslash \backslash \{\text{six points in } \mathbb{P}^2 \text{ in general position}\} // S_6 \end{aligned}$$

given as $B \leftrightarrow \mathfrak{X}_{x,B}$.

The first part is classical and was established in works including ([22], etc.). Regarding the second part, for any set of six points in general position

$$(p_1^t, p_2^t, \dots, p_6^t) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \lambda & \alpha \\ 0 & 0 & 1 & 1 & \mu & \beta \end{pmatrix},$$

one can explicitly construct a $3 \times 3 \times 4$ form such that $Y_{B,2}$ becomes above six points.

For non-smooth forms, [38] gives a complete classification of orbits of $3 \times 3 \times 4$ forms, using properties of $\mathfrak{X}_{w,B}$. Among all such cases, we list all cases where $\mathfrak{X}_{x,B}$

becomes distinct six points as follows. ([Ng, Theorem 3], cases #1, #3, #9, #20)

$$A_1(a) : (p_1^t, p_2^t, \dots, p_6^t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & \nu & -\lambda \\ 0 & 0 & 1 & -1 & -1 & \mu \end{pmatrix}$$

(We have $(\lambda, \mu, \nu) \in (\mathbb{P}^1)^3$ with $\lambda \neq 0, 1, \mu, -\nu, \mu - \nu + 1, \mu\nu, \infty, \mu \neq 0, -1, \infty, \nu \neq 0, 1, \infty$)

$$A_1^2(a) : (p_1^t, p_2^t, \dots, p_6^t) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -\lambda \\ 0 & 0 & 1 & -1 & -1 & \mu \end{pmatrix}$$

$((\lambda, \mu) \in (\mathbb{P}^1)^2$ with $\lambda \neq 0, \mu, \mu + 1, \infty, \mu \neq 0, -1, \infty$)

$$A_1^3(a) : (p_1^t, p_2^t, \dots, p_6^t) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -\lambda \\ 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}$$

$(\lambda \in \mathbb{P}^1, \lambda \neq 0, 1, \infty)$

$$A_1^4(a) : (p_1^t, p_2^t, \dots, p_6^t) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix}$$

Remark. All above classes are included in forms of *Duval of type $A_1^{n_1}$* , the cases where $\mathfrak{X}_{w,B}$ has n_1 Du Val singularities of type A_1 . For $\mathfrak{X}_{w,B}$ only having Du Val singularities, [38] first classifies forms by singularity types of $\mathfrak{X}_{w,B}$, and same singularity types are distinguished by labels $a, b, c, d, \dots$, ($A_1^4(b)$ etc.) For *non-Duval* forms, forms which $W_{B,2}$ have singularities of other type, $\mathfrak{X}_{w,B}$ never becomes distinct six points.

We claim that those cases can be incorporated to equivalence classes of smooth forms in the following way. Define $V_{n.d.}^{3,3,4}(\bar{k})$ as the union of forms in four equivalence classes of $A_1^i(a)$ and equivalence class of smooth forms.

Proposition 3.2.3. *1. For a $3 \times 3 \times 4$ form B , $\mathfrak{X}_{x,B}$ and $\mathfrak{X}_{y,B}$ becomes distinct six points if and only if $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_4(k)$ -equivalence class B is in $V_{n.d.}^{3,3,4}(\bar{k})$. $\mathfrak{X}_{x,B}$ and $\mathfrak{X}_{y,B}$ are the Gale transforms of each other. (One has $M_x M_y^t = 0$ for M_x, M_y be 3×6 matrices whose column vectors represent coordinates.) Six points of $\mathfrak{X}_{x,B}$ do not lie on a conic.*

2. There is one-to-one correspondence

$$\begin{aligned} & V_{n.d.}(\bar{k})/\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_4(\bar{k}) \\ & \Leftrightarrow \mathrm{PGL}_3(\bar{k}) \setminus \setminus \{ \text{six points in } \mathbb{P}^2, \text{ not on a conic} \} // S_6 \end{aligned}$$

given as $B \leftrightarrow \mathfrak{X}_{x,B}$.

Proof. For the first part, one simply needs to verify the claims for each of $A_1^i(a)$ cases, using the explicit representative $3 \times 3 \times 4$ forms in the equivalence class (suggested in ([38], Theorem 3)). For the second part, we can show that any six points not on a conic is PGL_3 -isomorphic to either one of $\mathfrak{X}_{x,B}$ for $B \in V_{n.d.}^{3,3,4}(\bar{k})$. If there are no lines through three of those points, the set of six points are in general position and thus they corresponds to smooth forms. If there are k distinct lines going through three of those points, one can manually check that the point set corresponds to $A_1^k(a)$ type. There can't be four points p_1, p_2, p_3, p_4 on a line: if so, all six points are contained in union of two lines (one going through p_1, p_2, p_3, p_4 and the other through remaining two points) thus they lie on a conic. $\square$

Remark. In ([38], Theorem 3), all Duval cases were obtained by all possible isomorphism class of zero-dimensional degree 6 subscheme of $\mathbb{P}^2$ not on a conic, and $A_1^i(a)$

types were all obtained from distinct six points.

Now going back to $3 \times 3 \times 5$ forms, we can define a set of forms $V_{n.d.} \subset V$ (remind that $V(k) = \bar{k}^3 \otimes \bar{k}^3 \otimes \bar{k}^5$, $G(\bar{k}) = \text{SL}_3 \times \text{SL}_3 \times \text{GL}_5(\bar{k})$) consisting of forms corresponding to $V_{n.d.}^{3,3,4}$ via castling-like correspondence. As we have shown that $\mathfrak{X}_{x,B} = \mathfrak{X}_{x,A}$ and $\mathfrak{X}_{y,B} = \mathfrak{X}_{y,A}$, we can apply Proposition 3.2.2 to forms in $V_{n.d.}$. For A not in $V_{n.d.}$, one can show there are no cases where X_A consists of distinct six points when Y_A and Z_A do not by examining all other equivalence classes of $3 \times 3 \times 4$ forms. This should complete the proof of the Proposition 3.2.1.

Remark. It is noted in [38] that for smooth forms B , we can construct $\mathfrak{X}_{w,B}$ as a blowup of $\mathbb{P}^2$ at six points of $\mathfrak{X}_{y,B}$. Over $\bar{k}$, all smooth cubic surfaces can be obtained in this way.

3.2.2 An explicit expression for forms representing six points

For $A \in V_{n.d.}(\bar{k})$, let $t_1, \dots, t_6 \in \bar{k}^5$ be some affine coordinates of six points of $\mathfrak{X}_{t,A}$ satisfying $t_1 + \dots + t_6 = 0$. Then matrices $M_i = A(t_i)$ satisfy

$$\text{rk}(M_i) = 1, \quad M_1 + M_2 + \dots + M_6 = 0.$$

Write $M_i = x_i y_i^t$ for $i = 1, \dots, 6$, then $\{x_i\}_i = \mathfrak{X}_{x,A}$ and $\{y_i\}_i = \mathfrak{X}_{y,A}$. As $x_1, x_2, \dots, x_6$ are not on a same conic, they are not one a same line. By permutating $M_1, \dots, M_6$, we can let x_1, x_2, x_3 be linearly independent. If y_4, y_5, y_6 are linearly dependent, then there exists nonzero $u \in k^3$ such that $y_4^t u = y_5^t u = y_6^t u = 0$. So we have

$$0 = (M_1 + \dots + M_6)u = x_1(y_1^t u) + x_2(y_2^t u) + x_3(y_3^t u) = 0.$$

As x_1, x_2, x_3 are linearly independent, this implies $y_1^t u = y_2^t u = y_3^t u = 0$, so we obtain the contradiction that $y_1, \dots, y_6$ are on a same line. Thus y_4, y_5, y_6 are linearly

independent. Let G_1 be the inverse of a matrix having x_1, x_2, x_3 as column vectors, and G_2 be the inverse of a matrix having y_4, y_5, y_6 as row vectors. Then for $N_i = G_1 M_i G_2$, we can see that

$$(N_1, N_2, N_3) = \left(\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{pmatrix} \right)$$

and

$$(N_4, N_5, N_6) = \left(\begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \right)$$

The condition $N_1 + \cdots + N_6 = 0$ essentially requires that its entries should be given as

$$\sum_{i=1}^6 N_i s_i = \begin{pmatrix} \lambda_{14}(s_1 - s_4) & \lambda_{15}(s_1 - s_5) & \lambda_{16}(s_1 - s_6) \\ \lambda_{24}(s_2 - s_4) & \lambda_{25}(s_2 - s_5) & \lambda_{26}(s_2 - s_6) \\ \lambda_{34}(s_3 - s_4) & \lambda_{35}(s_3 - s_5) & \lambda_{36}(s_3 - s_6) \end{pmatrix}$$

for some numbers λ_{ij} .

Proposition 3.2.4. *For any $A \in V_{n.d.}(\bar{k})$, there exists $g \in G(\bar{k})$ such that the action of g maps A to $N = (N_1, N_2, N_3, N_4, N_5)$ having form of*

$$\sum_{i=1}^6 N_i s_i = \begin{pmatrix} \lambda_{14}(s_1 - s_4) & \lambda_{15}(s_1 - s_5) & \lambda_{16}(s_1 - s_6) \\ \lambda_{24}(s_2 - s_4) & \lambda_{25}(s_2 - s_5) & \lambda_{26}(s_2 - s_6) \\ \lambda_{34}(s_3 - s_4) & \lambda_{35}(s_3 - s_5) & \lambda_{36}(s_3 - s_6) \end{pmatrix}.$$

In this expression, the six points of $\mathfrak{X}_{x,N}$ and $\mathfrak{X}_{y,N}$ are given as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = \begin{pmatrix} 1 & 0 & 0 & \lambda_{14} & \lambda_{15} & \lambda_{16} \\ 0 & 1 & 0 & \lambda_{24} & \lambda_{25} & \lambda_{26} \\ 0 & 0 & 1 & \lambda_{34} & \lambda_{35} & \lambda_{36} \end{pmatrix}$$

and

$$(y_1, y_2, y_3, y_4, y_5, y_6) = \begin{pmatrix} \lambda_{14} & \lambda_{24} & \lambda_{34} & -1 & 0 & 0 \\ \lambda_{15} & \lambda_{25} & \lambda_{35} & 0 & -1 & 0 \\ \lambda_{16} & \lambda_{26} & \lambda_{36} & 0 & 0 & -1 \end{pmatrix}.$$

This expression illustrates more directly that those two sets are the Gale transforms of each other. Six points of $\mathfrak{X}_{t,N}$ are given as

$$\{[1; 0; 0; 0; 0], [0; 1; 0; 0; 0], [0; 0; 1; 0; 0], [0; 0; 0; 1; 0], [0; 0; 0; 0; 1], [-1; -1; -1; -1; -1]\}$$

in projective coordinate.

As six points of $\mathfrak{X}_{x,N}$ are not on the same conic, we require

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \lambda_{14}^2 & \lambda_{24}^2 & \lambda_{34}^2 & \lambda_{14}\lambda_{24} & \lambda_{14}\lambda_{34} & \lambda_{24}\lambda_{34} \\ \lambda_{15}^2 & \lambda_{25}^2 & \lambda_{35}^2 & \lambda_{15}\lambda_{25} & \lambda_{15}\lambda_{35} & \lambda_{25}\lambda_{35} \\ \lambda_{16}^2 & \lambda_{26}^2 & \lambda_{36}^2 & \lambda_{16}\lambda_{26} & \lambda_{16}\lambda_{36} & \lambda_{26}\lambda_{36} \end{pmatrix} = \det \begin{pmatrix} \lambda_{14}\lambda_{24} & \lambda_{14}\lambda_{34} & \lambda_{24}\lambda_{34} \\ \lambda_{15}\lambda_{25} & \lambda_{15}\lambda_{35} & \lambda_{25}\lambda_{35} \\ \lambda_{16}\lambda_{26} & \lambda_{16}\lambda_{36} & \lambda_{26}\lambda_{36} \end{pmatrix} \neq 0$$

as a condition for λ_{ij} to satisfy. Denote $F(\lambda)$ as the 3×3 determinant in the above expression.

3.2.3 Symmetric invariants of six points

For six points in $\mathbb{P}^2$, we can consider PGL_3 -invariants of those six points which were discussed in [33]. We will calculate these invariants of six points of $\mathfrak{X}_{x,A}$, and discuss how it can be obtained from the expression constructed in the Proposition 3.2.4.

As those invariants are described by using an outer automorphism of S_6 , we first describe properties of outer automorphisms of S_6 and some related objects to them.

Definition 3.2.5. *Consier an outer automorphim φ of S_6 . For $1 \leq i \leq 6$, let G_i be the subgroup of S_6 which fixes the element i . We call six subgroups $H_i = \varphi(G_i)$ as sextic resolvent groups of S_6 . The set of the sextic resolvent subgroups $\{H_1, H_2, \dots, H_6\}$ does not depend on the choice of φ .*

Proposition 3.2.6. *A sextic resolvent group H satisfies following properties.*

1. *H acts transitively on $\{1, 2, 3, 4, 5, 6\}$.*
2. *For any $g \in S_6$, gHg^{-1} also becomes sextic resolvent group. All sextic resolvent groups appear as conjugations of H .*
3. *Denote $(a_1, a_2, \dots, a_k)$ as cycle class of a permutation having cycle decomposition into product of disjoint cycles of length $a_1, a_2, \dots, a_k$. Then H consists of permutations having cycle classes*

$$1, (2, 2), (4), (3, 3), (5), (6), (2, 2, 2).$$

Under an outer automorphism $H \simeq G_i$, each class can be corresponded one-to-one with the following cycle classes of $G_5 \simeq S_5$ respectively

$$1, (2, 2), (4), (3), (5), (3, 2), (2).$$

In particular, H contains 10 permutations having cycle decomposition $(ab)(cd)(ef)$,

out of 15 such permutations in S_6 .

We describe the invariants of six points in $\mathbb{P}^2$ using these properties of the sextic resolvent group.

Proposition 3.2.7 ([33], section 2.3). *Let $\{H_1, \dots, H_6\}$ be set of sextic resolvent groups. For six distinct points $p_i = [x_i; y_i; z_i]$ ($1 \leq i \leq 6$) in $\mathbb{P}^2$, consider the following six variables for $t = 1, 2, \dots, 6$:*

$$W_i = \sum_{a,b,c,d,e,f} N_{t,(a,b,c,d,e,f)} x_a x_b y_c y_d z_e z_f.$$

Here the sum is taken over indices satisfying $\{a, b, c, d, e, f\} = \{1, 2, 3, 4, 5, 6\}$ and $a < b, c < d, e < f$, and coefficients is defined as

$$N_{i,(a,b,c,d,e,f)} = \begin{cases} 2 & \text{if } (ab)(cd)(ef) \notin H_i \\ -1 & \text{if } (ab)(cd)(ef) \in H_i. \end{cases}$$

Also consider the following expression

$$V = \det \begin{pmatrix} x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 & z_6^2 \\ x_1 y_1 & x_2 y_2 & x_3 y_3 & x_4 y_4 & x_5 y_5 & x_6 y_6 \\ x_1 z_1 & x_2 z_2 & x_3 z_3 & x_4 z_4 & x_5 z_5 & x_6 z_6 \\ y_1 z_1 & y_2 z_2 & y_3 z_3 & y_4 z_4 & y_5 z_5 & y_6 z_6 \end{pmatrix}$$

which describes the condition whether $p_1, \dots, p_6$ lie on a conic. (V vanishes if and only if they lie on a conic.) Then

$$\sum_{i=1}^6 W_i = 0$$

and

$$\left(\sum_{i=1}^6 W_i^2\right)^2 - 4 \sum_{i=1}^6 W_i^4 + 324V^2 = 0.$$

The variables $W_1, \dots, W_6, V$ are invariant under SL_3 -action on coordinates (x_i, y_i, z_i) .

Moreover, the map

$$(\mathbb{P}^2)^6 \rightarrow \text{Proj } \bar{k}[W_1, \dots, W_6, V] / ((\sum W_i^2)^2 - 4 \sum W_i^4 + 324V^2)$$

which maps six points into corresponding $W_1, \dots, W_6, V$ under above equation, provides Geometric Invariant Theory quotient of six points on $\mathbb{P}^2$.

As we are considering points over an algebraically closed field k , this means that there is a one-to-one correspondence given as

$$\begin{aligned} & \text{PGL}_3(\bar{k}) \backslash \backslash \{\text{six (ordered) points in } \mathbb{P}^2, \text{ not on a conic}\} \\ & \Leftrightarrow \{(w_1, \dots, w_6, v) : \sum w_i = 0, (\sum w_i^2)^2 - 4 \sum w_i^4 = -324v^2 \neq 0\} / \bar{k}^\times \end{aligned}$$

where the action of $c \in k^\times$ is described as multiplication by c on w_i and multiplication by c^2 on v . We would like to connect this result to the correspondence between equivalence classes of forms in $V_{n.d.}$ and six points on $\mathbb{P}^2$ we described.

$$V_{n.d.}(\bar{k}) \Leftrightarrow \text{PGL}_3(\bar{k}) \backslash \backslash \{\text{six points in } \mathbb{P}^2, \text{ not on a conic}\} // S_6$$

Let σ be a permutation of S_6 permuting six points of $p_1, \dots, p_6$. Then for $W_i = W_i(p_1, \dots, p_6)$ corresponding to a sextic resolvent group H_i , value of W_i for permuted points $p_{\sigma(i)}$ is given as

$$W_i(p_{\sigma(1)}, \dots, p_{\sigma(6)}) = W_j(p_1, \dots, p_6)$$

where W_j corresponds to a sextic resolvent group $H_j = \sigma H_i \sigma^{-1}$. This means that σ permutes W_i in the way it permutes sextic resolvent groups via conjugation operation $H \mapsto \sigma H \sigma^{-1}$. Also one can see σ acts on V as multiplication by its signature $\text{sgn}(\sigma) \in \{-1, +1\}$. This establishes the correspondence

$$V_{n.d.}(\bar{k}) \Leftrightarrow \left\{ \begin{array}{c} (w_1, \dots, w_6, v) : \\ \sum w_i = 0, (\sum w_i^2)^2 - 4 \sum w_i^4 = -324v^2 \neq 0 \end{array} \right\} // \bar{k}^\times \times S_6.$$

Meanwhile, we can look at the involution acting on $3 \times 3 \times 5$ forms A which takes transpose of each 3×3 slices as

$$A = (A_1, A_2, A_3, A_4, A_5) \mapsto A^t := (A_1^t, A_2^t, A_3^t, A_4^t, A_5^t).$$

We have $\mathfrak{X}_{t,A^t} = \mathfrak{X}_{t,A}$, $\mathfrak{X}_{x,A^t} = \mathfrak{X}_{y,A}$ and $\mathfrak{X}_{y,A^t} = \mathfrak{X}_{x,A}$. We calculate the invariants W_i and V for $\mathfrak{X}_{x,A}$ and $\mathfrak{X}_{y,A}$, using the representative example of proposition 2.3: one can let N be a $\text{SL}_3 \times \text{SL}_3 \times \text{SL}_5(k)$ -equivalent element such that

$$\sum_{i=1}^6 N_i s_i = \begin{pmatrix} \lambda_{14}(s_1 - s_4) & \lambda_{15}(s_1 - s_5) & \lambda_{16}(s_1 - s_6) \\ \lambda_{24}(s_2 - s_4) & \lambda_{25}(s_2 - s_5) & \lambda_{26}(s_2 - s_6) \\ \lambda_{34}(s_3 - s_4) & \lambda_{35}(s_3 - s_5) & \lambda_{36}(s_3 - s_6) \end{pmatrix}.$$

Calculating invariants for

$$(x_1, x_2, x_3, x_4, x_5, x_6) = \begin{pmatrix} 1 & 0 & 0 & \lambda_{14} & \lambda_{15} & \lambda_{16} \\ 0 & 1 & 0 & \lambda_{24} & \lambda_{25} & \lambda_{26} \\ 0 & 0 & 1 & \lambda_{34} & \lambda_{35} & \lambda_{36} \end{pmatrix}$$

$$(y_1, y_2, y_3, y_4, y_5, y_6) = \begin{pmatrix} \lambda_{14} & \lambda_{24} & \lambda_{34} & -1 & 0 & 0 \\ \lambda_{15} & \lambda_{25} & \lambda_{35} & 0 & -1 & 0 \\ \lambda_{16} & \lambda_{26} & \lambda_{36} & 0 & 0 & -1 \end{pmatrix}$$

gives

$$\begin{aligned} & (W_1, W_2, W_3, W_4, W_5, W_6)^t(x_1, x_2, x_3, x_4, x_5, x_6) \\ &= \begin{pmatrix} -1 & 2 & 2 & -1 & -1 & -1 \\ 2 & -1 & 2 & -1 & -1 & -1 \\ 2 & 2 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 2 & 2 \\ -1 & -1 & -1 & 2 & -1 & 2 \\ -1 & -1 & -1 & 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda_{14}\lambda_{25}\lambda_{36} \\ \lambda_{15}\lambda_{26}\lambda_{34} \\ \lambda_{16}\lambda_{24}\lambda_{35} \\ \lambda_{14}\lambda_{26}\lambda_{35} \\ \lambda_{16}\lambda_{25}\lambda_{34} \\ \lambda_{15}\lambda_{24}\lambda_{36} \end{pmatrix} \\ &= -(W_1, W_2, W_3, W_4, W_5, W_6)^t(y_1, y_2, y_3, y_4, y_5, y_6) \end{aligned}$$

for a certain choice of sextic resolvent groups $H_1, \dots, H_6$ and

$$\begin{aligned} V(y_1, y_2, y_3, y_4, y_5, y_6) &= F(\lambda) = \det \begin{pmatrix} \lambda_{14}\lambda_{24} & \lambda_{14}\lambda_{34} & \lambda_{24}\lambda_{34} \\ \lambda_{15}\lambda_{25} & \lambda_{15}\lambda_{35} & \lambda_{25}\lambda_{35} \\ \lambda_{16}\lambda_{26} & \lambda_{16}\lambda_{36} & \lambda_{26}\lambda_{36} \end{pmatrix} \\ &= -V(z_1, z_2, z_3, z_4, z_5, z_6). \end{aligned}$$

This shows that the 'transpose' operation $A \mapsto A^t$, which acts on the $\mathfrak{X}_{x,A}$ as the Gale transform (equivalent to switching of $\mathfrak{X}x, A$ and $\mathfrak{X}_{y,A}$), keeps $(W_1, \dots, W_6)$ and changes sign of V . In particular, this shows that A and A^t are in different $\text{SL}_3 \times \text{SL}_3 \times \text{GL}_5(k)$ -orbits, and considering the equivalence $A \sim A^t$ in $V_{n.d.}$ gives following

one-to-one correspondence

$$V_{n.d.}(\bar{k})/\{A \sim A^t\} \Leftrightarrow \left\{ \begin{array}{c} (w_1, \dots, w_6, v) : \\ \sum w_i = 0, (\sum w_i^2)^2 - 4 \sum w_i^4 = -324v^2 \neq 0 \end{array} \right\} // \bar{k}^\times \times S_6.$$

This coincides with the description of [33] that on the GIT quotient space of six points in $\mathbb{P}^2$

$$\{(w_1, \dots, w_6, v) : \sum w_i = 0, (\sum w_i^2)^2 - 4 \sum w_i^4 = -324v^2\} // k^\times$$

which can be considered as double cover over the Igusa quartic

$$\sum w_i = 0, (\sum w_i^2)^2 - 4 \sum w_i^4 = 0,$$

the Gale transform on six points acts as involution of exchanging two sheets.

We would like to summarize all the correspondences obtained so far.

Theorem 3.2.8. *Let $V_{n.d.}(\bar{k})$ be the set of forms $A \in \bar{k}^3 \otimes \bar{k}^3 \otimes \bar{k}^5$ such that $\mathfrak{X}_A$ becomes distinct six points.*

1. *There is a one-to-one correspondence between $V_{n.d.}(\bar{k})/\bar{k}^\times$ and*

$$\{(X \subset \mathbb{P}^2, Y \subset \mathbb{P}^2, T \subset \mathbb{P}^4, \varphi_1 : T \rightarrow X :$$

X, Y, T are distinct 6 k -points,

T in a linearly general position,

X not on a single conic,

$\varphi_1 : \text{one-to-one correspondence,}$

X and Y Gale transform to each other}.

The group action of $G(k)$ ($G = \text{SL}_3 \times \text{SL}_3 \times \text{GL}_5$) on A corresponds to PGL-

action on $Y \subset \mathbb{P}^2, Z \subset \mathbb{P}^2, X \subset \mathbb{P}^4$ respectively. The transpose action $A \mapsto A^t$, which takes transpose of all 3×3 slices of A , acts as switching Y and Z and keeping X .

2. There is a one-to-one correspondence between $V_{n.d.}(\bar{k})/\mathrm{SL}_3 \times \mathrm{GL}_3(\bar{k})$ and

$$\begin{aligned} & \{(X \subset \mathbb{P}^2, T \subset \mathbb{P}^4, \varphi_1 : T \rightarrow X) : \\ & \quad X, T \text{ are distinct } 6 \text{ } k\text{-points,} \\ & \quad T \text{ in a lineraly general position,} \\ & \quad X \text{ not on a single conic,} \\ & \quad \varphi_1 : \text{one-to-one correspondence}\} \\ & // \mathrm{PGL}_3(\bar{k}) \end{aligned}$$

where $\mathrm{PGL}_3(\bar{k})$ acts on $X \subset \mathbb{P}^2$.

3. There is a one-to-one correspondence between $V_{n.d.}(\bar{k})/G(\bar{k})$ and

$$\mathrm{PGL}_3(\bar{k}) \setminus \setminus \{X \subset \mathbb{P}^2 \text{ not on a single conic}\} // S_6$$

and set of invariants

$$\{(w_1, w_2, \dots, w_6, v) : \sum w_i = 0, (\sum w_i^2)^2 - 4 \sum w_i^4 = -324v^2 \neq 0\} / \bar{k}^\times.$$

4. The transpose action is well-defined on $V_{n.d.}(\bar{k})/G(\bar{k})$ and becomes involution without fixed points. It acts on X as the Gale transform, and on invariants it acts as $(w_1, \dots, w_6, v) \mapsto (w_1, \dots, w_6, -v)$.

Proof. We only need the surjectivity proof of (1). Note that the for element N of the

Proposition 2.3

$$\sum_{i=1}^6 N_i s_i = \begin{pmatrix} \lambda_{14}(s_1 - s_4) & \lambda_{15}(s_1 - s_5) & \lambda_{16}(s_1 - s_6) \\ \lambda_{24}(s_2 - s_4) & \lambda_{25}(s_2 - s_5) & \lambda_{26}(s_2 - s_6) \\ \lambda_{34}(s_3 - s_4) & \lambda_{35}(s_3 - s_5) & \lambda_{36}(s_3 - s_6) \end{pmatrix},$$

six points of $\mathfrak{X}_{x,N}$ are given as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = \begin{pmatrix} 1 & 0 & 0 & \lambda_{14} & \lambda_{15} & \lambda_{16} \\ 0 & 1 & 0 & \lambda_{24} & \lambda_{25} & \lambda_{26} \\ 0 & 0 & 1 & \lambda_{34} & \lambda_{35} & \lambda_{36} \end{pmatrix}.$$

For any six points Y not on a conic, we can pick three points among them which are not on a line, and let them x_1, x_2, x_3 . Then a certain $\mathrm{PGL}_3(\bar{k})$ -action can transform X into $\mathfrak{X}_{x,N}$. As one can check that Gale transform of X is $\mathrm{PGL}_3(\bar{k})$ -equivalent to both Y and $\mathfrak{X}_{y,N}$, Y and $\mathfrak{X}_{x,N}$ are $\mathrm{PGL}_3(\bar{k})$ -equivalent. For X , any 6 points of $\mathbb{P}^4$ in general position are $\mathrm{PGL}_5(k)$ -equivalent to each other. This shows that any set of (X, Y, T) in above class is PGL -equivalent to $(\mathfrak{X}_{x,N}, \mathfrak{X}_{y,N}, \mathfrak{X}_{t,N})$ for some N . $\square$

3.2.4 Cubic threefolds having six nodes

Here we discuss the alternative way of calculating the W_i -and V -invariants out of the $3 \times 3 \times 5$ forms. This will later connect to Galois theory perspective for sextic resolvent rings for integral case. First we note the following.

Proposition 3.2.9. *Consider the cubic threefold $f_A(t) = \det(A(t))$. If $\mathrm{rk}(A(t_0)) \leq 1$, then $f_A(t)$ is singular at t_0 . If $A \in V_{n.d.}(\bar{k})$, then six points of X_A become simple nodes of f_A .*

Proof. For any $z \in k^5$, we show that $f_A(t_0 + \epsilon z) = \det(A(x_0) + \epsilon A(z))$ has ϵ^2 factor.

As rank of $A(x_0)$ is ≤ 1 , there exists $G_1, G_2 \in \text{SL}_3(\bar{k})$ and $c \in \bar{k}$ such that

$$G_1 A(x_0) G_2 = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\det(A(t_0) + \epsilon A(z)) = \det(G_1(A(t_0) + \epsilon A(z))G_2)$ can be represented as

$$\det \begin{pmatrix} c + a_{11}\epsilon & a_{12}\epsilon & a_{13}\epsilon \\ a_{21}\epsilon & a_{22}\epsilon & a_{23}\epsilon \\ a_{31}\epsilon & a_{32}\epsilon & a_{33}\epsilon \end{pmatrix}$$

and when we expand out this determinant, all terms are divisible by t^2 . For the second statement, it suffices to show for any element in the $G(\bar{k})$ -orbit. We pick the explicit representative of proposition 2.3.

$$M(t) = \begin{pmatrix} \lambda_{14}(t_1 - t_4) & \lambda_{15}(t_1 - t_5) & \lambda_{16}(t_1) \\ \lambda_{24}(t_2 - t_4) & \lambda_{25}(t_2 - t_5) & \lambda_{26}(t_2) \\ \lambda_{34}(t_3 - t_4) & \lambda_{35}(t_3 - t_5) & \lambda_{36}(t_3) \end{pmatrix}$$

For $\Lambda_{pqr} = \lambda_{1p}\lambda_{2q}\lambda_{3r}$, its determinant is

$$\begin{aligned} \det(M(t)) &= (\Lambda_{456} - \Lambda_{465} - \Lambda_{546} + \Lambda_{564} + \Lambda_{645} - \Lambda_{654})t_1t_2t_3 + (\Lambda_{654} - \Lambda_{564})t_1t_2t_4 \\ &\quad + (\Lambda_{465} - \Lambda_{645})t_1t_2t_5 + (\Lambda_{546} - \Lambda_{645})t_1t_3t_4 \\ &\quad + (\Lambda_{654} - \Lambda_{456})t_1t_3t_5 + (\Lambda_{645} - \Lambda_{654})t_1t_4t_5 \\ &\quad + (\Lambda_{465} - \Lambda_{456})t_2t_3t_4 + (\Lambda_{546} - \Lambda_{564})t_2t_3t_5 \\ &\quad + (\Lambda_{564} - \Lambda_{465})t_2t_4t_5 + (\Lambda_{456} - \Lambda_{546})t_3t_4t_5. \end{aligned}$$

We show that the point $t = [1; 0; 0; 0; 0]$ ($t_2 = t_3 = t_4 = t_5 = 0$) of $\mathfrak{X}_{t,M}$ becomes simple

node of $\det(M(t)) = 0$. t is simple node if and only if the following approximating quadric near t

$$\begin{aligned} & (\Lambda_{456} - \Lambda_{465} - \Lambda_{546} + \Lambda_{564} + \Lambda_{645} - \Lambda_{654})t_2t_3 + (\Lambda_{654} - \Lambda_{564})t_2t_4 \\ & + (\Lambda_{465} - \Lambda_{645})t_2t_5 + (\Lambda_{546} - \Lambda_{645})t_3t_4 \\ & + (\Lambda_{654} - \Lambda_{456})t_3t_5 + (\Lambda_{645} - \Lambda_{654})t_4t_5 \end{aligned}$$

is nonsingular quadric form, that is, nonzero and not square. If it is zero, it implies $\Lambda_{456} = \dots = \Lambda_{654}$, so

$$F(\lambda) = \Lambda_{456}\Lambda_{564} + \Lambda_{564}\Lambda_{645} + \Lambda_{645}\Lambda_{456} - \Lambda_{465}\Lambda_{546} - \Lambda_{546}\Lambda_{654} - \Lambda_{654}\Lambda_{465} = 0.$$

But $F(\lambda)$ is V -invariant for six points of Y_N so should be nonzero. Thus it can't be zero. It also can't be nonzero square form (by considering coefficient of forms $c(*t_2 + *t_3 + *t_4 + *t_5)^2$) so the statement is proven. $\square$

Remark. It is known that the converse holds: any cubic threefold having six nodes which are in general position is isomorphic to form of $\det(A(t))$ for some $3 \times 3 \times 5$ form A . (See [21], Theorem 3.5.)

Suppose that a cubic in variables t_1, t_2, t_3, t_4, t_5 is singular at six points

$$[1; 0; 0; 0; 0], [0; 1; 0; 0; 0], [0; 0; 1; 0; 0], [0; 0; 0; 1; 0], [0; 0; 0; 0; 1], [1; 1; 1; 1; 1].$$

The linear space of all such cubics has dimension 5, and we will exhibit some explicit basis for this space. Let H be a sextic resolvent group. For $t_6 = 0$, consider the expression

$$R_H(t_1, t_2, t_3, t_4, t_5, t_6) = \pm \frac{1}{6} \sum_{\sigma \in H} \text{sgn}(\sigma) t_{\sigma(1)} t_{\sigma(2)} t_{\sigma(3)}.$$

For example, for $H = \langle (14)(25)(36), (26345) \rangle$, we have

$$\begin{aligned} R_H = & \pm(t_1t_2t_3 + t_1t_2t_4 - t_1t_3t_4 - t_2t_3t_4 - t_1t_2t_5 + t_1t_3t_5 - t_2t_3t_5 - t_1t_4t_5 \\ & + t_2t_4t_5 + t_3t_4t_5 - t_1t_2t_6 - t_1t_3t_6 + t_2t_3t_6 + t_1t_4t_6 - t_2t_4t_6 + t_3t_4t_6 \\ & + t_1t_5t_6 + t_2t_5t_6 - t_3t_5t_6 - t_4t_5t_6). \end{aligned}$$

One can check that $R_H(x_1, x_2, x_3, x_4, x_5, 0)$ is singular at above 6 points, and also

$$R_H(t_1 + u, \dots, t_6 + u) = R_H(t_1, \dots, t_6).$$

The sign of R_H will be defined together when considering all other R_{H_i} for all sextic resolvent groups $\{H_1, \dots, H_6\}$. For $\sigma \in S_6$ acting on the variables $\{t_1, \dots, t_6\}$ as standard permutation, we have

$$\tau \in H \Rightarrow \tau(R_H) = \text{sgn}(\tau)R_H.$$

For $\sigma \in S^6 \setminus H$, one can see

$$\sigma(R_H) = \pm R_{\sigma H \sigma^{-1}}$$

We want to define the sign of $R_{H_1}, \dots, R_{H_6}$ such that

$$\sigma(R_H) = \text{sgn}(\sigma)R_{\sigma H \sigma^{-1}}$$

holds for any sextic resolvent group H and any $\sigma \in S^6$. As conjugation by $\sigma \in S_6$ acts transitively on six sextic resolvent groups, one can determine signs of all R_H using above rule, and show that it is consistent with above rule. I used the explicit

example of

$$\begin{aligned}
R_{H_1} = & t_1t_2t_3 + t_1t_2t_4 - t_1t_3t_4 - t_2t_3t_4 - t_1t_2t_5 + t_1t_3t_5 - t_2t_3t_5 - t_1t_4t_5 \\
& + t_2t_4t_5 + t_3t_4t_5 - t_1t_2t_6 - t_1t_3t_6 + t_2t_3t_6 + t_1t_4t_6 - t_2t_4t_6 + t_3t_4t_6 \\
& + t_1t_5t_6 + t_2t_5t_6 - t_3t_5t_6 - t_4t_5t_6
\end{aligned}$$

$$\begin{aligned}
R_{H_2} = & -t_1t_2t_3 - t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 + t_1t_2t_5 + t_1t_3t_5 - t_2t_3t_5 - t_1t_4t_5 \\
& + t_2t_4t_5 - t_3t_4t_5 + t_1t_2t_6 - t_1t_3t_6 + t_2t_3t_6 + t_1t_4t_6 - t_2t_4t_6 - t_3t_4t_6 \\
& - t_1t_5t_6 - t_2t_5t_6 + t_3t_5t_6 + t_4t_5t_6
\end{aligned}$$

$$\begin{aligned}
R_{H_3} = & -t_1t_2t_3 + t_1t_2t_4 - t_1t_3t_4 + t_2t_3t_4 - t_1t_2t_5 + t_1t_3t_5 + t_2t_3t_5 + t_1t_4t_5 \\
& - t_2t_4t_5 - t_3t_4t_5 + t_1t_2t_6 + t_1t_3t_6 - t_2t_3t_6 - t_1t_4t_6 - t_2t_4t_6 + t_3t_4t_6 \\
& - t_1t_5t_6 + t_2t_5t_6 - t_3t_5t_6 + t_4t_5t_6
\end{aligned}$$

$$\begin{aligned}
R_{H_4} = & -t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 - t_2t_3t_4 + t_1t_2t_5 - t_1t_3t_5 + t_2t_3t_5 - t_1t_4t_5 \\
& - t_2t_4t_5 + t_3t_4t_5 - t_1t_2t_6 + t_1t_3t_6 + t_2t_3t_6 - t_1t_4t_6 + t_2t_4t_6 - t_3t_4t_6 \\
& + t_1t_5t_6 - t_2t_5t_6 - t_3t_5t_6 + t_4t_5t_6
\end{aligned}$$

$$\begin{aligned}
R_{H_5} = & t_1t_2t_3 - t_1t_2t_4 + t_1t_3t_4 - t_2t_3t_4 - t_1t_2t_5 - t_1t_3t_5 + t_2t_3t_5 + t_1t_4t_5 \\
& + t_2t_4t_5 - t_3t_4t_5 + t_1t_2t_6 - t_1t_3t_6 - t_2t_3t_6 - t_1t_4t_6 + t_2t_4t_6 + t_3t_4t_6 \\
& + t_1t_5t_6 - t_2t_5t_6 + t_3t_5t_6 - t_4t_5t_6
\end{aligned}$$

$$\begin{aligned}
R_{H_6} = & t_1t_2t_3 - t_1t_2t_4 - t_1t_3t_4 + t_2t_3t_4 + t_1t_2t_5 - t_1t_3t_5 - t_2t_3t_5 + t_1t_4t_5 \\
& - t_2t_4t_5 + t_3t_4t_5 - t_1t_2t_6 + t_1t_3t_6 - t_2t_3t_6 + t_1t_4t_6 + t_2t_4t_6 - t_3t_4t_6 \\
& - t_1t_5t_6 + t_2t_5t_6 + t_3t_5t_6 - t_4t_5t_6
\end{aligned}$$

for the choice of sextic resolvent groups

$$\begin{aligned} & (H_1, H_2, H_3, H_4, H_5, H_6) \\ & = (\langle (14)(25)(36), (26345) \rangle, \langle (16)(23)(45), (15436) \rangle, \langle (12)(34)(56), (16342) \rangle, \\ & \quad \langle (12)(34)(56), (24365) \rangle, \langle (12)(34)(56), (15634) \rangle, \langle (12)(34)(56), (14362) \rangle) \end{aligned}$$

Then one can verify

$$\sum_{i=1}^6 R_{H_i}(t) = \sum_{i=1}^6 R_{H_i}(t)^3 = 0.$$

One can also check any 5 of the 6 forms of $R_{H_1}, \dots, R_{H_6}$. So for any cubic form f which is singular at six points

$$[1; 0; 0; 0; 0], [0; 1; 0; 0; 0], [0; 0; 1; 0; 0], [0; 0; 0; 1; 0], [0; 0; 0; 0; 1], [1; 1; 1; 1; 1],$$

there exists number $w_1, w_2, \dots, w_6$ such that $\sum w_i = 0$ and $f = \sum w_i R_{H_i}$. Meanwhile if we let $f = \det(M(t))$ for

$$M(t) = \begin{pmatrix} \lambda_{14}(t_1 - t_4) & \lambda_{15}(t_1 - t_5) & \lambda_{16}(t_1) \\ \lambda_{24}(t_2 - t_4) & \lambda_{25}(t_2 - t_5) & \lambda_{26}(t_2) \\ \lambda_{34}(t_3 - t_4) & \lambda_{35}(t_3 - t_5) & \lambda_{36}(t_3) \end{pmatrix},$$

then we have

$$(w_1, w_2, \dots, w_6) = \frac{1}{6}(W_1, \dots, W_6)(\mathfrak{X}_{x,M}).$$

By applying suitable PGL_5 -action on six points, we can generalize this result to arbitrary six points in general position and determinantal equations $\det(A(t))$.

Proposition 3.2.10. *For any six points $T \subset \mathbb{P}^4$ in general position, there exists nonzero six cubic forms $R_{T,H_i}(x)$ satisfying following properties:*

1. $R_{T,i}$ is singular at six points of X .

2. We have

$$\sum_{i=1}^6 R_{T,H_i}(t) = \sum_{i=1}^6 R_{T,H_i}(t)^3 = 0.$$

3. The automorphism group of six points, which sits inside $\mathrm{PGL}_5(\bar{k})$ and is isomorphic to S_6 , acts transitively on the set of six cubic forms $R_{T,H_i}(t)$ (up to constant).

The list $(R_{T,H_i})_{i=1,\dots,6}$ is determined uniquely up to constant multiplication and permutation, and any five of R_{T,H_i} form basis for cubic forms which is singular at all points of T .

Proposition 3.2.11. *For a $A \in V_{n.d.}(\bar{k})$, we have*

$$\det(A(t)) = c \sum_{i=1}^6 w_i(\mathfrak{X}_{x,A}) R_{\mathfrak{X}_{t,A},H_i}(t)$$

for some $c \neq 0$.

We will use this description of $w_i(\mathfrak{X}_{x,A})$ in later in the Section 3.4.

3.3 Construction of the sextic ring

In this section, we will construct our ring $\mathcal{R}(A)$ for $A \in V(\mathbb{Z})$. We will first give explicit descriptions of the multiplication coefficients $c_{ij}^k(A)$ in polynomials of A . By considering how the action of $G(\mathbb{Z})$ on A transforms those coefficients, we can show the isomorphism class of $\mathcal{R}(A)$ only depends on the $G(\mathbb{Z})$ -equivalence class of A .

Then we will describe the construction of $\mathcal{R}(A)$ via hypercohomology construction described in the Chapter 2. We will construct two complexes $\mathcal{C}_{t,A}$ and $\mathcal{C}_{x,A}$ of sheaves on $\mathbb{P}^4$ and $\mathbb{P}^2$ respectively, each becoming free resolutions for $\mathfrak{X}_{t,A}$ and $\mathfrak{X}_{x,A}$ respectively for generic A . It turns out that hypercohomology rings for both complexes are all isomorphic to $\mathcal{R}(A)$. This property will be used to establish many properties of

$\mathfrak{X}_A$ including the proof of Theorem 3.1.2, which will be achieved by showing that condition for $V_{n,d}$ ($\Leftrightarrow \mathfrak{X}_A$ being distinct six points) is equivalent to $\text{Disc}(\mathcal{R}(A)) \neq 0$.

Before start, we would like to point out how multiplication coefficients changed when basis elements are shifted by integers. In a rank n ring, its multiplication coefficients c_{ij}^k depend on the choice of basis $1 = \alpha_0, \alpha_1, \dots, \alpha_{n-1}$, and should satisfy

$$\alpha_i \alpha_j = \sum_{k=0}^{n-1} c_{ij}^k \alpha_k.$$

If we shift each α_i by integral constant C_i (so that $\alpha_i \mapsto \alpha_i + C_i$), then multiplication coefficients change accordingly. Under those changes, only the following values are invariant (here $1 \leq i, j, k \leq n-1$ are all distinct)

$$c_{ij}^k, c_{ii}^j, c_{ij}^j - c_{ik}^k, c_{ii}^i - c_{ij}^j - c_{ik}^k.$$

Conversely, if we are given those values, one can recover all multiplication coefficients by specifying $n-1$ coefficients c_{21}^1 and $c_{12}^2, c_{13}^3, \dots, c_{1(n-1)}^{n-1}$. Thus in constructing $\mathcal{R}(A)$, we will only specify above invariant values, equivalent to specifying the $\mathbb{Z}$ -basis of $\mathcal{R}(A)/\mathbb{Z}$. When this ambiguity is considered, the correct statement for the Theorem 3.1.1 would rather take the following form.

Theorem 3.3.1 (Restatement of the Theorem 3.1.1). *For a form $A \in V(\mathbb{Z})$, there exists a sextic ring $\mathcal{R}(A)$ such that following values of the form*

$$c_{ij}^k, c_{ii}^j, c_{ij}^j - c_{ik}^k, c_{ii}^i - c_{ij}^j - c_{ik}^k$$

($1 \leq i, j, k \leq 5$ are all distinct) are $\text{SL}_3 \times \text{SL}_3$ -invariants of A of degree 6.

For an action of $g = (g_1, g_2, g_3) \in G(\mathbb{Z}) = \text{SL}_3(\mathbb{Z}) \times \text{SL}_3(\mathbb{Z}) \times \text{GL}_5(\mathbb{Z})$, $\mathcal{R}(gA)$ is isomorphic to $\mathcal{R}(A)$ in the following way. If $(1 = \alpha_0, \alpha_1, \dots, \alpha_5)$ is basis of $\mathcal{R}(A)$, then $\mathcal{R}(gA)$ has basis $1, \tilde{\alpha}_1, \dots, \tilde{\alpha}_5$ where $\tilde{\alpha}_i = \det(g_3) \sum_{j=1}^5 (g_3)_{ij} \alpha_j$, up to shifting of

$\tilde{\alpha}_i \mapsto \tilde{\alpha}_i + C_i$ by integer C_i .

Remark. The ring parametrizations in the Bhargava's Higher composition laws ([1],[2],[3]) encounters the exact same problem. In there, a basis of rank n ring is called 'standardized' if it satisfies $c_{21}^1 = c_{12}^2 = c_{13}^3 = \dots = c_{1(n-1)}^{n-1} = 0$. We will not use this notion here.

3.3.1 Multiplication table

Before we give explicit expressions for the multiplication coefficients c_{ij}^k of $\mathcal{R}(A)$, we define some $\mathrm{SL}_3 \times \mathrm{SL}_3$ -covariants for 3×3 matrices.

For $X, Y \in \mathrm{Mat}_{3 \times 3}$, let $\mathrm{adj}(X)$ be a classical adjugate matrix

$$\mathrm{adj}(X) = \begin{pmatrix} X_{22}X_{33} - X_{23}X_{32} & -X_{12}X_{33} + X_{13}X_{32} & X_{12}X_{23} - X_{13}X_{22} \\ -X_{21}X_{33} + X_{23}X_{31} & X_{11}X_{33} - X_{13}X_{31} & -X_{11}X_{23} + X_{13}X_{21} \\ X_{21}X_{32} - X_{22}X_{31} & -X_{11}X_{32} + X_{12}X_{31} & X_{11}X_{22} - X_{21}X_{12} \end{pmatrix}$$

and define

$$B(X, Y) = \mathrm{adj}(X + Y)^t - \mathrm{adj}(X)^t - \mathrm{adj}(Y)^t.$$

Then for any $G_1, G_2 \in \mathrm{Mat}_{3 \times 3}$ we have

$$B(G_2^t X G_1, G_2^t Y G_1) = \mathrm{adj}(G_2) B(X, Y) \mathrm{adj}(G_1)^t.$$

This can be proved by noting that

$$\mathrm{adj}(GX) = \mathrm{adj}(X) \mathrm{adj}(G), \quad \mathrm{adj}(X^t) = \mathrm{adj}(X)^t$$

for any matrices X, G . Now for $X_1, \dots, X_6 \in \mathrm{Mat}_{3 \times 3}$,

$$\Phi(X_1, X_2, X_3, X_4, X_5, X_6) = \mathrm{tr}(B(X_1, X_2) X_6^t B(X_3, X_4) X_5^t).$$

We can see that it is invariant under $\text{SL}_3 \times \text{SL}_3$ -action from the above property of B as

$$\begin{aligned}
& \Phi(G_2^t X_1 G_1, \dots, G_2^t X_6 G_1) \\
&= \text{tr}(\text{adj}(G_2)B(X_1, X_2)\text{adj}(G_1)^t(G_1^t X_6^t G_2)\text{adj}(G_2)B(X_3, X_4)\text{adj}(G_1)^t(G_1^t X_5^t G_2)) \\
&= \text{tr}(\text{adj}(G_2)B(X_1, X_2)(\det(G_1)I)X_6^t(\det(G_2)I)B(X_3, X_4)(\det(G_1)I)X_5^t G_2)) \\
&= (\det G_1)^2(\det G_2)\text{tr}(\text{adj}(G_2)B(X_1, X_2)X_6^t B(X_3, X_4)X_5^t G_2) \\
&= (\det G_1)^2(\det G_2)\text{tr}(G_2\text{adj}(G_2)B(X_1, X_2)X_6^t B(X_3, X_4)X_5^t) \\
&= (\det G_1)^2(\det G_2)^2\Phi(X_1, \dots, X_6).
\end{aligned}$$

The invariant Φ can be also interpreted as follows. In the following determinant of block diagonal matrix

$$\det \begin{pmatrix} X_1 + X_2 & X_5 \\ X_6 & X_3 + X_4 \end{pmatrix},$$

collecting all terms whose degree in X_i is 1 for all $i = 1, \dots, 6$ gives Φ .

Define

$$\begin{aligned}
M(x) &= M(x_1, x_2, x_3, x_4, x_5, x_6) \\
&= \begin{pmatrix} \lambda_{14}(x_1 - x_4) & \lambda_{15}(x_1 - x_5) & \lambda_{16}(x_1 - x_6) \\ \lambda_{24}(x_2 - x_4) & \lambda_{25}(x_2 - x_5) & \lambda_{26}(x_2 - x_6) \\ \lambda_{34}(x_3 - x_4) & \lambda_{35}(x_3 - x_5) & \lambda_{36}(x_3 - x_6) \end{pmatrix}.
\end{aligned}$$

Lemma. For x, y, z, u, w elements $x = (x_1, \dots, x_6)$ of the ring $\mathbb{C}^6$ (with its multipli-

cation naturally given as $x \cdot y = (x_1 y_1, \dots, x_6 y_6)$, define

$$\text{Ind}(x, y, \dots, w) = \det \begin{pmatrix} x_1 & x_2 & \cdots & x_6 \\ y_1 & y_2 & \cdots & y_6 \\ \cdots & \cdots & \cdots & \cdots \\ w_1 & w_2 & \cdots & w_6 \end{pmatrix}$$

Then we have

$$\begin{aligned} & \Phi(M(x), M(x), M(y), M(z), M(u), M(w)) - \Phi(M(x), M(x), M(y), M(z), M(w), M(u)) \\ &= 2F(\lambda)(\text{Ind}(1, x, y, u, w, xz) + \text{Ind}(1, x, z, u, w, xy)) \end{aligned}$$

and

$$\begin{aligned} & \Phi(M(x), M(y), M(x), M(z), M(u), M(w)) - \Phi(M(x), M(y), M(x), M(z), M(w), M(u)) \\ &= F(\lambda)(\text{Ind}(1, y, z, u, w, x^2) - \text{Ind}(1, x, y, u, w, xz) + \text{Ind}(1, x, z, u, w, xy)) \end{aligned}$$

where $F(\lambda)$ is given as

$$F(\lambda) = \det \begin{pmatrix} \lambda_{14}\lambda_{24} & \lambda_{14}\lambda_{34} & \lambda_{24}\lambda_{34} \\ \lambda_{15}\lambda_{25} & \lambda_{15}\lambda_{35} & \lambda_{25}\lambda_{35} \\ \lambda_{16}\lambda_{26} & \lambda_{16}\lambda_{36} & \lambda_{26}\lambda_{36} \end{pmatrix}.$$

Proof. Explicit calculation. □

We are now ready to define $\mathcal{R}(A)$ for $A = (A_1, \dots, A_5)$. Denote

$$[abcdef] = \Phi(A_a, A_b, A_c, A_d, A_e, A_f)$$

for $a, \dots, f \in \{1, 2, 3, 4, 5\}$. Define $c_{pq}^r = c_{pq}^r(A)$ for $p, q, r \in \{1, 2, 3, 4, 5\}$ as any

integers satisfying

$$\begin{aligned}
c_{ij}^k &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i & j & l & m & k \end{pmatrix} ([iijjlm] - [iijjml])/4 \\
c_{ii}^j &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i & k & l & m & j \end{pmatrix} ([iiklm] - [iikml])/2 \\
c_{ij}^j - c_{ik}^k &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i & k & l & m & j \end{pmatrix} ([iijklm] - [iijkml])/2 \\
c_{ii}^i - c_{ij}^j - c_{ik}^k &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ j & k & l & m & i \end{pmatrix} ([ijiklm] - [ijikml])
\end{aligned} \tag{*}$$

for i, j, l, m, k any permutation of $\{1, 2, 3, 4, 5\}$. Here the notation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i & j & l & m & k \end{pmatrix}$ means the signature of permutation. Remaining coefficients c_{ij}^0 of constant terms can be determined by equations

$$c_{ij}^0 = \sum_{l=1}^5 (c_{jk}^l c_{li}^k - c_{ij}^l c_{lk}^k) \tag{**}$$

for $i, j \in \{1, 2, 3, 4, 5\}$. The number k can be chosen as any number in $\{1, 2, 3, 4, 5\} \setminus i$. This expression is obtained by comparing coefficient of α_k in $(\alpha_i \alpha_j) \alpha_k$ and $\alpha_i (\alpha_j \alpha_k)$, supposing that there exists a ring with $\mathbb{Z}$ -basis $1 = \alpha_0, \dots, \alpha_5$ whose multiplication coefficients are c_{ij}^k .

Theorem 3.3.2. *1. For $A \in V(\mathbb{Z})$, there exist integral polynomials $c_{pq}^r(A)$ in entries of A satisfying (*) and (**). Define $\mathcal{R}(A)$ as a free $\mathbb{Z}$ -module with basis $1 = \alpha_0, \alpha_1, \dots, \alpha_5$. The multiplication structure on $\mathcal{R}(A)$ defined as*

$$\alpha_i \alpha_j = \sum_{k=0}^5 c_{ij}^k(A) \alpha_k$$

establishes $\mathcal{R}(A)$ as well-defined rank 6 ring over $\mathbb{Z}$. The choice of multiplication coefficients and α_i is unique up to integer translations $\alpha_i \mapsto \alpha_i + C_i$.

2. If $A \in V_{n.d.}(\mathbb{Z})$, then $\mathcal{R}(A)$ is nondegenerate. Let $\alpha_0^*, \dots, \alpha_5^* \in \mathcal{R}(A) \otimes \mathbb{Q}$ be dual basis of $\alpha_0, \dots, \alpha_5$, and $\rho_1, \dots, \rho_6 : \mathcal{R}(A) \otimes \mathbb{Q} \rightarrow \mathbb{C}$ be distinct ring morphisms. Then six points of $\mathfrak{X}_{t,A}$ are given as

$$[\rho_i(\alpha_1^*); \rho_i(\alpha_2^*); \rho_i(\alpha_3^*); \rho_i(\alpha_4^*); \rho_i(\alpha_5^*)]$$

for $1 \leq i \leq 6$.

Proof. (1) As pointed out in the beginning of this section, one can recover all c_{ij}^k from (*) and (**) after we specify $(c_{12}^2, c_{21}^1, c_{31}^1, c_{41}^1, c_{51}^1)$. Take them to be all zero, and explicitly calculate all c_{ij}^k . One can explicitly check the resulting multiplication on $\mathcal{R}(A)$ is well-defined by polynomial calculation. (I used computer algebra system.)

(2) To provide more motivation for the construction of $\mathcal{R}(A)$, we will construct a ring whose multiplication coefficients become $c_{pq}^r(A)$ using another method, and exhibit how this ring relates to coordinates of the points of $\mathfrak{X}_{t,A}$.

