From the holdings of the Department of Special Collections , Princeton University Library, Princeton, NJ

These documents are either in the public domain, or can only be used for educational and research purposes (Fair use) as per U.S. Copyright law (text below). By accessing this file, all users agree that their use falls in the public domain or within fair use as defined by the copyrightlaw. We do not charge any permission or usage fees for the publication of images of material inour collections, including those provided by Princeton University Library, via our website, catalog, or directly from staff.For more information regarding copyright, credit, and citation please review the information onthe Copyright, Credit and Citation Guidelines page of our website.
https://library.princeton.edu/special-collections/policies/copyright-credit-and-citation-guidelines. If you have any questions, including how to order higher-quality images from the collection, contact Special Collections Public Services via our Ask Us! Form. https://library.princeton.edu/special-collections/ask-us

---

Doctoral Dissertations, 2011-2026 Mathematics

2014

p-adic approaches to the Langlands program

Shah, Shrenik Nitin

Princeton, NJ : Princeton University

---

http://arks.princeton.edu/ark:/88435/dsp011831cn17q

U.S. Copyright law: The copyright law of the United States (Title 17, §108, United States Code) governs the making of photocopies or other reproductions of copyrighted material. Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be used for any purpose other than private study, scholarship, or research. If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of fair use that user may be liable for copyright infringement. This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve a violation of copyright law. https://www.copyright.gov/title17/title17.pdf

p-adic approaches to the Langlands program

Shrenik Nitin Shah

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Christopher McLean Skinner

September 2014

© Copyright by Shrenik Nitin Shah, 2014.

All Rights Reserved

Abstract

We develop and utilize p-adic Hodge theory in families in the context of local-global aspects of the Langlands program.

Our first result allows one to interpolate Hodge-Tate and de Rham periods when some Hodge-Tate-Sen weights are fixed. This is a common generalization of results of Kisin in the case where one weight is fixed and of Berger-Colmez where every weight is fixed. Our main technique is to systematically prove interpolation results for the first cohomology group, where it is possible to obtain base change results regardless of the geometry of the family, and then use algebraic methods to deduce results for periods. We also obtain vanishing for higher cohomology. Varma has applied the main result to show that the Galois representations constructed by Harris-Lan-Taylor-Thorne are de Rham.

Our second result concerns the transfer of regular cuspidal automorphic representations on unitary similitude groups to general linear groups. We build on work of Morel, Shin, and Skinner by proving compatibility at places where the unitary group is ramified. We first obtain compatibility up to monodromy by purely automorphic methods. By applying a crystalline period interpolation result of Kisin and Nakamura to a family constructed using Urban's eigenvariety, we are able to improve this to full compatibility. Two subtleties that arise are (1) the construction of a suitable p-stabilization in the ramified setting, which uses work of Reeder and Lusztig, and (2) the placement of a suitable finite slope representation into a p-adic family, which requires studying the Eisenstein cohomology of the unitary group. We obtain new cases of the generalized Ramanujan conjecture and produce the first examples of strong functorial transfers to generalized linear groups for cuspidal automorphic representations on unitary similitude groups. These strong transfers are used by Skinner-Urban in their work on the Bloch-Kato conjecture.

Acknowledgements

First and foremost, I thank Christopher Skinner for his thoughtful mentorship and guidance over the past five years. He patiently answered my many questions about mathematics, generously shared his many ideas about p-adic families, the Langlands program, and more, and advised me on many other matters as well. He was constantly encouraging regardless of whether I was struggling or succeeding at my work. He suggested all the problems in this thesis, and no version of this work would have been possible without his input and support.

I thank Richard Taylor for his support over the years; he advised my senior thesis at Harvard, which was my first foray into the Langlands program, and has provided guidance on innumerable occasions since then.

I thank Eric Urban for his support, for answering many questions about eigenvarieties and related matters, and for updating his paper on eigenvarieties to cover the ramified case I consider here. He has kindly agreed to supervise me during my postdoctoral research at Columbia, and I am very excited to work with him and the rest of the Columbia faculty.

I thank Sophie Morel for her advice and guidance on many matters over the years as well as for reading this thesis.

Many more professors have offered their support and guidance over the years. Ken Ono ran an REU I attended many years ago and has thoughtfully kept up with my progress since then. I also thank Brian Conrad for his advice on a number of occasions over the years. Thanks also to the many Princeton professors who have supported me in graduate school, including Manjul Bhargava, David Gabai, Robert Gunning, Peter Sarnak (who kindly agreed to be on my FPO committee), Nicolas Templier, Andrew Wiles, and Shou-Wu Zhang. I have returned to Harvard a number of times over the past few years, and I would particularly like to thank Noam Elkies, Benedict Gross, Joe Harris, Mark Kisin, Barry Mazur, and Curtis McMullen for their advice and continued interest in my development.

My friends at Princeton have made my time here incredible. There are too many to

name, but I would especially like to thank Aaron for an amazing continuing collaboration and friendship, Stefan for his guidance on all things mathematical and extra-mathematical, Dan for three great years in Lawrence, Andrei for teaching a crash course on the Langlands program in my first year, and Michael for conversations about representation theory and a fun time in Hibben. I would also like to thank Bhargav Bhatt, David Geraghty, Wei Ho, Tasho Kaletha, Kai-Wen Lan, and Jack Thorne for their advice on many things. Thanks to Alison, Ana, Arul, Florian, Ila, Jacob, Jerry, and Xin for many great discussions about algebraic number theory. Thanks to Andy, Bela, Boris, Cotton, Daniel, Gabriele, Heather, Kevin[2], Matthew, Nathan, Philippe, Ryan, Sam[2], Sasha, Tom, Will[2], Yu-Chao, and any other friends I have forgotten to mention. Finally I would like to thank John for his continuing friendship over many years.

Jill has offered me advice on countless occasions over the years and has made tireless efforts to make my time here (and that of every other graduate student) smooth and enjoyable. I also thank Gale for her invaluable assistance with running the second number theory seminar and on many other matters.

Specific to the work in this thesis, I would like to thank Brian Conrad, David Geraghty, Kiran Kedlaya, Mark Kisin, Aaron Pollack, Richard Taylor, and Eric Urban for helpful discussions about Chapter 1, and I thank Benedict Gross, Tasho Kaletha, Sophie Morel, Stefan Patrikis, Aaron Pollack, Sean Rostami, Jack Thorne, and Eric Urban for discussions about Chapter 2. I thank Jack Thorne for explaining Lusztig's technique used below.

My work was funded by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE 1148900 and the Department of Defense (DoD) through the National Defense Science & Engineering Graduate Fellowship.

Finally, I would like to thank Monica, my brother Shreepal, my grandparents, and my parents for their love, support, and encouragement over the years.

To the memory of my grandmother Chanda.

Contents

Abstract iii

Acknowledgements iv

0 Overview **1**

0.1 Interpolating periods in a p-adic family 3

0.2 Constructing strong base changes 5

1 Interpolating Periods **8**

1.1 Introduction 8

1.1.1 History and motivation 8

1.1.2 Statement and discussion of main results 11

1.1.3 Method of proof 16

1.2 Interpolating de Rham and Hodge-Tate periods and 1-cocycles 17

1.2.1 Commutative algebra 18

1.2.2 Some results of Sen and Kisin 21

1.2.3 $\mathbf{C}_p$-periods in families 25

1.2.4 Bounded de Rham periods 34

1.2.5 Geometric specializations 39

1.2.6 Unbounded periods 45

1.2.7 Essentially self-dual and decomposable specializations 56

1.3 Stratification by de Rham data . . . 58
1.3.1 Definition of a de Rham datum . . . 59
1.3.2 The de Rham datum of a family . . . 61
1.3.3 Globalization . . . 65
1.3.4 Stratification of families of Galois representations . . . 69
1.4 Higher cohomology . . . 75
1.4.1 Higher H_K-cohomology . . . 76
1.4.2 Γ_K-cohomology and inflation-restriction . . . 79
1.4.3 Vanishing of cohomology for bounded and unbounded periods . . . 84
1.5 Conditional base change for invariants . . . 85
1.5.1 Base change and invariants . . . 86
1.5.2 Application to Hodge-Tate periods . . . 90

2 A p-adic approach to local-global compatibility 94
2.1 Introduction . . . 94
2.2 Weak base change, L-functions, and γ-factors . . . 100
2.2.1 Unitary groups and base change . . . 100
2.2.2 Weak base changes and Galois representations attached to automorphic representations of G . . . 103
2.2.3 A relation between Satake parameters . . . 108
2.2.4 Local L-factors for tori . . . 110
2.2.5 Unitary groups over p-adic fields . . . 111
2.2.6 Local L-factors for ramified unitary groups . . . 112
2.2.7 γ-factors . . . 116
2.3 Possibilities for τ_v and π_v . . . 118
2.3.1 Local L-factors and γ-factors of representations of GL_n . . . 119

2.3.2 Roots and poles of γ-factors 122
2.3.3 Possibilities for π_v 123
2.4 Iwahori-Hecke algebras of quasi-split ramified unitary groups 126
2.4.1 The Kottwitz homomorphism 126
2.4.2 Extended affine Weyl groups 129
2.4.3 Iwahori-Hecke algebras 131
2.4.4 Ramified unitary groups in odd dimension and symplectic groups . . 133
2.4.5 Ramified quasi-split unitary groups in even dimension and odd special orthogonal groups 138
2.4.6 Similitude groups and restriction 143
2.5 Weight structure of almost unramified representations of p-adic unitary groups 146
2.5.1 Weight spaces of unramified representations of split groups 146
2.5.2 Proof of Theorem 2.5.1 150
2.6 Crystalline periods in families of Galois representations 151
2.6.1 Purity 152
2.6.2 Interpolating crystalline periods 155
2.6.3 Hecke algebras and slopes 160
2.6.4 Slopes on ramified unitary similitude groups 164
2.6.5 Behavior of finite slope automorphic representations with very regular weight 168
2.6.6 Properties of a suitable family 172
2.7 Families of automorphic forms and Galois representations 174
2.7.1 Urban's eigenvarieties 175
2.7.2 Pseudorepresentations and p-adic families of Galois representations . 177
2.7.3 Proof of Theorem 2.2.5 in the non-critical case 179
2.8 Classical and overconvergent automorphic multiplicities 182

2.8.1 Automorphic cohomology 182
2.8.2 Critical slopes 189
2.8.3 Finite slope Eisenstein cohomology 194

Chapter 0

Overview

The Langlands program predicts a relationship between certain automorphic representations on reductive groups and motives. There has been a staggering amount of progress on this conjecture in recent years, thanks in part to the successful efforts of many mathematicians to realize a strategy of Langlands to use the cohomology of Shimura varieties as an intermediary between the automorphic and motivic worlds. In moving from an automorphic representation to a motive, one uses a comparison of geometric and automorphic trace formulas to relate the action of Frobenius on the cohomology of a suitable Shimura variety to the action of a Hecke operator. To achieve this requires a precise understanding of all the terms contributing to these formulas, including the so-called "stabilization" of the trace formula.

This approach as described thus far ignores some structure present in the automorphic side of the picture as well as in each p-adic Galois realization of the motivic side. Namely, there is a rich theory of p-adic congruences between automorphic representations and between Galois representations. Since the work of Deligne-Serre on attaching Galois representations to weight 1 modular forms, many mathematicians have exploited these congruences to access some "fringe" representations that are just out of the range of accessibility of the preceding strategy. These congruence-based, or "p-adic", methods do not produce new cases of

Langlands from nowhere – rather, they interpolate known cases in order to obtain new ones.

In fact, such p-adic strategies have been pushed even further recently. One direction is in the work of Harris-Lan-Taylor-Thorne [27] and Scholze [64], where non-self-dual automorphic representations and even torsion classes have had Galois representations attached. Another is in the program of various authors, including Ribet, Mazur, Wiles, Skinner-Urban [76, 75, 83], and Bellaïche-Chenevier [3], to obtain cases of the Iwasawa main conjecture and Bloch-Kato conjecture by utilizing the p-adic structure of both automorphic representations and Galois representations.

In this work we are interested both in expanding the toolset of available p-adic techniques in light of these developments, and also proving new cases of Langlands functoriality that are critical for enabling these p-adic approaches to succeed.

One step towards characterizing the image of the Galois representation construction of Harris-Lan-Taylor-Thorne [27] is to establish their expected p-adic Hodge theoretic properties. Thanks to work of Varma [84], one understands in a precise p-adic analytic sense just how these representations are realized in the "fringe" of the locus of Galois representations whose p-adic Hodge theoretic properties are understood; namely, half the Hodge-Tate-Sen weights are moving while the other half remain constant. Families of this sort also arise in the work of Skinner-Urban [75], who refer to them as "finite slope families". One of our goals is to study the variation of Hodge-Tate and de Rham periods in this situation. Bellovin [4] has independently given an another approach to understanding this variation using different methods; see Section 1.1 for a brief discussion of the similarities and differences between our works.

Skinner-Urban [76, 75, 83] study certain non-semisimple p-adic Galois representations that are constructed using p-adic families of cusp forms on unitary similitude groups of non-compact signature degenerating to an Eisenstein series at a point. Bellaïche-Chenevier [3] consider similar constructions as well. These non-semi-simple representations correspond

to nontrivial cohomology classes inside a very large cohomology group $H^1(G_{\mathbf{Q}}, V)$. For applications to the Bloch-Kato conjecture, one is interested only in classes that satisfy local conditions at each place, which form a Selmer group denoted $H^1_f(G_{\mathbf{Q}}, V)$. In order to check containment in $H^1_f(G_{\mathbf{Q}}, V)$, one must apply strong local-global compatibility results between cusp forms in the family and their associated Galois representations. Therefore it is essential, especially in the most recent results [83], to know strong compatibility between cusp forms on unitary similitude groups and their p-adic Galois representations, at least for the generic members of a p-adic family. This is our second motivating goal.

The main obstruction to obtaining strong compatibility actually lies in understanding the transfer of this cusp form to an automorphic representation on $\mathbf{G}_m \times \mathrm{GL}_n$. Such a transfer has been constructed by Morel [49], Skinner [73], and Shin [71] – see Theorem 2.2.2 for a statement of their result. If E is the imaginary quadratic field used in the definition of the unitary similitude group, Shin proves local-global compatibility at split places of E as well as unramified places of E where π is also unramified. Since E always has a place of ramification, this result never produces a strong base change. Obtaining compatibility for π that are as unramified as possible at p serves as the focus of Chapter 2, and allows one to produce strong base changes as needed for the arguments above. The main ingredient is Urban's construction of eigenvarieties for reductive groups that have discrete series [82].

The recurring theme in both these works is that in order to best take advantage of the extra structure granted by the relevant eigenvariety, one should develop and utilize p-adic Hodge theory in families.

0.1 Interpolating periods in a p-adic family

If a p-adic family of Galois representations has some number of fixed integral Hodge-Tate-Sen weights, it is plausible that there are the same number of Hodge-Tate, de Rham, semistable,

or crystalline periods that vary over the family. In Chapter 1, we study the variation of Hodge-Tate and de Rham periods. We prove a kind of interpolation result: if there is a dense subset of points possessing the requisite periods, then they must vary over the whole family. Our results are stated precisely in Main Theorems 1, 2, and 3 in the introduction, but we highlight part of these results here. Let $K/\mathbf{Q}_p$ be a finite extension. If $\mathfrak{X}$ is a rigid space and $\mathfrak{M}$ is a finite locally free coherent sheaf over $\mathfrak{M}$ with a continuous action of G_K, we prove that for integers $i \leq j$ and $r \in \{0, 1\}$, $\dim_{\kappa(x)} H^r(G_K, t^i B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}} \hat{\otimes}_{\mathbf{Q}_p} \overline{\mathfrak{M}}_x)$ is a lower semi-continuous function with respect to the Zariski topology, where $\overline{\mathfrak{M}}_x$ denotes the Galois representation over the residue field $\kappa(x)$ at x.

Varma [84] produces a family $\mathfrak{M}$ over $\mathfrak{X}$ as above where a dense set of points have half their Hodge-Tate-Sen weights in a fixed range $[i, j]$ and are de Rham, while the other weights can be assumed to lie outside this interval. Moreover, the sum of ρ and its conjugate dual appears as the specialization $\overline{\mathfrak{M}}_x$ at some point $x \in \mathfrak{X}$. Applying this semicontinuity result to this situation immediately establishes that $\rho \oplus \rho^{c\vee}$ has n de Rham periods, where n is the dimension of ρ, and by calculating the Hodge-Tate-Sen weights of ρ and $\rho^{c\vee}$ she deduces that ρ itself possesses n de Rham periods.

We also mention some of the ideas that go into the proof of the semicontinuity mentioned above. In the following, H_K denotes the kernel of the $p - 1^{\text{th}}$ power (or 2^{nd} if $p = 2$) of the cyclotomic character.

- In Section 1.2.6, we use some basic results from the p-adic functional analysis of LF-spaces to relate the cohomology of B_{dR} with that of $t^i B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}}$; this reduces us to the study of bounded periods.

- We interlace many of our results with induction on $j - i$ in order to reduce ourselves to statements about $\mathbf{C}_p$-periods. This induction usually takes the form of examining

6-term exact sequences obtained in a simple manner from ones of the form

$$0 \to t^j B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \to t^i B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \to t^i B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}} \to 0$$

for $i \leq j \leq k$. For instance, see Sections 1.2.4 and 1.2.5. At one point (Lemma 1.2.25), we need to use a result of Berger-Colmez [5, Lemma 4.3.1] generalizing Sen's work to the de Rham case.

- We apply Sen's theory to the study of $\mathbf{C}_p$ periods. One key application (Lemma 1.2.24) is to show that the tensor product $(\mathbf{C}_p \hat{\otimes} \mathfrak{R})^{H_K} \otimes_{\mathfrak{R}} \mathfrak{N}$ is complete for a finite $\mathfrak{R}$-module $\mathfrak{N}$. Also see Section 1.2.3.

- In Section 1.2.5, the aforementioned completeness and some additional results of Sen allow us to prove base change statements for H_K-invariants with respect to a map $\mathfrak{R} \to \mathfrak{R}'$ with $\mathfrak{R}'$ a finite Noetherian $\mathfrak{R}$-algebra. The cyclicity of G_K/H_K allows us to deduce base change statements for $H^1(G_K, \mathbf{C}_p \hat{\otimes} \cdot)$ as a consequence.

- Passing from the above to global semicontinuity and base change results uses standard methods from commutative algebra and algebraic geometry. See Proposition 1.2.26 and Section 1.3.3.

We also study higher H_K-cohomology and base change for G_K-invariants; see Main Theorem 3, which is proved in Section 1.4, as well as Section 1.5.2.

0.2 Constructing strong base changes

The first step – compatibility up to monodromy – is a purely combinatorial exercise using properties of γ-factors proved in other works; the argument is based on one of Skinner [73]. Approaches based on looking at traces have difficulty seeing the monodromy of a represen-

tation, so it is difficult to improve on this using purely automorphic methods. Previous approaches in other settings have used intricate studies of the geometry and cohomology of Shimura varieties, such the works of Taylor-Yoshida, Clozel, or Caraiani. We have the benefit of being able to rely on these works as an input, but also have to deal with a difficulty not present in other cases: ramification of the group.

The output of the γ-factor argument shows that if the cuspidal representation π on the unitary similitude group is unramified but does *not* satisfy local-global compatibility at p, there must be two Satake parameters, α and β, that are related by $\alpha = p^k\beta$, where k is a positive integer depending only on the group. The key idea is to exploit two observations about this relation.

1. The quotient α/β has a p-adic valuation that, roughly speaking, can only occur infrequently in an eigenvariety.

2. The quotient α/β has an archimedean valuation larger than 1, which one expects should only occur for very special cuspidal automorphic representations.

The first of these says that for most points in a suitable family, we must have compatibility. The second of these gives us a hope of finding a contradiction using the purity of the attached Galois representation. This is where p-adic variation comes in – if we can construct a crystalline period at the dense set of points where compatibility holds, we can use work of Kisin [36] and Nakamura [50] to construct one in the Galois representation ρ_π attached to our original representation π. See Sections 2.6 and 2.7.

For the aforementioned argument to produce a period contradicting the purity of ρ_π, we need to carefully choose a p-stabilization of π; this choice relies on a trick of Lusztig [44] to deal with the ramification and work of Reeder [58] to understand the structure of unramified representations of a related split group. This is carried out in Sections 2.4 and 2.5.

To produce a family, we work with the multiplicities that appear in Urban's construction

of eigenvarieties [82] in Section 2.8. The key is to use twists appearing in non-classical weight and to eliminate the possibility that π arises from an Eisenstein class attached to an overconvergent form on a proper Levi subgroup.

For a precise statement of the main result, see Theorem 2.2.5, which is an enhancement of the previously known Theorem 2.2.2 to include compatibility at ramified places.

Chapter 1

Interpolating Periods

1.1 Introduction

1.1.1 History and motivation

Fontaine introduced period rings such as B_{HT}, the ring of Hodge-Tate periods, and B_{dR}, the ring of de Rham periods, in order to algebraically detect and study geometric properties of p-adic representations. Let B be a ring of periods. If K is a finite extension of $\mathbf{Q}_p$ and V is a finite dimensional vector space over $\mathbf{Q}_p$ with a continuous linear action of G_K, one defines $D_B(V) = H^0(G_K, B \otimes_{\mathbf{Q}_p} V)$ to be the B^{G_K}-vector space of B-periods of V. For the ring $B = B_{\mathrm{dR}}$, we have $B^{G_K} = K$. The dimension $\dim_K D_B(V)$ is conjecturally involved in detecting whether V may be a restriction of a global Galois representation "arising from geometry," i.e., appearing inside the étale cohomology of a variety over a number field. It is therefore important to be able to calculate $\dim_K D_B(V)$. A result of Tsuji achieves this for V arising from geometry, but in many cases, the representation V is constructed via the use of congruences. In such situations, V can often be realized as the specialization of a p-adic family.

In this paper, we investigate the variation of periods in a p-adic family. In particular, we

will work with a $\mathbf{Q}_p$-Banach algebra $\mathfrak{R}$ and a finite free module $\mathfrak{M}$ over it, equipped with a continuous $\mathfrak{R}$-linear action of G_K. We study $D_B(\mathfrak{M}) = H^0(G_K, B\hat{\otimes}_{\mathbf{Q}_p}\mathfrak{M})$, which has the structure of a module over $B^{G_K} \otimes_{\mathbf{Q}_p} \mathfrak{R}$. (Here $\hat{\otimes}$ denotes completed tensor product.)

The first understanding of $D_B(\mathfrak{M})$ was achieved for $B = B_{\mathrm{HT}}$ in the seminal work of Sen [67, 68]. Sen attaches p-adically varying Hodge-Tate-Sen weights to the family $\mathfrak{M}$. These are generalized eigenvectors of an endomorphism ϕ, called the Sen operator, which acts on a finite $\mathfrak{R}$-module $\mathfrak{E}$. Since ϕ is not necessarily semi-simple, a continuous G_K-representation on a $\mathbf{Q}_p$-vector space V with a multiple integral Hodge-Tate-Sen weight k need not have a multiple Hodge-Tate weight k.

The vanishing locus of the m lowest order coefficients of the characteristic polynomial of ϕ is a closed subspace of $\operatorname{Spec}\mathfrak{R}$ corresponding to the existence of m Hodge-Tate-Sen weights of weight 0. Intersecting over weights in a bounded range, we obtain a "Sen stratification" of $\operatorname{Spec}\mathfrak{R}$.

We can ask the following questions about $D_B(\mathfrak{M})$ in the more general setting where $\mathfrak{M}$ is a coherent locally free sheaf with a continuous G_K-action over a rigid space $\mathfrak{X}$. In the "bounded" case, we mean $B = t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}}$, whereas in the "unbounded" case, we mean that $B = B_{\mathrm{dR}}$.

1. Is there a refinement of the aforementioned Sen stratification of $\mathfrak{X}$, where one can find closed analytic subvarieties $\mathfrak{S} \subseteq \mathfrak{X}$ corresponding exactly to existence of (at least) a certain number of Hodge-Tate or bounded de Rham periods for the specialization of $\mathfrak{M}$ at $x \in \mathfrak{S}$?

2. For bounded periods, is it possible to decompose $\mathfrak{X}$ into a disjoint union of locally closed strata $\pi_{\mathfrak{L}} : \mathfrak{L} \hookrightarrow \mathfrak{X}$ so that $D_B(\pi^*_{\mathfrak{L}}\mathfrak{M})$ is a coherent locally free sheaf of known rank? How does the formation of this sheaf behave with respect to change of $\mathfrak{X}$?

3. Can one construct unbounded periods at all geometric or even thickened geometric

specializations?

In Theorems 1.3.19, 1.3.20, and 1.3.21, we answer these questions when $\mathfrak{X}$ is reduced. In the most important setting, when a de Rham period is interpolated for every fixed Hodge-Tate-Sen weight, we even provide an answer for (2) and (3) in the non-reduced affinoid case. In Section 1.4, we also prove vanishing results for higher cohomology of $B\hat{\otimes}_{\mathbf{Q}_p}\mathfrak{M}$.

The primary impetus for this study is a growing need for methods to prove properties of p-adic representations constructed by congruences. Our main results are suited to, for example, studying the Galois representations recently constructed by Harris, Lan, Taylor, and Thorne [27] – this application has already been carried out by I. Varma [84], who puts their representations into p-adic families with very nice properties. Another application is to extensions constructed by specializing a p-adic family; these arise, for example, in the work of Skinner and Urban [76]. We develop a method for proving potential semi-stability of an extension of a representation V by a twist of its dual when this extension arises as the specialization of a p-adic family (see Theorem 1.2.40).

We are also interested in the geometry of families, including the global eigenvariety of a reductive group G over $\mathbf{Q}$. The structure of the eigenvariety is central, for instance, to the work of Bellaïche-Chenevier [3]. One interpretation of the Hodge-Tate-Sen theory is that the weight space for $\mathfrak{X}$ is intrinsic to the G_K-representation on $\mathfrak{M}$. Our results refine this statement, implying that for each choice of some fixed Hodge-Tate-Sen weights, there is a stratification of the corresponding subspace of $\mathfrak{X}$ defined entirely in terms of existence of de Rham periods.

Interpolation results of this sort have already been essential to work on the Langlands program. Berger and Colmez [5] interpolate periods when *all* of the Hodge-Tate-Sen weights are fixed. Their work is used by Chenevier and Harris [15] to establish local-global compatibility of the Langlands correspondence at p for p-adic Galois representations that are constructed via p-adic interpolation rather than in the cohomology of a Shimura variety. Kisin [36] proved

an important and widely applied interpolation result for a single Hodge-Tate, de Rham, or crystalline period of fixed Hodge-Tate-Sen weight, where the remaining weights are required to vary. Recently Liu proved an interpolation theorem for multiple semistable periods using different methods, which implies the interpolation of the corresponding de Rham periods in that setting [42]. However, this result does not allow one to interpolate de Rham periods that do not come from semistable periods, except in the aforementioned bounded weight situation of Berger and Colmez [5].

An earlier version of this paper had a slightly weaker form of Lemma 1.2.25, which was not quite strong enough in the de Rham case to apply to Proposition 1.2.26. The stronger form follows easily from the original argument once one applies the result [5, Lemma 4.3.1] of Berger-Colmez on D_{dif} to the situation under consideration.

After posting the first version of this paper, we learned that Bellovin [4] independently proved related work in her dissertation. She proves base change results on the (φ, Γ)-module side that include a statement analogous to Proposition 1.2.26 below, and uses these to stratify the base as in Theorem 1.3.19 and study its properties as we do. Our result is based on a p-adic analytic observation (Lemma 1.2.24) whereas hers are based on calculations with Tor groups and spectral sequences. Unlike our Section 1.3, her work applies in a nonreduced setting (using a freeness assumption at *all* Artinian specializations), though we prove a slightly less general result in the nonreduced setting under a less stringent density hypothesis in Theorems 1.2.13 and 1.2.19. We work with Galois representations rather than (φ, Γ)-modules, so there is little direct intersection between her results and the results and methods of Section 1.2, and Section 1.4 is entirely distinct.

1.1.2 Statement and discussion of main results

For the finite flat interpolation of Hodge-Tate and bounded de Rham periods, our work generalizes both Kisin's work and Berger-Colmez; we allow any number of fixed Hodge-

Tate-Sen weights and any number of varying weights. More precisely, in Theorem 1.3.20, we provide for any bounded interval $[i,j]$ a decomposition $\mathfrak{X} = \coprod \mathfrak{L}$ of the reduced space $\mathfrak{X}$ into locally closed strata $\pi_{\mathfrak{L}} : \mathfrak{L} \hookrightarrow \mathfrak{X}$ so that $D_B(\pi_{\mathfrak{L}}^* \mathfrak{M})$ is a coherent locally free sheaf if B is bounded in this range. If $\mathfrak{X}$ is a non-reduced affinoid and we interpolate a de Rham period for every fixed Hodge-Tate-Sen weight, we obtain finite flatness after localizing to fix multiplicities (as Kisin does).

We also study specializations of the family. This presents some serious difficulties in general. For example, Mazur and Wiles [47] have studied cases where a family of de Rham representations specialize to one that does not even have a semisimple Sen operator; it is this kind of behavior that we avoid by localization in the result just mentioned. However, in Theorem 1.2.27, we construct Hodge-Tate and de Rham periods at every specialization that was originally removed by this localization. Mazur and Wiles' example demonstrates that our construction gives an optimal bound on the number of periods of the specialization in both the Hodge-Tate and de Rham cases.

The first main result of this paper is Theorem 1.2.19. We provide here a simplified statement of this result as well as Propositions 1.2.36, 1.2.37, and 1.2.38. See Definition 1.2.11 for the notion of stable density, which generalizes Zariski density to the setting of non-reduced rings. See Remark 1.2.30 for the definitions of the topology on B_{dR} and of $\otimes_{\mathbf{Q}_p}$ for non-Fréchet spaces. Note that much of the following holds even if $\mathfrak{R}$ is only a Noetherian $\mathbf{Q}_p$-Banach algebra; see Section 1.2 for precise statements.

Main Theorem 1. *Fix positive integers n and k and a multiset $\{w_1, \dots, w_n\}$ of integers in the interval $[0, k-1]$. Let $S(T) = \prod_{j=1}^{n}(T + w_j)$. Let K be a finite extension of $\mathbf{Q}_p$, let $\mathfrak{R}$ be an affinoid $\mathbf{Q}_p$-algebra, and let $\mathfrak{M}$ be a finitely generated free $\mathfrak{R}$-module equipped with a continuous G_K-action. Suppose that the Sen polynomial $P(T)$ of $\mathfrak{M}$ factors as $P(T) = Q(T)S(T)$, and define $Q_k = \prod_{j=0}^{k-1} Q(-j)$. Assume that $\{\xi_i\}_{i \in I}$ is a set of maps $\xi_i : \mathfrak{R} \to \mathfrak{R}_i$ of $\mathbf{Q}_p$-Banach algebras, where $\mathfrak{R}_i$ is local Artinian of finite dimension over $\mathbf{Q}_p$, with the*

following properties.

(i) *For $i \in I$,* $\dim_K H^0(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}_i)) = n \dim_{\mathbf{Q}_p} \mathfrak{R}_i$.

(ii) *For $i \in I$, the image of Q_k in $\mathfrak{R}_i$ is a unit.*

(iii) *The collection $\{\xi_i\}$ is stably dense in $\mathfrak{R}$.*

We then have the following.

(a) *For $r \in \{0,1\}$, the $K \otimes_{\mathbf{Q}_p} \mathfrak{R}_{Q_k}$-module $H^r(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{\mathbf{Q}_p} \mathfrak{M})_{Q_k}$ is finite flat of rank n.*

(b) *For a map $\xi : \mathfrak{R} \to \mathfrak{R}'$ of affinoid $\mathbf{Q}_p$-algebras, the kernel and cokernel of the map*

$$H^r(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{\mathbf{Q}_p} \mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}' \to H^r(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{\mathbf{Q}_p} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))$$

are killed by a power of $\xi(Q_k)$ for $r \in \{0,1\}$. If $\xi(Q(-j))$ is a unit for $j \notin [0, k-1]$, then $H^r(G_K, B_{\mathrm{dR}} \hat{\otimes}_{\mathbf{Q}_p} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))$ and $H^r(G_K, B^+_{\mathrm{dR}} \hat{\otimes}_{\mathbf{Q}_p} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))$ are finite flat of rank n over $K \otimes_{\mathbf{Q}_p} \mathfrak{R}'$ for $r \in \{0,1\}$.

Let $\xi' : \mathfrak{R} \to \mathfrak{R}'$ be a map of $\mathbf{Q}_p$-Banach algebras with $\mathfrak{R}'$ local Artinian of finite $\mathbf{Q}_p$-dimension. If either $\mathfrak{R}'$ is a field or $\xi'(Q_k)$ is a unit, then for $B \in \left\{B_{\mathrm{dR}}, B^+_{\mathrm{dR}}, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\right\}$ we have

$$\dim_K H^0(G_K, B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \geq \dim_K H^0(G_K, B^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))$$
$$\geq \dim_K H^0(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \geq n \dim_{\mathbf{Q}_p} \mathfrak{R}'$$

$$\textit{and} \quad \dim_K H^0(G_K, B \otimes_{\mathbf{Q}_p} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) = \dim_K H^1(G_K, B \otimes_{\mathbf{Q}_p} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')).$$

To illustrate a possible application of this result, we comment further on the aforementioned Galois representations $\rho_\pi : G_F \to \mathrm{GL}_n(E)$ associated to regular cuspidal algebraic

automorphic representations π on $\mathrm{GL}_{n/F}$ by Harris, Lan, Taylor, and Thorne [27], where F is a CM field and E is a finite extension of $\mathbf{Q}_p$. I. Varma [84] has constructed a family $\mathfrak{M}$ with a dense set of specializations to de Rham representations as well as a specialization to the Galois representation $\rho_\pi \oplus \rho_\pi^{c\vee}(k)$, where $(\cdot)(k)$ denotes a Tate twist and c denotes complex conjugation. Moreover, in $\mathfrak{M}$, she is able to fix the Hodge-Tate-Sen weights associated to the factor ρ_π and have k be such that the weights of $\rho_\pi^{c\vee}(k)$ are negative while those of ρ_π are nonnegative. (Our convention is that $\mathbf{Q}_p(1)$ has the Hodge-Tate weight -1.) In this setting, Theorem 1.2.19 applies, and using Proposition 1.2.36, one shows that $\dim_F D_{B_{\mathrm{dR}}^+}(\rho_\pi \oplus \rho_\pi^{c\vee}(k)) = n[E : \mathbf{Q}_p]$ has exactly the expected dimension.

If $\mathfrak{R}$ is reduced, one can think of Main Theorem 1 as showing that if there are n fixed Hodge-Tate-Sen weights, having n de Rham periods defines a closed condition on $\operatorname{Spm}\mathfrak{R}$. Requiring that these are the same number turns out to be unnecessarily restrictive; one could imagine trying to interpolate m Hodge-Tate periods of weight 0 even though $n > m$ Hodge-Tate weights 0 are fixed, or a similar statement in the de Rham setting.

To achieve this, we define a *de Rham datum* $\mathbf{D} = (\Omega, \Delta)$ in Section 1.3.1, where $\Omega : \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ and $\Delta : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ keep track of the number of fixed Hodge-Tate-Sen weights and the generic dimension of $H^0(G_K, t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} \overline{\mathfrak{M}}_x)$, respectively. (We write $\overline{\mathfrak{M}}_x = \mathfrak{M}_x \otimes_{\mathfrak{O}_{\mathfrak{X},x}} \kappa(x)$, where $\mathfrak{O}_{\mathfrak{X}}$ is the structure sheaf, the subscript x on a sheaf denotes localization at x, and $\kappa(x)$ is the residue field.) The definition encodes the natural conditions that are satisfied by these quantities. We say $\mathbf{D} \geq \mathbf{D}'$ if this inequality holds pointwise for Ω and Δ. We show that on an irreducible rigid space $\mathfrak{X}$ with a coherent locally free module $\mathfrak{M}$ equipped with a continuous G_K action, there is a naturally associated de Rham datum that can be detected locally.

Associated to $\mathbf{D}$ are the Hodge-Tate dimension in the interval (k, ℓ), given by $d_{\mathrm{HT}}^{(k,\ell)}(\mathbf{D}) = \sum_{i=k}^{\ell-1} \Delta(i, i+1)$, and the de Rham dimension, given by $d_{\mathrm{dR}}(\mathbf{D}) = \max_{k,\ell \in \mathbf{Z}} \Delta(k, \ell)$. We also write $\mathrm{Supp}(\mathbf{D})$ for the smallest interval containing the support of Ω. A summary of

Theorems 1.3.19, 1.3.20, and 1.3.21 appears below. We write $\mathbf{D}_{\mathfrak{M},x}$ for the de Rham datum of the representation $\overline{\mathfrak{M}}_x$ over the base $\operatorname{Spm}\kappa(x)$. By "thickened geometric point", we mean a local Artinian $\mathbf{Q}_p$-Banach algebra of finite $\mathbf{Q}_p$-dimension.

Main Theorem 2. *Suppose that $\mathfrak{X}$ is a reduced rigid space over $\mathbf{Q}_p$ and $\mathfrak{M}$ is a coherent locally free sheaf of $\mathfrak{O}_{\mathfrak{X}}$-modules equipped with a homomorphism $G_K \to \operatorname{End}_{\mathfrak{O}_{\mathfrak{X}}}\mathfrak{M}$ that is continuous when restricted to any affinoid.*

For any de Rham datum $\mathbf{D}$, the points $x \in \mathfrak{X}$ such that $\mathbf{D}_{\mathfrak{M},x} \geq \mathbf{D}$ form a closed analytic subvariety $\mathfrak{S}_{\mathbf{D}}$. If $\mathbf{D} \geq \mathbf{D}'$, then $\mathfrak{S}_{\mathbf{D}} \subseteq \mathfrak{S}_{\mathbf{D}'}$.

For any interval $[i,j] \subseteq \mathbf{Z}$, there is a decomposition of $\mathfrak{X}$ into a finite disjoint union of locally closed subspaces $\pi_{\mathbf{D}}^{[i,j]} : \mathfrak{L}_{\mathbf{D}}^{[i,j]} \hookrightarrow \mathfrak{X}$ indexed by de Rham data $\mathbf{D} = (\Omega, \Delta)$ with $\operatorname{Supp}(\mathbf{D}) \subseteq [i,j]$. Let $\mathfrak{M}_{\mathbf{D}}^{[i,j]} = \pi_{\mathbf{D}}^{[i,j]*}\mathfrak{M}$. If $i \leq k < \ell \leq j+1$, there is a coherent locally free sheaf $\mathfrak{H}^r_{(k,\ell)}(\mathfrak{M}_{\mathbf{D}}^{[i,j]})$ such that for any affinoid subdomain $\mathfrak{U} \subseteq \mathfrak{L}_{\mathbf{D}}^{[i,j]}$, there is a canonical isomorphism $\mathfrak{H}^r_{(k,\ell)}(\mathfrak{M}_{\mathbf{D}}^{[i,j]})(\mathfrak{U}) \cong H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}_{\mathbf{D}}^{[i,j]}(\mathfrak{U}))$. Moreover, the formation of $\mathfrak{H}^r_{(k,\ell)}(\mathfrak{M}_{\mathbf{D}}^{[i,j]})$ is compatible with pullback along any morphism $\pi : \mathfrak{Y} \to \mathfrak{L}_{\mathbf{D}}^{[i,j]}$ of reduced rigid spaces.

For any thickened geometric point $\xi : \mathfrak{x} \to \mathfrak{L}_{\mathbf{D}}^{[i,j]}$, where $\mathfrak{x} = \operatorname{Spm}\mathfrak{R}_{\mathfrak{x}}$, we have

(a) $H^r(G_K, t^k B^+_{\mathrm{HT}}/t^\ell B^+_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} \xi^* \mathfrak{M}_{\mathbf{D}}^{[i,j]})$ *is flat over $K \otimes_{\mathbf{Q}_p} \mathfrak{R}_{\mathfrak{x}}$ of rank $d_{\mathrm{HT}}^{(k,\ell)}(\mathbf{D})$,*

(b) $H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} \xi^* \mathfrak{M}_{\mathbf{D}}^{[i,j]})$ *is flat over $K \otimes_{\mathbf{Q}_p} \mathfrak{R}_{\mathfrak{x}}$ of rank $\Delta(k,\ell)$, and*

(c) $\dim_K H^r(G_K, B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} \xi^* \mathfrak{M}_{\mathbf{D}}^{[i,j]}) \geq \dim_K H^{r^+}(G_K, t^i B^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} \xi^* \mathfrak{M}_{\mathbf{D}}^{[i,j]}) \geq d_{\mathrm{dR}}(\mathbf{D}) \dim_{\mathbf{Q}_p} \mathfrak{R}_{\mathfrak{x}}$

for all $i \leq k < \ell \leq j+1$ and $r, r^+ \in \{0,1\}$.

We can recover the reduced case of Main Theorem 1 by setting $\mathbf{D}$ to be a *full* de Rham datum, which is one for which $\Omega(i) = \Delta(i, i+1)$ and $\Delta(i,k) = \Delta(i,j) + \Delta(j,k)$ for all $i \leq j \leq k \in \mathbf{Z}$.

In Section 1.4, we prove strong vanishing results for higher continuous Galois cohomology of modules of Hodge-Tate or de Rham periods in a family. The key ingredient, Theorem 1.4.1, generalizes a result of Sen [67, Proposition 2]. The main result, Theorem 1.4.6, is the following.

Main Theorem 3. *Suppose that K is a finite extension of $\mathbf{Q}_p$, $\mathfrak{R}$ is a Noetherian $\mathbf{Q}_p$-Banach algebra, and the finitely generated $\mathfrak{R}$-Banach module $\mathfrak{M}$ is equipped with a continuous $\mathfrak{R}$-linear action of G_K. Then for $n \geq 2$, $k \leq \ell \in \mathbf{Z}$, and $B \in \left\{B_{\mathrm{HT}}, t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+, t^k B_{\mathrm{dR}}^+, B_{\mathrm{dR}}\right\}$, we have $H^n(G_K, B\hat{\otimes}_{\mathbf{Q}_p}\mathfrak{M}) = 0$.*

1.1.3 Method of proof

Berger-Colmez [5] use an approach very different from ours; they use the theory of (φ, Γ)-modules while we work directly with the G_K-module itself. Moreover, there is a series of works using the (φ, Γ)-module point of view that obtain strong results regarding the interpolation of crystalline (and even semistable) periods [41, 29, 35, 42].