Using the Proposition 2.4, we can find a $3 \times 3 \times 5$ form $N = (N_1, N_2, N_3, N_4, N_5)$ which is $\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_5(\mathbb{C})$ -equivalent to A and has form

$$\sum_{i=1}^5 N_i s_i = \begin{pmatrix} \lambda_{14}(s_1 - s_4) & \lambda_{15}(s_1 - s_5) & \lambda_{16}(s_1) \\ \lambda_{24}(s_2 - s_4) & \lambda_{25}(s_2 - s_5) & \lambda_{26}(s_2) \\ \lambda_{34}(s_3 - s_4) & \lambda_{35}(s_3 - s_5) & \lambda_{36}(s_3) \end{pmatrix}.$$

We only consider the $\mathrm{SL}_3 \times \mathrm{SL}_3(\mathbb{C})$ -action by the element $(G_1, G_2) \in \mathrm{SL}_3 \times \mathrm{SL}_3(\mathbb{C})$. Consider another set of coordinates $(t_1, \dots, t_5)$ satisfying that

$$G_2^t \left(\sum_{i=1}^5 A_i t_i \right) G_1 = \sum_{i=1}^5 N_i s_i$$

and represent s_i as linear expression in t_1, t_2, t_3, t_4, t_5 . Explicitly we can write $s_i = \sum_{j=1}^5 g_{ij}t_j$ for some $(g_{ij})_{i,j} \in \text{GL}_5(\mathbb{C})$. If we bring the notation of the previous lemma

$$M(z_1, z_2, z_3, z_4, z_5, z_6) = \begin{pmatrix} \lambda_{14}(z_1 - z_4) & \lambda_{15}(z_1 - z_5) & \lambda_{16}(z_1 - z_6) \\ \lambda_{24}(z_2 - z_4) & \lambda_{25}(z_2 - z_5) & \lambda_{26}(z_2 - z_6) \\ \lambda_{34}(z_3 - z_4) & \lambda_{35}(z_3 - z_5) & \lambda_{36}(z_3 - z_6) \end{pmatrix},$$

then we can see

$$G_2^t A_i G_1 = M(g_{1i}, g_{2i}, \dots, g_{5i}, 0).$$

If we think elements $\alpha'_0, \alpha'_1, \dots, \alpha'_5 \in \mathbb{C}^6$ defined as

$$\alpha'_0 = (1, 1, 1, 1, 1, 1), \alpha'_i = (g_{1i}, g_{2i}, \dots, g_{5i}, 0),$$

we can represent values of Φ at A_i as

$$\begin{aligned} [abcdef] &= \Phi(A_a, A_b, A_c, A_d, A_e, A_f) \\ &= \Phi(G_2^t A_a G_1, \dots, G_2^t A_f G_1) \\ &= \Phi(M(\alpha'_a), M(\alpha'_b), M(\alpha'_c), M(\alpha'_d), M(\alpha'_e), M(\alpha'_f)). \end{aligned}$$

Meanwhile, as $\alpha'_0, \dots, \alpha'_5$ are linearly independent, (which equivalent to that tg_{ij} s form 5×5 invertible matrix) there exist unique numbers $d_{pq}^r \in \mathbb{C}$ ($1 \leq p, q \leq 5, 0 \leq r \leq 5$) such that

$$\alpha'_i \alpha'_j = \sum_{k=0}^5 d_{ij}^k \alpha'_k \text{ in } \mathbb{C}^6.$$

We can use the preceding lemma to show

$$\begin{aligned}
& [iijjlm] - [iijjml] \\
&= 4F(\lambda) \text{Ind}(1, \alpha'_i, \alpha'_j, \alpha'_l, \alpha'_m, \alpha'_i \alpha'_j) \\
&= 4F(\lambda) \text{Ind}(1, \alpha'_i, \alpha'_j, \alpha'_l, \alpha'_m, \alpha'_k d_{ij}^k + \sum_{t \neq k} d_{ij}^t \alpha'_t) \\
&= 4F(\lambda) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i & j & l & m & k \end{pmatrix} \text{Ind}(1, \alpha'_1, \dots, \alpha'_5) \cdot d_{ij}^k.
\end{aligned}$$

Denote $h = F(\lambda) \text{Ind}(\alpha'_0, \dots, \alpha'_5)$. We can similarly show the following equations

$$\begin{aligned}
hd_{ij}^k &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i & j & l & m & k \end{pmatrix} ([iijjlm] - [iijjml])/4 \\
hd_{ii}^j &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i & k & l & m & j \end{pmatrix} ([iiklm] - [iikml])/2 \\
h(d_{ij}^j - d_{ik}^k) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i & k & l & m & j \end{pmatrix} ([iijkml] - [iijkml])/2 \\
h(d_{ii}^i - d_{ij}^j - d_{ik}^k) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ j & k & l & m & i \end{pmatrix} ([ijiklm] - [ijikml]).
\end{aligned}$$

Compare them with (*) to obtain

$$\begin{aligned}
hd_{ij}^k &= c_{ij}^k \\
hd_{ii}^j &= c_{ii}^j \\
h(d_{ij}^j - d_{ik}^k) &= (c_{ij}^j - c_{ik}^k) \\
h(d_{ii}^i - d_{ij}^j - d_{ik}^k) &= (c_{ii}^i - c_{ij}^j - c_{ik}^k).
\end{aligned}$$

Here we assume that we have chosen $c_{pq}^r = c_{pq}^r(A)$ satisfying $(*)$ and $(**)$ and

$$(c_{12}^2, c_{21}^1, c_{31}^1, c_{41}^1, c_{51}^1) = (0, 0, 0, 0, 0).$$

Now define $\alpha_0, \alpha_1, \dots, \alpha_5 \in \mathbb{C}^6$ as follows.

$$\alpha_0 = \alpha'_0 = (1, 1, 1, 1, 1, 1), \quad \alpha_i = h\alpha'_i + u_i\alpha_0$$

($u_i \in \mathbb{C}$) As those are linearly independent in $\mathbb{C}^6$, we can similarly define $e_{pq}^r \in \mathbb{C}$ ($1 \leq p, q \leq 5, 0 \leq r \leq 5$) satisfying

$$\alpha_i \alpha_j = \sum_{k=0}^5 e_{ij}^k \alpha_k \text{ in } \mathbb{C}^6.$$

Then it follows

$$\begin{aligned} hd_{ij}^k &= e_{ij}^k \\ hd_{ii}^j &= e_{ii}^j \\ h(d_{ij}^j - d_{ik}^k) &= (e_{ij}^j - e_{ik}^k) \\ h(d_{ii}^i - d_{ij}^j - d_{ik}^k) &= (e_{ii}^i - e_{ij}^j - e_{ik}^k) \end{aligned}$$

regardless of u_i . We can pick u_i such that e_{pq}^r satisfies

$$(e_{12}^2, e_{21}^1, e_{31}^1, e_{41}^1, e_{51}^1) = (0, 0, 0, 0, 0)$$

then it follows $e_{pq}^r = c_{pq}^r$ for all $p, q, r \in \{1, 2, 3, 4, 5\}$. For $r = 0$, both c_{pq}^0 and e_{pq}^0 should satisfy $(**)$, the equation which follows from associativity, so $e_{pq}^r = c_{pq}^r$ for any p, q, r . This means that the $\mathbb{Z}$ -module

$$\alpha_0\mathbb{Z} + \alpha_1\mathbb{Z} + \dots + \alpha_5\mathbb{Z} \subset \mathbb{C}^6$$

forms a subring of $\mathbb{C}^6$ which is isomorphic to $\mathcal{R}(A)$. In this perspective, $\mathcal{R}(A)$ allows six embeddings into $\mathbb{C}$ given as composition of inclusion $\mathcal{R}(A) \hookrightarrow \mathbb{C}^6$ and projection $\mathbb{C}^6 \rightarrow \mathbb{C}$ to six coordinates. As $\alpha_0, \dots, \alpha_5$ are linearly independent, those morphisms are distinct on $\mathcal{R}(A)$. This shows that $\mathcal{R}(A)$ is nondegenerate, and we can consider dual basis elements $\alpha_i^* \in \mathcal{R}(A) \otimes \mathbb{Q}$. Also if we call those homomorphisms $\rho_1, \dots, \rho_6$, then we can write

$$\alpha_i = (\rho_1(\alpha_i), \rho_2(\alpha_i), \dots, \rho_6(\alpha_i)).$$

Now we go back to description

$$G_2^t A_i G_1 = M(\alpha'_i)$$

where

$$M(z_1, z_2, z_3, z_4, z_5, z_6) = \begin{pmatrix} \lambda_{14}(z_1 - z_4) & \lambda_{15}(z_1 - z_5) & \lambda_{16}(z_1 - z_6) \\ \lambda_{24}(z_2 - z_4) & \lambda_{25}(z_2 - z_5) & \lambda_{26}(z_2 - z_6) \\ \lambda_{34}(z_3 - z_4) & \lambda_{35}(z_3 - z_5) & \lambda_{36}(z_3 - z_6) \end{pmatrix}.$$

As $M(1, 1, 1, 1, 1, 1) = 0$ and $M(cz) = cM(z)$ for $z \in \mathbb{C}^6$, we have

$$M(\alpha_i) = M(h \cdot \alpha'_i + u_i \alpha'_0) = h G_2^t A_i G_1.$$

Using this, we can finally show the following for any $i = 1, \dots, 6$,

$$\text{rk} \left(\sum_{j=1}^5 \rho_i(\alpha_j^*) A_j \right) \leq 1$$

as

$$\begin{aligned}
\sum_{j=1}^5 \rho_i(\alpha_j^*) A_j &= h^{-1} (G_2^{-1})^t \left(\sum_{j=1}^5 \rho_i(\alpha_j^*) M(\alpha_j) \right) G_1^{-1} \\
&= h^{-1} (G_2^{-1})^t M \left(\sum_{j=1}^5 \rho_i(\alpha_j^*) \alpha_j \right) G_1^{-1} \\
&= h^{-1} (G_2^{-1})^t M \left(\sum_{j=0}^5 \rho_i(\alpha_j^*) \alpha_j \right) G_1^{-1} \\
&= h^{-1} (G_2^{-1})^t M ((\delta_{ik})_{k=1, \dots, 6}) G_1^{-1}
\end{aligned}$$

and it is clear that $M(1, 0, 0, 0, 0, 0), M(0, 1, 0, 0, 0, 0), \dots, M(0, 0, 0, 0, 0, 1)$ are all of rank 1. Thus the following six points

$$[\rho_i(\alpha_1^*); \rho_i(\alpha_2^*); \rho_i(\alpha_3^*); \rho_i(\alpha_4^*); \rho_i(\alpha_5^*)]$$

are all in $\mathfrak{X}_{t,A}$. □

We describe how the natural action of $G(\mathbb{Z})$ on A acts on $\mathcal{R}(A)$ and its basis in covariant way. In particular, we show that the ring isomorphism class of $\mathcal{R}(A)$ is invariant under this action of $G(\mathbb{Z})$.

Proposition 3.3.3. *1. The ring $\mathcal{R}(A)$ and basis $\alpha_1, \dots, \alpha_5$ of $\mathcal{R}(A)/\mathbb{Z}$ are invariant under the $\mathrm{SL}_3 \times \mathrm{SL}_3$ -action and the transpose action $A \mapsto A^t$.*

2. For $G = (g_{ij})_{i,j=1, \dots, 5} \in \mathrm{Mat}_{5 \times 5}(\mathbb{Z}) \cap \mathrm{GL}_5(\mathbb{Q})$, it acts on α_i as standard action times $\det(G)$. For $\alpha_1, \dots, \alpha_5$ the basis of $\mathcal{R}(A)/\mathbb{Z}$, consider

$$\alpha'_i = \sum_{j=1}^5 \det(G) g_{ij} \alpha_j.$$

Then α'_i s form a subring of $\mathcal{R}(A)$ which is isomorphic to basis of $\mathcal{R}(GA)$.

The first part can be checked from invariance of Φ under $\mathrm{SL}_3 \times \mathrm{SL}_3$ -action and

transpose, and the second part can be checked from explicit calculation. As G acts on α_i as standard action times $\det(G)$, one can see G acts on $[\rho_i(\alpha_1^*); \dots; \rho_i(\alpha_5^*)]$ as standard projective action by G^{-1} . It is consistent with how $\mathfrak{X}_{t,GA}$ is related to $\mathfrak{X}_{t,A}$.

Definition 3.3.4. *Define the discriminant of $3 \times 3 \times 5$ form A as discriminant of the ring $\mathcal{R}(A)$, and denote it as $\text{Disc}(A)$.*

From the above proposition and descriptions of $c_{ij}^k(A)$, we can see that $\text{Disc}(A)$ is degree 60 homogeneous polynomial in entries of A which is $\text{SL}_3 \times \text{SL}_3 \times \text{SL}_5$ -invariant. Theorem 3.3.1 shows that $\text{Disc}(A)$ is nonzero for $A \in V_{n.d.}(\mathbb{Z})$. We will soon prove that the converse holds.

3.3.2 Hypercohomology ring of a complex on $\mathbb{P}^4$

We describe an alternative way of defining $\mathcal{R}(A)$ as hypercohomology ring of certain complexes of sheaves on projective space. We will construct two complexes $\mathcal{C}_{t,A}$ and $\mathcal{C}_{x,A}$, each of them becoming minimal free resolutions for $\mathfrak{X}_{t,A}$ and $\mathfrak{X}_{x,A}$ respectively for generic A .

The complex $\mathcal{C}_{t,A}$ will have the following form ($\mathcal{O} = \mathcal{O}_{\mathbb{P}^4}$)

$$\mathcal{C}_{t,A} : \mathcal{O}(-6) \rightarrow \mathcal{O}(-4)^9 \rightarrow \mathcal{O}(-3)^{16} \rightarrow \mathcal{O}(-2)^9 \rightarrow \mathcal{O}.$$

The image of $\mathcal{O}(-2)^9 \rightarrow \mathcal{O}$ will become ideal sheaf $\mathcal{I}_{t,A}$ for $\mathfrak{X}_{t,A}$ for generic A . We will first describe $\mathcal{C}_{t,A}$ geometrically, and apply the hypercohomology construction established in the Chapter 2.

The condition $\text{rk}(A(t)) \leq 1$ which defines $\mathfrak{X}_{t,A}$ is equivalent to that $\text{adj}(A(t)) = 0$, where adj denotes the classical adjugate. Write $\alpha_{ij} = \alpha_{ij}(t)$ as (i, j) -entry of $\text{adj}(A(t))$, and define $\mathcal{I}_{t,A}$ as the ideal sheaf spanned by quadrics α_{ij} . Then $\mathcal{I}_{t,A}$ becomes ideal sheaf for $\mathfrak{X}_{t,A}$.

Our complex $\mathcal{C}_{t,A}$ will be constructed as minimal free resolution of $\mathcal{I}_{t,A}$ over formal variables. We give its description as follows:

$$\mathcal{C}_{t,A} : 0 \rightarrow \mathcal{O}(-6) \otimes \Lambda_6 \rightarrow \mathcal{O}(-4) \otimes \Lambda_4 \rightarrow \mathcal{O}(-3) \otimes \Lambda_3 \rightarrow \mathcal{O}(-2) \otimes \Lambda_2 \rightarrow \mathcal{O} \rightarrow 0$$

We define $\Lambda_6, \Lambda_4, \Lambda_3, \Lambda_2$ as free $\mathbb{Z}$ -modules of rank 1, 9, 16, 9 respectively, and describe their basis elements and their images under complex map of $\mathcal{C}_{t,A}$ as follows.

- Λ_2 has 9 basis elements $L_2^{i,j}$ ($1 \leq i, j \leq 3$), and those elements are mapped to d as $L_2^{i,j} \mapsto \alpha_{ij}$. If we write L_2 as matrix having $L_2^{i,j}$ as (i, j) -entry,

$$L_2 \mapsto \text{adj}(A(t)).$$

- Λ_3 is constructed as follows. Consider a free $\mathbb{Z}$ -module Λ_3^{pre} having 18 elements $L_{3,l}^{i,j}, L_{3,r}^{i,j}$ ($1 \leq i, j \leq 3$) as $\mathbb{Z}$ -basis. Then consider a complex

$$\mathbb{Z} \rightarrow \Lambda_3^{pre} \rightarrow \mathbb{Z}$$

where two maps are given as

$$d_1 : 1 \mapsto \sum_{i=1}^3 (L_{3,l}^{i,i} + L_{3,r}^{i,i}), \quad d_2 : L_{3,l}^{i,j} \mapsto \delta_{ij}, L_{3,r}^{i,j} \mapsto -\delta_{ij}.$$

Define Λ_3 as cohomology at the Λ_3^{pre} place, which becomes free $\mathbb{Z}$ -module of rank 16. Denote $L_{3,l}, L_{3,r}$ be 3×3 matrices having $L_{3,l}^{i,j}, L_{3,r}^{i,j}$ as their (i, j) -entries respectively. Under these notations, the complex maps d from Λ_3 can be described as

$$L_{3,l} \mapsto A(t)L_2, \quad L_{3,r} \mapsto L_2A(t).$$

Here juxtaposition means the matrix multiplication.

- Λ_4 has 9 basis elements $L_4^{i,j}$ ($1 \leq i, j \leq 3$). If we write L_4 as matrix having $L_4^{i,j}$ as (i, j) -entry, then image of d can be represented as

$$L_4 \mapsto A(t)L_{3,r} - L_{3,l}A(t).$$

- Λ_6 has 1 basis element L_6 , which is mapped to d by

$$L_6 \mapsto \text{tr}(L_4 \text{adj}(A(t))).$$

We check that $\mathcal{C}_{t,A}$ is a well-defined complex. We first note the Cramer's rule

$$\text{adj}(A(t))A(t) = A(t)\text{adj}(A(t)) = \det(A(t))I.$$

Then we can calculate

$$\begin{aligned} d^2(L_6) &= d(\text{tr}(L_4 \text{adj}(A(t)))) \\ &= \text{tr}(A(t)L_{3,r} \text{adj}(A(t))) - \text{tr}(L_{3,l}A(t) \text{adj}(A(t))) \\ &= \text{tr}(\text{adj}(A(t))A(t)L_{3,r}) - \text{tr}(L_{3,l}A(t) \text{adj}(A(t))) \\ &= \det(A(t))\text{tr}(L_{3,r}) - \det(A(t))\text{tr}(L_{3,l}) = 0 \end{aligned}$$

and

$$\begin{aligned} d^2 L_4 &= d(A(t)L_{3,r} - L_{3,l}A(t)) \\ &= A(t)L_2 A(t) - A(t)L_2 A(t) = 0. \end{aligned}$$

For Λ_2 place, we have

$$d^2 L_{3,l} = d(A(t)L_2) = \det(A(t))I, \quad d^2 L_{3,r} = d(L_2 A(t)) = \det(A(t))I.$$

and this implies that d^2 is zero on Λ_3 .

The complex $\mathcal{C}_{t,A}$ has form of minimal free resolution for 6 points in $\mathbb{P}^4$, (See Chapter 2, Proposition 2.5.4) so it satisfies the WTM conditions (Definition 2.3.4). Thus we can apply the Theorem 2.3.3 to $\mathcal{C}_{t,A}$ to obtain the following exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(\mathcal{C}_{t,A}) \rightarrow H^4(\mathcal{O}(-6) \otimes \Lambda_6) \rightarrow 0.$$

The cohomology $H^4(\mathcal{O}(-6) \otimes \Lambda_6)$ can be identified with free $\mathbb{Z}$ -module with basis $L_6/(t_1 \cdots t_5 t_i)$ for $i = 1, \dots, 5$. Also by considering the Čech complex

$$C_\bullet(\mathbb{P}^k) : \bigoplus \mathcal{O}_{t_i} \rightarrow \bigoplus \mathcal{O}_{t_i t_j} \rightarrow \cdots \rightarrow \bigoplus \mathcal{O}_{t_1 t_2 t_3 t_4 t_5},$$

we can identify $H^0 R\Gamma(\mathcal{C}_{t,A})$ with $H^0 \text{Tot}\Gamma(\mathcal{C}_{t,A} \otimes C_\bullet(\mathbb{P}^k))$ as in the Theorem 2.3.7, especially using the algorithm in the Proposition 2.3.8. Consider the map

$$\varphi : H^0 R\Gamma(\mathcal{C}_{t,A}) \simeq H^0 \text{Tot}\Gamma(\mathcal{C}_{t,A} \otimes C_\bullet(\mathbb{P}^k)) \rightarrow H^0 \Gamma(\mathcal{O} \otimes \bigoplus \mathcal{O}_{t_i}) \simeq \bigoplus \mathbb{Z}[t_1/t_i, \dots, t_5/t_i].$$

For any u , $\varphi(u)$ consists of 5 Laurent polynomials whose differences between them are in ideal $\mathcal{I}_{t,A}$. This follows from that the image of $\varphi(u)$ under the complex map of $C_\bullet(\mathbb{P}^k)$ sits inside image of the map $\mathcal{O}(-2) \otimes \Lambda_2 \rightarrow \mathcal{O}$ of $\mathcal{C}_{t,A}$. For $i = 1, \dots, 5$, denote α_i as a preimage of $L_6/(t_1 \cdots t_5 t_i)$ under the map $H^0 R\Gamma(\mathcal{C}_{t,A}) \rightarrow H^4(\mathcal{O}(-6) \otimes \Lambda_6)$ (so that it can vary by integers). Then one can show that for some appropriate choice of $\alpha_1, \dots, \alpha_5$, those satisfy the following relations

$$\varphi(\alpha_i)\varphi(\alpha_j) \equiv \sum_{k=1}^5 \varphi(\alpha_k)c_{ij}^k(A) + c_{ij}^0(1, 1, 1, 1, 1) \pmod{\mathcal{I}_{t,A}}$$

where $c_{ij}^k(A)$ are multiplication coefficients of $\mathcal{R}(A)$. This can be checked by some explicit polynomial calculations.

The right way to define ring structure on $H^0 R\Gamma(\mathcal{C}_{t,A})$ is to use DG-algebra structure on $\mathcal{C}_{t,A}$. While we haven't specified a DG-algebra structure for $\mathcal{C}_{t,A}$ explicitly, it is possible to give it in indirect way so that it induces the multiplicative structure of $\mathcal{R}(A)$ on hypercohomology.

Proposition 3.3.5. *There exists a DG-algebra structure on $\mathcal{C}_{t,A}$ such that the induced ring structure on $H^0 R\Gamma(\mathcal{C}_{t,A})$ is isomorphic to $\mathcal{R}(A)$.*

Proof. We first note that $\mathcal{C}_{t,A}$ becomes minimal free resolution for six points of $\mathfrak{X}_{t,A}$ for A in generic subset of $V_{n,d}$. Generally, a minimal free resolution is constructed by repeatedly adding all lowest degree syzygies of previous map. If $\mathfrak{X}_{t,A}$ consists of six points in linearly general position, then $\mathcal{C}_{t,A}$ is constructed exactly in this way, provided that all of its maps are exact at lowest degree. As it can be expressed as the condition of matrices having maximal rank, the set of such A is Zariski open. To show such set is nonempty, it suffices to pick one numerical element $A \in V(\mathbb{Z})$ satisfying the condition (the example I used was of form Proposition 3.2.4, letting λ_{ij} as some integers.)

Then we can apply ([46], Theorem 136) for sextic cover $\mathfrak{X}_{t,A}$ over base of universal variables $\mathbb{Q}[A]$, and it allows us to construct a DG-algebra structure on $\mathcal{C}_{t,A}$ with coefficients in $\mathbb{Q}[A]$. The hypercohomology ring $H^0 R\Gamma(\mathcal{C}_{t,A})$ with multiplication induced by this DG-algebra becomes rank 6 algebra over base $\mathbb{Q}[A]$.

When A takes an explicit numerical value of $A = A_0$, this ring specializes to the ring structure $H^0 R\Gamma(\mathcal{C}_{t,A_0})$ naturally, as we showed that hypercohomology rings for complexes whose entries consisting of $\mathcal{O}(k)$ behave consistently with the base change. Meanwhile, in the Theorem 2.5.1, we showed that $H^0 R\Gamma(\mathcal{C}_{t,A_0})$ injects into $\Gamma(\mathcal{O}/\mathcal{I})$ if $\mathcal{C}_{t,A_0}$ is exact over $\mathbb{Q}$. The image of $H^0 R\Gamma(\mathcal{C}_{t,A_0})$ inside $\Gamma(\mathcal{O}/\mathcal{I})$ is described exactly by φ , so it follows that $H^0 R\Gamma(\mathcal{C}_{t,A_0})$ is isomorphic to $\mathcal{R}(A_0)$ for generic A_0 . As multiplication coefficients of $\mathcal{R}(A)$ are given as polynomials in A , this shows that $H^0 R\Gamma(\mathcal{C}_{t,A})$ is isomorphic to $\mathcal{R}(A)$ over universal variables $\mathbb{Q}[A]$ (thus over $\mathbb{Z}[A]$). $\square$

3.3.3 Hypercohomology ring of a complex on $\mathbb{P}^2$

We similarly define $\mathcal{C}_{x,A}$ of sheaves on $\mathbb{P}^2$, which is related to six points of $\mathfrak{X}_{x,A} \subset \mathbb{P}^2$. We will similarly first describe $\mathcal{C}_{t,A}$ from geometry of $\mathfrak{X}_{x,A}$, and apply the hypercohomology construction.

To find a system of equations for $\mathfrak{X}_{x,A}$, we want to eliminate t and y from the relation $A(t) = xy^t$, which is equivalent to

$$\sum_{i=1}^5 t_i A_i - y_1(xe_1^t) - y_2(xe_2^t) - y_3(xe_3^t) = 0.$$

Here $(e_1, e_2, e_3) = ((1, 0, 0)^t, (0, 1, 0)^t, (0, 0, 1)^t)$. Thus it is again equivalent to that $A_1, \dots, A_5, xe_1^t, xe_2^t, xe_3^t$ are linearly dependent. We identify the space of 3×3 matrix with k^9 and consider the following determinants

$$\psi_{ij} = \det(A_1, A_2, A_3, A_4, A_5, xe_1^t, xe_2^t, xe_3^t, e_i e_j^t).$$

It follows that the vanishing ideal of $\mathfrak{X}_{x,A}$ is spanned by ψ_{ij} . Also we can find the following syzygies between them:

$$\begin{aligned} \det(A_1, \dots, A_5, xe_1^t, xe_2^t, xe_3^t, A_k) &= \sum_{i,j=1}^3 (A_k)_{ij} \psi_{ij} = 0 \\ \det(A_1, \dots, A_5, xe_1^t, xe_2^t, xe_3^t, xe_l^t) &= \sum_{i=1}^3 x_i \psi_{il} = 0. \end{aligned}$$

We construct $\mathcal{C}_{x,A}$ using generators ψ_{ij} of the vanishing ideal and their syzygies above. For $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2}$, it has the form

$$0 \rightarrow \mathcal{O}(-3) \otimes \Lambda_{3,2} \oplus \mathcal{O}(-4) \otimes \Lambda_{4,2} \rightarrow \mathcal{O}(-3) \otimes \Lambda_3 \rightarrow \mathcal{O} \rightarrow 0$$

where

$$\begin{aligned}\Lambda_{3,2} &= \mathbb{Z}L_{3,2}^1 \oplus \mathbb{Z}L_{3,2}^2 \oplus \cdots \oplus \mathbb{Z}L_{3,2}^5 \simeq \mathbb{Z}^5, \\ \Lambda_{4,2} &= \mathbb{Z}L_{4,2}^1 \oplus \mathbb{Z}L_{4,2}^2 \oplus \mathbb{Z}L_{4,2}^3 \simeq \mathbb{Z}^3, \\ \Lambda_3 &= \bigoplus_{i,j=1}^3 \mathbb{Z}L_3^{i,j} \simeq \mathbb{Z}^9.\end{aligned}$$

and its complex maps are defined as

$$\begin{aligned}L_3^{i,j} &\mapsto \psi_{ij}, \\ L_{4,2}^i &\mapsto \sum_{k=1}^3 y_k L_3^{i,k}, \\ L_{3,2}^t &\mapsto \sum_{i,j=1}^3 (A_t)_{ij} L_3^{i,j}.\end{aligned}$$

If we denote L_3 as a length 9 vector identified with 3×3 matrix $L_3^{i,j}$, then we can give simpler descriptions as follows:

$$L_{4,2}^i \mapsto (xe_i^t) \cdot L_3, \quad L_{3,2}^t \mapsto A_t \cdot L_3.$$

One can also explicitly describe a DG-algebra structure on $\mathcal{C}_{x,A}$ where $\mathcal{O}$ becomes unit. We only need to describe multiplications between L_3 , as all other multiplications are trivial. For M_1, M_2 length 9 vectors identified with 3×3 matrix with integral entries,

multiplication between $M_1 \cdot L_3$ and $M_2 \cdot M_3$ is given as

$$\begin{aligned}
(M_1 \cdot L_3) \cdot (M_2 \cdot L_3) = & -\det(M_1, A_2, A_3, A_4, A_5, xe_1^t, xe_2^t, xe_3^t, M_2)L_{3,2}^1 \\
& -\det(A_1, M_1, A_3, A_4, A_5, xe_1^t, xe_2^t, xe_3^t, M_2)L_{3,2}^2 \\
& -\det(A_1, A_2, M_1, A_4, A_5, xe_1^t, xe_2^t, xe_3^t, M_2)L_{3,2}^3 \\
& -\det(A_1, A_2, A_3, M_1, A_5, xe_1^t, xe_2^t, xe_3^t, M_2)L_{3,2}^4 \\
& -\det(A_1, A_2, A_3, A_4, M_1, xe_1^t, xe_2^t, xe_3^t, M_2)L_{3,2}^5 \\
& -\det(A_1, A_2, A_3, A_4, A_5, M_1, xe_2^t, xe_3^t, M_2)L_{4,2}^1 \\
& -\det(A_1, A_2, A_3, A_4, A_5, xe_1^t, M_2, xe_3^t, M_2)L_{4,2}^2 \\
& -\det(A_1, A_2, A_3, A_4, A_5, xe_1^t, xe_3^t, M_1, M_2)L_{4,2}^3.
\end{aligned}$$

To check that this is well-defined DG-algebra, we note that

$$d(M \cdot L_3) = \det(A_1, A_2, A_3, A_4, A_5, xe_1^t, xe_2^t, xe_3^t, M)$$

so

$$\begin{aligned}
& d(M_1 \cdot L_3)(M_2 \cdot L_3) - d(M_2 \cdot L_3)(M_1 \cdot L_3) \\
& = \det(A_1, A_2, A_3, A_4, A_5, xe_1^t, xe_2^t, xe_3^t, M_1)(M_2 \cdot L_2) \\
& - \det(A_1, A_2, A_3, A_4, A_5, xe_1^t, xe_2^t, xe_3^t, M_2)(M_1 \cdot L_2)
\end{aligned}$$

and

$$\begin{aligned}
d((M_1 \cdot L_3) \cdot (M_2 \cdot L_3)) = & -\det(M_1, A_2, A_3, A_4, A_5, xe_1^t, xe_2^t, xe_3^t, M_2)(A_1 \cdot L_3) \\
& -\det(A_1, M_1, A_3, A_4, A_5, xe_1^t, xe_2^t, xe_3^t, M_2)(A_2 \cdot L_3) \\
& -\det(A_1, A_2, M_1, A_4, A_5, xe_1^t, xe_2^t, xe_3^t, M_2)(A_3 \cdot L_3) \\
& -\det(A_1, A_2, A_3, M_1, A_5, xe_1^t, xe_2^t, xe_3^t, M_2)(A_4 \cdot L_3) \\
& -\det(A_1, A_2, A_3, A_4, M_1, xe_1^t, xe_2^t, xe_3^t, M_2)(A_5 \cdot L_3) \\
& -\det(A_1, A_2, A_3, A_4, A_5, M_1, xe_2^t, xe_3^t, M_2)(xe_1^t) \cdot L_3 \\
& -\det(A_1, A_2, A_3, A_4, A_5, xe_1^t, M_2, xe_3^t, M_2)(xe_2^t) \cdot L_3 \\
& -\det(A_1, A_2, A_3, A_4, A_5, xe_1^t, xe_3^t, M_1, M_2)(xe_3^t) \cdot L_3.
\end{aligned}$$

Showing that their difference is zero can be interpreted as a special case of the Plücker relations.

Proposition 3.3.6. *The hypercohomology ring $H^0 R\Gamma(\mathcal{C}_{x,A})$ is isomorphic to $\mathcal{R}(A)$.*

Proof. We would like to apply the Theorem 2.3.3 to $\mathcal{C}_{x,A}$. The complex $\mathcal{C}_{x,A}$ has form

$$\mathcal{O}(-3)^5 \oplus \mathcal{O}(-4)^3 \rightarrow \mathcal{O}(-3)^9 \rightarrow \mathcal{O} \rightarrow 0$$

so it satisfies the WTM conditions if the following $\mathbb{Z}$ -module

$$T = \ker(H^2(\mathcal{O}(-3)^5 \oplus \mathcal{O}(-4)^3) \rightarrow H^2(\mathcal{O}(-3)^9))$$

is isomorphic to $\mathbb{Z}^5$. We note

$$H^2(\mathcal{O}(-3)) \simeq \frac{1}{x_1 x_2 x_3} \mathbb{Z}, \quad H^2(\mathcal{O}(-4)) \simeq \frac{1}{x_1^2 x_2 x_3} \mathbb{Z} \oplus \frac{1}{x_1 x_2^2 x_3} \mathbb{Z} \oplus \frac{1}{x_1 x_2 x_3^2} \mathbb{Z}.$$

More exactly, representatives of $H^2(\mathcal{O}(-k))$ can be considered as Laurent polynomials

in x_1, x_2, x_3 of degree $-k$ whose degree in each y_i is less than zero. A particular element of $H^2(\mathcal{O}(-3)^5 \oplus \mathcal{O}(-4)^3)$ of $\mathcal{C}_{x,A}$ can be described as

$$\gamma = \sum_{t=1}^5 \frac{\alpha_t L_{3,2}^t}{x_1 x_2 x_3} + \sum_{i,j=1}^3 \frac{\mu_{i,j} L_{4,2}^i}{x_j x_1 x_2 x_3}$$

and it is mapped to

$$d\gamma = \frac{1}{x_1 x_2 x_3} \sum_{i,j=1}^3 \left(\mu_{i,j} L_3^{i,j} + \sum_{t=1}^5 \alpha_t (A_t e_i \cdot e_j) L_3^{i,j} \right) + \frac{1}{x_1 x_2 x_3} \sum_{i,j=1}^3 \sum_{k \neq j} \frac{x_k}{x_j} \mu_{i,j} L_3^{i,k}$$

and the later term vanishes in $H^2(\mathcal{O}(-3) \otimes \Lambda_3)$. To make $d\alpha = 0$ in $H^2(\mathcal{O}(-3) \otimes \Lambda_3)$ we require

$$\mu_{i,j} = - \sum_{t=1}^5 \alpha_t (A_t)_{ij}$$

and this shows T is isomorphic to $\mathbb{Z}^5$ as $\mathbb{Z}$ -module, with basis elements

$$\gamma_t = \frac{1}{x_1 x_2 x_3} \left(L_{3,2}^t - \sum_{i,j=1}^3 (A_t e_i \cdot e_j) \frac{L_{4,2}^i}{x_j} \right), \quad 1 \leq t \leq 5.$$

Thus Theorem 2.3.3 gives the following exact sequence of $\mathbb{Z}$ -module

$$0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(\mathcal{C}_{x,A}) \rightarrow T \rightarrow 0.$$

One can apply the Theorem 2.4.2 to explicitly calculate the ring structure on $H^0 R\Gamma(\mathcal{C}_{x,A})$.

If we denote α_t as a preimage of γ_t under $H^0 R\Gamma(\mathcal{C}_{x,A}) \rightarrow T$, then together with $\alpha_0 = 1$ they satisfy the relation

$$\alpha_i \alpha_j = \sum_{k=0}^5 c_{ij}^k(A) \alpha_k$$

for appropriate choices of α_i . This shows $H^0 R\Gamma(\mathcal{C}_{x,A}) \simeq \mathcal{R}(A)$. □

3.3.4 Proof of the Theorem 3.1.2

We are now finally ready to prove that $\text{Disc}(A) \neq 0$ suffices to characterize forms determining 6 distinct points.

Theorem 3.3.7. *The discriminant $\text{Disc}(A)$ is nonzero if and only if $A \in V_{n.d.}$. In that case, $\mathcal{C}_{t,A} \otimes \mathbb{Q}$ and $\mathcal{C}_{x,A} \otimes \mathbb{Q}$ each becomes minimal free resolution of $\mathfrak{X}_{t,A} \otimes \mathbb{Q}$ and $\mathfrak{X}_{x,A} \otimes \mathbb{Q}$ respectively.*

Proof. We first apply the Theorem 2.5.5 on $\mathcal{C}_{t,A}$. As $\mathcal{R}(A) \simeq H^0 R\Gamma(\mathcal{C}_{t,A})$ is nondegenerate ring, $V(\mathcal{I}_{t,A}) = \mathfrak{X}_{t,A}$ consists of 6 points in $\mathbb{P}^4$ in linearly general position. This shows $\text{Disc}(A) \neq 0$ implies $A \in V_{n.d.}$. We have already shown that $A \in V_{n.d.}$ implies $\text{Disc}(A) \neq 0$ in Theorem 3.3.1, so those two statements are equivalent.

It remains to show that $\mathcal{C}_{t,A} \otimes \mathbb{Q}$ and $\mathcal{C}_{x,A} \otimes \mathbb{Q}$ are minimal free resolutions of $\mathfrak{X}_{t,A} \otimes \mathbb{Q}$ and $\mathfrak{X}_{x,A} \otimes \mathbb{Q}$ respectively. First we note it suffices to prove statement over algebraically closed field $\overline{\mathbb{Q}}$. We use Proposition 3.2.4 to find an element N in the same $G(\overline{\mathbb{Q}})$ -orbit of A having the following form

$$\sum_{i=1}^6 N_i t_i = \begin{pmatrix} \lambda_{14}(t_1 - t_4) & \lambda_{15}(t_1 - t_5) & \lambda_{16}(t_1 - t_6) \\ \lambda_{24}(t_2 - t_4) & \lambda_{25}(t_2 - t_5) & \lambda_{26}(t_2 - t_6) \\ \lambda_{34}(t_3 - t_4) & \lambda_{35}(t_3 - t_5) & \lambda_{36}(t_3 - t_6) \end{pmatrix}.$$

As the actions of G on $\mathcal{C}_{t,A}$ and $\mathcal{C}_{x,A}$ should be given as linear isomorphisms, proving our statement for N will suffice. From now, we can simply assume that $A = N$ takes the above form.

We can observe that $\mathcal{C}_{t,A}$ has same Betti numbers as a minimal free resolution of 6 points in $\mathbb{P}^4$ in general position. The minimal free resolution of P has form

$$\mathcal{C}_P : \mathcal{O}(-6) \rightarrow \mathcal{O}(-4)^9 \rightarrow \mathcal{O}(-3)^{16} \rightarrow \mathcal{O}(-2)^9 \rightarrow \mathcal{O}$$

where each map is given by a complete set of syzygies of lowest possible degree - map

from $\mathcal{O}(-3)^{16}$ given as all degree 1 syzygies of 9 quadrics, map from $\mathcal{O}(-4)^9$ given as all degree 1 syzygies of those 16 syzygies, etc.. Note that complex $\mathcal{C}_{t,A}$

$$\mathcal{O}(-6) \otimes \Lambda_6 \rightarrow \mathcal{O}(-4) \otimes \Lambda_4 \rightarrow \mathcal{O}(-3) \otimes \Lambda_3 \rightarrow \mathcal{O}(-2) \otimes \Lambda_2 \rightarrow \mathcal{O}$$

has exactly same Betti numbers. Thus if one checks the maps $\mathcal{O}(-n_k) \otimes \Lambda_{n_k} \rightarrow \mathcal{O}(-n_{k-1}) \otimes \Lambda_{n_{k-1}}$ are injective at degree n_k for any k , it follows that the maps of $\mathcal{C}_{t,A}$ are also given as complete set of lowest possible degree syzygies. This shows that $\mathcal{C}_{t,A}$ is isomorphic to $\mathcal{C}_P$ via linear transformation and $\mathcal{C}_{t,A}$ should also become minimal free resolution for P . Checking injectiveness of map $\mathcal{O}(-n_k) \otimes \Lambda_{n_k} \rightarrow \mathcal{O}(-n_{k-1}) \otimes \Lambda_{n_{k-1}}$ at degree n_k can be done in elementary way by calculating minors of matrices having entries which are polynomials in λ_{ij} and showing some of them is nonzero.

We can proceed similarly for $\mathcal{C}_{x,A}$. The minimal free resolution of six points in $\mathbb{P}^2$ which are not on a conic has form

$$\mathcal{O}(-4)^3 \rightarrow \mathcal{O}(-3)^4 \rightarrow \mathcal{O}$$

where each map is given as complete set of syzygies of lowest possible degree. The complex $\mathcal{C}_{x,A}$ given as

$$\mathcal{O}(-3) \otimes \Lambda_{3,2} \oplus \mathcal{O}(-4) \otimes \Lambda_{4,2} \rightarrow \mathcal{O}(-3) \otimes \Lambda_3 \rightarrow \mathcal{O}$$

is not minimal, because of linear relations between cubics coming from $\mathcal{O}(-3) \otimes \Lambda_{3,2}$. But if one can prove that $\mathcal{O}(-3) \otimes \Lambda_{3,2}$ is mapped into injectively, then above complex is homotopic to

$$\mathcal{C}'_{x,A} : \mathcal{O}(-4) \otimes \Lambda_{4,2} \rightarrow \mathcal{O}(-3) \otimes (\Lambda_3/d(\Lambda_{4,2})) \rightarrow \mathcal{O}$$

which has same Betti number as minimal free resolution. Thus if we can show $\mathcal{O}(-4) \otimes \Lambda_{4,2}$ is mapped injectively at degree 4 and map from $\mathcal{O}(-3) \otimes \Lambda_3$ spans all vanishing cubics, then $\mathcal{C}'_{x,A}$ becomes minimal free resolution for $\mathfrak{X}_{x,A}$. Again those injectiveness statements at finite degree can be verified from elementary calculation of matrix minors. $\square$

While the statement of the above theorem was about forms over $\mathbb{Q}$ or $\overline{\mathbb{Q}}$, the same argument should hold for any algebraically closed field of characteristic zero. As we have observed that $H^0 R\pi_*(\mathcal{C}_{t,A}) \simeq \Gamma(\mathfrak{X}_{t,A})$ if $\mathcal{C}_{t,A}$ becomes free resolution (for $A \in V(k)$ and $\mathcal{C}_{t,A}, X_A$ are all considered over k), Theorem 3.1.3 follows immediately from the result of the Theorem 3.3.2.

3.4 Descriptions of the invariants

For this section, we will only consider forms over field $k = \mathbb{Q}$, while most of the arguments should also hold for any fields having characteristic zero. We want to examine the behavior of the map

$$V_{D \neq 0}(\mathbb{Q})/G(\mathbb{Q}) \rightarrow \{\text{nondegenerate sextic } \mathbb{Q}\text{-algebras}\}, \quad A \mapsto \mathcal{R}(A).$$

It can be checked that it is surjective, but there are infinitely many forms giving same $\mathbb{Q}$ -algebras. One possible attempt to convert this map into a well-established correspondence is to look for an additional arithmetic object corresponding to $V_{D \neq 0}(\mathbb{Q})/G(\mathbb{Q})$. We remind the invariants of six points of $\mathfrak{X}_{x,A}$ discussed in Theorem 3.2.1, which implied the one-to-one correspondence between $V_{D \neq 0}(\mathbb{C})/G(\mathbb{C})$ and

$$\{(w_1, \dots, w_6, v) \in \mathbb{C}^7 : \sum_{i=1}^6 w_i = 0, \left(\sum_{i=1}^6 w_i^2 \right)^2 - 4 \sum_{i=1}^6 w_i^4 = -324v^2 \neq 0\} / \mathbb{C}^\times.$$

The main goal of this section is to construct an element s in a sextic resolvent algebra S of $\mathcal{R}(A)$ (which we will soon define) which can represent $(w_1, \dots, w_6)$ and prove the Theorem 3.1.4.

Theorem 3.4.1 (Restatement of the Theorem 3.1.4). *Let R be a nondegenerate sextic $\mathbb{Q}$ -algebra, and S be its sextic resolvent algebra (with respect to outer automorphism of S_6). For $A \in V(\mathbb{Q})$ such that $\mathcal{R}(A) \simeq R$, one can construct $s \in S$ corresponding to A satisfying following two properties.*

(1) *Let $(w_1, \dots, w_6) \in \mathbb{C}^6$ be values of the relative invariants of Y_A given via the correspondence of Theorem 2.1. For $\psi_1, \psi_2, \dots, \psi_6 : S \rightarrow \mathbb{C}$ be the distinct embeddings, we have*

$$[\psi_1(s); \dots; \psi_6(s)] = [w_1; \dots; w_6] \quad \text{in } \mathbb{P}_{\mathbb{C}}^5.$$

(2) *We have*

$$16\text{tr}_{S/\mathbb{Q}}(s^4) - 4\text{tr}_{S/\mathbb{Q}}(s^2)^2 = \text{Disc}(A).$$

For such A , the mapping $A \mapsto s$ is exactly 2-to-1, up to $G(\mathbb{Q})$ -equivalence.

3.4.1 Preliminaries: definition of the sextic resolvent algebra

For an outer automorphism φ of S_6 , we remind the definition of the sextic resolvent groups: we have $H_i = \varphi(G_i)$ where $G_i \leq S_6$ is a subgroup of S_6 fixing element i for $i = 1, \dots, 6$. As $G_1, \dots, G_6$ are conjugate to each other, $H_1, \dots, H_6$ are also conjugate to each other. Also one can check that $\{H_1, \dots, H_6\}$ does not depend on choice of outer automorphism φ . Throughout this section we will fix the choice of φ

and $(H_1, \dots, H_6)$ such that

$$\begin{aligned} & (H_1, H_2, H_3, H_4, H_5, H_6) \\ &= (\langle (14)(25)(36), (26345) \rangle, \langle (16)(23)(45), (15436) \rangle, \langle (12)(34)(56), (16342) \rangle, \\ & \quad \langle (12)(34)(56), (24365) \rangle, \langle (12)(34)(56), (15634) \rangle, \langle (12)(34)(56), (14362) \rangle). \end{aligned}$$

We will use these explicit representatives when it comes to calculate the invariants of six points in $\mathbb{P}^2$.

If our sextic $\mathbb{Q}$ algebra is a degree 6 number field having Galois group S_6 , its sextic resolvent algebra for a sextic resolvent group H can be constructed as a subalgebra of the Galois closure of R consisting of elements fixed by H . In order to generalize this construction to any sextic nondegenerate algebras, we need a notion called S_n closure, which can be found in [3] or [4]

Definition 3.4.2. *Let R be a nondegenerate ring of rank n . The S_n -closure of R is defined as $\overline{R} = R^{\otimes n} / I_R$. We define I_R^{pre} as ideal in $R^{\otimes n}$ generated by elements of the form (for $x \in R$)*

$$\begin{aligned} & (x \otimes 1 \otimes \dots \otimes 1) + (1 \otimes x \otimes \dots \otimes 1) + \dots + (1 \otimes 1 \otimes \dots \otimes x) \\ & - \text{tr}_{R/\mathbb{Z}}(x)(1 \otimes 1 \otimes \dots \otimes 1) \end{aligned}$$

and define $I_R = (I_R^{pre} \otimes_{\mathbb{Z}} \mathbb{Q}) \cap R^{\otimes k}$ as its $\mathbb{Z}$ -closure.

If R is $\mathbb{Q}$ -algebra of rank n , then its S_n -closure is $\mathbb{Q}$ -algebra defined analogously as $\overline{R} = R^{\otimes n} / I_R$ where $I_R = I_R^{pre}$ is $\mathbb{Q}$ -ideal of $R^{\otimes n}$ generated by elements of the form

$$\begin{aligned} & (x \otimes 1 \otimes \dots \otimes 1) + (1 \otimes x \otimes \dots \otimes 1) + \dots + (1 \otimes 1 \otimes \dots \otimes x) \\ & - \text{tr}_{R/\mathbb{Q}}(x)(1 \otimes 1 \otimes \dots \otimes 1). \end{aligned}$$

It holds that S_n -closure of a nondegenerate ring is ring of rank $n!$, with natural action of S_n induced from its permutation action on $R^{\otimes n}$. Also the fixed elements of $\overline{R}$ under action of $S_{n-1} \leq S_n$ (permutation group on $\{1, 2, \dots, n\} \setminus i$) is isomorphic to R . The notion of S_n -closure serves as an analogue of S_n -Galois extension of R defined for rings instead of fields.

Denote $\rho_i : R \rightarrow \overline{R}$ as composition of the projection map $p_i : R \rightarrow R^n$ defined as

$$p_i(x) = 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1 \quad (x \text{ located at } i\text{th component})$$

and $R^n \rightarrow R^n/I_R$. Then the natural action of S_n on $\overline{R}$ permutes those ρ_i : for $\sigma \in S_n$ we have $\sigma \circ \rho_i = \rho_{\sigma(i)}$. For $r \in R$, write $r_i = \rho_i(r)$ for shorthand. Then we can write $\sigma r_i = r_{\sigma(i)}$ for $\sigma \in S_n$.

Also If we are given a ring homomorphism $\psi : \overline{R} \rightarrow \mathbb{C}$, then $\psi \circ \rho_i$ becomes six distinct morphisms of $\overline{R}$ into $\mathbb{C}$. We will fix ψ and abusively denote $\psi \circ \rho_i : R \rightarrow \mathbb{C}$ just as ρ_i .

Definition 3.4.3. *The sextic resolvent algebra of a sextic nondegenerate $\mathbb{Q}$ -algebra R is defined as set of elements in $\overline{R}$ which are fixed by action of outer automorphism group.*

As $|S_6 : H_i| = 6$, a sextic resolvent algebra indeed becomes sextic algebra. Also six sextic resolvent algebras for $H_1, \dots, H_6$ are all isomorphic to each other with respect to S_n -action on $\overline{R}$.

We want to find some ‘resolvent map’ - a map involving R and S satisfying certain invariance with respect to action of H . We remind the construction of

$$R_H(x_1, x_2, x_3, x_4, x_5, x_6) = \pm \frac{1}{6} \sum_{\sigma \in H} \text{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}.$$

in Section 2.4. For $H_1, \dots, H_6$ our given set of sextic resolvent groups, R_{H_i} were all

defined together so that

$$R_H(x_{\sigma(1)}, \dots, x_{\sigma(6)}) = \text{sgn}(\sigma) R_{\sigma H \sigma^{-1}}(x_1, \dots, x_6)$$

and those satisfied

$$\sum_{i=1}^6 R_{H_i} = \sum_{i=1}^6 R_{H_i}^3 = 0.$$

Those covariance properties of R_i under S_6 allows us to consider them as a resolvent map as follows.

Proposition 3.4.4. *Let R be an nondegenerate sextic $\mathbb{Q}$ -algebra. For $r \in R$, denote $r_i = \rho_i(r) \in \overline{R}$. For fixed sextic resolvent groups $H_1, \dots, H_6$, denote*

$$\begin{aligned} \text{res}_{R, H_i}(r) &= \frac{1}{\sqrt{\text{Disc}(R)}} R_{H_i}(r_1, \dots, r_6) \\ &= \pm \frac{1}{6\sqrt{\text{Disc}(R)}} \sum_{\sigma \in H} \text{sgn}(\sigma) r_{\sigma(1)} r_{\sigma(2)} r_{\sigma(3)}. \end{aligned}$$

Here we consider $1/\sqrt{\text{Disc}(R)} = (\text{Disc}(R))^{-1/2}$ as an element of $\overline{R}$ such that

$$\sigma(\text{Disc}(R))^{-1/2} = \text{sgn}(\sigma)(\text{Disc}(R))^{-1/2}$$

for $\sigma \in S_6$. Then we have:

- (1) $\text{res}_{R, H_i}(r)$ is in sextic resolvent algebra of R for H_i .
- (2) Action of $\sigma \in S_6$ on $\overline{R}$ permutes res_{R, H_i} . In particular, we have $\sigma \circ \text{res}_{R, H} = \text{res}_{R, \sigma H \sigma^{-1}}$ for sextic resolvent group H .

Proof. By noting that action of $\sigma \in S_n$ on r_i is given as $\sigma(r_i) = r_{\sigma(i)}$, both statements can be checked by explicit calculation. □

3.4.2 Construction of an invariant element

Going back to our form $A \in V(\mathbb{Q})$, we want to define the invariant element s in the sextic resolvent algebra S of $\mathcal{R}(A)$ in case $\mathcal{R}(A)$ is nondegenerate. Let $1 = \alpha_0, \dots, \alpha_5$ be the basis of $\mathcal{R}(A)$ canonically defined together with $c_{ij}^k(A)$.

Proposition 3.4.5. *Let A be a form such that $\mathcal{R}(A)$ is nondegenerate. Denote S_{H_i} as sextic resolvent algebra of $\mathcal{R}(A)$ with respect to H_i . Let $S = S_{H_1}$, and consider six ring embeddings $\psi_i : S \rightarrow \overline{R}$ whose image is S_{H_i} .*

There exists an element $s \in S$ such that: for $s_i = \psi_i(s)$, the following expression holds

$$\det(t_1 A_1 + \dots + t_5 A_5) = \sum_{i=1}^6 (s_i \cdot \text{res}_{R,i}(t_1 \alpha_1 + \dots + t_5 \alpha_5)),$$

$$\sum_{i=1}^6 s_i = 0$$

for all $x_1, \dots, x_5 \in \mathbb{Q}$. Moreover, this s satisfies

$$16 \sum_{i=1}^6 s_i^6 - 4 \left(\sum_{i=1}^6 s_i^2 \right)^2 = 16 \text{tr}_{S/\mathbb{Q}}(s^4) - 4 \text{tr}_{S/\mathbb{Q}}(s^2)^2 = \text{Disc}(A).$$

For such s , image of $(s_1, \dots, s_6) \in \overline{R}^6$ mapped by $\mathbb{C}^6$ coincides with invariant $(w_1, \dots, w_6)$ of six geometric points of $\mathfrak{X}_{x,A}$ (up to $\mathbb{C}$ -multiple).

This proposition is essentially the beginning part of the Theorem 3.4.1, which states the existence of $s \in S$. The main idea for its proof is to apply the Proposition 3.2.8 and Proposition 3.2.9 for $T = \mathfrak{X}_{t,A}$, and observe that the special six cubics $R_{T,H_i}(x)$ of the Proposition 3.2.8 actually coincides with $\text{res}_{R,i}$ we defined in Proposition 3.4.1.

Let $1 = \alpha_0, \dots, \alpha_5$ be the basis of $R(A)$ canonically defined with $c_{ij}^k(A)$. If

$\text{Disc}(A) \neq 0$, we can think distinct ring morphisms $\rho_1, \dots, \rho_6 : R(A) \rightarrow \overline{R} \rightarrow \mathbb{C}$ and dual basis $\alpha_0^*, \dots, \alpha_5^*$ of $\alpha_0, \dots, \alpha_5$. In Theorem 3.3.2, we have shown that the six points of

$$\mathfrak{X}_{t,A} = \{t \in \mathbb{P}^4 : \text{rk}(t_1 A_1 + \dots + t_5 A_5) \leq 1\}$$

are given as

$$p_i = [\rho_i(\alpha_1^*); \rho_i(\alpha_2^*); \rho_i(\alpha_3^*); \rho_i(\alpha_4^*); \rho_i(\alpha_5^*)]$$

for $1 \leq i \leq 6$.

We remind the statement of the Proposition 2.8: for $T = \mathfrak{X}_{t,A}$, there exists nonzero six cubic forms $R_{T,i}(x)$ such that

1. R_{T,H_i} is singular at six points of X .
2. $\sum_{i=1}^6 R_{T,H_i}(x) = \sum_{i=1}^6 R_{T,H_i}(x)^3 = 0$
3. The group $\text{Aut}(T) \subset \text{PGL}_5(k)$ isomorphic to S_6 acts transitively on $R_{T,H_i}(x)$,

and such list of six of $R_{T,H_i}(x)$ is determined uniquely up to constant multiplication and permutation.

We want to show that those R_{T,H_i} can be given as $\text{res}_{R,i}$. The map

$$t \mapsto \text{res}_{R,i}(t_1 \alpha_1 + \dots + t_5 \alpha_5)$$

is well-defined for $t \in \mathbb{Q}^5$ and can be represented as cubic polynomial in $t_1, \dots, t_5$ with $\mathbb{C}$ -coefficients. Denote $\alpha_t = t_1 \alpha_1 + \dots + t_5 \alpha_5$, and denote the above cubic form as $\text{res}_{R,H_i}(\alpha_t)$.