Our approach to Theorem 1.2.19 is similar to Kisin's work in [36, §2]. Kisin shows that in *any* family of representations with a single fixed Hodge-Tate-Sen weight 0, the $\mathbf{C}_p$-periods form a finite flat module of rank 1 after localizing to remove possible multiplicity of this weight. However, if a Hodge-Tate-Sen weight occurs with multiplicity or if one has distinct Hodge-Tate-Sen weights and would like to study de Rham periods, this property is no longer automatic. One needs to use the existence of Hodge-Tate or de Rham periods at a dense set of points. To interpolate these periods, we devise an algebraic situation in which finite flatness of $D_B(\mathfrak{M})$ will follow from the density of the given specializations. Since the formation of $D_B(\mathfrak{M})$ is not always compatible with specialization, we use Sen's theory to study a simpler module $\mathfrak{E}$ associated to $\mathfrak{M}$ instead [67, 68].

Another consideration is that in attempting to pass to unbounded de Rham periods, the

direct approach requires one to localize at all integral weights above the fixed range. For thickenings of geometric points, we control the dimension of the space of periods as one passes to the limit. This requires some functional analysis.

We also need to retain information when multiplicities change. For this, we study the behavior of a morphism under base change in Section 1.2.5. The series of lemmas proved there use Sen's theory to show that certain tensor products of Banach spaces are already complete, allowing us to apply homological algebra. By various arguments that allow us to switch from 1-cocycles to 0-cocycles, we are able to show existence of periods at specializations anywhere on the base.

For the stratification result, we apply the same techniques, but since we no longer have fine control over the localization needed to make the module of periods finite flat, we need a slightly different approach. In order to globalize in this setting, we first study the affinoid case, and then apply those interpolation results to analytically continue across intersections.

1.2 Interpolating de Rham and Hodge-Tate periods and 1-cocycles

We study the interpolation of Hodge-Tate and de Rham periods over a Noetherian Banach algebra. We prove some useful lemmas in commutative algebra, and then summarize the work of Sen [67, 68] and Kisin [36, §2] on the construction and properties of the Sen operator. We proceed to the proofs of Theorem 1.2.13 in the Hodge-Tate setting and Theorem 1.2.19 in the de Rham setting for periods over an open set $\operatorname{Spec} \mathfrak{R}_{Q_{\sigma,k}}$. We then study specializations on all of $\operatorname{Spec} \mathfrak{R}$ under the additional assumption that $\mathfrak{R}$ is affinoid, pass from bounded periods to the full ring B_{dR}, and give an application to essentially self-dual specializations.

1.2.1 Commutative algebra

All rings are commutative. When we say a module is finite, we mean that it is finitely generated.

Lemma 1.2.1. *Suppose that G is a finite group, K is a field of characteristic 0, and M is a module over a K-algebra R equipped with an R-linear G-action.*

(a) *If M is finite (resp. flat, projective), then M^G is finite (resp. flat, projective).*

(b) *For any R-module N, the natural map $\varphi : N \otimes_R M^G \to (N \otimes_R M)^G$ is an isomorphism.*

Proof. There is a functorial decomposition $M = M^G \oplus M^{G,\perp}$ as the sum of the image and kernel of the projection $\frac{1}{|G|}\sum_{g\in G} g : M \to M$. Part (a) follows, as well as that φ is injective. For any $K[G]$-module P, the image of $\frac{1}{|G|}\sum_{g\in G} g : P \to P$ is P^G, and the image of this map for the module $N \otimes_R M$ lands in the submodule $N \otimes_R M^G$. Thus φ is also surjective.

□

We will use the next lemma to decompose a module into finite flat summands. We state it in a slightly more general form in case it may be helpful in other contexts.

Lemma 1.2.2. *Suppose M is a module over the ring R. Let $r_1, \ldots, r_n \in R$, let $k_1, \ldots, k_n \in \mathbf{Z}_{\geq 1}$, let $S(x) = \prod_{i=1}^n (x - r_i)^{k_n}$, let $Q(x) \in R[x]$, let $P(x) = Q(x)S(x)$, and assume that $r = \prod_i Q(r_i) \prod_{i<j}(r_i - r_j)$ is a unit in R. Let $\varphi : M \to M$ be a morphism of R-modules such that $P(\varphi)M = 0$.*

Define $S_i(x) = \prod_{j\neq i}(x - r_j)^{k_j}$ and $P_i(x) = Q(x)S_i(x)$ for $1 \leq i \leq n$. We define the following submodules of M: $M_0 = \ker Q(\varphi), M^0 = S(\varphi)M, M_i = \ker(\varphi - r_i)^{k_i}$, and $M^i = P_i(\varphi)M$. Then we have $M_i = M^i$ for $0 \leq i \leq n$ and a φ-compatible splitting $M = \oplus_{i=0}^n M_i = \oplus_{i=0}^n M^i$.

Proof. Given a factorization $U(x) = (x-r)^k T(x)$, where $U, T \in R[x]$ and $T(r)$ is a unit, $T(x)$ is a unit in $R[x]/(x-r)$ and thus in $R[x]/((x-r)^k)$. So $T(x)$ and $((x-r)^k)$ are coprime, and

$$R[x]/U(x) = R[x]/T(x) \times R[x]/(x-r)^k$$

by the Chinese remainder theorem. Setting $U(x) = Q(x)\prod_{j=i}^n (x-r_j)^{k_j}$, $r = r_i$, $k = k_i$, and $T(x) = Q(x)\prod_{j=i+1}^n (x-r_j)^{k_j}$ for each i, we deduce

$$R[x]/P(x) = \underbrace{R[x]/Q(x)}_{R_0} \times \prod_{i=1}^n \underbrace{R[x]/(x-r_i)^{k_i}}_{R_i}.$$

Since $P(\varphi)M = 0$, M can be regarded as an $R[x]/P(x)$-module with x acting as φ. All claims above follow from the observations that each $P_i(x)$ (resp. each $(x-r_i)^{k_i}$) is a unit multiple of the idempotent projecting onto R_i (resp. $R_0 \times \prod_{i\neq j} R_i$), and $S(x)$ (resp. $Q(x)$) is a unit multiple of the the idempotent projecting onto R_0 (resp. $\prod_{i=1}^n R_i$).

□

We will also use the following easy lemma.

Lemma 1.2.3. *Let M be a flat module over a ring R equipped with an endomorphism $\psi : M \to M$. If ψ has flat cokernel, then it has flat kernel and image. Moreover, the formation of* $\ker\psi$, $\operatorname{im}\psi$, *and* $\operatorname{coker}\psi$ *commutes with taking the tensor product with an R-module N. If M is also finitely generated, and R is Noetherian, then the kernel, image, and cokernel of ψ are finitely generated, with the rank of the kernel equal to the rank of the cokernel on each connected component of* $\operatorname{Spec} R$.

If we drop the hypotheses that M is flat and ψ has flat cokernel, but instead ask that the R-module N is flat, then tensoring with N again commutes with formation of $\ker\psi$, $\operatorname{im}\psi$, *and* $\operatorname{coker}\psi$.

Proof. We have exact sequences

$$0 \to \ker \psi \to M \to \operatorname{im} \psi \to 0 \quad \text{and} \quad 0 \to \operatorname{im} \psi \to M \to \operatorname{coker} \psi \to 0. \tag{1.1}$$

It follows that if $\operatorname{coker} \psi$ is flat, so are $\operatorname{im} \psi$ and $\ker \psi$. The finite generation claim is clear, and equality of ranks follows from examining the localization of (1.1) at a prime of R in each connected component.

Let N be an R-module, and assume that either M and $\operatorname{coker} \psi$ or N are flat. By tensoring (1.1) with N and using the flatness of either $\operatorname{coker} \psi$ and $\operatorname{im} \psi$ or N, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\ker \psi) \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & (\operatorname{coker} \psi) \otimes_R N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{=} & & \downarrow{=} & & \downarrow & & \\ 0 & \longrightarrow & \ker(\psi \otimes_R N) & \longrightarrow & M \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & \operatorname{coker}(\psi \otimes_R N) & \longrightarrow & 0 \end{array}$$

with both rows exact. By the 5-lemma, we obtain isomorphisms

$$(\ker \psi) \otimes_R N \xrightarrow{\sim} \ker(\psi \otimes_R N) \text{ and } (\operatorname{coker} \psi) \otimes_R N \xrightarrow{\sim} \operatorname{coker}(\psi \otimes_R N)$$

via the natural maps. The last part of the claim follows similarly from the exact sequence

$$0 \to (\operatorname{im} \psi) \otimes_R N \to M \otimes_R N \to (\operatorname{coker} \psi) \otimes_R N \to 0$$

and the isomorphism $(\operatorname{coker} \psi) \otimes_R N \xrightarrow{\sim} \operatorname{coker}(\psi \otimes_R N)$ above.

□

We can say more in the case of the cokernel.

Lemma 1.2.4. *Suppose $\psi : M \to N$ is a map of R-modules and P is an R-module. Then the natural map $(\operatorname{coker} \psi) \otimes_R P \to \operatorname{coker}(\psi \otimes_R P)$ is an isomorphism.*

Proof. Tensor products commute with colimits.

□

We will need an explicit description of continuous $\mathbf{Z}_p$-cohomology for certain nice modules.

Lemma 1.2.5. *Suppose that Γ is the additive topological group $\mathbf{Z}_p$, and write γ for the topological generator $1 \in \mathbf{Z}_p$. Let M be a continuous Γ-module with the property that there exist Γ-invariant submodules M_i for $i \in \mathbf{Z}_{\geq 0}$ such that $M_i \subseteq M_{i-1}$, $M \cong \varprojlim_i M/M_i$ with the inverse limit topology, and each M/M_i is a discrete p^∞-torsion module, meaning that each $m \in M/M_i$ is killed by a finite power of p. Then there is an isomorphism $H^1(\Gamma, M) \cong \operatorname{coker}(\gamma - 1 : M \to M)$.*

Proof. It suffices to show that for any $m \in M$, there is a continuous 1-cocycle $\varphi : \Gamma \to M$ with $\varphi(\gamma) = m$. The cocycle relation forces $\varphi(\gamma^k) = \sum_{j=0}^{k-1} \gamma^j m$ for $k \in \mathbf{Z}_{>0}$. Since a continuous $\varphi : \Gamma \to M$ is determined by $\varphi|_{\mathbf{Z}_{>0}}$, we need only check that this has a continuous extension to Γ. In fact, we need only check that $\varphi|_{\mathbf{Z}_{>0}} : \Gamma \to M_i$ has a continuous extension for each i; these are automatically compatible and give $\varphi : \Gamma \to M$ by the universal property. This is proved in [51, Proposition 1.7.7].

□

1.2.2 Some results of Sen and Kisin

Fix a separable algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and denote by $\mathbf{C}_p$ its completion. For any finite extension $K \subseteq \overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, let $G_K = \operatorname{Gal}(\overline{\mathbf{Q}}_p/K)$ be the absolute Galois group of K, and let $\chi_K : G_K \to \mathbf{Z}_p^\times$ denote the cyclotomic character. Denote the kernel of χ_K^{p-1} (or χ_K^2 if $p = 2$) by H_K, and write $\Gamma_K = G_K/H_K$. Note that Γ_K is isomorphic to $\mathbf{Z}_p$. We define $K_\infty = \overline{\mathbf{Q}}_p^{H_K}$ and $\widehat{K}_\infty = \mathbf{C}_p^{H_K}$.

Let $K/\mathbf{Q}_p$ and $E/\mathbf{Q}_p$ be finite extensions such that E contains the normal closure of K. Define $\Sigma = \{\sigma : K \to E\}$ to be the set of embeddings of K in E. Let $\mathfrak{R}$ be a Noetherian Banach algebra over E and let $\mathfrak{M}$ be a free $\mathfrak{R}$-module of finite rank equipped with a continuous G_K-action.

We denote the completed tensor product of Banach spaces or modules by $\hat{\otimes}$, and we always require any map between topological algebras or modules to be continuous. In this paper, rings are always commutative (except for endomorphism rings of modules) and cohomology is always continuous cohomology. We *never* regard a cohomology group $H^r(G, M)$ for $r \geq 1$ as having a topology, even if it is possible to do so by virtue of it being a finitely generated module over a Banach algebra. (But we do equip $H^0(G, M)$ with the subspace topology when M is topological.)

Using the work of Sen [67, 68], after a base change in K, Kisin relates the study of G_K-cohomology of $\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}$ to the action of a linear operator on a finite free module $\mathfrak{E}$ over $\mathfrak{R}$. Kisin works under the hypothesis $E \subseteq K$, but we have instead reformulated his results for E containing the normal closure of K, and a choice of embedding $\sigma : K \to E$. To deduce the forms of his results given below, one sets Kisin's E to be K and regards the E-algebra $\mathfrak{R}$ as a K-algebra via the choice of σ. In this paper, we will always fix σ and regard $\mathfrak{R}$ as a K-algebra in this way. We also fix a choice of topological generator $\gamma \in \Gamma_K$.

Proposition 1.2.6 ([36, §2])**.** *We have the following facts regarding the H_K-invariants of $\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}$.*

(a) *After replacing E and K with sufficiently large finite Galois extensions, the $\widehat{K}_\infty\hat{\otimes}_{K,\sigma}\mathfrak{R}$-module $(\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K}$ is free with a basis $\mathbf{e} = \{e_1, \ldots, e_n\}$ such that the free $\mathfrak{R}$-module $\mathfrak{E}$ generated by $\mathbf{e}$ is stable by Γ_K.*

(b) *We may define a map $\phi \in \mathrm{End}_{\mathfrak{R}}\,\mathfrak{E}$ by $\phi = (\log\chi(\gamma^{p^r}))^{-1} \cdot \log(\gamma^{p^r}|_{\mathfrak{E}})$ for any sufficiently large integer r, where $\log(\gamma^{p^r}|_{\mathfrak{E}})$ is defined using the standard series expansion in $\gamma^{p^r} - 1$.*

(c) *Denote also by ϕ the extension of scalars to $(\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K}$, and call the characteristic polynomial of ϕ on this module $P_\sigma(T) \in \widehat{K}_\infty\hat{\otimes}_{K,\sigma}\mathfrak{R}[T]$. Then in fact, we have $P_\sigma(T) \in \mathfrak{R}[T]$, even before increasing E and K. Moreover, this polynomial is canonically attached to $\mathfrak{M}$.*

(d) *The preceding constructions are all compatible with base change along a map of Noetherian E-Banach algebras $\mathfrak{R} \to \mathfrak{R}'$, without requiring a further extension of K. In particular, we have*

$$\mathfrak{E}_{\mathfrak{R}'} = \mathfrak{E} \otimes_{\mathfrak{R}} \mathfrak{R}' \text{ and } (\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K}\hat{\otimes}_{\mathfrak{R}}\mathfrak{R}' \xrightarrow{\sim} (\mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))^{H_K},$$

and these are equipped with the Sen operator $\phi\hat{\otimes}_{\mathfrak{R}}\mathfrak{R}'$.

(e) *If $\mathfrak{R}' = E'$ is a finite extension field of E, then $\phi\hat{\otimes}_{\mathfrak{R}}E'$ coincides with the Sen operator of [66].*

We check that the kernel and cokernel of the Sen operator calculate G_K-cohomology of $\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}$.

Proposition 1.2.7 ([36, Proposition 2.3])**.** *Write*

$$\psi = \gamma - 1 : (\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \to (\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K}.$$

After replacing E and K with finite extensions, we have $\ker(\psi) = \ker(\phi|_{\mathfrak{E}})$ *and* $\operatorname{coker}(\psi) = \operatorname{coker}(\phi|_{\mathfrak{E}})$. *Moreover, if $\mathfrak{R} \to \mathfrak{R}'$ is a map of Noetherian E-Banach algebras, the same is true for $\mathfrak{R}'$ without a further extension of K.*

Proof. In the proof of [36, Proposition 2.3], Kisin shows that after increasing K beyond that used in Proposition 1.2.6 if necessary, the kernel and cokernel of $\phi|_{\mathfrak{E}}$ are equal, respectively,

to the kernel and cokernel of $\gamma - 1$ on all of $(\mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M})^{H_K}$, so we are left to check the change of ring.

Kisin uses a construction by Tate of a topological isomorphism $\widehat{K}_\infty = K \oplus X_0$ such that X_0 is Γ_K-stable and $1 - \gamma : X_0 \to X_0$ is an isomorphism [77]. We then find that

$$(\mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M})^{H_K} \cong \widehat{K}_\infty \hat{\otimes}_{K,\sigma} \mathfrak{E} \cong \mathfrak{E} \oplus X_0 \hat{\otimes}_{K,\sigma} \mathfrak{E}.$$

Kisin shows that on the first factor, $-\psi = 1 - \exp(\log(\chi(\gamma))\phi) = \log(\chi(\gamma))\phi G(\phi)$, where $G(\phi) \in \mathrm{GL}_n(\mathfrak{R})$ for sufficiently large K (which replaces γ by γ^{p^r}). A base change will not alter this property.

On the second factor, $X_0 \hat{\otimes}_{K,\sigma} \mathfrak{E}$, Kisin rewrites the action of $-\psi = 1 - \gamma$ as

$$1 - \gamma \otimes \gamma = (\gamma(1-\gamma)^{-1} \otimes (1-\gamma) + 1)((1-\gamma) \otimes 1),$$

and shows that if we replace K by a sufficiently large extension, the action of $1 - \gamma$ is topologically nilpotent on $\mathfrak{E}$. This calculation then implies that $1 - \gamma$ acts as an isomorphism on $X_0 \hat{\otimes}_{K,\sigma} \mathfrak{E}$. For a ring extension of Noetherian E-Banach algebras, $1 - \gamma$ will remain an isomorphism, so there is no need to extend K further.

□

Corollary 1.2.8. *For E and K sufficiently large, we have*

$$H^0(G_K, \mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M}) \cong \ker(\phi|_{\mathfrak{e}}) \text{ and } H^1(G_K, \mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M}) \cong \mathrm{coker}(\phi|_{\mathfrak{e}}).$$

Proof. The first follows from Proposition 1.2.7. For the second, by Proposition 1.2.7, the inflation-restriction exact sequence, and the vanishing $H^1(H_K, \mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M}) = 0$ [67, Proposition 2], we are reduced to checking that $H^1(\Gamma_K, (\mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M})^{H_K}) \cong \mathrm{coker}(\gamma - 1)$. This follows from Lemma 1.2.5.

□

Remark 1.2.9. For the remainder of this paper, we always use ϕ for the map on $\mathfrak{E}$. We also use ϕ only when K is large enough so that the conclusions of Proposition 1.2.6, Proposition 1.2.7, and Corollary 1.2.8 hold. However, we make use of $P_\sigma(T)$ more generally, since it is defined as an element of $\mathfrak{R}[T]$.

Let $P_\sigma(T) \in \mathfrak{R}[T]$ denote the Sen polynomial of $\mathfrak{M}$.

Proposition 1.2.10 ([36, Proposition 2.3])**.** *Fix $\sigma \in \Sigma$. Then the $\mathfrak{R}$-modules*

$$H^0(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \text{ and } H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})$$

are finitely generated and killed by $\det(\phi) = P_\sigma(0) \in \mathfrak{R}$.

1.2.3 $\mathbf{C}_p$-periods in families

Kisin [36, Proposition 2.4] considers a family with a single fixed Hodge-Tate-Sen weight 0 of multiplicity 1, localizes to fix this multiplicity, and shows that the $\mathbf{C}_p$-periods form a finite flat module of rank 1 that is compatible with base change. We generalize this result, though we require an extra hypothesis on the family if the multiplicity is greater than 1: there exist a dense set of specializations with m Hodge-Tate weights equal to 0. On the open set where the weight 0 has fixed multiplicity m, we show that both the $\mathbf{C}_p$-periods and 1-cocycles interpolate in a finite flat module and are compatible with base change.

In order to interpolate periods over E-Banach algebras that are possibly non-reduced, we will need to define a robust version of density that generalizes Zariski density. For specializations $\xi_i : \mathfrak{R} \to \mathfrak{R}_i$, we want the map $\mathfrak{R} \to \prod_i \mathfrak{R}_i$ to be injective. When each $\mathfrak{R}_i$ is a finite field extension of E, this is Zariski density of $\{\ker \xi_i\}$. Limiting ourselves to this case would force $\mathfrak{R}$ to be reduced. However, when the $\mathfrak{R}_i$ are permitted to be local Artinian rings

of finite E-dimension, the condition that $\mathfrak{R} \hookrightarrow \prod_i \mathfrak{R}_i$ is no longer stable under localization. This motivates the following definition.

Definition 1.2.11. For a local Artinian algebra R with maximal ideal $\mathfrak{m}$, we define the *breadth* of R, $\mathrm{br}(R) \in \mathbf{Z}_{\geq 1}$, by $\mathrm{br}(R) = \min\{n : \mathfrak{m}^n = 0\}$. We say that a set $\{\xi_i\}$ of specializations $\xi_i : R \to R_i$ of a ring R to local Artinian rings R_i is *stably dense* if $R \hookrightarrow \prod_i R_i$ is injective and $\{\mathrm{br}(R_i)\}$ is bounded above.

The term "stable" is justified by the following observation.

Lemma 1.2.12. *Suppose that $\{\xi_i\}$ is a stably dense set of specializations of the ring R to local Artinian rings R_i. Then for any $f \in R$, the localizations $\{\xi_{i,f}\}$ are stably dense in R_f.*

Proof. Let $\mathfrak{m}_i$ be the maximal ideal of R_i. Let f be an element of R, and suppose that $\frac{a}{f^k} \in \ker(R_f \to \prod_i R_{i,f})$. Then a is also in the kernel. We have $R_{i,f} = 0$ if $\xi_i(f) \in \mathfrak{m}_i$ and $R_{i,f} = R_i$ if $\xi_i(f) \notin \mathfrak{m}_i$. Write $\prod_i R_i = \prod_{\{i:\xi_i(f)\in\mathfrak{m}_i\}} R_i \times \prod_{\{i:\xi_i(f)\notin\mathfrak{m}_i\}} R_i$. Then the image of a is 0 in the second factor, while the image of f is nilpotent in the first by the bounded breadth condition. Therefore $f^b a = 0$ for $b = \max\{\mathrm{br}(R_i)\}$, so $a = 0 \in R_f$. The bounded breadth condition remains satisfied, so we conclude that $\{\xi_{i,f}\}$ is stably dense.

□

We proceed to the proof of the main theorem of this section.

Theorem 1.2.13. *Maintain the notation of Section 1.2.2. Fix $\sigma \in \Sigma$. Suppose that $P_\sigma(T) = T^m Q_\sigma(T)$. If $m \geq 2$, assume that there exist maps $\xi_i : \mathfrak{R} \to \mathfrak{R}_i$ of Banach E-algebras for $i \in I$, where $\mathfrak{R}_i$ is local Artinian of finite E-dimension, such that the following hold.*

(i) *For each i, $\dim_E H^0(G_K, \mathbf{C}_p \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}_i)) = m \dim_E \mathfrak{R}_i$.*

(ii) *For each i, the image of $Q_\sigma(0)$ in $\mathfrak{R}_i$ is a unit.*

(iii) *The set $\{\xi_i\}$ is stably dense in $\mathfrak{R}$.*

Then we have the following. We use subscripts to denote localization.

(a) *For $r \in \{0,1\}$, the $\mathfrak{R}_{Q_\sigma(0)}$ module $H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})_{Q_\sigma(0)}$ is finite flat of rank m.*

(b) *For any map $\xi : \mathfrak{R} \to \mathfrak{R}'$ of Noetherian E-Banach algebras, the cokernel and kernel of the map $H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}' \to H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))$ are killed by a power of $\xi(Q_\sigma(0))$ if $r = 0$, and vanish if $r = 1$.*

Remark 1.2.14. Whenever we localize a ring $\mathfrak{R}$ or $\mathfrak{R}$-module, we do so in the categories of rings and their modules, with no topology or norm structure. When we write $(\cdot)^G_a$, we always mean the localization of $(\cdot)^G$ at a.

Remark 1.2.15. The rings $\mathfrak{R}_i$ will usually be geometric points, or possibly thickened geometric points of the form $\mathfrak{R}/\mathfrak{m}^n$.

To apply the results of Section 1.2.2, we will need the following.

Lemma 1.2.16. *For a finite Galois extension E' of E, a profinite group G, an integer $r \geq 0$, and a continuous $E[G]$-module M, we have a natural isomorphism*

$$H^r(G, M) \otimes_E E' \xrightarrow{\sim} H^r(G, M \otimes_E E'). \tag{1.2}$$

For a finite Galois extension K' of K whose normal closure is contained in E, a fixed choice of extension $\sigma' : K' \to E$ of σ, and $r \in \{0,1\}$, we have a natural isomorphism

$$H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \xrightarrow{\sim} H^r(G_{K'}, \mathbf{C}_p\hat{\otimes}_{K',\sigma'}\mathfrak{M}). \tag{1.3}$$

Proof. Let $C^\cdot(G, M)$ denote the complex of continuous cochains. The natural map

$$C^\cdot(G, M) \otimes_E E' \xrightarrow{\sim} C^\cdot(G, M \otimes_E E')$$

is an isomorphism since there is a topological isomorphism $E' \cong E^k$ for some k. The isomorphism (1.2) follows from this and Lemma 1.2.3, using the flat E-module E' for N.

By inflation-restriction and the vanishing of finite group cohomology in characteristic 0, we have for any finite Galois extension K'/K contained in E and $r \in \{0,1\}$ a natural isomorphism

$$H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \xrightarrow{\sim} H^r(G_{K'}, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{\mathrm{Gal}(K'/K)},$$

where $\mathrm{Gal}(K'/K)$ acts on both $\mathbf{C}_p$ and $\mathfrak{M}$.

For clarity, we write K' for the field with its usual $\mathrm{Gal}(K'/K)$ action and K'_{triv} for K' with the trivial action. We claim that for any $\sigma' : K'_{\mathrm{triv}} \to E$ extending σ, we have

$$\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{R} = \mathbf{C}_p\hat{\otimes}_K(K'_{\mathrm{triv}}\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{R}) = (\mathbf{C}_p\hat{\otimes}_K K'_{\mathrm{triv}})\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{R} \cong \mathbf{C}_p^{\{\tau|\tau\in\mathrm{Gal}(K'/K)\}}\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{R}, \tag{1.4}$$

where the action of G_K on the right acts on each copy of $\mathbf{C}_p$ as well as by permutation via its quotient $\mathrm{Gal}(K'/K)$ on the factors, and acts trivially on $\mathfrak{R}$. In particular, σ sends $(c_\tau)_\tau$ to $(\sigma(c_{\sigma^{-1}\tau}))_\tau$. The last map is obtained from the G_K-equivariant map $\mathbf{C}_p\hat{\otimes}_K K'_{\mathrm{triv}} \to \mathbf{C}_p^{\{\tau|\tau\in\mathrm{Gal}(K'/K)\}}$ defined by $c\otimes k \mapsto (c\cdot\tau(k))_\tau$ (where $\tau(k)$ denotes the usual action of τ on k). It follows from independence of characters that if $\{k_j\}$ are a K-basis of K' and $\{\tau_i\} = \mathrm{Gal}(K'/K)$, the matrix $(\tau_i(k_j))_{i,j}$ is invertible, so this is an isomorphism. We deduce

$$\begin{aligned} H^r(G_{K'}, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{\mathrm{Gal}(K'/K)} &\cong H^r(G_{K'}, \mathbf{C}_p^{\{\tau|\tau\in\mathrm{Gal}(K'/K)\}}\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{\mathrm{Gal}(K'/K)} \\ &\cong \left[H^r(G_{K'}, \mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{\{\tau|\tau\in\mathrm{Gal}(K'/K)\}}\right]^{\mathrm{Gal}(K'/K)}. \end{aligned}$$

The second map is induced by the natural G_K-equivariant Banach space isomorphism

$$\mathbf{C}_p^{\{\tau\in\mathrm{Gal}(K'/K)\}}\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M} \xrightarrow{\sim} (\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{\{\tau\in\mathrm{Gal}(K'/K)\}} \tag{1.5}$$

defined by sending $(c_\tau)_\tau \otimes m$ to $(c_\tau \otimes m)_\tau$ and extending by continuity.

We construct an isomorphism

$$H^r(G_{K'}, \mathbf{C}_p \hat{\otimes}_{K'_{\mathrm{triv}},\sigma'} \mathfrak{M}) \xrightarrow{\sim} [H^r(G_{K'}, \mathbf{C}_p \hat{\otimes}_{K'_{\mathrm{triv}},\sigma'} \mathfrak{M})^{\{\tau | \tau \in \mathrm{Gal}(K'/K)\}}]^{\mathrm{Gal}(K'/K)}$$

by mapping the cocycle ψ to $(\tau\psi)_{\tau \in \mathrm{Gal}(K'/K)}$. Combining the above, we obtain (1.3); we write K' on the right in that expression since there is no $\mathrm{Gal}(K'/K)$-action. □

Proof of Theorem 1.2.13. We claim that may replace the pair K, E with any Galois extensions K', E' such that E' contains the normal closure of K'. More precisely, if we replace $\mathfrak{R}$ with $\mathfrak{R} \otimes_E E'$, $\mathfrak{M}$ with $\mathfrak{M} \otimes_E E'$, E with E', and K with K' in Theorem 1.2.13, the old conditions (i)-(iii) imply the new conditions and the new conclusions (a) and (b) imply the original (a) and (b). In fact, for all these conditions and conclusions, the ability to swap K for K' and vice-versa follows immediately from (1.3), so we need only check the change in E.

If we replace $\mathfrak{R}_i \otimes_E E'$ with the set of its local Artinian factors, the claims regarding E and conditions (ii) and (iii) are clear. For (i), we apply Lemma 1.2.16 to calculate the new E-dimension for the base change to $\mathfrak{R}_i \otimes_E E'$. We will obtain (i) for each local Artinian factor by showing in Claim 1.2.17 below that no Artinian local $\mathfrak{R} \otimes_E E'$-algebra $\mathfrak{R}'$ of finite E'-dimension with unit image of $Q_\sigma(0)$ can have $\dim_{E'} H^0(G_K, \mathbf{C}_p \hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) > m \dim_{E'} \mathfrak{R}'$. Note that will not make use of condition (i) before proving this claim.

Suppose that E'/E is a finite Galois extension and M is a $\mathfrak{R}$-module. By Lemma 1.2.1.(b), $(M \otimes_E E')^{\mathrm{Gal}(E'/E)} = M$, where $\mathrm{Gal}(E'/E)$ acts only on E'. If $M \otimes_E E'$ is finite flat (when viewed as a $\mathfrak{R} \otimes_E E'$-module), it is finite flat over $\mathfrak{R}$, and so by Lemma 1.2.1.(a), M is finite flat over $\mathfrak{R}$ as well. Moreover, we have $\mathrm{rank}_{\mathfrak{R}} M = \mathrm{rank}_{\mathfrak{R} \otimes_E E'} M \otimes_E E'$. We set

$M = H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})_{Q_\sigma(0)}$. By (1.2),

$$\begin{aligned} H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})_{Q_\sigma(0)} \otimes_E E' &\cong (H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_E E')_{Q_\sigma(0)} \\ &\cong H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_E E'))_{Q_\sigma(0)}. \end{aligned}$$

The reduction of Theorem 1.2.13.(a) from E to E' follows.

For the reduction of Theorem 1.2.13.(b), we suppose a change of ring $\mathfrak{R} \to \mathfrak{R}'$ is given, and consider the exact sequence

$$\begin{aligned} 0 \to \ker\psi \to H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_E E')) \otimes_{\mathfrak{R}\otimes_E E'} (\mathfrak{R}' \otimes_E E') \\ \xrightarrow{\psi} H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}((\mathfrak{M} \otimes_E E') \otimes_{\mathfrak{R}\otimes_E E'} \mathfrak{R}' \otimes_E E')) \to \operatorname{coker}\psi \to 0. \end{aligned} \tag{1.6}$$

For any $\mathfrak{R}$-module P, there is a natural identification

$$(P \otimes_E E') \otimes_{\mathfrak{R}\otimes_E E'} (\mathfrak{R}' \otimes_E E') = (P \otimes_{\mathfrak{R}} \mathfrak{R}') \otimes_E E', \tag{1.7}$$

so by using this and (1.2) we may rewrite ψ as

$$(H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}') \otimes_E E' \to H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \otimes_E E'.$$

Take $\mathrm{Gal}(E'/E)$-invariants of (1.6) and apply Lemma 1.2.1.(b) to obtain

$$\begin{aligned} 0 \to (\ker\psi)^{\mathrm{Gal}(E'/E)} \to H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}' \xrightarrow{\psi^{\mathrm{Gal}(E'/E)}} H^r(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \\ \to (\operatorname{coker}\psi)^{\mathrm{Gal}(E'/E)} \to 0. \end{aligned} \tag{1.8}$$

By vanishing of cohomology in characteristic 0, the sequence remains exact. Finally, if a

finite power of $\xi(Q_\sigma(0))$ kills $\ker\psi$ and $\operatorname{coker}\psi$, it kills their $\mathrm{Gal}(E'/E)$-invariants.

Replace K and E by extensions so that we may use Proposition 1.2.6 and Corollary 1.2.8. We have a map $\phi : \mathfrak{E} \to \mathfrak{E}$ of finite free $\mathfrak{R}$-modules such that $\ker\phi \cong H^0(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})$ and $\operatorname{coker}\phi \cong H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})$. Moreover, the operator $P_\sigma(\phi)$ annihilates $\mathfrak{E}$.

By Lemma 1.2.2, there is a $\phi_{Q_\sigma(0)}$-compatible splitting

$$\mathfrak{E}_{Q_\sigma(0)} = \ker(\phi^m_{Q_\sigma(0)}) \oplus \ker(Q_\sigma(\phi_{Q_\sigma(0)})), \tag{1.9}$$

so both of these modules are finite flat over the (non-topological) ring $\mathfrak{R}_{Q_\sigma(0)}$.

For any $\mathfrak{R}$-module P and $\mathfrak{R}$-algebra $\mathfrak{R}'$, we have natural identifications

$$P_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}'_{Q_\sigma(0)} = P \otimes_{\mathfrak{R}} \mathfrak{R}_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}'_{Q_\sigma(0)} = P \otimes_{\mathfrak{R}} \mathfrak{R}'_{Q_\sigma(0)}. \tag{1.10}$$

In particular, for $\mathfrak{R}'$ in which $Q_\sigma(0)$ is a unit, we have

$$\mathfrak{E} \otimes_{\mathfrak{R}} \mathfrak{R}' = \mathfrak{E}_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}' \text{ and } \phi \otimes_{\mathfrak{R}} \mathfrak{R}' = \phi_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}'. \tag{1.11}$$

We will use this to switch between the specialization of the localized module (which allows us to use its flatness) and the specialization of the original module (for which the properties in Proposition 1.2.6 hold).

Claim 1.2.17. *We have* $\operatorname{rank}_{\mathfrak{R}'} \ker((\phi \otimes_{\mathfrak{R}} \mathfrak{R}')^m) = m$ *for any* $\mathfrak{R}$*-algebra* $\mathfrak{R}'$ *such that* $Q_\sigma(0)$ *is a unit in* $\mathfrak{R}'$*.*

Proof. For each connected component of $\operatorname{Spec}\mathfrak{R}_{Q_\sigma(0)}$, fix one of the specializations $\mathfrak{R}_i$ whose spectrum maps to it and consider the further specialization $\xi'_i : \mathfrak{R} \to E_i$ to its residue field, which is a finite extension of E. By Proposition 1.2.6.(e), the specialization $\phi \otimes_{\mathfrak{R}} E_i$ is the classical Sen operator, and has Sen polynomial $\xi'_i(P_\sigma(T)) \in E_i[T]$. By condition (ii),

$\xi_i'(P_\sigma(T))$ has a root of order exactly m at 0. Therefore the generalized eigenspace of $\phi \otimes_{\mathfrak{R}} E_i$ with eigenvalue 0 has dimension exactly equal to m, so $\dim_{E_i} \ker((\phi \otimes_{\mathfrak{R}} E_i)^m) = m$. By Lemma 1.2.2, $\operatorname{coker}(\phi^m_{Q_\sigma(0)})$ is flat. Applying Lemma 1.2.3 to the base change $\phi^m_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} E_i = \phi^m \otimes_{\mathfrak{R}} E_i$ of $\phi^m_{Q_\sigma(0)}$ for the chosen E_i in each connected component of $\operatorname{Spec} \mathfrak{R}_{Q_\sigma(0)}$, we find $\operatorname{rank}_{\mathfrak{R}_{Q_\sigma(0)}} \ker(\phi^m_{Q_\sigma(0)}) = \dim_{E_i} \ker((\phi \otimes_{\mathfrak{R}} E_i)^m) = m$. We apply Lemma 1.2.3 again with $M = \mathfrak{E}_{Q_\sigma(0)}$, $\varphi = \phi^m_{Q_\sigma(0)}$, and $N = \mathfrak{R}'$ to obtain the claim.

□

Claim 1.2.18. *The inclusion* $\ker(\phi_{Q_\sigma(0)}) \subseteq \ker(\phi^m_{Q_\sigma(0)})$ *is an equality.*

Proof. This is tautological if $m = 1$, so assume $m \geq 2$. Let $N = \ker(\phi^m_{Q_\sigma(0)})$ and let $N' = \ker(\phi_{Q_\sigma(0)})$. By $\phi_{Q_\sigma(0)}$-equivariance of (1.9), we obtain a map $\phi_N : N \to N$ by restricting $\phi_{Q_\sigma(0)}$ to N. Moreover, we have $N' = \ker \phi_N$, since $\phi^m_{Q_\sigma(0)}|_{\ker(Q_\sigma(\phi_{Q_\sigma(0)}))}$ is an isomorphism by Lemma 1.2.2 and therefore $\phi_{Q_\sigma(0)}|_{\ker(Q_\sigma(\phi_{Q_\sigma(0)}))}$ is one as well.

Since N is a locally free $\mathfrak{R}_{Q_\sigma(0)}$-module, we can pick $f_1, \ldots, f_n \in \mathfrak{R}_{Q_\sigma(0)}$ such that the Zariski opens $\operatorname{Spec} \mathfrak{R}_{Q_\sigma(0)f_j}$ cover $\operatorname{Spec} \mathfrak{R}_{Q_\sigma(0)}$ and N_{f_j} is free for each j. Write $\phi_{N_{f_j}} = (\phi_N)_{f_j}$. Then $\phi_{N_{f_j}}$ can be written as an $n \times n$ matrix over $\mathfrak{R}_{Q_\sigma(0)f_j}$. Let $\mathfrak{m}_i$ be the maximal ideal of $\mathfrak{R}_i$ and let $I_j = \{i : \xi_i(f_j) \notin \mathfrak{m}_i\}$. Fix j and $i \in I_j$. We have identifications

$$\begin{aligned}
\ker(\phi \otimes_{\mathfrak{R}} \mathfrak{R}_i) &= \ker(\phi_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}_i) = \ker(\phi_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}_i|_{\ker(\phi^m_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}_i)}) \\
&= \ker(\phi_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}_i|_{(\ker \phi^m_{Q_\sigma(0)}) \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}_i}) = \ker(\phi_{Q_\sigma(0)}|_{\ker(\phi^m_{Q_\sigma(0)})} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}_i) \\
&= \ker(\phi_N \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}_i) = \ker(\phi_{N_{f_j}} \otimes_{\mathfrak{R}_{Q_\sigma(0)f_j}} \mathfrak{R}_i),
\end{aligned} \tag{1.12}$$

where we use (1.11) for the first identification, Lemma 1.2.3 with $M = \mathfrak{E}_{Q_\sigma(0)}$, $\varphi = \phi_{Q_\sigma(0)}$, and $N = \mathfrak{R}_i$ for the third, the fact that $\phi_{Q_\sigma(0)}$ is an isomorphism on the second factor of the $\phi_{Q_\sigma(0)}$-compatible splitting (1.9) for the fourth, and (1.10) for the sixth.

By Proposition 1.2.6 and Corollary 1.2.8, $\ker(\phi \otimes_{\mathfrak{R}} \mathfrak{R}_i) = H^0(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}_i))$. We have $\ker(\phi \otimes_{\mathfrak{R}} \mathfrak{R}_i) \subseteq \ker((\phi \otimes_{\mathfrak{R}} \mathfrak{R}_i)^m)$, and the latter has dimension exactly $m \dim_E \mathfrak{R}_i$ by Claim 1.2.17. By condition (i) above, we must have $\ker(\phi \otimes_{\mathfrak{R}} \mathfrak{R}_i) = \ker((\phi \otimes_{\mathfrak{R}} \mathfrak{R}_i)^m)$, so by (1.12) the specialization of $\phi_{N_{f_j}}$ to $\mathfrak{R}_i$ vanishes.

By Lemma 1.2.12, there is an injection $\mathfrak{R}_{Q_\sigma(0)f_j} \to \prod_{i\in I_j} \mathfrak{R}_i$. We deduce that $\phi_{N_{f_j}} = 0$ for each j and thus that $\phi_N = 0$.

□

We therefore have $\mathfrak{E}_{Q_\sigma(0)} = \ker(\phi_{Q_\sigma(0)}) \oplus \ker(Q_\sigma(\phi_{Q_\sigma(0)}))$ and, using Lemma 1.2.2,

$$\mathrm{im}(\phi_{Q_\sigma(0)}) = \mathrm{im}(\phi^m_{Q_\sigma(0)}) = \ker(Q_\sigma(\phi_{Q_\sigma(0)})).$$

We deduce that $\ker(\phi_{Q_\sigma(0)}) \cong \mathrm{coker}(\phi_{Q_\sigma(0)})$ and obtain Theorem 1.2.13.(a) from Claims 1.2.17 and 1.2.18.