Proposition 3.4.6. *For A nondegenerate and $R = R(A)$, the six cubic forms $\text{res}_{R,H_i}(\alpha_t)$ defined as above satisfies all properties of the Proposition 2.8 of the cubic forms $R_{T,H_i}(t)$ corresponding to the point set*

$$T = \mathfrak{X}_{t,A} = \{[\rho_i(\alpha_1^*); \dots; \rho_i(\alpha_5^*)] : i = 1, \dots, 6\}.$$

Proof. The forms $\text{res}_{R,i}(\alpha_t)$ are obtained by first considering forms

$$R_{H_i}(z_1, \dots, z_6) = \pm \frac{1}{6} \sum_{\sigma \in H_i} \text{sgn}(\sigma) z_{\sigma(1)} z_{\sigma(2)} z_{\sigma(3)}$$

and applying the linear change of variable

$$z_i = x_1 \rho_i(\alpha_1) + \dots + x_5 \rho_i(\alpha_5)$$

to them. For forms R_{H_i} , we already showed that

$$R_{H_i}(z_1, \dots, z_6) = R_{H_i}(z_1 - z_6, z_2 - z_6, \dots, z_5 - z_6, 0)$$

and $R_{H_i}(z_1, \dots, z_5, 0)$ satisfies properties of $R_{X,i}$ the Proposition 2.7 for X being the following specific set

$$\begin{aligned} T = T_{std} = \{ & [1; 0; 0; 0; 0], [0; 1; 0; 0; 0], [0; 0; 1; 0; 0], \\ & [0; 0; 0; 1; 0], [0; 0; 0; 0; 1], [1; 1; 1; 1; 1] \}. \end{aligned}$$

So it suffices to show that the coordinate change

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (z_1 - z_6, \dots, z_5 - z_6),$$

$$z_i = x_1 \rho_i(\alpha_1) + \dots + x_5 \rho_i(\alpha_5)$$

maps six points of $\mathfrak{X}_{t,A}$ to T_{std} . Indeed for a point $p_k = (\rho_k(\alpha_j^*))_{j=1, \dots, 5} \in X_A$, its

coordinates are $x_j = \rho_k(\alpha_j^*)$ and corresponding coordinate change is given as

$$\begin{aligned} z_i &= \sum_{j=1}^5 \rho_k(\alpha_j^*) \rho_i(\alpha_j) \\ &= \left(\sum_{j=0}^5 \rho_k(\alpha_j^*) \rho_i(\alpha_j) \right) - \rho_k(\alpha_0^*) \rho_i(\alpha_0) \\ &= \delta_{ik} - \rho_k(\alpha_0^*) \end{aligned}$$

so $z_i - z_6 = \delta_{ik} - \delta_{6k}$. This corresponds to coordinate of k th point of T_{std} , so $\mathfrak{X}_{t,A}$ is mapped to T_{std} . $\square$

We can apply the Proposition 2.9 to obtain

$$\det(t_1 A_1 + \cdots + t_5 A_5) = \sum_{i=1}^6 (cw_i(\mathfrak{X}_{x,A})) \text{res}_{R,H_i}(\alpha_t)$$

for some $c \in \mathbb{C}$ nonzero. Let $s_i = cw_i(\mathfrak{X}_{x,A})$. We claim that there exists $s \in S$ such that $s_i = \psi_i(s)$ are given as images of distinct embeddings $\psi_i : S \rightarrow \mathbb{C}$. This can be checked from the following argument. For any $x \in \mathbb{Q}^5$, we have $\sum_{i=1}^6 s_i \text{res}_{R,H_i}(\alpha_t) \in \mathbb{Q}$. Note that $\text{res}_{R,H_i}(\alpha_t)$ are conjugate images of $\text{res}_{R,H_1}(\alpha_x) \in S$. So we should have

$$\sum_{i=1}^6 s_i \psi_i(\text{res}_{R,H_1}(\alpha_x)) \in \mathbb{Q}.$$

If elements $\text{res}_{R,H_1}(\alpha_x)$ span set of trace zero elements of S over $\mathbb{Q}$, it follows that $\sum_{i=1}^6 s_i \psi_i(u) \in \mathbb{Q}$ for any $u \in S$, so we should have $s_i = \psi_i(s)$ for some $s \in S$. Suppose on the contrary that elements $\text{res}_{R,H_1}(\alpha_x)$ do not span all trace zero elements. Then there exists some $t \in S$ with $\text{tr}_{S/\mathbb{Q}} = 0$ such that $\text{tr}_{S/\mathbb{Q}}(t \cdot \text{res}_{R,H_1}(\alpha_t)) = 0$. But it implies

$$\sum_{i=1}^6 \psi_i(u) \text{res}_{R,H_i}(\alpha_t) = 0,$$

which contradicts that six cubics $\text{res}_{R,H_i}(\alpha_x)$ have only one linear relation that their

sum is zero. Thus $s_i = \psi_i(s)$ for some $s \in S$, and we can define such s as the element of S corresponding to A . As $\sum w_i(Y_A) = 0$, we should have $\text{tr}_{S/\mathbb{Q}}(s) = 0$ too.

To complete the proof of the Proposition 3.4.2, we need to show

$$16 \sum_{i=1}^6 s_i^4 - 4 \left(\sum_{i=1}^6 s_i^2 \right)^2 = 16 \text{tr}_{S/\mathbb{Q}}(s^4) - 4 \text{tr}_{S/\mathbb{Q}}(s^2)^2 = \text{Disc}(A).$$

Denote $R = \mathcal{R}(A)$.

Proof. Denote $u_i = t_1 \rho_i(\alpha_1) + \cdots + t_5 \rho_i(\alpha_5)$, $i = 1, \dots, 6$. Also denote

$$B_i = \sum_{j=1}^5 \rho_i(\alpha_j^*) A_j, \quad i = 1, \dots, 6.$$

As the Theorem 3.3.1 states that $[\rho_i(\alpha_1^*); \dots; \rho_i(\alpha_5^*)] \in X_A$, the equation $\text{rk}(t_1 A_1 + \cdots + t_5 A_5) \leq 1$ has $u_j = \rho_i(\alpha_j^*)$ as solution. This implies $\text{rk}(B_i) \leq 1$. Also, one can check

$$\sum_{i=1}^6 B_i = 0, \quad \sum_{i=1}^5 t_i A_i = \sum_{i=1}^6 u_i B_i.$$

So we have

$$\det \left(\sum_{i=1}^5 t_i A_i \right) = \det \left(\sum_{i=1}^6 u_i B_i \right) = \sum_{i=1}^6 s_i \text{res}_{R, H_i}(\alpha_t).$$

Meanwhile, we remind our construction in Proposition 3.4.1

$$\text{res}_{R, H_i}(\alpha_t) = \frac{1}{\sqrt{\text{Disc}(R)}} R_{H_i}(u_1, u_2, \dots, u_6)$$

where

$$R_H(u_1, \dots, u_6) = \pm \frac{1}{6} \sum_{\sigma \in H} \text{sgn}(\sigma) u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)}$$

and $\pm$ sign are defined in explicit way as of Section 3.2.4. So we have

$$\det \left(\sum_{i=1}^6 u_i B_i \right) = \frac{1}{\sqrt{\text{Disc}(R)}} \sum_{i=1}^6 s_i R_{H_i}(u_1, \dots, u_6).$$

By the Proposition 2.4, there exists $(g_1, g_2) \in \mathrm{SL}_3 \times \mathrm{SL}_3(\overline{\mathbb{Q}})$ such that

$$g_1 \left(\sum_{i=1}^6 u_i B_i \right) g_2 = \begin{pmatrix} \lambda_{14}(u_1 - u_4) & \lambda_{15}(u_1 - u_5) & \lambda_{16}(u_1 - u_6) \\ \lambda_{24}(u_2 - u_4) & \lambda_{25}(u_2 - u_5) & \lambda_{26}(u_2 - u_6) \\ \lambda_{34}(u_3 - u_4) & \lambda_{35}(u_3 - u_5) & \lambda_{36}(u_3 - u_6) \end{pmatrix}.$$

By expanding out both sides of

$$\det \begin{pmatrix} \lambda_{14}(u_1 - u_4) & \lambda_{15}(u_1 - u_5) & \lambda_{16}(u_1 - u_6) \\ \lambda_{24}(u_2 - u_4) & \lambda_{25}(u_2 - u_5) & \lambda_{26}(u_2 - u_6) \\ \lambda_{34}(u_3 - u_4) & \lambda_{35}(u_3 - u_5) & \lambda_{36}(u_3 - u_6) \end{pmatrix} = \frac{1}{\sqrt{\mathrm{Disc}(R)}} \sum_{i=1}^6 s_i R_{H_i}(u_1, \dots, u_6),$$

we can express s_i in terms of λ_{pq} as

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{pmatrix} = \frac{1}{6\sqrt{\mathrm{Disc}(R)}} \begin{pmatrix} -1 & 2 & 2 & -1 & -1 & -1 \\ 2 & -1 & 2 & -1 & -1 & -1 \\ 2 & 2 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 2 & 2 \\ -1 & -1 & -1 & 2 & -1 & 2 \\ -1 & -1 & -1 & 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda_{14}\lambda_{25}\lambda_{36} \\ \lambda_{15}\lambda_{26}\lambda_{34} \\ \lambda_{16}\lambda_{24}\lambda_{35} \\ \lambda_{14}\lambda_{26}\lambda_{35} \\ \lambda_{16}\lambda_{25}\lambda_{34} \\ \lambda_{15}\lambda_{24}\lambda_{36} \end{pmatrix}$$

and we can use this expression to explicitly calculate

$$16 \sum_{i=1}^6 s_i^4 - 4 \left(\sum_{i=1}^6 s_i^2 \right)^2 = F(\lambda)^2 \mathrm{Disc}(R)^2$$

where

$$F(\lambda) = \det \begin{pmatrix} \lambda_{14}\lambda_{24} & \lambda_{14}\lambda_{34} & \lambda_{24}\lambda_{34} \\ \lambda_{15}\lambda_{25} & \lambda_{15}\lambda_{35} & \lambda_{25}\lambda_{35} \\ \lambda_{16}\lambda_{26} & \lambda_{16}\lambda_{36} & \lambda_{26}\lambda_{36} \end{pmatrix}.$$

Thus showing that $F(\lambda) = \mathrm{Disc}(R)^{-1/2}$ would suffice.

Let $B = (B_1, B_2, B_3, B_4, B_5) \in V(\mathbb{C})$. Using

$$g_1 \left(\sum_{i=1}^6 u_i B_i \right) g_2 = \begin{pmatrix} \lambda_{14}(u_1 - u_4) & \lambda_{15}(u_1 - u_5) & \lambda_{16}(u_1 - u_6) \\ \lambda_{24}(u_2 - u_4) & \lambda_{25}(u_2 - u_5) & \lambda_{26}(u_2 - u_6) \\ \lambda_{34}(u_3 - u_4) & \lambda_{35}(u_3 - u_5) & \lambda_{36}(u_3 - u_6) \end{pmatrix}$$

$(g_1, g_2 \in \text{SL}_3(\mathbb{C}))$, we can explicitly calculate $\mathcal{R}(B) = \mathcal{R}(g_1 B g_2)$. The multiplication coefficients of $\mathcal{R}(g_1 B g_2)$ are given as

$$c_{ij}^k(g_1 B g_2) = \begin{cases} F(\lambda) & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}.$$

Thus ring $\mathcal{R}(B)$ and its basis $(1, \alpha_1(B), \dots, \alpha_5(B))$ is isomorphic as $\mathcal{R}(B) \simeq \mathbb{C}^6$ and

$$(\alpha_1(B), \dots, \alpha_5(B)) = F(\lambda)(e_1, e_2, e_3, e_4, e_5),$$

where $e_i \in \mathbb{C}^6$ is the standard basis element ($e_1 = (1, 0, 0, 0, 0, 0)$, $e_2 = (0, 1, 0, 0, 0, 0)$, etc.). Meanwhile, we note that

$$B_i = \sum_{j=1}^5 \rho_i(\alpha_j^*) A_j \quad \Rightarrow \quad A_j = \sum_{i=1}^5 B_i (\rho_i(\alpha_j) - \rho_6(\alpha_j)).$$

This GL_5 transform $(B_1, \dots, B_5) \mapsto (A_1, \dots, A_5)$ has determinant

$$\det \begin{pmatrix} \rho_1(\alpha_1) - \rho_6(\alpha_1) & \rho_1(\alpha_2) - \rho_6(\alpha_2) & \cdots & \rho_1(\alpha_5) - \rho_6(\alpha_5) \\ \rho_2(\alpha_1) - \rho_6(\alpha_1) & \rho_2(\alpha_2) - \rho_6(\alpha_2) & \cdots & \rho_2(\alpha_5) - \rho_6(\alpha_5) \\ \rho_3(\alpha_1) - \rho_6(\alpha_1) & \rho_3(\alpha_2) - \rho_6(\alpha_2) & \cdots & \rho_3(\alpha_5) - \rho_6(\alpha_5) \\ \rho_4(\alpha_1) - \rho_6(\alpha_1) & \rho_4(\alpha_2) - \rho_6(\alpha_2) & \cdots & \rho_4(\alpha_5) - \rho_6(\alpha_5) \\ \rho_5(\alpha_1) - \rho_6(\alpha_1) & \rho_5(\alpha_2) - \rho_6(\alpha_2) & \cdots & \rho_5(\alpha_5) - \rho_6(\alpha_5) \end{pmatrix} \\ = \sqrt{\text{Disc}(R)},$$

so it acts on $\alpha_i(B)$ as $\sqrt{\text{Disc}(R)}$ times standard action. If we apply this action in the above identification $\mathcal{R}(B) \simeq \mathbb{C}^6$ and $(\alpha_1(B), \dots, \alpha_5(B)) = F(\lambda)(e_1, e_2, e_3, e_4, e_5)$, we obtain

$$\begin{aligned}\alpha_k(A) &= F(\lambda)\sqrt{\text{Disc}(R)} \sum_{i=1}^5 (\rho_i(\alpha_k) - \rho_6(\alpha_k))e_i \\ &= F(\lambda)\sqrt{\text{Disc}(R)} \cdot ((\rho_1(\alpha_k), \rho_2(\alpha_k), \dots, \rho_6(\alpha_k)) - \rho_6(\alpha_k) \cdot (1, 1, 1, 1, 1)).\end{aligned}$$

But the elements $\alpha_k(A)$ are should be same as the following image of α_k in $\mathbb{C}^6$ (via embedding $\mathcal{R}(A) \rightarrow \mathbb{C}^6$)

$$\alpha_k \mapsto (\rho_1(\alpha_k), \rho_2(\alpha_k), \dots, \rho_6(\alpha_k))$$

up to shifting by constant times 1, so we should have $F(\lambda)\sqrt{\text{Disc}(R)} = 1$. $\square$

To complete the proof of Theorem 3.4.1, we show the following statement.

Proposition 3.4.7. *Suppose $A, B \in V_{D \neq 0}(\mathbb{Q})$ satisfies $R = \mathcal{R}(A) = \mathcal{R}(B)$ and their corresponding elements $s \in S$ are same. Then B is either in $G(\mathbb{Q})$ -orbit of A or A^t .*

Proof. We have $X_A = X_B$. Also Y_A and Y_B has same W -invariant, as W -invariant of Y_A can be determined by conjugate images of s . Thus the Theorem 2.1 shows that Y_B is either equivalent to Y_A or Y_{A^t} , so either A, B or A^t, B are $\text{SL}_3 \times \text{GL}_3(\overline{\mathbb{Q}})$ -equivalent. We prove the following lemma.

Lemma. *If $A, B \in V(\mathbb{Q})$ are $\text{GL}_3 \times \text{GL}_3(\overline{\mathbb{Q}})$ -equivalent, then those are $\text{GL}_3 \times \text{GL}_3(\mathbb{Q})$ -equivalent.*

Proof. Let X, Y be 3×3 matrices, and consider a system of equations

$$A_i X = Y B_i, \quad i = 1, \dots, 5.$$

If we consider it as a system of linear homogeneous equations on 18 variables of entries of X and Y , its solution space will be some vector space over $\mathbb{Q}$ (as A, B were defined over $\mathbb{Q}$). The $\mathrm{GL}_3 \times \mathrm{GL}_3(\overline{\mathbb{Q}})$ -equivalence of A and B suggests that there is a nontrivial $\overline{\mathbb{Q}}$ -solution, so this vector space is nontrivial. On this vector space, we consider the polynomial $\det(X) \det(Y)$. As there are solutions in $X, Y \in \mathrm{GL}_3(\overline{\mathbb{Q}})$, this polynomial is not identically zero on the vector space, so there should be $\mathbb{Q}$ -solution such that $\det(X) \det(Y)$ does not vanish. This shows there exists $X, Y \in \mathrm{GL}_3(\mathbb{Q})$ such that $A_i X = Y B_i$ for all $i = 1, \dots, 5$, so $Y^{-1} A X = B$. □

□

Chapter 4

Sextic rings associated to $2 \times 2 \times 2 \times 5$ forms

4.1 Introduction

In this chapter, we construct a sextic ring from a $2 \times 2 \times 2 \times 5$ form $A \in V(\mathbb{Z}) = \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^5$, where (G, V) is a representation of an algebraic group given as $(G(\mathbb{Z}), V(\mathbb{Z})) = (\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{GL}_5(\mathbb{Z}), \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^5)$. Similar to the case of the Chapter 3 which constructs sextic rings from $3 \times 3 \times 5$ forms, our ring $\mathcal{R}(A)$ has its multiplication coefficients $c_{ij}^k = c_{ij}^k(A)$ determined as degree 6 polynomials in entries of A .

Theorem 4.1.1. *For a form $A \in \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^5$, there exists a sextic ring $\mathcal{R}(A)$ with $\mathbb{Z}$ -basis of $\mathcal{R}(A)/\mathbb{Z}$ fixed, such that its multiplication coefficients $c_{ij}^k(A)$ are given as integral polynomials in entries of A . Those c_{ij}^k are $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ -invariant of degree 6. Under the action of $g \in \mathrm{GL}_5(\mathbb{Z})$, $\mathcal{R}(gA)$ is isomorphic to $\mathcal{R}(A)$ in the following way: if $(1 = \alpha_0, \dots, \alpha_5)$ is basis of $\mathcal{R}(A)$, then $\mathcal{R}(gA)$ is isomorphic to the ring having basis $1, \tilde{\alpha}_1, \dots, \tilde{\alpha}_5$ where $\tilde{\alpha}_i = \det(g) \sum_{j=1}^5 g_{ij} \alpha_j$.*

Thus the isomorphism class of $\mathcal{R}(A)$ is invariant under the $G(\mathbb{Z}) = \mathrm{SL}_2 \times \mathrm{SL}_2 \times$

$\mathrm{SL}_2 \times \mathrm{GL}_5(\mathbb{Z})$ -action on A . Call the discriminant of this ring $\mathrm{Disc}(\mathcal{R}(A))$ as the *discriminant* $\mathrm{Disc}(A)$ of the element A , which becomes a degree 60 relative G -invariant. This construction can be generalized over any base ring S : for $A \in V(S)$ we can construct $\mathcal{R}(A)$ as a sextic S -algebra using same polynomial formula for $c_{ij}^k(A)$.

Similar to the ring parametrization for $3 \times 3 \times 5$ forms, construction of $\mathcal{R}(A)$ is related to a certain $\mathbb{Z}$ -scheme which has six geometric points for generic A . For $A \in V$, write A as 5-tuple of $(A_1, A_2, A_3, A_4, A_5)$ for A_l being 5 slices of $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ forms, and write each coordinates of A_l as $A_l = \sum_{i,j,k=1}^2 a_{ijk,l} e_i \otimes e_j \otimes e_k$, where e_i s are standard basis vectors $(e_1, e_2) = ((1, 0), (0, 1))$. Our scheme will take the form

$$\mathfrak{X}_A : A_1 t_1 + A_2 t_2 + A_3 t_3 + A_4 t_4 + A_5 t_5 = (x_1, x_2) \otimes (y_1, y_2) \otimes (z_1, z_2)$$

which is equivalent to that

$$\sum_{l=1}^5 a_{ijk,l} t_l = x_i y_j z_k, \quad i, j, k \in \{1, 2\}.$$

These equations can be considered as projective equations in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^4$, where $[x_1; x_2], [y_1; y_2], [z_1; z_2], [t_1; t_2; t_3; t_4; t_5]$ are considered as projective coordinates of the first three $\mathbb{P}^1$ parts and $\mathbb{P}^4$ part respectively. One can show that for generic choice of A , the above equations cut out 6 geometric points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^4$. The set of elements $(x_1, x_2) \otimes (y_1, y_2) \otimes (z_1, z_2)$ on the righthand side can be considered as image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^7$, which is degree 6, and the six points can be considered as fiber of this image above generic hyperplane.

We consider the projection of $\mathfrak{X}_A$ to $\mathbb{P}_4^t$ and $\mathbb{P}_x^1 \times \mathbb{P}_y^1$, and denote them as $\mathfrak{X}_{t,A}$ and $\mathfrak{X}_{(x,y),A}$ respectively. (One can similarly think of $\mathfrak{X}_{(y,z),A}$, $\mathfrak{X}_{(z,x),A}$ too.)

Theorem 4.1.2. *Let A be a $2 \times 2 \times 2 \times 5$ form over an algebraically closed field k of characteristic zero. The condition $\mathrm{Disc}(A) \neq 0$ is equivalent to $\mathfrak{X}_{(x,y),A}$ satisfying the following condition: $\mathfrak{X}_{(x,y),A}$ consists of six points of $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying that no single*

global section of $\mathcal{O}(1, 2)$ and $\mathcal{O}(2, 1)$ vanishes on all of those six points. In that case, $\mathfrak{X}_{t,A}$ consists of six points of $\mathbb{P}^4$ in linearly general position (any 5 of them not on a hyperplane.)

Conversely, for any distinct six k -points in $\mathbb{P}^1 \times \mathbb{P}^1$ such that no single global section of $\mathcal{O}(1, 2)$ and $\mathcal{O}(2, 1)$ vanishes on all of those six points, there exists A over k such that $\mathfrak{X}_{(x,y),A}$ becomes set of those six points, and such A is determined uniquely up to $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{GL}_5(k)$ -equivalence.

The sheaf $\mathcal{O}(k, l)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is defined as $\mathcal{O}(k) \boxtimes \mathcal{O}(l)$, and its global sections consist of homogeneous polynomials of bidegree (k, l) . The condition that six points not on a bidegree $(2, 1)$ - or $(1, 2)$ -form will appear repeatedly.

One might ask the problem of figuring out which $A \in V(\mathbb{Q})$ gives $\mathcal{R}(A) \simeq R$ for fixed sextic nondegenerate $\mathbb{Q}$ -algebra R , and discuss the orbit of $V_{D \neq 0}(\mathbb{Q})/G(\mathbb{Q})$. Using the irrational invariants of six points in $\mathbb{P}^1 \times \mathbb{P}^1$, which can be obtained from knowledge of six points in $\mathbb{P}^1$ (see [33] etc.), we can construct the following invariant element in the sextic resolvent algebra of $\mathcal{R}(A)$ which characterizes the orbit of $V_{D \neq 0}(\mathbb{Q})/G(\mathbb{Q})$.

Theorem 4.1.3. *Let R be a nondegenerate sextic $\mathbb{Q}$ -algebra, and S be its sextic resolvent algebra (with respect to outer automorphism of S_6), For $A \in V(\mathbb{Q})$ such that $\mathcal{R}(A) \simeq R$, one can construct $s_1, s_2 \in S$ corresponding to A satisfying the following property: for $\psi_1, \dots, \psi_6 : S \rightarrow \mathbb{C}$ be distinct embeddings, $[\psi_1(s_1); \dots; \psi_6(s_1)]$ and $[\psi_1(s_2); \dots; \psi_6(s_2)]$ provides invariants of $\mathfrak{X}_{(x,y),A}$ (as defined in Section 4.3).*

For such A , elements s_1, s_2 are determined uniquely up to $\mathbb{Q}^\times$ -multiple, and the mapping $A \mapsto (s_1, s_2) \in (S/\mathbb{Q}^\times)^2$ is injective.

The construction of $\mathcal{R}(A)$ relies on the hypercohomology construction we discussed in the Chapter 2 of this thesis. We will construct a complex $\mathcal{C}_{t,A}$ of sheaves on $\mathbb{P}^4$, which becomes a minimal free resolution of $\mathfrak{X}_{t,A} \subset \mathbb{P}^4$ over $\mathbb{Q}$ for generic A .

One can explicitly calculate $H^0 R\Gamma(\mathcal{C}_{t,A})$ and define it as $\mathcal{R}(A)$. Checking that the multiplication coefficients $c_{ij}^k(A)$ of $\mathcal{R}(A)$ satisfy certain invariance properties under the G -action will prove the Theorem 4.1.1.

The construction of $\mathcal{C}_{t,A}$ relies on the geometry of $\mathfrak{X}_{t,A}$, which was studied in [15]. Casnati in [15] discussed sextic covers of an arbitrary Noetherian scheme T over a field associated to a $2 \times 2 \times 2 \times 5$ global form, and named them ‘Anglo-American covers’. It also provides the construction of $\mathcal{C}_{t,A}$ in terms of the Eagon-Northcott complex (see [24]), though finding explicit formula is original work of the author.

We will construct a complex $\mathcal{C}_{(x,y),A}$ of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, and calculate its hypercohomology ring $H^0 R\Gamma(\mathcal{C}_{(x,y),A})$ and show it is isomorphic to $\mathcal{R}(A)$. Likewise, $\mathcal{C}_{(x,y),A}$ is related to the geometry of $\mathfrak{X}_{(x,y),A}$, but the geometry of six points in $\mathbb{P}^1 \times \mathbb{P}^1$ was not studied as much as six points in $\mathbb{P}^m$. Other than using the property of invariants of six points in $\mathbb{P}^1 \times \mathbb{P}^1$, properties of $\mathfrak{X}_{(x,y),A}$ and six points in $\mathbb{P}^1 \times \mathbb{P}^1$ will be proved independently of other works.

We briefly describe the outline of this chapter as follows. In the Section 4.2 we will start with showing elementary properties of six points in $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathfrak{X}_{(x,y),A}$ using elementary methods, which will suffice to prove the first part of the Theorem 4.1.2. In the Section 4.3 we will construct $\mathcal{C}_{t,A}$ and $\mathcal{C}_{(x,y),A}$ and their hypercohomology ring, and show our main theorems (Theorem 4.1.1, 4.1.2) using properties of $\mathcal{R}(A)$ and hypercohomology. In the Section 4.4 we will construct the irrational invariant in sextic resolvent algebra of $\mathcal{R}(A)$ from invariants of six points in $\mathbb{P}^1 \times \mathbb{P}^1$, and show the Theorem 4.1.3 using their properties.

4.2 Geometry of $2 \times 2 \times 2 \times 5$ forms

In this section, we consider $2 \times 2 \times 2 \times 5$ forms over an algebraically closed field k of characteristic zero (which includes the case of $A \in V(\mathbb{Z})$ when $k = \overline{\mathbb{Q}}$). Denote A as

$A = (A_1, A_2, A_3, A_4, A_5)$ where each $A_i \in k^2 \otimes k^2 \otimes k^2$ is slice of A in k^5 direction. Using the following the shorthand notation

$$A(t) = A_1 t_1 + A_2 t_2 + A_3 t_3 + A_4 t_4 + A_5 t_5, \quad x \otimes y \otimes z = (x_1, x_2) \otimes (y_1, y_2) \otimes (z_1, z_2)$$

we denote our variety for six points as

$$\mathfrak{X}_A = \{(x, y, z, t) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^4 : A(t) = x \otimes y \otimes z\}.$$

Denote $\mathfrak{X}_{t,A}$, $\mathfrak{X}_{(x,y),A}$ as the projections of $\mathfrak{X}_A$ onto t -space of $\mathbb{P}^4$ and onto (x, y) -space of $\mathbb{P}^1 \times \mathbb{P}^1$ respectively.

The variety $\mathfrak{X}_A$ can be associated to the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^7, \quad (x, y, z) \mapsto x \otimes y \otimes z.$$

As we know that image of the Segre embedding has dimension 3 and degree 6 inside $\mathbb{P}^7$, the generic intersection of the image and linear subspace of $\mathbb{P}^7$ spanned by A_1, A_2, A_3, A_4, A_5 becomes distinct six points.

Our goal is to characterize the exact condition for A for which $\mathfrak{X}_A$ consists of six points, and to determine which six points of $\mathbb{P}^1 \times \mathbb{P}^1$ can be represented as $\mathfrak{X}_{(x,y),A}$, as discussed in conclusion of Theorem 4.1.2. In this section, we will focus on some elementary geometric properties of six points in $\mathbb{P}^1 \times \mathbb{P}^1$, and show the later half of Theorem 4.1.2. The rest of Theorem 4.1.2 will be proved in the Section 4.3, where we will define a sextic ring $\mathcal{R}(A)$ and its discriminant as $\text{Disc}(A) = \text{Disc}(\mathcal{R}(A))$.

4.2.1 Preliminaries: six points in $\mathbb{P}^1$

It is known that the moduli space of 6 points in $\mathbb{P}^1$ can be described as a Segre cubic threefold, ([33], [23] etc.) which is often described as a cubic threefold determined by

the following equation

$$p_1 + \cdots + p_6 = p_1^3 + \cdots + p_6^3 = 0.$$

Here we consider it as cubic hypersurface in ambient space of $\mathbb{P}^4$, here having form of hyperplane $p_1 + \cdots + p_6 = 0$. While we picked the most convenient form of equation for Segre cubic, Segre cubic threefold includes all cubics which are PGL_4 -equivalent to above form, and such cubics are exactly characterized by the condition that it has 10 nodal singularities.

The map from six points to p_i can be described as follows. For six points $x_i = [x_{i,1}; x_{i,2}]$, p_i are the sextic resolvent maps having following form

$$p_H = \sum_{\{a,b,c\} \cup \{d,e,f\} = \{1,\dots,6\}} \pm (x_{a,1}x_{b,1}x_{c,1})(x_{d,2}y_{e,2}y_{f,2})$$

where H corresponds to one of six sextic resolvent subgroups of S_6 . The exact equation has following form (here $\sigma \in H \subset S_6$ is considered as permutation of $\{1, 2, \dots, 6\}$)

$$p_H = \pm \frac{1}{6} \sum_{\sigma \in H} \mathrm{sgn}(\sigma) x_{\sigma(1),1} x_{\sigma(2),1} x_{\sigma(3),1} x_{\sigma(4),2} x_{\sigma(5),2} x_{\sigma(6),2}.$$

The signatures $\pm$ of p_H are chosen such that $\sum p_H = 0$ is satisfied. It turns out that there are exactly two such choices of giving signs, and those two are equivalent under $p_H \mapsto -p_H$. Furthermore, with these choices of signs, it also follows that $\sum p_H^3 = 0$. So the mapping from six points to a projective point $[p_{H_1}; \dots; p_{H_6}]$ of Segre cubic is well-defined. In the appendix section of Section 4.5, we will give the definition and explicit descriptions of sextic resolvent groups H and explicit formula of all p_H .

For six points in $\mathbb{P}^1$ having affine coordinates $(0, 1, \infty, \alpha_1, \alpha_2, \alpha_3)$ ($\alpha \in k$ is considered as point $[k; 1]$ and ∞ is considered as $[1; 0]$), then the six invariants p_H can be

given as follows for some choice of sextic resolvent groups H_i .

$$\begin{pmatrix} p_{H_1} \\ p_{H_2} \\ p_{H_3} \\ p_{H_4} \\ p_{H_5} \\ p_{H_6} \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_2 \alpha_3 \\ \alpha_1 \alpha_3 \\ \alpha_1 \alpha_2 \end{pmatrix}.$$

If we have six points such that three of them are distinct, we can map those three points to $0, 1, \infty$ by certain $\text{PGL}_2(k)$ -transformation, and calculate the invariants using the above form.

If more than four of x_i s are same, then all invariants vanish. This coincides with Geometric Invariant Theory of n points in $\mathbb{P}^m$ stating that that more than $n/2$ identical points are in degenerate orbit ([37], Chapter 3). If three of x_i s are same, the six point set is mapped to one of 10 singular points of the Segre cubic (of form $[1; 1; 1; -1; -1; -1]$), depending on which three points are same.

If two x_i s are same, the image of all points $x_1 = x_2$ form one of 15 planes of Segre cubic, obtained by S_6 -permutations of $p_1 - p_2 = p_3 - p_4 = p_5 - p_6 = 0$, corresponding to 15 synthemes. Those 15 lines are all planes contained in Segre cubic and they form abstract configuration $(15_4, 10_6)$ configuration with singular points. ([21], Proposition 2.2.)

In conclusion, if we denote the subset $S \subset (\mathbb{P}^1)^6$ of points such that no four of x_i s are same, then we have the correspondence

$$S/\text{PGL}_2(k) \simeq \text{Seg} = \{[p_1; \dots; p_6] : \sum p_i = \sum p_i^3 = 0\}.$$

Remark. The action of $\sigma \in S_6$ permuting six points of $x_1, \dots, x_6$ acts on p_H as follows.

$$\sigma p_H = \text{sgn}(\sigma) p_{\sigma H \sigma^{-1}}$$

Suppose that our six points $x_1, \dots, x_6$ are Galois conjugates to each other - action of $\text{Gal}(\bar{k}/k)$ permutes those six points - or equivalently, those points are cut out by a binary sextic form f whose coefficients are in k . Those six points correspond to a sextic k -algebra of global functions of those points $R_f = \Gamma(V(f))$ if we consider the perspective of the parametrizations of rank n algebra using binary n -ic forms. In this setting, the invariants p_H of $x_1, \dots, x_6$ can be expressed as an element α of the sextic resolvent algebra of R_f , in the sense that the following holds

$$[\rho_1(\alpha); \rho_2(\alpha); \dots; \rho_6(\alpha)] = [p_{H_1}; \dots; p_{H_6}]$$

where $\rho_1, \dots, \rho_6$ are embeddings of the sextic resolvent algebra into $\bar{k}$.

This is analogous to case of the parametrization model of the Chapter 3 using $3 \times 3 \times 5$ forms and six points in $\mathbb{P}^2$, where we could find an invariant element in the sextic resolvent algebra of the sextic ring parametrized which expresses all invariants of the six points. We will soon see that we can similarly construct an element in the sextic resolvent algebra in the parametrization model of this chapter which uses six points in $\mathbb{P}^1 \times \mathbb{P}^1$.

4.2.2 Six points in $\mathbb{P}^1 \times \mathbb{P}^1$

The space $\mathbb{P}^1 \times \mathbb{P}^1$ has automorphism group $\mathbb{Z}/2 \rtimes \text{PGL}_2 \times \text{PGL}_2(k)$, with each copy of $\text{PGL}_2(k)$ acting on each factor of $\mathbb{P}^1$ respectively and $\mathbb{Z}/2$ acting as switching two $\mathbb{P}^1$ parts. (This can be seen from considering that $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \text{Pic}(\mathbb{P}^1) \times \text{Pic}(\mathbb{P}^1)$ and that automorphisms should act on Picard group in the trivial way.)

Suppose that we have six geometric points $(x_i, y_i) = ([x_{i,1}; x_{i,2}], [y_{i,1}; y_{i,2}]) \in \mathbb{P}^1 \times$

$\mathbb{P}^1$. If $(x_1, \dots, x_6)$ and $(y_1, \dots, y_6)$ are all in S , then we can define a map from S^2 to the Seg^2 (Seg:Segre cubic) defined as pairs of $(p_i(x_1, \dots, x_6))_i$ and $(p_i(y_1, \dots, y_6))_i$ (for $(p_1, \dots, p_6) : \mathbb{P}^1 \rightarrow \text{Seg}$ defined above). This map $\text{PGL}_2 \times \text{PGL}_2$ -equivalent, so if we compose it with the quotient $\text{Seg}^2 \rightarrow \text{Seg}^2 / (\mathbb{Z}/2)$ then it defines a map invariant under $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}/2 \rtimes \text{PGL}_2 \times \text{PGL}_2(k)$. In conclusion, the map

$$S^2 \subset (\mathbb{P}^1 \times \mathbb{P}^1)^6 \rightarrow \text{Seg}^2 / (\mathbb{Z}/2)$$

can be considered as Geometric Invariant Theory quotient of six points describing $\mathbb{P}^1 \times \mathbb{P}^1$. By considering the $\mathbb{Z}/2$ -invariant functions on Seg^2 , we can see that all projective invariants for $\{(x_i, y_i)\}_{i=1, \dots, 6}$ are homogeneous polynomials of $p_i(x)p_j(y) + p_j(x)p_i(y)$ (where $x = (x_1, \dots, x_6)$, $y = (y_1, \dots, y_6)$, $(p_1, \dots, p_6) : \mathbb{P}^1 \rightarrow \text{Seg}$ defined above).

The set S^2 is yet too large, and we want to impose stronger conditions on set of six points. Let P_{gen} be the set of lists of six points $((x_1, y_1), \dots, (x_6, y_6)) \in (\mathbb{P}^1 \times \mathbb{P}^1)^6$ which satisfies the following two conditions: the six points

$$[x_{i,1}^2 y_{i,1}; x_{i,1} x_{i,2} y_{i,1}; x_{i,2}^2 y_{i,1}; x_{i,1}^2 y_{i,2}; x_{i,1} y_{i,2} y_{i,2}; x_{i,2}^2 y_{i,2}] \in \mathbb{P}^5$$

are not on a single hyperplane, and the six points

$$[y_{1,1}^2 x_{1,1}; y_{1,1} y_{1,2} x_{1,1}; y_{1,2}^2 x_{1,1}; y_{1,1}^2 x_{1,2}; y_{1,1} x_{1,2} x_{1,2}; y_{1,2}^2 x_{1,2}] \in \mathbb{P}^5$$

are not on a single hyperplane. Those points can be considered naturally as images of the six points under the composition of the Veronese map $\mathbb{P}^1 \rightarrow \mathbb{P}^2$ of degree 2 and Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^5$.

For $((x_i, y_i))_i \in P_{\text{gen}}$, it follows that (x_i, y_i) are all distinct. Also we can show P_{gen} does not contain lists satisfying $x_1 = x_2 = x_3$, which implies $P_{\text{gen}} \subset S^2$. (For such case, the above six points are on hyperplane.) It is possible that $x_1 = x_2$.

We can consider an polynomial expression describing P_{gen} . In coordinatewise, it is

$$\det \begin{pmatrix} x_{1,1}^2 y_{1,1} & x_{1,1} x_{1,2} y_{1,1} & x_{1,2}^2 y_{1,1} & x_{1,1}^2 x_{1,2} & x_{1,1} y_{1,2} y_{1,2} & x_{1,2}^2 y_{1,2} \\ x_{2,1}^2 y_{2,1} & x_{2,1} x_{2,2} y_{2,1} & x_{2,2}^2 y_{2,1} & x_{2,1}^2 x_{2,2} & x_{2,1} y_{2,2} y_{2,2} & x_{2,2}^2 y_{2,2} \\ x_{3,1}^2 y_{3,1} & x_{3,1} x_{3,2} y_{3,1} & x_{3,2}^2 y_{3,1} & x_{3,1}^2 x_{3,2} & x_{3,1} y_{3,2} y_{3,2} & x_{3,2}^2 y_{3,2} \\ x_{4,1}^2 y_{4,1} & x_{4,1} x_{4,2} y_{4,1} & x_{4,2}^2 y_{4,1} & x_{4,1}^2 x_{4,2} & x_{4,1} y_{4,2} y_{4,2} & x_{4,2}^2 y_{4,2} \\ x_{5,1}^2 y_{5,1} & x_{5,1} x_{5,2} y_{5,1} & x_{5,2}^2 y_{5,1} & x_{5,1}^2 x_{5,2} & x_{5,1} y_{5,2} y_{5,2} & x_{5,2}^2 y_{5,2} \\ x_{6,1}^2 y_{6,1} & x_{6,1} x_{6,2} y_{6,1} & x_{6,2}^2 y_{6,1} & x_{6,1}^2 x_{6,2} & x_{6,1} y_{6,2} y_{6,2} & x_{6,2}^2 y_{6,2} \end{pmatrix}$$

times

$$\det \begin{pmatrix} y_{1,1}^2 x_{1,1} & y_{1,1} y_{1,2} x_{1,1} & y_{1,2}^2 x_{1,1} & y_{1,1}^2 y_{1,2} & y_{1,1} x_{1,2} x_{1,2} & y_{1,2}^2 x_{1,2} \\ y_{2,1}^2 x_{2,1} & y_{2,1} y_{2,2} x_{2,1} & y_{2,2}^2 x_{2,1} & y_{2,1}^2 y_{2,2} & y_{2,1} x_{2,2} x_{2,2} & y_{2,2}^2 x_{2,2} \\ y_{3,1}^2 x_{3,1} & y_{3,1} y_{3,2} x_{3,1} & y_{3,2}^2 x_{3,1} & y_{3,1}^2 y_{3,2} & y_{3,1} x_{3,2} x_{3,2} & y_{3,2}^2 x_{3,2} \\ y_{4,1}^2 x_{4,1} & y_{4,1} y_{4,2} x_{4,1} & y_{4,2}^2 x_{4,1} & y_{4,1}^2 y_{4,2} & y_{4,1} x_{4,2} x_{4,2} & y_{4,2}^2 x_{4,2} \\ y_{5,1}^2 x_{5,1} & y_{5,1} y_{5,2} x_{5,1} & y_{5,2}^2 x_{5,1} & y_{5,1}^2 y_{5,2} & y_{5,1} x_{5,2} x_{5,2} & y_{5,2}^2 x_{5,2} \\ y_{6,1}^2 x_{6,1} & y_{6,1} y_{6,2} x_{6,1} & y_{6,2}^2 x_{6,1} & y_{6,1}^2 y_{6,2} & y_{6,1} x_{6,2} x_{6,2} & y_{6,2}^2 x_{6,2} \end{pmatrix}$$

this polynomial $D(x, y)$ becomes degree $(18, 18)$ -polynomial in $x_{i,j}, y_{i,j}$, and one can show it is $SL_2 \times SL_2$ -invariant. As we have defined $p_i(x), p_j(y)$ as degree 6 polynomial in $x_{i,j}$ and $y_{i,j}$ respectively, we should be able to express it as degree $(3, 3)$ -polynomial in $(p_i(x), p_i(y))$, and indeed it becomes

$$D(x, y) = \frac{1}{64} \left(\sum_{i=1}^6 p_i(x)^2 p_i(y) \right) \left(\sum_{i=1}^6 p_i(x) p_i(y)^2 \right).$$

The first factor $(\sum_{i=1}^6 p_i(x)^2 p_i(y)) / 8$ is same as first determinant (and so for second too).

We will later show that the set P_{gen} arises naturally when we consider $2 \times 2 \times 2 \times 5$ forms and resolutions of six points in $\mathbb{P}^1 \times \mathbb{P}^1$.

4.2.3 Free resolution of six points in $\mathbb{P}^1 \times \mathbb{P}^1$

We consider the free resolution of six points in $\mathbb{P}^1 \times \mathbb{P}^1$, especially those in P_{gen} . In the space of $\mathbb{P}^1 \times \mathbb{P}^1$, let $[X_1; X_2]$ and $[Y_1; Y_2]$ be the coordinates of the first and the second component of $\mathbb{P}^1$, then all coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ can be written as $X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2$. Explicitly, we can write

$$\begin{aligned}\mathbb{P}^1 \times \mathbb{P}^1 &\simeq \text{Proj}k[X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2] \\ &\simeq \text{Proj}k[t_{11}, t_{12}, t_{21}, t_{22}]/(t_{11}t_{22} - t_{12}t_{21})\end{aligned}$$

where $B = k[X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2]$ is given a grading which gives x_iy_j grading 1.

Let $B = k[X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2]$ and let $I \subset B$ be the vanishing ideal of given six points in $\mathbb{P}^1 \times \mathbb{P}^1$. We want to find a free resolution for B/I , whose summand consists of direct sums of $B(-m)$.

Proposition 4.2.1. *For six points of $\mathbb{P}^1 \times \mathbb{P}^1$ in P_{gen} , the vanishing ideal I for those six points has free resolution of the form*

$$0 \rightarrow B(-3)^2 \rightarrow B(-2)^3 \rightarrow B \rightarrow B/I \rightarrow 0.$$

Proof. Let $([x_{i,1}; x_{i,2}], [y_{i,1}; y_{i,2}])$ be the some coordinate of those six points. By applying some $\text{PGL}_1 \times \text{PGL}_1(k)$ -transform ($k = \bar{k}$ is our base field), we can assume none of $x_{i,1}, x_{i,2}, y_{i,1}, y_{i,2}$ are zero. (Note that P_{gen} is invariant under $\text{PGL}_1 \times \text{PGL}_1(k)$ -transform.) Under this condition, we can use the affine coordinate

$$u = X_1/X_2, v = Y_1/Y_2$$

on points of I . Let $u_i = x_{i,1}/x_{i,2}$ and $v_i = y_{i,1}/y_{i,2}$, then u_i, v_i are nonzero in k .

As the six points are distinct (from the above observation on P_{gen}) we can consider the scheme $\mathfrak{X} = V(I) = \text{Proj}(B/I)$ as union of six points, and it follows that the global

function ring $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is isomorphic to k^6 . Also all those six points are contained in affine subset $X_2Y_2 \neq 0$ which is isomorphic to $\text{Spec}k[X_1/X_2, Y_1/Y_2]$, so for I' be the image of $I \subset$ in $k[u, v]$ we have

$$\mathfrak{X} \simeq \text{Spec}k[u, v]/I', \quad \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \simeq k[u, v]/I'.$$

Denote B_d, I_d as the set of all degree d forms in B and I respectively. For $f \in B_d$, the rational function $f/(X_2Y_2)^d$ is well-defined on $\mathfrak{X}$, so it can be considered as global function on $\mathfrak{X}$. If we consider this evaluation map

$$\varphi_d : B_d \rightarrow \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}), \quad f \mapsto \frac{f}{(X_2Y_2)^d},$$

its kernel becomes I_d . This map φ_d can be seen as composition of $B_d \rightarrow k[u, v]$ given as $f \mapsto f/(X_2Y_2)^d$ and the quotient $k[u, v] \rightarrow k[u, v]/I'$.

Step 1) We show that φ_d is injective when $d = 1$, and surjective when $d \geq 2$. As $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \simeq k^6$, showing these would imply

$$\dim_k I_1 = 0, \quad \dim_k I_2 = 3.$$

The condition $([x_{i,1}; x_{i,2}], [y_{i,1}; y_{i,2}]) \in P_{gen}$ is equivalent to that the matrix having six rows of

$$(x_{i,1}^2 y_{i,1}, x_{i,1} x_{i,2} y_{i,1}, x_{i,2}^2 y_{i,1}, x_{i,1}^2 y_{i,2}, x_{i,1} x_{i,2} y_{i,2}, x_{i,2}^2 y_{i,2})$$

has rank 6 and the matrix having six rows of

$$(y_{i,1}^2 x_{i,1}, y_{i,1} y_{i,2} x_{i,1}, y_{i,2}^2 x_{i,1}, y_{i,1}^2 x_{i,2}, y_{i,1} y_{i,2} x_{i,2}, y_{i,2}^2 x_{i,2})$$

has rank 6. Those conditions are equivalent to that

$$\det((u_i^2 v_i, u_i v_i, v_i, u_i^2, u_i, 1)_{i=1, \dots, 6}) \neq 0$$

and

$$\det((v_i^2 u_i, v_i u_i, u_i, v_i^2, v_i, 1)_{i=1, \dots, 6}) \neq 0.$$

If we regard u, v as global functions of $\mathfrak{X}$, u_i, v_i corresponds to their evaluations at point i . So above is equivalent to that

$$(u^2 v, uv, v, u^2, u, 1) \text{ becomes basis of } \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$$

and

$$(uv^2, uv, u, v^2, v, 1) \text{ becomes basis of } \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}).$$

The map φ_d maps the degree d monomial $x_1^{k_1} x_2^{d-k_1} y_1^{k_2} y_2^{d-k_2}$ to $u^{k_1} v^{k_2}$ for $0 \leq k_1, k_2 \leq d$. So for $d = 1$, we have $uv, u, v, 1$ linearly independent in $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, so φ_d is injective. For $d \geq 2$, the list $u^{k_1} v^{k_2}$ for $0 \leq k_1, k_2 \leq 2$ spans $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, so φ_d is surjective. In particular, this shows that I_1 has dimension 0 and I_2 has dimension 3. (B_2 has dimension 9 and $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ has dimension 6.)

Step 2) As $\dim_k I_2 = 3$, there are three linear independent forms $q_1, q_2, q_3 \in I_2$. We show that I is generated by q_1, q_2, q_3 .

First we can show we can consider a particular form of q_1, q_2, q_3 as

$$\begin{aligned} \frac{q_1}{(X_2 Y_2)^2} &= u^2 v^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{1,ij} u^i v^j \\ \frac{q_2}{(X_2 Y_2)^2} &= u^2 v + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{2,ij} u^i v^j \\ \frac{q_3}{(X_2 Y_2)^2} &= u^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{3,ij} u^i v^j. \end{aligned}$$

As elements $\{u^i v^j\}_{0 \leq i \leq 1, 0 \leq j \leq 2} = \{uv^2, uv, u, v^2, v, 1\}$ can be considered as k -basis of $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \simeq k[u, v]/I'$, (I' : image of I in $B[(X_2 Y_2)^{-1}] \simeq k[u, v]$) there exist unique $c_{1,ij}$ s satisfying

$$u^2 v^2 \equiv - \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{1,ij} u^i v^j \pmod{I'}.$$

This implies that

$$\begin{aligned} q_1 &= (X_2 Y_2)^2 \left(u^2 v^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{1,ij} u^i v^j \right) \\ &= \left(X_1^2 Y_1^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{1,ij} X_1^i X_2^{2-i} Y_1^j Y_2^{2-j} \right) \in I \end{aligned}$$

and q_1 is of degree 2, so $q_1 \in I_2$. We can similarly construct q_2 and q_3 . By considering the $u^2 v^2$, $u^2 v$, u^2 -coefficients in the equation of $q_i/(X_2 Y_2)^2$, one can observe that q_1, q_2, q_3 are linearly independent. Let $\tilde{q}_i(u, v) = q_i/(X_2 Y_2)^2$. Then it suffices to show that I' is generated by $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$.

For any polyomial $f \in k[u, v]$, we can reduce any monomials of form $u^k v^l$ with $k \geq 2$ with terms of lower degree in u modulo $(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$, by replacing $u^2 v^2$ by $u^2 v^2 - \tilde{q}_1 = - \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{1,ij} u^i v^j$ etc. Meanwhile it is also possible to find k -basis r_1, r_2, r_3 of I_2 and $\tilde{r}_i = r_i/(X_2 Y_2)^2 \in I_2$ such that

$$\begin{aligned} \tilde{r}_1 &= v^2 u^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} d_{1,ij} v^i u^j \\ \tilde{r}_2 &= v^2 u + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} d_{1,ij} v^i u^j \\ \tilde{r}_3 &= v^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} d_{1,ij} v^i u^j, \end{aligned}$$

and we can reduce any monomials which degree in v is ≥ 2 with terms of lower degree in v modulo $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)$. As both (q_1, q_2, q_3) and (r_1, r_2, r_3) are k -basis of I_2 , $\tilde{r}_1, \tilde{r}_2, \tilde{r}_3$ can be represented as k -span of $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ in I' . So modulo $(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$, we can also

reduce monomials of degree ≥ 2 in v into terms of degree ≤ 1 in v .

Suppose $f \in I'$. We first reduce all monomials of degree ≥ 3 in v into terms of degree ≤ 2 in v . So we can find polynomials f_0, f_1, f_2 such that

$$f(u, v) \equiv f_0(u) + v f_1(u) + v^2 f_2(u) \pmod{(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)}.$$

For $d \geq 2$, we can replace monomials of the kind $u^d v^2, u^d v, u^d$ by span of $u^{d-1} v^2, u^{d-1} v, u^{d-1}, u^{d-2} v^2, u^{d-2} v, u^{d-2}$ modulo $(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$. We repeat this procedure to reduce all terms of righthand side into the form

$$f(u, v) \equiv \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} f_{ij} u^i v^j \pmod{(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)}.$$

As we have seen that elements $uv^2, uv, u, v^2, v, 1$ form basis of $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \simeq k[u, v]/I'$, $f(u, v) \in I'$ implies that $f_{ij} = 0$, so $f \equiv 0 \pmod{(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)}$. This shows that $I' = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$, so I is generated by q_1, q_2, q_3 .

Step 3) We show that there are two linearly independent syzygies of degree 1 between q_1, q_2, q_3 which span all syzygies between q_1, q_2, q_3 .

If we consider the linear map $I_2 \times B_1 \rightarrow I_3$ (B_d, I_d denotes degree d parts of B and I respectively), this map is surjective linear map (as I is generated by I_2) from dimension $3 \times 4 = 12$ space to 10 space ($\dim_k I_3 = \dim_k B_3 - 6 = 16 - 6 = 10$). Thus there are exactly two linearly independent syzygies.

Linear syzygies between q_1, q_2, q_3 are equivalent to syzygies between $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ with coefficients $g_{11}uv + g_{12}u + g_{21}v + g_{22}$. Suppose that

$$\sum_{i=1}^3 (g_{i,11}uv + g_{i,12}u + g_{i,21}v + g_{i,22}) \tilde{q}_i(u, v) = 0.$$

We remind that

$$\begin{aligned}\tilde{q}_1 &= u^2v^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{1,ij} u^i v^j \\ \tilde{q}_2 &= u^2v + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{2,ij} u^i v^j \\ \tilde{q}_3 &= u^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{3,ij} u^i v^j.\end{aligned}$$

We consider the coefficient of u^3 . The monomials in $\tilde{q}_i$ which degree in u is < 2 do not contribute to coefficient of u^3 , so

$$(g_{1,11}v + g_{1,12})v^2 + (g_{2,11}v + g_{2,12})v + (g_{3,11}v + g_{3,12}) = 0.$$

Solving this gives

$$g_{1,11} = g_{3,12} = 0, \quad g_{2,11} = -g_{1,12}, \quad g_{3,11} = -g_{2,12}.$$

We can consider the linear map from the set of linear syzygies to k^2 given as $(g_{2,11}, g_{3,11})$.

We claim that this map is one-to-one correspondence. As we know that the dimension of linear syzygy is 2, it suffices to show this map is injective. Suppose $g_{2,11} = g_{3,11} = 0$.

Then

$$\sum_{i=1}^3 (g_{i,21}v + g_{i,22})\tilde{q}_i(u, v) = 0.$$

We consider the coefficient of u^2 , which becomes

$$(g_{1,21}v + g_{1,22})v^2 + (g_{2,21}v + g_{2,22})v + (g_{3,21}v + g_{3,22}) = 0$$

so we have

$$g_{1,21} = g_{3,22} = 0, \quad g_{2,21} = -g_{1,22}, \quad g_{3,21} = -g_{2,22}.$$

Let $g_{2,21} = \alpha, g_{3,21} = \beta$. The equation for syzygy becomes

$$v(\alpha\tilde{q}_2 + \beta\tilde{q}_3) - (\alpha\tilde{q}_1 + \beta\tilde{q}_2) = 0,$$

We consider the coefficient of v^3 . As $(\alpha\tilde{q}_1 + \beta\tilde{q}_2)$ has degree in $v \leq 2$, only v^2 -coefficient of $(\alpha\tilde{q}_2 + \beta\tilde{q}_3)$ contributes to coefficient of v^3 , so it should be zero. This implies that in the expression

$$\alpha\tilde{q}_2 + \beta\tilde{q}_3 = \alpha u^2 v + \beta u^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} (\alpha c_{2,ij} + \beta c_{3,ij}) u^i v^j$$

we have

$$\alpha c_{2,12} + \beta c_{3,12} = \alpha c_{2,02} + \beta c_{3,02} = 0.$$

But this implies that $\alpha\tilde{q}_2 + \beta\tilde{q}_3$ is in linear span of $u^2 v, u^2, uv, u, v, 1$. As those six elements form basis of $k[u, v]/I'$ and $\alpha\tilde{q}_2 + \beta\tilde{q}_3 \in I'$, we should have $\alpha = \beta = 0$ and whole syzygy should be zero.

This shows that there are two linear syzygies such that $(g_{2,11}, g_{3,11}) = (1, 0)$ and $(g_{2,11}, g_{3,11}) = (0, 1)$. Such two syzygies have form of

$$(u + h_{1,1}(v))\tilde{q}_1 + (-uv + h_{1,2}(v))\tilde{q}_2 + h_{1,3}(v)\tilde{q}_3 = 0$$

for $(g_{2,11}, g_{3,11}) = (1, 0)$ and

$$h_{2,1}(v)\tilde{q}_1 + (u + h_{2,2}(v))\tilde{q}_2 + (-uv + h_{2,3}(v))\tilde{q}_3 = 0$$

for $(g_{2,11}, g_{3,11}) = (0, 1)$, where $h_{i,j}$ are polynomials of form $av + b$. It remains to show that any syzygy between $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ can be generated over $k[u, v]$ by those linear

syzygies. Denote J as the set of all syzygies

$$J = \{(f_1(u, v), f_2(u, v), f_3(u, v)) \in k[u, v]^3 : f_1\tilde{q}_1 + f_2\tilde{q}_2 + f_3\tilde{q}_3 = 0\}$$

and

$$j_1 = (u + h_{1,1}(v), -uv + h_{1,2}(v), h_{1,3}(v)), \quad j_2 = (h_{2,1}(v), u + h_{2,2}(v), -uv + h_{2,3}(v)).$$

We can consider

$$j_3 = j_1 + vj_2 = (u + h_{3,1}(v), h_{3,2}(v), -uv^2 + h_{3,3}(v))$$

where $h_{3,1}, h_{3,2}, h_{3,3}$ are some polynomials in v . For $f = (f_1, f_2, f_3) \in J$, suppose that degrees of u in f_1 and f_2 are both $\leq k$. Let $g_1(v), g_2(v)$ be the coefficient of u^k in f_1 and f_2 respectively (when we consider those as polynomials in u). We first subtract

$$u^{k-1}g_1(v)j_3 = (u^k g_1(v) + u^{k-1}g_1(v)h_{3,1}, u^{k-1}g_1(v)h_{3,2}(v), -u^k v^2 g_1(v) + u^{k-1}g_1(v)h_{3,3}(v))$$

from f . This operation makes u^k -coefficient of f_1 zero, while leaving u^k -coefficient of f_2 same. Next we subtract

$$u^{k-1}g_2(v)j_2 = (u^{k-1}g_2(v)h_{2,1}(v), u^k g_2(v) + u^{k-1}g_2(v)h_{2,2}(v), -u^k v g_2(v) + u^{k-1}g_2(v)h_{2,3}(v))$$

from f , and this operation leaves u^k -coefficient of f_1 unchanged and makes u^k -coefficient of f_2 zero. Then for $f' = f - u^{k-1}g_1(v)j_3 - u^{k-1}g_2(v)j_2 = (f'_1, f'_2, f'_3)$, degrees of u in f'_1 and f'_2 are both $\leq k-1$. We repeat this process till $k=0$, and we can find the following reduced form of f modulo (j_1, j_2) :

$$f \equiv (g_1(v), g_2(v), h(u, v)) \pmod{(j_1, j_2)}.$$

We claim that

$$g_1(v)\tilde{q}_1(u, v) + g_2(v)\tilde{q}_2(u, v) + h(u, v)\tilde{q}_3(u, v) = 0$$

implies $g_1 = g_2 = h = 0$. We remind that

$$\begin{aligned}\tilde{q}_1 &= u^2v^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{1,ij}u^i v^j \\ \tilde{q}_2 &= u^2v + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{2,ij}u^i v^j \\ \tilde{q}_3 &= u^2 + \sum_{0 \leq i \leq 1, 0 \leq j \leq 2} c_{3,ij}u^i v^j.\end{aligned}$$

Write $h(u, v) = uh_1(u, v) + g_3(v)$, then the above sum is of form

$$g_1(v)\tilde{q}_1 + g_2(v)\tilde{q}_2 + g_3(v)\tilde{q}_3 = u^3h_1(u, v) + (\text{terms which degree in } u \text{ is } \leq 2)$$

so $h_1(u, v) = 0$. We expand out the rest of the expression as

$$\begin{aligned}0 &= g_1(v)\tilde{q}_1(u, v) + g_2(v)\tilde{q}_2(u, v) + h(u, v)\tilde{q}_3(u, v) \\ &= u^2(v^2g_1 + vg_2 + g_3) \\ &\quad + u \left(\left(\sum_{j=0}^2 c_{1,1j}v^j \right) g_1 + \left(\sum_{j=0}^2 c_{2,1j}v^j \right) g_2 + \left(\sum_{j=0}^2 c_{3,1j}v^j \right) g_3 \right) \\ &\quad + \left(\left(\sum_{j=0}^2 c_{1,0j}v^j \right) g_1 + \left(\sum_{j=0}^2 c_{2,0j}v^j \right) g_2 + \left(\sum_{j=0}^2 c_{3,0j}v^j \right) g_3 \right).\end{aligned}$$

If not all of g_1, g_2, g_3 are zero, this implies

$$\det \begin{pmatrix} v^2 & v & 1 \\ \sum_{j=0}^2 c_{1,1j}v^j & \sum_{j=0}^2 c_{2,1j}v^j & \sum_{j=0}^2 c_{3,1j}v^j \\ \sum_{j=0}^2 c_{1,0j}v^j & \sum_{j=0}^2 c_{2,0j}v^j & \sum_{j=0}^2 c_{3,0j}v^j \end{pmatrix} = 0.$$

If we look at the coefficient of v^6 in this expression, we have

$$\det \begin{pmatrix} 1 & 0 & 0 \\ c_{1,12} & c_{2,12} & c_{3,12} \\ c_{1,02} & c_{2,02} & c_{3,02} \end{pmatrix} = 0,$$

so there exists $d_1, d_2, d_3 \in k$ not all zero, such that

$$d_1(1, c_{1,12}, c_{1,02}) + d_2(0, c_{2,12}, c_{2,02}) + d_3(0, c_{3,12}, c_{3,02}) = 0.$$

Consider

$$d_1\tilde{q}_1 + d_2\tilde{q}_2 + d_3\tilde{q}_3 \in I'.$$

Its v^2 -coefficient is

$$d_1(u^2 + c_{1,12}u + c_{1,02}) + d_2(c_{2,12}u + c_{2,02}) + d_3(c_{3,12}u + c_{3,02}) = 0$$

so its degree in v is at most 1 and it should be contained in $\text{span}(u^2v, uv, v, u^2, u, 1)$.

But those elements form k -basis of $k[u, v]/I' \simeq k^6$, so $d_1\tilde{q}_1 + d_2\tilde{q}_2 + d_3\tilde{q}_3 = 0$, therefore $d_1 = d_2 = d_3 = 0$. This contradiction leads to that all g_1, g_2, g_3 should be zero, so

$$f \equiv (g_1, g_2, g_3) \equiv 0 \pmod{(j_1, j_2)}$$

for any $f \in J$. Thus the set J of all syzygies between $\tilde{q}_i$ is generated by two linear syzygies j_1, j_2 , and similar properties should also hold for corresponding syzygies of q_i .

To summarize, we have shown that $\dim_k I_2 = 3$ and I is generated by I_2 , and syzygies between forms in I_2 are generated by two syzygies of degree 1. Those two

linear syzygies

$$j_1 = (u + h_{1,1}(v), -uv + h_{1,2}(v), h_{1,3}(v)), \quad j_2 = (h_{2,1}(v), u + h_{2,2}(v), -uv + h_{2,3}(v))$$

are linearly independent over $k[u, v]$, so there are no syzygies between them. Those three generators of I in I_2 and their two linear syzygies gives the minimal free resolution of the form

$$0 \rightarrow B(-3)^2 \rightarrow B(-2)^3 \rightarrow B \rightarrow B/I \rightarrow 0.$$

□

Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ be the structure sheaf, and denote $\mathcal{O}(a, b)$ as $\mathcal{O}_{\mathbb{P}^1}(a) \boxtimes \mathcal{O}_{\mathbb{P}^1}(b)$. Then $B(-m)$ can be corresponded to sheaves $\mathcal{O}(-m, -m)$. For six points of $\mathbb{P}^1 \times \mathbb{P}^1$ which are in P_{gen} , the above free resolution corresponds to following free resolution of the ideal sheaf $\mathcal{I}$ of vanishing functions on those six points.

$$0 \rightarrow \mathcal{O}(-3, -3)^2 \rightarrow \mathcal{O}(-2, -2)^3 \rightarrow \mathcal{O} \rightarrow \mathcal{O}/\mathcal{I} \rightarrow 0$$

4.2.4 Six points in $\mathbb{P}^1 \times \mathbb{P}^1$ associated to $2 \times 2 \times 2 \times 5$ forms

We would like to discuss which set of points (x_i, y_i) can be given as $\mathfrak{X}_{(x,y),A}$ for some $2 \times 2 \times 2 \times 5$ form A . We claim that the condition $((x_i, y_i))_i \in P_{gen}$ is sufficient to guarantee the existence of such A . Later we will show this is necessary condition.

Proposition 4.2.2. *For $\mathfrak{X} = ((x_i, y_i))_{i=1,\dots,6} \in P_{gen}$ ($x_i, y_i \in \mathbb{P}^1$), there exists $(z_i)_{i=1,\dots,6} \in (\mathbb{P}^1)^6$, $2 \times 2 \times 2 \times 5$ form A , and $t_1, \dots, t_6 \in \mathbb{P}^4$ over $k = \bar{k}$ such that*

$$(x_i, y_i, z_i, t_i) \in \mathfrak{X}_A \quad \text{for } i = 1, 2, \dots, 6.$$

Such choice of (z_i) is unique up to $\text{PGL}_2(k)$ -action on $\mathbb{P}^1$.

Points $t_1, \dots, t_6$ are required to be in general position (no five of them on a hyperplane), and slices $A_1, \dots, A_5$ are linearly independent. The choice of such $(t_i)_i$ and A is in one-to-one correspondence with choice of 6 points $(t_i)_i$ in general position.

Proof. Denote the coordinates of x_i and y_i as $x_i = [x_{i,1}; x_{i,2}]$ and $y_i = [y_{i,1}; y_{i,2}]$. For the coordinate variables of the space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, use capital letters $[X_1; X_2], [Y_1; Y_2], [Z_1; Z_2]$ etc. Also denote $p_i = (x_i, y_i)$.

Before we start, we note that the condition $\mathfrak{X} \in P_{gen}$ implies the following two properties.

- Any 5 points among $\mathfrak{X}$ are not on a degree 1 form of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$

$$c_{11}X_1Y_1 + c_{12}X_1Y_2 + c_{21}X_2Y_1 + c_{22}X_2Y_2.$$

Suppose the above form vanishes at $p_1, \dots, p_5$. Then the following form in $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1)$ vanishes on all six p_i , contradiction to $\mathfrak{X} \in P_{gen}$.

$$(c_{11}X_1Y_1 + c_{12}X_1Y_2 + c_{21}X_2Y_1 + c_{22}X_2Y_2)(x_{6,2}X_1 - x_{6,1}X_2)$$

- Any 5 points among x_i consists of at least three distinct points. If all of x_1, x_2, x_3, x_4, x_5 are same as one of two points of $[\mu_1; \mu_2]$ and $[\lambda_1; \lambda_2]$, then the following form in $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1)$ vanishes on all six p_i .

$$(\mu_2X_1 - \mu_1X_2)(\lambda_2X_1 - \lambda_1X_2)(y_{6,2}Y_1 - y_{6,1}Y_2)$$

- None of three of x_i are same. If $x_1 = x_2 = x_3 = [\mu_1; \mu_2]$, consider a linear form $h(X, Y)$ of $\mathcal{O}(1, 1)$ vanishing on p_4, p_5, p_6 . Then the form $(\mu_2X_1 - \mu_1X_2)h(X, Y)$ of $\mathcal{O}(2, 1)$ vanishes on all six p_i , contradiction to $\mathfrak{X} \in P_{gen}$.

Suppose there exists z_i, A, t_i such that

$$\mathfrak{X}_A = \{(x_i, y_i, z_i, t_i) : i = 1, 2, 3, 4, 5, 6\}.$$

Consider the following shorthand notation

$$x_i \otimes y_i \otimes z_i = (x_{i,j_1} y_{i,j_2} z_{i,j_3})_{1 \leq j_1, j_2, j_3 \leq 2} \in k^2 \otimes k^2 \otimes k^2 \simeq k^8.$$

$$v_i = x_i \otimes y_i = (x_{i,j_1} y_{i,j_2})_{1 \leq j_1, j_2 \leq 2} \in k^2 \otimes k^2 \simeq k^4.$$

Choice of those affine vectors can vary by up to constant multiple . For $t_i = [t_{i,1}; t_{i,2}; t_{i,3}; t_{i,4}; t_{i,5}]$, the condition $(x_i, y_i, z_i, t_i) \in \mathfrak{X}_A$ is equivalent to that

$$x_i \otimes y_i \otimes z_i = A(t_i) = \sum_{j=1}^5 t_{i,j} A_j$$

for some appropriate choice of $t_{i,j}$ (which can vary by up to constant multiple). We also need to consider the condition that $\mathfrak{X}_A$ does not contain points other than (x_i, y_i, z_i, t_i) .

Step 1) We start showing that following five elements are linearly independent.

$$x_1 \otimes y_1 \otimes z_1, x_2 \otimes y_2 \otimes z_2, x_3 \otimes y_3 \otimes z_3, x_4 \otimes y_4 \otimes z_4, x_5 \otimes y_5 \otimes z_5$$

Suppose that

$$\sum_{i=1}^5 u_i (x_i \otimes y_i \otimes z_i) = \sum_{i=1}^5 x_i \otimes y_i \otimes (u_i z_i) = 0$$

for $u_i \in k$, not all zero. For vectors

$$v_i = (x_{i,1} y_{i,1}, x_{i,1} y_{i,2}, x_{i,2} y_{i,1}, x_{i,2} y_{i,2}),$$

this equation is equivalent to that

$$\sum_{i=1}^5 u_i z_{i,1} v_i = \sum_{i=1}^5 u_i z_{i,2} v_i = 0.$$

But we know that $v_1, \dots, v_5$ span k^4 so their linear independences form 1-dimensional vector space. So $(u_1 z_{1,1}, \dots, u_5 z_{5,1})$ and $(u_1 z_{1,2}, \dots, u_5 z_{5,2})$ are linearly dependent, and there exists μ_1, μ_2 not all zero such that $u_i(\mu_2 z_{i,1} - \mu_1 z_{i,2}) = 0$.

We can show that v_1, v_2, v_3 are linearly independent, provided that pairs (x_i, y_i) are all distinct (as points of $\mathbb{P}^1 \times \mathbb{P}^1$), x_1, x_2, x_3 and y_1, y_2, y_3 are not all same in $\mathbb{P}^1$. (Those properties all follow from $(p_i) \in P_{gen}$.) Suppose $\sum_{i=1}^3 \alpha_i v_i = 0$. Then $\sum_{i=1}^3 \alpha_i (x_i \otimes y_i) = 0$, when we consider x_i and y_i as some affine vector. If $\alpha_3 = 0$, then $\alpha_1(x_1 \otimes y_1) = -\alpha_2(x_2 \otimes y_2)$ implies that x_2, y_2 are parallel to x_1, y_1 respectively. This means $(x_1, y_1) = (x_2, y_2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. So $\alpha_1, \alpha_2, \alpha_3 \neq 0$. In the expression

$$\alpha_1(x_1 \otimes y_1) + \alpha_2(x_2 \otimes y_2) = -\alpha_3(x_3 \otimes y_3),$$

the determinant of the lefthand side can be calculated as $\alpha_1 \alpha_2 \det(x_1, x_2) \det(y_1, y_2)$. As $\alpha_1 \alpha_2 \neq 0$, we have either $\det(x_1, x_2) = 0$ or $\det(y_1, y_2) = 0$. If x_1 and x_2 are parallel to each other, it follows that x_3 is parallel to them too and $x_1 = x_2 = x_3$ in $\mathbb{P}^1$. So we obtain contradiction and we can conclude v_1, v_2, v_3 can't be linearly independent.

In the expression $\sum_{i=1}^5 u_i z_{i,1} v_i = 0$, this suggests that at least four terms among $u_1 z_{1,1}, \dots, u_5 z_{5,1}$ are nonzero. Without loss of generality, assume that u_1, u_2, u_3, u_4 are nonzero. Then from $u_i(\mu_2 z_{i,1} - \mu_1 z_{i,2}) = 0$, it follows that $z_1 = z_2 = z_3 = z_4 = [\mu_1; \mu_2]$. But if $z_1 = z_2 = z_3 = z_4$, we can show that the set

$$\text{span}(x_1 \otimes y_1 \otimes z_1, x_2 \otimes y_2 \otimes z_2, x_3 \otimes y_3 \otimes z_3, x_4 \otimes y_4 \otimes z_4)$$

contains an element of the form $x' \otimes y' \otimes z_1$ from (x', y') distinct from all (x_i, y_i) . As $x_1 \otimes y_1, x_2 \otimes y_2, x_3 \otimes y_3$ are linearly independent, their span forms a subspace of dimension 3, and its intersection with all vectors corresponding to elements of form $x \otimes y$ is infinite. As $x_i \otimes y_i \otimes z_i = A(t_i)$ is in span of $A_1, \dots, A_5$, so is $x' \otimes y' \otimes z_1$. This implies that $\mathfrak{X}_A$ contains a point other than (x_i, y_i, z_i, t_i) , which is contradiction. Thus five elements $x_i \otimes y_i \otimes z_i \in k^2 \otimes k^2 \otimes k^2$ for $i = 1, 2, 3, 4, 5$ are linearly independent.

Step 2) We show that we can pick coordinates for $z_{i,1}, z_{i,2}$ such that

$$\sum_{i=1}^6 x_i \otimes y_i \otimes z_i = 0$$

while $z_i = [z_{i,1}; z_{i,2}]$ is given. As six elements $A(t_i)$ ($i = 1, \dots, 6$) are in span of $A_1, \dots, A_5$, those are linearly dependent. Suppose

$$\sum_{i=1}^6 u_i A(t_i) = \sum_{i=1}^6 u_i (x_i \otimes y_i \otimes z_i) = 0.$$

By the step 1, we know that $u_6 \neq 0$. Similarly we have u_1, u_2, u_3, u_4, u_5 are all nonzero.

Replace $(z_{i,1}, z_{i,2})$ to $(u_i z_{i,1}, u_i z_{i,2})$ then we obtain

$$\sum_{i=1}^6 x_i \otimes y_i \otimes z_i = \sum_{i=1}^6 A(t_i) = 0.$$

As $A(t_1), \dots, A(t_5)$ are linearly independent elements which are in span of A_1, A_2, A_3, A_4, A_5 , this implies that both $A(t_1), \dots, A(t_5)$ and $A_1, \dots, A_5$ both become basis of $\text{span}(A_1, \dots, A_5)$. This shows that affine vectors corresponding to $t_1, \dots, t_5$ are linearly independent, and from $\sum A(t_i) = 0$ we have $t_1 + \dots + t_6 = 0$. This shows that six points in $\mathbb{P}^4$ corresponding to $t_1, \dots, t_6$ are in general position and $A_1, \dots, A_5$ are linearly independent.

Step 3) We construct $z_{i,1}, z_{i,2} \in k$ such that $z_i = [z_{i,1}; z_{i,2}]$ defines points in $\mathbb{P}^1$ and

$\sum_i x_i \otimes y_i \otimes z_i = 0$. Then for $A_i = x_i \otimes y_i \otimes z_i$ ($i = 1, 2, 3, 4, 5$) and

$$t_1 = [1; 0, 0, 0, 0], t_2 = [0; 1; 0; 0; 0], \dots, t_5 = [0; 0; 0; 0; 1], t_6 = [-1; -1; -1; -1; -1],$$

we have $(x_i, y_i, z_i, t_i) \in \mathfrak{X}_A$.

The condition $\sum_i x_i \otimes y_i \otimes z_i = 0$ is basically equivalent to that

$$\sum_{i=1}^6 z_{i,1} v_i = \sum_{i=1}^6 z_{i,2} v_i = 0.$$

($v_i = (x_{i,1} y_{i,1}, x_{i,1} y_{i,2}, x_{i,2} y_{i,1}, x_{i,2} y_{i,2})$) We can naturally think of

$$L = \{(\alpha_1, \dots, \alpha_6) \in k^6 : \sum \alpha_i v_i = 0\}.$$

As $((x_i, y_i))_i \in P_{gen}$, the above observation suggests that any five vectors among $v_1, \dots, v_6$ span k^4 . In particular, six vectors $v_1, \dots, v_6$ span k^4 , so L is of dimension 2. Let $(z_{1,1}, \dots, z_{6,1})$ and $(z_{1,2}, \dots, z_{6,2})$ be two linearly independent vectors of L . If $z_{1,1} = z_{1,2} = 0$, then there are two linearly independent linear dependences between v_2, v_3, v_4, v_5, v_6 so it contradicts that v_2, v_3, v_4, v_5, v_6 span k^4 and therefore they have 1-dimensional linear dependences between them. So $(z_{1,1}, z_{1,2}) \neq (0, 0)$ and $z_1 = [z_{1,1}; z_{1,2}]$ is a well-defined point in $\mathbb{P}^1$. We can similarly show that $z_i = [z_{i,1}; z_{i,2}]$ are well-defined for $i = 2, 3, 4, 5, 6$.

We show that this choice of $(z_1, \dots, z_6) \in (\mathbb{P}^1)^6$ is unique up to $\mathrm{PGL}_2(k)$ -action on $\mathbb{P}^1$ when x_i and y_i are given. In the construction above, explicit affine coordinates $z_{i,j}$ depends on choice of basis vectors $(z_{1,1}, \dots, z_{6,1})$ and $(z_{1,2}, \dots, z_{6,2})$ of L and choice of affine coordinates $x_{i,j}$ and $y_{i,j}$. Two basis vectors of L can be chosen up to $\mathrm{GL}_2(k)$ -action, and this action operates on each z_i as $\mathrm{PGL}_2(k)$ -action. Meanwhile, the choice of $x_{i,1}, x_{i,2}, y_{i,1}, y_{i,2}$ is unique up to multiplications by nonzero constants $x_{i,j} \mapsto c_{i,x} x_{i,j}$ and $y_{i,j} \mapsto c_{i,y} y_{i,j}$. These operations change v_i by $c_{i,x} c_{i,y}$ -multiple

and change $z_{i,j}$ by $(c_{i,x}c_{i,y})^{-1}$ -multiple. It does not change $z_i = [z_{i,1}; z_{i,2}]$. So the choice of $(z_1, \dots, z_6) \in (\mathbb{P}^1)^6$ modulo $\mathrm{PGL}_2(k)$ -action is uniquely determined when x_i and y_i are given.

Step 4) So far, we constructed $(z_i)_{i=1,\dots,6} \in (\mathbb{P}^1)^6$ such that there exists A and t_i satisfying $\{(x_i, y_i, z_i, t_i)\} \in \mathfrak{X}_A$, and showed such choice of $(z_i)_i$ is unique up to $\mathrm{PGL}_2(k)$ -action on $\mathbb{P}^1$. Assume affine coordinates of z_i was chosen such that $\sum(x_i \otimes y_i \otimes z_i) = 0$. From the condition $x_i \otimes y_i \otimes z_i = A(t_i)$ and that $x_1 \otimes y_1 \otimes z_1, \dots, x_5 \otimes y_5 \otimes z_5$ are linearly independent, we require t_i to be in linearly general position and $A_1, \dots, A_5$ to be linearly independent. Conversely if $t_1, \dots, t_6$ are picked as points in linearly general position, we can pick affine coordinates for t_i such that $\sum t_i = 0$ in affine coordinates, and set A uniquely satisfying $x_i \otimes y_i \otimes z_i = A(t_i)$. $\square$

We will later prove that $\mathfrak{X}_A$ exactly consist of six points and there is one-to-one correspondence between set of points in P_{gen} and $\mathrm{PGL}_2 \times \mathrm{PGL}_5(\bar{k})$ -equivalence class of A , using properties of the sextic ring which we will define in the next section.

We note the following interesting relation.

Proposition 4.2.3. *For (x_i, y_i, z_i) constructed in proposition 4.2.2, we have*

$$\sum_{i=1}^6 p_i(x)p_i(y)p_i(z) = 0,$$

and

$$\begin{aligned} & \left(\sum_{i=1}^6 p_i(y)^2 p_i(x) \right) \left(\sum_{i=1}^6 p_i(z)^2 p_i(x) \right) \\ &= \left(\sum_{i=1}^6 p_i(x)^2 p_i(y) \right) \left(\sum_{i=1}^6 p_i(z)^2 p_i(y) \right) \\ &= \left(\sum_{i=1}^6 p_i(x)^2 p_i(z) \right) \left(\sum_{i=1}^6 p_i(y)^2 p_i(z) \right). \end{aligned}$$

Proof. It suffices to show the statement for generic enough choice of (x_i, y_i, z_i) . Also

p_i s are relative invariants in PGL_2 -action, so the statement is equivalent when we apply PGL_2 -action on either one set of (x_i) , (y_i) or (z_i) . For the case of $x = (0, 1, \infty, x_4, x_5, x_6)$, $y = (0, 1, \infty, y_4, y_5, y_6)$ where variables are generic enough, we can explicitly calculate (z_i) and check that above algebraic equations are satisfied. Those representatives can cover generic enough PGL_2 -equivalence classes of points in P_{gen} . $\square$

We can use this to prove the following.

Proposition 4.2.4. *For such x_i, y_i, z_i constructed in proposition 4.2.2, we have $((x_i, z_i))_{i=1, \dots, 6} \in P_{gen}$ and $((y_i, z_i))_{i=1, \dots, 6} \in P_{gen}$.*

Proof. First we can assume all $x_{i,j}$ and $y_{i,j}$ are nonzero, but applying suitable PGL_2 -coordinate change. We note the relation

$$\frac{1}{8} \sum_{i=1}^6 p_i(x)^2 p_i(y) = \det \begin{pmatrix} x_{1,1}^2 y_{1,1} & x_{1,1} x_{1,2} y_{1,1} & x_{1,2}^2 y_{1,1} & x_{1,1}^2 x_{1,2} & x_{1,1} y_{1,2} y_{1,2} & x_{1,2}^2 y_{1,2} \\ x_{2,1}^2 y_{2,1} & x_{2,1} x_{2,2} y_{2,1} & x_{2,2}^2 y_{2,1} & x_{2,1}^2 x_{2,2} & x_{2,1} y_{2,2} y_{2,2} & x_{2,2}^2 y_{2,2} \\ x_{3,1}^2 y_{3,1} & x_{3,1} x_{3,2} y_{3,1} & x_{3,2}^2 y_{3,1} & x_{3,1}^2 x_{3,2} & x_{3,1} y_{3,2} y_{3,2} & x_{3,2}^2 y_{3,2} \\ x_{4,1}^2 y_{4,1} & x_{4,1} x_{4,2} y_{4,1} & x_{4,2}^2 y_{4,1} & x_{4,1}^2 x_{4,2} & x_{4,1} y_{4,2} y_{4,2} & x_{4,2}^2 y_{4,2} \\ x_{5,1}^2 y_{5,1} & x_{5,1} x_{5,2} y_{5,1} & x_{5,2}^2 y_{5,1} & x_{5,1}^2 x_{5,2} & x_{5,1} y_{5,2} y_{5,2} & x_{5,2}^2 y_{5,2} \\ x_{6,1}^2 y_{6,1} & x_{6,1} x_{6,2} y_{6,1} & x_{6,2}^2 y_{6,1} & x_{6,1}^2 x_{6,2} & x_{6,1} y_{6,2} y_{6,2} & x_{6,2}^2 y_{6,2} \end{pmatrix}.$$

So it suffices to prove

$$\sum p_i(x)^2 p_i(z), \sum p_i(x) p_i(z)^2, \sum p_i(y)^2 p_i(z), \sum p_i(y) p_i(z)^2$$

are all nonzero. From the proposition 2.3, one can see proving $\sum p_i(x)^2 p_i(z), \sum p_i(y)^2 p_i(z) \neq 0$ suffices, as

$$\left(\sum p_i(x) p_i(z)^2 \right) \left(\sum p_i(x) p_i(y)^2 \right) = \left(\sum p_i(x)^2 p_i(z) \right) \left(\sum p_i(y)^2 p_i(z) \right) \neq 0.$$

Without loss of generality we only need to prove one of them. Suppose $\sum p_i(x)^2 p_i(z) = 0$. This implies

$$\det \begin{pmatrix} x_{1,1}^2 z_{1,1} & x_{1,1} x_{1,2} z_{1,1} & x_{1,2}^2 z_{1,1} & x_{1,1}^2 x_{1,2} & x_{1,1} z_{1,2} z_{1,2} & x_{1,2}^2 z_{1,2} \\ x_{2,1}^2 z_{2,1} & x_{2,1} x_{2,2} z_{2,1} & x_{2,2}^2 z_{2,1} & x_{2,1}^2 x_{2,2} & x_{2,1} z_{2,2} z_{2,2} & x_{2,2}^2 z_{2,2} \\ x_{3,1}^2 z_{3,1} & x_{3,1} x_{3,2} z_{3,1} & x_{3,2}^2 z_{3,1} & x_{3,1}^2 x_{3,2} & x_{3,1} z_{3,2} z_{3,2} & x_{3,2}^2 z_{3,2} \\ x_{4,1}^2 z_{4,1} & x_{4,1} x_{4,2} z_{4,1} & x_{4,2}^2 z_{4,1} & x_{4,1}^2 x_{4,2} & x_{4,1} z_{4,2} z_{4,2} & x_{4,2}^2 z_{4,2} \\ x_{5,1}^2 z_{5,1} & x_{5,1} x_{5,2} z_{5,1} & x_{5,2}^2 z_{5,1} & x_{5,1}^2 x_{5,2} & x_{5,1} z_{5,2} z_{5,2} & x_{5,2}^2 z_{5,2} \\ x_{6,1}^2 z_{6,1} & x_{6,1} x_{6,2} z_{6,1} & x_{6,2}^2 z_{6,1} & x_{6,1}^2 x_{6,2} & x_{6,1} z_{6,2} z_{6,2} & x_{6,2}^2 z_{6,2} \end{pmatrix} = 0,$$

so there exists nonzero vector $(a_1, a_2, \dots, a_6)$ in left null space. This implies

$$\begin{aligned} \sum_{i=1}^6 a_i x_{i,1}^2 z_{i,1} &= \sum_{i=1}^6 a_i x_{i,1}^2 z_{i,2} = \sum_{i=1}^6 a_i x_{i,1} x_{i,2} z_{i,1} = \sum_{i=1}^6 a_i x_{i,1} x_{i,2} z_{i,2} \\ &= \sum_{i=1}^6 a_i x_{i,2}^2 z_{i,1} = \sum_{i=1}^6 a_i x_{i,2}^2 z_{i,2} = 0. \end{aligned}$$

If we denote

$$v_1 = (a_i x_{i,1}^2)_{i=1, \dots, 6}, v_2 = (a_i x_{i,1} x_{i,2})_{i=1, \dots, 6}, v_3 = (a_i x_{i,2}^2)_{i=1, \dots, 6},$$

then $v_t \cdot (z_{1,1}, \dots, z_{6,1}) = v_t \cdot (z_{1,2}, \dots, z_{6,2}) = 0$. Meanwhile, from the proof of the Proposition 4.2.2, we know that two vectors $(z_{i,j})_{i=1, \dots, 6}$ are given as basis of orthogonal complement of space having basis $(x_{i,1} y_{i,1})_i, (x_{i,1} y_{i,2})_i, (x_{i,2} y_{i,1})_i, (x_{i,2} y_{i,2})_i$. So this implies

$$v_t \in \text{span}((x_{i,1} y_{i,1})_i, (x_{i,1} y_{i,2})_i, (x_{i,2} y_{i,1})_i, (x_{i,2} y_{i,2})_i)$$

for any $t = 1, 2, 3$. For $t = 2$ in particular, we have

$$(a_i x_{i,1} x_{i,2})_i \in \text{span}((x_{i,1} y_{i,1})_i, (x_{i,1} y_{i,2})_i, (x_{i,2} y_{i,1})_i, (x_{i,2} y_{i,2})_i).$$

Suppose that

$$(a_i x_{i,1} x_{i,2})_i = c_{11}(x_{i,1} y_{i,1})_i + c_{12}(x_{i,1} y_{i,2})_i + c_{21}(x_{i,2} y_{i,1})_i + c_{22}(x_{i,2} y_{i,2})_i.$$

Then from the assumption that all $x_{i,j}, y_{i,j}$ are nonzero, we can express a_i as

$$a_i = c_{11} \frac{y_{i,1}}{x_{i,2}} + c_{12} \frac{y_{i,2}}{x_{i,2}} + c_{21} \frac{y_{i,1}}{x_{i,1}} + c_{22} \frac{y_{i,2}}{x_{i,1}}.$$

We consider v_1 , which becomes

$$\begin{aligned} v_1 &= \left(x_{i,1}^2 \cdot \left(c_{11} \frac{y_{i,1}}{x_{i,2}} + c_{12} \frac{y_{i,2}}{x_{i,2}} + c_{21} \frac{y_{i,1}}{x_{i,1}} + c_{22} \frac{y_{i,2}}{x_{i,1}} \right) \right)_i \\ &= \left(c_{11} \frac{x_{i,1}^2}{x_{i,2}} y_{i,1} + c_{12} \frac{x_{i,1}^2}{x_{i,2}} y_{i,2} + c_{21} x_{i,1} y_{i,1} + c_{22} x_{i,1} y_{i,2} \right)_i. \end{aligned}$$

This shows

$$\left(c_{11} \frac{x_{i,1}^2}{x_{i,2}} y_{i,1} + c_{12} \frac{x_{i,1}^2}{x_{i,2}} y_{i,2} \right)_i \in \text{span}((x_{i,1} y_{i,1})_i, (x_{i,1} y_{i,2})_i, (x_{i,2} y_{i,1})_i, (x_{i,2} y_{i,2})_i).$$

If we multiply $x_{i,2}$ at i th coordinate, we have

$$(c_{11} x_{i,1}^2 y_{i,1} + c_{12} x_{i,1}^2 y_{i,2})_i \in \text{span}((x_{i,1} x_{i,2} y_{i,1})_i, (x_{i,1} x_{i,2} y_{i,2})_i, (x_{i,2}^2 y_{i,1})_i, (x_{i,2}^2 y_{i,2})_i).$$

From the condition $((x_i, y_i)) \in P_{gen}$ implying no nonzero degree $(2, 1)$ form on (x, y)

vanishes on six point, we should have $c_{11} = c_{12} = 0$. If we consider v_3 . we have

$$v_3 = \left(c_{21} \frac{x_{i,2}^2}{x_{i,1}} y_{i,1} + c_{22} \frac{x_{i,2}^2}{x_{i,1}} y_{i,2} \right)_i \in \text{span}((x_{i,1} y_{i,1})_i, (x_{i,1} y_{i,2})_i, (x_{i,2} y_{i,1})_i, (x_{i,2} y_{i,2})_i)$$

so multiplying $x_{i,1}$ at i th coordinate gives

$$(c_{21}x_{i,2}^2y_{i,1} + c_{22}x_{i,2}^2y_{i,2})_i \in \text{span} \left((x_{i,1}^2y_{i,1})_i, (x_{i,1}^2y_{i,2})_i, (x_{i,1}x_{i,2}y_{i,1})_i, (x_{i,1}x_{i,2}y_{i,2})_i \right)$$

which implies $c_{21} = c_{22} = 0$ similarly. Thus $a_i = 0$ for all i , which contradicts our construction. Therefore we should have $\sum p_i(x)^2 p_i(z) \neq 0$. $\square$

4.3 Construction of the sextic ring

In this section, we will construct a sextic ring $\mathcal{R}(A)$ associated to a $2 \times 2 \times 2 \times 5$ form $A \in \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^5$ using the method of hypercohomology construction which was discussed in the Chapter 2.

We will first construct a complex $\mathcal{C}_{t,A}$ of sheaves on $\mathbb{P}^4$ which becomes minimal free resolution of $\mathfrak{X}_{t,A} \subset \mathbb{P}^4$ over $\mathbb{Q}$ for generic A . We can also construct $\mathcal{C}_{(x,y),A}$ of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ which becomes minimal free resolution of $\mathfrak{X}_{(x,y),A} \subset \mathbb{P}^1 \times \mathbb{P}^1$ for generic A , and it turns out that the hypercohomology rings for $\mathcal{C}_{t,A}$ and $\mathcal{C}_{(x,y),A}$ are isomorphic sextic rings. One can naturally correspond this ring $\mathcal{R}(A)$ to A , and denote its discriminant as $\text{Disc}(A) = \text{Disc}(\mathcal{R}(A))$. Under the $G = \text{SL}_2 \times \text{SL}_2 \times \text{SL}_2 \times \text{GL}_5$ -action on A , $\mathcal{R}(A)$ is $\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2$ -invariant, and GL_5 acts on the basis elements of $\mathcal{R}(A)/\mathbb{Z}$ as standard action times determinant. Thus $\text{Disc}(A)$ is $\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2 \times \text{SL}_5$ -invariant polynomial.

Lastly we will show that the condition $\text{Disc}(A) \neq 0$ is equivalent to that $\mathfrak{X}_A$ consists of distinct 6 points, and complete the proof of Theorem 4.1.2.

4.3.1 Complex for six points in $\mathbb{P}^4$

We will consider a complex $\mathcal{C}_{t,A}$ having form ($\mathcal{O} = \mathcal{O}_{\mathbb{P}^4}$)

$$0 \rightarrow \mathcal{O}(-6) \otimes \Lambda_6 \rightarrow \mathcal{O}(-4) \otimes \Lambda_4 \rightarrow \mathcal{O}(-3) \otimes \Lambda_3 \rightarrow \mathcal{O}(-2) \otimes \Lambda_2 \rightarrow \mathcal{O} \rightarrow 0$$

such that $\Lambda_6, \Lambda_4, \Lambda_3, \Lambda_2$ are free $\mathbb{Z}$ -modules of rank 1, 6, 16, 9 respectively. As complete description of $\mathcal{C}_{t,A}$ is rather lengthy, it will be given in the appendix section (Section 4.5). The following two properties are enough to characterize $\mathcal{C}_{t,A}$ up to isomorphism.

Denote α_{ijk} as (i, j, k) -entries of the cube $A(t) = t_1 A_1 + \cdots + t_5 A_5$ ($i, j, k = 1, 2$)

- The quadrics of the image of $\mathcal{O}(-2)^9 \rightarrow \mathcal{O}$ are given as follows:

$$\begin{aligned} \alpha_{111}\alpha_{122} - \alpha_{112}\alpha_{121}, & \quad \alpha_{111}\alpha_{212} - \alpha_{112}\alpha_{211}, & \quad \alpha_{111}\alpha_{221} - \alpha_{121}\alpha_{211}, \\ \alpha_{211}\alpha_{222} - \alpha_{212}\alpha_{221}, & \quad \alpha_{121}\alpha_{222} - \alpha_{122}\alpha_{221}, & \quad \alpha_{112}\alpha_{222} - \alpha_{122}\alpha_{212}, \\ \alpha_{111}\alpha_{222} - \alpha_{112}\alpha_{221}, & \quad \alpha_{111}\alpha_{222} - \alpha_{121}\alpha_{212}, & \quad \alpha_{111}\alpha_{222} - \alpha_{211}\alpha_{122} \end{aligned}$$

The first six of them are determinants of six faces of the cube $A(t)$, and the last three are determinants of the rectangle formed by elements in two diagonals of the cube.

- When we consider maps of $\mathcal{C}_{t,A}$ as polynomial maps by identifying $\mathcal{O}(l)$ with graded module $B(l)$ ($B = \mathbb{Z}[t_1, \dots, t_5]$ with degree shifted by l), those are injective at their lowest nonzero degree for generic value of A . For example, the map $\mathcal{O}(-3)^{16} \rightarrow \mathcal{O}(-2)^9$ can be considered as map $B(-3)^{16} \rightarrow B(-2)^9$, and this map at degree 3 is injective. (Degree k elements of $B(-3)$ are degree $k - 3$ homogeneous polynomials, 3 is lowest nonzero degree)

Remark. In [15], the above complex is constructed using the generalized Eagon-Northcott complex (see [24]), although the explicit descriptions of the all maps of $\mathcal{C}_{t,A}$ were obtained independently by the author.

The complex $\mathcal{C}_{t,A}$ has form of minimal free resolution for 6 points in $\mathbb{P}^4$, (Proposition 2.5.4) so it satisfies the WTM conditions (Definition 2.3.4). Thus we can apply the Theorem 2.3.3 to $\mathcal{C}_{t,A}$ to deduce that hypercohomology is isomorphic to $\mathbb{Z}^6$ as

$\mathbb{Z}$ -module. In particular, we have an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(\mathcal{C}_{t,A}) \rightarrow H^4(\mathcal{O}(-6) \otimes \Lambda_6) \simeq 0$$

where $H^4(\mathcal{O}(-6) \otimes \Lambda_6) \simeq \mathbb{Z}^5$ with $\mathbb{Z}$ -basis $L_6/(t_1 t_2 t_3 t_4 t_5 t_i)$ ($i = 1, \dots, 5$).

Proposition 4.3.1. *For generic A , complex $\mathcal{C}_{t,A} \otimes \mathbb{Q}$ is minimal free resolution for distinct six points of $\mathfrak{X}_{t,A}$.*

Proof. As we pointed out, the image of the map $\mathcal{O}(-2)^9 \rightarrow \mathcal{O}$ of $\mathcal{C}_{t,A}$ is spanned by 9 quadrics - 6 of them given as determinants of six faces of $2 \times 2 \times 2$ cube $A(t)$ and 3 of them given as differences of products of elements located at diagonals of $A(t)$. Suppose that all those 9 quadrics vanish at $t = t_0$. This implies that two vectors $(\alpha_{111}, \alpha_{112}, \alpha_{121}, \alpha_{122})$ and $(\alpha_{211}, \alpha_{212}, \alpha_{221}, \alpha_{222})$ are parallel, all 2×2 minors of

$$\begin{pmatrix} \alpha_{111} & \alpha_{112} & \alpha_{121} & \alpha_{122} \\ \alpha_{211} & \alpha_{212} & \alpha_{221} & \alpha_{222} \end{pmatrix}$$

are spanned by those 9 quadrics. Meanwhile

$$\alpha_{111}\alpha_{122} - \alpha_{112}\alpha_{121} = \alpha_{211}\alpha_{222} - \alpha_{212}\alpha_{221} = 0$$

as those determinants are also in 9 quadrics. This suggests that α_{ijk} should be of form $x_i y_j z_k$ for $t = t_0$, so $t_0 \in \mathfrak{X}_{t,A}$. Conversely, one can check that $t_0 \in \mathfrak{X}_{t,A}$ implies all 9 quadrics vanish at $t = t_0$, by substituting $\alpha_{ijk} = x_i y_j z_k$ and showing all quadrics vanish. Therefore we have $\mathfrak{X}_{t,A} = V(\mathcal{I})$ for $\mathcal{I}$ the image of $\mathcal{O}(-2)^9 \rightarrow \mathcal{O}$.

Proposition 4.2.2 suggests that there exists a Zariski open subset consisting of A such that $\mathfrak{X}_{t,A}$ contains distinct six points in linearly general position. As we established that maps of $\mathcal{C}_{t,A}$ are injective at their lowest degree for generic A , we can pick Zariski open subset S which also satisfies this property. The condition

$\mathcal{O}(-2)^9 \rightarrow \mathcal{O}$ being injective requires that $\mathfrak{X}_{t,A}$ should not contain more than six points: otherwise dimension of the vanishing quadric would be less than 9.

We show that the above condition that $\mathcal{C}_{t,A}$ is exact suffices to show it becomes free resolution of $\mathcal{C}_{t,A}$. First we observe that the minimal free resolution of six points in $\mathbb{P}^4$ has same form as $\mathcal{C}_{t,A}$. The image of $\mathcal{O}(-2)^9 \rightarrow \mathcal{O}$ at degree 2 is of dimension 9 due to exactness condition, and is contained in dimension 9 quadrics vanishing on six points of $\mathfrak{X}_{t,A}$. So the map $\mathcal{O}(-2)^9 \rightarrow \mathcal{O}$ describes the first syzygy. The image of $\mathcal{O}(-3)^{16} \rightarrow \mathcal{O}(-2)^9$ at degree 3 is injective and sits inside dimension 16 syzygies of quadrics in the map $\mathcal{O}(-2)^9 \rightarrow \mathcal{O}$ (provided by the Betti number of resolution), so $\mathcal{O}(-3)^{16} \rightarrow \mathcal{O}(-2)^9$ describes the second syzygy. Similarly we can show that maps of $\mathcal{C}_{t,A}$ describes third, fourth and fifth syzygy, and conclude that $\mathcal{C}_{t,A}$ is minimal free resolution of $\mathfrak{X}_{t,A}$. $\square$

Corollary 4.3.2. *The complex $\mathcal{C}_{t,A}$ becomes minimal free resolution over ring of universal variables $\mathbb{Q}[A]$, and has induced DG-algebra structure with coefficients in rational polynomials in entries of A which is identity on zeroth entry $\mathcal{O}$.*

Proof. The above proposition states that the set of A where $\mathcal{C}_{t,A}$ is exact is Zariski open. If $\mathcal{C}_{t,A}$ is not exact over $\mathbb{Q}[A]$, then there exists a nontrivial section in cohomology, and the condition that this section is not in the cycle is Zariski open. As those sets should not intersect, $\mathcal{C}_{t,A}$ is exact over $\mathbb{Q}[A]$.

The construction of DG-algebra structure on $\mathcal{C}_{t,A}$ can be done using ([46], Theorem 136). We can view $\mathfrak{X}_{t,A} = V(\mathcal{I})$ universally as rank 6 cover over $\text{Spec } \mathbb{Q}[A]$ which has $\mathcal{C}_{t,A}$ over general variables $\mathbb{Q}[A]$ as minimal free resolution. One can observe that this cover is Gorenstein having symmetric Betti numbers. Also, the previous proposition shows that for generic value of A its fiber $\mathfrak{X}_{t,A}$ becomes distinct six points, so this cover is generically étale. Thus ([46], Theorem 136) can be applied to construct a DG-algebra. $\square$

I expect that one can give explicit DG-algebra structure on $\mathcal{C}_{t,A}$ which is defined over $\mathbb{Z}[A]$, but I could not find elegant description yet. Anyway, we can define the ring $\mathcal{R}(A)$ as $H^0 R\Gamma(\mathcal{C}_{t,A})$ with algebra structure induced from DG-algebra structure on $\mathcal{C}_{t,A}$.

We apply the Theorem 2.3.7 to $\mathcal{C}_{t,A}$ to obtain explicit description of the hypercohomology element as global functions on $\mathfrak{X}_{t,A}$. For the Čech complex

$$C_\bullet(\mathbb{P}^k) : \bigoplus \mathcal{O}_{t_i} \rightarrow \bigoplus \mathcal{O}_{t_i t_j} \rightarrow \cdots \rightarrow \bigoplus \mathcal{O}_{t_1 t_2 t_3 t_4 t_5},$$

Theorem 2.3.7 states that we can identify $H^0 R\Gamma(\mathcal{C}_{t,A})$ with $H^0 \text{Tot}\Gamma(\mathcal{C}_{t,A} \otimes C_\bullet(\mathbb{P}^k))$.

Consider the map

$$\varphi : H^0 R\Gamma(\mathcal{C}_{t,A}) \simeq H^0 \text{Tot}\Gamma(\mathcal{C}_{t,A} \otimes C_\bullet(\mathbb{P}^k)) \rightarrow H^0 \Gamma(\mathcal{O} \otimes \bigoplus \mathcal{O}_{t_i}) \simeq \bigoplus \mathbb{Z}[t_1/t_i, \dots, t_5/t_i].$$

By tracking how the multiplication on $H^0 \text{Tot}\Gamma(\mathcal{C}_{t,A} \otimes C_\bullet(\mathbb{P}^k))$ given by Theorem 2.4.2 operates, we can observe this map should preserve multiplication. Meanwhile, if we denote α_i as a preimage of $L_6/(t_1 \cdots t_5 t_i) \in H^4(\mathcal{O}(-6))$ under the map $H^0 R\Gamma(\mathcal{C}_{t,A}) \rightarrow H^4(\mathcal{O}(-6))$, then one can explicitly compute $\varphi(\alpha_i)$ as elements in $\bigoplus \mathbb{Z}[t_1/t_i, \dots, t_5/t_i]$, and find some integral polynomials $c_{ij}^k(A)$ satisfying following polynomial equations.

$$\varphi(\alpha_i)\varphi(\alpha_j) \equiv \sum_{k=1}^5 \varphi(\alpha_k)c_{ij}^k(A) + c_{ij}^0(1, 1, 1, 1, 1) \pmod{\mathcal{I}_{t,A}}$$

One needs to be careful that not all $c_{ij}^k(A)$ are uniquely determined, as choice of φ_i and $\varphi_{i,0}$ can be shifted by integers. For distinct $i, j, k = 1, \dots, 5$, the following expressions are the complete invariant under the $\mathbb{Z}^5$ -action of shifting basis elements by integers $\varphi_i \mapsto \varphi_i + C_i$

$$c_{ij}^k, c_{ii}^j, c_{ij}^j - c_{ik}^k, c_{ii}^i - c_{ij}^j - c_{ik}^k.$$

This is because the action $\varphi_i \mapsto \varphi_i + C_i$ acts on multiplication coefficients as $c_{ij}^j \mapsto c_{ij}^j + C_i$ and $c_{ii}^i \mapsto c_{ii}^i + 2C_i$. (c_{ij}^k and c_{ii}^j are unchanged.) If one needs to fix the basis of $\mathcal{R}(A)$, then one can standardize the basis by fixing some of the values c_{ij}^j , for example letting $c_{12}^1 = c_{13}^1 = c_{14}^1 = c_{15}^1 = c_{12}^2 = 0$. This choice determines all c_{ij}^k , and all c_{ij}^0 from the equation (obtained from associativity, coefficient of α_k in $\alpha_i(\alpha_j\alpha_k) - (\alpha_i\alpha_j)\alpha_k$)

$$c_{ij}^0 = \sum_{l=1}^5 (c_{jk}^l c_{li}^k - c_{ij}^l c_{lk}^k).$$

Proposition 4.3.3. *The invariants from multiplication coefficients*

$$c_{ij}^k, c_{ii}^j, c_{ij}^j - c_{ik}^k, c_{ii}^i - c_{ij}^j - c_{ik}^k$$

are degree 6 homogeneous polynomials in $a_{ijk,t}$, and are invariant under the action of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ on A . Under the action of $g \in \mathrm{GL}_5(\mathbb{Z})$, $\mathcal{R}(gA)$ is isomorphic to $\mathcal{R}(A)$ in the following way: if $(1 = \alpha_0, \dots, \alpha_5)$ is basis of $\mathcal{R}(A)$, then $\mathcal{R}(gA)$ is isomorphic to the ring having basis $1, \tilde{\alpha}_1, \dots, \tilde{\alpha}_5$ where $\tilde{\alpha}_i = \det(g) \sum_{j=1}^5 g_{ij} \alpha_j$.

It can be proven via some explicit polynomial calculations.

Remark. The GL_5 -representation spanned by elements

$$c_{ij}^k, c_{ii}^j, c_{ij}^j - c_{ik}^k, c_{ii}^i - c_{ij}^j - c_{ik}^k$$

is isomorphic to the irreducible representation of GL_5 having highest weight $(3, 1, 1, 1, 0)$. Among all subrepresentations of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ -invariants of degree 6, this representation is unique irreducible representation having weight $(3, 1, 1, 1, 0)$. This provides one way to find explicit formulas for $c_{ij}^k(A)$ using computer algebra system.

4.3.2 Complex for six points in $\mathbb{P}^1 \times \mathbb{P}^1$

The complex $\mathcal{C}_{(x,y),A}$ for $\mathfrak{X}_{(x,y),A}$ in $\mathbb{P}_x^1 \times \mathbb{P}_y^1$ can be described more easily. Before we give definition of the complex, we will show how one can obtain equations for $\mathfrak{X}_{(x,y),A}$ in elementary way. This is basically about eliminating z and t from the equation of $\mathfrak{X}_A$

$$A_1 t_1 + A_2 t_2 + A_3 t_3 + A_4 t_4 + A_5 t_5 = (x_1, x_2) \otimes (y_1, y_2) \otimes (z_1, z_2).$$

It is equivalent to that

$$\sum_{i=1}^5 t_i A_i - z_1 \cdot (x_1, x_2) \otimes (y_1, y_2) \otimes (1, 0) - cz_2 \cdot (x_1, x_2) \otimes (y_1, y_2) \otimes (0, 1) = 0$$

so it is again equivalent to that seven $2 \times 2 \times 2$ cubes of $A_1, \dots, A_5, (x_1, x_2) \otimes (y_1, y_2) \otimes (1, 0), (x_1, x_2) \otimes (y_1, y_2) \otimes (0, 1)$ are linearly dependent. We identify the space of $2 \times 2 \times 2$ cubes with k^8 and consider the following determinants $((e_1, e_2) = ((1, 0), (0, 1))$ are standard basis vectors)

$$\psi_{ijk} = \det(A_1, A_2, A_3, A_4, A_5, (x_1, x_2) \otimes (y_1, y_2) \otimes e_1, (x_1, x_2) \otimes (y_1, y_2) \otimes e_2, e_i \otimes e_j \otimes e_k)$$

for $(i, j, k) \in \{1, 2\}^3$. Those eight ψ_{ijk} s all vanish if and only if $A_1, \dots, A_5, (x_1, x_2) \otimes (y_1, y_2) \otimes (1, 0), (x_1, x_2) \otimes (y_1, y_2) \otimes (0, 1)$ are linearly dependent. Thus the ideal $I = (\psi_{ijk})$ becomes vanishing ideal (over field) of six points in $\mathbb{P}_x^1 \times \mathbb{P}_y^1$. Denote

$$\epsilon_{ijk} = e_i \otimes e_j \otimes e_k, \quad \epsilon_{xy1} = (x_1, x_2) \otimes (y_1, y_2) \otimes e_1, \quad \epsilon_{xy2} = (x_1, x_2) \otimes (y_1, y_2) \otimes e_2.$$

We can find the following syzygies between them:

$$\begin{aligned}\det(A_1, \dots, A_5, \epsilon_{xy1}, \epsilon_{xy2}, A_t) &= \sum_{i,j,k=1}^2 (A_t)_{i,j,k} \psi_{ijk} = 0 \\ \det(A_1, \dots, A_5, \epsilon_{xy1}, \epsilon_{xy2}, \epsilon_{xy1}) &= \sum_{i,j=1}^2 \psi_{ij1} x_i y_j = 0 \\ \det(A_1, \dots, A_5, \epsilon_{xy1}, \epsilon_{xy2}, \epsilon_{xy2}) &= \sum_{i,j=1}^2 \psi_{ij2} x_i y_j = 0\end{aligned}$$

We construct $\mathcal{C}_{(x,y),A}$ using generators ψ_{ijk} of the vanishing ideal and their syzygies above. For $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$, it has the form

$$0 \rightarrow \mathcal{O}(-3, -3) \otimes \Lambda_{3,2} \oplus \mathcal{O}(-2, -2) \otimes \Lambda_{2,2} \rightarrow \mathcal{O}(-2, -2) \otimes \Lambda_{2,1} \rightarrow \mathcal{O} \rightarrow 0$$

where $\Lambda_{3,2}, \Lambda_{2,2}, \Lambda_{2,1}$ are free $\mathbb{Z}$ modules of rank 2, 5, 8 respectively. Denote their $\mathbb{Z}$ -basis as

$$\{L_{3,2}^k\}_{k=1,2}, \quad \{L_{2,2}^t\}_{t=1,2,3,4,5}, \quad \{L_{2,1}^{i,j,k}\}_{i,j,k=1,2}$$

and construct the maps of $\mathcal{C}_{(x,y),A}$ as

$$\begin{aligned}L_{3,2}^k &\mapsto \sum_{i,j=1}^2 x_i y_j L_{2,1}^{i,j,k} \\ L_{2,2}^t &\mapsto \sum_{i,j,k=1}^2 a_{ijk,t} L_{2,1}^{i,j,k} \\ L_{2,1}^{i,j,k} &\mapsto \psi_{ijk}.\end{aligned}$$

If we denote L_2 as length 8 vector identified with $2 \times 2 \times 2$ cube $L_{2,1}^{i,j,k}$, then we can simplify above notations to

$$L_{3,2}^k \mapsto \epsilon_{xyk} \cdot L_2, \quad L_{2,2}^t \mapsto A_t \cdot L_2.$$

One can explicitly describe a DG-algebra structure on $\mathcal{C}_{(x,y),A}$ where $\mathcal{O}$ becomes unit. We only need to describe multiplications between $L_{2,1}$, as all other multiplications are trivial. For M_1, M_2 length 8 vectors identified with $2 \times 2 \times 2$ cube with integral entries, multiplication between $M_1 \cdot L_2$ and $M_2 \cdot L_2$ is given as

$$\begin{aligned}
(M_1 \cdot L_2) \cdot (M_2 \cdot L_2) = & -\det(M_1, A_2, A_3, A_4, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M_2) L_{2,2}^1 \\
& -\det(A_1, M_1, A_3, A_4, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M_2) L_{2,2}^2 \\
& -\det(A_1, A_2, M_1, A_4, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M_2) L_{2,2}^3 \\
& -\det(A_1, A_2, A_3, M_1, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M_2) L_{2,2}^4 \\
& -\det(A_1, A_2, A_3, A_4, M_1, \epsilon_{xy1}, \epsilon_{xy2}, M_2) L_{2,2}^5 \\
& -\det(A_1, A_2, A_3, A_4, A_5, M_1, \epsilon_{xy2}, M_2) L_{3,2}^1 \\
& -\det(A_1, A_2, A_3, A_4, A_5, \epsilon_{xy1}, M_1, M_2) L_{3,2}^2.
\end{aligned}$$

To check that this is well-defined DG-algebra, we note that

$$d(M \cdot L_2) = \det(A_1, A_2, A_3, A_4, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M)$$

so

$$\begin{aligned}
& d(M_1 \cdot L_2)(M_2 \cdot L_2) - d(M_2 \cdot L_2)(M_1 \cdot L_2) \\
& = \det(A_1, A_2, A_3, A_4, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M_1)(M_2 \cdot L_2) \\
& - \det(A_1, A_2, A_3, A_4, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M_2)(M_1 \cdot L_2)
\end{aligned}$$

and

$$\begin{aligned}
d((M_1 \cdot L_2) \cdot (M_2 \cdot L_2)) = & -\det(M_1, A_2, A_3, A_4, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M_2)(A_1 \cdot L_2) \\
& -\det(A_1, M_1, A_3, A_4, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M_2)(A_2 \cdot L_2) \\
& -\det(A_1, A_2, M_1, A_4, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M_2)(A_3 \cdot L_2) \\
& -\det(A_1, A_2, A_3, M_1, A_5, \epsilon_{xy1}, \epsilon_{xy2}, M_2)(A_4 \cdot L_2) \\
& -\det(A_1, A_2, A_3, A_4, M_1, \epsilon_{xy1}, \epsilon_{xy2}, M_2)(A_5 \cdot L_2) \\
& -\det(A_1, A_2, A_3, A_4, A_5, M_1, \epsilon_{xy2}, M_2)(\epsilon_{xy1} \cdot L_2) \\
& -\det(A_1, A_2, A_3, A_4, A_5, \epsilon_{xy1}, M_1, M_2)(\epsilon_{xy2} \cdot L_2).
\end{aligned}$$

Showing that their differences is zero can be interpreted as a special case of the Plücker relations.