For Theorem 1.2.13.(b), we observe that if $\mathfrak{R} \to \mathfrak{R}'$ is a map of Noetherian E-Banach algebras, Proposition 1.2.6.(d) and Corollary 1.2.8 imply that

$$\ker(\phi\otimes_{\mathfrak{R}}\mathfrak{R}') \cong H^0(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}')) \text{ and } \mathrm{coker}(\phi\otimes_{\mathfrak{R}}\mathfrak{R}') \cong H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}')).$$

The $r = 1$ case of Theorem 1.2.13.(b) follows from Lemma 1.2.4, so we are left to check the $r = 0$ case.

Using Theorem 1.2.13.(a) with Lemma 1.2.3 for $\psi = \phi_{Q_\sigma(0)}$ and $N = \mathfrak{R}'_{Q_\sigma(0)}$, we have a natural isomorphism

$$\ker(\phi_{Q_\sigma(0)}) \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}'_{Q_\sigma(0)} \xrightarrow{\sim} \ker(\phi_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}'_{Q_\sigma(0)}). \tag{1.13}$$

Using Lemma 1.2.3 with the flat $\mathfrak{R}$-module $N = \mathfrak{R}_{Q_\sigma(0)}$, we obtain a natural isomorphism

$$\ker(\phi)_{Q_\sigma(0)} \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}'_{Q_\sigma(0)} \xrightarrow{\sim} \ker(\phi_{Q_\sigma(0)}) \otimes_{\mathfrak{R}_{Q_\sigma(0)}} \mathfrak{R}'_{Q_\sigma(0)}. \tag{1.14}$$

Composing (1.13) and (1.14) and reasoning as in (1.10), we obtain

$$\ker(\phi) \otimes_{\mathfrak{R}} \mathfrak{R}'_{Q_\sigma(0)} \xrightarrow{\sim} \ker(\phi \otimes_{\mathfrak{R}} \mathfrak{R}'_{Q_\sigma(0)}). \tag{1.15}$$

Using Lemma 1.2.3 again with $\psi = \phi \otimes_{\mathfrak{R}} \mathfrak{R}'$ and $N = \mathfrak{R}'_{Q_\sigma(0)}$ on the right-hand side of (1.15), we have a natural isomorphism

$$\ker(\phi \otimes_{\mathfrak{R}} \mathfrak{R}')_{Q_\sigma(0)} \xrightarrow{\sim} \ker(\phi \otimes_{\mathfrak{R}} \mathfrak{R}'_{Q_\sigma(0)}). \tag{1.16}$$

The composition of (1.15) with the inverse of (1.16) gives Theorem 1.2.13.(b) for $r = 0$ by the finite generation in Proposition 1.2.10.

□

1.2.4 Bounded de Rham periods

After localizing the modules $(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})^{G_K}$ and $(\mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M})^{G_K}$ to disallow all integral Hodge-Tate-Sen weights up to $k-1$, and also localizing to force multiplicity 1 at weight 0, Kisin [36, Proposition 2.5 and Corollary 2.6] proves an isomorphism between these modules, which reduces a rank 1 de Rham version of Theorem 1.2.13 to the Hodge-Tate case. This is possible only in families with a unique fixed Hodge-Tate-Sen weight of multiplicity 1. Our results will require a dense set of points with de Rham periods for the fixed Hodge-Tate weights.

Let B^+_{dR} be equipped with its canonical topology, and let $t \in B^+_{\mathrm{dR}}$ denote the usual choice of uniformizer. Fix a positive integer n, an element $\sigma \in \Sigma$, and a multiset $\{w_{1,\sigma}, \ldots, w_{n,\sigma}\}$

of nonnegative integers. We require that the Sen polynomial of $\mathfrak{M}$ factors as $P_\sigma(T) = S_\sigma(T)Q_\sigma(T)$, where $S_\sigma(T) = \prod_{i=1}^{n}(T + w_{i,\sigma}) \in \mathbf{Z}[T]$ and $Q_\sigma(T)$ is arbitrary. Define $Q_{\sigma,k} = \prod_{j=0}^{k-1} Q_\sigma(-j)$. Notice that localization at $Q_{\sigma,k}$ fixes the multiplicity of each integral Hodge-Tate-Sen weight in the interval $[0, k-1]$.

Theorem 1.2.19. *Fix a positive integer k and let $d_k = \#\{w_{j,\sigma} | w_{j,\sigma} < k\}$. Suppose that there exists a collection $\{\xi_i\}_{i\in I}$ of maps $\xi_i : \mathfrak{R} \to \mathfrak{R}_i$ of E-Banach algebras, where $\mathfrak{R}_i$ is local Artinian of finite dimension over E, satisfying the following properties.*

(i) *For $i \in I$, $\dim_E H^0(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}_i)) = d_k \dim_E \mathfrak{R}_i$.*

(ii) *For $i \in I$, the image of $Q_{\sigma,k}$ in $\mathfrak{R}_i$ is a unit.*

(iii) *The collection $\{\xi_i\}$ is stably dense in $\mathfrak{R}$.*

Then we have the following.

(a) *For $r \in \{0, 1\}$, the $\mathfrak{R}_{Q_{\sigma,k}}$-module $H^r(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})_{Q_{\sigma,k}}$ is finite flat of rank d_k.*

(b) *For any map $\xi : \mathfrak{R} \to \mathfrak{R}'$ of Noetherian E-Banach algebras, the cokernel and kernel of the map*

$$H^r(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}' \to H^r(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \qquad (1.17)$$

are killed by a power of $\xi(Q_{\sigma,k})$ for $r \in \{0, 1\}$.

Proof. We prove Theorem 1.2.19.(a) and (b) by induction on k; we will also need the k case of (a) to prove the k case of (b). If $k = 1$, the results follow from Theorem 1.2.13.

Claim 1.2.20. *We have $H^1(H_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}) = 0$.*

Proof. This follows from the result of Sen [67, Proposition 2] mentioned in Corollary 1.2.8 and induction using the exact sequence

$$0 \to \mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M} \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M} \to B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M} \to 0. \tag{1.18}$$

□

Remark 1.2.21. We record for future reference that a continuous additive section to the surjection in the exact sequence (1.18) exists by [30, Remark 3.2].

Fix a topological generator $\gamma \in \Gamma_K$. The diagram

$$\begin{array}{ccccccc} 0 \to (\mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} & \to & (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} & \to & (B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} & \to & 0 \\ \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} & & \\ 0 \to (\mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} & \to & (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} & \to & (B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} & \to & 0 \end{array}$$

has exact rows by the vanishing of $H^1(H_K, \mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})$. By inflation-restriction, Claim 1.2.20, and Lemma 1.2.5, the exact sequence associated to the above diagram using the snake lemma is

$$\begin{aligned} 0 \to (\mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})^{G_K} &\to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{G_K} \\ &\to (B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{G_K} \xrightarrow{\psi} H^1(G_K, \mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M}) \\ &\to H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}) \to H^1(G_K, B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}) \to 0. \end{aligned} \tag{1.19}$$

Claim 1.2.22. *The map $\psi_{Q_{\sigma,k}}$ is 0.*

Proof. Consider the 6-term exact sequence associated to

$$0 \to \mathbf{C}_p(k-1)\otimes_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}_i) \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\otimes_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}_i) \to B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\otimes_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}_i) \to 0$$

by the method that produced (1.19). Dimension counting over E using condition (i) above,

$$(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}_i))^{G_K} \to (B^+_{\mathrm{dR}}/t^{k-1} B^+_{\mathrm{dR}} \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}_i))^{G_K}$$

is surjective, so

$$(B^+_{\mathrm{dR}}/t^{k-1} B^+_{\mathrm{dR}} \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}_i))^{G_K} \xrightarrow{\psi_i} H^1(G_K, \mathbf{C}_p(k-1) \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}_i))$$

is the 0 map.

The commutative diagram

$$\begin{array}{ccc}
(B^+_{\mathrm{dR}}/t^{k-1} B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})^{G_K}_{Q_{\sigma,k}} & \xrightarrow{\psi_{Q_{\sigma,k}}} & H^1(G_K, \mathbf{C}_p(k-1) \hat{\otimes}_{K,\sigma} \mathfrak{M})_{Q_{\sigma,k}} \\
\downarrow & & \downarrow \\
(B^+_{\mathrm{dR}}/t^{k-1} B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})^{G_K} \otimes_{\mathfrak{R}} \mathfrak{R}_i & \longrightarrow & H^1(G_K, \mathbf{C}_p(k-1) \hat{\otimes}_{K,\sigma} \mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}_i \\
\downarrow\wr & & \downarrow\wr \\
(B^+_{\mathrm{dR}}/t^{k-1} B^+_{\mathrm{dR}} \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}_i))^{G_K} & \xrightarrow{\psi_i} & H^1(G_K, \mathbf{C}_p(k-1) \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}_i))
\end{array}$$

has the marked downward isomorphisms by the inductive hypothesis and Theorem 1.2.13.(b). By Theorem 1.2.13.(a), the $\mathfrak{R}_{Q_{\sigma,k}}$-module $H^1(G_K, \mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})_{Q_{\sigma,k}}$ is locally free, so we may pick elements $f_j \in \mathfrak{R}_{Q_{\sigma,k}}$ such that $H^1(G_K, \mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})_{Q_{\sigma,k}f_j}$ is free for each j, and such that the $\operatorname{Spec} \mathfrak{R}_{Q_{\sigma,k}f_j}$ cover $\operatorname{Spec} \mathfrak{R}_{Q_{\sigma,k}}$.

Let i be such that $f_j \notin \mathfrak{m}_i$, where $\mathfrak{m}_i$ denotes the maximal ideal of $\mathfrak{R}_i$. Looking at the diagram localized at f_j (which only affects the top row), the image of $\psi_{f_j Q_{\sigma,k}}$ lies in the submodule $\ker \xi_i \cdot H^1(G_K, \mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})_{f_j Q_{\sigma,k}}$. By Lemma 1.2.12, there is an injection $\mathfrak{R}_{Q_{\sigma,k}f_j} \to \prod_{\{i:f_j \notin \mathfrak{m}_i\}} \mathfrak{R}_i$. By this and freeness, $\bigcap_{\{i:f_j \notin \mathfrak{m}_i\}} \ker \xi_i \cdot H^1(G_K, \mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})_{f_j Q_{\sigma,k}} = 0$, so $\psi_{f_j Q_{\sigma,k}} = 0$ for all j and thus $\psi_{Q_{\sigma,k}} = 0$.

□

From the claim, we have exact sequences

$$0 \to (\mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})^{G_K}_{Q_{\sigma,k}} \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{G_K}_{Q_{\sigma,k}} \tag{1.20}$$
$$\to (B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{G_K}_{Q_{\sigma,k}} \to 0$$

$$\text{and} \quad 0 \to H^1(G_K, \mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})_{Q_{\sigma,k}} \to H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})_{Q_{\sigma,k}} \tag{1.21}$$
$$\to H^1(G_K, B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})_{Q_{\sigma,k}} \to 0.$$

The k case of Theorem 1.2.19.(a) follows immediately from these, Theorem 1.2.13.(a), and the inductive hypothesis.

For Theorem 1.2.19.(b), we begin by observing that by Proposition 1.2.10, the fact that $\mathfrak{R}$ is Noetherian, and induction using (1.18), both sides of (1.17) are finitely generated, so it suffices to prove that (1.17) becomes an isomorphism after localization at $\xi(Q_{\sigma,k})$.

We have a natural map between 6-term complexes of finitely generated $\mathfrak{R}$-modules

$$0 \to (\mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})^{G_K} \otimes_{\mathfrak{R}} \mathfrak{R}' \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{G_K} \otimes_{\mathfrak{R}} \mathfrak{R}' \tag{1.22}$$
$$\to (B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{G_K} \otimes_{\mathfrak{R}} \mathfrak{R}' \xrightarrow{\psi_1} H^1(G_K, \mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}'$$
$$\to H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}' \to H^1(G_K, B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}' \to 0$$

and

$$0 \to (\mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))^{G_K} \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))^{G_K} \tag{1.23}$$
$$\to (B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))^{G_K} \xrightarrow{\psi_2} H^1(G_K, \mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))$$
$$\to H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \to H^1(G_K, B^+_{\mathrm{dR}}/t^{k-1}B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \to 0,$$

where the second complex is exact. Localizing at $Q_{\sigma,k}$ and applying part (a) and (1.10), the first complex becomes exact as well, and by Claim 1.2.22, the map $\psi_{1,Q_{\sigma,k}}$ is 0. By the

inductive hypothesis, the first, third, fourth, and sixth downward maps are isomorphisms. It follows that $\psi_{2,Q_{\sigma,k}} = 0$, and by the 5-lemma, we obtain (1.17).

□

1.2.5 Geometric specializations

We can improve Theorem 1.2.19.(b). The key step is to show that the formation of the cohomology group $H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})$ is compatible with finite base change. We need the following lemma.

Lemma 1.2.23. *Maintain the notation of Section 1.2.2. If $\mathfrak{R}$ is an Artinian E-Banach algebra of finite dimension over E, then for any k,*

$$\dim_E H^0(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{K,\sigma} \mathfrak{M}) = \dim_E H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \otimes_{K,\sigma} \mathfrak{M}). \tag{1.24}$$

Proof. If $k = 1$, using Lemma 1.2.16, we may replace E, K with finite Galois extensions E', K' so that Corollary 1.2.8 holds and deduce the original result. After doing so, we have $H^0(G_K, \mathbf{C}_p \otimes_{K,\sigma} \mathfrak{M}) \cong \ker \phi$ and $H^1(G_K, \mathbf{C}_p \otimes_{K,\sigma} \mathfrak{M}) \cong \operatorname{coker} \phi$ for a map $\phi : \mathfrak{E} \to \mathfrak{E}$ of finite dimensional E-vector spaces. Lemma 1.2.3 implies that $\dim_E \ker \phi = \dim_E \operatorname{coker} \phi$ as needed. We obtain (1.24) by induction on k using (1.19) and dimension counting over E.

□

In the remainder of this section, we will always assume that $\mathfrak{R}$ is an affinoid E-algebra.

Lemma 1.2.24. *Maintain the notation of Section 1.2.2, and assume that $\mathfrak{R}$ is E-affinoid. For any finite $\mathfrak{R}$-module $\mathfrak{N}$, the tensor product $(\mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M})^{H_K} \otimes_{\mathfrak{R}} \mathfrak{N}$ is already complete.*

Proof. By Proposition 1.2.6, if E and K are sufficiently large, $(\mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M})^{H_K}$ is a finite $\widehat{K}_\infty \hat{\otimes}_{K,\sigma} \mathfrak{R}$-module. We claim that the same holds for the original E and K. Using (1.2)

and Lemma 1.2.1.(a), one sees that finiteness holds over the original E if it holds over an extension E'.

Assume that K' is a Galois extension of K and $\sigma' : K' \to E$ extends σ. Observe that $\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M} \cong \mathbf{C}_p^{\{\tau|\tau\in\mathrm{Gal}(K'/K)\}}\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M}$ by (1.4), where the action of $H_{K'}$ on the right is trivial on the τ, so

$$(\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \cong ((\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_{K'}})^{H_K/H_{K'}} \cong ((\mathbf{C}_p{}^{\{\tau|\tau\in\mathrm{Gal}(K'/K)\}}\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{H_{K'}})^{H_K/H_{K'}} \tag{1.25}$$

$$\cong (((\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{H_{K'}})^{\{\tau|\tau\in\mathrm{Gal}(K'/K)\}})^{H_K/H_{K'}}.$$

One may similarly identify $\widehat{K}_\infty\hat{\otimes}_{K,\sigma}\mathfrak{R}$ with $((\widehat{K}'_\infty\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{R})^{\{\tau|\tau\in\mathrm{Gal}(K'/K)\}})^{H_K/H_{K'}}$. It now follows from the finiteness of $(\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{H_{K'}}$ over $\widehat{K}'_\infty\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{R}$ and thus finiteness (by Lemma 1.2.1.(a)) of $((\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{H_{K'}})^{\{\tau|\tau\in\mathrm{Gal}(K'/K)\}}$ over $\widehat{K}_\infty\hat{\otimes}_{K,\sigma}\mathfrak{R}$ that $(\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K}$ is a finite $\widehat{K}_\infty\hat{\otimes}_{K,\sigma}\mathfrak{R}$-module.

We have a presentation $\mathfrak{R}^n \xrightarrow{\psi_\mathfrak{R}} \mathfrak{R}^m \to \mathfrak{N} \to 0$. Writing $\psi = (\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \otimes_\mathfrak{R} \psi_\mathfrak{R}$, the morphism $((\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K})^n \xrightarrow{\psi} ((\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K})^m$ is strict, since it is a $\widehat{K}_\infty\hat{\otimes}_{K,\sigma}\mathfrak{R}$-linear homomorphism of finite $\widehat{K}_\infty\hat{\otimes}_{K,\sigma}\mathfrak{R}$-modules, where the ring $\widehat{K}_\infty\hat{\otimes}_{K,\sigma}\mathfrak{R}$ is Noetherian since $\mathfrak{R}$ is affinoid [8, Proposition 3.7.3/5 and Corollary 6.1.1/9].

By right-exactness of the tensor product, the top row of the diagram

$$\begin{array}{ccccccc}
((\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K})^n & \xrightarrow{\psi} & ((\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K})^m & \twoheadrightarrow & (\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \otimes_\mathfrak{R} \mathfrak{N} & \to & 0 \\
\downarrow\| & & \downarrow\| & & \downarrow & & \\
((\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K})^n & \xrightarrow{\hat{\psi}} & ((\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K})^m & \twoheadrightarrow & (\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K}\hat{\otimes}_\mathfrak{R}\mathfrak{N} & \to & 0
\end{array}$$

is exact. We just checked that the first map of the top row is strict, and the second is strict by [8, Proposition 2.1.8/6]. The bottom row is exact by [8, Corollary 1.1.9/6], giving the needed isomorphism by the 5-lemma.

□

We use this to prove a base change result for H_K-invariants.

Lemma 1.2.25. *Maintain the notation of Section 1.2.2, and assume that $\mathfrak{R}$ is E-affinoid. Let $\mathfrak{R}'$ be a finite $\mathfrak{R}$-algebra. We have a natural Γ_K-equivariant isomorphism of topological $\mathfrak{R}'$-modules*

$$(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K}\otimes_{\mathfrak{R}}\mathfrak{R}' \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))^{H_K}. \tag{1.26}$$

Proof. By using the analysis of (1.6) in the proof of Theorem 1.2.13 with G_K replaced by H_K and $r=0$, we can replace E by any finite Galois extension and deduce the original result.

We may therefore replace K by a sufficiently large finite Galois extension K', equipped with an extension of σ to $\sigma' : K' \to E$, so that Proposition 1.2.6 holds. By Proposition 1.2.6.(d), we have an isomorphism

$$(\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{H_{K'}}\hat{\otimes}_{\mathfrak{R}}\mathfrak{R}' \xrightarrow{\sim} (\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))^{H_{K'}}.$$

Using Lemma 1.2.24, we can replace the left hand side by $(\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{H_{K'}}\otimes_{\mathfrak{R}}\mathfrak{R}'$. We take a direct sum over $\tau\in\mathrm{Gal}(K'/K)$ to obtain a G_K-equivariant isomorphism

$$((\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{H_{K'}}\otimes_{\mathfrak{R}}\mathfrak{R}')^{\{\tau\in\mathrm{Gal}(K'/K)\}} \xrightarrow{\sim} ((\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))^{H_{K'}})^{\{\tau\in\mathrm{Gal}(K'/K)\}}. \tag{1.27}$$

We define a G_K-equivariant isomorphism

$$((\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{H_{K'}})^{\{\tau\in\mathrm{Gal}(K'/K)\}}\otimes_{\mathfrak{R}}\mathfrak{R}' \xrightarrow{\sim} ((\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{H_{K'}}\otimes_{\mathfrak{R}}\mathfrak{R}')^{\{\tau\in\mathrm{Gal}(K'/K)\}} \tag{1.28}$$

as in (1.5). By (1.25), $(\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \cong (((\mathbf{C}_p\hat{\otimes}_{K'_{\mathrm{triv}},\sigma'}\mathfrak{M})^{H_{K'}})^{\{\tau|\tau\in\mathrm{Gal}(K'/K)\}})^{H_K/H_{K'}}$. We finish the descent to K using Lemma 1.2.1.(b) on the $H_K/H_{K'}$ invariants of the composition

of (1.27) and (1.28).

Let H'_K denote the intersection of H_K with the kernel of the cyclotomic character and write $L^+_{\mathrm{dR}} = (B^+_{\mathrm{dR}})^{H'_K}$. For general k, there is a construction (after replacing K with a sufficiently large finite Galois extension, which in the notation of [5, Lemma 4.3.1] corresponds to taking L and n sufficiently large) of a finite free module $D^+_{\mathrm{dif}}(\mathfrak{M})$ over $K[[t]]\hat{\otimes}_{K,\sigma}\mathfrak{R}$ such that

$$\begin{aligned}(D^+_{\mathrm{dif}}(\mathfrak{M}) \otimes_{K[[t]]\hat{\otimes}_{K,\sigma}\mathfrak{R}} B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{R})^{H'_K} &= D^+_{\mathrm{dif}}(\mathfrak{M}) \otimes_{K[[t]]\hat{\otimes}_{K,\sigma}\mathfrak{R}} (L^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{R}) \\ &= (B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H'_K}.\end{aligned}$$

Here the first equality uses the fact that $D^+_{\mathrm{dif}}(\mathfrak{M})$ is free and has no H'_K action to move the H'_K invariants to $B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{R}$ and a standard Schauder basis argument (c.f. [34, Lemma 2.6]) to move the H'_K invariants onto B^+_{dR}. For simplicity, we have dropped the n from the notation of [5, Lemma 4.3.1] and substituted K for L_n. In particular, $(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H'_K}$ is identified with

$$D^+_{\mathrm{dif}}(\mathfrak{M})/t^k D^+_{\mathrm{dif}}(\mathfrak{M}) \otimes_{K_\infty[t]/(t^k)\hat{\otimes}_{K,\sigma}\mathfrak{R}} (L^+_{\mathrm{dR}}/t^k L^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{R}).$$

The construction of $D^+_{\mathrm{dif}}(\mathfrak{M})$ is compatible with base change along any morphism $\mathfrak{R} \to \mathfrak{R}'$ of affinoid algebras with respect to taking completed tensor products (as follows, for instance, from [34, Theorem 3.11.(4)]). We obtain an isomorphism

$$(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H'_K}\hat{\otimes}_{\mathfrak{R}}\mathfrak{R}' \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))^{H'_K}, \tag{1.29}$$

which by descent arguments as before holds for the original K and H_K.

We now show that (1.26) is an isomorphism by induction on k. Consider the natural

morphism of complexes between

$$\begin{aligned} 0 \to (\mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \otimes_{\mathfrak{R}} \mathfrak{R}' &\to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \otimes_{\mathfrak{R}} \mathfrak{R}' \\ &\to (B^+_{\mathrm{dR}}/t^{k-1} B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \otimes_{\mathfrak{R}} \mathfrak{R}' \to 0 \\ \text{and} \quad 0 \to (\mathbf{C}_p(k-1)\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))^{H_K} &\to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))^{H_K} \\ &\to (B^+_{\mathrm{dR}}/t^{k-1} B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))^{H_K} \to 0. \end{aligned}$$

The first row is right exact and the second is left exact by Claim 1.2.20 and right-exactness of the tensor product, and the outer maps are isomorphisms by the inductive hypothesis and $k = 1$ case above. The middle map is an algebraic isomorphism by the 5-lemma. By the universal property of completion, the middle map factors through the isomorphism (1.29). Since the map from a space to its completion is homeomorphic onto its image, the map (1.26) is also a topological isomorphism.

□

We get the de Rham case of Lemma 1.2.24 as a corollary. We use these to obtain a base change result for cohomology.

Proposition 1.2.26. *Maintain the notation of Section 1.2.2, and assume that $\mathfrak{R}$ is E-affinoid. Let $\mathfrak{R}'$ be a finite $\mathfrak{R}$-algebra. The natural map*

$$H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}' \to H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \tag{1.30}$$

is an isomorphism of (non-topological) $\mathfrak{R}$-modules.

Proof. Write $\psi = \gamma - 1 : (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K}$. We have a natural isomorphism $\mathrm{coker}(\psi) \otimes_{\mathfrak{R}} \mathfrak{R}' \xrightarrow{\sim} \mathrm{coker}(\psi \otimes_{\mathfrak{R}} \mathfrak{R}')$ by Lemma 1.2.4. The $\mathfrak{R}'$-module

isomorphism

$$(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K}\otimes_{\mathfrak{R}}\mathfrak{R}'\xrightarrow{\sim}(B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))^{H_K}$$

of Lemma 1.2.25 takes ψ to the natural action of $\gamma-1$ on the right hand side. Using the inflation-restriction exact sequence, Lemma 1.2.5, and Claim 1.2.20, we obtain (1.30) from the isomorphisms

$$\begin{aligned}&\operatorname{coker}(\psi)\otimes_{\mathfrak{R}}\mathfrak{R}'\cong H^1(G_K,B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})\otimes_{\mathfrak{R}}\mathfrak{R}'\\ \text{and}\quad&\operatorname{coker}(\psi\otimes_{\mathfrak{R}}\mathfrak{R}')\cong H^1(G_K,B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}')).\end{aligned}$$

□

We obtain de Rham periods at all geometric specializations.

Theorem 1.2.27. *Maintain the notation and hypotheses of Theorem 1.2.19 and assume that either $k=1$ or $\mathfrak{R}$ is E-affinoid. If $\xi:\mathfrak{R}\to E'$ is a specialization to a finite field extension of E, then we have $\dim_{E'}H^r(G_K,B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}E'))\geq d_k$ for $r\in\{0,1\}$.*

Proof. Let $\mathfrak{m}=\ker\xi$ and $\mathfrak{p}\subseteq\mathfrak{m}$ be a minimal prime. By Proposition 1.2.26 (or Theorem 1.2.13.(b) if $k=1$), we have identifications

$$H^1(G_K,B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})\otimes_{\mathfrak{R}}\mathfrak{R}/\mathfrak{p}\xrightarrow{\sim}H^1(G_K,B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}/\mathfrak{p}\mathfrak{M})\quad(1.31)$$

$$\text{and}\quad H^1(G_K,B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}/\mathfrak{p}\mathfrak{M})\otimes_{\mathfrak{R}/\mathfrak{p}}E'\xrightarrow{\sim}H^1(G_K,B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}E')).\quad(1.32)$$

Picking f in every minimal prime of $\mathfrak{R}$ except $\mathfrak{p}$, we obtain injections $\mathfrak{R}/\mathfrak{p}\hookrightarrow\mathfrak{R}_f\hookrightarrow\prod_{\{i:f\notin\mathfrak{m}_i\}}\mathfrak{R}_i$ by condition (iii) of Theorem 1.2.19. By condition (ii) of Theorem 1.2.19, the image of $Q_{\sigma,k}$ in $\mathfrak{R}/\mathfrak{p}$ is nonzero, so $H^1(G_K,B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}/\mathfrak{p}\mathfrak{M})_{Q_{\sigma,k}}$ is finite flat of rank

d_k by Theorem 1.2.19.(a), (1.10), and (1.31). Let $L = \mathrm{Frac}(\mathfrak{R}/\mathfrak{p}) = \mathrm{Frac}((\mathfrak{R}/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}})$. It follows that

$$\dim_L H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}/\mathfrak{p}\mathfrak{M})_{\mathfrak{m}/\mathfrak{p}} \otimes_{(\mathfrak{R}/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}}} L = d_k.$$

By Nakayama's lemma, $H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}/\mathfrak{p}\mathfrak{M})_{\mathfrak{m}/\mathfrak{p}}$ can be generated by

$$\dim_{E'} H^1(G_K, B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}/\mathfrak{p}\mathfrak{M}) \otimes_{\mathfrak{R}/\mathfrak{p}} E'$$

elements, so this dimension must be at least d_k, giving the desired bound for $r = 1$ by (1.32). Lemma 1.2.23 implies the $r = 0$ case.

□

1.2.6 Unbounded periods

To pass from bounded to unbounded de Rham periods, we will use exact sequences exchanging projective limits with cohomology. Proposition 1.2.28 has the appearance of [51, Theorem 2.7.5], but their result makes a profiniteness assumption on the module. We instead consider a module M that is the projective limit of topological modules M/M_i such that the transition maps $M/M_i \to M/M_{i-1}$ have continuous topological sections. The category of topological abelian groups is not abelian, so we cannot use the spectral sequence for a composition of functors as they do. We instead work with the groups of cochains, cocycles, and coboundaries "by hand" to obtain a similar result.

Proposition 1.2.28. *Let G be a profinite group. Let M be a continuous G-module, and suppose that there exist G-submodules $M_i \subseteq M$ such that $M \cong \varprojlim_i M/M_i$ as topological G-modules, and such that there exist continuous sections $M/M_{i-1} \to M/M_i$. Then for $n \geq 1$*

there exists an exact sequence

$$0 \to \varprojlim_i{}^1 H^{n-1}(G, M/M_i) \to H^n(G, M) \to \varprojlim_i H^n(G, M/M_i) \to 0. \tag{1.33}$$

Remark 1.2.29. Note that $B^+_{\mathrm{dR}} = \varprojlim_k B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}$, where the left-hand side has the inverse limit topology [31, Remark 1.1]. As mentioned in Remark 1.2.21, sections to the transition maps exist. We will primarily use Proposition 1.2.28 to pass from modules over $B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}$ to modules over B^+_{dR}.

Proof of Proposition 1.2.28. We first note that there exists an exact sequence

$$0 \to (Z^{n-1}(G, M/M_i))_i \to (C^{n-1}(G, M/M_i))_i \xrightarrow{d} (B^n(G, M/M_i))_i \to 0 \tag{1.34}$$

of projective systems of abelian groups. Here Z^{n-1}, C^{n-1}, and B^n denote the usual groups of continuous cocycles, cochains, and coboundaries. By the hypothesis that continuous sections exist, the transition maps of the middle term and therefore (by commutativity) the last term are surjective, so they satisfy the Mittag-Leffler condition. Moreover, it follows from the definition of the cochains and cocycles that

$$\varprojlim_i C^n(G, M/M_i) = C^n(G, M) \text{ and } \varprojlim_i Z^n(G, M/M_i) = Z^n(G, M). \tag{1.35}$$

Passing to the limit in (1.34) and using (1.35) to simplify the expressions, we obtain

$$0 \to Z^{n-1}(G, M) \to C^{n-1}(G, M) \xrightarrow{d} \varprojlim_i B^n(G, M/M_i) \to \varprojlim_i{}^1 Z^{n-1}(G, M/M_i) \to 0,$$

where the 0 on the right is by the Mittag-Leffler property of $(C^{n-1}(G, M/M_i))_i$. If we define

$(C/Z)^{n-1}(G,M) = C^{n-1}(G,M)/Z^{n-1}(G,M)$, we deduce an isomorphism

$$(\varprojlim_i B^n(G, M/M_i))/d((C/Z)^{n-1}(G,M)) \cong \varprojlim_i{}^1 Z^{n-1}(G, M/M_i). \tag{1.36}$$

By (1.35), the limit of the exact sequence

$$0 \to (B^n(G, M/M_i))_i \to (Z^n(G, M/M_i))_i \to (H^n(G, M/M_i))_i \to 0 \tag{1.37}$$

of projective systems of abelian groups is

$$0 \to \varprojlim_i B^n(G, M/M_i) \to Z^n(G,M) \to \varprojlim_i (H^n(G, M/M_i))_i \to 0,$$

where we have used the surjectivity of the transition maps of $(B^n(G, M/M_i))_i$ to see that the last term vanishes. We obtain an exact sequence

$$\begin{aligned} 0 \to (\varprojlim_i B^n(G, M/M_i))/d((C/Z)^{n-1}(G,M)) &\to Z^n(G,M)/d((C/Z)^{n-1}(G,M)) \\ &\to \varprojlim_i (H^n(G, M/M_i))_i \to 0, \end{aligned}$$

which can be rewritten by (1.36) and the definition of cohomology as

$$0 \to \varprojlim_i{}^1 Z^{n-1}(G, M/M_i) \to H^n(G,M) \to \varprojlim_i H^n(G, M/M_i)_i \to 0.$$

By [51, Proposition 2.7.4], the (not very) long exact sequence of (1.37) for $n-1$ in place of n gives an isomorphism $\varprojlim_i{}^1 Z^{n-1}(G, M/M_i) \cong \varprojlim_i{}^1 H^{n-1}(G, M/M_i)$, which yields (1.33) above.

□

Remark 1.2.30. In this paper, we equip B_{dR} with the locally convex final topology as the

strict inductive limit of its K-locally convex subspaces $t^k B_{\mathrm{dR}}^+$ [63, §I.5.E2]. By definition, the topology on B_{dR} is induced by the seminorms $B_{\mathrm{dR}} \to \mathbf{R}$ with continuous restriction to each $t^k B_{\mathrm{dR}}^+$. (Also see the equivalent definitions in [55, Theorem 11.1.1].) With respect to this topology, B_{dR} is K-locally convex, complete (with respect to Cauchy nets), bornological, barrelled, and Hausdorff [63, Proposition 5.5.(ii), §6, and Lemma 7.9]. Moreover, $t^k B_{\mathrm{dR}}^+$ is closed in B_{dR} and inherits its original topology from that of B_{dR} [63, Proposition 5.5.(i) and (iii)]. It follows from the definition that the G_K-action is continuous. Although the $t^k B_{\mathrm{dR}}^+$ are Fréchet spaces, this topology on B_{dR} is likely not metrizable; see the discussion in [55, §11.2].

We always use $\mathfrak{A} \otimes_K \mathfrak{B}$ to denote the *inductive* tensor product of two locally convex K-vector spaces, which is induced by those seminorms $p : \mathfrak{A} \otimes_K \mathfrak{B} \to \mathbf{R}$ such that the induced map $\mathfrak{A} \times \mathfrak{B} \to \mathbf{R}$ is separately continuous, i.e. continuous when restricted to $\{a\} \times \mathfrak{B}$ or $\mathfrak{A} \times \{b\}$ for any $a \in \mathfrak{A}$ or $b \in \mathfrak{B}$. It follows from the definition of this topology and the universal property of completion that if $\mathfrak{A}$ and $\mathfrak{B}$ are equipped with continuous actions of G_K, then $\mathfrak{A} \otimes_K \mathfrak{B}$ and $\mathfrak{A} \hat{\otimes}_K \mathfrak{B}$ are as well [63, Proposition 7.5].

Note that since $t^k B_{\mathrm{dR}}^+$ is a Fréchet space, by [63, Proposition 17.6] the inductive and projective topologies coincide on $t^k B_{\mathrm{dR}}^+ \otimes_K \mathfrak{M}$ for any K-Banach (or even Fréchet) space $\mathfrak{M}$.

We use the following observation to work with ordinary tensor products in several of the results that follow.

Lemma 1.2.31. *Maintain the notation of Section 1.2.2, and suppose that $\mathfrak{R}$ is Artinian of finite dimension over E. For $B \in \left\{B_{\mathrm{dR}}, t^k B_{\mathrm{dR}}^+, t^k B_{\mathrm{dR}}^+ / t^\ell B_{\mathrm{dR}}^+\right\}$, where $k \leq \ell \in \mathbf{Z}$, the natural map $B \otimes_{K,\sigma} \mathfrak{M} \to B \hat{\otimes}_{K,\sigma} \mathfrak{M}$ is an isomorphism.*

Proof. Fix a K-basis $\{m_i\}$ of the finite dimensional K-vector space $\mathfrak{M}$. Note that B is a complete Hausdorff locally convex K-vector space. Then by [63, Lemma 5.2.(ii)], the usual product topology on $B \otimes_{K,\sigma} \mathfrak{M} \cong \oplus_{m_i} B$ is also the locally convex direct sum topology. By

[63, Lemma 7.8], $\oplus_{m_i} B$ is complete and Hausdorff.

□

We will need the following fact.

Lemma 1.2.32. *Let K be a finite extension of $\mathbf{Q}_p$. Suppose that $\mathfrak{A}$ is a strict inductive limit of locally convex K-vector spaces $\mathfrak{A}_i$ for $i \in \mathbf{Z}_{\geq 0}$ and $\mathfrak{B}$ is any locally convex K-vector space. Then the natural map $\varinjlim_i (\mathfrak{A}_i \otimes_K \mathfrak{B}) \to \mathfrak{A} \otimes_K \mathfrak{B}$, where the left-hand side has the locally convex final topology, is an isomorphism of locally convex K-vector spaces.*

Proof. Note that the map is an isomorphism of K-vector spaces, so we need only check that it is a homeomorphism.

The topology on $\varinjlim_i (\mathfrak{A}_i \otimes_K \mathfrak{B})$ is defined by all seminorms p such that the composite map $\mathfrak{A}_i \otimes_K \mathfrak{B} \to \mathfrak{A} \otimes_K \mathfrak{B} \to \mathbf{R}$ is continuous for all i. By definition of the inductive topology, this holds if and only if for each $i \in \mathbf{Z}_{\geq 0}$, $a \in \mathfrak{A}_i$, and $b \in \mathfrak{B}$, the composite maps $\{a\} \times \mathfrak{B} \to \mathfrak{A}_i \otimes_K \mathfrak{B} \to \mathbf{R}$ and $\mathfrak{A}_i \times \{b\} \to \mathfrak{A}_i \otimes_K \mathfrak{B} \to \mathbf{R}$ are continuous.

The topology of $\mathfrak{A} \otimes_K \mathfrak{B}$ is defined by those seminorms p such that for any $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$, the induced maps $\{a\} \times \mathfrak{B} \to \mathbf{R}$ and $\mathfrak{A} \times \{b\} \to \mathbf{R}$ are continuous. By definition, the latter is equivalent to saying that for each $b \in \mathfrak{B}$ and $i \in \mathbf{Z}_{\geq 0}$, the induced map $\mathfrak{A}_i \times \{b\} \to \mathbf{R}$ is continuous, which matches the second condition on p for $\varinjlim_i (\mathfrak{A}_i \otimes_K \mathfrak{B})$. For every i and $a' \in \mathfrak{A}_i$ mapping to a, there are maps $\{a'\} \times \mathfrak{B} \to \{a\} \times \mathfrak{B} \xrightarrow{p} \mathbf{R}$, where the first map is a homeomorphism. In particular, the composite map is continuous if and only if the second map is continuous.

Since p needs to satisfy the same conditions for continuity in either case, we deduce that the topologies are the same.

□

As a consequence, we can identify the topology on $B_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}$ with a strict inductive limit topology.

Lemma 1.2.33. *Maintain the notation of Section 1.2.2. There is a natural isomorphism of locally convex K-vector spaces $\varprojlim_k (t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M}) \xrightarrow{\sim} B_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}$. Moreover, the limit is strict with closed subspaces $t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M}$.*

Proof. The inclusion $t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M} \to t^{k-1} B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}$ is strict by [63, Corollary 17.5.(ii) and Proposition 17.6]. We obtain strict injections $t^{k-1} B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M} \to t^{k-1} B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M}$ and $t^{k-1} B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M} \to t^{k-1} B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M}$ from two applications of [63, Proposition 7.5]. The image of the latter map is closed [63, Remark 7.1.(v)]. The second claim now follows from [63, Proposition 5.5.(iii)].

By Lemma 1.2.32, there is a natural isomorphism $(\varprojlim_k (t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}))^\wedge \xrightarrow{\sim} B_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}$ of locally convex K-vector spaces, where $\wedge$ denotes completion. By the universal property, we have a continuous map $\varprojlim_k (t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}) \to \varprojlim_k (t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M})$. Since the range is complete and Hausdorff by [63, Proposition 5.5.(ii) and Lemma 7.9], this uniquely extends to a continuous map $\psi : (\varprojlim_k (t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}))^\wedge \to \varprojlim_k (t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M})$ by [63, Lemma 7.3]. For each k, we have a map $t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M} \to (\varprojlim_k (t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}))^\wedge$, where the image is Hausdorff and complete, so this extends to a map $t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M} \to (\varprojlim_k (t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}))^\wedge$. Using the universal property again, we obtain a continuous inverse to ψ. The needed isomorphism follows.

□

To calculate higher cohomology of B_{dR}, we will use the following observation.

Lemma 1.2.34. *Maintain the notation of Section 1.2.2. For $r \geq 0$, the natural maps $t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M} \to B_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}$ induce an isomorphism*

$$\varinjlim_k H^r(G_K, t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M}) \to H^r(G_K, B_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}).$$

Proof. We claim that every continuous map $\psi : G_K^r \to B_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}$ has image contained in $t^k B_{\mathrm{dR}}^+$ for some k. Both injectivity and surjectivity will follow from this.

A bounded subset of a locally convex K-vector space V is one such that for any open lattice $L \subseteq V$, there is $a \in K$ such that $B \subseteq aL$. Any finite set is bounded [63, §I.4]. It follows that for any open lattice L, each point of $\psi(G_K^r)$ is contained in some aL. Since $\psi(G_K^r)$ is compact, it is contained in a single aL, so $\psi(G_K^r)$ is bounded.

By Lemma 1.2.33, the completed tensor product $B_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}$ is the strict inductive limit of its closed subspaces $t^k B_{\mathrm{dR}}^+\hat{\otimes}_{K,\sigma}\mathfrak{M}$. The result now follows from [63, Proposition 5.6], which shows that any bounded subset of $B_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}$ is contained in some $t^k B_{\mathrm{dR}}^+\hat{\otimes}_{K,\sigma}\mathfrak{M}$.

□

We can deduce the following result.

Proposition 1.2.35. *Maintain the notation of Section 1.2.2. If $\mathfrak{R}$ is an Artinian E-Banach algebra of finite dimension over E, then for $\ell < k \in \mathbf{Z}$ and $B \in \left\{B_{\mathrm{dR}}, t^k B_{\mathrm{dR}}^+, t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+\right\}$ we have*

$$\dim_E H^0(G_K, B \otimes_{K,\sigma} \mathfrak{M}) = \dim_E H^1(G_K, B \otimes_{K,\sigma} \mathfrak{M}). \tag{1.38}$$

Writing $d(B)$ for this common dimension, we have inequalities

$$d(B_{\mathrm{dR}}) \geq d(t^k B_{\mathrm{dR}}^+) \geq d(t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+). \tag{1.39}$$

The first is an equality if k is sufficiently small, as is the second if ℓ is sufficiently large.