Proposition 4.3.4. *The hypercohomology ring $H^0 R\Gamma(\mathcal{C}_{(x,y),A})$ is isomorphic to $\mathcal{R}(A)$.*

Proof. We would like to apply the Proposition 2.6.4. Note that the complex $\mathcal{C}_{(x,y),A}$ has form

$$0 \rightarrow \mathcal{O}(-3, -3)^2 \oplus \mathcal{O}(-2, -2)^5 \rightarrow \mathcal{O}(-2, -2)^8 \rightarrow \mathcal{O} \rightarrow 0.$$

We remind that for $m_1, m_2 < 0$, we have $H^i(\mathcal{O}(m_1, m_2))$ zero except when $i = 2$, and

$$H^2(\mathcal{O}(m_1, m_2)) \simeq (\text{degree } (m_1, m_2) \text{ part of } \frac{1}{x_1 x_2 y_1 y_2} \mathbb{Z}[x_1^{-1}, x_2^{-1}, y_1^{-1}, y_2^1]).$$

Thus one can apply the Proposition 2.6.4 to obtain the exact sequence

$$0 \rightarrow H^0(\mathcal{O}) = \mathbb{Z} \rightarrow H^0 R\Gamma(\mathcal{C}_{(x,y),A}) \rightarrow T \rightarrow 0$$

where

$$T = \ker(H^2(\mathcal{O}(-3, -3)^2 \oplus \mathcal{O}(-2, -2)^5) \rightarrow H^2(\mathcal{O}(-2, -2))).$$

A particular element in the cohomology

$$\alpha \in H^2((\mathcal{C}_{(x,y),A})_{-2}) \simeq H^2(\mathcal{O}(-3, -3)) \otimes \Lambda_{3,2} \oplus H^2(\mathcal{O}(-2, -2)) \otimes \Lambda_{2,2}$$

can be described as

$$\alpha = \frac{\sum_{t=1}^5 \alpha_t L_{2,2}^t}{x_1 x_2 y_1 y_2} + \sum_{l,m,k=1}^2 \frac{L_{3,2}^k \mu_{l,m,k} x_l^{-1} y_m^{-1}}{x_1 x_2 y_1 y_2}.$$

We note that map of $\mathcal{C}_{(x,y),A}$ was set to be

$$\begin{aligned} L_{3,2}^k &\mapsto \sum_{i,j=1}^2 x_i y_j L_{2,1}^{i,j,k} \\ L_{2,2}^t &\mapsto \sum_{i,j,k=1}^2 a_{ijk,t} L_{2,1}^{i,j,k} \\ L_{2,1}^{i,j,k} &\mapsto \psi_{ijk}. \end{aligned}$$

The image of α in $H^2((\mathcal{C}_{(x,y),A})_{-1})$ can be calculated as

$$\begin{aligned} d\alpha &= \frac{1}{x_1 x_2 y_1 y_2} \sum_{t=1}^5 \sum_{i,j,k=1}^2 \alpha_t a_{ijk,t} L_{2,1}^{i,j,k} + \frac{1}{x_1 x_2 y_1 y_2} \sum_{i,j,k,l,m=1}^2 x_i y_j L_{2,1}^{i,j,k} \mu_{l,m,k} x_l^{-1} y_m^{-1} \\ &= \frac{1}{x_1 x_2 y_1 y_2} \sum_{i,j,k=1}^2 L_{2,1}^{i,j,k} \left(\sum_{t=1}^5 \alpha_t a_{ijk,t} + \mu_{i,j,k} \right) \\ &\quad + \sum_{(i,j) \neq (l,m), k=1,2} \frac{x_i y_j}{x_1 x_2 x_l y_1 y_2 y_m} \mu_{l,m,k} L_{2,1}^{i,j,k} \end{aligned}$$

and the later terms with $\frac{x_i y_j}{x_1 x_2 x_l y_1 y_2 y_m}$ vanish in $H^2((\mathcal{C}_{(x,y),A})_{-1}) \simeq H^2(\mathcal{O}(-2, -2)) \otimes \Lambda_{2,1}$. Thus in order to satisfy $\alpha \in T$ or $d\alpha = 0$, we need

$$\mu_{i,j,k} = - \sum_{t=1}^5 \alpha_t a_{ijk,t}$$

therefore we have

$$\alpha = \sum_{t=1}^5 \alpha_t \cdot \left(\frac{L_{2,2}^t}{x_1 x_2 y_1 y_2} - \sum_{l,m,k=1}^2 \frac{a_{lmk,t} L_{3,2}^k}{x_1 x_2 x_l y_1 y_2 y_m} \right).$$

This establishes $T \simeq \mathbb{Z}^5$ as $\mathbb{Z}$ -module with $\mathbb{Z}$ -basis

$$\frac{L_{2,2}^t}{x_1 x_2 y_1 y_2} - \sum_{l,m,k=1}^2 \frac{a_{lmk,t} L_{3,2}^k}{x_1 x_2 x_l y_1 y_2 y_m}.$$

From those cohomology elements, one can obtain explicit description of the map

$$H^0 R\Gamma(\mathcal{C}_{(x,y),A}) \rightarrow H^0 \text{Tot} \gamma(M_{\bullet} \otimes \text{Tot}(C_{\bullet}(\mathbb{P}^1)^{(1)} \boxtimes C_{\bullet}(\mathbb{P}^1)^{(2)}))$$

which comes from the Theorem 2.6.5 and the Proposition 2.6.7, and compute the algebra structure on $H^0 R\Gamma(\mathcal{C}_{(x,y),A})$ using multiplication structure on right side induced by total tensor product. For $\alpha_t \in H^0 R\Gamma(\mathcal{C}_{(x,y),A})$ which is preimage of an element

$$\frac{L_{2,2}^t}{x_1 x_2 y_1 y_2} - \sum_{l,m,k=1}^2 \frac{a_{lmk,t} L_{3,2}^k}{x_1 x_2 x_l y_1 y_2 y_m} \in T$$

via $H^0 R\Gamma(\mathcal{C}_{(x,y),A}) \rightarrow T$, one can show that

$$\alpha_i \alpha_j = \sum_{k=1}^5 c_{ij}^k(A) \alpha_k + c_{ij}^0(A)$$

for appropriate choice of α_i (which can be changed up to translation by integer constant). This shows $H^0 R\Gamma(\mathcal{C}_{(x,y),A}) \simeq \mathcal{R}(A)$. $\square$

4.3.3 Hypercohomology ring and geometry of six points

So far we constructed $\mathcal{R}(A)$ as hypercohomology ring for both $\mathcal{C}_{t,A}$ and $\mathcal{C}_{(x,y),A}$. We are ready to prove the Theorem 4.1.2 using properties of $\mathcal{R}(A)$ and those two complexes.

In Proposition 4.2.2 we already established that for a set of x_i, y_i in P_{gen} there exists A such that $\mathfrak{X}_{(x,y),A}$ contains (x_i, y_i) , and such choice of A is unique up to $SL_2 \times GL_5$ -equivalence. We first show that we have $\text{Disc}(A) \neq 0$ for such A .

Proposition 4.3.5. *Let $x_i, y_i, z_i \in \mathbb{C}^2$ ($i = 1, \dots, 6$) such that $\sum_{i=1}^6 x_i \otimes y_i \otimes z_i = 0$. Denote the invariant in Proposition 4.2.3 as*

$$\left(\sum_{i=1}^6 p_i(y)^2 p_i(x) \right) \left(\sum_{i=1}^6 p_i(z)^2 p_i(x) \right) = \delta(x, y, z).$$

Then for $A_i = x_i \otimes y_i \otimes z_i$ ($i = 1, \dots, 5$), we have $\text{Disc}(A) = -\delta(x, y, z)^5$.

Proof. One can explicitly calculate multiplication coefficients of $\mathcal{R}(A)$ as

$$c_{ij}^k = \begin{cases} \gamma(x, y, z) & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}$$

for some polynomial $\gamma(x, y, z)$ satisfying $\gamma(x, y, z)^2 = -\delta(x, y, z)$. It follows that $\text{Disc}(A) = \text{Disc}(\mathcal{R}(A)) = \gamma(x, y, z)^{10} = -\delta(x, y, z)^5$. $\square$

Proposition 4.3.6. *For $((x_i, y_i))_{i=1, \dots, 6} \in P_{gen}$, any A such that $(x_i, y_i) \in \mathfrak{X}_{(x,y),A}$ should satisfy $\text{Disc}(A) \neq 0$.*

Proof. We showed in Proposition 4.2.2 that such choice of A is unique up to GL_5 -transform. As $\text{Disc}(A)$ is relative GL_5 -invariant, $\text{Disc}(A) \neq 0$ for choice of above Proposition implies $\text{Disc}(A) \neq 0$ for all such A . $\square$

We show converse statements following from $\text{Disc}(A) \neq 0$.

Proposition 4.3.7. *If $\text{Disc}(A) \neq 0$, then*

1. $\mathfrak{X}_{t,A}$ consists of distinct six points in $\mathbb{P}^4$ general position.
2. $\mathfrak{X}_{(x,y),A}$ consists of distinct six points in $\mathbb{P}^1 \times \mathbb{P}^1$ in P_{gen} (not on any bidegree $(1, 2)$ -or $(2, 1)$ -form).

Proof. (1) We observed that $\mathcal{C}_{t,A}$ has form of minimal free resolution of 6 points in $\mathbb{P}^4$ in general position. For such complexes such that its hypercohomology ring has nonzero discriminant, Theorem 2.5.5 implies that $V(\mathcal{I})$ consists of 6 points in $\mathbb{P}^4$ in general position, where $\mathcal{I}$ is image of $\mathcal{O}(-2)^9 \rightarrow \mathcal{O}$. Meanwhile, in the proof of the Proposition 4.3.1, we have shown that $V(\mathcal{I}) = \mathfrak{X}_{t,A}$. So $\mathfrak{X}_{t,A}$ consists of 6 points in $\mathbb{P}^4$ in general position.

(2) The idea is similar to Theorem 2.5.5, except that we need the following different property regarding global function elements corresponding to $\mathcal{R}(A)$.

Lemma. *Let*

$$\alpha_t = \frac{L_{2,2}^t}{x_1 x_2 y_1 y_2} - \sum_{l,j,k=1}^2 \frac{a_{lmk,t} L_{3,2}^k}{x_1 x_2 x_l y_1 y_2 y_m}$$

be the basis element of the Tschirnhausen module of $\mathcal{C}_{(x,y),A}$ described in Section 4.3.2. One can choose $\varphi_t \in \bigoplus \mathcal{O}_{x_i y_j}$ as representative of image of α_t under the map $H^0 R\Gamma(\mathcal{C}_{(x,y),A}) \rightarrow \Gamma(\mathcal{O}/\mathcal{I})$, such that for all $i, j = 1, 2$, φ_t on $\mathcal{O}_{x_i y_j}$ has form of $1/(x_i^2 y_j)$ times polynomial.

Proof. We remind

$$\alpha_t = \frac{L_{2,2}^t}{x_1 x_2 y_1 y_2} - \sum_{l,j,k=1}^2 \frac{a_{lmk,t} L_{3,2}^k}{x_1 x_2 x_l y_1 y_2 y_m}.$$

We can calculate the φ_t using the lifting algorithm described in Section 3.2, and one can choose lifting such that φ_t is of above form. $\square$

This implies that for $x_{2/1} = x_2/x_1$ and $y_{2/1} = y_2/y_1$, φ_t on $\mathcal{O}_{x_1 y_1}$ is span of following elements

$$1, x_{2/1}, x_{2/1}^2, y_{2/1}, x_{2/1} y_{2/1}, x_{2/1}^2 y_{2/1}.$$

As the map $\mathcal{R}(A) \rightarrow \Gamma(\mathcal{O}/\mathcal{I})$ is injective (Theorem 2.5.2), we should have $1, \varphi_1, \dots, \varphi_5$ linearly independent. So $1, \varphi_1, \dots, \varphi_5$ become $\mathbb{Q}$ -basis of $\mathbb{Q}$ -span of $1, x_{2/1}, x_{2/1}^2, y_{2/1}, x_{2/1} y_{2/1}, x_{2/1}^2 y_{2/1}$.

For distinct embeddings $\rho_1, \dots, \rho_6 : \mathcal{R}(A) \rightarrow \overline{\mathbb{Q}}$, we can construct geometric points $p_1, \dots, p_6$ such that φ_i takes value $\rho_j(\varphi_i)$ at p_j , using same idea as Theorem

2.5.3. Consider kernel of ρ_j in $\mathcal{R} \otimes \overline{\mathbb{Q}}$, which is ideal over $\overline{\mathbb{Q}}$. As $\mathcal{R} \otimes \overline{\mathbb{Q}} \subset \Gamma(X \otimes \overline{\mathbb{Q}})$, one can construct a maximal ideal I of $\Gamma(X \otimes \overline{\mathbb{Q}})$ containing kernel of ρ_j . By the weak Nullstellensatz, there exists a point p_j of X such that functions on I vanishes on p_j . Then $\varphi_i - \rho_j(\varphi_i)$ vanishes on p_j , which implies $\varphi_i(p_j) = \rho_j(\varphi_i)$.

Suppose that $p_1, \dots, p_6$ are on a bidegree $(2, 1)$ -form $F(x, y)$. Then $F(x, y)/(x_1^2 y_1)$ is in span of $1, x_{2/1}, x_{2/1}^2, y_{2/1}, x_{2/1} y_{2/1}, x_{2/1}^2 y_{2/1}$, so it is spanned by $1, \varphi_1, \dots, \varphi_5$. Let $F(x, y)/(x_1^2 y_1) = c_0 + \sum c_t \varphi_t$. Evaluating this on p_j gives

$$c_0 + \sum_{t=1}^5 c_t \varphi(p_j) = c_0 + \sum_{t=1}^5 c_t \rho_j(\varphi_t) = \rho_j(c_0 + \sum_{t=1}^5 c_t \varphi_t) = 0$$

for all j , so $c_0 = c_1 = \dots = c_5 = 0$. This implies $F = 0$. By symmetry, we can also show $p_1, \dots, p_6$ are not on a bidegree $(1, 2)$ -form.

It remains to show that there are no additional points in $\mathfrak{X}_{(x,y),A}$ other than $p_1, \dots, p_6$. Suppose that there is a point q . We can consider the morphism $\mathcal{R}(A) \rightarrow \overline{\mathbb{Q}}$ given as evaluation at q . It maps 1 to 1 so it is nontrivial morphism identical to one of ρ_i , so $\varphi_j(q) = \rho_i(\varphi_j) = \varphi_j(p_i)$ for all j . As φ_j and 1 spans $x_{2/1}$ and $y_{2/1}$, this implies $q = p_i$. $\square$

Combining those statements leads of proof of the Theorem 4.1.2.

Theorem 4.3.8 (Theorem 4.1.2, repeated). *Let A be a $2 \times 2 \times 2 \times 5$ form over algebraically closed field k of characteristic zero. The condition $\text{Disc}(A) \neq 0$ is equivalent to $\mathfrak{X}_{(x,y),A}$ satisfying the following condition: $\mathfrak{X}_{(x,y),A}$ consists of six points of $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying that no single global section of $\mathcal{O}(1, 2)$ and $\mathcal{O}(2, 1)$ vanishes on all of those six points. In that case, $\mathfrak{X}_{t,A}$ consists of six points of $\mathbb{P}^4$ in linearly general position (any 5 of them are not on a hyperplane.)*

Conversely, for any distinct six k -points in $\mathbb{P}^1 \times \mathbb{P}^1$ such that no single global section of $\mathcal{O}(1, 2)$ and $\mathcal{O}(2, 1)$ vanishes on all of those six points, there exists A over k such that $\mathfrak{X}_{(x,y),A}$ becomes set of those six points, and such A is determined uniquely up to

$\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{GL}_5(k)$ -equivalence.

We can establish the complete correspondence between forms and $\mathfrak{X}_A$ over algebraically closed field of characteristic zero, together with the invariants of six points in $\mathbb{P}^1 \times \mathbb{P}^1$ coming from Geometric Invariant Theory interpretation.

Call a list of points $x_i, y_i, z_i \in \mathbb{P}^1$ ($i = 1, \dots, 6$) *triply related* if $(x_i, y_i) \in P_{gen}$ and $x_i \otimes y_i \otimes z_i$ are linearly dependent. Proposition 4.2.4 suggests that this relation is symmetric with respect to switching x, y, z . The uniqueness of z_i established in Proposition 2.2 can be interpreted as that Gale transform of six points $x_i \otimes y_i \in \mathbb{P}^3$ are $z_i \in \mathbb{P}^1$.

Theorem 4.3.9. *Let k be algebraically closed field of characteristic zero. Let $V_{D \neq 0}(k)$ be the set of $2 \times 2 \times 2 \times 5$ forms such that $\mathrm{Disc}(A) \neq 0$. There is a one-to-one correspondence between $V_{D \neq 0}(k)/k^\times$ and*

$$\begin{aligned} & \{(x_i, y_i, z_i, t_i)_{i=1, \dots, 6} \in (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^4(k))^6 : \\ & (x_i, y_i, z_i)_{i=1, \dots, 6} \text{ triply related,} \\ & (t_i)_{i=1, \dots, 6} \text{ in general position}\}. \end{aligned}$$

If we consider group action of $g = (g_x, g_y, g_z, g_t) \in G(k) = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{GL}_5(k)$ ($g_x, g_y, g_z \in \mathrm{SL}_2(k)$ and $g_t \in \mathrm{GL}_5(k)$) on A , the corresponding point lists $(x_i), (y_i), (z_i), (t_i)$ are transformed by g_x, g_y, g_z, g_t respectively, according to natural PGL_2 - or PGL_5 -actions. The S_3 -symmetry of forms A switching x, y, z -axes (symmetry of the $2 \times 2 \times 2$ cube) acts as switching the corresponding point lists $(x_i), (y_i), (z_i)$.

By considering equivalence classes by of A subgroups of $G(k)$, one can obtain the following correspondences.

Corollary 4.3.10. *1. There is a one-to-one correspondence between $V_{D \neq 0}(k)/\mathrm{GL}_5(k)$ and triply related lists of six points.*

2. *There is a one-to-one correspondence between $V_{D \neq 0}(k)/\mathrm{SL}_2 \times \mathrm{GL}_5(k)$ and lists of k -points in P_{gen} .*
3. *There is a one-to-one correspondence between $V_{D \neq 0}(k)/\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{GL}_5(k)$, $\mathrm{PGL}_2 \times \mathrm{PGL}_2(k)$ -equivalence classes of k -points in P_{gen} , and its relative invariants*

$$\begin{aligned} & \{((u_1, \dots, u_6), (w_1, \dots, w_6)) : \\ & \sum u_i = \sum u_i^3 = \sum w_i = \sum w_i^3 = 0, (\sum u_i^2 w_i)(\sum u_i w_i^2) \neq 0\} \\ & /k^\times \times k^\times. \end{aligned}$$

4.4 Descriptions of the invariants

Now let A be a $2 \times 2 \times 2 \times 5$ form with rational entries such that $\mathrm{Disc}(A) \neq 0$, such that $\mathcal{R}(A)$ becomes rank 6 nondegenerate $\mathbb{Q}$ -algebras. Similar to the work on $3 \times 3 \times 5$ form, we want to examine how the following correspondence behaves:

$$V_{D \neq 0}(\mathbb{Q})/G(\mathbb{Q}) \rightarrow \{\text{nondegenerate sextic } \mathbb{Q}\text{-algebras}\}, \quad A \mapsto \mathcal{R}(A)$$

here $G = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{GL}_5$. For given nondegenerate sextic $\mathbb{Q}$ -algebra R , we want to characterize which A gives $\mathcal{R}(A) \simeq R$.

To make this question precise, we first note that our construction $\mathcal{R}(A)$ for $A \in V(\mathbb{Q})$ not only produces a $\mathbb{Q}$ -algebra, but it comes with with specified basis elements $\alpha_1, \dots, \alpha_5$ of $\mathcal{R}(A)/\mathbb{Q}$. We can say that two algebras with specified basis elements are isomorphic if and only if specified basis elements correspond with respect to the ring isomorphism. One can consider following questions:

- Given a nondegenerate sextic $\mathbb{Q}$ -algebra R , which $A \in V_{D \neq 0}(\mathbb{Q})$ satisfy $\mathcal{R}(A) \otimes \mathbb{Q} \simeq R$?

- Given a nondegenerate sextic $\mathbb{Q}$ -algebra R with basis elements $1 = \alpha_0, \dots, \alpha_5$ specified, which $A \in V_{D \neq 0}(\mathbb{Q})$ satisfy $\mathcal{R}(A) \simeq R$ (as algebra with specified basis elements?)
- Given a nondegenerate sextic $\mathbb{Z}$ -algebra R , which $A \in V_{D \neq 0}(\mathbb{Z})$ satisfy $\mathcal{R}(A) \simeq R$?

The third question is out of our reach yet and will be the subject of author's future study. We will attempt to provide a partial answer for the first and second questions, showing that there exists certain invariant elements in sextic resolvent algebra of R which can be used to characterize such A .

Theorem 4.4.1. *Let R be a sextic $\mathbb{Q}$ -algebra with specified basis element, and let S be its sextic resolvent $\mathbb{Q}$ -algebra. Suppose $A \in V_{D \neq 0}(\mathbb{Q})$ satisfies $\mathcal{R}(A) \simeq R$. There exists $s_1, s_2 \in S$ satisfying following two properties.*

(1) *Let $(u_1, \dots, u_6)$ and $(w_1, \dots, w_6)$ be values of the relative invariants of $\mathfrak{X}_{x,A}$ and $\mathfrak{X}_{y,A}$, projection of $\mathfrak{X}_A$ to $\mathbb{P}_x^1$ and $\mathbb{P}_y^1$. For $\psi_1, \dots, \psi_6 : S \rightarrow \mathbb{C}$ be the distinct embeddings, we have*

$$[\psi_1(s_1); \dots; \psi_6(s_1)] = [u_1; \dots; u_6], \quad [\psi_1(s_2); \dots; \psi_6(s_2)] = [v_1; \dots; v_6].$$

(2) *We have*

$$\mathrm{tr}_{S/\mathbb{Q}}(s_1) = \mathrm{tr}_{S/\mathbb{Q}}(s_1^3) = \mathrm{tr}_{S/\mathbb{Q}}(s_2) = \mathrm{tr}_{S/\mathbb{Q}}(s_2^3) = 0, \quad \mathrm{tr}_{S/\mathbb{Q}}(s_1^2 s_2) \mathrm{tr}_{S/\mathbb{Q}}(s_1 s_2^2) \neq 0.$$

The choice of s_1, s_2 is unique up to multiplication by $\mathbb{Q}^\times$. The mapping from A such that $\mathcal{R}(A) \simeq R$ to $\mathbb{Q}^\times$ -equivalence class of (s_1, s_2) is one-to-one, up to $G(\mathbb{Q})$ -equivalence in A .

We refer the definition and various properties of the sextic resolvent algebra to the Section 3.4.1.

4.4.1 Invariants for six points in $\mathbb{P}^1 \times \mathbb{P}^1$

Proposition 4.4.2. *Let $\alpha_1, \dots, \alpha_5$ be the element of $R = \mathcal{R}(A) \simeq H^0 R\Gamma(\mathcal{C}_{t,A})$ which are preimages of $1/(t_1 t_2 \dots t_5 t_i)$ under $R \rightarrow H^4(\mathcal{O}(-6))$, constructed in Section 4.3.1. Suppose R is nondegenerate, and let $\rho_1, \dots, \rho_6$ be distinct embeddings $R \otimes \mathbb{Q} \rightarrow \mathbb{C}$. Then coordinates of the six points of $\mathfrak{X}_{t,A}$ are given as*

$$p_i = [\rho_i(\alpha_1^*); \rho_i(\alpha_2^*); \dots; \rho_i(\alpha_5^*)].$$

Here $\alpha_0^*, \dots, \alpha_5^* \in R \otimes \mathbb{Q}$ is the dual basis of $1 = \alpha_0, \alpha_1, \dots, \alpha_5$.

This is basically application of the the Proposition 2.5.6 to $\mathcal{C}_{t,A}$.

Corollary 4.4.3. *Let $R = \mathcal{R}(A)$ be nondegenerate, and let $\rho_1, \dots, \rho_6$ be distinct embeddings $R \otimes \mathbb{Q} \rightarrow \mathbb{C}$. There exists elements $u_1, u_2, v_1, v_2, w_1, w_2 \in R \otimes \mathbb{Q}$ such that coordinates of $\mathfrak{X}_A$ are given as*

$$([\rho_i(u_1); \rho_i(u_2)], [\rho_i(v_1); \rho_i(v_2)], [\rho_i(w_1); \rho_i(w_2)] p_i) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^4.$$

where p_i are points described in the above Proposition.

Proof. As $p_t = [\rho_t(\alpha_1^*); \dots; \rho_t(\alpha_5^*)]$, the point of $\mathfrak{X}_A$ above p_t (under the projection $\mathfrak{X}_A \rightarrow \mathfrak{X}_{t,A}$) has its coordinates satisfying the equation

$$x_i y_j z_k = \sum_{l=1}^5 \rho_t(\alpha_l^*)(A_l)_{ijk}, \quad i, j, k = 1, 2.$$

Denote $\mu_{ijk} = \sum_l \alpha_l^*(A_l)_{ijk} \in R \otimes \mathbb{Q}$. We show that there exists element $u_1, u_2, v_1, v_2, w_1, w_2 \in R \otimes \mathbb{Q}$ such that $\mu_{ijk} = u_i v_j w_k$. As the coordinates of the point above p_t is uniquely determined by above question, our desired statement follows from noting that $\rho_t(\mu_{ijk}) = \rho_t(u_i) \rho_t(v_j) \rho_t(w_k)$.

For fixed t , we consider the above equation $x_i y_j z_k = \rho_t(\mu_{ijk})$ for coordinates of

the point above p_t to deduce $\rho_t(\mu_{111})\rho_t(\mu_{122}) = \rho_t(\mu_{112})\rho_t(\mu_{121})$. As this holds for all t , we have $\mu_{111}\mu_{122} = \mu_{112}\mu_{121}$ in R . Similarly, for all quadrics appearing in $\mathcal{C}_{t,A}$ - determinants of the face and differences of product of diagonals - their evaluation at μ_{ijk} vanishes like

$$\begin{aligned} \mu_{211}\mu_{222} - \mu_{212}\mu_{221} &= \mu_{111}\mu_{212} - \mu_{112}\mu_{211} = \mu_{121}\mu_{222} - \mu_{122}\mu_{221} \\ &= \mu_{111}\mu_{221} - \mu_{121}\mu_{211} = \mu_{112}\mu_{222} - \mu_{122}\mu_{212} = \mu_{111}\mu_{222} - \mu_{112}\mu_{211} \\ &= \mu_{111}\mu_{222} - \mu_{121}\mu_{212} = \mu_{111}\mu_{222} - \mu_{211}\mu_{122} = 0. \end{aligned}$$

If $R \otimes \mathbb{Q}$ is a field, then above equations are enough to show there exists $u_i, v_j, w_k \in R \otimes \mathbb{Q}$ satisfying $\mu_{ijk} = u_i v_j w_k$. (Start from representing rank 1 2×2 slices as $\mu_{ij1} = a_i b_j$, $\mu_{ij2} = c_i d_j$, then deduce $[a_1; a_2] = [c_1; c_2]$ and $[b_1; b_2] = [d_1; d_2]$ in $\mathbb{P}^1$ by considering other quadrics.) For the general case of nondegenerate R such that $R \otimes \mathbb{Q}$ becomes direct sum of fields, solve the equation $\mu_{ijk} = u_i v_j w_k$ over each field component of $R \otimes \mathbb{Q}$ and combine those results to obtain $u_i, v_j, w_k \in R \otimes \mathbb{Q}$. $\square$

This corollary shows that the coordinates of the six points of $\mathfrak{X}_A$ are determined by elements in R . (We will use abusive notation denoting $R \otimes \mathbb{Q}$ just as R , while still specifying its basis elements.) If we calculate the relative invariant of six points in $\mathbb{P}^1$ we gave in Section 2.1 as

$$p_H = \pm \frac{1}{6} \sum_{\sigma \in H} \text{sgn}(\sigma) x_{\sigma(1),1} x_{\sigma(2),1} x_{\sigma(3),1} x_{\sigma(4),2} x_{\sigma(5),2} x_{\sigma(6),2}$$

when $x_{i,j} = \rho_i(u_j)$, we can expect p_H can be related to the resolvent map from R to sextic resolvent algebra S corresponding to resolvent group H given as

$$\text{res}_{R,H}(r) = \pm \frac{1}{6\sqrt{\text{Disc}(R)}} \sum_{\sigma \in H} \text{sgn}(\sigma) \rho_{\sigma(1)}(r) \rho_{\sigma(2)}(r) \rho_{\sigma(3)}(r).$$

Here $\rho_1, \dots, \rho_6$ are considered as embeddings into the S_6 closure $R \rightarrow \overline{R}$, and $\sqrt{\text{Disc}(\overline{R})} \in \overline{R}$ is generator of A_6 -invariant algebra $\mathbb{Q}(\sqrt{\text{Disc}(\overline{R})}) \simeq \overline{R}^{A_6} \leq \overline{R}$ such that $S_6 \simeq \text{Gal}(\overline{R}/R)$ acts as $\sigma(\sqrt{\text{Disc}(\overline{R})}) = \text{sgn}(\sigma)\sqrt{\text{Disc}(\overline{R})}$. If R is field and $u_1, u_2 \neq 0$, then we have

$$\begin{aligned} p_H &= \pm \frac{1}{6} \sum_{\sigma \in H} \text{sgn}(\sigma) \rho_{\sigma(1)}(u_1) \rho_{\sigma(2)}(u_1) \rho_{\sigma(3)}(u_1) \rho_{\sigma(4)}(u_2) \rho_{\sigma(5)}(u_2) \rho_{\sigma(6)}(u_2) \\ &= \pm N_{R/\mathbb{Q}}(u_2) \frac{1}{6} \sum_{\sigma \in H} \text{sgn}(\sigma) \rho_{\sigma(1)}(u_1/u_2) \rho_{\sigma(2)}(u_1/u_2) \rho_{\sigma(3)}(u_1/u_2) \\ &= \sqrt{\text{Disc}(\overline{R})} N_{R/\mathbb{Q}}(u_2) \text{res}_{R,H}(u_1/u_2). \end{aligned}$$

Even if R is not field and u_1, u_2 might be zerodivisors, it still holds that $p_H/\sqrt{\text{Disc}(\overline{R})}$ is invariant under action of $\sigma \in H$. If we consider other sextic resolvent subgroups $H_1, H_2, \dots, H_6$ together, action of $\sigma \in S_6 \simeq \text{Gal}(\overline{R}/R)$ maps $p_H/\sqrt{\text{Disc}(\overline{R})}$ to $p_{\sigma H \sigma^{-1}}/\sqrt{\text{Disc}(\overline{R})}$. This basically shows the existence of s_1, s_2 and their properties.

Proposition 4.4.4. *Let R be nondegenerate. The relative invariants of six points $[\rho_i(u_1); \rho_i(u_2)]$ can be represented by an element s in sextic resolvent algebra S in the sense that*

$$[p_{H_1}; \dots; p_{H_6}] = [\psi_1(s); \dots; \psi_6(s)]$$

where $\psi_1, \dots, \psi_6 : S \rightarrow \mathbb{C}$ are distinct embeddings. Also s satisfies

$$\text{tr}_{S/\mathbb{Q}}(s) = \text{tr}_{S/\mathbb{Q}}(s^3) = 0.$$

Such s is unique up to multiplication by nonzero rational number.

Proof. We showed that $p_H/\sqrt{\text{Disc}(\overline{R})} \in \overline{R}$ is H -invariant and $p_{H_i}/\sqrt{\text{Disc}(\overline{R})}$ are all conjugate, which implies the existence of s . The second statement follows from $\sum p_H = \sum p_H^3 = 0$. The uniqueness of s up to constant follows from that $[\psi_1(s); \dots; \psi_6(s)]$ is fixed in $\mathbb{P}^5$.

Lemma. *If $[\psi_1(s); \dots; \psi_6(s)] = [\psi_1(s'); \dots; \psi_6(s')]$ and $s, s' \neq 0$, then s is $\mathbb{Q}$ -multiple of s' .*

Proof. As R is nondegenerate, $R \otimes \mathbb{Q} \simeq \oplus F_j$ is direct sum of fields. If $\psi_i(s') \neq 0$ for all i , then s' is invertible in all F_j , so s' itself is invertible. In this case, $\psi_i(s)/\psi_i(s') = \psi_i(s/s')$ does not change by i , so $s/s' \in \mathbb{Q}$.

Now suppose that there exists i such that $\psi_i(s') = 0$. The embeddings $\psi_1, \dots, \psi_6$ all has form of $R \otimes \mathbb{Q} \rightarrow F_j \rightarrow \mathbb{C}$, first map being projection to F_j and second map being certain embedding of F_j .

If $\psi_i(s') = 0$ for some i , then s' is zero in F_j where ψ_i is an embedding through F_j . Meanwhile $\psi_i(s') = 0$ implies $\psi_i(s) = 0$ (provided that there exists j such that $\psi_j(s') \neq 0$ -follows from $s' \neq 0$) so s is also zero in F_j . We can consider sum of F_j such that s, s' are not zero in F_j , and show the statement for such nondegenerate $\mathbb{Q}$ -algebra R' where $\psi_i(s') \neq 0$ for embeddings $R' \rightarrow \psi_i$. □

□

This shows the first part of Theorem 4.4.1 regarding existence of s_1, s_2 representing invariant of six points of $\mathfrak{X}_{x,A}$ and $\mathfrak{X}_{y,A}$, some equations on their traces at part (2) ($\text{tr}_{S/\mathbb{Q}}(s_1^2 s_2) \text{tr}_{S/\mathbb{Q}}(s_1 s_2^2) \neq 0$ follows from that $\mathfrak{X}_{(x,y)} \in P_{gen}$), and uniqueness of s_1 and s_2 up to multiplication by $\mathbb{Q}^\times$. It only remains to prove this construction of s_1, s_2 is injective.

Proposition 4.4.5. *Suppose that $A, B \in V_{D \neq 0}(\mathbb{Q})$ satisfies $\mathcal{R}(A) \simeq \mathcal{R}(B)$ as algebra with specified basis element, and $\mathfrak{X}_{(x,y),A}$ and $\mathfrak{X}_{(x,y),B}$ has same invariants. Then A and B are $G(\mathbb{Q})$ -equivalent.*

Proof. Let $\alpha_1, \dots, \alpha_5$ be the common specified basis elements for $R = \mathcal{R}(A) = \mathcal{R}(B)$. By Corollary 4.1, one can find $u, v, w, p, q, r \in R \otimes \mathbb{Q}$ such that

$$\sum \alpha_t^* A_t = (u_i v_j w_k)_{i,j,k=1,2}, \quad \sum \alpha_t^* B_t = (p_i q_j r_k)_{i,j,k=1,2}.$$

As invariants for $[\rho_i(u_1); \rho_i(u_2)]$ and $[\rho_i(p_1); \rho_i(p_2)]$ are same, those two sets of points are $\mathrm{GL}_2(\mathbb{Q})$ -invariant. If we apply $\mathrm{GL}_2(\mathbb{Q})$ -action on A acting on x -direction, we can assume that $[\rho_i(u_1); \rho_i(u_2)] = [\rho_i(p_1); \rho_i(p_2)]$ for all i . As applying GL_2 -action on x -direction alters basis of $\mathcal{R}(A)$ by multiplication by $(\det)^3$, any $\mathrm{GL}_2(\mathbb{Q})$ -action on z -direction can complement this effect, so we can both assume $\mathcal{R}(A) \simeq \mathcal{R}(B)$ and $[\rho_i(u_1); \rho_i(u_2)] = [\rho_i(p_1); \rho_i(p_2)]$, $[\rho_i(v_1); \rho_i(v_2)] = [\rho_i(q_1); \rho_i(q_2)]$ by applying appropriate $G(\mathbb{Q})$ -operation on A and B .

We can also let $\rho_i(u_2), \rho_i(p_2) \neq 0$ by applying common $\mathrm{GL}_2(\mathbb{Q})$ -operation on both A and B . Assuming all of that, we have $\rho_i(u_1/u_2) = \rho_i(p_1/p_2)$ for all i , so $u_1/u_2 = p_1/p_2$ in R . Similarly we have $v_1/v_2 = q_1/q_2$. By replacing (u_i, v_j, w_k) by $(p_i, q_j, (u_i v_j w_k)/(p_i q_j))$ we can let $u_i = p_i$ and $v_j = q_j$. At this point, $w_1, w_2 \in R$ are set to be linearly independent elements satisfying $\mathrm{tr}(u_i v_j w) = 0$, so both (w_1, w_2) and (r_1, r_2) form $\mathbb{Q}$ -basis of this 2-dimensional vector space over $\mathbb{Q}$. Thus there is a $\mathrm{GL}_2(\mathbb{Q})$ -operation transforming (w_1, w_2) into (r_1, r_2) , and applying this action on A acting on z -direction produces $A = B$. $\square$

4.5 Appendix: explicit formulas

4.5.1 Descriptions of invariants p_H of six points and sextic resolvent groups

Let φ be an outer automorphism $S_6 \rightarrow S_6$. *Sextic resolvent groups* of S_6 are defined as image of the six subgroups $S_{\{1,2,3,4,5\}}, S_{\{1,2,3,4,6\}}, \dots, S_{\{2,3,4,5,6\}}$ by φ , and they do not depend on choice of φ . We give a following choice of sextic resolvent subgroups,

which we will use in describing the invariants of six points.

$$\begin{aligned}
& (H_1, H_2, H_3, H_4, H_5, H_6) \\
& = (\langle (14)(25)(36), (26345) \rangle, \langle (16)(23)(45), (15436) \rangle, \langle (12)(34)(56), (16342) \rangle, \\
& \quad \langle (12)(34)(56), (24365) \rangle, \langle (12)(34)(56), (15634) \rangle, \langle (12)(34)(56), (14362) \rangle).
\end{aligned}$$

To give the explicit description for following invariants of six points

$$p_H = \sum_{\{a,b,c\} \cup \{d,e,f\} = \{1, \dots, 6\}} \pm (x_{a,1} x_{b,1} x_{c,1}) (x_{d,2} x_{e,2} x_{f,2}),$$

one only needs to fix the choice of sextic resolvent groups, and choose the appropriate signs $\pm$ for each terms. Below we give one choice of p_{H_i} as homogeneous degree 3 polynomials in $x_{a,1}/x_{a,2} = x_1$. The above form of degree 6 polynomial in $x_{a,j}$ is given by multiplying $x_{1,2}x_{2,2}x_{3,2}x_{4,2}x_{5,2}x_{6,2}$ to below expressions.

$$\begin{aligned}
p_{H_1} &= x_1x_2x_3 + x_1x_2x_4 - x_1x_3x_4 - x_2x_3x_4 - x_1x_2x_5 + x_1x_3x_5 - x_2x_3x_5 - x_1x_4x_5 \\
&+ x_2x_4x_5 + x_3x_4x_5 - x_1x_2x_6 - x_1x_3x_6 + x_2x_3x_6 + x_1x_4x_6 - x_2x_4x_6 + x_3x_4x_6 \\
&+ x_1x_5x_6 + x_2x_5x_6 - x_3x_5x_6 - x_4x_5x_6 \\
p_{H_2} &= -x_1x_2x_3 - x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + x_1x_2x_5 + x_1x_3x_5 - x_2x_3x_5 - x_1x_4x_5 \\
&+ x_2x_4x_5 - x_3x_4x_5 + x_1x_2x_6 - x_1x_3x_6 + x_2x_3x_6 + x_1x_4x_6 - x_2x_4x_6 - x_3x_4x_6 \\
&- x_1x_5x_6 - x_2x_5x_6 + x_3x_5x_6 + x_4x_5x_6 \\
p_{H_3} &= -x_1x_2x_3 + x_1x_2x_4 - x_1x_3x_4 + x_2x_3x_4 - x_1x_2x_5 + x_1x_3x_5 + x_2x_3x_5 + x_1x_4x_5 \\
&- x_2x_4x_5 - x_3x_4x_5 + x_1x_2x_6 + x_1x_3x_6 - x_2x_3x_6 - x_1x_4x_6 - x_2x_4x_6 + x_3x_4x_6 \\
&- x_1x_5x_6 + x_2x_5x_6 - x_3x_5x_6 + x_4x_5x_6
\end{aligned}$$

$$\begin{aligned}
p_{H_4} &= -x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 - x_2x_3x_4 + x_1x_2x_5 - x_1x_3x_5 + x_2x_3x_5 - x_1x_4x_5 \\
&\quad - x_2x_4x_5 + x_3x_4x_5 - x_1x_2x_6 + x_1x_3x_6 + x_2x_3x_6 - x_1x_4x_6 + x_2x_4x_6 - x_3x_4x_6 \\
&\quad + x_1x_5x_6 - x_2x_5x_6 - x_3x_5x_6 + x_4x_5x_6 \\
p_{H_5} &= x_1x_2x_3 - x_1x_2x_4 + x_1x_3x_4 - x_2x_3x_4 - x_1x_2x_5 - x_1x_3x_5 + x_2x_3x_5 + x_1x_4x_5 \\
&\quad + x_2x_4x_5 - x_3x_4x_5 + x_1x_2x_6 - x_1x_3x_6 - x_2x_3x_6 - x_1x_4x_6 + x_2x_4x_6 + x_3x_4x_6 \\
&\quad + x_1x_5x_6 - x_2x_5x_6 + x_3x_5x_6 - x_4x_5x_6 \\
p_{H_6} &= x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4 + x_2x_3x_4 + x_1x_2x_5 - x_1x_3x_5 - x_2x_3x_5 + x_1x_4x_5 \\
&\quad - x_2x_4x_5 + x_3x_4x_5 - x_1x_2x_6 + x_1x_3x_6 - x_2x_3x_6 + x_1x_4x_6 + x_2x_4x_6 - x_3x_4x_6 \\
&\quad - x_1x_5x_6 + x_2x_5x_6 + x_3x_5x_6 - x_4x_5x_6
\end{aligned}$$

4.5.2 Descriptions of the complex $\mathcal{C}_{t,A}$

We give full descriptions of all maps of $\mathcal{C}_{t,A}$. Denote α_{ijk} as (i, j, k) -entry of $A(t)$.

- Λ_2 term: basis elements are following 9 of them ($i = 1, 2, u = x, y, z$)

$$L_2^{f,u,i}, \quad L_2^{d,111} - L_2^{d,112}, \quad L_2^{d,111} - L_2^{d,121}, \quad L_2^{d,111} - L_2^{d,211}$$

Maps are described as

$$\begin{aligned}
L_2^{f,x,1} &\mapsto \alpha_{111}\alpha_{122} - \alpha_{112}\alpha_{121}, & L_2^{f,y,1} &\mapsto \alpha_{111}\alpha_{212} - \alpha_{112}\alpha_{211}, \\
L_2^{f,z,1} &\mapsto \alpha_{111}\alpha_{221} - \alpha_{121}\alpha_{211}, & L_2^{f,x,2} &\mapsto \alpha_{211}\alpha_{222} - \alpha_{212}\alpha_{221}, \\
L_2^{f,y,2} &\mapsto \alpha_{121}\alpha_{222} - \alpha_{122}\alpha_{221}, & L_2^{f,z,2} &\mapsto \alpha_{112}\alpha_{222} - \alpha_{122}\alpha_{212} \\
L_2^{d,111} &\mapsto \alpha_{111}\alpha_{222}, & L_2^{d,112} &\mapsto \alpha_{112}\alpha_{221}, & L_2^{d,121} &\mapsto \alpha_{121}\alpha_{212}, & L_2^{d,211} &\mapsto \alpha_{211}\alpha_{122}
\end{aligned}$$

We introduce following auxillary variables for convenience.

$$L_2^{d,222} = L_2^{d,111}, \quad L_2^{d,221} = L_2^{d,112}, \quad L_2^{d,212} = L_2^{d,121}, \quad L_2^{d,122} = L_2^{d,211}$$

- Λ_3 term: basis elements are following 16 of them ($i, j, k = 1, 2$)

$$L_3^{ijk,x} - L_3^{ijk,y}, \quad L_3^{ijk,x} - L_3^{ijk,z}$$

and their images are described as

$$\begin{aligned} L_3^{111,x} &\mapsto \alpha_{111}L_2^{d,211} - \alpha_{211}L_2^{f,x,1}, & L_3^{111,y} &\mapsto \alpha_{111}L_2^{d,121} - \alpha_{121}L_2^{f,y,1}, \\ L_3^{111,z} &\mapsto \alpha_{111}L_2^{d,112} - \alpha_{112}L_2^{f,z,1}, & L_3^{112,x} &\mapsto \alpha_{112}L_2^{d,212} - \alpha_{212}L_2^{f,x,1}, \\ L_3^{112,y} &\mapsto \alpha_{112}L_2^{d,122} - \alpha_{122}L_2^{f,y,1}, & L_3^{112,z} &\mapsto \alpha_{112}L_2^{d,111} - \alpha_{111}L_2^{f,z,2}, \\ L_3^{121,x} &\mapsto \alpha_{121}L_2^{d,221} - \alpha_{211}L_2^{f,x,1}, & L_3^{121,y} &\mapsto \alpha_{121}L_2^{d,111} - \alpha_{111}L_2^{f,y,2}, \\ L_3^{121,z} &\mapsto \alpha_{121}L_2^{d,122} - \alpha_{122}L_2^{f,z,1}, & L_3^{122,x} &\mapsto \alpha_{122}L_2^{d,222} - \alpha_{222}L_2^{f,x,1}, \\ L_3^{122,y} &\mapsto \alpha_{122}L_2^{d,112} - \alpha_{112}L_2^{f,y,2}, & L_3^{122,z} &\mapsto \alpha_{122}L_2^{d,121} - \alpha_{121}L_2^{f,z,2}, \\ L_3^{211,x} &\mapsto \alpha_{211}L_2^{d,111} - \alpha_{111}L_2^{f,x,2}, & L_3^{211,y} &\mapsto \alpha_{211}L_2^{d,221} - \alpha_{221}L_2^{f,y,1}, \\ L_3^{211,z} &\mapsto \alpha_{211}L_2^{d,212} - \alpha_{212}L_2^{f,z,1}, & L_3^{212,x} &\mapsto \alpha_{212}L_2^{d,112} - \alpha_{112}L_2^{f,x,2}, \\ L_3^{212,y} &\mapsto \alpha_{212}L_2^{d,222} - \alpha_{222}L_2^{f,y,1}, & L_3^{212,z} &\mapsto \alpha_{212}L_2^{d,211} - \alpha_{211}L_2^{f,z,2}, \\ L_3^{221,x} &\mapsto \alpha_{221}L_2^{d,121} - \alpha_{121}L_2^{f,x,2}, & L_3^{221,y} &\mapsto \alpha_{221}L_2^{d,211} - \alpha_{211}L_2^{f,y,2}, \\ L_3^{221,z} &\mapsto \alpha_{221}L_2^{d,222} - \alpha_{222}L_2^{f,z,1}, & L_3^{222,x} &\mapsto \alpha_{222}L_2^{d,122} - \alpha_{122}L_2^{f,x,2}, \\ L_3^{222,y} &\mapsto \alpha_{222}L_2^{d,212} - \alpha_{212}L_2^{f,y,2}, & L_3^{222,z} &\mapsto \alpha_{222}L_2^{d,221} - \alpha_{221}L_2^{f,z,2}. \end{aligned}$$

- Λ_4 term: basis elements are following 9 of them ($i = 1, 2, u = x, y, z$)

$$L_4^{f,u,i}, \quad L_4^{d,111}, \quad L_4^{d,122}, \quad L_4^{d,212}$$

and their images are described as

$$\begin{aligned}
L_4^{f,x,1} &\mapsto \alpha_{111}(L_3^{122,y} - L_3^{122,z}) + \alpha_{112}(L_3^{121,y} - L_3^{121,z}) \\
&\quad + \alpha_{121}(L_3^{112,y} - L_3^{112,z}) + \alpha_{122}(L_3^{111,y} - L_3^{111,z}), \\
L_4^{f,x,2} &\mapsto \alpha_{211}(L_3^{222,y} - L_3^{222,z}) + \alpha_{212}(L_3^{221,y} - L_3^{221,z}) \\
&\quad + \alpha_{221}(L_3^{212,y} - L_3^{212,z}) + \alpha_{222}(L_3^{211,y} - L_3^{211,z}), \\
L_4^{f,y,1} &\mapsto \alpha_{111}(L_3^{212,z} - L_3^{212,x}) + \alpha_{112}(L_3^{211,z} - L_3^{211,x}) \\
&\quad + \alpha_{211}(L_3^{112,z} - L_3^{112,x}) + \alpha_{212}(L_3^{111,z} - L_3^{111,x}), \\
L_4^{f,y,2} &\mapsto \alpha_{121}(L_3^{222,z} - L_3^{222,x}) + \alpha_{122}(L_3^{221,z} - L_3^{221,x}) \\
&\quad + \alpha_{221}(L_3^{122,z} - L_3^{122,x}) + \alpha_{222}(L_3^{121,z} - L_3^{121,x}), \\
L_4^{f,z,1} &\mapsto \alpha_{111}(L_3^{221,x} - L_3^{221,y}) + \alpha_{121}(L_3^{211,x} - L_3^{211,y}) \\
&\quad + \alpha_{211}(L_3^{121,x} - L_3^{121,y}) + \alpha_{221}(L_3^{111,x} - L_3^{111,y}), \\
L_4^{f,z,2} &\mapsto \alpha_{112}(L_3^{222,x} - L_3^{222,y}) + \alpha_{122}(L_3^{212,x} - L_3^{212,y}) \\
&\quad + \alpha_{212}(L_3^{122,x} - L_3^{122,y}) + \alpha_{222}(L_3^{112,x} - L_3^{112,y})
\end{aligned}$$

and

$$\begin{aligned}
L_4^{d,111} &\mapsto \alpha_{112}(L_3^{221,x} - L_3^{221,y}) + \alpha_{121}(L_3^{212,z} - L_3^{212,x}) + \alpha_{122}(L_3^{211,z} - L_3^{211,y}) \\
&\quad + \alpha_{211}(L_3^{122,y} - L_3^{122,z}) + \alpha_{212}(L_3^{121,x} - L_3^{121,z}) + \alpha_{221}(L_3^{112,y} - L_3^{112,x}) \\
L_4^{d,122} &\mapsto \alpha_{111}(L_3^{222,z} - L_3^{222,y}) + \alpha_{112}(L_3^{221,z} - L_3^{221,x}) + \alpha_{121}(L_3^{212,x} - L_3^{212,y}) \\
&\quad + \alpha_{212}(L_3^{121,y} - L_3^{121,x}) + \alpha_{221}(L_3^{112,x} - L_3^{112,z}) + 4\alpha_{222}(L_3^{111,y} - L_3^{111,z}) \\
L_4^{d,212} &\mapsto \alpha_{111}(L_3^{222,x} - L_3^{222,z}) + \alpha_{112}(L_3^{221,y} - L_3^{221,z}) + \alpha_{122}(L_3^{211,y} - L_3^{211,x}) \\
&\quad + \alpha_{211}(L_3^{122,x} - L_3^{122,y}) + \alpha_{221}(L_3^{112,z} - L_3^{112,y}) + \alpha_{222}(L_3^{111,z} - L_3^{111,x}) \\
L_4^{d,221} &\mapsto \alpha_{111}(L_3^{222,y} - L_3^{222,x}) + \alpha_{121}(L_3^{212,y} - L_3^{212,z}) + \alpha_{122}(L_3^{211,x} - L_3^{211,z}) \\
&\quad + \alpha_{211}(L_3^{122,z} - L_3^{122,x}) + \alpha_{212}(L_3^{121,z} - L_3^{121,y}) + \alpha_{222}(L_3^{111,x} - L_3^{111,y})
\end{aligned}$$

Here $L_4^{d,221}$ is considered as an auxiliary variable satisfying $L_4^{d,111} + L_4^{d,122} + L_4^{d,212} + L_4^{d,221} = 0$.

- Λ_6 term: there is a single basis element L_6 , whose image under the complex map is

$$\begin{aligned}
L_6 \mapsto & \alpha_{111}\alpha_{222}L_4^{d,111} + \alpha_{122}\alpha_{211}L_4^{d,122} + \alpha_{212}\alpha_{121}L_4^{d,212} + \alpha_{221}\alpha_{112}L_4^{d,221} \\
& + (\alpha_{211}\alpha_{222} - \alpha_{212}\alpha_{221})L_4^{f,x,1} - (\alpha_{111}\alpha_{122} - \alpha_{112}\alpha_{121})L_4^{f,x,2} \\
& + (\alpha_{121}\alpha_{222} - \alpha_{122}\alpha_{221})L_4^{f,y,1} - (\alpha_{111}\alpha_{212} - \alpha_{112}\alpha_{211})L_4^{f,y,2} \\
& + (\alpha_{112}\alpha_{222} - \alpha_{122}\alpha_{212})L_4^{f,z,1} - (\alpha_{111}\alpha_{221} - \alpha_{121}\alpha_{211})L_4^{f,z,2}.
\end{aligned}$$

Chapter 5

Sextic algebras associated to the Segre cubic

5.1 Introduction

In this chapter, we will construct a cubic threefold in $\mathbb{P}^4$ associated to a nondegenerate sextic ring R from this construction. The cubic threefold constructed is Segre cubic threefold, a set of cubic surfaces of $\mathbb{P}^4$ which are characterized by the condition that it should have 10 singularity points. Our main result takes the following form.

Theorem 5.1.1. *1. There is a one-to-one correspondence between primitive nondegenerate sextic rings over $\mathbb{Z}$ and $\mathrm{GL}_5(\mathbb{Z})$ -equivalence class of primitive Segre cubic forms with $\mathbb{Z}$ -coefficients.*

2. There is a one-to-one correspondence between nondegenerate sextic $\mathbb{Q}$ -algebras and $\mathrm{GL}_5(\mathbb{Q})$ -equivalence class of Segre cubic forms with $\mathbb{Q}$ -coefficients.

We note that a rank n ring R is *primitive* if there is no sextic subring $R' \subset R$ satisfying $R \simeq \mathbb{Z} + nR'$ with $n > 1$. Meanwhile, a primitive Segre cubic form can be defined as a Segre cubic which becomes primitive polynomial (gcd of all coefficients is 1).

Segre cubics has been studied extensively in classical algebraic geometry (see [21] and [20]) and many properties are known. It is known that all Segre cubics are isomorphic under $\mathrm{GL}_5(\mathbb{C})$ -action, and the automorphism group of single Segre cubic is isomorphic to S_6 and it sits inside $\mathrm{PGL}_5(\mathbb{C})$ of automorphisms group of ambient $\mathbb{P}^4$ space. This suggests that all Segre cubics with $\mathbb{Q}$ -coefficients are $\mathrm{GL}_5(\overline{\mathbb{Q}})$ -equivalent.

Thus for a particular Segre cubic F_0 , any Segre cubic F can be written as $F = gF_0$ for $g \in \mathrm{GL}_5(\overline{\mathbb{Q}})$ which is determined uniquely up to multiplication by nonzero constant and automorphism group S_6 of F_0 . The correspondence of the Theorem 5.1.1 relies on the construction of six points in $\mathbb{P}^4$ related to a Segre cubic, which determines the choice of g . Those six points - which we will name as *pivot points* - can be characterized by the property that the set consisting of those six points is invariant under automorphism group of the Segre cubic. While pivot points do not lie on the Segre cubic, those fit into the hyperplane configuration constructed from ten singularity points of Segre cubic.

Finding a sextic algebra corresponding to six pivot points uses the explicit construction of n geometric points in $\mathbb{P}^{n-2}$ from rank n nondegenerate ring. This construction which we call as *canonical points* was already described in [3], and the general idea of associating n points in $\mathbb{P}^{n-2}$ to rank n algebra was considered in [52] and [1], [2], [3] as one of the main ideas for parametrizing rank n rings and algebras for $n \leq 5$. In such models parametrizing rings of rank $n \leq 5$, nondegenerate rank n algebras correspond to orbits $V'(\mathbb{Q})/G(\mathbb{Q})$ for some prehomogeneous vector space (G, V) specified by the parametrization model. (Here V' denotes the single nondegenerate open orbit, which exists uniquely in any prehomogeneous vector space.) Our parametrization of sextic algebras can be considered as an extension of this idea, in the sense that it considers rational orbits of certain variety of Segre cubics which form a single orbit with respect to the $\mathrm{PGL}_5(\overline{\mathbb{Q}})$ -action. Unfortunately, the subvariety of Segre cubic forms is not well understood yet, and the explicit construction of the

multiplication table of the sextic ring is beyond our reach.