Proof. Using Lemma 1.2.31, we may work with ordinary tensor products. Let $\kappa(\mathfrak{R})$ denote the residue field. The image of $P_\sigma(T)$ in $\kappa(\mathfrak{R})[T]$ has finitely many roots. For j sufficiently large or small, we deduce from Proposition 1.2.10 that for $r \in \{0, 1\}$, we have

$$\dim_E H^r(G_K, \mathbf{C}_p(j) \otimes_{K,\sigma} \mathfrak{M}) = 0. \tag{1.40}$$

For $k < \ell$, we have exact sequences

$$0 \to \mathbf{C}_p(\ell) \to t^k B_{\mathrm{dR}}^+/t^{\ell+1}B_{\mathrm{dR}}^+ \to t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \to 0, \tag{1.41}$$

$$0 \to t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \to t^{k-1} B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \to \mathbf{C}_p(k-1) \to 0 \tag{1.42}$$

$$\text{and} \quad 0 \to t^k B_{\mathrm{dR}}^+ \to t^{k-1} B_{\mathrm{dR}}^+ \to \mathbf{C}_p(k-1) \to 0. \tag{1.43}$$

For k sufficiently small in (1.45) and (1.46) and ℓ sufficiently large in (1.44), the natural maps

$$H^r(G_K, t^k B_{\mathrm{dR}}^+/t^{\ell+1}B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}) \to H^r(G_K, t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}), \tag{1.44}$$

$$H^r(G_K, t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}) \to H^r(G_K, t^{k-1} B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}), \tag{1.45}$$

$$\text{and} \quad H^r(G_K, t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}) \to H^r(G_K, t^{k-1} B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}) \tag{1.46}$$

are isomorphisms for $r \in \{0, 1\}$ by (1.40) and, for (1.44) and $r = 1$, the existence of a continuous section from Remark 1.2.21.

Now suppose $k < \ell \in \mathbf{Z}$ are given. By Lemma 1.2.23,

$$\dim_E H^0(G_K, B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}) = \dim_E H^1(G_K, B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}).$$

We induct using the 6-term exact sequence similar to (1.19) obtained from (1.42) to find

$$\dim_E H^0(G_K, t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}) = \dim_E H^1(G_K, t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M}).$$

This is (1.38) for $B = t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+$.

Fix ℓ large enough so that (1.40) holds for all $j \geq \ell$. Also note that by right-exactness of the tensor product, $t^k B_{\mathrm{dR}}^+/t^i B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M} \cong (t^k B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M})/(t^i B_{\mathrm{dR}}^+ \otimes_{K,\sigma} \mathfrak{M})$. Using the isomorphisms (1.44) with both $r = 0$ and $r = 1$, we see that in the exact sequence (1.33) with

$n = 1$, $M = t^k B^+_{\mathrm{dR}} \otimes_{K,\sigma} \mathfrak{M}$, and $M/M_i = t^k B^+_{\mathrm{dR}}/t^i B^+_{\mathrm{dR}} \otimes_{K,\sigma} \mathfrak{M}$, the left-hand term vanishes and the right-hand term has dimension equal to $\dim_E H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \otimes_{K,\sigma} \mathfrak{M})$. Using the isomorphisms (1.44) with $r = 0$ and the $B = t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}}$ case of (1.38), we deduce the equality (1.38) for $B = t^k B^+_{\mathrm{dR}}$.

Choose k so that (1.40) holds for all $j \leq k$. Beginning with the $B = t^k B^+_{\mathrm{dR}}$ case of (1.38), we form the inductive limit over k, and use that (1.46) is an isomorphism for $r \in \{0, 1\}$ and $j \leq k$ together with Lemmas 1.2.31 and 1.2.34 to deduce (1.38) for $B = B_{\mathrm{dR}}$.

We obtain the first inequality in (1.39) for 0-cocycles by applying left-exactness of G_K-invariants to the injection $t^k B^+_{\mathrm{dR}} \otimes_{K,\sigma} \mathfrak{M} \hookrightarrow B_{\mathrm{dR}} \otimes_{K,\sigma} \mathfrak{M}$. For the second, we observe that

$$H^1(G_K, t^k B^+_{\mathrm{dR}}/t^{\ell+1} B^+_{\mathrm{dR}} \otimes_{K,\sigma} \mathfrak{M}) \to H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \otimes_{K,\sigma} \mathfrak{M})$$

is always surjective by the sequence similar to (1.19) obtained from (1.41), so we have $d(t^k B^+_{\mathrm{dR}}/t^{\ell+1} B^+_{\mathrm{dR}}) \geq d(t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}})$. If we apply this inequality $\ell' - \ell$ times, where ℓ' is large enough for (1.40) to always hold, we obtain $d(t^k B^+_{\mathrm{dR}}/t^{\ell'} B^+_{\mathrm{dR}}) \geq d(t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}})$. It now follows from the proof of (1.38) for $B = t^k B^+_{\mathrm{dR}}$ that

$$d(t^k B^+_{\mathrm{dR}}) = \dim_E H^1(G_K, t^k B^+_{\mathrm{dR}} \otimes_{K,\sigma} \mathfrak{M}) = d(t^k B^+_{\mathrm{dR}}/t^{\ell'} B^+_{\mathrm{dR}}) \geq d(t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}}).$$

□

We use Proposition 1.2.28 with Theorem 1.2.19 to deduce the existence of de Rham periods for certain base changes of a family. We remark that in the Berger-Colmez case (i.e. when $Q_\sigma(T)$ is a unit), we find that $H^r(G_K, B_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})$ is finite flat of rank n for $r \in \{0, 1\}$.

Proposition 1.2.36. *Maintain the hypotheses of Theorem 1.2.19. Suppose that $\mathfrak{R}'$ is a Noetherian Banach $\mathfrak{R}$-algebra such that the image of $Q_{\sigma,k}$ is a unit in $\mathfrak{R}'$ for all $k \geq 0$. Then for $r \in \{0, 1\}$, the $\mathfrak{R}'$-module $H^r(G_K, B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'))$ is finite flat of rank n and*

$H^0(G_K, B_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))$ has a finite flat $\mathfrak{R}'$-submodule of rank n.

If, in addition, $Q_\sigma(k)$ is a unit in $\mathfrak{R}'$ for all $k>0$, then $H^r(G_K, B_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))$ are finite flat of rank n for $r\in\{0,1\}$.

Proof. By [23, Proposition 1.1.29], we have a natural isomorphism

$$B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}')\cong\varprojlim_k(B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}')). \tag{1.47}$$

It follows that $H^0(G_K, B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))=\varprojlim_k H^0(G_K, B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))$. The discussion of the exact sequences (1.22) and (1.23) in the proof of Theorem 1.2.19 implies that the transition maps on the right-hand side are surjective maps of finite flat modules. For k sufficiently large, combining Theorem 1.2.19.(a) and (b),

$$\mathrm{rank}_{\mathfrak{R}'}\,H^0(G_K, B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))=d_k=n$$

is independent of k, so the transition maps are isomorphisms and the limit is finite flat of rank n as well.

From Proposition 1.2.28 and (1.47), we have an exact sequence

$$\begin{aligned}0\to\varprojlim_i{}^1H^0(G_K, B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))\to H^1(G_K, B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))\\ \to\varprojlim_i H^1(G_K, B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))\to 0.\end{aligned}$$

By the aforementioned discussion of (1.22) and (1.23), the transition maps of the first and third terms are surjective maps of finite flat $\mathfrak{R}'$-modules, so the first term vanishes. As before, we have

$$\mathrm{rank}_{\mathfrak{R}'}\,H^1(G_K, B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))=d_k=n$$

for k sufficiently large by Theorem 1.2.19.(a) and (b), so the transition maps are eventually isomorphisms and the limit is finite flat of rank n over $\mathfrak{R}'$.

We take G_K-invariants of the natural injection $B_{\mathrm{dR}}^+\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}') \hookrightarrow B_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}')$ to deduce that $H^0(G_K, B_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'))$ has a finite flat rank n submodule.

If $Q_\sigma(k)$ is a unit in $\mathfrak{R}'$ for all $k > 0$, then the isomorphisms (1.46) hold. We deduce the final claim from Lemma 1.2.34.

□

If we no longer ask for flatness, we can relax the strong hypothesis on $Q_{\sigma,k}$. Observe that the bound $d_k \dim_E \mathfrak{R}'$ below is compatible with the expectation of d_k interpolated periods.

Proposition 1.2.37. *Maintain the hypotheses of Theorem 1.2.19. Suppose that we are given a local Artinian Banach $\mathfrak{R}$-algebra $\mathfrak{R}'$ of finite E-dimension with structure morphism $\xi : \mathfrak{R} \to \mathfrak{R}'$. Fix $k \geq 0$, and suppose that $\xi(Q_{\sigma,k})$ is a unit. Then for $r \in \{0,1\}$ we have*

$$\dim_E H^r(G_K, B_{\mathrm{dR}} \otimes_{K,\sigma} (\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}')) \geq \dim_E H^r(G_K, B_{\mathrm{dR}}^+ \otimes_{K,\sigma} (\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}')) \geq d_k \dim_E \mathfrak{R}'.$$

Using Theorem 1.2.27, we can also prove a result without the hypothesis that $Q_{\sigma,k}$ vanishes, but we need to assume that $\mathfrak{R}$ is E-affinoid.

Proposition 1.2.38. *Maintain the hypotheses of Theorem 1.2.19, assume that $\mathfrak{R}$ is E-affinoid, and let $\mathfrak{R} \to E'$ be a map to a finite extension E' of E. Then for $r \in \{0,1\}$ we have*

$$\dim_{E'} H^r(G_K, B_{\mathrm{dR}} \otimes_{K,\sigma} (\mathfrak{M}\otimes_{\mathfrak{R}} E')) \geq \dim_{E'} H^r(G_K, B_{\mathrm{dR}}^+ \otimes_{K,\sigma} (\mathfrak{M}\otimes_{\mathfrak{R}} E')) \geq d_k.$$

Proof of Propositions 1.2.37 and 1.2.38. Proposition 1.2.37 follows from Theorem 1.2.19 and Proposition 1.2.35. For Proposition 1.2.38, we apply Theorem 1.2.27 to the specialization $\mathfrak{R} \to E'$ and use Proposition 1.2.35.

□

1.2.7 Essentially self-dual and decomposable specializations

We study specializations V that are essentially self-dual in the sense that $V \cong V^\vee(s)$ for some s. Examples of such representations include polarizable representations with trivial complex conjugation. In the Hodge-Tate case, we will be able to recover $2m$ periods of weight w of V from just m fixed periods of weight w in a family, except for when the weight w satisfies $w = s - w$.

Theorem 1.2.39. *Maintain the notation and hypotheses of Theorem 1.2.13. Assume that a morphism of E-Banach algebras $\xi : \mathfrak{R} \to E'$ is given, where E' is a finite extension of E. Let $V = \mathfrak{M} \otimes_{\mathfrak{R}} E'$, and suppose that there is an isomorphism $V \cong V^\vee(s)$ for some $s \in \mathbf{Z}$. Then for $r \in \{0, 1\}$,*

$$\dim_{E'} H^r(G_K, \mathbf{C}_p \otimes_{K,\sigma} V) + \dim_{E'} H^r(G_K, \mathbf{C}_p(-s) \otimes_{K,\sigma} V) \geq 2m. \tag{1.48}$$

Proof. The module $\mathfrak{M} \oplus \mathfrak{M}^\vee$ satisfies the same hypotheses as $\mathfrak{M}$. We apply Theorem 1.2.27 (with $k = 1$) to this module and the map ξ to derive the conclusion.

□

To study the de Rham case, we will impose stronger hypotheses. We will be interested in specializations that decompose as an extension of a representation V by its twisted dual $V^\vee(s)$ or vice-versa, for $s \geq 1$. We will also need to know the determinant of V.

Theorem 1.2.40. *Maintain the hypotheses of Theorem 1.2.19. Assume that* $\mathrm{rank}_{\mathfrak{R}} \mathfrak{M} = 2d_k$. *Let $\xi : \mathfrak{R} \to E'$ be a map of E-Banach algebras, with E' is a finite extension of E. Suppose*

that there is a short exact sequence

$$0 \to V \to \mathfrak{M} \otimes_{\mathfrak{R}} E' \to V^\vee(s) \to 0 \tag{1.49}$$

$$\text{or} \quad 0 \to V^\vee(s) \to \mathfrak{M} \otimes_{\mathfrak{R}} E' \to V \to 0, \tag{1.50}$$

where $s \in \mathbf{Z}_{\geq 1}$ *and* V *is an* E'*-vector space equipped with a continuous action of* G_K*. Assume in addition that* $\det(\mathbf{C}_p \otimes_{K,\sigma} V) \cong \mathbf{C}_p\left(-\sum_i w_{i,\sigma}\right) \otimes_{K,\sigma} E'$*, where* $\mathbf{C}_p \otimes_{K,\sigma} V$ *is viewed as a module over* $\mathbf{C}_p \otimes_{K,\sigma} E'$*. For* $r \in \{0,1\}$*, we have*

$$\dim_{E'} H^r(G_K, B_{\mathrm{dR}}^+ \otimes_{K,\sigma} V) = d_k, \quad \dim_{E'} H^r(G_K, B_{\mathrm{dR}} \otimes_{K,\sigma} V) = d_k, \tag{1.51}$$

and, in the case of (1.50),

$$\dim_{E'} H^r(G_K, B_{\mathrm{dR}} \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} E')) = 2d_k. \tag{1.52}$$

Proof. With respect to the fixed $\sigma : K \to E$, let V have Sen polynomial $P_{\sigma,V}(T) \in E'[T]$. We will use the subscript E' to denote the specialization of an element of $\mathfrak{R}[T]$ to E'. Then from (1.49) or (1.50), the Sen polynomial of $\mathfrak{M} \otimes_{\mathfrak{R}} E'$ satisfies $P_{\sigma,E'}(T) = S_{\sigma,E'}(T)Q_{\sigma,E'}(T) = P_{\sigma,V}(T)P_{\sigma,V}(-s-T)$. Since the $w_{i,\sigma}$ are nonnegative, $w_{i,\sigma} \neq -s - w_{i',\sigma}$ for any i, i'. The zeros of $P_{\sigma,V}(T)P_{\sigma,V}(-s-T)$ must be the multiset $\coprod_j \{w_{i,\sigma}, -s - w_{i,\sigma}\}$, so we must have $Q_{\sigma,E'}(T) = S_{\sigma,E'}(-s-T)$.

The determinant condition ensures that $P_{\sigma,V}(T) = S_{\sigma,E'}(T)$ and $P_{\sigma,V}(-s-T) = Q_{\sigma,E'}(T)$. For any vector space V' over E' equipped with a linear G_K-action,

$$\dim_{E'} H^0(G_K, B_{\mathrm{dR}}^+ \otimes_{K,\sigma} V') \leq \dim_{E'} H^0(G_K, B_{\mathrm{HT}}^+ \otimes_{K,\sigma} V').$$

From this and our identification of $P_{\sigma,V}(-s-T)$, $\dim_{E'} H^0(G_K, B_{\mathrm{dR}}^+ \otimes_{K,\sigma} V^\vee(s)) = 0$. By

Proposition 1.2.36, $\dim_{E'} H^0(G_K, B^+_{\mathrm{dR}} \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} E')) \geq n$. Tensoring (1.49) or (1.50) with B^+_{dR} and using the exact sequence in G_K-cohomology, if V is a submodule, we are done. If it is a quotient, we have an exact sequence

$$0 \to H^0(G_K, B^+_{\mathrm{dR}} \otimes_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} E')) \to H^0(G_K, B^+_{\mathrm{dR}} \otimes_{K,\sigma} V) \to H^1(G_K, B^+_{\mathrm{dR}} \otimes_{K,\sigma} V^\vee(s)).$$

By Proposition 1.2.35, the last entry in the exact sequence vanishes, giving $\dim_{E'}(B^+_{\mathrm{dR}} \otimes_{K,\sigma} V) = d_k$ and thus (1.51). For the case (1.50), we obtain (1.52) from [11, Example 6.3.6] and Proposition 1.2.35.

□

1.3 Stratification by de Rham data

Given a reduced rigid space $\mathfrak{X}$ with a G_K-representation on a coherent locally free sheaf $\mathfrak{M}$ of $\mathfrak{O}_{\mathfrak{X}}$-modules, we are now interested in constructing closed subvarieties $\mathfrak{S}_{\mathbf{D}} \subseteq \mathfrak{X}$ indexed by *de Rham data* $\mathbf{D}$. Such a datum consists of lower bounds for the dimensions of bounded de Rham periods in all intervals of weights at geometric specializations of $\mathfrak{S}_{\mathbf{D}}$. Moreover, we want to decompose the entirety of $\mathfrak{X}$ into locally closed strata over which sheaves of bounded de Rham periods in a fixed range are locally free. In the terminology of this section, Section 1.2 is a study of the interpolation and specialization properties of a *full* de Rham datum for possibly non-reduced bases.

We will make use of the methodology of the previous sections. Our approach is as follows. We define the notion of a de Rham datum in Section 1.3.1 and study their existence, specialization, and interpolation on integral affinoid algebras $\mathfrak{R}$ in Section 1.3.2. In Section 1.3.3, we do the same for a reduced rigid space $\mathfrak{X}$. In Section 1.3.4, we state and prove the main theorems using the previous results.

1.3.1 Definition of a de Rham datum

We describe all of the terminology we will use to describe a de Rham datum.

We define a *de Rham datum* $\mathbf{D} = (\Omega, \Delta)$ as a tuple consisting of functions $\Omega : \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ and $\Delta : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ such that

(i) the function Ω is finitely supported,

(ii) we have $\Delta(i, j) = 0$ if $i \geq j$,

(iii) we have $\min(\Omega(i), 1) \leq \Delta(i, i+1) \leq \Omega(i)$ for $i \in \mathbf{Z}$, and

(iv) we have the inequality $\max(\Delta(i, j), \Delta(j, k)) \leq \Delta(i, k) \leq \Delta(i, j) + \Delta(j, k)$ for all $i \leq j \leq k \in \mathbf{Z}$.

Remark 1.3.1. One should think of $\Omega(i)$ as keeping track of the number of Hodge-Tate-Sen weights equal to i and $\Delta(i, j)$ as keeping track of the dimension of de Rham periods in the weight range $[i, j)$ of a representation. The conditions (i)-(iv) encode the relationships among these values implied by properties of the Sen operator and the structure of B_{dR}.

We say that $\mathbf{D}$ is *full* if $\Delta(i, i+1) = \Omega(i)$ and $\Delta(i, k) = \Delta(i, j) + \Delta(j, k)$ for all $i \leq j \leq k \in \mathbf{Z}$.

We say that $\mathbf{D}$ is a *Hodge-Tate datum* if $\max(\Delta(i, j), \Delta(j, k)) = \Delta(i, k)$ for all $i \leq j \leq k \in \mathbf{Z}$. We say that $\mathbf{D}$ is a *Sen datum* if $\Delta(i, i+1) = \min(\Omega(i), 1)$ and it is Hodge-Tate. For a de Rham datum $\mathbf{D}$, we define its *associated Hodge-Tate datum* $\mathrm{HT}(\mathbf{D}) = (\omega_{\mathrm{HT}}, \Delta_{\mathrm{HT}})$ by $\Omega_{\mathrm{HT}} = \Omega$ and $\Delta_{\mathrm{HT}}(i, j) = \max_{i \leq k < j} \Delta(k, k+1)$ and its *associated Sen datum* $\mathrm{Sen}(\mathbf{D}) = (\omega_{\mathrm{Sen}}, \Delta_{\mathrm{Sen}})$ by $\Omega_{\mathrm{Sen}} = \Omega$ and $\Delta_{\mathrm{Sen}}(i, j) = \max_{i \leq k < j} \min(\Omega(k), 1)$.

Remark 1.3.2. A Sen datum is determined by Ω, which as mentioned above keeps track of the Hodge-Tate-Sen weights, and a Hodge-Tate datum is determined by $\Delta(i, i+1)$ for $i \in \mathbf{Z}$, which keeps track of the Hodge-Tate weights.

For a de Rham datum $\mathbf{D}$, we define

- the *Sen dimension* by $d_{\mathrm{Sen}}(\mathbf{D}) = \sum_{i \in \mathbf{Z}} \Omega(i)$,
- the *Hodge-Tate dimension* by $d_{\mathrm{HT}}(\mathbf{D}) = \sum_{i \in \mathbf{Z}} \Delta(i, i+1)$ and its bounded variant $d_{\mathrm{HT}}^{(k,\ell)}(\mathbf{D}) = \sum_{i=k}^{\ell-1} \Delta(i, i+1)$, and
- the *de Rham dimension* by $d_{\mathrm{dR}}(\mathbf{D}) = \max_{i,j \in \mathbf{Z}} \Delta(i,j)$.

Remark 1.3.3. The sums defining the first two are finite by conditions (i) and (iv), and satisfy $d_{\mathrm{HT}}(\mathbf{D}) \leq d_{\mathrm{Sen}}(\mathbf{D})$. Observe that for any i, j, $\Delta(i,j) \leq \sum_{\nu=i}^{j-1} \Delta(i, i+1)$ by (iv), so $d_{\mathrm{dR}}(\mathbf{D}) \leq d_{\mathrm{HT}}(\mathbf{D})$.

We define the n^{th} twist of $\mathbf{D} = (\Omega, \Delta)$ by $\mathbf{D}(n) = (\Omega', \Delta')$, where $\Omega'(i) = \Omega(i+n)$ and $\Delta'(i,j) = \Delta(i+n, j+n)$.

We define the *lower and upper bounds* of $\mathbf{D}$ by

$$L(\mathbf{D}) = \min\{i : \Omega(i) > 0\} \text{ and } U(\mathbf{D}) = \max\{i : \Omega(i) > 0\},$$

respectively. By conditions (ii), (iii), and (iv), $\mathbf{D}$ is determined by $\Omega|_{[L(\mathbf{D}),U(\mathbf{D})]}$ and $\Delta(i,j)$ for $L(\mathbf{D}) \leq i < j \leq U(\mathbf{D}) + 1$. We write $\mathrm{Supp}(\mathbf{D}) = [L(\mathbf{D}), U(\mathbf{D})] \subseteq \mathbf{Z}$ for the *support* of $\mathbf{D}$.

We define a various notions of ordering on de Rham data $\mathbf{D} = (\Omega, \Delta)$ and $\mathbf{D}' = (\Omega', \Delta')$ as follows. The definition of $\leq$ is the most important, but $<_{\min}^{[i,j]}$ will be used in Theorem 1.3.20 to study flatness of period sheaves.

- We write $\mathbf{D} \leq \mathbf{D}'$ if $\Omega(i) \leq \Omega'(i)$ and $\Delta(i,j) \leq \Delta'(i,j)$ for $i, j \in \mathbf{Z}$.
- We write $\mathbf{D} < \mathbf{D}'$ if $\mathbf{D} \leq \mathbf{D}'$ but not $\mathbf{D}' \leq \mathbf{D}$.
- We write $\mathbf{D} <^{[i,j]} \mathbf{D}'$ if $\mathbf{D} < \mathbf{D}'$ with $\mathrm{Supp}(\mathbf{D}') \subseteq [i,j]$, and write $\mathbf{D} <_{\min}^{[i,j]} \mathbf{D}'$ if there is additionally no de Rham datum $\mathbf{D}''$ with $\mathbf{D} <^{[i,j]} \mathbf{D}'' <^{[i,j]} \mathbf{D}'$.

We also define the $[i,j]$-*truncation* $\mathbf{D}^{[i,j]} = (\Omega^{[i,j]}, \Delta^{[i,j]})$ of a de Rham datum $\mathbf{D}$ by $\Omega^{[i,j]} = \chi_{[i,j]}\Omega$ and $\Delta^{[i,j]}(k,\ell) = \Delta(\max(k,i), \min(\ell, j+1))$, where $\chi_{[i,j]}$ is the characteristic function of $[i,j]$. Note that $\mathbf{D}^{[i,j]}$ is a de Rham datum and $\mathbf{D}^{[i,j]} \leq \mathbf{D}$.

1.3.2 The de Rham datum of a family

We attach a de Rham datum to any G_K-representation on a free module over an integral affinoid E-algebra, and study its specialization and interpolation properties.

Remark 1.3.4. The properties of affinoid E-algebras $\mathfrak{R}$ we will need are that they are Jacobson, Noetherian, compatible with analytic field extension, and that their residue fields at maximal points are finite extensions of E.

We will need the following.

Lemma 1.3.5. *Maintain the notation of Section 1.2.2. If $\mathfrak{R}$ is an integral Noetherian E-Banach algebra with fraction field L, then for any $k < \ell \in \mathbf{Z}$,*

$$\dim_L H^0(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}) \otimes_{\mathfrak{R}} L = \dim_L H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}) \otimes_{\mathfrak{R}} L. \quad (1.53)$$

Proof. First assume $\ell - k = 1$. By symmetry we can just assume $k = 0$ and $\ell = 1$. As before, using Lemma 1.2.16, we may replace E, K with finite Galois extensions E', K' so that Corollary 1.2.8 holds. Extending E might cause $\mathfrak{R}$ to no longer be integral, but if we prove (1.53) for L replaced by the localization at each minimal prime of $\mathfrak{R} \otimes_E E'$ the conclusion for $\mathfrak{R}$ follows from (1.2) and (1.7).

So assume Corollary 1.2.8 holds, but that $\mathfrak{R}$ is only reduced, and let $\mathfrak{p}$ be a minimal prime of $\mathfrak{R}$. We have $\phi : \mathfrak{E} \to \mathfrak{E}$ with $\mathfrak{E}$ finitely generated over $\mathfrak{R}$,

$$H^0(G_K, \mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M}) \cong \ker \phi, \text{ and } H^1(G_K, \mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M}) \cong \operatorname{coker} \phi.$$

Localizing at $\mathfrak{p}$, we obtain (1.53) with $L = \mathfrak{R}_{\mathfrak{p}}$ for $\ell - k = 1$ by dimension counting. The equation (1.53) for general k follows by induction from the sequence similar to (1.19) but instead constructed from (1.41) and localized at $\mathfrak{p}$.

□

Proposition 1.3.6. *Suppose $\mathfrak{R}$ is an integral affinoid E-algebra with $L = \mathrm{Frac}(\mathfrak{R})$ and $\mathfrak{M}$ is a free $\mathfrak{R}$-module equipped with a continuous G_K-action. Let $P_\sigma(T) \in \mathfrak{R}[T]$ be the associated Sen polynomial. We define $\Omega_{\mathfrak{M}}(i) = \max\{n : (T+i)^n | P_\sigma(T)\}$ and*

$$\Delta(i,j) = \dim_L H^0(G_K, t^i B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}) \otimes_{\mathfrak{R}} L$$

for $i, j \in \mathbf{Z}$ with $i < j$. If $i \geq j$, we set $\Delta_{\mathfrak{M}}(i,j) = 0$. Then $\mathbf{D}_{\mathfrak{M}} = (\Omega_{\mathfrak{M}}, \Delta_{\mathfrak{M}})$ is a de Rham datum.

We have $\mathbf{D}_{\mathfrak{M}(n)} = \mathbf{D}_{\mathfrak{M}}(n)$ and $\Delta_{\mathfrak{M}}(i,j) = \dim_L H^1(G_K, t^i B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}) \otimes_{\mathfrak{R}} L$.

Proof. The $(T+i)^n$ are coprime for distinct i, so condition (i) is clear and (ii) is true by definition. The modules $H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})$ for $r \in \{0,1\}$ are finitely generated by induction using Proposition 1.2.10 and the 6-term exact sequences constructed as in (1.19) from the sequences (1.41) and (1.42). Condition (iv) follows from tensoring L with the long exact sequence similar to (1.19), but constructed from

$$0 \to t^j B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \to t^i B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \to t^i B^+_{\mathrm{dR}}/t^j B^+_{\mathrm{dR}} \to 0$$

instead of (1.18). The equality $\mathbf{D}_{\mathfrak{M}(n)} = \mathbf{D}_{\mathfrak{M}}(n)$ is clear. The last claim follows from Lemma 1.3.5.

We check condition (iii). We may let $i = 0$ by symmetry. We write $P_\sigma(T) = S_\sigma(T) Q_\sigma(T)$, where $S_\sigma(T) = \prod_{i \in \mathbf{Z}} (T+i)^{\Omega_{\mathfrak{M}}(i)}$. By Proposition 1.2.26, for any maximal ideal $\mathfrak{m} \subseteq \mathfrak{R}$ with

residue field $\kappa(\mathfrak{m})$, we have an isomorphism

$$H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} \kappa(\mathfrak{m}) \cong H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \kappa(\mathfrak{m}))).$$

Since $H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})$ is finitely generated, there exists a nonzero element $f \in \mathfrak{R}$ so that $H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})_f$ is free. By definition, $Q_\sigma(0) \neq 0$, so by the Jacobson property, $fQ_\sigma(0)$ is nonvanishing in $\kappa(\mathfrak{m})$ for some $\mathfrak{m}$. By (1.10) we have an identification

$$H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} \kappa(\mathfrak{m}) \cong H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})_{fQ_\sigma(0)} \otimes_{\mathfrak{R}_{fQ_\sigma(0)}} \kappa(\mathfrak{m}).$$

In particular, $\dim_{\kappa(\mathfrak{m})} H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \kappa(\mathfrak{m}))) = \Delta_{\mathfrak{M}}(0,1) \leq \Omega_{\mathfrak{M}}(0)$ by Lemma 1.2.23 and compatibility of the Sen polynomial with specialization. If the bound $\Omega_{\mathfrak{M}}(0) \geq 1$ holds, then $\dim_{\kappa(\mathfrak{m})} H^1(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M}\otimes_{\mathfrak{R}}\kappa(\mathfrak{m}))) \geq 1$ since T divides the image of $P_\sigma(T)$ in $\kappa(\mathfrak{m})[T]$, so $\Delta_{\mathfrak{M}}(0,1) \geq 1$.

□

We study the behavior of a de Rham datum under specialization.

Proposition 1.3.7. *Suppose that $\mathfrak{R}$, $\mathfrak{M}$, and $\mathbf{D}_{\mathfrak{M}}$ are as in Proposition 1.3.6. Then for any prime ideal $\mathfrak{p} \subseteq \mathfrak{R}$ with quotient $\mathfrak{R}' = \mathfrak{R}/\mathfrak{p}$, $\mathbf{D}_{\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'} \geq \mathbf{D}_{\mathfrak{M}}$.*

Proof. The inequality $\Omega_{\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'} \geq \Omega_{\mathfrak{M}}$ is clear. For any $k, \ell \in \mathbf{Z}$ we have an isomorphism

$$H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}' \cong H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \qquad (1.54)$$

by Proposition 1.2.26 (and twisting). Let $L = \mathrm{Frac}(\mathfrak{R})$ and $\kappa = \mathrm{Frac}(\mathfrak{R}')$. By (1.54) and

(1.10) we have

$$\dim_\kappa H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})_{\mathfrak{p}} \otimes_{\mathfrak{R}_{\mathfrak{p}}} \kappa = \Delta_{\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'}(k,\ell)$$
$$\text{and} \quad \dim_L H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})_{\mathfrak{p}} \otimes_{\mathfrak{R}_{\mathfrak{p}}} L = \Delta_{\mathfrak{M}}(k,\ell).$$

We deduce the inequality $\Delta_{\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'}(k,\ell) \geq \Delta_{\mathfrak{M}}(k,\ell)$ from Nakayama's lemma.

□

We prove a converse to Proposition 1.3.7. Taken together, they form an interpolation result for de Rham data. If $x \in \operatorname{Spm} \mathfrak{R}$ has residue field $\kappa(x)$, we write $\mathbf{D}_{\mathfrak{M},x}$ for $\mathbf{D}_{\mathfrak{M}\otimes_{\mathfrak{R}}\kappa(x)}$.

Proposition 1.3.8. *Maintain the notation and hypotheses of Proposition 1.3.6. Let* $\mathfrak{X} = \operatorname{Spm} \mathfrak{R}$, *let* $\mathbf{D} = (\Omega, \Delta)$ *be a de Rham datum, assume that* $\Xi \subseteq \mathfrak{X}$ *is Zariski dense, and assume that* $\mathbf{D}_{\mathfrak{M},x} \geq \mathbf{D}$ *for all* $x \in \Xi$. *Then* $\mathbf{D}_{\mathfrak{M}} \geq \mathbf{D}$.

Proof. To see that $\Omega_{\mathfrak{M}} \geq \Omega$, write the Sen polynomial $P_\sigma(T)$ of $\mathfrak{M}$ as a function of $T+i$ and observe that the images of the lowest $\Omega_{\mathfrak{M}}(i)$ coefficients vanish in $(\prod_{x\in\Xi} \kappa(x))[T]$.

By generic freeness (and the finite generation from the proof of Proposition 1.3.6), there is a nonzero element $f \in \mathfrak{R}$ such that $H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})_f$ is free of rank $\Delta_{\mathfrak{M}}(k,\ell)$.

By Proposition 1.2.26 and (1.10), for any $x \in \operatorname{Spm} \mathfrak{R} \setminus V((f))$, we have a natural isomorphism

$$H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})_f \otimes_{\mathfrak{R}_f} \kappa(x) \xrightarrow{\sim} H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \kappa(x))$$

By picking $x \in \Xi \cap (\operatorname{Spm} \mathfrak{R} \setminus V((f)))$, it follows that $\Delta_{\mathfrak{M}}(k,\ell) = \Delta_{\mathfrak{M},x}(k,\ell) \geq \Delta(k,\ell)$.

□

We can deduce the following.

Corollary 1.3.9. *Maintain the hypotheses of Proposition 1.3.6. Suppose $\xi : \mathfrak{R} \to \mathfrak{R}'$ is a finite map of integral E-affinoid algebras, and let $\mathfrak{M}' = \mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}'$. Then $\mathbf{D}_{\mathfrak{M}'} \geq \mathbf{D}_{\mathfrak{M}}$. If $\Xi \subseteq \operatorname{Spm} \mathfrak{R}$ is Zariski-dense in the image of ξ^* and $\mathbf{D}$ is a de Rham datum with $\mathbf{D}_{\mathfrak{M},x} \geq \mathbf{D}$ for $x \in \Xi$, then $\mathbf{D}_{\mathfrak{M}'} \geq \mathbf{D}$.*

Proof. We may factor our map as $\mathfrak{R} \to \mathfrak{R}'' \to \mathfrak{R}'$, where the first map is surjective and the second is injective, so in particular $\mathfrak{R}''$ is also integral. We apply Proposition 1.3.7 to $\mathfrak{R} \to \mathfrak{R}''$ to obtain $\mathbf{D}_{\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}''} \geq \mathbf{D}_{\mathfrak{M}}$ and use the definition of the de Rham datum to find $\mathbf{D}_{\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}''} = \mathbf{D}_{\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'}$, since $\operatorname{Frac}(\mathfrak{R}')$ is just a field extension of $\operatorname{Frac}(\mathfrak{R}'')$ and the equality $\Omega_{\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}''} = \Omega_{\mathfrak{M}\otimes_{\mathfrak{R}}\mathfrak{R}'}$ is clear. For the second claim, we apply Proposition 1.3.8 to $\mathfrak{R}''$ and the first claim to the map $\mathfrak{R}'' \to \mathfrak{R}'$.

□

1.3.3 Globalization

Specialization of a de Rham datum is a local phenomenon, but one may ask for a global version of the definition and interpolation of a de Rham datum.

For a rigid analytic space $\mathfrak{X}$, we use $\mathfrak{O}_{\mathfrak{X}}$ to denote the structure sheaf, $\mathfrak{O}_{\mathfrak{X},x}$ for the stalk at $x \in \mathfrak{X}$, and $\kappa(x)$ for the residue field. For a coherent sheaf $\mathfrak{M}$ on $\mathfrak{X}$, we write $\mathfrak{M}_x$ for the stalk at x and define $\overline{\mathfrak{M}}_x = \mathfrak{M}_x \otimes_{\mathfrak{O}_{\mathfrak{X},x}} \kappa(x)$. We define $\mathbf{D}_{\mathfrak{M},x} = \mathbf{D}_{\overline{\mathfrak{M}}_x}$.

If $\mathfrak{X}$ is a rigid analytic space and $\mathfrak{M}$ is a coherent locally free sheaf on $\mathfrak{X}$, we say that a continuous G_K-representation on $\mathfrak{M}$ is a map $G_K \to \operatorname{End}_{\mathfrak{O}_{\mathfrak{X}}}(\mathfrak{M})$ such that the restriction to any affinoid is continuous with respect to the canonical topology.

Since the Sen polynomial $P_\sigma(T)$ attached to a finite free module $\mathfrak{M}$ over an E-Banach algebra $\mathfrak{R}$ and $\sigma \in \Sigma$ as before is canonical, if $\mathfrak{M}$ is instead a coherent locally free sheaf over a rigid space $\mathfrak{X}$, one can patch together locally defined Sen polynomials on a sufficiently fine admissible affinoid cover to obtain $P_\sigma(T) \in \mathfrak{O}_{\mathfrak{X}}(\mathfrak{X})$.

We will use the notion of an irreducible component of a rigid analytic space $\mathfrak{X}$ [20]. Following Conrad, we say that a closed analytic subvariety $\mathfrak{Z} \subseteq \mathfrak{X}$ is nowhere dense in $\mathfrak{X}$ if it contains no non-empty admissible open of $\mathfrak{X}$. Equivalently, $\dim \mathfrak{O}_{\mathfrak{X},z} > \dim \mathfrak{O}_{\mathfrak{Z},z}$ for all $z \in \mathfrak{Z}$ [20, §2]. This local definition implies that the finite union of nowhere dense closed analytic subvarieties is again nowhere dense. If $\mathfrak{X}$ is irreducible, then every proper closed analytic subvariety is nowhere dense [20, Lemma 2.2.3].

We attach a de Rham datum to any reduced, irreducible rigid analytic space $\mathfrak{X}$ and a G_K-representation on a coherent locally free sheaf $\mathfrak{M}$. We begin with the case where $\mathfrak{X}$ is normal.

Proposition 1.3.10. *Let $\mathfrak{X}$ be a reduced, normal, irreducible rigid analytic space and let $\mathfrak{M}$ be a coherent locally free sheaf equipped with a continuous G_K-action. Then there is a naturally associated de Rham datum $\mathbf{D}_{\mathfrak{M}}$ such that for any connected (thus integral) affinoid subdomain $\mathfrak{U} \subseteq \mathfrak{X}$ with $\mathfrak{M}(\mathfrak{U})$ free, we have $\mathbf{D}_{\mathfrak{M}(\mathfrak{U})} = \mathbf{D}_{\mathfrak{M}}$.*

Proof. It suffices to check that if connected affinoid subdomains $\mathfrak{U}, \mathfrak{V} \subseteq \mathfrak{X}$ have $\mathfrak{M}(\mathfrak{U})$ and $\mathfrak{M}(\mathfrak{V})$ free, then $\mathbf{D}_{\mathfrak{M}(\mathfrak{U})} = \mathbf{D}_{\mathfrak{M}(\mathfrak{V})}$. Fix an admissible covering $\{\mathfrak{U}_j\}_{j\in J}$ of $\mathfrak{X}$ by connected open affinoid subdomains over which $\mathfrak{M}$ is free. We claim that there exists a sequence $\mathfrak{U}_{j_0}, \mathfrak{U}_{j_1}, \ldots, \mathfrak{U}_{j_n}$ of affinoid subdomains from the covering such that $\mathfrak{U} \cap \mathfrak{U}_{j_0} \neq \emptyset$, $\mathfrak{U}_{j_n} \cap \mathfrak{V} \neq \emptyset$, and $\mathfrak{U}_{j_\nu} \cap \mathfrak{U}_{j_{\nu+1}} \neq \emptyset$ for each $\nu \in [0, n-1]$. Indeed, observe that the union $\mathfrak{X}_{\mathfrak{U}}$ of all $\mathfrak{U}_j$ for $j \in J$ such that $\mathfrak{U}_j$ can be reached in finitely many steps by such a sequence starting from $\mathfrak{U}$ is closed, since it either contains or does not intersect each $\mathfrak{U}_j$. Since $\mathfrak{X}_{\mathfrak{U}}$ contains an affinoid and $\mathfrak{X}$ is irreducible, we must have $\mathfrak{X}_{\mathfrak{U}} = \mathfrak{X}$.

By the existence of such a sequence, we are reduced to the case where $\mathfrak{U} \cap \mathfrak{V} \neq 0$. Write $\mathfrak{U} = \operatorname{Spm} \mathfrak{R}_1$ and $\mathfrak{V} = \operatorname{Spm} \mathfrak{R}_2$. Fix any connected affinoid subdomain $\operatorname{Spm} \mathfrak{R} = \mathfrak{W} \subseteq \mathfrak{U} \cap \mathfrak{V}$. As in the proof of Proposition 1.3.8, there exist nonzero $f \in \mathfrak{R}_1$ and $g \in \mathfrak{R}_2$ so that the localizations $H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}(\mathfrak{U}))_f$ and $H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}(\mathfrak{V}))_g$ are free. The proof of Proposition 1.3.8 also shows that for any $x \in \mathfrak{U} \setminus V((f))$, we have $\Delta_{\mathfrak{M},x}(k,\ell) =$

$\Delta_{\mathfrak{M}(\mathfrak{U})}(k,\ell)$, and similarly for $x \in \mathfrak{V} \setminus V((g))$. Both f and g define nonzero elements of $\mathfrak{R}$. Since $\mathfrak{R}$ is integral, we may choose $x \in \mathfrak{W} \setminus V((fg))$. This gives the equality $\Delta_{\mathfrak{M}(\mathfrak{U})} = \Delta_{\mathfrak{M}(\mathfrak{V})}$.

By irreducibility of $\mathfrak{U}$ and $\mathfrak{V}$ and by examining the vanishing locus of the lowest order coefficients of $P_\sigma(T)|_{\mathfrak{U}}$ and $P_\sigma(T)|_{\mathfrak{V}}$ on $\mathfrak{W}$, we find $\Omega_{\mathfrak{M}(\mathfrak{U})}(0) = \Omega_{\mathfrak{M}(\mathfrak{V})}(0)$; $\Omega_{\mathfrak{M}(\mathfrak{U})} = \Omega_{\mathfrak{M}(\mathfrak{V})}$ follows similarly.

□

We obtain the general case by applying Corollary 1.3.9 to the normalization of $\mathfrak{X}$.