We give the outline of this brief chapter as follows. In Section 5.2 we will describe the property of the Segre cubic coming from classical algebraic geometry. In Section 5.3 we will construct a Segre cubic associated to rank 6 sextic ring or $\mathbb{Q}$ -algebra, and show this construction is consistent with the canonical point construction.

5.2 Preliminaries: Properties of Segre Cubic

In this section, we define the Segre cubic and briefly review its properties established in classical algebraic geometry of surfaces. Then we will define six pivot points, and show that those six points in $\mathbb{P}^4$ uniquely characterizes the Segre cubic. Throught this section and the next section (section 5.3), we will only consider equations over $\mathbb{C}$. For more detailed information on the Segre cubic, one can consult [21] and ([20], Section 9.4.4), both of them providing extensive coverage for the Segre cubic.

The **Segre cubic** (called Segre cubic primal in [21]) can be defined as cubic threefold on $\mathbb{P}^4$, which are GL_5 -isomorphic to the following equation

$$F_0 = x_1^3 + \cdots + x_5^3 - (x_1 + \cdots + x_5)^3.$$

The most used form for Segre cubic is given as following symmetric expression, which was found by Richmond ([41])

$$F_0 = \sum_{i=1}^6 x_i^3, \quad \sum_{i=1}^6 x_i = 0$$

Here F_0 is considered as a cubic surface sitting inside ambient space of hyperplane $\sum x_i = 0$ which is isomorphic to $\mathbb{P}^4$. This description allows us to observe the following properties of F_0 more easily.

Proposition 5.2.1. *(1) Any Segre cubic has 10 nodal singular points. On F_0 , they*

appear as all S_6 -orbits of the point $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, -1, -1, -1)$.

(2) Any irreducible cubic form in 5 variables having 10 nodal singular points should be Segre cubic.

(3) Automorphism group of Segre cubic surface $F = 0$ is isomorphic to S_6 , and all such automorphisms appear as element of projective coordinate change $\mathrm{PGL}_5(\mathbb{C})$. In particular, automorphism group of F_0 operates as permutations on $(x_1, \dots, x_6)$.

Proof. (1) Singularity points of F_0 can be calculated from explicit equation of F_0 , and singularities of general Segre cubic should be GL_5 -transformations of those points.

(2) See ([21], Proposition 2.1).

(3) As any two 10-nodal cubic hypersurfaces are $\mathrm{PGL}_5(\mathbb{C})$ -isomorphic, it follows that automorphism group of Segre cubic should sit inside $\mathrm{PGL}_5(\mathbb{C})$. We can explicitly compute all matrices of $\mathrm{GL}_5(\mathbb{C})$ which fix 10 singular points, and show only the permutations of $x_1, \dots, x_6$ give valid automorphism of Segre cubic. $\square$

A general Segre cubic F allows the expression

$$F = \sum_{i=1}^6 y_i^3, \quad \sum_{i=1}^6 y_i = 0, \quad (y_1, y_2, y_3, y_4, y_5) \text{ linearly independent}$$

where y_i are linear functionals: if φ is coordinate change transforming F_0 to F , then we can let $y_i = \varphi x_i$. Under φ , automorphism group of $F_0 = 0$ acting as permutation on $(x_1, \dots, x_6)$ corresponds to automorphism group of $F = 0$ acting as permutation on $(y_1, \dots, y_6)$. We can also see $(y_1, \dots, y_6)$ are determined uniquely up to permutations and constant multiplication. We can consider six projective points determined by

linear forms

$$p_1 : y_2 = y_3 = y_4 = y_5 = y_6$$

$$p_2 : y_1 = y_3 = y_4 = y_5 = y_6$$

...

$$p_6 : y_1 = y_2 = y_3 = y_4 = y_5.$$

Call those points $\{p_1, \dots, p_6\}$ *pivot points* of F . For F_0 , six pivot points are given as S_6 -orbits of $[x_1 : x_2 : x_3 : x_4 : x_5 : x_6] = [-5 : 1 : 1 : 1 : 1 : 1]$. As S_6 -orbits of a point other than pivot points always consist of more than 6 points, pivot points can be characterized as points whose orbit under $\text{Aut}(F)$ -action consists exactly 6 points. (I couldn't find literature discussing those points, probably because they are not on the Segre cubic itself.)

Proposition 5.2.2. *Pivot points give one-to-one correspondence between Segre cubics up to constant multiples and six points of $\mathbb{P}^4$ in linearly general position (no 5 of them on a hyperplane).*

Proof. For two sets of six points in general position, there is unique projective transform mapping one set to another. Thus for any $\{p_1, \dots, p_6\}$ in linearly general position, we can consider a projective transform φ mapping pivot points of F_0 to those six points. Then image of F_0 under φ becomes Segre cubic having $\{p_1, \dots, p_6\}$ as set of pivot points.

Suppose F_1, F_2 are two Segre cubics having same pivot points. As both F_1, F_2 are some GL_5 -transform of F_0 , there exists projective transform φ mapping F_1 to F_2 . Then φ preserves set of pivot points, so it should act as their permutation. But automorphisms of $F_1 = 0$ are exactly projective transforms which permutes pivot points, so φ should be automorphism of $F_1 = 0$. That means $F_1 = cF_2$ for some c . □

Remark. Pivot points can characterize singular points of a Segre cubic as follows. Intersection of two planes going through p_1, p_2, p_3 and p_4, p_5, p_6 becomes a singular point, and S_6 -permutations of those 10 points become full set of all singular points.

5.3 Sextic ring over $\mathbb{Z}$ and th associated Segre cubic

Let R be a nondegenerate sextic ring over $\mathbb{Z}$. Let $(1 = \alpha_0, \alpha_1, \dots, \alpha_5)$ be a $\mathbb{Z}$ -basis of R , and let $(\alpha_0^*, \dots, \alpha_5^*)$ be its dual basis in $R \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e. elements satisfying

$$\mathrm{Tr}_R(\alpha_i \alpha_j^*) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Let $\rho_1, \dots, \rho_6 : R \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \overline{\mathbb{Q}}$ be distinct ring embeddings. Define the following six projective points of $\mathbb{P}_{\overline{\mathbb{Q}}}^4$

$$p_i = [\rho_i(\alpha_1^*) : \rho_i(\alpha_2^*) : \dots : \rho_i(\alpha_5^*)]$$

as *canonical points* for base $(1, \alpha_1, \dots, \alpha_5)$. (This construction was considered in [3] first without the name.)

Canonical points do not change under shifting of α_i by integers, so they only depend on $\mathbb{Z}$ -basis of $R/\mathbb{Z}$ corresponding to $(\alpha_1, \dots, \alpha_5)$. Under the standard action of $G \in \mathrm{GL}_5(\mathbb{Z})$ on $(\alpha_1, \dots, \alpha_5)$, canonical points change by PGL_5 -action of $(G^{-1})^t$. We remind that the primitive ring is defined as ring R such that no sextic ring R' satisfies $R \simeq \mathbb{Z} + nR'$ with $n > 1$.

Proposition 5.3.1. (1) *Two nondegenerate rings with basis $(R_1, (\alpha_i))$ and $(R_2, (\beta_i))$ give same canonical points if and only if $R_1 \otimes \mathbb{Q} \simeq R_2 \otimes \mathbb{Q}$ and there exists $c \in \mathbb{Q}$*

such that $\beta_i = c\alpha_i$ in $R_1 \otimes \mathbb{Q}/\mathbb{Q}$.

(2) For any six distinct Galois conjugate points, there exists a sextic ring with $\mathbb{Z}$ -basis which has those six points as canonical points.

(3) Canonical points give one-to-one correspondence between primitive sextic rings R with basis of $R/\mathbb{Z}$ and sets of six distinct Galois conjugate points.

Those properties of canonical points hold for all rank n nondegenerate rings and can be proven in elementary way, but their proofs can be rather lengthy. We cover the proof of these properties in Chapter 7 (Section 7.2.5 in particular).

For a sextic ring R and its canonical points given, we want to look for a Segre cubic having those canonical points as pivot points. Let $y_1, y_2, \dots, y_6$ be six functionals satisfying following conditions:

- $y_2(p_1) = y_3(p_1) = \dots = y_6(p_1)$, $y_1(p_2) = y_3(p_2) = \dots = y_6(p_2)$, $\dots$, etc. In other words, for i, j, k all distinct, p_i is in plane $y_j - y_k$.
- $y_1 + y_2 + \dots + y_6 = 0$.

Then $y_1^3 + y_2^3 + \dots + y_6^3$ becomes the desired Segre cubic having p_i as pivot points.

Let

$$z_i = \rho_i(\alpha_1)x_1 + \rho_i(\alpha_2)x_2 + \dots + \rho_i(\alpha_5)x_5$$

and

$$y_i = 6z_i - \sum_{l=1}^6 z_l.$$

We claim that those y_i s satisfy above condition. As $\sum y_i = 0$ follows almost immediately, we only need to show $(y_j - y_k)(p_i) = 6(z_j - z_k)(p_i) = 0$. As trace Tr_R is given as

$$\text{Tr}_R(x) = \rho_1(x) + \dots + \rho_6(x),$$

the condition for dual basis is equivalent to

$$\sum_{k=1}^6 \rho_k(\alpha_i) \rho_k(\alpha_j^*) = \delta_{ij}$$

and we can write it as following matrix form

$$\begin{pmatrix} \rho_1(\alpha_0) & \rho_2(\alpha_0) & \cdots & \rho_6(\alpha_0) \\ \rho_1(\alpha_1) & \rho_2(\alpha_1) & \cdots & \rho_6(\alpha_1) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1(\alpha_5) & \rho_2(\alpha_5) & \cdots & \rho_6(\alpha_5) \end{pmatrix} \begin{pmatrix} \rho_1(\alpha_0^*) & \rho_1(\alpha_1^*) & \cdots & \rho_1(\alpha_5^*) \\ \rho_2(\alpha_0^*) & \rho_2(\alpha_1^*) & \cdots & \rho_2(\alpha_5^*) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_6(\alpha_0^*) & \rho_6(\alpha_1^*) & \cdots & \rho_6(\alpha_5^*) \end{pmatrix} = I.$$

We switch two matrices

$$\begin{pmatrix} \rho_1(\alpha_0^*) & \rho_1(\alpha_1^*) & \cdots & \rho_1(\alpha_5^*) \\ \rho_2(\alpha_0^*) & \rho_2(\alpha_1^*) & \cdots & \rho_2(\alpha_5^*) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_6(\alpha_0^*) & \rho_6(\alpha_1^*) & \cdots & \rho_6(\alpha_5^*) \end{pmatrix} \begin{pmatrix} \rho_1(\alpha_0) & \rho_2(\alpha_0) & \cdots & \rho_6(\alpha_0) \\ \rho_1(\alpha_1) & \rho_2(\alpha_1) & \cdots & \rho_6(\alpha_1) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1(\alpha_5) & \rho_2(\alpha_5) & \cdots & \rho_6(\alpha_5) \end{pmatrix} = I$$

to obtain

$$\sum_{k=0}^5 \rho_i(\alpha_k) \rho_j(\alpha_k^*) = \delta_{ij}.$$

Thus for $i \neq j$, we have

$$\begin{aligned} z_j(p_i) &= \rho_j(\alpha_1) \rho_i(\alpha_1^*) + \rho_j(\alpha_2) \rho_i(\alpha_2^*) + \cdots + \rho_j(\alpha_5) \rho_i(\alpha_5^*) \\ &= \left(\sum_{k=0}^5 \rho_j(\alpha_k) \rho_i(\alpha_k^*) \right) - \rho_j(\alpha_0) \rho_i(\alpha_0^*) \\ &= -\rho_i(\alpha_0^*) \end{aligned}$$

as $\alpha_0 = 1 \Rightarrow \rho_j(\alpha_0) = 1$. This implies that $(z_j - z_k)(p_i) = 0$ for i, j, k all distinct, so $y_1, \dots, y_6$ satisfies above conditions for $y_1^3 + y_2^3 + \cdots + y_6^3$ to be Segre cubic having p_i

as pivot points.

We want to express $y_1^3 + y_2^3 + \cdots + y_6^3$ more explicitly. If we let $x_1, \dots, x_5$ be rational numbers, then we can write

$$z_i = \rho_i(\alpha_1)x_1 + \rho_i(\alpha_2)x_2 + \cdots + \rho_i(\alpha_5)x_5 = \rho_i(\alpha_1x_1 + \cdots + \alpha_5x_5).$$

If we denote $\alpha(x) = \sum \alpha_i x_i$, this can be shortened as

$$z_i = \rho_i(\alpha(x)), \quad y_i = 6\rho_i(\alpha(x)) - \text{Tr}_R(\alpha(x)).$$

Thus

$$\begin{aligned} \sum_{i=1}^6 y_i^3 &= \sum_{i=1}^6 \rho_i (6\alpha(x) - \text{Tr}_R(\alpha(x)))^3 \\ &= \sum_{i=1}^6 \rho_i ((6\alpha(x) - \text{Tr}_R(\alpha(x)))^3) \\ &= \text{Tr}_R ((6\alpha(x) - \text{Tr}_R(\alpha(x)))^3) \end{aligned}$$

and this expression makes clear that this cubic form has integral coefficients. To summarize, we obtain:

Proposition 5.3.2. *For nondegenerate sextic ring R with $\mathbb{Z}$ -basis $(1, \alpha_1, \dots, \alpha_5)$, the integral cubic form*

$$F_{R,(\alpha_i)}(x_1, \dots, x_5) = \text{Tr}_R ((6\alpha(x) - \text{Tr}_R(\alpha(x)))^3) \quad (\alpha(x) = \sum \alpha_i x_i)$$

becomes a Segre cubic, and its six pivot points coincide with the canonical points for ring R and basis $(\alpha_1, \dots, \alpha_5)$ of $R/\mathbb{Z}$.

Conversely, for a Segre cubic F with integral coefficients, we can correspond a ring which has its six points as canonical points, as those six pivot points should be

Galois conjugate. We claim that those two correspondences give desired bijections.

Proposition 5.3.3. *There is one-to-one correspondence between*

$$\mathcal{C}_1 = \{\text{Nondegenerate sextic ring with basis } (R, (1, \alpha_1, \dots, \alpha_5))\} / \mathcal{R}$$

$$(\mathcal{R} = \{(R_1, (\alpha_i)) \sim (R_2, (\beta_i)) | R_1 \otimes \mathbb{Q} \simeq R_2 \otimes \mathbb{Q}, \alpha_i = c\beta_i \text{ in } R \otimes \mathbb{Q}/\mathbb{Q}\})$$

and

$$\mathcal{C}_2 = \{\text{Segre cubic with integral coefficients}\} / \{\text{constant multiple}\}.$$

The correspondence of $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ is given as

$$\Phi_{12} : R \mapsto F_{R,(\alpha_i)}(x) = \text{Tr}_R(6\alpha x - \text{Tr}_R(\alpha x))^3, \quad \alpha x = \sum_{i=1}^5 \alpha_i x_i$$

and correspondence of $\mathcal{C}_2 \rightarrow \mathcal{C}_1$ is

$$\Phi_{21} : F \mapsto (\text{ring whose canonical points are same as pivot points of } F).$$

Proof. We first check Φ_{12} and Φ_{21} are well-defined on equivalence class. For Φ_{12} , we have $F_{R,(c\alpha_i)}(x) = c^3 F_{R,(\alpha_i)}$ and $F_{R,(\alpha_i)}$ does not change under shifting $\alpha_i \mapsto \alpha_i + c$ ($c \in \mathbb{Q}$), so $F_{R,(\alpha_i)}$ is determined uniquely up to constant for each equivalence class of $\mathcal{C}_1$. For Φ_{21} , it is enough to note that two Segre cubics differing by constant multiple have same pivot points and equivalence relation $\mathcal{R}$ is essentially equivalent to that two rings with basis have same canonical points.

We have established that pivot points of $F_{R,(\alpha_i)}$ is identical to canonical points of $(R, (\alpha_i))$, so $\Phi_{21} \circ \Phi_{12}$ is identity. Meanwhile, Φ_{21} is injective, as determining pivot points of the Segre cubic determines cubic uniquely up to constant multiple. This establishes $\Phi_{21} \circ \Phi_{12} \circ \Phi_{21} = \Phi_{21} \Rightarrow \Phi_{12} \circ \Phi_{21} = \text{id}$. So Φ_{12}, Φ_{21} are bijective inverses to each other. □

We can consider $F_{R,(\alpha_i)}$ for any linearly independent elements $\alpha_i \in R \otimes \mathbb{Q}/\mathbb{Q}$, not only $\mathbb{Z}$ -basis. In this perspective, we can discuss action of $G \in \mathrm{GL}_5(\mathbb{Q})$ on (α_i) . For $G \in \mathrm{GL}_5(\mathbb{Q})$ and $(\beta_1, \dots, \beta_5)^t = G(\alpha_1, \dots, \alpha_5)^t$, we have

$$\beta x = \sum_{i=1}^5 \left(\sum_{j=1}^5 g_{ij} \alpha_j \right) x_i = \sum_{j=1}^5 \alpha_j \left(\sum_{i=1}^5 x_i g_{ij} \right) = \alpha y$$

for $(y_1, \dots, y_5) = (x_1, \dots, x_5)G^t$, so we have

$$F_{R,(\beta_i)}(x_1, \dots, x_5) = F_{R,(\alpha_i)}(y_1, \dots, y_5).$$

Thus GL_5 -action on basis $(\alpha_1, \dots, \alpha_5)$ of $R \otimes \mathbb{Q}/\mathbb{Q}$ corresponds to coordinate change in $F_{R,(\alpha_i)}$. By considering $\mathrm{GL}_5(\mathbb{Z})$ and $\mathrm{GL}_5(\mathbb{Q})$ -equivalence respectively in above proposition, we can obtain the following.

- Corollary 5.3.4.** 1. *There is one-to-one correspondence between primitive nondegenerate sextic rings over $\mathbb{Z}$ and $\mathrm{GL}_5(\mathbb{Z})$ -equivalence class of integral primitive Segre cubic forms.*
2. *There is one-to-one correspondence between nondegenerate sextic extensions over $\mathbb{Q}$ and $\mathrm{GL}_5(\mathbb{Q})$ -equivalence class of rational Segre cubic forms.*

Chapter 6

Geometric interpretations for the quintic ring parametrization and sextic resolvent ring

6.1 Introduction

In this chapter, we seek to provide new a geometric interpretation of the Bhargava's parametrization of quintic (rank 5) rings in [3], particularly regarding construction of the sextic resolvent ring of the quintic ring. In the Bhargava's parametrization of quintic ring, we consider a form $A \in V(\mathbb{Z}) = \mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5$ with $G(\mathbb{Z}) = \mathrm{GL}_4 \times \mathrm{SL}_5(\mathbb{Z})$ acting, and is given as follows.

Theorem 6.1.1. (*[3], Theorem 8*) *There is a natural bijection between between the set of nondegenerate $\mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$ -equivalence classes of elements $A \in \mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5$ and the set of isomorphism classes of pairs (R, S) , where R is a nondegenerate quintic ring and S is a sextic resolvent ring of R . Under this bijection we have $\mathrm{Disc}(A) = \mathrm{Disc}(R) = \frac{1}{16} \mathrm{Disc}(S)^{1/3}$.*

The definition of a *sextic resolvent ring* of a nondegenerate quintic ring is described

in ([3], Section 5).

The construction of $(\mathcal{R}(A), \mathcal{S}(A)) = (R, S)$ from a form A can be given as explicit formula for the multiplication coefficients. The multiplication coefficients $c_{ij}^k(A)$ of $\mathcal{R}(A)$ and $d_{ij}^k(A)$ of $\mathcal{S}(A)$ can be described as degree 5 and 12 integral polynomials in entries of A respectively, and they satisfy certain covariance properties with respect to the action of G . Under $G(\mathbb{Z})$ -action, isomorphism classes of $\mathcal{R}(A)$ and $\mathcal{S}(A)$ are kept same. (Note: multiplication coefficients correspond to the choice of $\mathbb{Z}$ -basis $1, \alpha_1, \dots, \alpha_{n-1}$ of the ring, but basis elements α_i can be changed by shifting by integer constants $\alpha_i \mapsto \alpha_i + C$ for $C \in \mathbb{Z}$. So they can vary by this change, and we are considering following invariants under these actions

$$c_{ij}^k, \quad c_{ii}^j, \quad c_{ij}^j - c_{ik}^k, \quad c_{ii}^i - c_{ij}^j - c_{ik}^k$$

for distinct $i, j, k = 1, \dots, n-1$.)

For the quintic ring $\mathcal{R}(A) = R$ parametrized, one can consider five points in $\mathbb{P}^3$ determined by certain equations in A - Pfaffians of the 5 principal minors of 5×5 alternating matrix $x_1 A_1 + \dots + x_4 A_4$, where $A_i \in \wedge^2 \mathbb{Z}^5$ is i th ‘slice’ of A in direction of SL_4 and $x_1, \dots, x_4$ are coordinates of $\mathbb{P}^3$. Construction of those five points was already described in [52] and [3], and construction of $\mathcal{R}(A)$ was inspired by those Pfaffian equations. We will use the hypercohomology construction from the Chapter 2 to give a geometric construction of the $\mathcal{R}(A)$. To put more explicitly, we will construct a complex $\mathcal{C}_{x,A}$ of sheaves on $\mathbb{P}^3$ with DG-algebra structure such that it becomes minimal free resolution of 5 points over universal variables $\mathbb{Q}[A]$, and show its hypercohomology ring $H^0 R\Gamma(\mathcal{C}_{x,A})$ is isomorphic to $\mathcal{R}(A)$.

For the sextic resolvent ring $\mathcal{S}(A) = S$, it is natural to ask: can we construct 6 points in $\mathbb{P}^4$ associated to $\mathcal{S}(A)$? We will attempt to approach this question by exploring the geometry of $4 \times \wedge^2 5$ form $(\mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5)$ in more detail, and associating a

Segre cubic threefold we discussed in the Chapter 5. The 5 Pfaffians of $x_1A_1 + \cdots + x_4A_4$ can be used to define a quadratic map $\mathbb{P}^3 \rightarrow \mathbb{P}^4$, and it is classically known that the image of this map becomes isomorphic to a Segre cubic for generic A . ([21], [20]) By considering the transformation of GL_5 -action on A , one can deduce that those six points should be the ones corresponding to $\mathcal{S}(A)$. While we could not succeed in obtaining the ring $\mathcal{S}(A)$ itself via hypercohomology yet, we could instead construct a complex $\mathcal{C}_{y,A}$ and a sextic ring $H^0R\Gamma(\mathcal{C}_{y,A})$ out of it which is isomorphic to $\mathcal{S}(A)$ over $\mathbb{Q}$.

Our application of the hypercohomology construction to the quintic ring parametrization can be seen as generalization of Wood's works in [49], where the quartic ring parametrization of [2] is generalized as a parametrization model for rank 4 S -algebras for arbitrary scheme base S . It was conjectured that quintic ring parametrization should have similar generalizations into arbitrary base. Results in this chapter are yet insufficient to obtain such results, as it would require that the sextic resolvent ring $\mathcal{S}(A)$ to be given a natural definition (which works for arbitrary base) with explicit construction. Our construction of $H^0R\Gamma(\mathcal{C}_{y,A})$ at least would allow constructing $\mathcal{S}(A)$ over the domain where $60\mathrm{Disc}(A)$ is invertible. Our attempt to generalize quintic ring parametrization into arbitrary base can be also considered as a generalization of [12], a work which considered degree 5 covers of varieties over a field of characteristic zero.

This chapter is constructed as follows. In Section 6.2 we will briefly review the construction of 5 points in [52]. In Section 6.3, we will introduce the complex $\mathcal{C}_{x,A}$ immediately with its DG-algebra structure, and calculate that its hypercohomology coincides with $\mathcal{R}(A)$. In Section 6.4 we will start discussing additional properties of A and explore the geometric meaning of the sextic resolvent ring $\mathcal{S}(A)$. When interpreting this sextic ring, we will bring classical perspective of the Segre cubic threefold, and suggest geometric constructions of certain Segre cubic and associated six points in $\mathbb{P}^4$ related to $4 \otimes \wedge^2 5$ form. In Section 6.5, we will construct a complex

$\mathcal{C}_{y,A}$ and show its hypercohomology ring

6.2 Review of the quintic ring parametrization

In this section we summarize geometric results on $4 \otimes \wedge^2 5$ forms and associated 5 points in $\mathbb{P}^3$ established in [52]. Before starting, we would like to point out that the representation $k^4 \otimes \wedge^2 k^5$ of group $G = \mathrm{SL}_4 \times \mathrm{GL}_5$ is prehomogeneous vector space ([42]), i.e. a representation having one Zariski open orbit. Its ring of invariant is generated by a single G -invariant of degree 40 in entries of A , which we call the *discriminant* $\mathrm{Disc}(A)$. The *semisimple elements* are exactly those with $\mathrm{Disc}(A) \neq 0$, and all semisimple elements are in single $G(\bar{k})$ -orbit. The explicit calculation of the discriminant can be found in [53], or one can compute the discriminant using the ring $\mathcal{R}(A)$ we will discuss soon. For the rest of this section, we consider all forms over an infinite field k of characteristic $\neq 2, 3, 5$.

6.2.1 Five points in $\mathbb{P}^3$

Let $A \in k^4 \otimes \wedge^2 k^5$, on which $G = \mathrm{SL}_4 \times \mathrm{GL}_5$ acts standardly. For $x = (x_1, x_2, x_3, x_4) \in (k^4)^*$, on which GL_4 acts as dual representation, denote $A(x) = A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 x_4 \in \wedge^2 k^5$. Then we consider $A(x) \wedge A(x) \simeq \wedge^4 k^5$. When viewed as vector in $(k^5)^* \simeq \wedge^4 k^5$, its entries are 5 quadrics which are exactly the Pfaffians of five 4×4 principal minors (minors obtained by deleting i th row and i th column for $i = 1, \dots, 5$) of alternating matrix $A(x)$. Denote $I_5(A)$ as the ideal formed by those 5 entries of $A(x) \wedge A(x)$.

As $A(x) \wedge A(x)$ vanishes if and only if $A(x)$ has rank 0 or 2, such $A(x)$ can be regarded as intersection of Grassmannian $\mathrm{Gr}(2, 5) \subset \mathbb{P}(\wedge^2 k^5)$ (more exactly, image of Grassmannian under Plücker embedding) and codimension 4 hyperplane going through A_1, A_2, A_3, A_4 . This embedding of $\mathrm{Gr}(2, 5)$ has degree 5, so we can expect to

have 5 distinct solutions for generic A . In [52], it is established that all semisimple A satisfies this condition, and those five points gives the desired correspondence of $A \in k^4 \otimes \wedge^2 k^5$ and quintic extension of k .

Proposition 6.2.1. *(1) An element $A \in k^4 \otimes \wedge^2 k^5$ is semisimple if and only if $I_5(A)$ cuts out five distinct geometric points. In that case, those five points are in general position (no 4 of them in same codimension 1 hyperplane).*

(2) Consider a correspondence mapping A to the global function ring for five points of $I_5(A)$. This gives one-to-one correspondence between equivalence classes of semisimple elements up to $\mathrm{SL}_4 \times \mathrm{GL}_5(k)$ -action and degree 5 separable k -algebras up to isomorphism.

For the rest of this section, let A be semisimple.

6.2.2 Effects of the permutations of 5 points

For n points $p_1, \dots, p_n$ in $\mathbb{P}^{n-2}$ in general position, we can consider n affine vectors $v_1, \dots, v_n \in k^{n-1}$ such that projective image of v_i (under $\mathbb{A}^{n-1} \rightarrow \mathbb{P}^{n-2}$) coincides with p_i and $\sum v_i = 0$, and such choice of $(v_1, \dots, v_n)$ is unique up to constant multiplication. We consider those such affine vectors for five vanishing points of $I_5(A)$, and denote the corresponding $A(x)$ as $B_1, \dots, B_5$. Then B_i s become matrices which are all in linear span of $A_1, \dots, A_4$, and satisfy

$$\mathrm{rk}(B_i) \leq 2, \quad \sum_{i=1}^5 B_i = 0.$$

The set $\{B_i\}$ is uniquely determined up to multiplication by same constant, and invariant under GL_4 -action. Moreover, we can show that there exists vector $f_1, \dots, f_5 \in \overline{k}^5$ such that

$$B_i = f_i \wedge (f_{i-1} + f_{i+1}) \quad (f_0 = f_5, f_6 = f_1).$$

It can be verified in the following way. As any semisimple element A is in single $G(\bar{k})$ -orbit, we can let $A = gE$ for $g = (g_1, g_2) \in \mathrm{SL}_4 \times \mathrm{GL}_5(\bar{k})$ and a particular element $E = (E_1, E_2, E_3, E_4)$ defined as

$$E_i = e_i \wedge (e_{i-1} + e_{i+1}), \quad i = 1, 2, 3, 4$$

where e_i s are standard basis vectors

$$\begin{aligned} e_1 &= (1, 0, 0, 0, 0), e_2 = (0, 1, 0, 0, 0), e_3 = (0, 0, 1, 0, 0), e_4 = (0, 0, 0, 1, 0), \\ e_0 &= e_5 = (0, 0, 0, 0, 1). \end{aligned}$$

Let $f_i = g_2 e_i$, then we have $A = g_1 B$ for $B = (B_1, B_2, B_3, B_4)$ defined as $B_i = f_i \wedge (f_{i-1} + f_{i+1})$. We will keep this notation B_i and f_i for semisimple form $A \in k^4 \otimes \wedge^2 k^5$ over k .

We consider permutations of 5 zero points of $I_5(A)$. For any two sets of 5 points in $\mathbb{P}^3$ each in general position, there exists unique PGL_4 -automorphism mapping one set to the other one. Thus we can think $S_5 \rightarrow \mathrm{PGL}_4(\bar{k})$, which correspond to permutations of B_i .

6.2.3 Multiplication table for the quintic ring

We briefly describe the formulas for multiplication coefficients of the Bhargava's quintic ring in ([3], Chapter 4). For $i, j, k, l, m \in \{1, 2, 3, 4\}$, denote

$$\{ijklm\} = (A_i \wedge A_j)^* \cdot A_k \wedge (A_l \wedge A_m)^*$$

Here $*$ denotes the Hodge star operator $\wedge^4 k^5 \rightarrow k^5$, and $(X \wedge Y)^*$ becomes the symmetric bilinear form corresponding to the Pfaffian form we discussed above. Then the multiplication coefficients c_{ij}^k of the quintic ring is given by the following formulas

(formula (21) of [3])

$$\begin{aligned}
c_{ij}^k &= \pm \{iiljj\}/4, \\
c_{ii}^j &= \pm \{liiik\}, \\
c_{ij}^j - c_{ik}^k &= \pm \{jklji\}/2, \\
c_{ii}^i - c_{ij}^j - c_{ik}^k &= \pm \{ijlki\}.
\end{aligned}$$

Here $\{i, j, k, l\}$ are chosen as distinct elements of $\{1, 2, 3, 4\}$, and the $\pm$ sign denotes the signature of the permutation (i, j, k, l) of $(1, 2, 3, 4)$. Any integers satisfying above formula (with c_{ij}^0 defined appropriately) determine a quintic ring, and the quintic ring is unique up to translation of basis elements by integer constants. Call this ring $\mathcal{R}(A)$.

We point out that multiplication coefficients (more exactly, invariants of the multiplication coefficients with respect to translation of basis element by integer, all listed in above formulas) are degree 5 polynomials in A which are SL_5 -invariant. Under GL_4 -action, they form an representation isomorphic to the irreducible representation of highest weight $(3, 1, 1, 0)$. This property is consistent with that GL_4 -action acts on $\mathbb{Z}$ -basis of $\mathcal{R}(A)/\mathbb{Z}$ as determinant times standard action (of GL_4 on $\mathbb{Z}^4$).

The formulas for multiplication coefficients of the sextic resolvent ring $\mathcal{S}(A)$ are given in ([3], Chapter 8), and is more complicated to describe. We will only note that multiplication coefficients $d_{ij}^k(A)$ (more exactly, invariants of the coefficient with respect to translation) are of degree 12 polynomials in A which are SL_4 -invariant, and under GL_5 -action they form an representation isomorphic to the irreducible representation of highest weight $(6, 5, 5, 5, 3)$. The GL_5 -action on $\mathbb{Z}$ -basis of $\mathcal{S}(A)/\mathbb{Z}$ is given as $(\det)^5$ times dual action (of GL_5 on $\mathbb{Z}^5$), and discriminant of $\mathcal{S}(A)$ is given as $\mathrm{Disc}(\mathcal{S}(A)) = 2^{12} \cdot \mathrm{Disc}(\mathcal{R}(A))^3$.

6.3 Hypercohomology construction for the quintic ring

In this section we would like to construct a complex $\mathcal{C}_{x,A}$ of sheaves on $\mathbb{P}^3$ with a DG-algebra structure, which becomes the minimal free resolution for ideal of five points $I_5(A)$ over field. Then we will see its hypercohomology ring $H^0 R\Gamma(\mathcal{C}_{x,A})$ with ring structure given by our DG-algebra structure is isomorphic to the ring $\mathcal{R}(A)$ constructed in [3]. While the complex $\mathcal{C}_{x,A}$ itself can be obtained as a special case of the resolution constructed in [12] and [11], constructing an explicit DG-algebra structure on it (while [46] suggests that such DG-algebra structure should exist) and calculating its hypercohomology ring is original work of the author.

Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^3}$ be structure sheaf for $\mathbb{P}^3 \simeq \text{Proj } k[x_1, x_2, x_3, x_4]$. Our complex $\mathcal{C}_{x,A}$ is of following form

$$\mathcal{C}_{x,A} : \mathcal{O}(-5) \otimes \Lambda_5 \rightarrow \mathcal{O}(-3) \otimes \Lambda_3 \rightarrow \mathcal{O}(-2) \otimes \Lambda_2 \rightarrow \mathcal{O}$$

where $\Lambda_5, \Lambda_3, \Lambda_2$ are free $\mathbb{Z}$ -modules of rank 1, 5, 5 respectively. Denote ther $\mathbb{Z}$ -basis as

$$\{L_5\}, \quad \{L_3^i\}_{i=1, \dots, 5}, \quad \{L_2^i\}_{i=1, \dots, 5},$$

and let (used ^{t} to denote column vector)

$$L_2 = (L_2^1, L_2^2, L_2^3, L_2^4, L_2^5)^t, \quad L_3 = (L_3^1, L_3^2, L_3^3, L_3^4, L_3^5)^t$$

for following descriptions for the map of $\mathcal{C}_{x,A}$.

$$L_2 \mapsto \text{pf}(A(x)),$$

$$L_3 \mapsto (A(x))L_2,$$

$$L_5 \mapsto \text{pf}(A(x)) \cdot L_3$$

Here pf is defined as length 5 vector of Pfaffians of principal minors, whose i -th entry is $(-1)^{i-1}$ times of Pfaffian of principal minor with i th row and column eliminated. Or equivalently we can write

$$\text{pf}(B) = B \wedge B \simeq \wedge^4 k^5 \simeq (k^5)^*.$$

Also $A(x)L_2$ means multiplication of the alternating matrix $A(x)$ and length 5 column vector L_2 , and $\text{pf}(A(x)) \cdot L_3$ means dot product when considering both as length 5 vectors. It is possible to describe a DG-algebra structure on $\mathcal{C}_{x,A}$ such that the map $\mathcal{O} \rightarrow \mathcal{C}_{x,A}$ (simply given as mapping $\mathcal{O}$ into the rightmost entry $\mathcal{O}$ of $\mathcal{C}_{x,A}$ via identity) becomes unit of the algebra. The multiplication rules for this DG-algebra are given as

$$L_2^i \cdot L_3^j = \delta_{ij} L_5 = \begin{cases} L_5 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for any i and j and

$$L_2^i \cdot L_2^j = \sigma_{ijpqr} (A(x)_{qr} L_3^p + A(x)_{rp} L_3^q + A(x)_{pq} L_3^r).$$

for $i \neq j$ (for $i = j$ one should have $L_2^i \cdot L_2^i = 0$). Here p, q, r are indices satisfying $\{i, j, p, q, r\} = \{1, 2, 3, 4, 5\}$, and σ_{ijpqr} is signature of the permutation (i, j, p, q, r) (signature of $(1, 2, 3, 4, 5)$ is 1). Those rules can be also expressed in the following

abusive notations

$$L_2 \otimes L_3 = L_5 I_5, \quad L_2 \wedge L_2 = (A(x) \wedge L_3)^\star$$

where $L_2 \otimes L_3 = (L_2^i \cdot L_3^j)_{i,j}$ is 5×5 matrix, $L_2 \wedge L_2 = (L_2^i \cdot L_3^j)_{i,j}$ is 5×5 alternating matrix, and the Hodge star $\star$ operates as $\wedge^3 k^5 \rightarrow \wedge^2 k^5$. This notation allows us to check consistency of this DG-algebra more easily.

Proposition 6.3.1. (1) *The complex $\mathcal{C}_{x,A}$ provides a minimal free resolution of I_A over formal variables. In other words, $\mathcal{C}_{x,A}$ is exact over $k(A)[x]$ (except at rightmost place).*

(2) *If we pick A as an explicit element which is semisimple (nonzero discriminant) element, then $\mathcal{C}_{x,A}$ becomes a minimal free resolution for I_A .*

Proof. First we note the following property for the Pfaffian vector pf.

Lemma. *For B a 5×5 alternating matrix of rank 4, a vector $\text{pf}(B)$ spans both left kernel (set of vectors $w^t B = 0$) and right kernel (set of vectors $Bv = 0$) for B .*

(1) First we consider $\mathcal{Q}_A$ over fraction field $k(A)(x)$. As $A(x)$ is of rank 4, the above lemma suggests that the map of L_2 and L_5 each corresponds to right and left kernels of $A(x)$ respectively. Thus $\mathcal{Q}_A$ is exact over $k(A)(x)$.

To show that $\mathcal{Q}_A$ is exact over $k(A)[x]$, one needs to show that (need to fill in, shouldn't be that hard).

(2) We pick particular choice of A , for example, the

$$E_i = e_i \wedge (e_{i-1} + e_{i+1}), \quad i = 1, 2, 3, 4$$

where e_i s are standard basis vectors

$$\begin{aligned} e_1 &= (1, 0, 0, 0, 0), e_2 = (0, 1, 0, 0, 0), e_3 = (0, 0, 1, 0, 0), e_4 = (0, 0, 0, 1, 0), \\ e_0 &= e_5 = (0, 0, 0, 0, 1). \end{aligned}$$

and work out with numerical calculation to show the claim. We have shown that all semisimple elements are in same $\mathrm{SL}_4 \times \mathrm{GL}_5(\bar{k})$ -orbit, so the general result follows from covariance this particular case of $\mathcal{Q}_A$ under $\mathrm{SL}_4 \times \mathrm{GL}_5$ -transformation. $\square$

We observe that $\mathcal{C}_{x,A}$ has shape of minimal free resolution for five points in $\mathbb{P}^3$

$$\mathcal{O}(-5) \rightarrow \mathcal{O}(-3)^5 \rightarrow \mathcal{O}(-2)^5 \rightarrow \mathcal{O}.$$

Thus we can calculate the hypercohomology ring using the algorithm given in the Chapter 2 (Theorem 2.4.2), and check that multiplication coefficients coincide with $\mathcal{R}(A)$.

Proposition 6.3.2. *The hypercohomology ring $H^0 R\Gamma(\mathcal{C}_{x,A})$ is isomorphic to $\mathcal{R}(A)$.*

Corollary 6.3.3. *Let $1 = \alpha_0, \alpha_1, \dots, \alpha_5$ be basis of nondegenerate $\mathcal{R}(A)$. The five points determined the Pfaffian equation $A(x) \wedge A(x) = 0$ are canonical points for $\alpha_1, \dots, \alpha_5$: for embeddings $\rho_i \mathcal{R}(A) \rightarrow \mathbb{C}$ and dual basis elements $\alpha_i^* \in \mathcal{R}(A) \otimes \mathbb{Q}$ for $(\alpha_i)_i$ (elements satisfying $\mathrm{tr}(\alpha_i \alpha_j^*) = \delta_{ij}$), those points are given as*

$$p_i = [\rho_i(\alpha_1^*); \rho_i(\alpha_2^*); \rho_i(\alpha_3^*); \rho_i(\alpha_4^*); \rho_i(\alpha_5^*)].$$

Over field k , the closed subscheme $V(I_A)$ has global function ring isomorphic to $\mathcal{R}(A) \otimes \mathbb{Q}$.

Proof. Apply the Theorem 2.5.6 and the Theorem 2.5.1 to $\mathcal{C}_{x,A}$. $\square$

6.4 Geometric interpretation for sextic resolvent

Now going back to geometric discussion, we would like to consider semisimple forms in $V(k) = k^4 \otimes \wedge^2 k^5$ and explore their geometric properties on the space $\mathbb{P}^4$. Let k be an infinite field of characteristic $\neq 2, 3, 5$, and let A be a semisimple element.

6.4.1 Geometry for $\mathbb{P}^4$ space

We would like to consider equations of $I_5(A)$ together with k^5 space too. The condition $\text{rk}(A(x)) < 4$ is equivalent to that $\dim \ker(A(x)) > 1$, or linear equation $(A(x))y = 0$ (matrix multiplication) in y has more than unique solution in $\mathbb{P}^4$ of $(k^5)^*$ -space. In this perspective, it is natural to think the scheme

$$\{(x, y) \in (k^4)^* \times (k^5)^* : (A(x))y = 0\}$$

which can be really thought as subscheme of $\mathbb{P}^3 \times \mathbb{P}^4$. Let this subscheme be $\mathfrak{X} = \mathfrak{X}_A$ and consider the fiber of the projection onto x -space. As A is semisimple, $A(x)$ can't be zero, so rank of $A(x)$ is either 2 for five vanishing points of $I_5(A)$ or 4 for all other points. Thus the fiber over x given as $\{y : A(x)y = 0\}$ consists of one point for all x except five vanishing points of $I_5(A)$. On those points fiber becomes hypersurface of codimension 2, thus isomorphic to $\mathbb{P}^2$. These observations tell us the following.

Proposition 6.4.1. *For A semisimple element in $k^4 \times \wedge^2 k^5$, let $\{p_1, \dots, p_5\}$ be geometric points of the scheme of vanishing Pfaffian $\text{pf}(Ax) = 0$. Then $\mathfrak{X}_A \subset \mathbb{P}^3 \times \mathbb{P}^4$ is a blowup of $\mathbb{P}^3$ at those five points.*

Meanwhile, we can also consider the projection of $\mathfrak{X}_A$ to $\mathbb{P}^4$ of y -space. For generic point x , solution for $A(x)y = 0$ is given uniquely as $y = A(x) \wedge A(x) \in \wedge^4 k^5 \simeq (k^5)^*$. In classical setting of $k = \mathbb{C}$, the following is known:

Proposition 6.4.2. *([21], proposition 9.4.15) Let p_1, p_2, p_3, p_4, p_5 be points in $\mathbb{P}^3$ in*

general linear position. The linear system of quadrics through these points defines a rational map $\mathbb{P}^3 \rightarrow \mathbb{P}^4$ whose image is isomorphic to the Segre cubic primal.

Although this was stated over $\mathbb{C}$ in Dolgachev, we will soon prove the statement for over k using other method.

6.4.2 Properties of the Segre cubic

The following properties are known for cubics over $\mathbb{C}$:

- Define *Segre cubic* as cubic form on 5 variables which are GL_5 -isomorphic to

$$F_0 = x_1^3 + \cdots + x_5^3 - (x_1 + \cdots + x_5)^3.$$

The following symmetric expression is used more often.

$$F_0 = \sum_{i=1}^6 x_i^3, \quad \sum_{i=1}^6 x_i = 0$$

- Automorphism group of a Segre cubic surface $F = 0$ is isomorphic to S_6 , and all automorphisms appear as PGL_5 -coordinate change. On particular F_0 above, it operates as permutation on $(x_1, \dots, x_6)$.
- Any Segre cubic has 10 singular points. On F_0 , they appear as all S_6 -orbits of the point $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, -1, -1, -1)$.
- Any irreducible cubic form in 5 variables having 10 singular points should be Segre cubic.

We claim that those properties also hold over infinite fields k of characteristic $\neq 2, 3, 5$. (I plan to discuss it in my other work regarding Segre cubic and sextic fields.)

A general Segre cubic F allows to be expressed as

$$F = \sum_{i=1}^6 y_i^3, \quad \sum_{i=1}^6 y_i = 0$$

for linear forms $y_1, y_2, \dots, y_6$ such that y_1, y_2, y_3, y_4, y_5 are linearly independent, and choice of $(y_1, \dots, y_6)$ is unique up to permutations of y_i and multiplications of the same constant to all y_i . We can consider six projective points determined by those y_i given as

$$p_1 : y_2 = y_3 = y_4 = y_5 = y_6$$

$$p_2 : y_1 = y_3 = y_4 = y_5 = y_6$$

$$\dots$$

$$p_6 : y_1 = y_2 = y_3 = y_4 = y_5.$$

Call those points $\{p_1, \dots, p_6\}$ *pivot points* of F . Those six points are not on the Segre cubic itself, but can be characterized as points in $\mathbb{P}^4$ whose orbit under $\text{Aut}(F) \subset \text{PGL}_5$ -action consists of exactly 6 points.

Remark. The 10 singular points and 6 pivot points of a Segre cubic form are related in the following way. Suppose that we divided our 6 pivot points into two sets $\{q_1, q_2, q_3\}$ and $\{r_1, r_2, r_3\}$. Let α and β be dimension 2 hyperplanes each going through q_1, q_2, q_3 and r_1, r_2, r_3 respectively. Then α and β intersects at a point which becomes a singular point of the Segre cubic. All singular points of the Segre cubic can be obtained in this way, for each of 10 different ways of dividing 6 points into two sets of 3 points. To prove these statements, one can manually work for the case of $F = F_0$ and consider PGL_4 -transform to prove for the general case.

6.4.3 Explicit construction of Segre cubic and its pivot points

We go back to the equation $A(x)y = 0$, and now try to eliminate x to find equations in y . Expanding it in the form of

$$(A_1y)x_1 + (A_2y)x_2 + (A_3y)x_3 + (A_4y)x_4 = 0,$$

we can see it has solution if and only if A_1y, A_2y, A_3y, A_4y are linearly dependent. Thus we can naturally think of $A_1y \wedge A_2y \wedge A_3y \wedge A_4y \in \wedge^4 k^5$, which is given surprisingly as follows.

Proposition 6.4.3. *For $y \in (k^5)^*$ and $z \in k^5$, suppose that we identify $\wedge^5 k^5 \simeq k$. Then we have*

$$A_1y \wedge A_2y \wedge A_3y \wedge A_4y \wedge z = F(A, y)\langle y, z \rangle$$

for some homogeneous cubic $F(A, y)$ in y . ($\langle y, z \rangle$ is natural pairing $(k^5)^* \times k^5$, here given as normal dot product $\sum y_i z_i$.) Coefficients of $F(A, y)$ in y are degree 4 polynomials which are SL_4 -invariant (under action on (A_1, A_2, A_3, A_4)), and the cubic as a whole is invariant under $\mathrm{SL}_4 \times \mathrm{SL}_5$ -action.

Proof. Explicit calculation. □

We remind the existence of elements B_1, B_2, B_3, B_4, B_5 in span of A_1, A_2, A_3, A_4 satisfying

$$\mathrm{rk}(B_i) \leq 2, \quad \sum_{i=1}^5 B_i = 0$$

and vectors $f_i \in \bar{k}^5$ such that

$$B_i = f_i \wedge (f_{i-1} + f_{i+1}) \quad (f_0 = f_5, f_6 = f_1).$$

Those elements will be used in describing singular points of $F(A, y)$.

Proposition 6.4.4. *The cubic $F(A, y)$ is Segre cubic, having ten singular points $y = B_i \wedge B_j$ ($1 \leq i < j \leq 5$).*

Proof. The expression $A_1 y \wedge A_2 y \wedge A_3 y \wedge A_4 y$ is invariant under the SL_4 -action on (A_1, A_2, A_3, A_4) , so $F(A, y)$ is constant multiple of $F(B, y)$ ($B = (B_1, B_2, B_3, B_4)$) and we can only consider $F(B, y)$.

Lemma. *For M_1, M_2 two 5×5 alternating matrices of rank 2, we have*

$$M_1(M_1 \wedge M_2) = M_2(M_1 \wedge M_2) = 0.$$

Proof. Let $M_1 = p \wedge q$ and $M_2 = r \wedge s$ for some $p, q, r, s \in k^5$. As the matrix multiplication of M and functional (or dual vector) $(M_1 \wedge M_2)$ is really the natural pairing $(p \wedge q)\phi = \phi(p)q - \phi(q)p$, we have

$$(p \wedge q) \cdot (p \wedge q \wedge r \wedge s) = (p \wedge p \wedge q \wedge r \wedge s)q - (q \wedge p \wedge q \wedge r \wedge s)p = 0$$

so our claim is proven. □

This lemma implies $B_i(B_i \wedge B_j) = B_j(B_i \wedge B_j) = 0$ for any i, j . In the following expression

$$B_1 y \wedge B_2 y \wedge B_3 y \wedge B_4 y \wedge z = F(B, y)(y \cdot z),$$

we take directional derivative D_w in y . The lefthand side can be expanded according to Leibniz rule as

$$\begin{aligned} & B_1 w \wedge B_2 y \wedge B_3 y \wedge B_4 y \wedge z + B_1 y \wedge B_2 w \wedge B_3 y \wedge B_4 y \wedge z \\ & + B_1 y \wedge B_2 y \wedge B_3 w \wedge B_4 y \wedge z + B_1 y \wedge B_2 y \wedge B_3 y \wedge B_4 w \wedge z \end{aligned}$$

while the righthand side is

$$(y \cdot z)D_w F(B, y) + F(B, y)(w \cdot z).$$

By substituting $y = B_i \wedge B_j$, the lefthand side vanishes, and $F(B, y) = 0$ is also immediate. Therefore we have $D_w F(B, y) = 0$ for any w , so $y = B_i \wedge B_j$ should be a singularity point of F . $\square$

From this result for singular points, one can obtain the following description for pivot points.

Proposition 6.4.5. *The pivot points of $F(A, y)$ are given as projective points corresponding to following affine vectors*

$$\beta_1 = B_1 \wedge B_2 + B_2 \wedge B_3 + B_3 \wedge B_4 + B_4 \wedge B_5 + B_5 \wedge B_1$$

$$\beta_2 = B_1 \wedge B_2 + B_2 \wedge B_4 + B_4 \wedge B_5 + B_5 \wedge B_3 + B_3 \wedge B_1$$

$$\beta_3 = B_1 \wedge B_2 + B_2 \wedge B_5 + B_5 \wedge B_3 + B_3 \wedge B_4 + B_4 \wedge B_1$$

$$\beta_4 = B_1 \wedge B_3 + B_3 \wedge B_2 + B_2 \wedge B_5 + B_5 \wedge B_4 + B_4 \wedge B_1$$

$$\beta_5 = B_1 \wedge B_3 + B_3 \wedge B_4 + B_4 \wedge B_2 + B_2 \wedge B_5 + B_5 \wedge B_1$$

$$\beta_6 = B_1 \wedge B_4 + B_4 \wedge B_2 + B_2 \wedge B_3 + B_3 \wedge B_5 + B_5 \wedge B_1.$$

Proof. One can prove that $\sum \beta_i = 0$ and

$$\beta_1 + \beta_2 + \beta_3 - \beta_4 - \beta_5 - \beta_6 = 6(B_1 \wedge B_2)$$

by using the property $\sum B_i = 0$ and symmetricity $B_i \wedge B_j = B_j \wedge B_i$. This shows that

$$B_1 \wedge B_2 = 3(\beta_1 + \beta_2 + \beta_3) = -3(\beta_4 + \beta_5 + \beta_6)$$

is in two hyperplanes each going through $\beta_1, \beta_2, \beta_3$ and $\beta_4, \beta_5, \beta_6$ respectively. One can

show similar statements for any $B_i \wedge B_j$ for $i \neq j$, by represent $B_i \wedge B_j$ in the form of $\beta_p + \beta_q + \beta_r - \beta_s - \beta_t - \beta_u$ for some suitable partition $\{p, q, r\} \cup \{s, t, u\} = \{1, 2, 3, 4, 5, 6\}$. By comparing this phenomenon to the remark at the end of the Section 6.4.2, we can conclude that six projective points represented by β_i s should coincide with the six pivot points of $F(A, y)$. $\square$

Each choice of β_i corresponds to one of six of 5-Sylow subgroups of S_5 . For a 5-Sylow subgroup $H \leq S_5$, pick any $\sigma \in H \setminus \{1\}$. Then for $\sigma = (i_1 i_2 i_3 i_4 i_5)$, corresponding β is given as $\sum_t (B_{i_t} \wedge B_{i_{t+1}})$. Value of β stays same for picking σ^4 instead, and changes sign for picking σ^2 and σ^3 (as $\sum_{i,j} (B_i \wedge B_j) = 0$). So this process of associating β to an element of $\sigma \in H$ does not depend on the choice of σ , and we can say that six β_i correspond to six 5-Sylow subgroups of S_5 .

We can find similar expressions for six pivot functionals of F .

Corollary 6.4.6. *For $\sigma = (i_1 i_2 i_3 i_4 i_5)$ a 5-cycle of S_5 , define*

$$\gamma_\sigma = \sum_{t=1}^5 (B_{i_{t-1}} \wedge B_{i_{t+1}}) B_{i_t}.$$

For a 5-Sylow subgroup $H \leq S_5$, pick any $\sigma \in H \setminus \{1\}$ and let $\gamma_H = \gamma_\sigma$. It does not depend on choice of σ . Let β_H be the element β_i according to correspondence described above. Then we have

$$\langle \beta_{H_1}, \gamma_{H_2} \rangle = 2(f_1 \wedge f_2 \wedge f_3 \wedge f_4 \wedge f_5)^2 \cdot \begin{cases} 5 & \text{if } H_1 = H_2 \\ -1 & \text{otherwise.} \end{cases}$$

We can list them as follows.

$$\begin{aligned}
\gamma_1 &= (B_1 \wedge B_3)B_2 + (B_2 \wedge B_4)B_3 + (B_3 \wedge B_5)B_4 + (B_4 \wedge B_1)B_5 + (B_5 \wedge B_2)B_1 \\
\gamma_2 &= (B_1 \wedge B_4)B_2 + (B_2 \wedge B_5)B_4 + (B_4 \wedge B_3)B_5 + (B_5 \wedge B_1)B_3 + (B_3 \wedge B_2)B_1 \\
\gamma_3 &= (B_1 \wedge B_5)B_2 + (B_2 \wedge B_3)B_5 + (B_5 \wedge B_4)B_3 + (B_3 \wedge B_1)B_4 + (B_4 \wedge B_2)B_1 \\
\gamma_4 &= (B_1 \wedge B_2)B_3 + (B_3 \wedge B_5)B_2 + (B_2 \wedge B_4)B_5 + (B_5 \wedge B_1)B_4 + (B_4 \wedge B_3)B_1 \\
\gamma_5 &= (B_1 \wedge B_4)B_3 + (B_3 \wedge B_2)B_4 + (B_4 \wedge B_5)B_2 + (B_2 \wedge B_1)B_5 + (B_5 \wedge B_3)B_1 \\
\gamma_6 &= (B_1 \wedge B_2)B_4 + (B_4 \wedge B_3)B_2 + (B_2 \wedge B_5)B_3 + (B_3 \wedge B_1)B_5 + (B_5 \wedge B_4)B_1.
\end{aligned}$$

Remark. The coefficient of Segre cubic $F(A, y)$ can be considered in following representation theoretic way too. If we view A as $\mathrm{SL}_4 \times \mathrm{SL}_5$ -representation, all degree 4 SL_4 -invariant forms of A form SL_5 -representation $\wedge^4(\wedge^2 k^5)$. By calculating character, we can see $\wedge^4(\wedge^2 k^5)$ splits into two irreducible representations of highest weight $(3, 2, 2, 1, 0)$ and $(4, 1, 1, 1, 1)$. Representation corresponding to $(4, 1, 1, 1, 1)$ is isomorphic to $(\wedge^5 k^5) \otimes (\mathrm{Sym}^3 k^5)$, and coefficients of $F(A, y)$ can be interpreted as elements in this subrepresentations.

Meanwhile, one can also show that $\vee^{10}(\vee^3 k^5)$ contains exactly 1 copy of SL_5 -trivial representation. This degree 10 invariant of a quinary cubic $F(A, y)$ is $\mathrm{SL}_4 \times \mathrm{SL}_5$ -invariant of degree 40 in A , so it should be constant multiple of the discriminant.