Proposition 1.3.11. *Let $\mathfrak{X}$ be a reduced, irreducible rigid analytic space and let $\mathfrak{M}$ be a coherent locally free sheaf equipped with a continuous G_K-action. Then there is a naturally associated de Rham datum $\mathbf{D}_{\mathfrak{M}}$ such that for any irreducible component (with the reduced structure) $\mathfrak{V}$ of any affinoid subdomain $\mathfrak{U} \subseteq \mathfrak{X}$ with $\mathfrak{M}(\mathfrak{U})$ free, if we write $\mathfrak{M}'$ for the pullback to $\mathfrak{V}$, we have $\mathbf{D}_{\mathfrak{M}'} = \mathbf{D}_{\mathfrak{M}}$.*

Proof. Let $\pi : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ be the normalization of $\mathfrak{X}$. Let $\mathfrak{U}$ and $\mathfrak{V}$ be as in the statement of the proposition. The restriction $\pi|_{\pi^{-1}(\mathfrak{U})} : \widetilde{\mathfrak{U}} \to \mathfrak{U}$ is a finite map of reduced affinoids. Let $\mathfrak{R} \to \mathfrak{R}'$ be the corresponding map of reduced affinoid algebras. The map factors as $\mathfrak{R} \to \prod_{\mathfrak{p}} \mathfrak{R}/\mathfrak{p} \to \prod_{\mathfrak{p}} \mathfrak{R}'_{(\mathfrak{p})}$, where $\mathfrak{p}$ ranges over the minimal primes of $\mathfrak{R}$ and $\mathfrak{R}'_{(\mathfrak{p})}$ denotes the normalization of $\mathfrak{R}/\mathfrak{p}$. Choose $\mathfrak{p}$ corresponding to $\mathfrak{V}$. The proof of Corollary 1.3.9 shows that $\mathbf{D}_{\mathfrak{M}(\mathfrak{U})\otimes_{\mathfrak{R}}\mathfrak{R}/\mathfrak{p}} = \mathbf{D}_{\mathfrak{M}(\mathfrak{U})\otimes_{\mathfrak{R}}\mathfrak{R}'_{(\mathfrak{p})}}$. By Proposition 1.3.10, $\mathbf{D}_{\mathfrak{M}(\mathfrak{U})\otimes_{\mathfrak{R}}\mathfrak{R}'_{(\mathfrak{p})}} = \mathbf{D}_{\pi^*\mathfrak{M}}$, which we take to be $\mathbf{D}_{\mathfrak{M}}$.

□

We now globalize the interpolation of a de Rham datum. We will assume Zariski density for our interpolation result, which has the following definition for a general rigid analytic space.

Definition 1.3.12. We say that a subset $S \subseteq \mathfrak{X}$ of a rigid analytic space is *Zariski dense* if every closed analytic subvariety $\mathfrak{Z} \subseteq \mathfrak{X}$ containing S is equal to $\mathfrak{X}$.

Proposition 1.3.13. *Suppose that $\mathfrak{X}$, $\mathfrak{M}$, and $\mathbf{D}_{\mathfrak{M}}$ are as in Proposition 1.3.11. Let $\mathbf{D} = (\Omega, \Delta)$ be a de Rham datum, assume that $\Xi \subseteq \mathfrak{X}$ is Zariski dense, and assume that $\mathbf{D}_{\mathfrak{M},x} \geq \mathbf{D}$ for all $x \in \Xi$. Then $\mathbf{D}_{\mathfrak{M}} \geq \mathbf{D}$.*

Proof. Write $P_\sigma(T)$ for the Sen polynomial of $\mathfrak{M}$. To see that $\Omega_{\mathfrak{M}} \geq \Omega$, observe that the vanishing locus of any of the first $\Omega(i)$ coefficients of the Sen polynomial $P_\sigma(T)$ expanded around $T + i$ includes Ξ and is closed.

We claim that the locus $\mathfrak{Y}$ of points $x \in \mathfrak{X}$ such that $\mathbf{D}_{\mathfrak{M},x} \geq \mathbf{D}$ is closed. To see this, pick an admissible cover of $\mathfrak{X}$ by connected affinoid subsets $\mathfrak{U}$ small enough that $\mathfrak{M}(\mathfrak{U})$ is free, and let $\overline{\mathfrak{Y}}_{\mathfrak{U}}$ denote the Zariski closure of $\mathfrak{Y} \cap \mathfrak{U}$. We apply Corollary 1.3.9 to each irreducible component $\mathfrak{Y}' \to \overline{\mathfrak{Y}}_{\mathfrak{U}}$ to deduce that the pullback $\mathfrak{M}'$ of $\mathfrak{M}(\mathfrak{U})$ to $\mathfrak{Y}'$ satisfies $\mathbf{D}_{\mathfrak{M}'} \geq \mathbf{D}$. By Proposition 1.3.7, we deduce that $\mathfrak{Y} \cap \mathfrak{U} = \overline{\mathfrak{Y}}_{\mathfrak{U}}$.

Thus $\mathfrak{Y}$ is closed and contains the Zariski-dense subset Ξ, so $\mathfrak{Y} = \mathfrak{X}$. The result now follows from applying Proposition 1.3.8 to any affinoid of $\mathfrak{X}$ small enough that $\mathfrak{M}(\mathfrak{U})$ is free and the dense set consisting of all of its points.

□

We deduce the following globalization of Corollary 1.3.9.

Corollary 1.3.14. *Suppose $\pi : \mathfrak{Y} \to \mathfrak{X}$ is a finite map of reduced, irreducible rigid spaces and $\mathfrak{M}$ is a coherent locally free sheaf on $\mathfrak{X}$ equipped with a continuous G_K-action as before. Let $\mathfrak{M}' = \pi^*\mathfrak{M}$. Then $\mathbf{D}_{\mathfrak{M}'} \geq \mathbf{D}_{\mathfrak{M}}$.*

Moreover, if $\Xi \subseteq \mathfrak{X}$ is Zariski-dense in the image of $\mathfrak{Y}$ and $\mathbf{D}$ is a de Rham datum with $\mathbf{D}_{\mathfrak{M},x} \geq \mathbf{D}$ for $x \in \Xi$, then $\mathbf{D}_{\mathfrak{M}'} \geq \mathbf{D}$.

Proof. For the first claim, pick an affinoid open $\mathfrak{U} \subseteq \mathfrak{X}$ so that $\mathfrak{M}(\mathfrak{U})$ is free, and let $\mathfrak{V} \subseteq \mathfrak{Y}$ be its preimage. By [8, Corollary 9.4.4/2], $\mathfrak{V}$ is an affinoid open. Pick an irreducible component $\mathfrak{Z} \subseteq \mathfrak{U}$, pick an irreducible component $\mathfrak{Z}' \subseteq \pi^{-1}(\mathfrak{Z})$, and write $\xi : \mathfrak{Z} \to \mathfrak{X}$ and $\xi' : \mathfrak{Z}' \to \mathfrak{Y}$ for

the inclusions. Apply Corollary 1.3.9 and Proposition 1.3.11 to the map $\mathfrak{Z}' \to \mathfrak{Z}$ to find that $\mathbf{D}_{\mathfrak{M}'} = \mathbf{D}_{\xi'^*\mathfrak{M}'} \geq \mathbf{D}_{\xi^*\mathfrak{M}} = \mathbf{D}_{\mathfrak{M}}$.

For the second claim, the image of $\mathfrak{Y}$ is an irreducible closed analytic subvariety [8, Propositions 9.6.2/5 and 9.6.3/3], so we may factor our map as $\mathfrak{Y} \to \mathfrak{Z} \to \mathfrak{X}$, where the second map is a closed immersion. We then apply Proposition 1.3.13 to $\mathfrak{Z}$ and the first claim to the map $\mathfrak{Y} \to \mathfrak{Z}$.

□

1.3.4 Stratification of families of Galois representations

We collect the results of this section into a simple statement. The motivating case is where $\mathfrak{X}$ is the global eigenvariety on a group and $\mathfrak{M}$ is the associated Galois representation over $\mathfrak{X}$. A discussion of the importance of understanding the geometry of $\mathfrak{X}$ can be found in, for instance, Bellaïche-Chenevier [3].

We will use the following flatness criterion.

Lemma 1.3.15. *Suppose that R is a reduced Noetherian ring with total ring of fractions K and M is a finitely generated R-module. Suppose that $M \otimes_R K$ is free of rank r and that $\dim_{\kappa(\mathfrak{m})} M \otimes_R \kappa(\mathfrak{m}) = r$ for every maximal ideal $\mathfrak{m} \subseteq R$, denoting the residue field of $\mathfrak{m}$ by $\kappa(\mathfrak{m})$. Then M is locally free of rank r.*

Proof. It suffices to check that $M_{\mathfrak{m}}$ is free of rank r for all maximal $\mathfrak{m} \subseteq R$ by [9, Theorem II.5.2.1]. Also note that by hypothesis $M_{\mathfrak{p}}$ has dimension r for any minimal prime $\mathfrak{p}$ of R.

Let $L = \prod_{\mathfrak{p}} R_{\mathfrak{p}}$ be the total ring of fractions of $R_{\mathfrak{m}}$, where $\mathfrak{p}$ ranges over the minimal primes of $R_{\mathfrak{m}}$. Using Nakayama's lemma, let $\psi : R_{\mathfrak{m}}^r \twoheadrightarrow M_{\mathfrak{m}}$ be a surjection, and suppose that a is a nonzero element of its kernel. Since $R_{\mathfrak{m}}^r \to L^r$ is injective, there is some $\mathfrak{p}$ so that a is nonzero in $R_{\mathfrak{p}}^r$. Localizing ψ at $\mathfrak{p}$, we obtain $R_{\mathfrak{p}}^r \twoheadrightarrow M_{\mathfrak{p}}$. By dimension count, this map is an isomorphism, so a cannot have been nonzero. In particular, ψ is an isomorphism as needed.

□

We will use the following to reduce checking flatness and base change to the study of 1-cocycles.

Lemma 1.3.16. *Maintain the notation of Section 1.2.2. Let $[i,j] \subseteq \mathbf{Z}$ be a fixed interval, let $f \in \mathfrak{R}$, and suppose that $H^1(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})_f$ is flat for $i \leq k < \ell \leq j+1$. Then the following hold.*

(a) *The $\mathfrak{R}_f$-module $H^0(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M})_f$ is flat for $i \leq k < \ell \leq j+1$.*

(b) *If $\xi : \mathfrak{R} \to \mathfrak{R}'$ is a map of Noetherian E-Banach algebras, the natural maps*

$$H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}' \to H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \quad (1.55)$$

have kernel and cokernel annihilated by a power of $\xi(f)$ for $r \in \{0,1\}$.

Proof. For a fixed interval $[i,j]$, we prove both results by induction on $\ell - k$. If $\ell - k = 0$, after increasing E and K, both parts follow from Corollary 1.2.8 and Lemma 1.2.3. The reduction to the original E and K at the beginning of the proof of Theorem 1.2.13.(a) applies here for both parts, giving the base case.

For part (a), observe that if R is a ring, $0 \to A \to B \to C \to D \to E \to F \to 0$ is a 6-term exact sequence of R-modules, and A, C, D, E, and F are flat over R, then B is flat as well. For the inductive step, we localize the 6-term exact sequence similar to (1.19) but starting instead with (1.41) at f to reduce to a case with smaller $\ell - k$.

For part (b), first note that as in the proof of Theorem 1.2.19.(b), it suffices to check that the map is an isomorphism after localization at $\xi(f)$. We consider the map between two rows of 6-term complexes as in (1.22) and (1.23), except beginning with the exact sequence (1.41). In particular, the downward maps have the form of (1.55). The lower row is exact, while the top row becomes exact after localizing at $\xi(f)$ by (1.10) and the flatness

of $H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M})_f$ from part (a). By the inductive hypothesis, after localizing at $\xi(f)$, the first, third, fourth, and sixth downward maps are isomorphisms, which gives the result by the 5-lemma.

□

We also check a finiteness property of de Rham data. In fact, what is useful here is not the finiteness but the somewhat explicit description of the $\mathbf{D}'$ with $\mathbf{D} <^{[i,j]}_{\min} \mathbf{D}'$.

Lemma 1.3.17. *Let $\mathbf{D} = (\Omega, \Delta)$ be a de Rham datum. For a fixed interval $[i, j] \supseteq \mathrm{Supp}(\mathbf{D})$, there are finitely many de Rham data $\mathbf{D}'$ with $\mathbf{D} <^{[i,j]}_{\min} \mathbf{D}'$.*

Proof. It suffices to define de Rham data $\mathbf{D}_w = (\Omega_w, \Delta_w)$ for $w \in [i, j]$ with $\mathbf{D} <^{[i,j]}_{\min} \mathbf{D}_w$ and the additional property that if $\mathbf{D}' = (\Omega', \Delta')$ satisfies $\mathbf{D}' \geq \mathbf{D}$ and $\Omega'(w) > \Omega(w)$, then $\mathbf{D} \geq \mathbf{D}_w$. Indeed, for any $\mathbf{D}'$ with $\mathbf{D} <^{[i,j]}_{\min} \mathbf{D}'$, either $\Omega' = \Omega$ or $\mathbf{D}' = \mathbf{D}_w$ for some w. There are only finitely many possibilities for $\mathbf{D}'$ in the former case.

We define $\Omega_w(i) = \Omega(i)$ for $i \neq w$ and $\Omega_w(w) = \Omega(w) + 1$, and for $k < \ell$, we set $\Delta_w(k, \ell) = \max(\Delta(k, \ell), 1)$ if $k \leq w < \ell$ and $\Delta_w(k, \ell) = \Delta(k, \ell)$ otherwise. It is clear that $\mathbf{D}_w$ is a de Rham datum and has the needed properties.

□

We denote the set of such $\mathbf{D}'$ by $\mathrm{Min}(i, j)$. If $\mathbf{D}$ is a full de Rham datum, then the proof shows that $\{\mathbf{D}_w\}_{w\in[i,j]} = \mathrm{Min}(i, j)$.

We will use the following gluing result to globalize the construction of sheaves of periods and 1-cocycles.

Lemma 1.3.18. *Let $\mathfrak{X}$ be a rigid analytic space over E, let $\mathfrak{M}$ be a coherent locally free sheaf of $\mathfrak{O}_{\mathfrak{X}}$-modules equipped with a continuous G_K-action.*

Let $\mathfrak{V} \subseteq \mathfrak{U}$ be a pair of affinoid subdomains of $\mathfrak{X}$ such that $\mathfrak{M}(\mathfrak{U})$ is free, and let $\mathfrak{U} =$

$\operatorname{Spm}\mathfrak{R}_{\mathfrak{U}}$ *and* $\mathfrak{V} = \operatorname{Spm}\mathfrak{R}_{\mathfrak{V}}$. *Assume that for any such pair* $(\mathfrak{U}, \mathfrak{V})$, *the natural map*

$$H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}(\mathfrak{U})) \otimes_{\mathfrak{R}_{\mathfrak{U}}} \mathfrak{R}_{\mathfrak{V}} \to H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}(\mathfrak{V})) \tag{1.56}$$

is an isomorphism.

Then there are locally free coherent sheaves $\mathfrak{H}^r_{(k,\ell)}(\mathfrak{M})$ *on* $\mathfrak{X}$ *for* $r \in \{0, 1\}$ *such that for any affinoid subdomain* $\mathfrak{W}$ *with* $\mathfrak{M}(\mathfrak{W})$ *free,*

$$\mathfrak{H}^r_{(k,\ell)}(\mathfrak{W}) \cong H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}(\mathfrak{W})). \tag{1.57}$$

Proof. Let $\{\mathfrak{U}_j\}$ be an admissible affinoid cover of $\mathfrak{X}$ such that $\mathfrak{M}(\mathfrak{U}_j)$ is free for each j. We claim that if we define $\mathfrak{H}^{(k,\ell)}_r(\mathfrak{M})|_{\mathfrak{U}_j}$ to be the sheaf on $\mathfrak{U}_j$ associated to the module in (1.57) for each j, then these naturally patch together on the intersections to form a coherent sheaf. So let $\mathfrak{V} \subseteq \mathfrak{U}_{j_1} \cap \mathfrak{U}_{j_2}$, where all of these are affinoid, and write $\mathfrak{V} = \operatorname{Spm}\mathfrak{R}$ and $\mathfrak{U}_{j_\nu} = \operatorname{Spm}\mathfrak{R}_\nu$ for $\nu \in \{1, 2\}$. Our hypothesis (1.56) implies that the restrictions of each $\mathfrak{H}^{(k,\ell)}_r(\mathfrak{M})|_{\mathfrak{U}_{j_\nu}}$ to $\mathfrak{V}$ are naturally identified with the sheaf associated to $H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}(\mathfrak{V}))$. We pick an admissible affinoid cover of $\mathfrak{U}_{j_1} \cap \mathfrak{U}_{j_2}$ to define the gluing via this identification, and it is clear from the hypothesis (1.56) that these are compatible on the triple intersections (by again restricting to affinoids).

If $\mathfrak{W} = \operatorname{Spm}\mathfrak{R}$ is any affinoid subdomain of $\mathfrak{X}$ with $\mathfrak{M}(\mathfrak{W})$ free, then we claim that $\mathfrak{H}^{(k,\ell)}_r(\mathfrak{M})|_{\mathfrak{W}}$ is naturally isomorphic to the sheaf associated to the module in (1.57). This can be checked locally, so we pick an admissible affinoid cover that refines the cover $\{\mathfrak{W} \cap \mathfrak{U}_j\}$. For any member $\mathfrak{V}$ of this cover, we again use the hypothesis (1.56) to obtain the needed compatibility.

□

We break the main theorem up into three parts, which respectively discuss the closed

subvarieties of the base associated to de Rham data, behavior of periods on the corresponding stratification, and specializations. We maintain the same notation and hypotheses throughout.

Theorem 1.3.19. *Suppose that $\mathfrak{X}$ is a reduced rigid space over E and $\mathfrak{M}$ is a coherent locally free sheaf equipped with a continuous homomorphism $G_K \to \mathrm{End}_{\mathcal{O}_{\mathfrak{X}}} \mathfrak{M}$.*

Then for any de Rham datum $\mathbf{D}$, the points $x \in \mathfrak{X}$ such that $\mathbf{D}_{\mathfrak{M},x} \geq \mathbf{D}$ form a closed analytic subvariety $\mathfrak{S}_{\mathbf{D}}$, which we give its reduced structure. If $\mathbf{D} \geq \mathbf{D}'$, then $\mathfrak{S}_{\mathbf{D}} \subseteq \mathfrak{S}_{\mathbf{D}'}$. We write $\pi_{\mathbf{D}} : \mathfrak{S}_{\mathbf{D}} \to \mathfrak{X}$ for the inclusion and define $\mathfrak{M}_{\mathbf{D}} = \pi_{\mathbf{D}}^* \mathfrak{M}$.

If we range instead over the various Sen data or Hodge-Tate data, we obtain coarser stratifications. For any Sen datum $\mathbf{D}$, the various Hodge-Tate or de Rham data $\mathbf{D}'$ with $\mathrm{Sen}(\mathbf{D}') = \mathbf{D}$ give stratifications of $\mathfrak{S}_{\mathbf{D}}$, and the same holds for a Hodge-Tate datum $\mathbf{D}$ and the de Rham data $\mathbf{D}'$ with $\mathrm{HT}(\mathbf{D}') = \mathbf{D}$.

Proof. Let $\mathfrak{S}_{\mathbf{D}}$ be the Zariski closure of the set of $x \in \mathfrak{X}$ with $\mathbf{D}_{\mathfrak{M},x} \geq \mathbf{D}$, let $\mathfrak{Z}$ be any irreducible component of $\mathfrak{S}_{\mathbf{D}}$, and let $\mathfrak{Z}'$ be any irreducible component of $\mathfrak{X}$ containing $\mathfrak{Z}$. We apply the second part of Corollary 1.3.14 to the closed immersion of $\mathfrak{Z}$ into $\mathfrak{Z}'$ to deduce that $\mathbf{D}_{\mathfrak{M}_{\mathfrak{Z}}} \geq \mathbf{D}$, where $\mathfrak{M}_{\mathfrak{Z}}$ is the pullback of $\mathfrak{M}$ to $\mathfrak{Z}$. Using Proposition 1.3.7, we deduce that for any $x \in \mathfrak{Z}$, $\mathbf{D}_{\mathfrak{M},x} \geq \mathbf{D}$. The containment $\mathfrak{S}_{\mathbf{D}} \subseteq \mathfrak{S}_{\mathbf{D}'}$ is clear, as is the second claim. □

We next study flatness and base change. Note that the proof of Lemma 1.3.17 gives a somewhat explicit description of the possible $\mathfrak{L}_{\mathbf{D}}^{[i,j]}$ below. Moreover, $d_{\mathrm{Sen}}(\mathbf{D})$ is constrained by the rank of $\mathfrak{M}$.

Theorem 1.3.20. *For any interval $[i,j] \subseteq \mathbf{Z}$, we define Zariski open (in $\mathfrak{S}_{\mathbf{D}}$) subvarieties $\mathfrak{L}_{\mathbf{D}}^{[i,j]} = \mathfrak{S}_{\mathbf{D}} \setminus \cup_{\mathbf{D}' \in \mathrm{Min}(i,j)} \mathfrak{S}_{\mathbf{D}'}$ for $\mathbf{D}$ with $\mathrm{Supp}(\mathbf{D}) \subseteq [i,j]$. These are locally closed in $\mathfrak{X}$, and $\coprod_{\mathbf{D}} \mathfrak{L}_{\mathbf{D}}^{[i,j]} = \mathfrak{X}$. We write $\pi_{\mathbf{D}}^{[i,j]} : \mathfrak{L}_{\mathbf{D}}^{[i,j]} \to \mathfrak{X}$ for the inclusion. Let $\mathfrak{M}_{\mathbf{D}}^{[i,j]} = \pi_{\mathbf{D}}^{[i,j]*} \mathfrak{M}$. Then we have the following.*

(a) *If* $i \leq k < \ell \leq j+1$, $r \in \{0,1\}$, *and* $\mathbf{D} = (\Omega, \Delta)$ *has* $\mathrm{Supp}(\mathbf{D}) \subseteq [i,j]$, *then there exists a coherent locally free sheaf* $\mathfrak{H}^r_{(k,\ell)}(\mathfrak{M}^{[i,j]}_{\mathbf{D}})$ *of rank* $\Delta(k,\ell)$ *such that for any affinoid subdomain* $\mathfrak{U} \subseteq \mathfrak{L}^{[i,j]}_{\mathbf{D}}$, *there is a canonical isomorphism* $\mathfrak{H}^r_{(k,\ell)}(\mathfrak{M}^{[i,j]}_{\mathbf{D}})(\mathfrak{U}) \cong H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}^{[i,j]}_{\mathbf{D}}(\mathfrak{U}))$.

(b) *Suppose that* $i \leq k < \ell \leq j+1$, $r \in \{0,1\}$, *and* $\pi : \mathfrak{Y} \to \mathfrak{L}^{[i,j]}_{\mathbf{D}}$ *is a map of reduced rigid spaces. Then for any* $y \in \mathfrak{Y}$, $\mathbf{D}^{[i,j]}_{\pi^*\mathfrak{M}^{[i,j]}_{\mathbf{D}},y} = \mathbf{D}$ *and the natural map* $\pi^*\mathfrak{H}^r_{(k,\ell)}(\mathfrak{M}^{[i,j]}_{\mathbf{D}}) \to \mathfrak{H}^r_{(k,\ell)}(\pi^*\mathfrak{M}^{[i,j]}_{\mathbf{D}})$ *is an isomorphism.*

Proof. If $x \in \mathfrak{L}^{[i,j]}_{\mathbf{D}} \cap \mathfrak{L}^{[i,j]}_{\mathbf{D}'}$, then $\mathbf{D}^{[i,j]}_x \geq \mathbf{D}$ and $\mathbf{D}^{[i,j]}_x \geq \mathbf{D}'$, so by definition of the $\mathfrak{L}^{[i,j]}_{\mathbf{D}}$, we must have $\mathbf{D}^{[i,j]}_x = \mathbf{D} = \mathbf{D}'$. Moreover, for any $x \in \mathfrak{X}$, $x \in \mathfrak{L}^{[i,j]}_{\mathbf{D}^{[i,j]}_x}$, so $\coprod_{\mathbf{D}} \mathfrak{L}^{[i,j]}_{\mathbf{D}} = \mathfrak{X}$.

To prove (a), first let $\mathfrak{U} = \mathrm{Spm}\, \mathfrak{R}$ be any affinoid subdomain of $\mathfrak{L}^{[i,j]}_{\mathbf{D}}$ such that $\mathfrak{M}^{[i,j]}_{\mathbf{D}}(\mathfrak{U})$ is free. We define

$$M^r = H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}^{[i,j]}_{\mathbf{D}}(\mathfrak{U})) \text{ and } M^r(x) = H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \overline{\mathfrak{M}}^{[i,j]}_{\mathbf{D},x}),$$

for $x \in \mathfrak{U}$ and $r \in \{0,1\}$. We claim that for $i \leq k < \ell \leq j+1$, M^1 is finite flat of rank $\Delta(k,\ell)$. It will then follow from Lemma 1.3.16.(a) that the same is true for M^0.

For any $x \in \mathfrak{U}$, $M^1(x) \cong M^1 \otimes_{\mathfrak{R}} \kappa(x)$ by Proposition 1.2.26. By the bound $i \leq k < \ell \leq j+1$ and the definition of $\mathfrak{L}^{[i,j]}_{\mathbf{D}}$, we have $\mathbf{D}^{[i,j]}_x = \mathbf{D}$ for each x. By Proposition 1.3.11, for every irreducible component $\mathfrak{Z}$ of $\mathfrak{U}$, if $\mathfrak{Z}$ is equipped with the pullback of $\mathfrak{M}^{[i,j]}_{\mathbf{D}}$, it has de Rham datum $\mathbf{D}$. Thus the module M^1 satisfies the hypotheses of Lemma 1.3.15, so it is finite flat of rank $\Delta(k,\ell)$. It now follows from Lemma 1.3.16.(b) that the hypothesis (1.56) of Lemma 1.3.18 holds for $\mathfrak{L}^{[i,j]}_{\mathbf{D}}$, $\mathfrak{M}^{[i,j]}_{\mathbf{D}}$, and the pair (k,ℓ), which gives (a).

For (b), we pick an admissible affinoid cover $\{\mathfrak{U}_j\}$ of $\mathfrak{L}^{[i,j]}_{\mathbf{D}}$ with $\mathfrak{M}^{[i,j]}_{\mathbf{D}}(\mathfrak{U}_j)$ free. For each $y \in \mathfrak{Y}$, pick j so that y maps to $\mathfrak{U}_j$. We obtain $\mathbf{D}^{[i,j]}_{\pi^*\mathfrak{M}^{[i,j]}_{\mathbf{D}},y} = \mathbf{D}$ from the restriction of π to $y \to \mathfrak{U}_j$, part (a), and Lemma 1.3.16.(b). We use this and the construction of part (a) to define $\mathfrak{H}^r_{(k,\ell)}(\pi^*\mathfrak{M}^{[i,j]}_{\mathbf{D}})$.

For each j, we pick an admissible affinoid cover $\{\mathfrak{V}_{ij}\}$ of $\pi^{-1}(\mathfrak{U}_j)$ for each j. These form an admissible cover of $\mathfrak{Y}$ by (G_2) of [8, §9.1.2]. We now apply Lemma 1.3.16.(b) to each $\mathfrak{U}_j$ and $\mathfrak{V}_{ij}$, using the flatness in part (a), to deduce that the restriction of the natural map $\pi^*\mathfrak{H}^r_{(k,\ell)}(\mathfrak{M}_{\mathbf{D}}^{[i,j]}) \to \mathfrak{H}^r_{(k,\ell)}(\pi^*\mathfrak{M}_{\mathbf{D}}^{[i,j]})$ to each $\mathfrak{V}_{ij}$ is an isomorphism, as needed.

□

Theorem 1.3.21. *Suppose that we have an interval $[i,j] \subseteq \mathbf{Z}$ and a map $\xi : \mathfrak{x} \to \mathfrak{L}_{\mathbf{D}}^{[i,j]}$, where $\mathfrak{x} = \operatorname{Spm} \mathfrak{R}$ for a local Artinian E-algebra $\mathfrak{R}$ of finite dimension over E and $\operatorname{Supp}(\mathbf{D}) \subseteq [i,j]$. Then for $i \leq k < \ell \leq j+1$ we have $\operatorname{rank}_{\mathfrak{R}} H^r(G_K, t^k B^+_{\mathrm{HT}}/t^\ell B^+_{\mathrm{HT}} \otimes_{K,\sigma} \xi^*\mathfrak{M}) = d^{(k,\ell)}_{\mathrm{HT}}(\mathbf{D})$ and $\operatorname{rank}_{\mathfrak{R}} H^r(G_K, t^k B^+_{\mathrm{dR}}/t^\ell B^+_{\mathrm{dR}} \otimes_{K,\sigma} \xi^*\mathfrak{M}) = \Delta(k,\ell)$ for $r \in \{0,1\}$, where these modules are finite flat. Moreover, for $r, r^+ \in \{0,1\}$,*

$$\dim_E H^r(G_K, B_{\mathrm{dR}} \otimes_{K,\sigma} \xi^*\mathfrak{M}) \geq \dim_E H^{r^+}(G_K, t^i B^+_{\mathrm{dR}} \otimes_{K,\sigma} \xi^*\mathfrak{M}) \geq d_{\mathrm{dR}}(\mathbf{D}) \dim_E \mathfrak{R}. \quad (1.58)$$

Proof. The first two claims follow from Theorem 1.3.20.(b). We deduce (1.58) from Proposition 1.2.35.

□

1.4 Higher cohomology

The strategy for proving vanishing of higher cohomology of $B_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}\mathfrak{M}$ and $B_{\mathrm{HT}}\hat{\otimes}_{K,\sigma}\mathfrak{M}$ (in the notation above) is to first prove vanishing of H_K-cohomology and Γ_K-cohomology in the bounded setting, and then use a continuous cohomology version of the inflation-restriction exact sequence to obtain vanishing of G_K-cohomology. We apply Proposition 1.2.28 and Lemma 1.2.34 to pass to $t^k B^+_{\mathrm{dR}}$ and then B_{dR}.

1.4.1 Higher H_K-cohomology

We begin by proving a higher dimensional variant of a vanishing result of Sen [67, Proposition 2]. Sen proves the $n = 1$ case of the following result.

Theorem 1.4.1. *Let $\mathfrak{M}$ be a Banach space over $\mathbf{C}_p$ on which H_K acts continuously by semilinear automorphisms. Then $H^n(H_K, \mathfrak{M}) = 0$ for $n > 0$.*

Proof. For a continuous function $\psi : H_K^n \to \mathfrak{M}$, define $|\psi| = \max_{h_i \in H_K} \|\psi(h_1, \ldots, h_n)\|$. Fix $\delta > 1$. Given $\epsilon, \epsilon' > 0$, and a normalized cocycle $\psi : H_K^n \to \mathfrak{M}$ such that $|\psi| \leq \epsilon$, we will produce a normalized cocycle $\psi' = \psi - d\beta$ where $|\psi'| \leq \epsilon'$ and $d\beta$ is a normalized coboundary such that $|d\beta| \leq \delta\epsilon$. (Recall that a normalized cocycle has the property that $\psi(h_1, \ldots, h_n) = 0$ if any $h_i = 1$. Every cocycle is equivalent to a normalized cocycle [22, Lemma 6.1], and the coboundary of a normalized cochain is normalized.)

Assume that such a construction is possible. By compactness of H_K, any cocycle $\psi = \psi_1$ (which we may assume is normalized) has a bound ϵ_1. We choose a decreasing sequence $\epsilon_i \to 0$, and in iterating the construction, we set $\epsilon = \epsilon_i$, $\epsilon' = \epsilon_{i+1}$ at each stage. We obtain $\psi_i, d\beta_{i-1}$ with $\psi_i = \psi_{i-1} - d\beta_{i-1}$ with $|\psi_i| \leq \epsilon_i$ and $|d\beta_{i-1}| \leq \delta\epsilon_{i-1}$. Then the coboundary $\sum_i d\beta_i$ is defined and $\psi_1 = -\sum_i d\beta_i$, as needed.

Now suppose $\psi : H_K^n \to \mathfrak{M}$, $\epsilon > 0$, and $\epsilon' > 0$ as above are given. We claim that there exists an open normal subgroup $H' \subseteq H_K$ such that $\|\psi(h_1, \ldots, h_n)\| \leq \frac{\epsilon'}{\delta}$ for $h_1, \ldots, h_{n-1} \in H_K$ and $h_n \in H'$. Using [51, Proposition 1.1.3], let $\{H_i\}_{i \in I}$ be a set of open normal subgroups of H_K such that $\cap_i H_i = 1$. The images $\mathfrak{M}_i = \psi(H_K, \ldots, H_K, H_i)$ satisfy $\cap_i \mathfrak{M}_i = \{0\}$ (because ψ is normalized and continuous and $\mathfrak{M}$ is Hausdorff) and are compact. Let $Y_i = (\mathfrak{M} \setminus B_{\frac{\epsilon'}{\delta}}) \cap \mathfrak{M}_i$, where $B_{\frac{\epsilon'}{\delta}}$ denotes the open ball of radius $\frac{\epsilon'}{\delta}$. Since $\mathfrak{M} \setminus B_{\frac{\epsilon'}{\delta}}$ is closed, Y_i is compact. We have $\cap_i Y_i = \emptyset$. By compactness, some $\cap_{i \in I'} Y_i$ is empty with $I' \subseteq I$ finite, so we may set $H' = \cap_{i \in I'} H_i$.

Let S be a set of coset representatives of H_K/H', and as in [67, Proposition 2], we define

an element $z \in \mathbf{C}_p^{H'}$ such that $|z| \leq \delta$ and $\operatorname{Tr} z = \sum_{s \in S} s(z) = 1$.

We define a normalized cochain $\beta : H_K^{n-1} \to \mathfrak{M}$ by

$$\beta(h_1, \ldots, h_{n-1}) = (-1)^n \sum_{s \in S} h_1 \ldots h_{n-1} s(z) \psi(h_1, \ldots, h_{n-1}, s).$$

Note that we have the bound $|d\beta| \leq \delta\epsilon$. In the following calculation, we write $O(r)$ to denote any element $m \in \mathfrak{M}$ such that $\|m\| \leq r$. Since ψ is a cocycle, we have

$$h_1 \psi(h_2, \ldots, h_n, s) = \sum_{j=1}^{n} (-1)^{j+1} \psi(h_1, \ldots, h_j h_{j+1}, \ldots, h_n, s) + (-1)^n \psi(h_1, \ldots, h_n) \quad (1.59)$$

We have

$$\begin{aligned}(\psi - d\beta)(h_1, \ldots, h_n) = \psi(h_1, \ldots, h_n) &- h_1 \left((-1)^n \sum_{s \in S} h_2 \ldots h_n s(z) \psi(h_2, \ldots, h_n, s) \right) \\ &- \sum_{j=1}^{n-1} (-1)^j \left((-1)^n \sum_{s \in S} h_1 \ldots h_n s(z) \psi(h_1, \ldots, h_j h_{j+1}, \ldots, h_n, s) \right) \\ &+ (-1)^n \left((-1)^n \sum_{s \in S} h_1 \ldots h_{n-1} s(z) \psi(h_1, \ldots, h_{n-1}, s) \right).\end{aligned}$$

By (1.59) and semilinearity of the H_K-action, we have

$$
\begin{aligned}
&(\psi - d\beta)(h_1, \ldots, h_n) = \psi(h_1, \ldots, h_n) \\
&+ (-1)^{n+1} \sum_{s \in S} h_1 h_2 \ldots h_n s(z) \\
&\qquad \cdot \left(\sum_{j=1}^{n} (-1)^{j+1} \psi(h_1, \ldots, h_j h_{j+1}, \ldots, h_n, s) + (-1)^n \psi(h_1, \ldots, h_n) \right) \\
&+ \sum_{j=1}^{n-1} (-1)^{j+1} \left((-1)^n \sum_{s \in S} h_1 \ldots h_n s(z) \psi(h_1, \ldots, h_j h_{j+1}, \ldots, h_n, s) \right) \\
&- \sum_{s \in S} h_1 \ldots h_{n-1} s(z) \psi(h_1, \ldots, h_{n-1}, s) \\
&= \psi(h_1, \ldots, h_n) - \sum_{s \in S} h_1 h_2 \ldots h_n s(z) \psi(h_1, \ldots, h_n) \\
&+ (-1)^{n+1} \sum_{s \in S} h_1 h_2 \ldots h_n s(z) (-1)^{n+1} \psi(h_1, \ldots, h_{n-1}, h_n s) \\
&- \sum_{s \in S} h_1 \ldots h_{n-1} s(z) \psi(h_1, \ldots, h_{n-1}, s).
\end{aligned}
$$

Note that $\sum_{s \in S} h_1 h_2 \ldots h_n s(z) = \operatorname{Tr} z = 1$. We are left with

$$
\sum_{s \in S} h_1 \ldots h_n s(z) \psi(h_1, \ldots, h_{n-1}, h_n s) - \sum_{s \in S} h_1 \ldots h_{n-1} s(z) \psi(h_1, \ldots, h_{n-1}, s).
$$

Let $s'(h_n, s)$ and $h'(h_n, s)$ be the unique elements of S and H', respectively, such that $h_n s = s' h'$. Then by H'-invariance of z, this expression becomes

$$
\sum_{s \in S} h_1 \ldots h_{n-1} s'(z) \psi(h_1, \ldots, h_{n-1}, s' h') - \sum_{s \in S} h_1 \ldots h_{n-1} s(z) \psi(h_1, \ldots, h_{n-1}, s),
$$

which we may rewrite as

$$
\sum_{s' \in S} h_1 \ldots h_{n-1} s'(z) \left(\psi(h_1, \ldots, h_{n-1}, s' h') - \psi(h_1, \ldots, h_{n-1}, s') \right)
$$

since the s' are a permutation of the s. We use the cocycle relation

$$\begin{aligned}\psi(h_1,\dots,h_{n-1},s'h') &= (-1)^{n+1}\big[h_1\psi(h_2,\dots,h_{n-1},s',h')\\ &+\sum_{j=1}^{n-1}(-1)^j\psi(h_1,\dots,h_jh_{j+1},\dots,h_{n-1},s',h')+(-1)^{n+1}\psi(h_1,\dots,h_{n-1},s')\big]\\ &=\psi(h_1,\dots,h_{n-1},s')+O\left(\frac{\epsilon'}{\delta}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{s'\in S}h_1\dots h_{n-1}s'(z)\left(\psi(h_1,\dots,h_{n-1},s'h')-\psi(h_1,\dots,h_{n-1},s')\right)\\ &\qquad=\sum_{s'\in S}h_1\dots h_{n-1}s'(z)O\left(\frac{\epsilon'}{\delta}\right)=O(\epsilon')\end{aligned}$$

as needed, using $|z|\le\delta$.

□

1.4.2 Γ_K-cohomology and inflation-restriction

We now use Proposition 1.2.28 to find sufficient conditions for vanishing of higher continuous Γ_K-cohomology.

Proposition 1.4.2. *Let M be a continuous $\mathbf{Z}_p[\Gamma_K]$-module, and suppose that there exist $\mathbf{Z}_p[\Gamma_K]$-submodules $M_i\subseteq M$ for $i\in\mathbf{Z}_{\ge0}$ such that $M_i\subseteq M_{i-1}$, each M/M_i is a discrete p^∞-torsion module, and $M\cong\varprojlim_i M/M_i$ as topological $\mathbf{Z}_p[\Gamma_K]$-modules. Then $H^n(\Gamma_K,M)=0$ for $n\ge2$.*

Proof. Observe that $H^n(\Gamma_K,M/M_i)=0$ for $n\ge2$ and all i by [51, Proposition 1.7.7]. For $n\ge3$, the claim follows from (1.33). For $n=2$, we find instead that $\varprojlim_i{}^1 H^1(\Gamma_K,M/M_i)\cong H^2(\Gamma_K,M)$, so we need only show that $(H^1(\Gamma_K,M/M_i))_i$ has surjective transition maps. But

we have $(H^1(\Gamma_K, M/M_i))_i = (M/(M_i + (\gamma - 1)M))_i$ by Lemma 1.2.5, so the result follows.

□

We prove the inflation-restriction exact sequence for continuous cohomology of profinite groups acting on a special class of modules.

Proposition 1.4.3. *Suppose that $H \subseteq G$ are profinite groups, with H closed and normal in G. Assume that M is a continuous G-module with $H^k(H, M) = 0$ for $1 \leq k \leq n-1$. Moreover, assume that there exist open G-submodules $M_i \subseteq M$ for $i \in \mathbf{Z}_{\geq 0}$ such that $M_i \subseteq M_{i-1}$ for each i and $M \cong \varprojlim_i M/M_i$ as topological G-modules. Then the inflation and restriction maps induce an exact sequence*

$$0 \to H^n(G/H, M^H) \to H^n(G, M) \to H^n(H, M). \tag{1.60}$$

We use dimension shifting, so we need to construct an induced module for continuous cohomology.

Lemma 1.4.4. *Let G be profinite, and let M and M_i be as in Proposition 1.4.3. Define $\operatorname{Ind}^G M$ to be the set of continuous maps $\xi : G \to M$ equipped with the G-action $(g\xi)(g') = \xi(g^{-1}g')$ for $g, g' \in G$ and the projective limit topology coming from the identification $\operatorname{Ind}^G M \cong \varprojlim_i \operatorname{Ind}^G(M/M_i)$, where $\operatorname{Ind}^G(M/M_i)$ has the discrete topology. Then $H^n(G, \operatorname{Ind}^G M) = 0$ for $n \geq 1$. If $H \subseteq G$ is a closed subgroup, then $H^n(H, \operatorname{Ind}^G M) = 0$ for $n \geq 1$. If H is also normal, $H^n(G/H, (\operatorname{Ind}^G M)^H) = 0$ for $n \geq 1$ as well.*

Proof. By Proposition 1.2.28, we have an exact sequence

$$0 \to \varprojlim_i{}^1 H^{n-1}(G, \operatorname{Ind}^G(M/M_i)) \to H^n(G, \operatorname{Ind}^G M) \to \varprojlim_i H^n(G, \operatorname{Ind}^G(M/M_i)) \to 0.$$

The usual vanishing of cohomology of induced modules implies that $H^n(G, \operatorname{Ind}^G M) = 0$ for $n \geq 2$. If $n = 1$, the same follows from $H^0(G, \operatorname{Ind}^G(M/M_i)) = M/M_i$.