6.5 Resolution for 6 pivot points and its hypercohomology

We constructed the Segre cubic $F_A(y) = F(A, y)$ as cubic defining following property

$$A_1 y \wedge A_2 y \wedge A_3 y \wedge A_4 y \wedge z = F_A(y) \langle y, z \rangle$$

where $y \in (k^5)^*$, $z \in k^5$, and calculated its six pivot points in terms of solution for quintic equation $A(x) \wedge A(x) = 0$. We can expect the global function ring for pivot points relate to $\mathcal{S}(A)$ in ring-theoretic perspective too. But what are the equations determining 6 pivot points of $F_A(y)$?

In this section we would like to construct the a complex which becomes a minimal free resolution for six pivot points of F_A when A takes generic value, and construct a sextic algebra from hypercohomology of this complex. In this section we will describe a certain complex $\mathcal{C}_{y,A}$ in terms of coefficients of A , which is covariant under $\mathrm{GL}_4 \times \mathrm{SL}_5$ -action on A and x . Then we show that $\mathcal{C}_{y,A}$ becomes minimal free resolution for six pivot points of F_A under certain condition on k (including $\mathrm{char}(k) = 0$ and separable case). Finally we will compute its hypercohomology ring and show it coincides with Bhargava's explicit construction of sextic resolvent multiplication table.

Before we start, we would like to introduce several abusive notations for our convenience. We abuse the notation (e_1, e_2, e_3, e_4) to denote irreducible SL_4 -representation of highest weight (e_1, e_2, e_3, e_4) . We also denote standard SL_4 -representation of k^4 as 4, and standard SL_5 -representation of k^5 as 5.

For expression h on which $\mathrm{SL}_4 \times \mathrm{SL}_5$ acts upon, and for SL_4 -representation (or character) λ and SL_5 -representation μ , the notation $h \in (\lambda, \mu)$ means that representation V_h spanned by $\mathrm{SL}_4 \times \mathrm{SL}_5$ -orbit of h is isomorphic to $\lambda \otimes \mu$. We will also say that H is SL_4 -covariant with λ and SL_5 -covariant with μ . For example, we can say that

$$A \in (4, \wedge^2 5), \quad x \in (1, 5^*).$$

We also identify $\wedge^4 4 \simeq k$ and $\wedge^5 5 \simeq k$, as we are dealing with SL -representation.

6.5.1 Description of the complex

Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^4}$ be the structure sheaf of $\mathbb{P}^4 \simeq \text{Proj } k[y_1, y_2, y_3, y_4, y_5]$. Our complex has the following form:

$$\begin{aligned}
& \mathcal{C}_{y,A} : L_{28}\mathcal{O}(-6) \\
& \rightarrow \bigoplus_{i=1}^4 L_{17}^i \mathcal{O}(-4) \oplus \bigoplus_{i=1}^5 L_{20}^i \mathcal{O}(-4) \\
& \rightarrow \bigoplus_{i=1}^5 L_{12}^i \mathcal{O}(-3) \oplus \bigoplus_{1 \leq i < j \leq 4} L_{14}^{i,j} \mathcal{O}(-3) \oplus \bigoplus_{i=1}^5 L_{16}^i \mathcal{O}(-3) \\
& \rightarrow \bigoplus_{i=1}^5 L_8^i \mathcal{O}(-2) \oplus \bigoplus_{i=1}^4 L_{11}^i \mathcal{O}(-2) \\
& \rightarrow \mathcal{O}.
\end{aligned}$$

Here L_m denotes formal variables of degree zero, so if we don't distinguish them the above complex is just

$$\mathcal{O}(-6) \rightarrow \mathcal{O}(-4)^9 \rightarrow \mathcal{O}(-3)^{16} \rightarrow \mathcal{O}(-2)^9 \rightarrow \mathcal{O}$$

which is the shape of minimal free resolution of six points in general position. We would like to consider L_m as collection of variables L_m^i having degree m in entries of A , which behaves under $\text{SL}_4 \times \text{SL}_5$ -action as follows:

$$\begin{aligned}
L_8 &= (L_8^i)_{i=1, \dots, 5} \in (1, 5^*), & L_{11} &= (L_{11}^i)_{i=1, \dots, 4} \in (4^*, 1), \\
L_{12} &= (L_{12}^i)_{i=1, \dots, 5} \in (1, 5), & L_{14} &= (L_{14}^{i,j})_{1 \leq i < j \leq 4} \in (\wedge^2 4, 1), \\
L_{16} &= (L_{16}^i)_{i=1, \dots, 5} \in (1, 5^*), & L_{17} &= (L_{17}^i)_{i=1, \dots, 4} \in (4, 1), \\
L_{20} &= (L_{20}^i)_{i=1, \dots, 5} \in (1, 5), & L_{28} &\in (1, 1).
\end{aligned}$$

For the rest of the section we will describe the maps for $\mathcal{C}_{y,A}$, and show that those

are covariant maps under $\mathrm{SL}_4 \times \mathrm{SL}_5$ -action.

- Some intermediate covariants and covariant maps Denote the duality pairing as bracket:

$$\langle -, - \rangle : 5 \times 5^* \rightarrow k.$$

For $u, v \in 5^*$ and $A_i \in \wedge^2 5$, we can consider the natural map

$$\wedge^2 5 \otimes 5^* \rightarrow 5, \quad A_i \otimes u \mapsto A_i u$$

which can be described in terms of basis elements as

$$(a \wedge b) \otimes u \mapsto \langle a, u \rangle b - \langle b, u \rangle a.$$

If we represent A_i as alternating matrix and u as column vector, $A_i u$ is represented as column vector whose coordinate is given by matrix multiplication on right. Similarly we denote

$$v^t A_i u = \langle A_i u, v \rangle = -\langle A_i v, u \rangle$$

which can be interpreted as matrix multiplication on both left and right side. Upon identifying $\wedge^4 4 = 1$ and $\wedge^5 5 = 1$, we can also identify $\wedge^4 5 \simeq 5^*$ from the pairing $\wedge : \wedge^4 5 \times 5 \rightarrow k$. We define

$$\alpha_{ij,l}(y) = A_i \wedge A_j \wedge A_l y \in \wedge^5 5 \simeq k.$$

as a map from 5^* to $1 \simeq k$. It is symmetric in i, j , and we can check $\alpha_{ii,i} = 0$. So if we consider $\alpha_{ij,l}$ as trilinear form on i, j, l place, with respect to SL_4 action those elements form subrepresentation of $\vee^2 4 \otimes 4$, as α is symmetric in i, j .

Meanwhile we have

$$\vee^2 4 \otimes 4 \simeq (3, 0, 0, 0) \oplus (2, 1, 0, 0)$$

and those two factors can be described as image and kernel of natural map

$$\vee^2 4 \otimes 4 \rightarrow \vee^3 4 = (3, 0, 0, 0), \quad (a \vee b) \otimes c \mapsto (a \vee b \vee c).$$

From $\alpha_{ii,i} = 0$, we can see $\alpha_{ij,l}$ are mapped to zero by this symmetrization, so this implies

$$\alpha_{ij,l}(y) \in ((2, 1, 0, 0), \text{Hom}(5^*, 1)).$$

Of course it can be checked from direct computation too.

For $u_1, u_2, u_3, u_4 \in 5^*$, define

$$\Omega(u_1, u_2, u_3, u_4) = \sum_{i_1, i_2, i_3, i_4} (A_{i_1} u_1 \wedge A_{i_2} u_2 \wedge A_{i_3} u_3 \wedge A_{i_4} u_4) \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

One can check that this form is symmetric in u_i , and SL_4 -invariant with respect to action on A_i . Thus we can consider Ω as SL_4 -invariant element of $\text{Hom}(\vee^4 5^*, 5)$. This really comes from Segre cubic, as we have

$$\Omega(y, y, y, y) = 24F_A(y)y.$$

- Construction of degree 8 quadrics L_8 For $u_1, u_2 \in 5^*$, define the quadratic form

$$\Psi(u_1, u_2) = \frac{1}{3} \sum_{i_1, i_2, i_3, i_4} \sum_{j_1, j_2, j_3, j_4} \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \\ (A_{i_4} \wedge A_{j_4}) \alpha_{i_1 j_1, j_3}(u_1) \alpha_{i_2 j_2, i_3}(u_2).$$

One can check that it is symmetric in u_1, u_2 , and invariant under SL_4 -action. So it can be considered as SL_4 -invariant element of $\mathrm{Hom}(\vee^2 5^*, 5^*)$. Using it, we give the map of $\mathcal{C}_{y,A}$

$$L_8 \mapsto \frac{1}{2} \Psi(y, y).$$

This is covariant with specified $\mathrm{SL}_4 \times \mathrm{SL}_5$ -action of $L_8 \in (1, 5^*)$, and degree in A coincides as 8. We can check that both $\Psi(u_1, u_2)$ and $\Psi(y, y)/2$ are integral polynomials (in y, u_1 , coefficients of A_i) and primitive.

We can think of following motivation for construction of Ψ . The set of all forms

$$\sum (A_{\bullet} \wedge A_{\bullet}) \alpha_{\bullet\bullet, \bullet}(y) \alpha_{\bullet\bullet, \bullet}(y)$$

form SL_4 -representation of weight

$$(\vee^2 4) \otimes \vee^2(2, 1, 0, 0)$$

and this representation contains exactly one copy of trivial representation $(2, 2, 2, 2)$.

It should come with the pairing of $(\vee^2 4) = (2, 0, 0, 0)$ with unique subrepresentation of $(2, 2, 2, 0)$ in $\vee^2(2, 1, 0, 0)$ by $\vee^2 4 \times \vee^2 4^* \rightarrow 1$. Indeed the expressions of form $\sum \alpha_{\bullet\bullet, \bullet}(u_1) \alpha_{\bullet\bullet, \bullet}(u_2)$, which are multiplied to $(A_{i_4} \wedge A_{j_4})$ in formula of Ψ , are set to form SL_4 -representation of weight $(2, 2, 2, 0)$.

The formula for Ψ can be obtained from the symbolic invariant method.

- Construction of degree 11 quadrics L_{11} There exists four quadrics $\phi_1, \phi_2, \phi_3, \phi_4$ in y which is suitable sum of terms of form

$$\phi_i = \sum ((A_{\bullet} \wedge A_{\bullet}) A_{\bullet} (A_{\bullet} \wedge A_{\bullet})) \alpha_{\bullet\bullet,\bullet}(y) \alpha_{\bullet\bullet,\bullet}(y),$$

and $(\phi_1(y), \phi_2(y), \phi_3(y), \phi_4(y))$ has SL_4 -weight 4^* . Here we consider $(A_{\bullet} \wedge A_{\bullet}) \in \wedge^4 5$ as element of 5^* , and juxtaposition means matrix multiplication on left and right we described earlier. (It is same as the 5-term map introduced by Bhargava when giving the multiplication table for quintic HCL) As those elements are SL_5 -invariant, we can give the map of $\mathcal{C}_{y,A}$

$$L_{11} \mapsto (\phi_1, \phi_2, \phi_3, \phi_4)$$

which is covariant map of $(4^*, 1)$. In particular, the following expression its image

$$(L_{11} \cdot A) = \sum_{i=1}^4 L_{11}^i A_i \mapsto \sum_{i=1}^4 \phi_i A_i$$

are SL_4 -invariant, thus is in $(1, \wedge^2 5)$.

The explicit formula for ϕ_i can be given as follows.

$$\begin{aligned} \sum_{i=1}^4 \phi_i A_i &= \frac{1}{12} \sum_{i_1, i_2, i_3, i_4} \sum_{j_1, j_2, j_3, j_4} \sum_{k_1, k_2, k_3, k_4} \\ &\quad \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \cdot \\ &\quad \alpha_{i_1 j_1, j_3}(y) \alpha_{i_2 j_2, i_3}(y) ((A_{i_4} \wedge A_{k_2}) A_{k_3} (A_{j_4} \wedge A_{k_4})) A_{k_1}. \end{aligned}$$

The parts $\alpha_{i_1 j_1, j_3}(y) \alpha_{i_2 j_2, i_3}(y)$ are basis elements of subrepresentation $(2, 2, 2, 0)$ of $\vee^2(2, 1, 0, 0)$, same as ones used in constructing Ψ . Those parts should be

paired with basis elements representation $(3, 1, 1, 1)$ to give SL_4 -invariant element. Meanwhile, The space of all forms $(A_\bullet \wedge A_\bullet)A_\bullet(A_\bullet \wedge A_\bullet)$ appears as subrepresentation of

$$\wedge^2(\vee^2 4) \otimes 4 \simeq (4, 1, 0, 0) \oplus (3, 2, 0, 0) \oplus (3, 1, 1, 0)$$

as the form is antisymmetric in two entries $(A_\bullet \wedge A_\bullet)$ which are symmetric. One can check that its representation corresponds exactly to $(3, 1, 1, 0)$. Then one needs to find formula for basis elements of

$$(3, 1, 1, 1) \subset (3, 1, 1, 0) \otimes (1, 0, 0, 0)$$

and pair those elements with basis elements of $(2, 2, 2, 0)$.

The explicit formula can also be obtained from certain symbolic invariant method for this case too.

- Construction of degree 12 syzygies L_{12} We use the SL_4 -invariant map $\Omega \in \mathrm{Hom}(\vee^4 5^*, 5^*)$, and consider its restriction $\Omega(y, L_8, -, -) \in \mathrm{Hom}(\vee^2 5^*, 5^*)$. As we have

$$\mathrm{Hom}(\vee^2 5^*, 5^*) \simeq (\vee^2 5^*)^* \otimes 5^* \simeq \vee^2 5 \otimes 5^*,$$

composing it with the natural map

$$\vee^2 5 \otimes 5^* \rightarrow 5, \quad (a \vee b) \otimes w \mapsto \langle a, w \rangle b + \langle b, w \rangle a$$

gives element of 5, which we denote $\tilde{\Omega}$. If we want to put that in explicit form,

pick basis $\{e_i\}$ of 5 and its dual basis $\{e_i^*\}$ of 5^* . Then we can identify

$$\Omega(y, L_8, -, -) = \frac{1}{2} \sum_{i,j=1}^5 (e_i \vee e_j) \otimes \Omega(y, \Psi, e_i^*, e_j^*) \in \vee^2 5 \otimes 5^*$$

thus applying the above natural map gives

$$\tilde{\Omega} = \sum_{i,j=1}^5 \langle e_i, \Omega(y, L_8, e_i^*, e_j^*) \rangle e_j.$$

Our $\mathcal{C}_{y,A}$ map which maps L_{12} to degree 12 syzygy is given as

$$L_{12} \mapsto \frac{1}{8} \tilde{\Omega} + (L_{11} \cdot A)y \in (1, 5).$$

Even with constant $1/8$, the right hand side is integral. (Can be checked from explicit calculation.)

- Construction of degree 14 syzygies L_{14} We consider the following six elements

$$(L_{11} \cdot A) \wedge (A_i \wedge A_j y - A_j \wedge A_i y)$$

for $i < j$. As $(L_{11} \cdot A)$ is SL_4 -invariant and $(A_i \wedge A_j y - A_j \wedge A_i y)$ is antisymmetric with respect to i, j , those six elements form representation of weight $\wedge^2 4$. In SL_5 -action those are invariant, thus it is of weight $(\wedge^2 4, 1)$.

Meanwhile we also want to look for weight $(\wedge^2 4, 1)$ covariant made from L_8 . We first consider space of all forms

$$\sum \alpha_{\bullet\bullet\bullet}(u) \alpha_{\bullet\bullet\bullet}(v)$$

which are alternating in u and v . Those forms form SL_4 -representation of

$\wedge^2(2, 1, 0, 0)$. As we have

$$\wedge^2(2, 1, 0, 0) \simeq (3, 3, 0, 0) \oplus (4, 1, 1, 0) \oplus (3, 2, 1, 0) \oplus (2, 2, 1, 1),$$

we can pick six forms which form SL_4 -representation $(2, 2, 1, 1)$. Their explicit formulas can be given as

$$F_{p,q}(u, v) = \sum_{i,j,k,l} \begin{pmatrix} i & j & k & l \\ 1 & 2 & 3 & 4 \end{pmatrix} (\alpha_{pi,j}(u)\alpha_{qk,l}(v) - \alpha_{qi,j}(u)\alpha_{pk,l}(v))$$

Our $\mathcal{C}_{y,A}$ map which maps L_{14} to degree 14 syzygy is given as difference of those two expressions.

$$L_{14}^{i,j} \mapsto F_{i,j}(L_8, y) - 5(L_{11} \cdot A) \wedge (A_i \wedge A_j y - A_j \wedge A_i y)$$

- Construction of degree 16 syzygies L_{16} For $u, v \in 5^*$, we consider the following forms

$$\beta_{pqrs}(u, v) = (A_p u \wedge A_q v)(A_r \wedge A_s) - (A_q u \wedge A_p v)(A_r \wedge A_s)$$

where $(A_p u \wedge A_q v) \in \wedge^2 5$, $(A_r \wedge A_s) \in 5^*$, and juxtaposition is matrix multiplication. We can show that they are symmetric in u, v . As SL_4 -representation, those elements form representation $\vee^2 4 \otimes \wedge^2 4$, and it contains a single subrepresentation of highest weight $(2, 1, 1, 0)$. We will construct the basis element for the $(2, 1, 1, 0)$ subrepresentation explicitly. For elements

$$\gamma_{i,l} = \begin{pmatrix} i & j & k & l \\ 1 & 2 & 3 & 4 \end{pmatrix} (\beta_{ijik} - \beta_{ikij} + \beta_{jkii})$$

for $\{i, j, k, l\} = \{1, 2, 3, 4\}$, and

$$\gamma_{1,1} = -\beta_{2314} + \beta_{2413} - \beta_{3412}$$

$$\gamma_{2,2} = \beta_{1324} - \beta_{1423} + \beta_{3412}$$

$$\gamma_{3,3} = -\beta_{1234} - \beta_{2413} + \beta_{1423}$$

$$\gamma_{4,4} = \beta_{2314} + \beta_{1234} - \beta_{1324},$$

consider the linear map which maps trace zero 4×4 matrix to

$$M = (m_{i,j})_{1 \leq i,j \leq 4} \mapsto \sum_{i,j=1}^4 \gamma_{i,j} m_{ij}.$$

If $G \in \text{SL}_4$ acts on M as $M \mapsto (\det G)GMG^{-1}$, it is representation of highest weight $(2, 1, 1, 0)$. The SL_4 -action on $\gamma_{i,j}$ operates exactly in this way, such that above map becomes isomorphism of SL_4 -isomorphism. If we view $(2, 1, 1, 0)$ as subrepresentation of $4 \otimes 4^*$, we can pair $L_{11} \in (4^*, 1)$ and $A \in (4, \wedge^2 5)$ with $\gamma_{i,j}$ to give following SL_4 -invariant form

$$h(u, v) = \sum_{i,j=1}^4 L_{11}^i (\gamma_{i,j}(u, v) \wedge A_j) \in \text{Hom}(\vee^2 5^*, \wedge^3 5).$$

We pair $h \in \text{Hom}(\vee^2 5^*, \wedge^3 5)$ and $x \in 5^*$ with the series of mapping

$$\text{Hom}(\vee^2 5^*, \wedge^3 5) \otimes 5^* \simeq \vee^2 5 \otimes \wedge^3 5 \otimes 5^* \rightarrow 5 \otimes \wedge^3 5 \rightarrow \wedge^4 5 \simeq 5^*$$

where the first map is

$$\vee^2 5 \otimes 5^* \rightarrow 5, \quad (a \vee b) \otimes w \mapsto \langle a, w \rangle b + \langle b, w \rangle a$$

and the second map is wedge product. In explicit form, we can pick basis $\{e_i\}$

of 5-space and its dual basis $\{e_i^*\}$ of 5^* to identify

$$h = \frac{1}{2} \sum_{i,j=1}^5 (e_i \vee e_j) \otimes h(e_i^*, e_j^*) \in \vee^2 5 \otimes \wedge^3 5^*$$

and it becomes

$$\tilde{h} = \sum_{i,j=1}^5 (y, e_i) (h(e_i^*, e_j^*) \wedge e_j) \in \wedge^4 5 \simeq 5^*.$$

Our $\mathcal{C}_{y,A}$ map which maps L_{16} to degree 16 syzygy is given as

$$L_{16} \mapsto \tilde{h} - \Psi(y, L_8) \in (1, 5^*).$$

- Construction of degree 17 syzygies L_{17} We remind that

$$L_{12} \in (1, 5), \quad L_{14} \in (\wedge^2 4, 1), \quad L_{16} \in (1, 5^*).$$

We will look into contribution of L_{16}, L_{14}, L_{12} in that order.

For L_{16} , we have $L_{16}, y \in (1, 5^*)$. We give following form using matrix multiplication.

$$U_{17}^{16} = (L_{16}^t A_1 y, L_{16}^t A_2 y, L_{16}^t A_3 y, L_{16}^t A_4 y) \in (4, 1).$$

For $L_{14} \in (\wedge^2 4, 1)$, we first note that the following sum

$$\frac{1}{4} \sum_{i,j,k,l} \begin{pmatrix} i & j & k & l \\ 1 & 2 & 3 & 4 \end{pmatrix} L_{14}^{i,j} (A_k u \wedge A_l v - A_l u \wedge A_k v)$$

$(u, v \in 5^*)$ is SL_4 -invariant. We tensor it with (A_1, A_2, A_3, A_4) to obtain

$$\left(\frac{1}{4} \sum_{i,j,k,l} \begin{pmatrix} i & j & k & l \\ 1 & 2 & 3 & 4 \end{pmatrix} L_{14}^{i,j} (A_k u \wedge A_l v - A_l u \wedge A_k v) \wedge A_t \right)_{1 \leq t \leq 4}$$

which is of SL_4 -weight 4, and of SL_5 -weight $\text{Hom}(\wedge^2 5^*, 5^*)$. We substitute $u = y$ to give form $\text{Hom}(\wedge^2 5^*, 5^*)$ and take trace with respect to v . In explicit coordinates, it is given as

$$U_{17}^{14} = \left(\sum_{s=1}^5 \frac{1}{4} \sum_{i,j,k,l} \begin{pmatrix} i & j & k & l \\ 1 & 2 & 3 & 4 \end{pmatrix} L_{14}^{i,j} ((A_k y \wedge A_l e_s^* - A_l y \wedge A_k e_s^*) \wedge A_t \wedge e_s) \right)_{1 \leq t \leq 4}.$$

For $L_{12} \in (1, 5)$, we reuse the elements $\gamma_{i,l}(u, v) \in \text{Hom}(\vee^2 5^*, 5)$ I defined in construction of mapping of L_{16} . As those formed a SL_4 -representation of highest weight $(2, 1, 1, 0)$, we consider the following four elements

$$\left(\sum_{j=1}^4 \gamma_{k,j}(u, v) \wedge A_j \right)_{k=1, \dots, 4} \in \text{Hom}(\vee^2 5^*, \wedge^3 5)^4$$

which is of SL_4 -representation of highest weight $(2, 1, 1, 1)$. We substitute $u = y$ and take wedge with L_{12} to give

$$\left(\sum_{j=1}^4 \gamma_{k,j}(y, v) \wedge A_j \wedge L_{12} \right)_{k=1, \dots, 4} \in \text{Hom}(5^*, \wedge^4 5)^4 \simeq \text{Hom}(5^*, 5^*)^4$$

and taking its trace over v gives

$$U_{17}^{12} = \left(\sum_{j=1}^4 \sum_{i=1}^5 (\gamma_{k,j}(y, e_i^*) \wedge A_j \wedge L_{12} \wedge e_i) \right)_{k=1, \dots, 4} \in (4, 1).$$

Our $\mathcal{C}_{y,A}$ map which maps L_{17} to degree 17 syzygy is given as

$$L_{17} = 3U_{17}^{16} + U_{17}^{14} + 3U_{17}^{12}.$$

- Construction of degree 20 syzygies L_{20} Starting with L_{16} , we bring the SL_4 -invariant form $\Omega : \vee^4 5^* \rightarrow 5^*$ again. The form $\Omega(y, L_{16}, u, v)$ as map of u, v is in $\text{Hom}(\vee^2 5^*, 5^*)$. We map this to element of 5 by the following map

$$(\vee^2 5^*)^* \otimes 5^* \simeq \vee^2 5 \otimes 5^* \rightarrow 5.$$

If we express that in explicit coordinates, we obtain

$$U_{20}^{16} = \sum_{i,j=1}^5 \langle e_i, \Omega(y, L_{16}, e_i^*, e_j^*) \rangle e_j.$$

For L_{14} , we reuse the form

$$F_{p,q}(u, v) = \sum_{i,j,k,l} \begin{pmatrix} i & j & k & l \\ 1 & 2 & 3 & 4 \end{pmatrix} (\alpha_{pi,j}(u)\alpha_{qk,l}(v) - \alpha_{qi,j}(u)\alpha_{pk,l}(v))$$

which are in $(\wedge^2 4, \text{Hom}(\wedge^2 5^*, 1))$. Thus the following form in v

$$\frac{1}{4} \sum_{i,j,k,l} \begin{pmatrix} i & j & k & l \\ 1 & 2 & 3 & 4 \end{pmatrix} L_{14}^{i,j} F_{k,l}(y, v)$$

is SL_4 -invariant form in $\text{Hom}(5^*, 1) \simeq 5$. In explicit basis it can be represented

as

$$U_{20}^{14} = \sum_{t=1}^5 \left(\sum_{i,j,k,l} \begin{pmatrix} i & j & k & l \\ 1 & 2 & 3 & 4 \end{pmatrix} L_{14}^{i,j} F_{k,l}(y, e_t^*) \right) e_t \in (1, 5).$$

For L_{12} , we reuse bilinear form Ψ . The form

$$u \in 5^* \mapsto \Psi(y, u) \in 5^*$$

can be considered as linear map from 5^* to 5 , so we take transpose of this map and consider image of $L_{12} \in 5$. In explicit basis it can be written as

$$U_{20}^{12} = \sum_{i=1}^5 \langle L_{12}, \Psi(y, e_i^*) \rangle e_i.$$

The degree 20 syzygy can be described as

$$L_{20} \mapsto -15U_{20}^{12} - 2U_{20}^{14} + \frac{15}{8}U_{20}^{16} \in (1, 5).$$

Again the form is integral even with constant $15/8$.

- Construction of degree 28 syzygy L_{28}

The map for last syzygy is given as

$$L_{28} \mapsto 5 \sum_{i=1}^4 L_{17}^i \phi_i + \langle L_{20}, \frac{\Psi(y, y)}{2} \rangle \in (1, 1).$$

The covariance is justified from $L_{17} \in (4, 1)$, $\phi_i \in (4^*, 1)$, $\Psi(y, y) \in (1, 5^*)$, and $L_{20} \in (1, 5)$.

6.5.2 Properties of the hypercohomology ring

Proposition 6.5.1. *Let A be a semisimple element of $k^4 \otimes \wedge^2 k^5$.*

(1) *The complex $\mathcal{C}_{y,A}$ is well-defined and covariant under $\mathrm{SL}_4 \times \mathrm{SL}_5$ -action (with above specified rule for L_i).*

(2) *If k is field of characteristic zero, then $\mathcal{C}_{y,A}$ is exact, and it provides minimal free resolution for six pivot points of $F_A(y)$.*

Proof. For A being a particular explicit element (the example I used was

$$A_1 = e_1 \wedge (e_5 + e_2), \quad A_2 = e_2 \wedge (e_1 + e_3), \quad A_3 = e_3 \wedge (e_2 + e_4), \quad A_4 = e_4 \wedge (e_3 + e_5)$$

which has discriminant 1), one can show that the nine quadrics coming from the map $\mathcal{O}(-2)^9 \rightarrow \mathcal{O}$ of $\mathcal{C}_{y,A}$ all vanish on six pivot points and are linearly independent. Also one can show that all maps of the $\mathcal{C}_{y,A}$ consist of linearly independent syzygies of lowest possible degree. As $\mathcal{C}_{y,A}$ has shape of minimal free resolution for six points in $\mathbb{P}^4$, this suggests that $\mathcal{C}_{y,A}$ should be minimal free resolution for six pivot points. $\square$

As $\mathcal{C}_{y,A}$ becomes minimal free resolution of six points for semisimple A , it should be minimal free resolution over universal variable $\mathbb{Q}[A]$. Thus one can construct a DG-algebra structure on $\mathcal{C}_{y,A}$ ([46], Theorem 136) with coefficients in $\mathbb{Q}[A]$. (I expect that this DG-algebra structure actually has coefficients in $\mathbb{Z}[A]$.)

Now we apply the hypercohomology construction of the Chapter 2 to $\mathcal{C}_{y,A}$ to calculate the explicit ring structure. First we have the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow H^0 R\Gamma(\mathcal{C}_{y,A}) \rightarrow H^4(\mathcal{O}(-6)) \rightarrow 0$$

and $H^4(\mathcal{O}(-6))$ has $\mathbb{Z}$ -basis $1/(y_1 y_2 y_3 y_4 y_5 y_i)$ for $i = 1, \dots, 5$, so the hypercohomology is isomorphic to $\mathbb{Z}^6$ as $\mathbb{Z}$ -module. To compute the ring structure, we use the identification of $H^0 R\Gamma(\mathcal{C}_{y,A})$ with total cohomology of $\Gamma(\mathcal{C}_{y,A} \otimes C_\bullet(\mathbb{P}^4))$ ($C_\bullet(\mathbb{P}^4)$: Čech complex for $\mathbb{P}^4$) established in the Theorem 2.3.7, and its consistency with the natural multiplication structure on $H^0 R\Gamma(\mathcal{C}_{y,A})$. If we consider the map

$$\varphi : H^0 R\Gamma(\mathcal{C}_{t,A}) \simeq H^0 \text{Tot} \Gamma(\mathcal{C}_{t,A} \otimes C_\bullet(\mathbb{P}^k)) \rightarrow H^0 \Gamma(\mathcal{O} \otimes \bigoplus \mathcal{O}_{t_i}) \simeq \bigoplus \mathbb{Z}[t_1/t_i, \dots, t_5/t_i]$$

which can be computed explicitly by following the algorithm in the section 2.3, this map becomes injective ring homomorphism for semisimple A . Thus if we denote α_i as

a preimage of $1/(y_1y_2y_3y_4y_5y_i)$ under the map $H^0R\Gamma(\mathcal{C}_{y,A}) \rightarrow H^4(\mathcal{O}(-6))$ described above, then multiplication coefficients associated to α_i should satisfy

$$\varphi(\alpha_i)\varphi(\alpha_j) = \sum_{k=1}^5 e_{ij}^k(A)\varphi(\alpha_k) + e_{ij}^0(A)(1, 1, 1, 1, 1).$$

Solving this linear equations in e_{ij}^k with $\mathbb{Z}(A)$ -coefficients should yield $e_{ij}^k(A)$. One can expect that e_{ij}^k are polynomials in $\mathbb{Q}[A]$, as those can be also calculated from DG-algebra structure on $\mathcal{C}_{y,A}$. If we look at the construction of $\varphi(\alpha_i)$, one can see those consist of Laurent polynomials in $y_1, \dots, y_5$ of degree zero having polynomial coefficient of degree 28 in A . This suggests that all $e_{ij}^k(A)$ are degree 28 polynomials in A . One can check $e_{ij}^k(A)$ are indeed integral polynomials through explicit calculation, and thus the natural multiplication structure on $H^0R\Gamma(\mathcal{C}_{y,A})$ becomes the integral ring. Denote this ring by $\mathcal{R}(\mathcal{C}_{y,A})$.

Meanwhile, if we consider how SL_5 action affects the ring $\mathcal{R}(\mathcal{C}_{y,A})$, we note $y \in 5^*$ and $H^0(\mathcal{O}(m))$ is isomorphic to $\mathrm{Sym}^m(y)$. By considering the covariance of the pairing of Serre duality

$$H^4(\mathcal{O}(-6)) \times H^0(\mathcal{O}(1)) \rightarrow H^4(\mathcal{O}(-5)) \simeq \mathbb{Z},$$

we can deduce that SL_5 -action on $H^4(\mathcal{O}(-6))$ is dual to action on $H^0(\mathcal{O}(1))$. So $H^4(\mathcal{O}(-6)) \in 5$. We would like to compare $\mathcal{R}(\mathcal{C}_{y,A})$ with the sextic resolvent ring $\mathcal{S}(A)$ Bhargava constructed in the quintic higher composition law, which behaves as 5^* in SL_5 -action.

Proposition 6.5.2. *For a nondegenerate element $A \in 4 \otimes \wedge^2 5$, we have*

$$\mathcal{R}(\mathcal{C}_{y,A}) \otimes \mathbb{Q} \simeq \mathcal{S}(A) \otimes \mathbb{Q}.$$

Moreover, let $\alpha_1, \dots, \alpha_5$ be trace zero basis of $(\mathcal{R}(\mathcal{C}_{y,A})/\mathbb{Z}) \otimes \mathbb{Q}$, and let $\beta_1, \dots, \beta_5$ be trace zero basis of $\mathcal{S}(A) \otimes \mathbb{Q}$ which was constructed in Bhargava's quintic higher

composition law. Then we have

$$\mathrm{Tr}(\alpha_i \beta_j) = 2160 \delta_{ij} \cdot \mathrm{Disc}(A).$$

Here $\mathrm{Disc}(A)$ denotes the discriminant of $\mathcal{R}(A)$ (or equivalently, the degree 40 invariant generator of the prehomogeneous space $\mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5$)

Proof. We consider dual basis α_i^* of α_i , and show elements $2160 \mathrm{Disc}(A) \alpha_i^*$ with 1 form $\mathbb{Z}$ -basis of a ring which is isomorphic to a ring with $\mathbb{Z}$ -basis $1, \beta_1, \dots, \beta_5$. We can first check the statement holds for one case of A taking some specific value, through explicit calculations.

As $\mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5$ is prehomogeneous, showing that the action of $\mathrm{SL}_4 \times \mathrm{GL}_5$ transforms $2160 \mathrm{Disc}(A) \alpha_i^*$ and β_i in identical way should suffice for our proof. We already have shown that α_i is of $\mathrm{SL}_4 \times \mathrm{SL}_5$ -character $(1, 5^*)$, so dual basis elements α_i^* should have character $(1, 5)$. Meanwhile, all β_i were constructed using SL_4 -invariants so that SL_5 acts as standard representation on space 5. Finally, under GL_5 -action, both $2160 \mathrm{Disc}(A) \alpha_i^*$ and β_i has degree 12 in entries of A , so the group $\mathrm{SL}_4 \times \mathrm{GL}_5$ acts on them in identical way. $\square$

Remark. The proposition above suggests that one needs to consider six points dual to pivot points in order to obtain the ring behaving covariantly with $\mathcal{S}(A)$. For a Segre cubic F , one can consider the *pivot functionals* as linear functionals l_i satisfying

$$F = \sum_{i=1}^6 l_i^3, \quad \sum_{i=1}^6 l_i = 0.$$

Those $l_1, \dots, l_6$ are uniquely determined up to constant multiplication. By using the dual relation between $\mathcal{R}(\mathcal{C}_{y,A})$ and $\mathcal{S}(A)$ and the fact that six pivot points of F_A are canonical points of $\mathcal{R}(\mathcal{C}_{y,A})$ (Theorem 2.5.6), one can show that six points in $\mathbb{P}^4$ represented by pivot functionals are the canonical points of $\mathcal{S}(A)$.

Chapter 7

Schemes over $\mathbb{Z}$ related to n points in projective space

7.1 Introduction

This chapter takes a different approach on ring parametrizations from our earlier chapters. We would like to examine various scheme-theoretic objects occurring in the ring parametrization models, which can be considered as generalizations of the notion of n geometric points in $\mathbb{P}^m$ over integer base $\text{Spec } \mathbb{Z}$. We will discuss how those objects exhibit properties different from n geometric points, mainly by examining explicit examples including the case of n distinct $\mathbb{Q}$ -points of $\mathbb{P}^m$.

In many previous works on parametrizations of rank n rings and algebras, rank n rings and their parametrization methods were related to certain n points in $\mathbb{P}^m$. Wright and Yukie ([52]) considered certain systems of equations determining n points in $\mathbb{P}^{n-2}$ for $n = 4, 5$ and used them to parametrize rank n algebras over a field, and this perspective was generalized to the Bhargava's Higher Composition Laws ([2], [3]) which could construct rank $n = 4, 5$ integral rings from the same system of equations. In [52], a rank n $\mathbb{Q}$ -algebra is constructed as the ring of global functions on the union

of those n points.

This perspective of considering global functions of n points was further made explicit in Wood's works including [47], [49] (from [51]), where the hypercohomology construction of rank n ring could produce global functions corresponding to its elements which are expressed as polynomials in local coordinates of projective space. In the example of [49], which generalizes the quartic ring parametrization of Bhargava in [2], a ring corresponding to a pair of ternary quadrics $(f, g) = (\sum_{i \leq j} a_{ij} x_i x_j, \sum_{i \leq j} b_{ij} x_i x_j)$ is constructed as hypercohomology of the Koszul complex $\mathcal{K}_{f,g}$ associated to f, g . For the generic case where two quadrics f and g intersect at four points in $\mathbb{P}^2$, one can express any element of this hypercohomology ring as an expression of form

$$(h_1(x_2/x_1, x_3/x_1), h_2(x_1/x_2, x_3/x_2), h_3(x_1/x_3, x_2/x_3))$$

where h_1, h_2, h_3 are polynomials satisfying that all differences $h_i - h_j$ between them are in the ideal (f, g) . Then this element can represent a global function on a scheme $V(f, g)$. In the Chapter 2 of this thesis we discussed a generalized version of the hypercohomology construction which works for a broader class of complexes of sheaves.

There are other geometric perspectives which relate n geometric points to a rank n ring in a way different from the perspective of using global functions. In the Section 2 of [3], an explicit construction of n geometric points out of a basis of a rank n degenerate ring R is described. Those n points, which we name *canonical points* in this thesis, will be further discussed in the Section 7.2.4 of this chapter. Canonical points only depend on numerical values of the n distinct embeddings $\rho_1, \dots, \rho_n : R \rightarrow \mathbb{C}$ and can be constructed easily, but its inverse correspondence - finding a ring R whose canonical points become n points already given - is hard to compute.

The construction of [16] decomposing any degree n Gorenstein cover $X \rightarrow Y$ of

integral schemes over field k into an embedding $X \hookrightarrow \mathbb{P}_k^{n-2} \times_k Y$ and the natural map $\mathbb{P}_k^{n-2} \times_k Y \rightarrow Y$ can be considered as generalization of this canonical point construction, as those coincide when $Y = \operatorname{Spec} \mathbb{Q}$. It suggests that notion of ‘ n points in $\mathbb{P}^{n-2}$ ’ might be generalized over arbitrary base: the rest of [16] is devoted with $n = 3, 4$ case to give a ‘parametrization’ of those Gorenstein covers in terms of a global section of locally free sheaves (corresponding to global forms parametrizing rings) This is further generalized in [49] which provides parametrizations of cubic and quartic algebras over an arbitrary base scheme.

However, things do not go well when one tries to generalize those results over integral base $\mathbb{Z}$. It is possible to consider several objects which generalize the notion of n points in $\mathbb{P}^m$, but they all seem to be inconsistent with well-behaving ring parametrization models established. For example, in the quartic ring parametrization of [2] which corresponds a quartic ring to a pair of two quadrics $a(x_1, x_2, x_3), b(x_1, x_2, x_3)$ in three variables, the following rings turn out to be all different even for generic a and b .

- The quartic ring given in correspondence of [2], which was obtained from hypercohomology of the Koszul complex (see [49])

$$\mathcal{K}_{a,b} : \mathcal{O}(-4) \rightarrow \mathcal{O}(-2)^2 \rightarrow \mathcal{O}$$

- The global function ring of $V(a, b)$, vanishing scheme for homogeneous ideal (a, b)
- The global function ring of the smallest $\mathbb{Z}$ -scheme having 4 geometric points in $\mathbb{P}^2$ determined by the quartic ring

Among those, only the last ring can be made into a morphism $\operatorname{Spec} R \rightarrow \mathbb{P}_{\mathbb{Z}}^2$ such that its geometric points coincide with the solution of $a(x_1, x_2, x_3) = b(x_1, x_2, x_3) = 0$. This example illustrates that many of those objects arising naturally from algebraic geometry over $\mathbb{Z}$ are not consistent with ring parametrization models.

Another examples of properties no longer satisfied by objects over integral base $\mathbb{Z}$ would be finiteness properties stating that there are finitely many objects up to the bounded invariant. Classical results on the finiteness of nondegenerate rings with bounded discriminant and finiteness of its ideal class groups are examples of those. Birch and Merriman's work [9] suggests that binary n -ic forms with bounded discriminant (on a number field except finite places) are finite, and this implies there are finitely many embeddings $\mathrm{Spec} R \rightarrow \mathbb{P}^1$ up to PGL_2 -equivalence of a rank n ring R . It is not hard to show, however, that there are infinitely many $\mathrm{PGL}_{m+1}(\mathbb{Z})$ -nonequivalent embeddings $\mathrm{Spec} R \rightarrow \mathbb{P}^m$ for $m \geq 2$. This infinitude of embeddings implies that $\mathbb{Z}$ -schemes in $\mathbb{P}^m$ related to n geometric points contain more information than underlying rings and ideal classes. What does this additional information consist of, which might help classifying maps of $\mathrm{Spec} R$ into a projective space? Is it possible to establish analogue of Birch-Merriman finiteness result for some suitable limited class of embeddings? At this point we are not ready to give satisfactory answers for those kind of questions yet. In this chapter, we would like to consider various objects one can possibly relate to n geometric points in $\mathbb{P}^m$, and explore their behaviors different from geometric properties. This chapter is structured as follows.

In Section 7.2 we define several notions which generalizes n Galois conjugate $\overline{\mathbb{Q}}$ -points, including $\mathbb{Z}$ -subschemes and maps $\mathrm{Spec} R \rightarrow \mathbb{P}^m$. We will establish the condition when those two coincide, which allows us to provide a limited classification for those objects. Those all turns out to be different from the hypercohomology construction for ring parametrization models and the canonical point construction.

Section 7.3 will discuss particular example of n $\mathbb{Z}$ -valued points in $\mathbb{P}^m$, or maps $\mathrm{Spec} \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ equivalently. Those maps are easiest to classify among all maps $\mathrm{Spec} R \rightarrow \mathbb{P}_{\mathbb{Z}}^m$, and one can even give a criterion for such maps to be closed embedding. They are, however, still enough to provide many counterexamples to naive generalizations of properties of n geometric points. In particular we will show there

are infinitely many nonequivalent closed embeddings $\mathrm{Spec} \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ for $m \geq 2$ and $2 \leq n \leq 2^m - 1$, contrary to the finiteness of embeddings $\mathrm{Spec} R \rightarrow \mathbb{P}^1$ shown in [9].

7.2 Various notions related to n points in $\mathbb{P}^m$

We can consider the following objects as generalizations of notions of n points in $\mathbb{P}^m$ over integral base.

- Subschemes $X \subseteq \mathbb{P}_{\mathbb{Z}}^m$ such that X has n geometric points, and global functions $\Gamma(X)$
- Maps $\mathrm{Spec} R \rightarrow \mathbb{P}_{\mathbb{Z}}^m$
- Canonical points with respect to $\mathbb{Z}$ basis $1, \alpha_1, \dots, \alpha_{n-1}$ of R (will be defined in Section 7.2.4)
- Complex $\mathcal{C}$ having form of free resolution of n points, and its hypercohomology ring (as given in Chapter 2)

Over the rationals (when we consider base change $\otimes \mathbb{Q}$ for schemes and maps and we consider rank n $\mathbb{Q}$ -algebras instead of rank n rings), those objects all can be identified with n distinct $\overline{\mathbb{Q}}$ -points in $\mathbb{P}^m$ which are Galois conjugate. We will first briefly review properties of those geometric points over $\mathbb{Q}$, and move on to each notions listed above.

7.2.1 Geometric points

Proposition 7.2.1. *For the set $\{p_1, p_2, \dots, p_n\}$ of distinct $\overline{\mathbb{Q}}$ -points in $\mathbb{P}_{\mathbb{Z}}^m$, the followings are equivalent.*

1. *Those n points are Galois conjugate. That is, for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, σ acts as permutation on the set of points.*

2. There exists a nondegenerate $\mathbb{Q}$ -algebra R of rank n and elements $\alpha_0, \alpha_1, \dots, \alpha_m$ of R such that, for distinct embeddings $\rho_1, \dots, \rho_n : R \rightarrow \overline{\mathbb{Q}}$, coordinates of p_i are given as

$$p_i = [\rho_i(\alpha_0); \rho_i(\alpha_1); \dots; \rho_i(\alpha_m)].$$

3. There exists a closed embedding $\text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Q}}^m$ such that its image has $\overline{\mathbb{Q}}$ -points of $\{p_1, \dots, p_n\}$.

Proof. ((1) $\Rightarrow$ (2)) Denote the projective coordinates of the points as $p_i = [x_{i0}; \dots; x_{im}]$.

We can assume that all points have their first coordinates nonzero by applying some coordinate change by $\text{GL}_{m+1}(\mathbb{Q})$, and let $x_{i0} = 1$.

We construct R as subset of $\overline{\mathbb{Q}}^n$. First define the following $m+1$ elements in $\overline{\mathbb{Q}}^n$

$$\alpha_i = (x_{1i}, x_{2i}, \dots, x_{ni}), \quad 0 \leq i \leq m.$$

Especially we have $\alpha_0 = (1, 1, \dots, 1)$. Define $R = \mathbb{Q}[\alpha_0, \alpha_1, \dots, \alpha_m] \subset \overline{\mathbb{Q}}^n$ as subring of $\overline{\mathbb{Q}}^n$. We show that R is rank n nondegenerate algebra over $\mathbb{Q}$ in two steps.

- The dimension of R as $\mathbb{Q}$ -vector space is at most n :

First we note that Galois group acts on elements of F as permutation. Suppose $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on points p_i as permutation τ , that is, $\sigma p_i = p_{\tau(i)}$. If we consider explicit coordinates of p_i , we have

$$\sigma p_i = \sigma([1; x_{i1}; \dots; x_{im}]) = [1; x_{j1}; \dots; x_{jm}] \quad (j = \tau(i))$$

so it follows that $\sigma(x_{i1}) = x_{\tau(i)}$ and

$$\sigma(\alpha_k) = \sigma(x_{1i}, x_{2i}, \dots, x_{ni}) = (x_{\tau(1)i}, \dots, x_{\tau(n)i}).$$

Thus we have $\sigma \alpha_k = P_{\tau} \alpha_k$, where P_{τ} is the permutation of n coordinates by τ .

As P_τ is ring homomorphism, $\sigma v = P_\tau v$ should hold for every $v \in R$.

We show that any $n + 1$ elements of R are linearly dependent over $\mathbb{Q}$. For elements $v_1, v_2, \dots, v_{n+1} \in R$, consider the $\overline{\mathbb{Q}}$ -linear subspace

$$L = \{(c_1, \dots, c_{n+1}) : \sum c_i v_i = 0\}.$$

Then for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have

$$\sigma(\sum c_i v_i) = \sum \sigma(c_i) \sigma(v_i) = \sum \sigma(c_i) P_\tau(v_i) = P_\tau(\sum \sigma(c_i) v_i)$$

so if $\sum c_i v_i = 0$ then $\sum \sigma(c_i) v_i = 0$ for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This shows that L is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -invariant. As L is nonempty ($n + 1$ elements in $\overline{\mathbb{Q}}^n$ should be linearly dependent over $\overline{\mathbb{Q}}$), L contains at least one rational element.

- There are distinct n ring homomorphisms $\rho_i : R \rightarrow \overline{\mathbb{Q}}$:

Define ρ_i as composition of the inclusion $R \rightarrow \overline{\mathbb{Q}}^n$ and the projection to i th coordinate. We have $\rho_i(\alpha_j) = x_{ij}$, therefore

$$p_i = [1; \rho_i(\alpha_1); \dots; \rho_i(\alpha_m)].$$

As p_i s are distinct, ρ_i s should be distinct too.

Those imply $\dim_{\mathbb{Q}} R = n$. Also R is nondegenerate, as we have n distinct ring homomorphisms $\rho_i : R \rightarrow \overline{\mathbb{Q}}$.

((2) $\Rightarrow$ (1)) For $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the composition of $\rho_i : R \rightarrow \overline{\mathbb{Q}}$ and $\sigma : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}$ is another embedding ρ_j . Thus σ maps p_i to p_j .

((2) $\Leftrightarrow$ (3)) Represent R as direct sum of number fields $R \equiv \oplus F_i$. Then $\text{Spec } R$

consists of disjoint union of $\text{Spec } F_i$, and closed embedding $\text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Q}}^m$ corresponds to disjoint union of F_i -valued points of $\mathbb{P}_{\mathbb{Q}}^m$. Meanwhile, an F_i -valued point of $\mathbb{P}_{\mathbb{Q}}^m$ is form of $[\alpha_0; \alpha_1; \cdots; \alpha_m]$ for $\alpha_j \in F_i$, and its $\overline{\mathbb{Q}}$ -valued points are of form $[\rho(\alpha_0); \rho(\alpha_1); \cdots; \rho(\alpha_m)]$ for embeddings $\rho : F_i \rightarrow \overline{\mathbb{Q}}$. As all embeddings $\rho_1, \cdots, \rho_n$ of R embeddings of F_i for some i , one can see the union of those $\overline{\mathbb{Q}}$ -valued points exactly become points of the form $[\rho_i(\alpha_0); \rho_i(\alpha_1); \cdots; \rho_i(\alpha_m)]$. $\square$

Maps $\text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Q}}^m$ are in one-to-one correspondence with its set of $\overline{\mathbb{Q}}$ -points described as $p_i = [\rho_i(\alpha_0); \cdots; \rho_i(\alpha_m)]$ for $\alpha_j \in R$, and this map becomes closed embedding if and only if all p_i are distinct. In particular, if p_i s are all distinct, then their union as $\mathbb{Q}$ -scheme is affine.

7.2.2 $\mathbb{Z}$ -subschemes containing given geometric points

For the set $\{p_1, \cdots, p_n\}$ of Galois conjugate $\overline{\mathbb{Q}}$ -points of $\mathbb{P}^m$, denote $I_m = I_m(p_1, \cdots, p_n)$ as a homogeneous ideal of $\mathbb{Z}[x_0, \cdots, x_m]$ generated by all homogeneous polynomials vanishing at $p_1, \cdots, p_n$. It is clear that any closed subscheme $V(I)$ whose $\overline{\mathbb{Q}}$ -points include $p_1, \cdots, p_n$ satisfy $I \subset I_m$, as elements of I should be evaluated as zero at points of $V(I)$. This suggests that $V(I_m)$ is the smallest closed $\mathbb{Z}$ -subscheme of $\mathbb{P}_{\mathbb{Z}}^m$ whose $\overline{\mathbb{Q}}$ -points include $p_1, \cdots, p_n$.

Meanwhile we just observed that there exists a closed embedding of $\mathbb{Q}$ -scheme $\text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Q}}^m$ whose image consists of points $p_1, \cdots, p_n$, for some nondegenerate $\mathbb{Q}$ -algebra R . If we denote $I_{\mathbb{Q}}$ as vanishing homogeneous $\mathbb{Q}$ -ideal for image of this closed embedding, then $I_m = I_{\mathbb{Q}} \cap \mathbb{Z}[x_0, \cdots, x_m]$ and $I_m \otimes \mathbb{Q} = I_{\mathbb{Q}}$ holds, and it follows that $V(I_m) \otimes \mathbb{Q} \subset \mathbb{P}_{\mathbb{Q}}^m$ is isomorphic to $\text{Spec } R$.

Definition 7.2.2. Call $I_m(p_1, \cdots, p_n)$ as vanishing ($\mathbb{Z}$ -)ideal for $p_1, \cdots, p_n$, and denote $V(I_m)$ as smallest ($\mathbb{Z}$ -)scheme containing $p_1, \cdots, p_n$. Call the global function ring for $V(I_m)$ as global ($\mathbb{Z}$ -)functions for $p_1, \cdots, p_n$.

One can also think of homogeneous idelas I satisfying $I_m \otimes \mathbb{Q} \simeq I \otimes \mathbb{Q}$. In this case it also holds that $V(I) \otimes \mathbb{Q} \simeq V(I_m) \otimes \mathbb{Q} \simeq \text{Spec } R$. Call any such ideal I as *integral ideal* for $p_1, \dots, p_n$, and write $\mathcal{I}(p_1, \dots, p_n)$ as set of all integral ideals.

The correspondence from $\mathcal{I}$ to subschemes given as $I \mapsto V(I)$ is not one-to-one. If I_1, I_2 are homogeneous ideals such that their degree d components are all identical for large enough d , then we have $V(I_1) \simeq V(I_2)$. For example, two ideals $(xy), (2xy, x^2y, xy^2) \in \mathcal{I}([1; 0], [0; 1])$ have identical components for degrees $d \geq 3$, so $V(xy) = V((2xy, x^2y, xy^2))$ follows. It is not even clear that $V(I)$ for $I \in \mathcal{I}$ gives all $\mathbb{Z}$ -scheme X such that $X \otimes \mathbb{Q} \simeq V(I_m)$. For these reasons, integral ideals for $p_1, \dots, p_n$ may not be suitable for representing all such $\mathbb{Z}$ -schemes. Nevertheless they appear often in ring parametrization models together with their vanishing schemes.

Proposition 7.2.3. *Let I, I' be elements of $\mathcal{I}(p_1, \dots, p_n)$.*

1. *For R be the nondegenerate $\mathbb{Q}$ -algebra of global functions at $p_1, \dots, p_n$, we have $\Gamma(V(I)) \otimes \mathbb{Q} \simeq F$ and $\Gamma(V(I)) \subset F$ is nondegenerate ring of rank n .*
2. *If $I \subset I'$, then $\Gamma(V(I))$ is subring of $\Gamma(V(I'))$ of finite index. In particular, $\Gamma(V(I))$ is subring of $\Gamma(V(I_m))$ of finite index.*

Proof. (1) We have already seen that $V(I) \otimes \mathbb{Q} \simeq V(I_m) \otimes \mathbb{Q} \simeq \text{Spec } F$. As taking global function commutes with flat base change, $\Gamma(V(I)) \otimes \mathbb{Q} \simeq F$. We only need to show that $\Gamma(V(I))$ has no torsion. As it sits inside $\Gamma(V(I_m))$ (shown below), it suffices to show $\Gamma(V(I_m))$ has no torsion. A torsion element $u \in \Gamma(V(I_m))$ should be evaluated as zero at every $\overline{\mathbb{Q}}$ -points p_i . But I_m contains any homogeneous polynomials which vanishes on every points, so if u is expressed as $u = g/x_i^L$ on affine open cover $D(x_i)$ (g : homogeneous polynomial of degree L) we should have $g \in I[x_i^{-1}]$ and u is equivalent to zero on $D(x_i)$. Therefore u should be zero.

(2) If $I_1 \subset I_2$, then the induced map $V(I_2) \rightarrow V(I_1)$ is closed embedding, so the map

of global functions $\Gamma(V(I_1)) \rightarrow \Gamma(V(I_2))$ is inclusion. As both rings have rank n , inclusion should be of finite index. $\square$

The global function ring for I_m can be found using the following criterion.

Proposition 7.2.4. *Let $p_1, \dots, p_n$ be Galois conjugate n $\overline{\mathbb{Q}}$ -points in $\mathbb{P}_{\mathbb{Z}}^m$. Consider a map $\varphi : \Gamma(V(I_m)) \rightarrow \overline{\mathbb{Q}}^n$ whose j th coordinate is given as evaluation at p_j . Then φ is injective ring embedding whose image is dimension n lattice with its volume equal to $\sqrt{\text{Disc}(\Gamma(V(I_m)))}$.*

An element $(u_1, \dots, u_n) \in \overline{\mathbb{Q}}^n$ is in image of φ if and only if it satisfies the following condition for all $j = 0, \dots, m$; there exists a polynomial f_j in $x_0/x_j, \dots, x_m/x_j$ such that: evaluation of f_j at p_k coincides with u_k if x_j -coordinate of p_k is nonzero. ()*

Proof. Note that $V(I_m) \otimes \mathbb{Q} \simeq \text{Spec} R$ is reduced, so global function which is evaluated as zero at all points is zero. Therefore φ is injective. If we consider the base change $V(I_m) \otimes \overline{\mathbb{Q}} \simeq \text{Spec} \overline{\mathbb{Q}}$, then $\Gamma(V(I_m)) \otimes \overline{\mathbb{Q}} \simeq \overline{\mathbb{Q}}^n$, and the isomorphism can be given in the way that its j th coordinate corresponds to evaluation at p_j . This shows that $\mathbb{Z}$ -basis elements of $\Gamma(V(I_m))$ are mapped by φ to linearly independent elements of $\overline{\mathbb{Q}}^n$ (otherwise $\Gamma(V(I_m)) \otimes \overline{\mathbb{Q}}$ do not span all $\overline{\mathbb{Q}}^n$), so image of $\Gamma(V(I_m))$ form lattice inside $\overline{\mathbb{Q}}^n$. For the statement regarding volume of the image, it is known that the discriminant of subring of $\overline{\mathbb{Q}}^n$ forming lattice is same as square of volume of the lattice.

The condition for $(u_1, \dots, u_n) \in \overline{\mathbb{Q}}^n$ to be in image of φ is the equivalent statement for that global function on X comes from set of global functions $f_j \in \Gamma(X \cap D(x_j))$ coinciding on intersections of domain. Global functions of

$$V(I_m) \cap D(x_j) \simeq \text{Spec } \mathbb{Z}[x_0/x_j, \dots, x_m/x_j]/(I_m[x_j^{-1}])_0$$

can be represented as a polynomial in $x_0/x_j, \dots, x_m/x_j$. If $f_0, \dots, f_m$ determines global function taking values $(u_1, \dots, u_n)$ at points $p_1, \dots, p_n$ respectively, then the condition (*) is automatically satisfied. Conversely we show that (*) is sufficient: we

should show that (*) implies that $f_i - f_j \in I_m[x_i^{-1}, x_j^{-1}]$ for any i, j . Pick integer M such that $x_i^M f_i, x_j^M f_j$ both becomes polynomial in $x_0, \dots, x_m$. We claim

$$x_i x_j^{M+1} (x_i^M f_i) - x_i^{M+1} x_j (x_j^M f_j) \in I_m.$$

This can be proven by showing $x_i x_j^{M+1} (x_i^M f_i) - x_i^{M+1} x_j (x_j^M f_j)$ vanishes at p_k for every k . If either one of x_i -or x_j -coordinate of p_k vanishes, then it vanishes regardless of the polynomial $x_i^M f_i$ and $x_j^M f_j$. If both x_i -and x_j -coordinate of p_k does not vanish, we can divide by $x_i^{M+1} x_j^{M+1}$ on p_k and see its evaluation at p_k vanishes due to the condition (*). $\square$

For the case when all p_j s have nonzero x_i -coordinates for all i , we can describe the image of φ as intersection $\cap L_i$, where L_i is the image of

$$\varphi_i : \mathbb{Z}[x_0/x_i, \dots, x_m/x_i] \rightarrow \overline{\mathbb{Q}}^n$$

whose j th coordinate is given as evaluation at p_j .

Example. We calculate the global $\mathbb{Z}$ -function ring of two $\mathbb{Q}$ -points $(p_1, p_2) = ([1; 0], [a; b])$ of $\mathbb{P}_{\mathbb{Z}}^1 = \text{Proj } \mathbb{Z}[x, y]$. (a, b : coprime integers) Consider the evaluation map $\varphi : \Gamma(V(I_m)) \rightarrow \overline{\mathbb{Q}}^2$ as above. As evaluations of integral polynomials in x_i/x_j at $\mathbb{Q}$ -points should be rational, image of φ sits inside $\mathbb{Q}^2$. Also evaluations of global function elements should be integral, so the image of φ is subring of $\mathbb{Z}^2$. Call this image R .

A global function on $D(x)$ can be represented as form $f(y/x)$ for polynomial f , and its evaluations at p_1 and p_2 are given as $f(0)$ and $f(b/a)$ respectively. Then $f(b/a) - f(0)$ is divisible by b in $\mathbb{Z}[a^{-1}]$. As $f(b/a)$ and $f(0)$ are both integers and $\gcd(a, b) = 1$, $f(b/a) - f(0)$ is divisible by b in $\mathbb{Z}$. Therefore R is subset of $\{(u_1, u_2) \in \mathbb{Z}^2 : b|u_2 - u_1\}$. Meanwhile the pair $(0, b)$ is in R , as polynomials $f_1 = ay/x$

and $f_2 = 0$ can give these evaluations on $D(x)$ and $D(y)$ respectively. This shows

$$R = \{(u_1, u_2) \in \mathbb{Z}^2 : b|u_2 - u_1\}$$

and its discriminant is b^2 . One can also check $I_m = y(bx - ay)$, and applying the hypercohomology construction following the generalized Gauss composition procedure in [51] gives same results for $\Gamma(V(I_m))$.

Example. The four $\mathbb{Q}$ -points $[1; 0; 0]$, $[0; 1; 0]$, $[0; 0; 1]$, $[1; 1; 2]$ are $\mathbb{Q}$ -points of the scheme cut out by two quadrics $2xy - xz, xz - yz$. Let $I = (2xy - xz, xz - yz)$. It turns out that the ring of global $\mathbb{Z}$ -functions on those four points is isomorphic to $\mathbb{Z}^4$. For the following elements of $\vec{u} = (u_1, u_2, u_3, u_4) \in \mathbb{Z}^4$, we can give f_1, f_2, f_3 as representatives for global functions on $D(x), D(y), D(z)$ respectively such that their evaluations at $p_1, \dots, p_4$ (if evaluation can be defined) give $\vec{u}$.

$$\begin{aligned}\vec{u} &= (1, 0, 0, 0), & (f_1, f_2, f_3) &= \left(1 - \frac{y}{x}, 0, 0\right) \\ \vec{u} &= (0, 1, 0, 0), & (f_1, f_2, f_3) &= \left(0, 1 - \frac{x}{y}, 0\right) \\ \vec{u} &= (0, 0, 1, 0), & (f_1, f_2, f_3) &= \left(0, 0, 1 - \frac{2y}{z}\right) \\ \vec{u} &= (1, 1, 1, 1), & (f_1, f_2, f_3) &= (1, 1, 1)\end{aligned}$$

As global function ring should be subring of $\mathbb{Z}^4$ (evaluations of global functions should be integral), this shows the global function ring is exactly $\mathbb{Z}^4$.

Meanwhile $V(I)$ can be calculated from the hypercohomology of the Koszul complex obtained from $2xy - xz, xz - yz$, which gives the quartic ring corresponding to two quadrics according to the quartic Higher Composition Law of [2]. This quartic ring is different from $\mathbb{Z}^4$. This is because I_m is $(2xy - xz, xz - yz, x^2y - xy^2)$ and $V(I)$ is not isomorphic to $V(I_m)$.

7.2.3 Maps from $\text{Spec } R$ to $\mathbb{P}_{\mathbb{Z}}^m$

Our discussions so far were about schemes in $\mathbb{P}_{\mathbb{Z}}^m$ having n geometric points and its global function ring. For such schemes, for $R = \Gamma(V(I))$, natural maps $V(I) \rightarrow \text{Spec } R$ can be induced from ring map $R \rightarrow \Gamma(V(I))$. Now, we consider maps in opposite direction $\text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Z}}^m$. By combining two perspectives on maps in two direction, we would like to consider when these maps become isomorphism.

In general, maps from scheme X to $\mathbb{P}_{\mathbb{Z}}^m$ correspond to line bundle $\mathcal{L}$ and sections $s_0, \dots, s_m$ having no common zeros, up to isomorphism of line bundles. If $X = \text{Spec } R$ for R nondegenerate ring of rank n , line bundles on X correspond to R -module M which is locally of rank 1 over R , and we would like to classify all such R -modules.

Let $R \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \oplus F_i$ for F_i number fields. As $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is $R \otimes_{\mathbb{Z}} \mathbb{Q}$ -module, we can write $M \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \oplus V_i$ where V_i is F_i -vector space. Above the infinite prime of R above F_i (explicitly given as kernel of $R \rightarrow R \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow F_i$), the residual part of M on it is V_i , and it should be isomorphic to F_i (by locally rank 1 assumption). So $M \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \oplus F_i \simeq R \otimes_{\mathbb{Z}} \mathbb{Q}$ follows. As locally freeness implies flatness, M is flat thus torsion free, so we can identify M as subset of $R \otimes_{\mathbb{Z}} \mathbb{Q}$ via inclusion $M \hookrightarrow M \otimes_{\mathbb{Z}} \mathbb{Q} \simeq R \otimes_{\mathbb{Z}} \mathbb{Q}$.

For the general case when R is not a maximal order of a number field, there are R -submodules $M \subset R \otimes_{\mathbb{Z}} \mathbb{Q}$ having $\mathbb{Z}$ -rank n which are not rank 1 over some places of R . We give the following characterization for locally rank 1 R -modules.

Definition 7.2.5. *A R -submodule $M \subseteq R \otimes_{\mathbb{Z}} \mathbb{Q}$ having $\mathbb{Z}$ -rank n is invertible ideal if there exists another R -submodule N having $\mathbb{Z}$ -rank n which satisfies $MN = R$.*

Proposition 7.2.6. *Let R be a rank n nondegenerate ring. A R -submodule $M \subset R \otimes_{\mathbb{Z}} \mathbb{Q}$ is locally free of rank 1 if and only if M is invertible ideal.*

Proof. ($\Leftarrow$) Suppose if M is invertible ideal and let $MN = R$. Pick $a_i \in M, b_i \in N$

such that $1 = \sum a_i b_i$. For any prime $\mathfrak{p}$ of R there exists $a_i b_i$ which is not in $\mathfrak{p}$, thus invertible in $R_{\mathfrak{p}}$. The maps between $M_{\mathfrak{p}}$ and $R_{\mathfrak{p}}$ given as $x \mapsto (b_i x)$ and $r \mapsto ((a_i b_i)^{-1} r) a_i$ respectively become inverse on $\mathfrak{p}$.

($\Rightarrow$) Take $N = \{x \in R \otimes_{\mathbb{Z}} \mathbb{Q} : xM \subseteq R\}$ and observe it is R -submodule. We claim $MN = R$. Suppose $MN \neq R$, and pick a maximal ideal $\mathfrak{p}$ containing MN . As both M and N are of full rank, $\mathfrak{p}$ has finite residue field.

Let a be the image of $1 \in R$ under the identification of $R_{\mathfrak{p}} \simeq M_{\mathfrak{p}}$, and let $a_1, \dots, a_n$ be $\mathbb{Z}$ -basis of M . Then each a_i are $R_{\mathfrak{p}}$ -multiples of a . Write $a_i = (b_i/s_i)a$ for $s_i \notin \mathfrak{p}$ and set $s = s_1 s_2 \cdots s_n$. Then $sa^{-1}a_i \in R$ for all i so $sa^{-1} \in N$, $s = a(sa^{-1}) \in MN \subset \mathfrak{p}$ so contradiction. $\square$

Analogous to the case when R is order, invertible ideals form a group under ideal multiplication, and we can define *invertible ideal class group* as its quotient by principal invertible ideals. This group is naturally isomorphic to the Picard group of $\text{Spec } R$.

Let $M \subset R \otimes \mathbb{Q}$ be an invertible ideal, and let $s_0, s_1, \dots, s_m \in M$ be global sections. The 'zero' of section s can be considered as a prime ideal $\mathfrak{p} \subset R$ such that s is zero on $M \otimes (R/\mathfrak{p})$, or equivalently $(M/Rs) \otimes (R/\mathfrak{p}) \neq 0$. Thus the common zeros of $s_0, \dots, s_m$ are points on which $M/(Rs_0 + \dots + Rs_m)$ is nontrivial. Therefore the condition that $s_0, s_1, \dots, s_m$ having no common zeros is equivalent to that M is generated by $s_0, \dots, s_m$ over R . Thus the morphism from $\text{Spec } R$ to $\mathbb{P}_{\mathbb{Z}}^m$ correspond to the data

$$\{(M, (s_0, \dots, s_m)) : \\ M : \text{ideal class of } R, \quad (s_0, s_1, \dots, s_m) : R - \text{generators of } M.\}$$

Two M isomorphic as R -module give the same morphism, and changing (s_i) by $\text{GL}_{m+1}(\mathbb{Z})$ action would have same effect on $\mathbb{P}_{\mathbb{Z}}^m$ by $\text{PGL}_{m+1}(\mathbb{Z})$ action.

Proposition 7.2.7. *Suppose we are given a morphism $\text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ from an invertible ideal M and its sections $(s_0, \dots, s_m)$ generating M as R -module. Let X be its scheme-theoretic image. Then the vanishing ideal for X is given as*

$$I = (\phi \in \mathbb{Z}[x_0, \dots, x_m] : \phi \text{ homogeneous, } \phi(s_0, s_1, \dots, s_m) = 0 \text{ in } M \otimes_{\mathbb{Z}} \mathbb{Q}).$$

For n morphisms $\rho_i : R \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \overline{\mathbb{Q}}$, X has $\overline{\mathbb{Q}}$ -points

$$p_i = [\rho_i(s_0); \rho_i(s_1); \dots; \rho_i(s_m)].$$

Those points, distinct or not, are all $\overline{\mathbb{Q}}$ -points of X . And X is the smallest $\mathbb{Z}$ -scheme having those $\overline{\mathbb{Q}}$ -points.

Proof. We first consider the case $M = R$. In this case the map is factored as

$$\text{Spec } R \rightarrow \text{Spec } \mathbb{Z}[x_0, \dots, x_m] \rightarrow \text{Proj } \mathbb{Z}[x_0, \dots, x_m]$$

where the map $\text{Spec } R \rightarrow \mathbb{Z}[x_0, \dots, x_m]$ given by $x_i \mapsto s_i$. Thus the scheme-theoretic image is naturally obtained from homogeneous ideal

$$I = \{\phi \in \mathbb{Z}[x_0, \dots, x_m] : \phi \text{ homogeneous, } \phi(s_0, s_1, \dots, s_m) = 0\}.$$

For the general case, the vanishing ideal is obtained by combining all local information. On the open subset $\text{Spec } R[s^{-1}] \subset \text{Spec } R$ where M is trivial, we can consider isomorphism $\psi : M[s^{-1}] \simeq R[s^{-1}]$ and the map $\text{Spec } R[s^{-1}] \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ is given by composition of

$$\text{Spec } R[s^{-1}] \rightarrow \text{Spec } \mathbb{Z}[x_0, \dots, x_m] \rightarrow \text{Proj } \mathbb{Z}[x_0, \dots, x_m]$$

where $x_0 \mapsto \psi(s_i)$. Thus the image of $\text{Spec} R[s^{-1}]$ is given as

$$\{\phi \in \mathbb{Z}[x_0, \dots, x_m] : \phi \text{ homogeneous, } \phi(\psi(s_0), \psi(s_1), \dots, \psi(s_m)) = 0 \text{ in } R[s^{-1}]\}.$$

But if we regard M as subset of $R \otimes \mathbb{Q}$, the isomorphism $\psi : M[s^{-1}] \simeq R[s^{-1}]$ should be given as multiplication by invertible elements in $R \otimes \mathbb{Q}$. So the condition $\phi(\psi(s_0), \psi(s_1), \dots, \psi(s_m)) = 0$ is equivalent to $\phi(s_0, \dots, s_m) = 0$ in $R[s^{-1}] \otimes \mathbb{Q}$. Taking union for all such subset $\text{Spec } R[s^{-1}]$ covers $\text{Spec } R$, so the image of $\text{Spec } R$ is given as their union

$$I = (\phi \in \mathbb{Z}[x_0, \dots, x_m] : \phi \text{ homogeneous, } \phi(s_0, s_1, \dots, s_m) = 0 \text{ in } M \otimes_{\mathbb{Z}} \mathbb{Q}).$$

We can note that I is the vanishing ideal for points $p_1, \dots, p_n$. □

We would like to discuss when a map $\phi : \text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ becomes closed embedding. If it is closed embedding, its scheme-theoretic image X would be isomorphic to $\text{Spec } R$ via ϕ . Composing $\phi : \text{Spec } R \rightarrow X$ with $X \rightarrow \text{Spec } \Gamma(X, \mathcal{O}_X)$ gives a map $\text{Spec } R \rightarrow \text{Spec } \Gamma(X, \mathcal{O}_X)$, which induces ring homomorphism $\Gamma(X, \mathcal{O}_X) \rightarrow R$. If it is isomorphism, then ϕ should be isomorphism. Therefore ϕ is closed embedding if and only if $\Gamma(X, \mathcal{O}_X) \simeq R$.

If subscheme $X = V(I)$ is affine, then we have isomorphism $\text{Spec } \Gamma(X, \mathcal{O}_X) \rightarrow X$ and its composition with $X \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ is closed embedding. Thus $X = V(I)$ is affine if and only if there exists a map $\text{Spec } \Gamma(X, \mathcal{O}_X) \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ whose scheme-theoretic image is X . In particular, affine schemes should only be those which can correspond to vanishing ideal I_m . Conversely for $X = V(I_m)$, we can take $R = \Gamma(X, \mathcal{O}_X)$ and $M = \mathcal{O}_X(1)$ and give a map $\text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ using generators of M . One can check that its scheme-theoretic image is X .

Proposition 7.2.8. *For a set P of n Galois conjugate points and the vanishing ideal*

I_m for those points, let I be integral ideal such that $I \otimes_{\mathbb{Z}} \mathbb{Q} \simeq I_m \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $X = V(I)$ is affine if and only if $X \simeq V(I_m)$. In general, $\Gamma(X)$ is subring of $\Gamma(V(I_m))$, and equality holds if and only if $X \simeq V(I_m)$.

We give another criterion for a map to become closed embedding.

Proposition 7.2.9. *For an invertible ideal M and its sections $(s_0, \dots, s_m)$ of a rank n ring R , let X be a scheme-theoretic image of the map $\varphi : \text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ formed by M and $(s_0, \dots, s_m)$. Consider M as a subset of $R \otimes_{\mathbb{Z}} \mathbb{Q}$, and suppose that the following criterion*

$$R \subset \mathbb{Z} \left[\frac{s_0}{s_i}, \frac{s_1}{s_i}, \dots, \frac{s_m}{s_i} \right] \text{ in } (R \otimes_{\mathbb{Z}} \mathbb{Q})[s_i^{-1}]$$

holds for all i . Then φ is closed embedding and X is isomorphic to $\text{Spec } R$.

Proof. For any $r \in R$ we can pick polynomials $f_i(s_j/s_i)$ such that it represents r in $(\text{Spec } R)_{s_i}$, so that f_i and f_j glue over $(\text{Spec } R)_{s_i s_j}$. The set of f_i determines a global function which coincides with r . Thus we can give inclusion $R \rightarrow \Gamma(X, \mathcal{O}_X)$, where X is the image of the map $\text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Z}}^m$. But the composition $\text{Spec } R \rightarrow X \rightarrow \text{Spec}(\Gamma(X, \mathcal{O}_X))$ gives $\Gamma(X, \mathcal{O}_X) \rightarrow R$, so it should follow that $\Gamma(X, \mathcal{O}_X) = R$ and the map $\text{Spec } R \rightarrow X$ is isomorphism. $\square$

7.2.4 Canonical n points in $\mathbb{P}^{n-2}$

There is a special correspondence of associating n Galois conjugate points in $\mathbb{P}^{n-2}$ (for $n \geq 3$), which was first suggested in [3]. The correspondence does not involve any algebraic geometry and can be carried out only using numerical calculations.

For a nondegenerate ring R of rank n and its basis $(1 = \alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{n-1})$, we can construct its dual basis α_i^* in $R \otimes \mathbb{Q}$ with respect to trace pairing - elements satisfying $\text{Tr}_R(\alpha_i \alpha_j^*) = \delta_{ij}$. Let $\rho_1, \rho_2, \dots, \rho_n$ be distinct embeddings $R \rightarrow \overline{\mathbb{Q}}$. Then

we can associate n Galois conjugate $\overline{\mathbb{Q}}$ -points in $\mathbb{P}^{n-2}$ as follows.

$$p_i = [\rho_i(\alpha_1^*); \rho_i(\alpha_2^*); \cdots; \rho_i(\alpha_n^*)]$$

Call this *canonical points* of R associated with basis $1, \alpha_1, \cdots, \alpha_{n-1}$.

Proposition 7.2.10. *For canonical point associated with basis $1, \alpha_1, \cdots, \alpha_{n-1}$, we have the following.*

1. *The canonical points do not change if we replace α_i by $\alpha_i + k$ for $k \in \mathbb{Z}$.*
2. *If a matrix $G \in \mathrm{GL}_{n-1}(\mathbb{Z})$ acts on $(\alpha_1, \cdots, \alpha_{n-1})^t$ as left multiplication, its action on $(\alpha_1^*, \cdots, \alpha_{n-1}^*)^t$ is left multiplication by $(G^{-1})^t$, and canonical points are mapped by the PGL_{n-1} -action induced by $(G^{-1})^t$.*
3. *Canonical points are in linearly general position (no $(n-1)$ points lie on a hyperplane).*

Proof. (1), (2) would be simple. For (3), we additionally prove the following (for future reference). For 'affine' points

$$z_i = (\rho_i(\alpha_1^*), \rho_i(\alpha_2^*), \cdots, \rho_i(\alpha_{n-1}^*))$$

we claim

$$\sum_{i=1}^n z_i = 0, \quad \det(z_2, z_3, \cdots, z_n) = \mathrm{Disc}(R)^{-1/2}.$$

The first part follows from that $\mathrm{Tr}(\alpha_i^*) = \mathrm{Tr}(\alpha_0 \alpha_i^*) = 0$. For the second part, note

that $\det(z_2, \dots, z_n)$ is $(1, 1)$ -principal minor of a matrix

$$M = (\rho_j(\alpha_{i-1}^*))_{i,j} = \begin{pmatrix} \rho_1(\alpha_0^*) & \rho_2(\alpha_0^*) & \cdots & \rho_n(\alpha_0^*) \\ \rho_1(\alpha_1^*) & \rho_2(\alpha_1^*) & \cdots & \rho_n(\alpha_1^*) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1(\alpha_{n-1}^*) & \rho_2(\alpha_{n-1}^*) & \cdots & \rho_n(\alpha_{n-1}^*) \end{pmatrix}.$$

But M has inverse $(\rho_j(\alpha_{i-1}))_{i,j}$ with determinant $\text{Disc}(R)^{1/2}$, and its $(1, 1)$ -entry has value 1. So the result follows from the Cramer's rule. $\square$

The first two properties imply that canonical points only depend on the choice of $\mathbb{Z}$ -basis of $R/\mathbb{Z}$, and the GL_{n-1} -action on basis of $R/\mathbb{Z}$ translates to the PGL -action on canonical points. Thus the equivalence class of canonical points up to $\text{PGL}_{n-1}(\mathbb{Z})$ -action is uniquely determined for a ring R .

We show that its converse holds for primitive rings. Recall that a ring R of rank n is *primitive* if there are no rings R_1 of rank n and integers $m > 1$ such that $R \simeq \mathbb{Z} + mR_1$. A *primitive closure* of a rank n ring R is a unique primitive ring containing R .

Proposition 7.2.11. *Two rings have same canonical points up to $\text{PGL}_{n-1}(\mathbb{Z})$ -isomorphism if and only if their primitive closures are isomorphic.*

Proof. Suppose that T is primitive closure of R . Let (α_i) be the basis of T ($\alpha_0 = 1$), then basis of R can be given as $(1, c\alpha_1, \dots, c\alpha_n)$ where $c = c(R)$ is content of R . Then the dual basis corresponding to $(1, c\alpha_1, \dots, c\alpha_n)$ is $(\beta, \alpha_1^*/c, \dots, \alpha_n^*/c)$, so R has same canonical points as T .

It remains to prove two primitive rings with same canonical points are isomorphic. Suppose R_1, R_2 are primitive and has same canonical points. As we can represent coordinates of the points using both R_1 and R_2 , proposition 3.2 shows that $R_1 \otimes \mathbb{Q} \simeq R_2 \otimes \mathbb{Q}$. If we identify R_1 and R_2 as subset of $R_1 \otimes \mathbb{Q}$, we can let $G \in \text{GL}_{n-1}(\mathbb{Q})$

be a matrix transforming $\mathbb{Z}$ -basis of R_1 (including 1) to $\mathbb{Z}$ -basis of R_2 . As the PGL-transformation G induces on canonical point is identity, G should be multiple of identity.

Let $G = (p/q)I$ for p, q coprime, and let R_1 and R_2 have basis $\{1, p\beta_1, \dots, p\beta_{n-1}\}$ and $\{1, q\beta_1, \dots, q\beta_{n-1}\}$ respectively. For 'multiplication coefficients' $\beta_i\beta_j = c_{ij}^0 + \sum_k c_{ij}^k\beta_k$ which belong to in $\mathbb{Q}$, R_1 has pc_{ij}^k and $p^2c_{ij}^0$ for its multiplication coefficient, so $pc_{ij}^k, p^2c_{ij}^0 \in \mathbb{Z}$. But we also have $qc_{ij}^k, q^2c_{ij}^0 \in \mathbb{Z}$, so $c_{ij}^k, c_{ij}^0 \in \mathbb{Z}$ and $\mathbb{Z}$ -span of $\{1, q\beta_1, \dots, q\beta_{n-1}\}$ become subring inside it. As both R_1 and R_2 are primitive, the only possibility is $p = q = 1$. $\square$

Proposition 7.2.12. *Let $p_1, \dots, p_n$ be $\overline{\mathbb{Q}}$ -points in $\mathbb{P}_{\mathbb{Z}}^{n-2}$ which are Galois conjugate and are in general position. Then there exists a rank n ring R such that its canonical points are exactly $\{p_1, \dots, p_n\}$.*

Proof. Let X be union of $p_1, \dots, p_n$ as $\mathbb{Z}$ -subscheme. For $R_0 = \Gamma(X, \mathcal{O}_X)$, we can expect our desired ring to sit inside $R_0 \otimes \mathbb{Q}$. Let ρ_i ($i = 1, \dots, n$) be distinct homomorphisms $\rho_i : R_0 \otimes \mathbb{Q} \rightarrow \overline{\mathbb{Q}}$.

Let $\alpha_0 = 1, \alpha_1, \dots, \alpha_{n-1}$ be basis of R_0 , and let $q_i = [\rho_i(\alpha_1^*); \dots; \rho_i(\alpha_{n-1}^*)]$ be canonical points for R_0 . As X and union of q_i are isomorphic over $\mathbb{Q}$, there exists automorphism $G \in \text{PGL}_{n-1}(\mathbb{Q})$ which transforms union of q_i to X . Regard G as matrix in $\text{GL}_{n-1}(\mathbb{Q})$ and let $(\beta_1, \dots, \beta_{n-1})^t = G(\alpha_1, \dots, \alpha_{n-1})^t$.

Let $d_{ij}^k \in \mathbb{Q}$ be such that $\beta_i\beta_j = \sum d_{ij}^k\beta_k$ ($\beta_0 = 1$), and choose M such that $Md_{ij}^k \in \mathbb{Z}$ for all i, j, k . Then $(1, M\beta_1, M\beta_2, \dots, M\beta_{n-1})$ form a basis for a ring, and canonical point of R coincides with X . $\square$

Those two propositions show that the construction of canonical points gives a correspondence between a set of n Galois conjugate geometric points in $\mathbb{P}^{n-2}$ and rank n primitive rings R with $\mathbb{Z}$ -basis of $R/\mathbb{Z}$. If we consider equivalence classes by $\text{GL}_{n-1}(\mathbb{Z})$ -actions on $\mathbb{Z}$ -basis of $R/\mathbb{Z}$, it gives a correspondence between an $\text{PGL}_{n-1}(\mathbb{Q})$ -equivalence

class of n Galois conjugate geometric points in $\mathbb{P}^{n-2}$ and an isomorphism class of rank n primitive rings. This correspondence here, however, does not relate to scheme theoretic perspective directly. The global function ring corresponding to the vanishing ideal for canonical points of R usually differ from original R , and it may have different primitive closure than R too.

We give another justification on how those ‘canonical’ points arise naturally. For a nondegenerate rank n ring R , one can define its *inverse different* $\mathcal{D}_R^{-1}$ as

$$\mathcal{D}_R^{-1} = \{x \in R \otimes_{\mathbb{Z}} \mathbb{Q} : \forall y \in R \quad \text{Tr}_R(xy) \in \mathbb{Z}\}$$

Its $\mathbb{Z}$ -basis can be given as any dual basis $\alpha_0^*, \dots, \alpha_{n-1}^*$. When R is a ring of integers of a number field, this coincides with the classical definition of inverse different. However, for general R , inverse different may not be invertible ideal of R (so that we can’t define the *different* in this case, which is usually defined as inverse of the inverse different when R is a ring of integers).

If inverse different is invertible, one can consider a mapping $\text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Z}}^{n-2}$ using the elements $\alpha_1^*, \dots, \alpha_{n-1}^*$, provided that those span $\mathcal{D}_R^{-1}$ as R -module. Considering elements $\alpha_1^*, \dots, \alpha_{n-1}^*$ are equivalent to considering $\mathbb{Z}$ -basis of trace zero elements of $\mathcal{D}_R^{-1}$, so different choice of $\alpha_1^*, \dots, \alpha_{n-1}^*$ changes the mapping only by $\text{PGL}_{n-1}(\mathbb{Z})$ -action on $\mathbb{P}_{\mathbb{Z}}^{n-2}$.

Claim. *If $\mathcal{D}_R^{-1}$ is invertible, then $(\mathcal{D}_R^{-1}, (\alpha_1^*, \dots, \alpha_{n-1}^*))$ gives a well-defined closed embedding $\text{Spec } R \rightarrow \mathbb{P}_{\mathbb{Z}}^{n-2}$ whose $\overline{\mathbb{Q}}$ -points are same as canonical points.*

Proof. We need to show: (1) Well-definedness of embedding: $\alpha_1^*, \alpha_2^*, \dots, \alpha_{n-1}^*$ generate $\mathcal{D}_R^{-1}$ (as R -module). (2) It suffices to show that $\mathbb{Z}[\alpha_2^*/\alpha_1^*, \dots, \alpha_{n-1}^*/\alpha_1^*] = S$ contains $R[(\alpha_1^*)^{-1}]$, viewed as subset of $(R \otimes \mathbb{Q})[(\alpha_1^*)^{-1}]$.

(1) We first note that

$$x = \sum_{j=0}^{n-1} \text{Tr}_R(\alpha_j x) \alpha_j^* \Rightarrow \alpha_i \alpha_j^* = \sum_{k=0}^{n-1} \text{Tr}_R(\alpha_i \alpha_k \alpha_j^*) \alpha_k^* = \sum_{k=0}^{n-1} c_{ik}^j \alpha_k^*.$$

We have $c_{i0}^i = 1$, so $\alpha_0^* = \alpha_1 \alpha_1^* - \sum_{k=1}^{n-1} c_{1k}^1 \alpha_k^*$.

(2) For $x \in \mathcal{D}_R^{-1}$ we can utilize the identity

$$\frac{x}{\alpha_1^*} = \sum_{k=1}^{n-1} \text{Tr}(x \alpha_k) \frac{\alpha_k^*}{\alpha_1^*} + \text{Tr}(x) \frac{\alpha_0^*}{\alpha_1^*}, \quad \text{so } \frac{x}{\alpha_1^*} - \text{Tr}(x) \frac{\alpha_0^*}{\alpha_1^*} \in S.$$

Substitute $x = \alpha_1^* \alpha_i$. For $i \neq 1$, its trace is zero, so $x/\alpha_1^* = \alpha_i \in S$.

Substitute $x = \alpha_2^* \alpha_2$. As $\alpha_2 \in S$, $x/\alpha_1^* \in S$. Meanwhile $\text{Tr}(x) = 1$, so $\alpha_0^*/\alpha_1^* \in S$.

It follows that $\mathcal{D}/\alpha_1^* \subset S$, so $(\alpha_1^*)^{-1} \in S$ and $R \subset S$. $\square$

By the claim above, one can call *canonical embedding* as the embedding described above, provided that $\mathcal{D}_R^{-1}$ is invertible. Canonical embedding is unique up to $\text{PGL}_{n-1}(\mathbb{Z})$ -action, and its $\overline{\mathbb{Q}}$ -points are canonical points.

Remark. It generally holds that the global function ring of canonical points contain R . By observing that the proof of the above claim did not use any assumptions on $\mathcal{D}_R^{-1}$, one can see $\mathbb{Z}[\alpha_2^*/\alpha_1^*, \dots, \alpha_{n-1}^*/\alpha_1^*]$ contains $R[(\alpha_1^*)^{-1}]$ for any R . So any $r \in R$ can be represented as polynomial in $\alpha_1^*/\alpha_i^*, \dots, \alpha_{n-1}^*/\alpha_i^*$ for any i . This implies that r identified with $(\rho_1(r), \dots, \rho_n(r)) \in \overline{\mathbb{Q}}^n$ is in image of the evaluation map $\Gamma(V(I_m)) \rightarrow \overline{\mathbb{Q}}^n$.

Remark. Inverse different can be also naturally identified to the dual R -module $\text{Hom}_{\mathbb{Z}}(R, \mathbb{Z})$ by considering the trace pairing $\text{tr}_R : R \times \mathcal{D}_R \rightarrow \mathbb{Z}$. In geometric perspective, inverse different corresponds to the dualizing sheaf of $\text{Spec } R$, and the invertibility of $\mathcal{D}_R$ is equivalent to that $\text{Spec } R$ is Gorenstein.

7.3 Examples: $\mathbb{Q}$ -points in $\mathbb{P}_{\mathbb{Z}}^m$

We consider the case where our Galois conjugate $\overline{\mathbb{Q}}$ -points are all $\mathbb{Q}$ -points. A $\mathbb{Q}$ -point of $\mathbb{P}_{\mathbb{Z}}^m$ having the coordinate $[q_0; q_1; \cdots; q_m]$ ($q_i \in \mathbb{Q}$) has unique coordinate of the form $[a_0; a_1; \cdots; a_m]$ for $a_i \in \mathbb{Z}, \gcd(a_0, a_1, \cdots, a_m) = 1$. Thus every $\mathbb{Q}$ -point is included in the unique $\mathbb{Z}$ -point $\text{Spec } \mathbb{Z} \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ constructed from line bundle and generator $(\mathbb{Z}, (a_0, \cdots, a_m))$, and it gives one-to-one correspondence of $\mathbb{Q}$ -points and $\mathbb{Z}$ -points. As $\text{Spec } \mathbb{Z}^n$ is isomorphic to disjoint union of n $\text{Spec } \mathbb{Z}$, maps $\text{Spec } \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ simply correspond to n points (which may not be distinct) in $\mathbb{P}_{\mathbb{Z}}^m$. Thus an ordered list of n $\mathbb{Q}$ -points in $\mathbb{P}_{\mathbb{Z}}^m$ bijectively correspond to a map $\text{Spec } \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$. We observed that this map is closed embedding if and only if the global function ring for those points is isomorphic to $\mathbb{Z}^n$.

We would like to give a criterion for the global function ring to be $\mathbb{Z}^n$, and use this result to obtain many examples of closed embeddings $\text{Spec } \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$. For k points $p_1, p_2, \cdots, p_k$ whose coordinates is $p_i = [a_{i0}; a_{i1}; \cdots; a_{im}]$, the gcd of all $k \times k$ minors of matrix

$$\begin{pmatrix} a_{10} & a_{11} & \cdots & a_{1m} \\ a_{20} & a_{21} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k0} & a_{k1} & \cdots & a_{km} \end{pmatrix}$$

is invariant under $\text{GL}_m(\mathbb{Z})$ -action on coordinates. Call this *determinantal invariant* of $p_1, \cdots, p_k$. (Define it as zero if $k \geq m+1$.) This is a temporary notion for this section and will be only used for set of $\mathbb{Z}$ -points. (One might possibly find its analogue for F -valued points for some other number field F .)

Proposition 7.3.1. *Suppose $n \leq m+1$ n $\mathbb{Z}$ -points in $\mathbb{P}_{\mathbb{Z}}^m$ have their determinantal invariant 1. Then they are $\text{PGL}_m(\mathbb{Z})$ -isomorphic to $[1; 0; \cdots; 0; \cdots; 0], [0; 1; \cdots; 0 \cdots; 0], \cdots, [0; 0 \cdots; 1; \cdots; 0]$.*

Proof. We use induction on n . For $n = 1$, one point can be always mapped to $[1; 0; \cdots; 0]$.

For inductive step, the determinant invariant for first $n - 1$ points is also 1, so one can map those points to $[1; 0; \cdots; 0], [0; 1; \cdots; 0], \dots$. Let $p = [a_0 : a_1 : \cdots : a_m]$ be the coordinate of the remaining one point. As determinant invariant is 1, we should have $\gcd(a_{n-1}, a_n, \dots, a_m) = 1$. $\text{GL}(\mathbb{Z})$ -action on those entries $a_{n-1}, \dots, a_m$ can transform p to $[a_0 : \cdots : a_{n-2}; 1; 0; \cdots; 0]$ while keeping other points same. Lastly, applying the transform $x_i \mapsto x_i - a_i x_{n-1}$ ($i = 0, \dots, n - 2$) on $[x_0; \cdots; x_m]$ gives $[1; 0; \cdots; 0; \cdots; 0], [0; 1; \cdots; 0 \cdots; 0], \dots, [0; 0 \cdots; 1; \cdots; 0]$. $\square$

As we can calculate the global function ring of the n points $[1; 0; \cdots; 0], [0; 1; \cdots; 0], \dots, [0; 0; \cdots; 1]$ as $\mathbb{Z}^n$ (we will observe it as result of the criterion below), this tells that $n \leq m + 1$ points having determinantal invariant 1 have global $\mathbb{Z}^n$ as their function ring. For $n = 2$, we can show that the converse statement holds (closed embeddings $\text{Spec } \mathbb{Z}^2 \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ should have determinantal invariant 1), but it is not the case for $n \geq 3$.

We give the necessary and sufficient criterion for a map $\text{Spec } \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ to be closed embedding. Denote the Veronese embedding $v_d : \mathbb{P}_{\mathbb{Z}}^m \rightarrow \mathbb{P}_{\mathbb{Z}}^{\binom{m+d}{d}-1}$ of degree d as

$$v_d([a_0; a_1; \cdots; a_m]) = [a_0^d; \cdots; (\text{all degree } d \text{ monomials in } a_i); \cdots; a_m^d].$$

Proposition 7.3.2. *For $\mathbb{Z}$ -points $p_1, \dots, p_n$ in $\mathbb{P}_{\mathbb{Z}}^m$, their global function ring is $\mathbb{Z}^m$ if and only if there exists d such that their degree d Veronese embeddings $v_d(p_1), \dots, v_d(p_n)$ in $\mathbb{P}_{\mathbb{Z}}^{\binom{m+d}{d}-1}$ have determinantal invariant 1.*

Proof. ($\Leftarrow$) Suppose that points $p_1, \dots, p_n$ have their global function ring $\mathbb{Z}^n$. Denote their coordinates as $p_i = [a_{i0}; a_{i1}; \cdots; a_{im}]$, $\gcd(a_{i0}, \dots, a_{im}) = 1$. Let f_i be the global function which takes value 1 at p_j and 0 at other points. In the subset $D(x_k)$, f_i can be represented as polynomial in $x_0/x_k, \dots, x_m/x_k$. We pick L as an integer greater

than degree of the denominator of f_i in $D(x_k)$ for any i, k . Then one can write

$$f_i(x_0, \dots, x_m) = \frac{g_{ik}(x_0, \dots, x_m)}{x_k^L} \text{ in } D_{x_k}, \quad f_i(a_{j0}, \dots, a_{jm}) = \delta_{ij}$$

for homogeneous polynomials g_{ik} of degree L . Then we have

$$g_{ik}(a_{j0}, \dots, a_{jm}) = \delta_{ij} a_{jk}^L.$$

Claim. *For any integers $t_1, \dots, t_n$, there exists polynomial h of degree L such that $h(a_{j0}, a_{j1}, \dots, a_{jn}) = t_j$ for all j .*

Proof. Assume that our desired polynomial has form $h = \sum_{i=1}^n \sum_{k=0}^m c_{ik} g_{ik}$. Then we have

$$h(a_{j0}, \dots, a_{jn}) = \sum_{i=1}^n \sum_{k=0}^m c_{ik} \delta_{ij} a_{jk}^L = \sum_{k=0}^m c_{jk} a_{jk}^L = t_j.$$

As $a_{j0}^L, a_{j1}^L, \dots, a_{jm}^L$ are coprime, the equation $\sum_{k=0}^m c_{jk} a_{jk}^L = t_j$ has integral solution for any t_j . So we can find integers c_{ik} satisfying the condition. $\square$

The projective coordinates of the image of p_i under v_L are given as

$$v_L(p_i) = [a_{i0}^L; \dots; a_{i0}^{e_0} a_{i1}^{e_1} \dots a_{im}^{e_m}; \dots; a_{im}^L],$$

where $(e_0, e_1, \dots, e_m)$ ranges over all nonnegative solutions of $e_0 + \dots + e_m = L$. If we denote the vector

$$v_i = (a_{i0}^L, \dots, a_{i0}^{e_0} a_{i1}^{e_1} \dots a_{im}^{e_m}, \dots, a_{im}^L)$$

and denote the coefficient of h (of previous claim) and corresponding vectors as

$$h = \sum c_{e_0, \dots, e_m} x_0^{e_0} x_1^{e_1} \dots x_m^{e_m}, \quad c = (c_{e_0, \dots, e_m})_{(e_0, \dots, e_m)},$$

our claim tells that the system of equations $c \cdot v_j = t_j$ ($j = 1, \dots, n$) always has an integral solution for c . For the $n \times n$ matrix M having $v_1, \dots, v_n$ as its row vectors, $Mc = (t_1, \dots, t_n)^t$ can take any value in $\mathbb{Z}^n$, so the $\mathbb{Z}$ -span of column vectors is whole $\mathbb{Z}^n$. As the determinantal invariant can be interpreted as the index of column space inside $\mathbb{Z}^n$, this claim proves points $v_L(p_i)$ have determinantal invariant 1.

($\Rightarrow$) The property of points $v_L(p_i)$ having determinantal invariant 1 is equivalent to the claim, so we can construct homogeneous polynomials g_{ik} of degree L satisfying $g_{ik}(a_{j0}, \dots, a_{jm}) = \delta_{ij} a_{jk}^L$. Then we can define f_i as g_{ik}/x_k^L over D_{x_k} : the only requirements for consistency of those sections is that their valuations at points p_i coincide, and it follows from conditions on g_{ik} . It is immediate that f_i takes value 1 at p_j and 0 at other points, so we can embed $\mathbb{Z}^n$ in global function ring. $\square$

Example. In the example after the Proposition 2.3, we observed that the global function ring four points $[1; 0; 0], [0; 1; 0], [0; 0; 1], [1; 1; 2]$ is $\mathbb{Z}^4$. Their degree 2 Veronese embeddings have coordinates

$$[1; 0; 0; 0; 0; 0], [0; 0; 0; 1; 0; 0], [0; 0; 0; 0; 0; 1], [1; 1; 2; 1; 2; 4]$$

so their determinantal invariant is 1. Meanwhile, if we pick any three points among those, their determinantal invariant is also 1. This implies that the global function ring for three points $[1; 0; 0], [0; 1; 0], [1; 1; 2]$ is also $\mathbb{Z}^3$, while the determinantal invariant of original points in $\mathbb{P}^2$ is not 1.

More generally, consider the following $n = m + 1$ points in $\mathbb{P}^m$.

$$p_1 = [0; 1; 0; \dots; 0], p_2 = [0; 0; 1; \dots; 0], \dots, p_m = [0; 0; \dots; 1],$$

$$p_{m+1} = [a; b_1; b_2; \dots; b_m].$$

If we consider their images in degree 2 Veronese embedding, their determinantal

invariant is given as $\gcd(a, b_1b_2, b_1b_3, \dots, b_{m-1}b_m)$. So if $\gcd(a, b_1b_2) = 1$ then the global function ring for $p_1, \dots, p_{m+1}$ is $\mathbb{Z}^{m+1}$ and those points determine a closed embedding $\text{Spec} \mathbb{Z}^{m+1} \rightarrow \mathbb{P}_{\mathbb{Z}}^m$. Meanwhile, those points have determinantal invariant a . As determinantal invariant does not change by $\text{GL}_{m+1}(\mathbb{Z})$ -action, we can produce infinitely many embeddings $\text{Spec} \mathbb{Z}^{m+1} \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ which are $\text{PGL}_{m+1}(\mathbb{Z})$ -nonequivalent.

By considering the natural inclusion $\mathbb{P}^m \rightarrow \mathbb{P}^{m'}$ for $m' > m$, we can see the following.

Proposition 7.3.3. *Let $m \geq 2$ and $3 \leq n \leq m+1$. There are infinitely many closed embeddings $\text{Spec} \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ which are not equivalent up to automorphism of $\mathbb{P}_{\mathbb{Z}}^m$.*

In usual projective spaces over field, any $m+2$ (or less) points were isomorphic under automorphism of $\mathbb{P}_k^m$. But in $\mathbb{P}_{\mathbb{Z}}^m$, even three points (with rings isomorphic) can have different configurations!

Example. The above proposition does not work for $n = 2$ points, where maps are defined uniquely by the determinantal invariant of two points. Let p_1, p_2 be two points in $\mathbb{P}_{\mathbb{Z}}^m$. By $\text{GL}_{m+1}(\mathbb{Z})$ -coordinate change, p_1 can be set to $[1; 0; 0; \dots; 0]$ by coordinate change. If determinantal invariant of p_1 and $p_2 = [a_0; a_1; \dots; a_m]$ is f , then $\gcd$ of $a_1, \dots, a_m$ is f . By applying $\text{GL}_m(\mathbb{Z})$ -change on last m coordinates, one can assume two points are $\text{PGL}_{m+1}(\mathbb{Z})$ -equivalent to

$$p_1 = [1; 0; 0; \dots; 0], p_2 = [e; f; 0; \dots; 0].$$

Thus $\{p_1, p_2\}$ isomorphic to $X = [1; 0] \cup [e; f]$ in $\mathbb{P}_{\mathbb{Z}}^1$. We claim that X has $\mathbb{Z}[v]/(v^2 - fv)$ as ring of global function. It is subscheme of $\mathbb{P}_{\mathbb{Z}}^1 = \text{Proj} \mathbb{Z}[x_0, x_1]$ cut out by $x_2(ex_2 - fx_1)$, so we have $(x_{i/j} = x_i/x_j)$

$$\Gamma(X \cap D(x_0), \mathcal{O}) = \mathbb{Z}[x_{1/0}]/(x_{1/0}(ex_{1/0} - f)), \quad \Gamma(X \cap D(x_1), \mathcal{O}) = \mathbb{Z}[x_{0/1}]/(e - fx_{0/1}).$$

The global section of X should consist of a pair of functions on $D(x_0)$ and $D(x_1)$ which coincide on $D(x_0x_1)$. Suppose $f(x_{1/0}) \in D(x_0)$ and $g(x_{0/1}) \in D(x_1)$ coincide on $D(x_0x_1)$. Then on $D(x_0x_1)$, $f - g$ should be $\mathbb{Z}[x_{1/0}, x_{0/1}]$ -multiple of $e - fx_{0/1}$. Therefore we can assume that g is constant. Anything in f which is equivalent to zero when modded out by $ex_{1/0} - f$ should be generated by 1 and $ex_{1/0} - f \bmod x_1(ex_1 - fx_0)$, and $ex_{1/0} - f$ follows multiplication law

$$(ex_{1/0} - f)^2 = ex_1(ex_1 - fx_0)x_0^{-2} - f(ex_{1/0} - f) = -f(ex_{1/0} - f).$$

So we have $\Gamma(X, \mathcal{O}) \simeq \mathbb{Z}[v]/(v^2 - fv)$.

From this example, one can show that there are no closed embeddings $\text{Spec } \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ for $m \geq 2$ and $n \geq 2^{m+1}$. We consider $(\gcd 1)$ coordinates of all points in modulo 2. For $[a_0; \dots; a_m]$ with $\gcd(a_0, \dots, a_m) = 1$, there are $2^{m+1} - 1$ possibilities for $(a_0, \dots, a_m) \bmod 2$, so among 2^{m+1} points there are two points with all coordinates equivalent in mod 2. Then the determinantal invariant for those two points is divisible by 2.

If $\text{Spec } \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ is closed embedding, then the composition of $\text{Spec } \mathbb{Z}^2 \rightarrow \text{Spec } \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ should be also closed embedding, so for any pair of points the global function ring on those should be $\mathbb{Z}^2$. Thus such embeddings can't exist for $n \geq 2^{m+1}$.

For $n \leq 2^{m+1} - 1$, we construct such embedding using the following example.

Example. In $\mathbb{P}_{\mathbb{Z}}^m$, the following set of $n = 2^{m+1} - 1$ points

$$S = \{[e_0 : e_1 : \dots : e_m] : e_i \in \{0, 1\}, (e_0, \dots, e_m) \neq (0, \dots, 0)\}$$

has global function ring isomorphic to $\mathbb{Z}^n$.

Proof. First note that $\mathbb{Z}[x]/x(x-1) \simeq \mathbb{Z}^2$. By taking tensor product of k -tuples of $\mathbb{Z}[x]/x(x-1)$, we can see that 2^k $\mathbb{Z}$ -points $\{(e_1, e_2, \dots, e_k) : e_i \in \{0, 1\}\}$ in $\mathbb{A}_{\mathbb{Z}}^k$ has

global function ring isomorphic to $\mathbb{Z}^{2^k}$.

On the cover $D(x_0)$, $S \cap D(x_0)$ is isomorphic to affine 2^m points inside $D(x_0) \simeq \mathbb{A}_{\mathbb{Z}}^m$, so $\Gamma(S \cap D(x_0), \mathcal{O}) \simeq \mathbb{Z}^{2^m}$. This holds similar for $D(x_i), D(x_i x_j)$ etc., so the exact sequence

$$0 \rightarrow \Gamma(S, \mathcal{O}) \rightarrow \prod_{i=0}^m \Gamma(S \cap D(x_i), \mathcal{O}) \rightarrow \prod_{0 \leq i < j \leq m} \Gamma(S \cap D(x_i x_j), \mathcal{O})$$

takes the form

$$0 \rightarrow \Gamma(S, \mathcal{O}) \rightarrow \prod_{i=0}^m \mathbb{Z}^{|S \cap D(x_i)|} \rightarrow \prod_{0 \leq i < j \leq m} \mathbb{Z}^{|S \cap D(x_i x_j)|},$$

with each $\mathbb{Z}^l$ having basis as function taking 1 at single point inside corresponding subset. This shows $\Gamma(S, \mathcal{O})$ also contains elements taking 1 at single point of S and 0 otherwise, so $\Gamma(S, \mathcal{O}) \simeq \mathbb{Z}^n$. $\square$

Example. Now replace $[1 : 1 : \cdots : 1]$ to $[a_0 : a_1 : \cdots : a_m]$ in S , and call the altered set T . We show there are infinitely many integers $(a_0, a_1, \cdots, a_m)$ such that $(m+1)$ -th power Veronese embedding of T has determinantal invariant 1. By the criterion we established in proposition 7.3.2, this will show that T has global function ring $\mathbb{Z}^n$ for $n = 2^{m+1} - 1$.

For a nonempty proper subset $A \subset \{0, 1, 2, \cdots, m\}$, we can correspond a point $p_A = [e_0 : e_1 : \cdots : e_m]$ where $e_i = 1$ if $i \in A$, $e_i = 0$ otherwise. These points and the point $q = [a_0 : a_1 : \cdots : a_m]$ consist T .

Let M be the matrix consisting of n rows, corresponding to projective coordinate of $(m+1)$ -power Veronese embedding of points in T . Each columns correspond to degree $(m+1)$ monomial consisting of $x_0, x_1, \cdots, x_m$.

For a monomial t , we can consider a set of i such that x_i is involved in t . Call this set 'base' of t . If t has base B , then t -coordinate of $v_{m+1}(p_A)$ is 1 if $A \subseteq B$ and

0 otherwise.

Order all nonempty subsets $\{0, 1, 2, \dots, m\}$ by number of elements and denote them $(A_1, A_2, \dots, A_n)$. Then we have $A_i \subset A_j \Rightarrow i \leq j$. Now consider monomials t_i such that t_i has base A_i , then consider the following $n \times n$ submatrix of M

$$\begin{pmatrix} t_1(p_{A_1}) & t_2(p_{A_1}) & \cdots & t_n(p_{A_1}) \\ t_1(p_{A_2}) & t_2(p_{A_2}) & \cdots & t_n(p_{A_2}) \\ \vdots & \vdots & \ddots & \vdots \\ t_1(p_{A_{n-1}}) & t_2(p_{A_{n-1}}) & \cdots & t_n(p_{A_{n-1}}) \\ t_1(q) & t_2(q) & \cdots & t_n(q) \end{pmatrix}.$$

Here $t_i(p_{A_j})$ means t_i evaluated at p_{A_j} . As t_i has base A_i , for $i \geq j$ we have $A_i \not\subset A_j$ so $t_i(p_{A_j}) = 0$. Thus the matrix above is lower triangular and its diagonals are all 1.

So its determinant is $t_n(q) = a_0 a_1 \cdots a_m$ for only possible choice $t_n = x_0 x_1 \cdots x_m$.

Let $A_{n-1} = \{1, 2, \dots, m\}$, and just switch t_n to monomial having base A_{n-1} . Now then the matrix above is lower triangular except $(n-1, n)$ -entry whose value is 1. Thus the determinant becomes determinant of rightmost lower 2×2 part, which is $t_{n-1}(q) - t_n(q)$. As t_{n-1} and t_n were set to be monomial with basis $A_{n-1} = \{1, 2, \dots, m\}$, determinant can be $a_1 a_2 \cdots a_m (a_i - a_j)$ for all $1 \leq i < j \leq m$. We can also let $A_{n-1} = \{0, 1, \dots, m\} \setminus k$ and in that case determinant becomes $a_0 a_1 \cdots \hat{a}_k \cdots a_m (a_i - a_j)$ (a_k missing in the product, i, j, k all distinct)

From these we can see that determinantal invariant of M should divide $a_0 a_1 a_2 \cdots a_m$ and $a_0 a_1 \cdots \hat{a}_k \cdots a_m (a_i - a_j)$. We can choose infinitely many $a_0, \dots, a_m$ such that their gcd is 1 (for example one can choose a_i to be primes within $[N, 2N]$), and in that case determinantal invariant of M is 1.

Proposition 7.3.4. *Let $m \geq 2$ and $3 \leq n \leq 2^{m+1} - 1$. There are infinitely many closed embeddings $\text{Spec } \mathbb{Z}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^m$ which are not equivalent up to automorphism of $\mathbb{P}_{\mathbb{Z}}^m$.*

Proof. We covered $3 \leq n \leq m+2$ case above. For $m+3 \leq n \leq 2^{m+1}$, choose n points from above example T including $m+3$ points $[1 : 0 : \cdots : 0], [0 : 1 : \cdots : 0], \dots, [0 : 0 : \cdots : 1], [1 : 1 : \cdots : 1]$ and $[a_0 : a_1 : \cdots : a_m]$. Note that those $m+3$ points are not geometrically equivalent (under $\mathrm{PGL}_{m+1}(\mathbb{C})$), so for any finite set of $m+3$ point sets one can set $a_0, \dots, a_m$ as coprime integers such that those are not geometrically equivalent with any previous sets. This means that one can construct infinitely many n -point sets whose global function ring is $\mathbb{Z}^n$ and are not geometrically equivalent. $\square$

We expect to have infinitely many nonequivalent embeddings of a n -ic ring into $\mathbb{P}_{\mathbb{Z}}^m$ even for $n \geq 2^{m+1}$ too, in case of $m \geq 2$. This result does not hold for $m = 1$ however. Instead, the following contrary phenomenon holds for all ring, implied by the result of Birch-Merriman.

Proposition 7.3.5. *Let R be a nondegenerate ring of rank n . There are finitely many closed embeddings $\mathrm{Spec} R \rightarrow \mathbb{P}_{\mathbb{Z}}^1$ which are not equivalent up to automorphism of $\mathbb{P}_{\mathbb{Z}}^1$.*

Proof. Suppose that a map $\mathrm{Spec} R \rightarrow \mathbb{P}_{\mathbb{Z}}^1$ is closed embedding. Let I_m be the vanishing ideal for its scheme-theoretic image. Then we have $\mathrm{Spec} R \simeq V(I_m)$. If $\overline{\mathbb{Q}}$ -points of the embeddings are given as $[a_i; b_i]$, then any polynomial vanishing on all those points is multiple of $\prod (a_i y - b_i x)$. We can let this binary n -ic form have rational coefficients by considering the choice of a_i, b_i as $[a_i; b_i] = [\rho_i(r_1); \rho_i(r_2)]$ for $r_1, r_2 \in R \otimes \mathbb{Q}$. Therefore we have $I_m \otimes \mathbb{Q} = (f)$ for some rational binary n -ic form f . If we pick f as primitive integral form by taking $\mathbb{Q}$ -multiple, then $I_m = (f) \cap \mathbb{Z}[x, y] = (f)$: if $g \in \mathbb{Z}[x, y]$ is divisible by f in $\mathbb{Q}[x, y]$ then it is divisible by f in $\mathbb{Z}[x, y]$ by the Gauss' lemma.

The global function ring of $V(I_m) = V(f)$ can be found using the minimal free resolution of $V(f)$, given as $\mathcal{O}(-n) \rightarrow \mathcal{O}$ where the map is multiplication by f . Its hypercohomology ring was given in [39] and [47] etc. and its discriminant is same as the discriminant of f . So we have $\mathrm{Disc}(R) = \mathrm{Disc}(\Gamma(V(f))) = \mathrm{Disc}(f)$. Meanwhile, [9]

shows that there are only finitely many binary n -ic form with bounded discriminant, so there are finitely many forms f satisfying the condition. $\square$

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