Consider the exact sequence

$$0 \to \varprojlim_i{}^1 H^{n-1}(H, \mathrm{Ind}^G(M/M_i)) \to H^n(H, \mathrm{Ind}^G M) \to \varprojlim_i H^n(H, \mathrm{Ind}^G(M/M_i)) \to 0.$$

By [51, Proposition 1.3.6.(ii)], $\mathrm{Ind}^G(M/M_i)$ is also an induced H-module, so we obtain vanishing of $H^n(H, \mathrm{Ind}^G M)$ for $n \geq 2$. (The definition of induced module in [51] has the action $(g\xi')(g') = g\xi'(g^{-1}g')$ instead, but we can construct a G-equivariant isomorphism between these two definitions by mapping ξ' with $(g\xi')(g') = g\xi'(g^{-1}g')$ to the function $\xi(g) = g^{-1}\xi'(g)$.) If $n = 1$, we need to show that the transition maps $H^0(H, \mathrm{Ind}^G(M/M_{i+1})) \to H^0(H, \mathrm{Ind}^G(M/M_i))$ are surjective. Note that $(\mathrm{Ind}^G(M/M_{i+1}))^H$ just consists of the functions that factor through $H\backslash G$, so since any map $H\backslash G \to M/M_i$ can be lifted (by discreteness) to a map $H\backslash G \to M/M_{i+1}$, the surjectivity is clear.

If H is normal, we may identify $(\mathrm{Ind}^G M)^H$ with $\mathrm{Ind}^{G/H} M$, so $H^n(G/H, (\mathrm{Ind}^G M)^H) = 0$ for $n \geq 1$.

□

Proof of Proposition 1.4.3. For $n = 1$, this is well known, so assume $n \geq 2$.

There is a closed G-equivariant embedding $M_i \to \mathrm{Ind}^G M_i$ taking m to $g \mapsto g^{-1}m$, and similarly for M and M/M_i. (To see that the embedding is closed, recall from the proof of Lemma 1.4.4 that if we define the induced module using $(g\xi')(g') = g\xi'(g^{-1}g')$, the resulting module is isomorphic to the original. The G-equivariant embedding into the new module takes M, M_i, or M/M_i into the constant functions, which are easily seen to form a closed subspace.) Write $M_i' = (\mathrm{Ind}^G M_i)/M_i$ and $M' = (\mathrm{Ind}^G M)/M$ with the topologies induced from $\mathrm{Ind}^G M_i$ and $\mathrm{Ind}^G M$. Equip M'/M_i' with the discrete topology. The commutative

diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_i & \longrightarrow & \mathrm{Ind}^G(M_i) & \longrightarrow & M_i' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & \mathrm{Ind}^G(M) & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M/M_i & \longrightarrow & \mathrm{Ind}^G(M/M_i) & \longrightarrow & M'/M_i' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

has first and third column exact by definition. The horizontal maps are continuous, as are the vertical maps in the left two columns. The map $M_i' \to M'$ is continuous and open, since the composite map $\mathrm{Ind}^G(M_i) \to \mathrm{Ind}^G(M) \to M'$ is continuous and open and M_i' has the topology induced from the surjection $\mathrm{Ind}^G(M_i)$. It follows that every map is continuous. To see that the second column is also exact, note that since M/M_i is discrete, there is a continuous section to $M \to M/M_i$, so any continuous map $G \to M/M_i$ can be lifted to one into M. The first two rows are exact by definition of M_i' and M', so the third row is exact by the 9-lemma. If we take H-invariants of this diagram, left-exactness shows that M^H/M_i^H, $(\mathrm{Ind}^G(M))^H/(\mathrm{Ind}^G(M_i))^H$, and M'^H/M_i^H are discrete. The system M/M_i has the Mittag-Leffler property, so if we look at the morphism from the middle row of the diagram to the limit over i of the third row, we find that $M' \cong \varprojlim_i M'/M_i'$.

We have an exact sequence of G-modules

$$0 \to M \to \mathrm{Ind}^G M \to M' \to 0. \tag{1.61}$$

We construct a continuous section to $\mathrm{Ind}^G M \to M'$ by choosing a compatible family of sections $M'/M_i' \to \mathrm{Ind}^G(M/M_i)$. By the assumption $H^1(H, M) = 0$, we also have an exact sequence

$$0 \to M^H \to (\mathrm{Ind}^G M)^H \to M'^H \to 0. \tag{1.62}$$

We claim that there is again a continuous topological section to the surjection. For this, observe that by the above, $(\mathrm{Ind}^G M_i)^H$ (resp. $M_i'^H$) is open in $(\mathrm{Ind}^G M)^H$ (resp. M'^H), and $\cap_i (\mathrm{Ind}^G M_i)^H = 0$ (resp. $\cap_i M_i'^H = 0$). Since $(\mathrm{Ind}^G M)^H$ (resp. M'^H) is closed in $\mathrm{Ind}^G M$ (resp. M'), we have $(\mathrm{Ind}^G M)^H \cong \varprojlim_i (\mathrm{Ind}^G M)^H/(\mathrm{Ind}^G M_i)^H$ (resp. $M'^H \cong \varprojlim_i M'^H/M_i'^H$). We pick compatible sections $M'^H/M_i'^H \to (\mathrm{Ind}^G M)^H/(\mathrm{Ind}^G M_i)^H$ to construct a continuous section.

From vanishing of cohomology in Lemma 1.4.4 and the exact sequences (1.61) and (1.62), we obtain isomorphisms

$$H^n(G, M') \xrightarrow{\sim} H^{n+1}(G, M), H^n(H, M') \xrightarrow{\sim} H^{n+1}(H, M),$$
$$\text{and } H^n(G/H, M'^H) \xrightarrow{\sim} H^{n+1}(G/H, M^H)$$

for $n \geq 1$.

To obtain (1.60), we induct on n; the base case is the usual inflation-restriction sequence for 1-cocycles. If we assume the result for n, the upper row of

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^n(G/H, M'^H) & \longrightarrow & H^n(G, M') & \longrightarrow & H^n(H, M') \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ 0 & \longrightarrow & H^{n+1}(G/H, M^H) & \longrightarrow & H^{n+1}(G, M) & \longrightarrow & H^{n+1}(H, M) \end{array}$$

is exact by the inductive hypothesis using the isomorphisms $H^k(H, M') \xrightarrow{\sim} H^{k+1}(H, M) = 0$ for $k = 1, \ldots, n-1$. The downward maps are isomorphisms, so the lower row is exact as well.

□

1.4.3 Vanishing of cohomology for bounded and unbounded periods

We may now prove a vanishing theorem for continuous cohomology of Hodge-Tate and bounded de Rham periods.

Theorem 1.4.5. *Suppose that E and K are finite extensions of $\mathbf{Q}_p$ such that E contains the normal closure of K, $\mathfrak{R}$ is a Noetherian E-Banach algebra, and the finitely generated $\mathfrak{R}$-Banach module $\mathfrak{M}$ is equipped with a continuous $\mathfrak{R}$-linear action of G_K. Fix $\sigma \in \Sigma$ as before. Then for any $n \geq 2$, we have $H^n(G_K, t^k B_{\mathrm{dR}}^+/t^\ell B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M}) = 0$ for $k \leq \ell \in \mathbf{Z}$, including the special case $H^n(G_K, \mathbf{C}_p(k) \hat{\otimes}_{K,\sigma} \mathfrak{M}) = 0$ for $k \in \mathbf{Z}$.*

Proof. Note that $\mathbf{C}_p(j) \hat{\otimes}_{K,\sigma} \mathfrak{M}$ is an H_K-semilinear $\mathbf{C}_p$-Banach space. We write $N = (\mathbf{C}_p(j) \hat{\otimes}_{K,\sigma} \mathfrak{M})^{H_K}$ for any fixed $j \in \mathbf{Z}$. Let N_i be the open ball in N of radius p^{-i} centered at 0. Then $N = \varprojlim_i N/N_i$ and each N/N_i is a discrete p-torsion module.

By Theorem 1.4.1, $H^n(H_K, \mathbf{C}_p(j) \hat{\otimes}_{K,\sigma} \mathfrak{M}) = 0$ for $n \geq 1$. It follows from Proposition 1.4.3 that $H^n(G_K, \mathbf{C}_p(j) \hat{\otimes}_{K,\sigma} \mathfrak{M}) \cong H^n(\Gamma_K, N)$. By Proposition 1.4.2, $H^n(\Gamma_K, N) = H^n(G_K, \mathbf{C}_p(j) \hat{\otimes}_{K,\sigma} \mathfrak{M}) = 0$ for $n \geq 2$. This handles the $\mathbf{C}_p(j)$ case, which is also the $k - \ell = 1$ case for bounded de Rham periods. For larger $k - \ell$, we obtain the result using induction via the long exact sequence in cohomology associated to

$$0 \to \mathbf{C}_p(k-1) \hat{\otimes}_{K,\sigma} \mathfrak{M} \to t^\ell B_{\mathrm{dR}}^+/t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M} \to t^\ell B_{\mathrm{dR}}^+/t^{k-1} B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M} \to 0.$$

As noted in Remark 1.2.21, a continuous section to the surjection exists.

□

We pass to the limit using Proposition 1.2.28 and Lemma 1.2.34.

Theorem 1.4.6. *Maintain the hypotheses of Theorem 1.4.5. We have*

$$H^n(G_K, t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M}) = 0 \text{ and } H^n(G_K, B_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} \mathfrak{M}) = 0$$

for $n \geq 2$ and $k \in \mathbf{Z}$.

Proof. Note that $t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M} \cong \varprojlim_\ell (t^k B_{\mathrm{dR}}^+ / t^\ell B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M})$ by (1.47). By Proposition 1.2.28 and the continuous sections from Remark 1.2.21, we have short exact sequences

$$0 \to \varprojlim_\ell{}^1 H^{n-1}(G_K, t^k B_{\mathrm{dR}}^+ / t^\ell B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M}) \to H^n(G_K, t^k B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M})$$
$$\to \varprojlim_\ell H^n(G_K, t^k B_{\mathrm{dR}}^+ / t^\ell B_{\mathrm{dR}}^+ \hat{\otimes}_{K,\sigma} \mathfrak{M}) \to 0.$$

For $n \geq 3$, the vanishing of the middle term follows from Theorem 1.4.5. For $n = 2$, we use the exact sequence (1.19) to see that the transition maps of the left-hand term are surjective. The result for B_{dR} follows from Lemma 1.2.34.

□

1.5 Conditional base change for invariants

In this section, we describe an alternative approach to the results of the preceding sections by replacing the base change result Proposition 1.2.26 with a weaker statement for invariants. For convenience, we will restrict ourself to the Hodge-Tate setting. The main motivation is that the results on vanishing of higher H_K-cohomology are unavailable in other settings (such as the rings of semi-stable or crystalline periods, whose topologies are quite subtle), and so if one were to ask similar questions for these kinds of periods, it would be helpful to have an approach to base change that only uses group invariants (and thus involves less topology). Section 1.5.1 contains purely commutative algebraic results that we hope find

other applications.

1.5.1 Base change and invariants

If $\psi : M \to N$ is a morphism of R-modules and P is an R-module, the map $\phi : (\ker \psi) \otimes_R P \to \ker(\psi \otimes_R P)$ may be neither surjective nor injective. Moreover, if R, M, N, and P, are finite dimensional over a field k, $\dim_k(\ker \psi) \otimes_R P$ can be strictly greater than $\dim_k \ker(\psi \otimes_R P)$. We illustrate these behaviors with the following examples.

Example 1.5.1. Let k be a field, let $M = N = R = k[\delta, \epsilon]/(\delta^2, \delta\epsilon, \epsilon^2)$, let ψ be multiplication by ϵ, and let $P = k$ be the residue field. Then $(\ker \psi) \otimes_R P \to \ker(\psi \otimes_R P)$ is the 0 map from $k\epsilon \oplus k\delta$ to k.

Observe that ϵ is a zero divisor on N. The map $(\ker \psi) \otimes_R P \to \ker(\psi \otimes_R P)$ can be a strict inclusion, as can be seen in the following example.

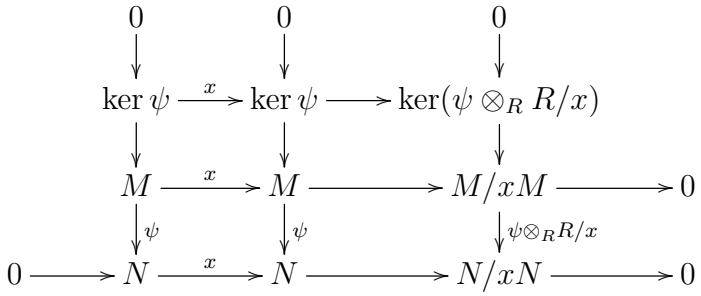
Example 1.5.2. If $M = N = R = \mathbf{Z}_p$, ψ is multiplication by p, and $P = \mathbf{Z}/p\mathbf{Z}$, then $(\ker \psi) \otimes_R P \to \ker(\psi \otimes_R P)$ is the map $0 \to \mathbf{Z}/p\mathbf{Z}$.

It turns out that the obstruction to injectivity of ϕ is related to the observation about zero-divisors above. In particular, we have the following lemma.

Lemma 1.5.3. *Suppose that R is a commutative ring, $\psi : M \to N$ is a morphism of R-modules, $x_1, \ldots, x_n$ is an N-sequence in R, and $P = P'/(x_1, \ldots, x_n)P'$, where P' is a flat R-module. Then the natural map $(\ker \psi) \otimes_R P \to \ker(\psi \otimes_R P)$ is injective.*

Proof. Since kernels commute with flat extension, it suffices to check that if x is a non-zero divisor on N, then $(\ker \psi) \otimes_R R/x \to \ker(\psi \otimes_R R/x)$ is injective.

The diagram



has exact columns and the lower two rows are exact. To see that the top row is also exact, observe that if $m \in \ker \psi \subseteq M$ maps to 0 in $\ker(\psi \otimes_R R/x)$, there must be $m' \in M$ with $xm' = m$. Then $0 = \psi(xm') = x\psi(m')$, so $\psi(m') = 0$ by our hypothesis on x. Thus $m' \in \ker(\psi)$ as needed.

□

Definition 1.5.4. We say that P as in the statement of Lemma 1.5.3 is an *N-admissible* module.

We now give an application to group invariants. We first note using an example that if M has an action by a group G, we cannot expect $M^G \otimes_R P \to (M \otimes_R P)^G$ to be surjective in general.

Example 1.5.5. Let $M = N = R^2$, let $\psi : R \to R$ be given, and let the action of $G = \mathbf{Z}$ be defined by letting $n \in \mathbf{Z}$ map (r_1, r_2) to $(r_1, n\psi(r_1) + r_2)$. Now let M, N, and ψ be as in Example 1.5.1 or 1.5.2. The G-invariants in this case are given by the kernel of $1 - \mathbf{1}_M$, or $\ker \psi \oplus R$. Now the lack of injectivity in those examples applies again here.

Under the admissibility condition, however, we obtain a positive result.

Proposition 1.5.6. *Suppose that M has an R-linear action of a group G. If P is M-admissible, then the natural map $M^G \otimes_R P \to (M \otimes_R P)^G$ is injective.*

Proof. Observe that if $\psi : M \to \prod_{g\in G} M$ sends m to $((g-1)m)_{g\in G}$, then $\ker\psi = M^G$. Note also that a regular sequence for M is also a regular sequence for $\prod_G M$, so P is $\prod_G M$-admissible. Write $P = P'/(x_1,\ldots,x_n)P' = P' \otimes_R R/(x_1,\ldots,x_n)$ with P' flat and $x_1,\ldots,x_n$ an M-sequence. Then $M^G \otimes_R R/(x_1,\ldots,x_n) \to M \otimes_R R/(x_1,\ldots,x_n)$ is injective by Lemma 1.5.3. By flatness of P' we deduce that $M^G \otimes_R P \to M \otimes_R P$ is injective.

□

We give an application to arithmetic invariant theory.

Corollary 1.5.7. *Let V be a free R-module equipped with an R-linear action of a group G. Then G acts on the polynomial ring $S = \mathrm{Sym}(V^\vee)$. If the ideal $I \subseteq R$ satisfies $\operatorname{depth} I = \operatorname{codim} I$, then $S^G \otimes_R R/I \to (S \otimes_R R/I)^G$ is injective. In particular, if $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, and $Z = \operatorname{Spec} R/I$, then the natural map $(Y \times_X Z)/G \to (Y/G) \times_X Z$ is dominant.*

Proof. We apply Proposition 1.5.6 to obtain the injectivity of $S^G \otimes_R R/I \to (S \otimes_R R/I)^G$.

□

We mention two more examples to indicate limitations on control over the dimension of the kernel of a morphism before and after base change.

Example 1.5.8. Let k be a field, let $M = R = k[\epsilon]/(\epsilon^2)$, let $N = k$, let ψ be the quotient by (ϵ), and let $P = k$ be the residue field. Then $(\ker\psi) \otimes_R P \to \ker(\psi \otimes_R P)$ is the 0 map from the ideal $(\epsilon) \subseteq k[\epsilon]/\epsilon^2$ to 0.

Example 1.5.9. Let $M = R = \mathbf{Z}_p$, let $N = \mathbf{Z}/p\mathbf{Z}$, let ψ be the quotient by (p), let $P = \mathbf{Z}/p\mathbf{Z}$, and let $L = \mathbf{Q}_p$. Then $(\ker\psi) \otimes_R P \to \ker(\psi \otimes_R P)$ is the 0 map from the ideal $(p)/(p^2)$ to 0. In particular, $\dim_\kappa \ker(\psi \otimes_R \kappa) < \dim_L \ker(\psi \otimes_R L)$.

Recall from Example 1.5.2 that at best we can hope that

$$\dim_\kappa \ker(\psi \otimes_R \kappa) \geq \dim_L \ker(\psi \otimes_R L)$$

even if M and N are finite flat and R is a Noetherian domain. Examples 1.5.8 and 1.5.9 illustrate that this inequality can fail if N is not flat. In a positive direction, if R is a Noetherian domain, N is finite flat, and M is finite, we have the following.

Proposition 1.5.10. *Let R be a Noetherian domain with fraction field L and let $\psi : M \to N$ be a map of finitely generated R-modules with N flat. Moreover, let $\mathfrak{p} \subseteq R$ be a prime ideal with residue field κ. Then* $\dim_\kappa \ker(\psi \otimes_R \kappa) \geq \dim_L \ker(\psi \otimes_R L)$.

Proof. We have an exact sequence

$$0 \to \ker(\psi \otimes_R L) \to M \otimes_R L \to N \otimes_R L \to \operatorname{coker}(\psi \otimes_R L) \to 0.$$

It follows that $\dim_L \ker(\psi \otimes_R L) = \dim_L(M \otimes_R L) - \dim_L(N \otimes_R L) + \dim_L \operatorname{coker}(\psi \otimes_R L)$. By flatness of localization, the sequence

$$0 \to (\ker \psi)_\mathfrak{p} \to M_\mathfrak{p} \to N_\mathfrak{p} \to (\operatorname{coker} \psi)_\mathfrak{p} \to 0$$

is exact, and by Nakayama's lemma, the $R_\mathfrak{p}$-module $(\operatorname{coker} \psi)_\mathfrak{p}$ (resp. $M_\mathfrak{p}$) can be generated by $d = \dim_\kappa(\operatorname{coker} \psi) \otimes_R \kappa$ elements (resp. $d' = \dim_\kappa M \otimes_R \kappa$ elements). It follows that $d \geq \dim_L \operatorname{coker}(\psi \otimes_R L)$ and $d' \geq \dim_L M \otimes_R L$. Since tensor products commute with cokernels, $d = \dim_\kappa \operatorname{coker}(\psi \otimes_R \kappa)$. Using the exact sequence

$$0 \to \ker(\psi \otimes_R \kappa) \to M \otimes_R \kappa \to N \otimes_R \kappa \to \operatorname{coker}(\psi \otimes_R \kappa) \to 0$$

and the flatness of N, we have

$$\begin{aligned}\dim_\kappa \ker(\psi \otimes_R \kappa) &= \dim_\kappa(M \otimes_R \kappa) - \dim_\kappa(N \otimes_R \kappa) + \dim_\kappa \operatorname{coker}(\psi \otimes_R \kappa) \\ &\geq \dim_L(M \otimes_R L) - \dim_L(N \otimes_R L) + \dim_L \operatorname{coker}(\psi \otimes_R L) \\ &= \dim_L \ker(\psi \otimes_R L).\end{aligned}$$

□

1.5.2 Application to Hodge-Tate periods

In this section we approach the lower semicontinuity aspect of the Hodge-Tate case of Theorem 2 via invariants as opposed to 1-cocycles. The reason for mentioning this alternative approach is that although we are able to approach the theory of bounded de Rham periods via 1-cocycles, it is very difficult to imagine how our arguments could apply to semi-stable or crystalline cases due to the subtle topologies on those period rings.

The techniques in this section show that if one is willing to desingularize the base of the family (via a result of Temkin, Theorem 1.5.16 below), one can avoid any discussion of the topology beyond a decompletion (which we proved in Lemma 1.2.24) of the underlying module. We stick to the Hodge-Tate case and affine families for simplicity.

We maintain the notation $\mathfrak{R}, \mathfrak{M}, G_K, \sigma$, and E from Section 1.2.2; assume $\mathfrak{R}$ is integral. Then we have the following extension of Theorem 1.2.13.

Theorem 1.5.11. *Suppose that $\xi : \mathfrak{R} \to E'$ is a specialization to a finite field extension of E. Let $L = \operatorname{Frac} \mathfrak{R}$. Then*

$$\dim_{E'} H^0(G_K, \mathbf{C}_p \hat{\otimes}_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} E')) \geq \dim_L H^0(G_K, \mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M}) \otimes_{\mathfrak{R}} L. \tag{1.63}$$

The argument of Theorem 1.2.13 produces an open subset where (1.63) is an equality.

(Replace $Q_\sigma(0)$ in the argument there by any $f \neq 0$ such that $\phi : \mathfrak{E} \to \mathfrak{E}$ has flat cokernel using generic flatness.) This immediately implies the semicontinuity of the dimension of Hodge-Tate periods by dévissage.

We will first give a proof under an additional hypothesis.

Proposition 1.5.12. *Suppose that $\mathfrak{R}$ is Cohen-Macaulay. Then Theorem 1.5.11 is true.*

Proof. Let $\mathfrak{m} = \ker \xi$. By Lemma 1.2.24 and Proposition 1.2.6, we have an isomorphism $(\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \otimes_{\mathfrak{R}} E' \xrightarrow{\sim} (\mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} E'))^{H_K}$. We deduce an isomorphism between the kernel of a generator $\gamma \in \Gamma_K$ (which we will denote by $\ker_\gamma$) acting on $(\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \otimes_{\mathfrak{R}} E'$ and $H^0(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} E'))$. The ideal $\mathfrak{m}$ has $\operatorname{depth} \mathfrak{m} = \operatorname{codim} \mathfrak{m}$ by the condition on $\mathfrak{R}$, so by Lemma 1.5.3 we have an injection

$$\begin{aligned} H^0(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} E' &= \ker_\gamma((\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K}) \otimes_{\mathfrak{R}} E' \\ &\hookrightarrow \ker_\gamma((\mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M})^{H_K} \otimes_{\mathfrak{R}} E') \xrightarrow{\sim} H^0(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} E')). \end{aligned}$$

But $\dim_{E'} H^0(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} E' \geq \dim_L H^0(G_K, \mathbf{C}_p\hat{\otimes}_{K,\sigma}\mathfrak{M}) \otimes_{\mathfrak{R}} L$ by the argument of Theorem 1.2.27.

□

For the general case, we will use some facts concerning blowups. The universal property for blowups is the same as in the schemes case; in particular, $\pi : \mathfrak{X}' \to \mathfrak{X}$ is the blowup along the coherent ideal sheaf $\mathfrak{I}$ if it is universal for the property that $\pi^{-1}(\mathfrak{I})\mathfrak{O}_{\mathfrak{X}'}$ is an invertible $\mathfrak{O}_{\mathfrak{X}'}$-sheaf.

Proposition 1.5.13 ([21, Definition 4.1.1])**.** *Let $\mathfrak{I}$ be a coherent sheaf of ideals on a rigid analytic space $\mathfrak{X}$. Then there exist a rigid analytic space $\mathfrak{X}'$ and a proper map $\pi : \mathfrak{X}' \to \mathfrak{X}$, such that $\mathfrak{X}'$ and π satisfy the universal property for the blowup of $\mathfrak{X}$ along $\mathfrak{I}$.*

As explained by Schoutens [65], one can deduce from this the following properties.

Proposition 1.5.14 ([65])**.** *Let $\pi : \mathfrak{X}' \to \mathfrak{X}$ be the blowing up of an irreducible rigid space $\mathfrak{X}$ along the ideal sheaf $\mathfrak{I}$ of a nowhere dense closed analytic subvariety $\mathfrak{Z}$. Then π is proper and surjective, and $\mathfrak{X}'$ is irreducible. Moreover, $\pi^{-1}(\mathfrak{Z})$ is nowhere dense in $\mathfrak{X}'$ and $\pi|_{\mathfrak{X}'\setminus\pi^{-1}(\mathfrak{Z})}$ is an isomorphism.*

Proof. Although Schoutens works over an algebraically closed field, the results [65, Proposition 1.4.4 and Corollary 1.4.5] use only the universal property of the blowup, and [65, Corollaries 3.2.2 and 3.2.4] use only the properness and universal property. While Schoutens is using a weaker notion of nowhere density than Conrad, [20, Lemma 2.2.3] shows that the stronger definition holds.

□

The following is an immediate corollary.

Corollary 1.5.15. *Suppose $\mathfrak{X}_n \to \cdots \to \mathfrak{X}_1$ is a sequence of blowups, each along a nowhere dense closed analytic subvariety, and denote by $\pi : \mathfrak{X}_n \to \mathfrak{X}_1$ the composite. Then there exists a nowhere dense closed analytic subvariety $\mathfrak{Z} \subseteq \mathfrak{X}_1$ such that $\pi|_{\mathfrak{X}_n\setminus\pi^{-1}(\mathfrak{Z})}$ is an isomorphism and $\pi^{-1}(\mathfrak{Z})$ is nowhere dense.*

The following resolution of singularities result is a very special case of what Temkin [80] proves, and not all of the good properties of this construction are listed here, but we will only need this statement.

Theorem 1.5.16 ([80, Theorem 5.2.2])**.** *Suppose that $\mathfrak{R}$ is a reduced affinoid algebra over E, and let $\mathfrak{X} = \operatorname{Spm}\mathfrak{R}$. Then there exist a regular rigid analytic space $\mathfrak{X}'$ over E and a proper birational morphism $\pi : \mathfrak{X}' \to \mathfrak{X}$. Moreover, π is the composition of at most finitely many blowups with nowhere dense center.*

We may now return to the proof of Theorem 1.5.11.

Proof of Theorem 1.5.11. Write $\mathfrak{X} = \operatorname{Spm} \mathfrak{R}$ and let $\pi : \mathfrak{X}' \to \mathfrak{X}$ be the map given by Proposition 1.5.16, so that $\mathfrak{X}'$ is regular. Then $\pi^* \mathfrak{M}$ is finite flat of the same rank on $\mathfrak{X}'$. Pick an integral affinoid neighborhood of a point y over $\mathfrak{m} = \ker \xi$ and let $\mathfrak{R}'$ be the resulting $\mathfrak{R}$-algebra. Note that the pullback of $\pi^* \mathfrak{M}$ to y is just an extension of coefficients of $\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}/\mathfrak{m}$, so it suffices to prove the inequality (1.63) for y instead. By Proposition 1.5.12 we are reduced to checking that

$$\dim_L H^0(G_K, \mathbf{C}_p \hat{\otimes}_{K,\sigma} \mathfrak{M}) \otimes_{\mathfrak{R}} L = \dim_{L'} H^0(G_K, \mathbf{C}_p \hat{\otimes}_{K,\sigma} (\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}')) \otimes_{\mathfrak{R}} L', \tag{1.64}$$

where $L' = \operatorname{Frac} \mathfrak{R}'$.

By Corollary 1.5.15, the map $\operatorname{Spm} \mathfrak{R}' \to \mathfrak{X}$ is an isomorphism on the complement of a closed analytic subspace, so by picking an affinoid subdomain in the complement it suffices to check that the equality (1.64) holds instead for a map $\mathfrak{R} \to \mathfrak{R}'$ where $\mathfrak{R}'$ is the coordinate ring of an affinoid subdomain of $\operatorname{Spm} \mathfrak{R}$. But in this case, $\mathfrak{R} \to \mathfrak{R}'$ is flat, so the formation of $\ker \phi : \mathfrak{E} \to \mathfrak{E}$ commutes with base change along $\mathfrak{R} \to \mathfrak{R}'$ and the required equality follows.

□

Chapter 2

A p-adic approach to local-global compatibility

2.1 Introduction

Let F/F^+ be the compositum of a totally real field F^+ with an imaginary quadratic field E and let $G = \mathrm{GU}(J)_{/\mathbf{Q}}$ denote the unitary similitude group acting on the Hermitian space (F^n, J) with similitudes in $\mathbf{Q}$, where J denotes the Hermitian pairing. If $J = J_{a,b} = \left(\begin{smallmatrix} \mathbf{1}_a & 0 \\ 0 & -\mathbf{1}_b \end{smallmatrix}\right)$, where a, b are nonzero positive integers, and $F^+ = \mathbf{Q}$, then Skinner [73] and Morel [49] associate to a regular cuspidal automorphic representation π on G a weak base change τ to $\mathrm{Res}^E_{\mathbf{Q}} \mathbf{G}_m \times \mathrm{Res}^F_{\mathbf{Q}} \mathrm{GL}_n$ and a Galois representation $\rho_\pi : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ with compatibility at any ℓ such that π_ℓ is unramified and ℓ does not divide the discriminant d_E. For general J and F^+, Shin [70] gives a similar construction that also generalizes work of Labesse [39] for the case $[F^+ : \mathbf{Q}] > 1$. Moreover, his weak base change τ has full compatibility with π at every place of $\mathbf{Q}$ that splits in E.

We improve on these results by proving cases of local-global compatibility for π with τ and ρ_π at primes $p|d_E$ with $p > 2$. We require that p is unramified in F^+. We study the

situation where $G(\mathbf{Q}_p)$ is quasi-split and π_p is spherical with respect to a certain special maximal compact subgroup $K \subseteq G(\mathbf{Q}_p)$, which is defined as follows. (See Section 2.4 for more details.) By our hypothesis that G is quasi-split at p, the group $G_{/\mathbf{Q}_p}$ is isomorphic to the unitary similitude group over $\mathbf{Q}_p$ stabilizing the form

$$J = \begin{pmatrix} & & & & & & -1 \\ & & & & & 1 & \\ & & & & \iddots & & \\ & & & -1 & & & \\ & & 1 & & & & \\ & -1 & & & & & \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} & & & & & & \varpi \\ & & & & & \iddots & \\ & & & & \varpi & & \\ & & & -\varpi & & & \\ & & \iddots & & & & \\ & -\varpi & & & & & \end{pmatrix}$$

in the case where n is odd or even, respectively. Here ϖ is a uniformizer at the place of E over p such that $\varpi^2 \in \mathbf{Q}_p$. Then K is the group of integral matrices in the group defined with respect to the form J.

The first step, carried out in Sections 2.2 and 2.3, is to establish compatibility up to monodromy at p. This uses the theory of γ-factors, and follows the strategy of the aforementioned work of Skinner [73]. The key ingredients are the classical theory of γ-factors on GL_n via the doubling method of Piatetski-Shapiro and Rallis [56, 24], Lapid and Rallis's adaptation of the doubling method [40] to construct γ-factors for unitary groups and prove their fundamental properties, and the stability of γ-factors on unitary groups and on GL_n, proved by Brenner [10] and Jacquet-Shalika [32], respectively.

To improve this to full compatibility, we introduce a new tool: variation in a p-adic family. Loosely speaking, the argument is three steps.

1. By assumption, π_p has trivial monodromy, so we have compatibility if τ has trivial monodromy. If instead, τ has non-trivial monodromy, the Satake parameters of π are

of a particular form. We use this to construct in Sections 2.4 and 2.5 a pathological p-stabilized overconvergent automorphic representation σ attached to π.

2. We show in Section 2.6 that if σ moves in a family of full dimension over weight space (or, more precisely, in a family that moves in a particular direction in weight space), we can construct a crystalline period in ρ_π that violates its known purity.

3. In Section 2.7, we explain how to construct a suitable family using Urban's eigenvarieties [82] under a non-criticality hypothesis on σ. Then in Section 2.8, we substantially weaken this hypothesis using a closer examination of the automorphic multiplicities appearing in Urban's work. We first show that whenever there exists $w \in W$, where W is the Weyl group of G, such that a certain overconvergent cuspidal automorphic multiplicity $m^\dagger_{G,0}(\sigma^{w,\lambda}, w * \lambda)$ is nonzero, the argument of Section 2.7 applies. We then check that such a nonvanishing multiplicity exists using a combination of the regularity of π_∞, the temperedness of π at unramified places, and the existence of an inductive formula for the classical multiplicity in terms of twisted overconvergent Eisenstein multiplicities. We also give a simpler argument for the case $[F^+ : \mathbf{Q}] \geq 2$ using a trivial bound on the defect of the Leopoldt conjecture.

A precise statement of the main theorem is given in Theorem 2.2.5 below. As a consequence of this result, one can produce the first examples of strong base changes τ of cuspidal automorphic representations π on G. (Since $E/\mathbf{Q}$ always has a place of ramification, there is no way to produce a strong base change by even applying, for instance, Shin's result [70].) To obtain such examples, one selects E so that it is split at places of ramification of F and selects the form J so that $G(\mathbf{Q}_p)$ is quasi-split at all p ramified in E. One then considers cuspidal automorphic representations π with level K at each ramified place of E, hyperspecial level at each inert place of E, and arbitrary level at split places. Then a strong base of τ exists.

In practice, one often is able to choose E and the form J of the unitary group, so it is not too difficult to satisfy the hypotheses above. One key application is to constructing elements of Selmer groups using the strategy of Skinner-Urban [76, 75]. Since these elements are constructed as extensions of Galois representations and must satisfy local conditions at every place, it is essential to have a strong base change. In fact, our result is applied in their recent work [83]. Another application is to the generalized Ramanujan conjecture for the group G. This conjecture states that any globally generic cuspidal automorphic representation should be everywhere tempered (up to the central character); we prove temperedness at p under the hypotheses above.

Our arguments apply without modification to cuspidal representations π on G that are possibly irregular discrete series at infinity if they satisfy two additional hypotheses. Namely, we require that the classical Euler-Poincaré characteristic of π is nonvanishing and that the weak base change τ of π constructed by Shin [70] is tempered at finite places. (See Remark 2.2.6.) We hope to relax the latter hypothesis in future work. Using the lifting from unitary groups to similitude groups[1], compatibility for unitary groups follows in the limited setting $F = EF^+$ considered above; see Corollary 2.2.7.

The strategy to use interpolation results for crystalline periods in order to prove local-global compatibility for Galois representations is due to Skinner [74], and our method can be seen as a higher rank generalization of his strategy. Jorza [33] and Luu [46] have also used strategies based on Skinner's for GSp_4 and GL_n. Three differences in our work are as follows.

- Instead of constructing a crystalline period in order to identify a Satake parameter or show a representation is crystalline, we use the interpolation in a negative way – we assume a representation fails local-global compatibility and use that failure to construct a crystalline period that we know cannot exist.

[1] I learned about this lifting from Stefan Patrikis.

- Since we are working on a ramified group, a substantial part of the argument deals with the local theory required to produce a suitable family.

- We are considering completely general unitary groups (rather than ones that are anisotropic over $\mathbf{Q}$), which makes it more challenging to produce a suitable family due to the presence of Eisenstein cohomology classes.

In the remainder of this introduction, we sketch some of the ideas that go into the proof. For compatibility up to monodromy, first write π_0 for an irreducible subrepresentation of the restriction of π to the unitary group $G_0 = U(J)_{/\mathbf{Q}}$ and write τ_0 for the restriction of τ to $H_0 = \mathrm{Res}^F_{\mathbf{Q}} \mathrm{GL}_n$. Then τ_0 is tempered at p [71]. Skinner shows that the γ-factors of $\pi_{0,p}$ and $\tau_{0,p}$ are equal. We check that if $\pi_{p,0}$ is a subquotient of the unramified principal series, the Satake parameters are of a particular form, and then $\tau_{0,p}$ is the unique tempered representation that agrees with $\pi_{p,0}$ up to monodromy; this uses a numerical invariant attached to the γ-factors and the Bernstein-Zelevinsky classification.

If $\tau_{0,p}$ has non-trivial monodromy, then for step (1) above we need the action of a certain Hecke operator U on the p-stabilized overconvergent automorphic representation σ to have an eigenvalue that is too large for a tempered representation. We require a very precise understanding of $\pi_{0,p}$ to produce such a σ. Since $G_0(\mathbf{Q}_p)$ is ramified, its structure theory is somewhat sophisticated. However, it turns out that the Iwahori-Hecke algebra of $G_0(\mathbf{Q}_p)$ can be identified with that of a split group $G'_0(\mathbf{Q}_p)$.[2] This observation is a case of a technique employed by Lusztig [44] to reduce the study of unipotent representations of certain possibly ramified groups to unramified cases, and was used to study Steinberg representations of $G_0(\mathbf{Q}_p)$ by Clozel-Thorne [17, 18]. To study the representation $\pi'_{0,p}$ of $G'_0(\mathbf{Q}_p)$ corresponding to $\pi_{0,p}$, we apply results of Reeder [58] that translates the structure of $\pi'_{0,p}$ into questions about orbits on a certain prehomogeneous vector space.

[2] I learned of this identification from Jack Thorne.

Assuming that σ can be p-adically deformed in a suitable family, in step (2) we construct a crystalline period in ρ_π that violates its known purity. For this, we apply a result of Kisin [36] and Nakamura [50] to analytically continue a period from points of very regular weight to ρ_π. We need to carefully choose a line such that

- it contains a dense set of points with arbitrarily regular algebraic weight,
- the eigenvalue of U varies analytically over the family,
- the eigenvalue of U is equal to a crystalline Frobenius eigenvalue in very regular weight, and
- the attached Galois representations have a fixed Hodge-Tate weight.

Finally, in step (3), to actually produce the required family of Galois representations, we apply the technique of pseudorepresentations due to Wiles [85] and Taylor [78] to the p-adic families constructed by Urban [82]. See Sections 2.7 and 2.8 for the subtleties involved here. Urban's main theorem applies when the multiplicity $m^\dagger_{G,0}(\sigma, \lambda) \neq 0$, which is true if σ has non-critical slope and regular weight. To construct the p-adic families in critical slope cases, we first study overconvergent automorphic multiplicities and look at avatars $\sigma^{w,\lambda}$ of classical representations in non-dominant weight $w * \lambda$ in Urban's eigenvariety, where $w \in W$ is an element of the Weyl group of G and $*$ is a normalized action on the weights. To our knowledge, this is the first time such a technique has been applied. If any of these $\sigma^{w,\lambda}$ can be p-adically interpolated in Urban's eigenvariety, the same argument as in the non-critical case applies. If all of their automorphic multiplicities vanish, we use Urban's work to produce a finite slope form on a proper Levi subgroup of G such that π is a subquotient of its parabolic induction to G. This Levi subgroup must have at least one $\mathrm{GL}_{1/F}$ factor. Although this form may not be classical, we check that the restriction to any of its $\mathrm{GL}_{1/F}$ factors is a Hecke character, and use a comparison of the weight of that character with the weight enforced by the temperedness of π at a split place to find a contradiction.

Our technique is not special to unitary groups, though many of the calculations here are only carried out for such groups. In subsequent work we will give many additional cases of local-global compatibility.

2.2 Weak base change, L-functions, and γ-factors

In this section, we use an approach similar to Skinner [73, §3] in order to compare the local Langlands parameters at a finite place of π and its weak base change τ.

2.2.1 Unitary groups and base change

Let F^+ be a totally real field, let $E \subseteq \mathbf{C}$ be a quadratic imaginary field (regarded as having a fixed embedding into $\mathbf{C}$), and let $F = EF^+$. Let J be a Hermitian form on F^n. We define $G = \mathrm{GU}(J)_{/\mathbf{Q}}$ as follows. For a $\mathbf{Q}$-algebra R,

$$G(R) = \left\{ g \in \mathrm{GL}_n(R \otimes_{\mathbf{Q}} F) | g J^t \overline{g} = \mu(g) J, \mu(g) \in R^\times \right\},$$

where we use $\bar{\cdot}$ for the action of the nontrivial element of $\mathrm{Gal}(E/\mathbf{Q})$. This also defines a homomorphism of $\mathbf{Q}$-groups $\mu : G \to \mathbf{G}_m$. Let G_0 denote the kernel of this homomorphism; it is precisely the restriction of scalars to $\mathbf{Q}$ of the usual unitary group of J over F^+. We define $H = \mathrm{Res}_{E/\mathbf{Q}} G_{/E}$ and $H_0 = \mathrm{Res}_{E/\mathbf{Q}} G_{0/E}$.

If R is an E-algebra, we may identify $R \otimes_{\mathbf{Q}} E$ with $R \times R$ by sending $r \otimes e \mapsto (er, \overline{e}r)$. We use this to identify $G_{/E}$ with $\mathbf{G}_m \times \mathrm{Res}_{F/E} \mathrm{GL}_n$ as follows. For any E-algebra R, we send an element $g = (g_1, g_2) \in G(R) \subseteq \mathrm{GL}_n(R \otimes F) \cong \mathrm{GL}_n(R \otimes F^+) \times \mathrm{GL}_n(R \otimes F^+)$ to the element $(\mu(g), g_1) \in R^\times \times \mathrm{GL}_n(R \otimes_{\mathbf{Q}} F^+)$. Since $J = {}^t\overline{J}$, the conditions in each factor for $g = (g_1, g_2) \in G(R)$ are equivalent to one another under transposition, and the condition on the first factor is $g_1 J^t g_2 = \mu(g) J$, or $g_2 = {}^t(J^{-1} g_1^{-1} \mu(g) J)$, so the (g_1, g_2)

are in bijection with the $(\mu(g), g_1)$. (We have $\mu(g) \in R^\times$ by definition.) Thus we have $H = \mathrm{Res}_{E/\mathbf{Q}} \mathbf{G}_m \times \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$ and $H_0 = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$. Using this identification, we define a map $\theta_H : H(R) \to H(R)$ by

$$\theta_H((x, g)) = (\overline{x}, \overline{x}^t \overline{g}^{-1}). \tag{2.1}$$

We also write θ_{H_0} for the restriction to H_0.

It follows from the preceding discussion that we have an identification $\widehat{G} = \mathbf{C}^\times \times \prod_{\nu : F^+ \to \mathbf{R}} \mathrm{GL}_n(\mathbf{C})$, where $\widehat{G}$ denotes the dual. Define an action of the nontrivial element $c \in \mathrm{Gal}(E/\mathbf{Q})$ to be the unique outer automorphism preserving the standard splitting; its action is given by

$$(x, (g_\nu)_\nu) \mapsto (x \prod_\nu \det g_\nu, (\Phi_n^{-1t} g_\nu^{-1} \Phi_n)_\nu),$$

where the $n \times n$-matrix Φ_n is defined by $(\Phi_n)_{ij} = (-1)^{i+1} \delta_{i,n+1-j}$ and δ_{ij} is the Kronecker δ function. There is also an action of $G_\mathbf{Q}$ by precomposition on the $\nu : F^+ \to \mathbf{R}$, so that it permutes the $\mathrm{GL}_n(\mathbf{C})$ factors. This action commutes with that of c. Then $G_\mathbf{Q}$ acts by the product of the action via its quotient $\mathrm{Gal}(E/\mathbf{Q})$ and its permutation action on the ν. We similarly define an action of $G_\mathbf{Q}$ on $\widehat{G}_0 = \prod_{\nu : F^+ \to \mathbf{R}} \mathrm{GL}_n(\mathbf{C})$ by the same product of actions, but where $c \in \mathrm{Gal}(E/\mathbf{Q})$ now acts by $c((g_\nu)_\nu) = (\Phi_n^{-1t} g_\nu^{-1} \Phi_n)_\nu$. We then define ${}^L G = \widehat{G} \rtimes W_\mathbf{Q}$ and ${}^L G_0 = \widehat{G}_0 \rtimes W_\mathbf{Q}$, where $W_\mathbf{Q}$ acts by its quotient $G_\mathbf{Q}$. There is a natural L-homomorphism ${}^L G \to {}^L G_0$ defined by $(x, g) \rtimes \sigma \mapsto g \rtimes \sigma$.

Since $H = \mathrm{Res}_{E/\mathbf{Q}} G_{/E}$, we have $\widehat{H} = \widehat{G} \times \widehat{G}$ with the action of $c \in \mathrm{Gal}(E/\mathbf{Q})$ given by $c(g, h) = (c(h), c(g))$, and the action of $W_\mathbf{Q}$ again factoring through $G_\mathbf{Q}$ and equal to the product of the $\mathrm{Gal}(E/\mathbf{Q})$ action just given with the permutation action in each factor. A similar definition holds for H_0. This defines ${}^L H$ and ${}^L H_0$, and the diagonal embeddings (using the identity map on $W_\mathbf{Q}$) yield L-homomorphisms $\mathrm{BC} : {}^L G \to {}^L H$ and $\mathrm{BC}_0 : {}^L G_0 \to {}^L H_0$. In the usual way, these global Langlands dual groups give rise to local Langlands dual groups using the embeddings $W_{\mathbf{Q}_p} \to W_\mathbf{Q}$.

Example 2.2.1. We discuss one particular case for $G(J)$ that served as the focus of Morel's work [49]. Let $a+b=n$ with $a \leq b$ and define $J_{a,b}$ by

$$J_{a,b} = \begin{pmatrix} & & A_a \\ & \mathbf{1}_{b-a} & \\ A_a & & \end{pmatrix}, \tag{2.2}$$

where $\mathbf{1}_{\cdot}$ is the identity matrix and A_m is the $m \times m$ matrix defined by

$$A_m = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}. \tag{2.3}$$

Then $G(J_{a,b})$, sometimes called $GU(a,b)$, is an example in the class of unitary groups under consideration. In the following, we write everything with respect to the form $J_{a,b}$. We fix a maximal torus defined by

$$T(R) = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in \mathrm{GL}_n(R \otimes_{\mathbf{Q}} F) \;\middle|\; \begin{array}{r} \lambda_1\overline{\lambda}_n = \cdots = \lambda_a\overline{\lambda}_{b+1} \\ = \lambda_{a+1}\overline{\lambda}_{a+1} = \cdots = \lambda_b\overline{\lambda}_b \end{array} \right\}.$$

A maximal $\mathbf{Q}$-split subtorus of T is

$$S(R) = \left\{ (\lambda \otimes 1)\,\mathrm{diag}(\lambda_1 \otimes 1, \ldots, \lambda_a \otimes 1, \underbrace{1 \otimes 1, \ldots, 1 \otimes 1}_{b-a}, \lambda_a^{-1} \otimes 1, \ldots, \lambda_1^{-1} \otimes 1) \right.$$
$$\left. \in \mathrm{GL}_n(R \otimes_{\mathbf{Q}} F) \right\}$$

if $b > a$ or

$$S(R) = \left\{ \mathrm{diag}(\lambda\lambda_1 \otimes 1, \ldots, \lambda\lambda_a \otimes 1, \lambda_a^{-1} \otimes 1, \ldots, \lambda_1^{-1} \otimes 1) \in \mathrm{GL}_n(R \otimes_{\mathbf{Q}} F) \right\}$$

if $b = a$. We let B_n denote the standard Borel subgroup of GL_n. Then a minimal parabolic subgroup of G is given by

$$P(R) = \left\{ \left(\begin{array}{ccc} M_1 & & * \\ & M_2 & \\ & & M_3 \end{array} \right) \in G(R) \middle| \begin{array}{l} \\ M_1, M_3 \in B_a(R \otimes_{\mathbf{Q}} F), M_2 \in \mathrm{GL}_{b-a}(R \otimes_{\mathbf{Q}} F) \\ \\ \end{array} \right\}.$$

For any $m_1, \ldots, m_k \in \mathbf{Z}_{>0}$ with $m_1 + \cdots + m_k = m \leq a$, we obtain a standard parabolic of G by intersecting G inside $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$ with the standard parabolic of type $(m_1, \ldots, m_k, b + a - 2m, m_k, \ldots, m_1)$.

2.2.2 Weak base changes and Galois representations attached to automorphic representations of G

It is a conjecture of Langlands and Clozel that certain automorphic representations – those that are *algebraic* – should be attached to motives (and thus compatible families of Galois representations). In this work we will be concerned only with cuspidal automorphic representations π on G that are regular discrete series at infinity. Such π satisfy Clozel's hypothesis. In fact, the Galois representation attached to π is the one attached to its weak base change τ to H. Since the Galois representations ρ_τ attached to such a τ have been constructed thanks to the work of many mathematicians, and nearly all the expected properties of ρ_τ are known, the relationship between π and τ is the main focus of this article.

When the unitary group is noncompact at infinity and $F^+ = \mathbf{Q}$, it is only possible to say anything about the group G because of the intricate study of the intersection cohomology of the Shimura variety attached to G by Morel [49], who attaches a very weak base change τ to a π on G. (By *very weak* here, we refer to the indeterminacy of the set of places where π is compatible with τ.) This case is the most important one for applications to the Bloch-Kato

conjecture and Iwasawa theory for elliptic curves over $\mathbf{Q}$.

Using different methodologies, Skinner [73] and Shin [70] describe additional compatibility for τ, yielding a *weak base change*, i.e. a base change that has compatibility at an explicit set of places including all the ones where the data is unramified. The former work requires $F^+ = \mathbf{Q}, ab \neq 0$, and gives a slightly weaker form of compatibility than the latter work at places of $\mathbf{Q}$ that split in E. For this reason, the statement below is based on Shin's result [70]. However, in what follows, we largely follow the notations and conventions of Skinner's paper [73]. Skinner and Shin have opposite conventions for the L-packet attached to the algebraic representation $\mathbf{V}_\lambda$ of weight λ – namely, Skinner asks for nonvanishing of the cohomology of $\pi_\infty \otimes \mathbf{V}_\lambda^\vee$ while Shin uses $\pi_\infty \otimes \mathbf{V}_\lambda$ instead (as does Morel's work [49]). We later rely on Urban's work [82], which uses Skinner's convention as we do.

We write $\mathrm{BC}(\pi_p)$ for the representation of $H(\mathbf{Q}_p)$ with L-parameter $\mathrm{BC} \circ \psi_{\pi_p}$, where ψ_{π_p} denotes the Langlands parameter of π_p. If τ is an irreducible admissible representation of $H(\mathbf{A_Q})$, then by using the identification of H with $\mathrm{Res}_{E/\mathbf{Q}} \mathbf{G}_m \times \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$ in Section 2.2.1, we can think of τ as being a pair (ψ, τ_0) of representations of $\mathrm{Res}_{E/\mathbf{Q}} \mathbf{G}_m(\mathbf{A_Q})$ and $H_0 = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n(\mathbf{A_Q})$.

Recall the involution θ_H defined in (2.1). We say that a τ as above is θ_H-stable if $\tau^{\theta_H} \cong \tau$, which is equivalent to $\tau_0^\vee \cong \tau_0^c$ and $\psi = \psi^c \chi_{\tau_0}^c$, where $\vee$ denotes the contragredient, χ_{τ_0} is the central character of τ_0, and c denotes the conjugate, i.e. the composition with the involution on $\mathrm{Res}_{E/\mathbf{Q}} \mathbf{G}_m \times \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$ induced by the non-trivial element of $\mathrm{Gal}(E/\mathbf{Q})$.

Theorem 2.2.2 ([49, 73, 71, 70])**.** *Let F^+ be a totally real field and let F/F^+ be the compositum of F^+ with an imaginary quadratic field E. Let J be a Hermitian form on F^n, suppose that π is a cuspidal automorphic representation on the unitary similitude group $G = GU(J)$, and let G_0, H, and H_0 be defined as in Section 2.2.1. Recall that via Weil restriction of scalars we regard all of these groups as being over $\mathbf{Q}$. Moreover, assume that there exists an algebraic representation $\mathbf{V}_\lambda$ of $G_{/F}$ such that π_∞ is a regular discrete series representation*

inside the L-packet attached to $\mathbf{V}_\lambda$*. Then there exists a possibly non-cuspidal automorphic representation* $\tau = (\psi, \tau_0)$ *on* H *with the following properties.*

1. *We have* $\tau_p = \mathrm{BC}(\pi_p)$ *for any prime* p *of* $\mathbf{Q}$ *that either*
   (a) *splits in* E*, or*
   (b) *is inert in* E *with* π_p *unramified and* p *not a prime of ramification of* F*.*
2. *The infinitesimal character of* τ_∞ *is associated to the algebraic representation* $\mathbf{V}_\lambda \otimes \mathbf{V}_\lambda^\theta$ *of* $H_{/F}$*, where this is regarded as a representation of* $H_{/F}$ *via the identifications (from Section 2.2.1) of* $G_{/F}$ *with* $\mathbf{G}_m \times \prod_{\nu: F^+ \to \mathbf{R}} \mathrm{GL}_n$ *and* $H_{/F}$ *with* $G_{/F} \times G_{/F}$ *via the map* $R \otimes_{\mathbf{Q}} E \to R \times R$ *(also defined in Section 2.2.1).*
3. *The representation* τ *is* θ_H*-stable. Moreover,* $\psi = \chi_\pi^c$ *and* $\chi_{\tau_0} = \chi_\pi / \chi_\pi^c$*, where* χ_π *denotes the central character of* π*.*
4. *The representation* τ_0 *is tempered at all finite places.*

Shin requires only that π_∞ is discrete series, and the resulting τ is an isobaric sum of discrete conjugate self-dual representations (rather than cuspidal ones). However, since we are assuming π_∞ is regular discrete series, these discrete representations are cuspidal and τ is tempered – this is proved in [72, Corollary 4.16].

We can be more precise about the possibilities for the representation $\mathbf{V}_\lambda$ mentioned above. If $T \subseteq G$ is the diagonal torus, then $T_{/F} \subseteq G_{/F}$ is identified with $\mathbf{G}_m \times \prod_{F^+ \to \mathbf{R}} \mathbf{G}_m^n$. Writing k for $[F^+ : \mathbf{Q}]$ and ν_i, $i = 1, \ldots, k$, for the distinct maps $F^+ \to \mathbf{R}$, we can identify the character group $X(T)$ with $\mathbf{Z}^{1+kn}$ as follows. We write $\underline{c} = (c, \underline{c}_{\nu_1}, \ldots, \underline{c}_{\nu_k}) \in \mathbf{Z}^{1+kn}$, where $\underline{c}_{\nu_i} = (c_{\nu_i,1}, \ldots, c_{\nu_i,n})$. Then $\lambda(\underline{c}) \in X(T)$ is given by $(t, \mathrm{diag}(t_{i,1}, \ldots, t_{i,n})) \mapsto t^c \prod_{i=1}^k \prod_{j=1}^n t_{i,j}^{c_{\nu_i,j}}$, where the group of elements of the form $\mathrm{diag}(t_{i,1}, \ldots, t_{i,n})$ is the torus of the factor $\mathbf{G}_m^n$ indexed by ν_i. Using the upper-triangular Borel, the dominant characters are exactly those satisfying $c_{\nu_i,1} \geq \cdots \geq c_{\nu_i,n}$ and regular dominant characters have strict inequalities. Then

a algebraic representation $\mathbf{V}_\lambda = \mathbf{V}_{\lambda(\underline{c})}$ as considered in Theorem 2.2.2 is determined by such a $\underline{c}$.

We will also need to know what the representation $\mathbf{V}_\lambda \otimes \mathbf{V}_\lambda^\theta$ is in this context. Letting $T_H \subseteq H$ be the usual maximal torus, we have $X(T_H) \cong X(T) \times X(T)$, where the first factor corresponds to the chosen embedding $E \subseteq \mathbf{C}$ and the second is the conjugate. So an algebraic representation of $H_{/F} \cong G_{/F} \times G_{/F}$ is determined by a pair $(\underline{c}, \underline{c}')$ of data in the format of $\underline{c}$ above. For $\mathbf{V}_\lambda \otimes \mathbf{V}_\lambda^\theta$, $\underline{c}_1 = \underline{c}$. The algebraic representation $\mathbf{V}_\lambda^\theta = \mathbf{V}_{\lambda^\theta}$ can be calculated by examining the effect of $\theta((x,g)) = (\overline{x}, \overline{x}^t\overline{g}^{-1})$ using the explicit form of λ on the torus. In particular, for each i, we have $(t, \mathrm{diag}(t_{i,1}, \dots, t_{i,n})) \overset{\theta}{\mapsto} (\overline{t}, \mathrm{diag}(\overline{tt_{i,1}}^{-1}, \dots, \overline{tt_{i,n}}^{-1}))$, so that $\underline{c}_2$ corresponds to the map

$$(t, \mathrm{diag}(t_{i,1}, \dots, t_{i,n})) \mapsto \overline{t}^c \prod_{i=1}^{k} \prod_{j=1}^{n} \overline{t}^{c_{\nu_i,j}} \overline{t}_{i,j}^{-c_{\nu_i,j}}.$$

Also note that θ takes the Borel to its opposite, so we need to conjugate by the longest element of the Weyl group, which interchanges $\overline{t}_i$ with $\overline{t}_{n+1-i}$. We deduce that $\underline{c}_2 = (c', \underline{c}'_{\nu_1}, \dots, \underline{c}'_{\nu_k})$, where $c' = c + \sum_{i=1}^k \sum_{j=1}^n c_{\nu_i,j}$ and $\underline{c}'_{\nu_i} = (-c_{\nu_i,n}, \dots, -c_{\nu_i,1})$. We note that if $\underline{c}$ is dominant or regular dominant, so is $\underline{c}_2$.

Definition 2.2.3. Suppose that $\lambda = \lambda(\underline{c})$. We define the *weight* of the algebraic representation $\mathbf{V}_\lambda$ to be $c + c' = 2c + \sum_{i=1}^k \sum_{j=1}^n c_{\nu_i,j}$. This can alternatively be defined as the weight of the induced algebraic representation of the center of G.

We now describe the Galois representation attached to π (which is just the one attached to $\tau = (\psi, \tau_0)$). Its construction is the culmination of works of many authors, including Shin [71, Theorem 1.2]. We note that usually one only attaches a Galois representation to τ_0; following Skinner [73, Theorem 10] we simply tensor that representation with the one attached to ψ. For the entire paper we will fix an isomorphism $\iota : \mathbf{C} \to \overline{\mathbf{Q}}_p$.

Theorem 2.2.4. *Suppose that π and $\tau = (\psi, \tau_0)$ are as described in Theorem 2.2.2. Then there is a continuous semi-simple representation $\rho_\pi : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following properties.*

1. *At places $v|p$, $\rho_\pi|_{G_{F_v}}$ is potentially semi-stable at v. The Hodge-Tate weights are given in terms of the aforementioned data $(\underline{c}_1, \underline{c}_2)$ attached to $\mathbf{V}_\lambda \otimes \mathbf{V}_\lambda^\theta$ as follows. For an embedding $\nu : F^+ \to \mathbf{R}$ (which then maps to $\overline{\mathbf{Q}}_p$ via ι), there is a set of Hodge-Tate weights attached to the chosen embedding $E \subseteq \mathbf{C}$ and a set of weights attached to its conjugate. This ordered pair of sets is given by*

$$\mathrm{HT}_\nu(\rho_\pi|_{G_{F_v}}) = (\{-c + j - 1 - c_{\nu,j}\}, \{-c' + j - 1 + c_{\nu,n+1-j}\}).$$

2. *The representation $\mathrm{WD}(\rho_\pi|_{G_{F_v}})$ is pure of weight $n - 1 - w$ for every place $v \in F$, where $\mathbf{V}_\lambda$ has weight w, and moreover*

$$\mathrm{WD}(\rho_\pi|_{G_{F_v}})^{\mathrm{Fr-ss}} = \iota \mathrm{Rec}_v(\tau_v \otimes \psi_v |\cdot|_v^{\frac{1-n}{2}}).$$

Here WD *denotes the Weil-Deligne representation and* $\mathrm{Fr} - \mathrm{ss}$ *denotes Frobenius semi-simplification.*

For the meaning of $\mathrm{WD}(\rho_\pi|_{G_{F_v}})$ when $v|p$ (at least in the semi-stable case, which is all we will use), see Section 2.6.1. We also remark that the weights in each of the two sets in $\mathrm{HT}_\nu(\rho_\pi|_{G_{F_v}})$ are all distinct when the data of $\underline{c}$ is dominant.

We now state our main theorem, which is an improvement to Theorem 2.2.2. It also follows that π has better compatibility with its Galois representation.

Theorem 2.2.5. *Maintain the notation of Theorem 2.2.2. Assume that the odd prime p ramifies in E but not in F, $G_{/\mathbf{Q}_p}$ is quasi-split, and π_p is K-spherical for the special maximal compact subgroup named in Section 2.4.4 or 2.4.5. Then in addition to the compatibility*

described in Theorem 2.2.2, we also have $\tau_p = \mathrm{BC}(\pi_p)$ *and* π_p *is tempered (up to its central character).*

In fact, our argument can apply to certain irregular π as well.

Remark 2.2.6. If π_∞ is discrete series but not regular, Shin still proves Theorem 2.2.2 but without the temperedness of τ_0 at all finite places. If we instead take this temperedness as an additional hypothesis and also assume that the classical Euler-Poincare characteristic of π is non-vanishing (which is automatic in the regular case), then the entirety of our argument here applies without modification.

We can deduce unitary cases as well.

Corollary 2.2.7. *Suppose that* E, F^+, F, *and* J *are as in Theorem 2.2.2, but consider the unitary group* G_0 *in place of the unitary similitude group* G. *Then given an automorphic representation* π_0 *on* G_0, *there exists a base change* τ_0 *to* H_0 *with compatibility as described in Theorems 2.2.2 and 2.2.5.*

Proof. The work of Langlands-Labesse [38, §6] shows that since G/G_0 is a torus, there exists an extension of the central character of π_0 to G and a lifting π of π_0 to G with that central character. The precise construction of such an extension of the central character is given by an argument of Patrikis [54, Proposition 3.1.4]. Then we just apply Theorems 2.2.2 and 2.2.5 and read off the compatibility between π_0 and τ_0.

□

2.2.3 A relation between Satake parameters

We now regard G_0 and H_0 (as defined in Section 2.2.1) as F^+-groups. Suppose that we are in the situation of Theorem 2.2.2. To avoid cluttering the notation, we let π denote the cuspidal automorphic representation of $G_0(\mathbf{A}_{F^+})$ given by choosing an irreducible subrepresentation

of the restriction of the π of Theorem 2.2.2 to G_0 and write τ in place of τ_0. Then τ is a weak base change of π to H_0. We write q_v for the size of the residue field of F_v. Note that if we study the base change properties of these F^+ groups in place of the $\mathbf{Q}$-groups, the effect on the Langlands dual group is only to ignore the permutation action of $G_{\mathbf{Q}}$ on the $\nu : F^+ \to \mathbf{R}$, which is harmless.

Suppose that the rational prime p ramifies in E and that F^+ is unramified at p. Fix a place $v|p$ of F^+. We also assume that $G_{0,v}$ is quasi-split and that π_v is a subquotient of the parabolic induction of an unramified character χ of the maximal torus of $G_{0,v}$.

Looking ahead, in Section 2.3 we will classify possibilities for τ_v and π_v. A consequence of the main result there is the following, which asserts compatibility as long as a special relationship between the Satake parameters of χ does not occur. We give a short, self-contained argument for this result at the end of the section. See Definition 2.2.9 for the definitions of Satake parameters and unramified principal series representations used below.

Theorem 2.2.8. *Let π be a cuspidal automorphic representation of $G_0(\mathbf{A}_{F^+})$ and let τ be a weak base change of π to $H_0(\mathbf{A}_{F^+})$. Suppose that the rational prime p ramified in E, $v|p$ is a place of F^+, and $G_{0,v}$ is quasi-split. Moreover, assume that π_v is a subquotient of $\mathrm{Ind}_B^{G_0} \chi$, where χ is an unramified character of the diagonal maximal F^+-rational torus of G_0. Let $\{\alpha_i\}_{i\in\{1,\dots,n\}}$ be the set of Satake parameters of π_v, and write ψ_{τ_v} and ψ_{π_v} for the Langlands parameters.*

1. *Suppose that π_v is an unramified principal series representation. Then if $\alpha_i \neq q_v\alpha_j$ for $i, j \in \{1, \dots, n\}$, we have $\psi_{\tau_v} = BC_0 \circ \psi_{\pi_v}$.*

2. *Suppose that $\alpha_i \neq \alpha_j$ and $\alpha_i \neq q_v\alpha_j$ for $i, j \in \{1, \dots, n\}$. Then we have $\psi_{\tau_v} = BC_0 \circ \psi_{\pi_v}$.*

Our first task will be to calculate the standard local L-factor of $\pi_v \times \omega_v$, where π_v is assumed to be an almost unramified principal series representation and ω_v is an unramified character of the group R_v, where $R = \mathrm{Res}^F_{F^+} \mathbf{G}_m$. We will then use a result of Skinner [73]

to compare the γ-factors of π_v and τ_v. Skinner's work employs results of Godemont-Jacquet [32], Lapid-Rallis [40], and Brenner [10] on the construction, properties, and stability of γ-factors for unitary and general linear groups.

2.2.4 Local L-factors for tori

Let $w|v$ be the place of F over v. Yu [86] calculates the local Langlands correspondence for an induced torus $T = \mathrm{Res}_{F_w/F_v^+} F_w^\times$ to be the composition

$$\mathrm{Hom}(F_w^\times, \mathbf{C}^\times) \xrightarrow{\sim} \mathrm{Hom}(W_{F_w}, \mathbf{C}^\times) \xrightarrow{\sim} H^1(W_{F_v^+}, \mathrm{Ind}_{W_{F_w}}^{W_{F_v^+}} \mathbf{C}^\times) = H^1(W_{F_v^+}, \widehat{T}),$$

where the first map is via local class field theory and the second is the isomorphism of Shapiro's lemma. (We use geometric normalizations, so a uniformizer in $F_w^\times$ maps to a geometric Frobenius element.) Note that the inverse of this second map is restriction to W_{F_w} followed by evaluation at $1_{W_{F_v}}$. (See, e.g., [69, Proposition 10, §2.5]).

The image of the unramified character $\chi_\alpha : F_w^\times \to \mathbf{C}^\times$ sending a uniformizer ϖ to $\alpha \in \mathbf{C}^\times$ is, under the first map, the unramified character sending Frob_{F_w} to α, where Frob_{F_w} denotes a geometric Frobenius element. We obtain this upon restriction to W_{F_w} and evaluation at $1_{W_{F_v^+}}$ of the homomorphism

$$\varphi_\alpha : W_{F_v^+} \to \mathrm{Ind}_{W_{F_w}}^{W_{F_v^+}} \mathbf{C}^\times$$

defined as follows. Let $I_{F_v^+}$ denote the inertia subgroup and set $\varphi_\alpha(I_{F_v^+}) = \mathbf{1}$, so that φ_α factors through $W_{F_v^+}/I_{F_v^+}$. Then define $\varphi_\alpha(\mathrm{Frob}_{F_v}^m) = \boldsymbol{\alpha}^m$ on W_{F_v}/I_{F_v}, where $m \in \mathbf{Z}$ and $\boldsymbol{\alpha}$ denotes the constant function $\sigma \mapsto \alpha$ for $\sigma \in W_{F_v^+}$. The homomorphism φ_α is an element of $H^1(W_{F_v^+}, \mathrm{Ind}_{W_{F_w}}^{W_{F_v^+}} \mathbf{C}^\times)$ since $W_{F_v^+}$ acts trivially on the constant functions in $\mathrm{Ind}_{W_{F_w}}^{W_{F_v^+}} \mathbf{C}^\times$. Thus, χ_α and φ_α correspond to each other under the local Langlands correspondence for T.

We also note that for the torus $T = U(1)$ over F_v^+, there is only the trivial unramified character, so the local Langlands correspondence takes this character to the trivial element

of $H^1(W_{F_v}, {}^L T)$.

2.2.5 Unitary groups over p-adic fields

We summarize some basic facts regarding unitary groups over p-adic fields. One reference for these is an article of Minguez [48]. We define a unitary group U for a quadratic extension L/L^+ of p-adic fields in the same way as the global case. However, in the p-adic case, if the dimension n of the Hermitian space is odd, there is only one possible unitary group U up to isomorphism, and it is quasi-split. (There are two non-isomorphic Hermitian forms, but the associated unitary groups are isomorphic.) If the dimension n is even, there are two possibilities for U, but only one is quasi-split. In both cases, the quasi-split unitary group can be given by the Hermitian form defined by A_n above. In Section 2.4, we will use a different choice of form in order to simplify the discussion of the finer integral structure of U, but for now A_n will suffice.

Let $G = U(A_n)_{/L^+}$. The maximal torus T and Borel B of G can be defined by requiring $\mu(g) = 1$ in the formulas in Example 2.2.1 for T and P, where we set $a = b$ or $a + 1 = b$ depending on whether n is even or odd. The description of parabolics containing B is also the same as in the global case with these values of a and b. If $n = 2m$ is even, $T \cong (\mathrm{Res}_{L/L^+} L^\times)^m$, whereas if $n = 2m + 1$ is odd, $T \cong (\mathrm{Res}_{L/L^+} L^\times)^m \times U(1)$. The spherical Weyl group is isomorphic to $S_m \rtimes (\mathbf{Z}/2\mathbf{Z})^m$, where S_m permutes the matrix entries $\lambda_1, \ldots, \lambda_m$ and their inverses in the description of T in Example 2.2.1, and the i^{th} cyclic factor switches λ_i with $\overline{\lambda_i}^{-1}$.

The L-group is defined using the same action as provided in the global case.

2.2.6 Local L-factors for ramified unitary groups

Let T_v be the aforementioned maximal torus in $G_{0,v}$ over F_v. We are assuming $G_{0,v}$ is quasi-split, so

$$T_v \cong \prod_{i=1}^{m} \mathrm{Res}_{F_w/F_v^+} F_w^\times \qquad \text{or} \qquad T_v \cong \left(\prod_{i=1}^{m} \mathrm{Res}_{F_w/F_v^+} F_w^\times \right) \times U(1)_{/F_v^+}.$$

In particular, we have dual groups

$$\widehat{T}_v \cong \prod_{i=1}^{m} \mathrm{Ind}_{W_{F_w}}^{W_{F_v^+}} \mathbf{C}^\times \qquad \text{or} \qquad \widehat{T}_v \cong \left(\prod_{i=1}^{m} \mathrm{Ind}_{W_{F_w}}^{W_{F_v^+}} \mathbf{C}^\times \right) \times \mathbf{C}^\times.$$

This defines the L-groups: the action of W_{F_v} factors through $\mathrm{Gal}(F_w/F_v^+)$, and the action of the nontrivial element $c \in \mathrm{Gal}(F_w/F_v^+)$ inverts the $U(1)_{/F_v^+}$ factor and acts in the usual way on the induction spaces.

Let $\underline{\beta} = (\beta_1, \ldots, \beta_m) \in (\mathbf{C}^\times)^m$. Under the local Langlands correspondence calculated in Section 2.2.4, the unramified character $\chi_{\underline{\beta}}$, defined by sending uniformizers in each non-$U(1)_{/F_v^+}$ factor to $\beta_1, \ldots, \beta_m$ respectively, maps to

$$((\beta_1, \beta_1), \ldots, (\beta_m, \beta_m)) \in \widehat{T}_v \quad \text{and} \quad ((\beta_1, \beta_1), \ldots, (\beta_m, \beta_m), 1) \in \widehat{T}_v$$

in the respective cases above. Here, we have denoted an element of $\mathrm{Ind}_{W_{F_w}}^{W_{F_v^+}} \mathbf{C}^\times$ by the ordered pair giving the values of a function at $\mathbf{1}_{W_{F_v^+}}$ and an arbitrary fixed lift of c to $W_{F_v^+}$.

In order to calculate the local Langlands parameter $W_{F_v^+} \to {}^L G_{0,v}$ of the corresponding unramified principal series representations, we need to determine how these tori embed into the L-group of $G_{0,v}$. We find that the morphisms $\widehat{T}_v \hookrightarrow \widehat{G}_{0,v}$ given by

$$((t_1, t_2), \ldots, (t_{2m-1}, t_{2m})) \mapsto \mathrm{diag}(t_1, t_3, \ldots, t_{2m-1}, t_{2m}^{-1}, \ldots, t_2^{-1})$$

and

$$((t_1,t_2),\dots,(t_{2m-3},t_{2m-2}),t_{2m+1}) \mapsto \mathrm{diag}(t_1,t_3,\dots,t_{2m-1},t_{2m+1},t_{2m}^{-1},\dots,t_2^{-1})$$

give embeddings ${}^LT_v \hookrightarrow {}^LG_{0,v}$ in the even and odd cases, respectively, since these maps are W_{F_v}-equivariant.

If K is a special maximal compact subgroup, these calculations and unramified functoriality determine the local Langlands parameter $\psi_{\underline{\beta}} : W_{F_v^+} \times \mathrm{SL}_2(\mathbf{C}) \to \widehat{G}_{0,v}$ attached to a K-spherical subquotient of the normalized induction $\mathrm{Ind}_B^U \chi_{\underline{\beta}}$ to be the map that kills $\mathrm{SL}_2(\mathbf{C})$ and sends any $f \in W_{F_v^+}$ lifting $\mathrm{Frob}_{F_v^+}^k$ to

$$(\mathrm{diag}(\beta_1,\dots,\beta_m,\beta_m^{-1},\dots,\beta_1^{-1}))^k \rtimes f \quad \text{and} \quad (\mathrm{diag}(\beta_1,\dots,\beta_m,1,\beta_m^{-1},\dots,\beta_1^{-1}))^k \rtimes f,$$

respectively. The map $\psi_{\underline{\beta}} : W_{F_v^+} \to {}^LG_{0,v}$ is nearly unramified in the sense that the projection of $\psi_{\underline{\beta}}(I_{F_v^+})$ to $\widehat{G}_{0,v}$ is trivial.

Let $R = \mathrm{Res}_{F/F^+}\mathbf{G}_m$, so that $\hat{R} = \mathbf{C}^\times \times \mathbf{C}^\times$ with the action of the nontrivial element $c \in \mathrm{Gal}(F/F^+)$ defined by $c(\alpha,\beta) = (\beta,\alpha)$. The standard representation $r_{\mathrm{st},G_0} : {}^LG_0 \times_{W_{F^+}} {}^LR \to \mathrm{GL}_{2n}(\mathbf{C})$ is defined by

$$r_{\mathrm{st},G_0}(g \rtimes 1,(\alpha,\beta)\rtimes 1) = \begin{pmatrix} \alpha g & \\ & \beta\Phi_n^{-1t}g^{-1}\Phi_n \end{pmatrix}, r_{\mathrm{st},G_0}(1\rtimes c, 1\rtimes c) = \begin{pmatrix} & \mathbf{1}_n \\ \mathbf{1}_n & \end{pmatrix}.$$

The standard representation $r_{\mathrm{st},H_0} : {}^LH_0 \times_{W_{F^+}} {}^LR \to \mathrm{GL}_{2n}(\mathbf{C})$ is defined by

$$r_{\mathrm{st},H_0}((g_1,g_2)\rtimes 1,(\alpha,\beta)\rtimes 1) = \begin{pmatrix} \alpha g_1 & \\ & \beta\Phi_n^{-1t}g_2^{-1}\Phi_n \end{pmatrix}, r_{\mathrm{st},H_0}(1\rtimes c, 1\rtimes c) = \begin{pmatrix} & \mathbf{1}_n \\ \mathbf{1}_n & \end{pmatrix}.$$

We have $r_{\mathrm{st},G_0} = r_{\mathrm{st},H_0} \circ (\mathrm{BC}_0 \times \mathbf{1}_{{}^LR})$.

Let π_v be the K-spherical subquotient of $\mathrm{Ind}_B^U \chi_{\underline{\beta}}$ as before and let ω_v be an unramified character mapping uniformizers to β. We consider the representation $r_{\mathrm{st},G_0}(\psi_{\pi_v}, \psi_{\omega_v}) : W_{F_v} \to \mathrm{GL}_{2n}(\mathbf{C})$. (We may ignore the $\mathrm{SL}_2(\mathbf{C})$ factor.) Observe that the image of I_{F_v} is the two element subgroup generated by $\left(\begin{smallmatrix} & \mathbf{1}_n \\ \mathbf{1}_n & \end{smallmatrix}\right)$, so that we are interested in the action of $\mathrm{Frob}_{F_v^+}$ on the subspace $V^{I_{F_v^+}}$ of vectors of the form ${}^t(v, v)$, $v \in \mathbf{C}^n$. Correspondingly, the image of $\mathrm{Frob}_{F_v^+}$ has the form

$$\mathrm{diag}(\beta\beta_1, \dots, \beta\beta_m, \beta\beta_m^{-1}, \dots, \beta\beta_1^{-1}, \beta\beta_1, \dots, \beta\beta_m, \beta\beta_m^{-1}, \dots, \beta\beta_1^{-1})$$

or

$$\mathrm{diag}(\beta\beta_1, \dots, \beta\beta_m, 1, \beta\beta_m^{-1}, \dots, \beta\beta_1^{-1}, \beta\beta_1, \dots, \beta\beta_m, 1, \beta\beta_m^{-1}, \dots, \beta\beta_1^{-1}).$$

We calculate in the even rank case

$$\begin{aligned} L(\pi_v \times \omega_v) &= \det(1 - q_v^{-s} r_{\mathrm{st},G_0}(\psi_{\underline{\beta}}(\mathrm{Frob}_{F_v^+}), \psi_{\omega_v}(\mathrm{Frob}_{F_v^+}))|_{V^{I_{F_v^+}}})^{-1} \\ &= \det(\mathrm{diag}(1 - q_v^{-s}\beta\beta_1, \dots, 1 - q_v^{-s}\beta\beta_m, 1 - q_v^{-s}\beta\beta_m^{-1}, \dots, 1 - q_v^{-s}\beta\beta_1^{-1}))^{-1} \\ &= \prod_{i=1}^{m} (1 - q_v^{-s}\beta\beta_i)^{-1}(1 - q_v^{-s}\beta\beta_i^{-1})^{-1}, \end{aligned}$$

or, in the odd rank case,

$$\begin{aligned} L(\pi_v \times \omega_v) &= \det(\mathrm{diag}(1 - q_v^{-s}\beta\beta_1, \dots, 1 - q_v^{-s}\beta\beta_m, 1 - q_v^{-s}\beta, \\ &\qquad\qquad 1 - q_v^{-s}\beta\beta_m^{-1}, \dots, 1 - q_v^{-s}\beta\beta_1^{-1}))^{-1} \\ &= (1 - q_v^{-s}\beta)^{-1} \prod_{i=1}^{m} (1 - q_v^{-s}\beta\beta_i)^{-1}(1 + q_v^{-s}\beta\beta_i^{-1})^{-1}. \end{aligned}$$

Definition 2.2.9. Suppose that $\chi_{\underline{\beta}}$ is an unramified character of the maximal torus. Then if π_v is *any* subquotient of $\mathrm{Ind}_B^U \chi_{\underline{\beta}}$, we say that the *Satake parameters* of π_v are the multiset $\{\beta_i, \beta_i^{-1}\}$ or $\{\beta_i, 1, \beta_i^{-1}\}$ in the even or odd case, respectively. If additionally π_v is the K-

spherical subquotient for a special maximal compact K, we say π_v is an *unramified principal series representation* with Satake parameters $\{\alpha_i\}$.

Using the same approach as above, once can carry out the calculation of L-factors for an unramified principal series representation τ_v with Satake parameters $\{\gamma_1, \ldots, \gamma_n\}$ and ω_v sending a uniformizer to β. (By unramified principal series, we again mean that it is the subquotient of the relevant parabolic induction that is spherical for a hyperspecial maximal compact.) Using the diagonal torus, the embedding

$$\prod_{i=1}^{m} \mathrm{Ind}_{W_{F_w}}^{W_{F_v^+}} \mathbf{C}^\times \to \widehat{H}_{0,v}$$

sends

$$\mathrm{diag}((t_1, t_2), \ldots, (t_{2n-1}, t_{2n})) \mapsto (\mathrm{diag}(t_1, t_3, \ldots, t_{2n-1}), \mathrm{diag}(t_{2n}^{-1}, \ldots, t_2^{-1})).$$

Thus ψ_{τ_v} sends Frob_{F_v} to $(\mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n), \mathrm{diag}(\gamma_n^{-1}, \ldots, \gamma_1^{-1}))$. In particular, the image of Frob_{F_v} under the composite map $r_{\mathrm{st},H_0}(\psi_{\tau_v}, \psi_{\omega_v}) : W_{F_v} \to \mathrm{GL}_{2n}(\mathbf{C})$ is given by

$$\mathrm{diag}(\beta\gamma_1, \ldots, \beta\gamma_n, \beta\gamma_1, \ldots, \beta\gamma_n),$$

which yields the L-factor

$$L(\tau_v \times \omega_v) = \prod_{i=1}^{n} (1 - q_v^{-s}\gamma_i\beta)^{-1}.$$

The observation above that $r_{\mathrm{st},G_0} = r_{\mathrm{st},H_0} \circ (\mathrm{BC}_0 \times \mathbf{1}_{L_R})$ implies that $L(s, \pi_v) = L(s, \tau_v)$ if $\mathrm{BC}_0 \circ \psi_{\pi_v} = \psi_{\tau_v}$. We can give a converse to this as follows.

Proposition 2.2.10. *Suppose that π_v and τ_v are unramified principal series representations. Then if $L(s, \pi_v) = L(s, \tau_v)$, we have $\mathrm{BC}_0 \circ \psi_{\pi_v} = \psi_{\tau_v}$.*

Proof. Indeed, the image of the map ψ_{π_v} defined above under the base change map BC_0

sends Frob_{F_v} to

$$(\mathrm{diag}(\beta_1, \ldots, \beta_m, \beta_m^{-1}, \ldots, \beta_1^{-1}), \mathrm{diag}(\beta_1, \ldots, \beta_m, \beta_m^{-1}, \ldots, \beta_1^{-1})) \in \widehat{H}$$

or

$$(\mathrm{diag}(\beta_1, \ldots, \beta_m, 1, \beta_m^{-1}, \ldots, \beta_1^{-1}), \mathrm{diag}(\beta_1, \ldots, \beta_m, 1, \beta_m^{-1}, \ldots, \beta_1^{-1})) \in \widehat{H}$$

in the respective cases. If

$$\{\gamma_1, \ldots, \gamma_n\} = \left\{\beta_1, \ldots, \beta_m, \beta_m^{-1}, \ldots, \beta_1^{-1}\right\} \text{ or } \left\{\beta_1, \ldots, \beta_m, 1, \beta_m^{-1}, \ldots, \beta_1^{-1}\right\},$$

respectively, then $\mathrm{BC}_0 \circ \psi_{\pi_v}$ is immediately seen to be equivalent to ψ_{τ_v}. □

2.2.7 γ-factors

Lapid and Rallis [40] define γ-factors $\gamma(s, \pi_v \times \omega_v, \psi_v)$, where v is a place of F and ψ_v is an additive character of F_v. We fix ψ_v to be the standard additive character of F_v, denoting the corresponding γ-factor by $\gamma(s, \pi_v \times \omega_v)$.

Skinner [73, §3] proves the following result using the results of Lapid and Rallis [40, Theorem 4] on γ-factors of unitary groups, their analogues in the general linear case [25], and stability of γ-factors in both the general linear [32] and unitary [10] cases.

Lemma 2.2.11. *Let π and τ be as in the statement of Theorem 2.2.8, ignoring the condition on Satake parameters. Then $\gamma(s, \pi_v \times \omega_v) = \gamma(s, \tau_v \times \omega_v)$ for all v and ω.*

We may now prove Theorem 2.2.8. We use various facts about the Bernstein-Zelevinsky classification that will be introduced in greater detail in Section 2.3.1.

Proof of Theorem 2.2.8. For now, assume only that π_v is a subquotient of the induction of an unramified character. We write $\gamma(s, \sigma) = \gamma(s, \sigma \times 1)$ for any admissible representation σ

of a local group, and we let $\{\alpha_i\}_{i=1}^n$ denote the multiset of Satake parameters of π_v. (This multiset has the form $\{\beta_i\} \cup \{\beta_i^{-1}\}$ or $\{\beta_i\} \cup \{\beta_i^{-1}\} \cup \{1\}$ in the notation above.) We first note that

$$\gamma(s, \pi_v) = \frac{L(1-s, \widetilde{\pi}_v)}{L(s, \pi_v)} = \frac{\prod_{i=1}^n (1 - q_v^{-s}\alpha_i)}{\prod_{i=1}^m (1 - q_v^{-1+s}\alpha_i^{-1})}$$

by multiplicativity of γ-factors [40, Theorem 4.2]. Let

$$\Omega = \left\{ z \,\middle|\, -\frac{\pi}{\log q_v} \leq \mathrm{im}(z) < \frac{\pi}{\log q_v} \right\} \subseteq \mathbf{C}.$$

The hypotheses of the proposition imply that this function has n roots and n poles as a function of s in Ω, counted with multiplicity, and moreover, these roots and poles determine the function $\gamma(s, \pi_v)$.

There exist $\{n_i\}_{i=1}^k$ such that $n = \sum_i n_i$ and supercuspidal representations τ_i on GL_{n_i} such that τ_v is a subrepresentation of $\mathrm{Ind}_P^{H_0} \oplus_i \tau_i$, where P is the parabolic subgroup associated to the decomposition $(n_1, \ldots, n_k)$. By multiplicativity and additivity of γ factors, $\gamma(s, \tau_v \times \omega_v) = \prod_i \gamma(s, \tau_i \times \omega_v)$.

Observe that

$$\gamma(s, \tau_i) = \frac{L(1-s, \widetilde{\tau}_i)}{L(s, \tau_i)} \epsilon(s, \tau_i)$$

where $\epsilon(s, \tau_v)$ is holomorphic and nonvanishing as a function of s. In particular, Lemma 2.2.11 implies that

$$\frac{L(1-s, \widetilde{\tau}_v)}{L(s, \tau_v)} = \gamma(s, \tau_v)\epsilon(s, \tau_v)^{-1} = \gamma(s, \pi_v)\epsilon(s, \tau_v)^{-1}$$

has n roots and n poles as a function of s in Ω. Each $\gamma(s, \tau_i)$ has at most one root and one pole in Γ, as follows, for example, from the calculations in [25, Proposition 5.11]. Thus $k = n$ and τ_v is a subrepresentation of the parabolic induction from a Borel of the character defined by the τ_i. In fact, $\gamma(s, \tau_i)$ is holomorphic for a ramified character τ_i [25, Proposition

5.11], so each τ_i must be unramified. As a consequence,

$$\gamma(s, \tau_i) = \prod_{i=1}^{n} \frac{L(1-s, \widetilde{\tau}_i)}{L(s, \tau_i)} = \frac{\prod_{i=1}^{n}(1 - q_v^{-s}\gamma_i)}{\prod_{i=1}^{m}(1 - q_v^{-1+s}\gamma_i^{-1})},$$

where the multiset $\{\gamma_i\}_{i=1}^{n}$ is the set of Satake parameters of τ_i. In this situation, the Satake parameters of both π_v and τ_v are determined by the poles and roots of their γ-factors in Ω, so this shows that $\{\gamma_i\} = \{\alpha_i\}$. Finally, τ_v is an unramified principal series representation by Theorem 4.2 of [7] and the condition $\alpha_i \neq q_v\alpha_j$ of the proposition, which now applies to the γ_i. Thus, if π_v is an unramified principal series representation, by Proposition 2.2.10, the Langlands parameter of τ_v is the base change of that of π_v, proving part (1).

For part (2), it suffices to show that π_v is an unramified principal series representation under the additional hypothesis that $\alpha_i \neq \alpha_j$ for all i, j. For this, we first note that τ_v is a *tempered* unramified principal series representation by Theorem 2.2.2, so that the γ_i have complex absolute value 1. In this situation, since the character $\chi_{\underline{\beta}}$ is both unitary and regular, we may apply Bruhat irreducibility [14, Theorem 6.6.1] to see that the full parabolic induction with Satake parameters $\{\alpha_i\}$ is already irreducible. It follows that π_v is an unramified principal series representation.

□

2.3 Possibilities for τ_v and π_v

In this section we use the Bernstein-Zelevinsky classification of automorphic representations and the calculation by Godemont-Jacquet of their L-factors in order to derive consequences for π_v and τ_v in case the conditions of Theorem 2.2.8 fail to hold. In particular, we prove the following result.

Theorem 2.3.1. *Let π be a cuspidal automorphic representation of $G_0(\mathbf{A}_{F^+})$ and let τ be a*

weak base change of π to $H_0(\mathbf{A}_{F^+})$. Suppose that $G_0(\mathbf{Q}_p)$ is quasi-split and ramified, and we assume that π_v is a subquotient of the parabolic induction of an unramified character of the Borel. Then the multiset of Satake parameters of π_v are the disjoint union of any number of sets of the form

$$\coprod_{i=0}^{m-1} \left\{ q_v^{i-\frac{m-1}{2}} \gamma \right\} \cup \coprod_{i=0}^{m-1} \left\{ q_v^{i-\frac{m-1}{2}} \gamma^{-1} \right\}, \gamma \in S^1 \setminus \pm 1,$$

any number of sets of the form

$$\coprod_{i=0}^{m-1} \left\{ q_v^{i-\frac{m-1}{2}} \right\} \quad or \quad \coprod_{i=0}^{m-1} \left\{ -q_v^{i-\frac{m-1}{2}} \right\} \tag{2.4}$$

with m even, an even (if n is even) or odd (if n is odd) number of sets of the form

$$\coprod_{i=0}^{m-1} \left\{ q_v^{i-\frac{m-1}{2}} \right\} \tag{2.5}$$

with m odd, and an even number of sets of the form

$$\coprod_{i=0}^{m-1} \left\{ -q_v^{i-\frac{m-1}{2}} \right\} \tag{2.6}$$

with m odd.

2.3.1 Local L-factors and γ-factors of representations of GL_n

In this section, we state some results that calculate local L-factors for representations of $\mathrm{GL}_n(L)$, where $L/\mathbf{Q}_p$ is a finite extension. Let q denote the order of the residue field of L. The essential facts regarding the classification of representations are due to Bernstein and Zelevinsky [6], while the results on L-factors we need were originally computed by Godemont and Jacquet [25]. We will use as our primary reference a recent paper of Cogdell and Piatetski-Shapiro [19].

For $P \subseteq \mathrm{GL}_n(L)$ a parabolic subgroup of type $(n_1, \ldots, n_k)$ and representations ρ_i of $\mathrm{GL}_{n_i}(L)$, we write $\mathrm{Ind}_P^{\mathrm{GL}_n(L)}(\rho_1 \times \cdots \times \rho_k)$ for the normalized induction. By the Bernstein-Zelevinsky classification, every irreducible admissible tempered representation ρ can be expressed as the unique irreducible quotient of $\mathrm{Ind}_P^{\mathrm{GL}_n(L)}(\rho_1 \times \cdots \times \rho_k)$, where each ρ_i is square-integrable. We write $\rho = Q(\rho_1 \times \cdots \times \rho_k)$, where the right hand side denotes the unique irreducible quotient of the normalized induction.

Theorem 2.3.2 ([19, §4])**.** *Let $\rho = Q(\rho_1 \times \cdots \times \rho_k)$ be any irreducible admissible tempered representation of $\mathrm{GL}_n(L)$, and let ω be any character of $\mathrm{GL}_1(L)$. Then we have*

$$L(s, \rho \times \omega) = \prod_{i=1}^{k} L(s, \rho_i \times \omega) \quad \text{and} \quad \gamma(s, \rho \times \omega) = \prod_{i=1}^{k} \gamma(s, \rho_i \times \omega).$$

Now let ρ be any irreducible square-integrable representation, and let ν denote the determinant character $|\det|$ on $\mathrm{GL}_n(L)$ for any n. Then

$$\rho = Q(\rho' \nu^{-\frac{m-1}{2}} \times \rho' \nu^{-\frac{m-1}{2}+1} \times \cdots \times \rho' \nu^{\frac{m-1}{2}}),$$

where ρ' is a supercuspidal representation of $\mathrm{GL}_{\frac{n}{m}}(L)$.

Theorem 2.3.3 ([19, §4])**.** *Let $\rho = Q(\rho' \nu^{-\frac{m-1}{2}} \times \rho' \nu^{-\frac{m-1}{2}+1} \times \cdots \times \rho' \nu^{\frac{m-1}{2}})$ be any square-integrable representation of $\mathrm{GL}_n(L)$, and let ω be any character of $\mathrm{GL}_1(L)$. Then we have*

$$L(s, \rho \times \omega) = L\left(s + \frac{m-1}{2}, \rho' \times \omega\right).$$

Lemma 2.3.4. *In the situation of Theorem 2.3.3, we have*

$$\gamma(s, \rho \times \omega) = \prod_{i=0}^{m-1} \gamma(s, \rho' \nu^{i-\frac{m-1}{2}} \times \omega).$$

Proof. Let P be the parabolic used in the induction defining ρ, and let ρ'' be the represen-

tation of the Levi factor given by the ordered sequence

$$\rho'\nu^{\frac{m-1}{2}}, \rho'\nu^{\frac{m-1}{2}-1}, \ldots, \rho'\nu^{-\frac{m-1}{2}}.$$

Using Theorem 4.2 of [40], we find that

$$\gamma(s, \rho \times \omega) = \gamma(s, \rho'' \times \omega) = \prod_{i=0}^{m-1} \gamma(s, \rho'\nu^{i-\frac{m-1}{2}} \times \omega),$$

as needed.

□

We are reduced to L-functions and γ-factors for supercuspidal representations.

Theorem 2.3.5 ([19, §2])**.** *Let ρ be a supercuspidal representation of $\mathrm{GL}_n(L)$ and ω a character of $\mathrm{GL}_1(L)$. Then if $n > 1$, $L(s, \rho \times \omega) = 1$. If $n = 1$,*

$$L(s, \rho \times \omega) = \prod_{\alpha} (1 - \alpha q^{-s})^{-1},$$

where the product is taken over $\alpha = q^{s_0}$ such that $\widetilde{\rho} \cong \omega\nu^{s_0}$, where $\widetilde{\cdot}$ denotes the contragredient.

In the case where ω is unramified, the calculation is the familiar one used above. In particular, $L(s, \rho \times \omega) = 1$ unless ρ also unramified, and if $\rho(\varpi) = \alpha$ and $\omega(\varpi) = \beta$, then $L(s, \rho \times \omega) = (1 - \alpha\beta q^{-s})^{-1}$. We also have

$$\gamma(s, \rho \times \omega) = \frac{L(1-s, \widetilde{\rho} \times \widetilde{\omega})}{L(s, \rho \times \omega)} = \frac{1 - \alpha\beta q^{-s}}{1 - \alpha^{-1}\beta^{-1}q^{-1+s}}.$$

2.3.2 Roots and poles of γ-factors

We are interested in the roots and poles of the function

$$\gamma(s, \pi_v) = \frac{\prod_{i=1}^{n}(1 - q_v^{-s}\alpha_i)}{\prod_{i=1}^{n}(1 - q_v^{-1+s}\alpha_i^{-1})},$$

where $\{\alpha_i\}$ is the multiset of Satake parameters of π_v. We always consider values of $\log_{q_v}$ in the region Ω defined in the proof of Theorem 2.2.8. The denominator vanishes when $1 = q_v^{-1+s}\alpha_i^{-1}$, or $s = 1 + \log_{q_v} \alpha_i$, while the numerator vanishes at $s = \log_{q_v} \alpha_i$. Let S_N and S_D denote the multisets of zeros of the numerator and denominator, respectively, counted with multiplicity. Observe that

$$\sum_{s \in S_D} s - \sum_{s \in S_N} s = n, \tag{2.7}$$

and that this quantity depends on the function $\gamma(s, \pi_v)$ only up to multiplication by a nowhere vanishing holomorphic function.

We have some additional information coming from the fact that $\frac{\prod_{i=1}^{n}(1-q_v^{-s}\alpha_i)}{\prod_{i=1}^{n}(1-q_v^{-1+s}\alpha_i^{-1})} = \gamma(s, \tau_v)$, where τ_v is tempered. Suppose that $\tau_v = Q(\rho_1, \ldots, \rho_m)$, where

$$\rho_i = Q(\rho_i' \nu^{-\frac{m_i-1}{2}} \times \cdots \times \rho_i' \nu^{\frac{m_i-1}{2}}).$$

Then $\gamma(s, \tau_v) = \prod_{i=1}^{m} \gamma(s, \rho_i)$, and $\gamma(s, \rho_i)$ is holomorphic unless ρ_i' is an unramified character. So let $J \subseteq \{1, \ldots, m\}$ denote the set of indices $j \in J$ where ρ_j' is an unramified character, and write γ_j for $\rho_j'(\varpi)$. Then $|\gamma_j| = 1$ by temperedness, and we may compute the γ-factor as

follows, using $F_1(s)$ and $F_2(s)$ to denote nonvanishing holomorphic functions of s. We have

$$\begin{aligned}
\gamma(s,\tau_v) &= F_1(s)\prod_{j\in J}\prod_{k=0}^{m_j-1}\gamma(s,\rho_j'\nu^{k-\frac{m_k-1}{2}}) = F_1(s)\prod_{j\in J}\prod_{k=0}^{m_j-1}\frac{L(1-s,\widetilde{\rho}_j'\nu^{-k+\frac{m_k-1}{2}})}{L(s,\rho_j'\nu^{k-\frac{m_k-1}{2}})}\\
&= F_1(s)\prod_{j\in J}\prod_{k=0}^{m_j-1}\frac{1-q_v^{-s-k+\frac{m_j-1}{2}}\gamma_j}{1-q_v^{-1+s+k-\frac{m_j-1}{2}}\gamma_j^{-1}}\\
&= F_1(s)\prod_{j\in J}\left(\frac{1-q_v^{-s-\frac{m_j-1}{2}}\gamma_j}{1-q_v^{-1+s-\frac{m_j-1}{2}}\gamma_j^{-1}}\prod_{k=1}^{m_j-1}\frac{1-q_v^{-s-k+1+\frac{m_j-1}{2}}\gamma_j}{1-q_v^{-1+s+k-\frac{m_j-1}{2}}\gamma_j^{-1}}\right)\\
&= F_1(s)\prod_{j\in J}\left(\frac{1-q_v^{-s-\frac{m_j-1}{2}}\gamma_j}{1-q_v^{-1+s-\frac{m_j-1}{2}}\gamma_j^{-1}}\prod_{k=1}^{m_j-1}\left(-\gamma_j q_v^{1-s-k+\frac{m_j-1}{2}}\right)\right)\\
&= F_2(s)\prod_{j\in J}\frac{1-q_v^{-s-\frac{m_j-1}{2}}\gamma_j}{1-q_v^{-1+s-\frac{m_j-1}{2}}\gamma_j^{-1}}.
\end{aligned}$$

Note that in the fourth equality, we pulled out the $k = m_j - 1$ term of the numerator and the $k = 0$ term of the denominator, shifting the indices appropriately. The roots of the numerator (with multiplicity) are $-\frac{m_j-1}{2}+\log_q\gamma_j$ for $j \in J$, and the roots of the denominator are $1+\frac{m_j-1}{2}+\log_q\gamma_j$. Using the equality $\gamma(s,\pi_v) = \gamma(s,\tau_v)$ and the quantity calculated in (2.7), we calculate

$$n = \sum_{j\in J}\left(\left(1+\frac{m_j-1}{2}\right)-\left(-\frac{m_j-1}{2}\right)\right) = \sum_{j\in J} m_j.$$

From this we immediately deduce that $J = \{1,\dots,m\}$, so that in fact τ_v is a tempered representation appearing as a subquotient inside an induction from a character of the Borel.

2.3.3 Possibilities for π_v

Using the calculations above, we observe that the roots and poles of $\gamma(s,\tau_v)$ completely determine all of the Satake parameters of π_v, since for each $j \in \{1,\dots,n\}$ and each factor

of the form

$$\frac{1 - q_v^{-s-\frac{m_j-1}{2}}\gamma_j}{1 - q_v^{-1+s-\frac{m_j-1}{2}}\gamma_j^{-1}},$$

each value (taken with multiplicities across different j)

$$s = -\frac{m_j - 1}{2} + \log_q \gamma_j, -\frac{m_j - 1}{2} + 1 + \log_q \gamma_j, \ldots, \frac{m_j - 1}{2} + \log_q \gamma_j$$

must appear as a root of the numerator of $\gamma(s, \pi_v)$.

If τ_v is a unitary principal series, we cannot have $\alpha_i = q_v \alpha_j$ for some $i \neq j$ (and so compatibility follows from Theorem 2.2.8. Otherwise, we have the following constraint on the Satake parameters α_i: the set

$$\left\{\alpha_1, \ldots, \alpha_m, \alpha_1^{-1}, \ldots, \alpha_m^{-1}\right\} \quad \text{or} \quad \left\{\alpha_1, \ldots, \alpha_m, 1, \alpha_1^{-1}, \ldots, \alpha_m^{-1}\right\}$$

(if n is even or odd, respectively) is equal to

$$\coprod_{j=1}^{m} \coprod_{i=0}^{m_j-1} \left\{q_v^{i-\frac{m_j-1}{2}}\gamma_j\right\}.$$

Observe that the inverses of $\coprod_{i=0}^{m_j-1} \left\{q_v^{i-\frac{m_j-1}{2}}\gamma_j\right\}$ are $\coprod_{i=0}^{m_j-1} \left\{q_v^{i-\frac{m_j-1}{2}}\gamma_j^{-1}\right\}$. No Satake parameters of the form $q_v^{i-\frac{m_j-1}{2}}\gamma_j$ can be inverse to one another unless the m_j's have the same parity, so we may consider them separately, grouping the extra Satake parameter 1 into the odd m_j case if n is odd.

In either case, each $\coprod_{i=0}^{m_j-1} \left\{q_v^{i-\frac{m_j-1}{2}}\gamma_j\right\}$ is either exactly equal to its set of inverses (if $\gamma_j = \pm 1$) or is disjoint from its set of inverses. Now fix a Satake parameter $\gamma \neq \pm 1$, $|\gamma| = 1$.

Claim 2.3.6. *We can pair up each j with $\gamma_j = \gamma$ with a j' with $\gamma_{j'} = \gamma^{-1}$ such that $m_{j'} = m_j$, using each $j'' \in \{1, \ldots, m\}$ with $\gamma_{j''} = \gamma^{\pm 1}$ exactly once.*

Proof. The key observation is that for j with $\gamma_j = \gamma$, if we consider the term of largest absolute value, we see that there must be a j' such that $\gamma_{j'} = \gamma_j^{-1}$ and $m_{j'} \geq m_j$.

Now note that by considering $\mathbf{C}/q_v^{\mathbf{Z}}$-equivalence classes, the number of Satake parameters of the form $q_v^{\frac{1}{2}+k}\gamma_j$ or $q_v^k\gamma_j$, $k \in \mathbf{Z}$ (depending on whether m_j is even or odd), is equal to the number of the form $q_v^{\frac{1}{2}+k}\gamma_j^{-1}$ or $q_v^k\gamma_j^{-1}$, respectively. Let j_1 be such that m_{j_1} is maximal among those j_1 with $\gamma_{j_1} = \gamma_j^{\pm 1}$. By the key observation and maximality, there exists j_1' with $m_{j_1'} = m_{j_1}$ and $\gamma_{j_1'} = \gamma_{j_1}^{-1}$. We now throw out these values and find $j_2, j_2', j_3, j_3', \ldots$ iteratively in this way, yielding the claim.

□

We may now complete the proof of the main result of the section.

Proof of Theorem 2.3.1. The classification of Satake parameters associated to j with $\gamma_j \neq \pm 1$ follows. Namely, these Satake parameters are the disjoint union of sets of the form

$$\coprod_{i=0}^{m'-1} \left\{ q_v^{i-\frac{m'-1}{2}}\gamma \right\} \cup \coprod_{i=0}^{m'-1} \left\{ q_v^{i-\frac{m'-1}{2}}\gamma^{-1} \right\},$$

again with even m'; the latter use two of the values j.

Now consider j such that $\gamma_j = \pm 1$. Note that each set of the form $\coprod_{i=0}^{m'} \left\{ q_v^{i-\frac{m'-1}{2}}\gamma \right\}$ with $\gamma = \pm 1$ for even m' has each element matched to a distinct inverse, and if m' is odd, each element is matched to a distinct inverse except for γ itself. We may thus include any number of sets of this form if m' is even. If m' is odd, and n is even, we require that there be an even number of sets of this form with $\gamma = 1$ and an even number with $\gamma = -1$. If n is odd, we require an odd number of sets of this form with $\gamma = 1$ and an even number with $\gamma = -1$.

□

2.4 Iwahori-Hecke algebras of quasi-split ramified unitary groups

In this section, we perform some calculations to identify the Iwahori-Hecke algebras of quasi-split ramified unitary groups with those of split groups. None of the results in this section are novel. Iwahori-Matsumoto and Bernstein have given different presentations of these algebras. Recently Rostami has generalized these constructions so that they are valid for any reductive group [62]. The identification with algebras for split groups is based on an observation of Lusztig [45, §10.13]. This technique was also used by Clozel and Thorne [17, 18] to study the same unitary groups we consider here in their proof of a case of Ihara's lemma, and it is from Jack Thorne that we learned of this identification. The purpose of our writing out these calculations is to make the identification precise, which will be needed in the next section.

In his survey of the theory of Bruhat-Tits buildings, Tits calculates the building of a quasi-split special unitary group of odd rank in great detail [81, §1.15, §2.10, §3.11]. We will write out the calculations to determine the apartment and affine Weyl group in one additional case (the even rank ramified unitary group) and quote the results of the calculations in split cases from the classification [81, §4]. Both of these cases are also treated in a paper of Pappas and Rapoport [52, §2.d]. We will also calculate the Kottwitz homomorphism for the split special orthogonal group. In the next three sections, we will recall the theory of the Kottwitz homomorphism following [37, §7] and of Iwahori-Hecke algebras following Rostami [62].

2.4.1 The Kottwitz homomorphism

Let $F/\mathbf{Q}_p$ be a finite extension and let G be a connected reductive group over F. We define the Kottwitz group Ω_G to be equal to $X^*(Z(\widehat{G})^I)^{\mathrm{Frob}}$, where $\widehat{G}$ is the complex reductive dual of G and I is the inertia group. The Kottwitz homomorphism is a surjective map

$\kappa_G : G(F) \to \Omega_G$. One defines this homomorphism by first passing to the completion $\widehat{F}^{\mathrm{ur}}$ of the maximal unramified extension of F; then the group is quasi-split over $\widehat{F}^{\mathrm{ur}}$ and becomes split over a totally ramified extension. In fact, one can alternatively use a finite unramified extension over which the group becomes split. One then defines the homomorphism for such quasi-split groups, and then takes invariants of the resulting map by Frobenius. We give this definition following Kottwitz [37, §7] and work out some key examples to be used later. In what follows, we always assume that G splits over a totally ramified extension.

Induced tori: If $G = T$ is a torus, there is an identification $X_*(T)_I$ with $X^*(\widehat{T}^I)$ using the restriction map $X_*(T) = X^*(\widehat{T}) \to X^*(\widehat{T}^I)$. If, moreover, $G = T = \mathrm{Res}_{E/F} \mathbf{G}_m$ where E/F is totally ramified, there is a natural map $T(F) \to X_*(T)_I$ given as follows. $X_*(T)$ is a free $\mathbf{Z}$-algebra with basis permuted by I, so $X_*(T)_I \cong \mathbf{Z}$. Then we simply send the element $e \in T(F) = E^\times$ to $\mathrm{ord}_\varpi e$, where ϖ is a uniformizer in E.

General tori: If $G = T$ is a torus, there is a presentation

$$R \to S \to T \to 1$$

of T where R and S are induced tori, meaning that they are of the form $\prod_i \mathrm{Res}_{E_i/F} \mathbf{G}_m^{n_i}$ for some choices of E_i and n. Then there is a unique way to choose $T(F) \to X_*(T)_I$ making the associated diagram of Kottwitz homomorphisms commute.

Simply connected derived subgroup: If G has a simply connected derived subgroup G^{der}, $T = G/G^{\mathrm{der}}$ is a torus and $X^*(Z(\widehat{G})^I) = X^*(\widehat{T}^I)$, so we define the Kottwitz homomorphism to factor through κ_T.

General case: We pick a z-extension

$$1 \to T \to \widetilde{G} \to G \to 1$$

with $\widetilde{G}^{\mathrm{der}}$ simply connected. (Such a sequence is a z-extension if T is an induced torus that is central in $\widetilde{G}$.) Then $\kappa_{\widetilde{G}}$ determines κ_G uniquely.

Example 2.4.1. Suppose E/F is a tamely ramified extension of degree 2, $G = U(1)$ is the unitary group associated to the form $J = (1)$, and ϖ is a uniformizer in E with $\varpi^2 \in F$. Then $G(F)$ consists of the norm 1 elements of E. There is an exact sequence

$$1 \to F^\times \to \mathrm{Res}_{E/F}\, E^\times \overset{\sigma(e)/e}{\to} U(1) \to 1,$$

where σ is the nontrivial element of $\mathrm{Gal}(E/F)$. Note that $X_*(U(1)) \cong \mathbf{Z}$ with σ acting by multiplication by -1, so $X_*(U(1))_I = \mathbf{Z}/2\mathbf{Z}$. We identify $X_*(F^\times) \cong \mathbf{Z}$ with trivial action and $X_*(\mathrm{Res}_{E/F}\, E^\times) \cong \mathbf{Z}^2$ with I interchanging the factors. Then the exact sequence of cocharacters is given by

$$X_*(F^\times) \overset{x\mapsto(x,x)}{\to} X_*(\mathrm{Res}_{E/F}\, E^\times) \overset{(x,y)\mapsto x-y}{\to} X_*(U(1)),$$

and upon taking coinvariants we find that the element $\sigma(e)/e$ maps to the nontrivial element of $\mathbf{Z}/2\mathbf{Z} \cong X_*(U(1))_I$ exactly when e has odd valuation in $E^\times$. For instance, taking $k = \varpi$, we find that $\kappa_{U(1)}(\sigma(\varpi)/\varpi) = \kappa_{U(1)}(-1)$ maps to $1 \in \mathbf{Z}/2\mathbf{Z}$. It follows that an element of $U(1)$ is sent to $1 \in \mathbf{Z}/2\mathbf{Z}$ exactly when it is -1 modulo ϖ.

Example 2.4.2. For either even or odd ramified quasi-split unitary groups $G = U(n,n)$ or $G = U(n,n+1)$, the determinant gives a map $G \to U(1)$ with simply connected semi-simple kernel. We deduce that the Kottwitz homomorphism factors through the determinant map and is given by the previous example.

Example 2.4.3. Let $F/\mathbf{Q}_p$ be a finite extension with uniformizer ϖ. Let $G = \mathrm{SO}_{2n+1}$ be a quasi-split orthogonal group of odd rank over F. Then G is semi-simple but not simply connected, so we need to find a z-extension of it. The dual group of G is Sp_{2n}, so it is

promising to consider the exact sequence

$$1 \to \mathrm{Sp}_{2n} \to \mathrm{GSp}_{2n} \xrightarrow{\mu} \mathbf{G}_m \to 1,$$

where μ is the similitude. Indeed, the dual exact sequence

$$1 \to \mathbf{G}_m \to \mathrm{GSpin}_{2n+1} \to \mathrm{SO}_{2n+1} \to 1$$

gives the needed z-extension. Since SO_{2n+1} is split, I acts trivially, so the Kottwitz group $X^*(Z(\widehat{\mathrm{SO}_{2n+1}}))$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. The map $\mathrm{GSpin}_{2n+1} \to \Omega_{\mathrm{GSpin}_{2n+1}}$ factors though the quotient $\mathrm{GSpin}_{2n+1}/\mathrm{Spin}_{2n+1} \cong \mathbf{G}_m$; this map is the spinor norm. Then the map $\Omega_{\mathrm{GSpin}_{2n+1}} \to \Omega_{\mathrm{SO}_{2n+1}}$, which must be surjective, can only be reduction modulo 2. So to calculate the image of an element of SO_{2n+1}, we pick any lift to GSpin_{2n+1}, compute the image under the spinor norm, and take ord_ϖ of the result, reducing modulo 2. More directly, we can use the spinor norm $\mathrm{SO}_{2n+1} \to F^\times/F^{\times 2}$ followed by ord_ϖ modulo 2. By definition, the spinor norm on SO_{2n+1} is the restriction of the unique map on the orthogonal group O_{2n+1} that sends reflection along a vector v to the element $\langle v, v\rangle$ modulo squares.

2.4.2 Extended affine Weyl groups

The notion of an extended affine Weyl group is somewhat subtle. There are several different definitions in the literature; the definitive reference on this topic is the aforementioned preprint of Rostami [62]; we give here a summary of his results.

For this section only, let G be any connected reductive group over F, let A be a maximal F-split torus, let $S \supseteq A$ be a maximal F^{ur}-split torus (where F^{ur} denotes the maximal unramified extension of F), let T be a maximal torus containing S and defined over F, and let $M = Z_G(A)$. We also fix an alcove $\mathbf{a}$ corresponding to the Iwahori subgroup I.

The main role of the extended affine Weyl group is to act as a system of representatives for $G\backslash I/G$. From this point of view, *the* extended affine Weyl group is the quotient $\widetilde{W} = N_G(S)(F)/T(F)_1$, where the notation $H(F)_1$ means to take the kernel of the Kottwitz homomorphism $H(F) \to \Omega_H$ [37, §7]. Rostami [62, Lemma 3.1.1] and Richarz [60, §1.2] showed that $N_G(S)(F)/T(F)_1$ is isomorphic to $N_G(A)(F)/M(F)_1$.

Tits [81, §1.2] defines the group $W' = N_G(A)(F)/M(F)^1$ instead (which he denotes $\widetilde{W}$), where $H(F)^1 \supseteq H(F)_1$ is the kernel of the Bruhat-Tits homomorphism $\nu_H : H(F) \to \mathrm{Hom}_{\mathrm{Gps}}(X^*(H)_F, \mathbf{Z})$. Here, $X^*(H)_F$ denotes characters defined over F and the map ν_H sends h to the map $\chi \mapsto -|\chi(h)|_F$. Then W' is naturally a quotient of $\widetilde{W}$ and the kernel is identified with $M(F)^1/M(F)_1$, which is precisely the torsion subgroup of Ω_M. There is also a root system Ψ naturally associated to G, to which Lusztig [44] associates another group $W(\Psi)$. Rostami shows [62, Lemma 3.4.1] that this is naturally isomorphic to W'.

Finally, there is the actual affine Weyl group W^{aff} associated to the affine Dynkin diagram of G, which is a subgroup of W' and of $\widetilde{W}$. As a subgroup, it is generated by the reflections in the walls of $\mathbf{a}$. Thus there are three different groups in play. The key point to keep in mind is that W^{aff} is the basic object that is being extended (non-canonically, as one must choose $\mathbf{a}$), and W' is as far as one can reach using the very natural construction of an extended affine Hecke algebra by Lusztig; for these algebras, the representation theory can be translated into that of a related graded algebra [43]. For some ramified groups, one must consider the larger group $\widetilde{W}$ to fully understand the representation theory at Iwahori level.

Example 2.4.4. In the case of a ramified unitary group of odd rank, one finds that $W' = W^{\mathrm{aff}}$ and $\widetilde{W}$ is an extension of W^{aff} by $\mathbf{Z}/2\mathbf{Z}$. We will check in Section 2.4.5 that in the case of a ramified unitary group of even rank, $\widetilde{W} = W'$ is an extension of W^{aff} by $\mathbf{Z}/2\mathbf{Z}$.

Remark 2.4.5. In our study of odd rank unitary groups later, we will restrict ourselves (as Clozel-Thorne do [18]) to the study of representations with vectors fixed under a special

maximal compact subgroup rather than a special maximal parahoric subgroup. As a consequence, we will be able to enlarge the Iwahori subgroup under consideration to one whose associated Weyl group is the ordinary affine Weyl group of the unitary group.

2.4.3 Iwahori-Hecke algebras

Given a reductive group G over a local field F with Iwahori subgroup I, one may ask what information about G determines the Hecke algebra of its Iwahori subgroup. Fix a maximal split F-torus A and a maximal F-torus S containing A split over $\widehat{F}^{\mathrm{ur}}$ as before, and assume that the alcove $\mathbf{a}$ corresponding to I is in the apartment of A. Rostami [62] shows that this data is

1. the local affine Dynkin diagram (ignoring orientations), which determines the Coxeter system $(W^{\mathrm{aff}}, \Sigma)$,
2. the Kottwitz group Ω_G together with its action on W^{aff}, and
3. the characteristic dimensions $d(v)$ attached to vertices v of the local affine Dynkin diagram.

The first two determine a presentation of the extended affine Weyl group as the external semi-direct product

$$\widetilde{W} \cong W^{\mathrm{aff}} \rtimes \Omega_G. \tag{2.8}$$

Given our choice of I, we can identify subgroups of $\widetilde{W}$ with W^{aff} and Ω_G. In particular, W^{aff} is embedded as the subgroup of $\widetilde{W}$ generated by the reflections across the walls of $\mathbf{a}$. The map $\widetilde{W} \to \Omega_G$ is induced by the Kottwitz homomorphism. The subgroup of $\widetilde{W}$ that fixes $\mathbf{a}$ gives us a splitting of this map.

Remark 2.4.6. The Σ in the pair $(W^{\mathrm{aff}}, \Sigma)$ can always be taken to be *reduced*, and we make this choice. To be more precise, Σ should be the reduced root system so that the set of root

hyperplanes of the affine root system Φ^{aff} of G are the same as the set of root hyperplanes of the functions in $\alpha\Sigma + \mathbf{Z}$ for some scalar α. The relative root system of G with respect to A has roots that are proportional to the roots in Σ, but may be a different root system (and can be non-reduced). In fact, this issue will occur for the unitary groups we consider below.

One presentation of the Iwahori-Hecke algebra is based on determining the relations between a set of generators corresponding to (2.8). Let $\ell : \widetilde{W} \to \mathbf{N}$ denote the trivial extension of the length function of W^{aff}. Also let $q : \widetilde{W} \to \mathbf{N}$ be defined by $q(w) = [IwI : I]$ and write Δ^{aff} for the simple reflections. The Iwahori-Matsumoto presentation [62, §1] has for a basis elements T_w for $w \in \widetilde{W}$ and relations

1. $T_w T_{w'} = T_{ww'}$ if $\ell(w) + \ell(w') = \ell(ww')$,

2. $T_s T_s = (q(s) - 1)T_s + q(s)T_1$ for $s \in \Delta^{\text{aff}}$, and

3. $T_w T_\tau = T_{w\tau} = T_\tau T_{\tau^{-1}w\tau}$ for all $w \in \widetilde{W}$ and $\tau \in \Omega_G$.

Note that the $q(s)$ are determined by the numbers $d(v)$.

Another presentation of the extended affine Weyl group is given by the external semi-direct product

$$\widetilde{W} = \Omega_M \rtimes W^\circ, \tag{2.9}$$

where M is the centralizer of the maximal F-split torus and W° is the finite rational Weyl group of G. We write $\Delta^\circ \subseteq \Delta^{\text{aff}}$ for the Coxeter generating set of W°. The map $\Omega_M \to \widetilde{W}$ is induced by the natural inclusion

$$N_M(S)(F)/T(F)_1 \hookrightarrow N_G(S)(F)/T(F)_1.$$

The map $\widetilde{W} \to W^\circ$ is obtained by taking the vector part of the action on the apartment of S, but it will be useful to choose an explicit splitting. This splitting will depend on the

choice of a special maximal parahoric subgroup K associated to a vertex of $\mathbf{a}$. Then an embedding of W° in $\widetilde{W}$ is given by $(N_G(S)(F) \cap K)/T(F)_1 \subseteq \widetilde{W}$; that the vector part maps isomorphically to W° is proved in [26, Lemma 5.0.1]. In the following, we refer to (2.9) as the Bernstein presentation of $\widetilde{W}$.

The Bernstein presentation of the Iwahori-Hecke algebra, based on (2.9), has basis elements $\Theta_\lambda T_w$ for $\lambda \in \Omega_M$ and $w \in W^\circ$. Here Θ_λ is defined by representing λ as a difference $\lambda_a - \lambda_b$ of elements λ_a, λ_b in the positive Weyl chamber and setting $\theta_\lambda = [I\lambda_a I][I\lambda_b I]^{-1}$ (cf. the discussion in Section 2.5.1). We refer to [62] for the details and the relations in general, but we explain them in a special case. In general, we have $\Theta_\lambda \Theta_\mu = \Theta_{\lambda+\mu}$ for $\lambda, \mu \in \Omega_M$. In the cases we consider below, all of the numbers $d(v)$ will be equal to 1. If this is the case, the remaining relation takes the form

$$T_s \Theta_\lambda = \Theta_{s(\lambda)} T_s + \sum_{j=0}^{\langle \alpha(s), \lambda \rangle_{\mathbf{R}} - 1} (q(s) - 1) \Theta_{\lambda - j\alpha(s)^\vee} \tag{2.10}$$

for $s \in \Delta^\circ$ and $\lambda \in \Omega_M$, where $\alpha(s)$ denotes the root of Σ corresponding to s.

2.4.4 Ramified unitary groups in odd dimension and symplectic groups

This section is brief for two reasons: the discussion in [18] is nearly sufficient for our purposes and the main result here is substantially simpler in this case than in the even case because we will not consider the full Iwahori-Hecke algebra, but instead a subalgebra built from a normal subgroup W^{aff} of $\widetilde{W}$, which has of index 2. Accordingly, we will be interested in K-spherical representations where K is a special maximal compact subgroup rather than a special maximal parahoric.

Let F/F^+ be a ramified extension of local fields of residue characteristic other than 2, and let ϖ be a uniformizer of F such that $\varpi^2 \in F^+$. Denote the residue field by κ_{F^+}. Following

Clozel-Thorne, we will let J be the $(2m+1)\times(2m+1)$-dimensional Hermitian form

$$J = \begin{pmatrix} & & & & -1 \\ & & & 1 & \\ & & \iddots & & \\ & -1 & & & \\ 1 & & & & \\ -1 & & & & \end{pmatrix} \tag{2.11}$$

and write $U = U(J)$ for the unitary group. We assume $m \geq 1$.

In this section, we prove the following result.

Proposition 2.4.7. *The Iwahori-Hecke algebra of U is isomorphic to the Iwahori-Hecke algebra of a split symplectic group over F acting on a space of dimension $2m$. Moreover, we may explicitly identify the Bernstein presentations by sending*

$$[Iu_iI] = [I\,\mathrm{diag}(1,\dots,1,\underbrace{\varpi}_{-i},1,\dots,1,\underbrace{\frac{1}{\varpi}}_{i},1\dots,1)I]$$

to

$$[Iu_{i,\mathrm{Sp}}I] = [I\,\mathrm{diag}(1,\dots,1,\underbrace{\varpi}_{-i},1,\dots,1,\underbrace{\frac{1}{\varpi}}_{i},1\dots,1)I]$$

for $i \in \{1,\dots,m\}$ and identifying the rational Weyl groups compatibly.

Remark 2.4.8. In order to identify a representation of U (or Sp_{2m}) with a representation of the Bernstein presentation of the Iwahori-Hecke algebra we need to choose a splitting of the presentation of the extended affine Weyl group as described in Section 2.4.3, which corresponds to a choice of special maximal parahoric subgroup of U. Whenever we apply Proposition 2.4.7 or the later Proposition 2.4.10 in the even rank case to identify a repre-

sentation of U with one of a split group, we will do so with respect to the special maximal parahorics given by the integral matrices with respect to the various forms J, J_{Sp}, or J_{SO} defined in Section 2.4.4 and Section 2.4.5, except in the odd rank unitary group case of this section, in which we need to consider the index 2 subgroup K' of the group of integral matrices described below.

We write $e_{-m}, \ldots, e_m$ for the corresponding basis of the Hermitian space so that the pairing is given by $\langle e_i, e_{-i} \rangle = (-1)^{i+n+1}$. Then a maximal F-split torus is given by

$$S(F) = \{\mathrm{diag}(s_{-n}, \ldots, s_{-1}, s_0, s_1, \ldots, s_n) | s_i \in F, s_i s_{-i} = 1, s_0 = 1\}.$$

Let $T \supset S$ be the diagonal maximal torus. The full Iwahori-Weyl group has a presentation of the form $\Omega_T \rtimes W^\circ$. Here, one can show that $\Omega_T \cong \mathbf{Z}^m \times \mathbf{Z}/2\mathbf{Z}$; the second factor comes from the subgroup $U(1) \subseteq T$ given by matrices of the form $\mathrm{diag}(1, \ldots, 1, t_0, 1, \ldots, 1)$. The integral points of U give a special maximal compact subgroup K, whose reduction modulo ϖ lands in an orthogonal group $\mathrm{O}(\kappa_{F^+})$. Since O is disconnected, the usual Borel subgroup (i.e. maximal connected solvable subgroup) consists of upper triangular matrices *of determinant 1*, and accordingly, there is a special maximal parahoric subgroup $K' \subseteq K$ of index 2 that is given by the preimage of $\mathrm{SO}(\kappa_{F^+}) \subseteq \mathrm{O}(\kappa_{F^+})$. In the definition of the affine Weyl group, we replace the true Iwahori subgroup I by the preimage I' of the full group of upper triangular matrices in $\mathrm{O}(\kappa_{F^+})$, which contains I as a subgroup of index 2. Then the modified Weyl group $\widetilde{W}'$ used to define the Iwahori-Hecke algebra of I' is given by $\Omega'_T \rtimes W^\circ$, where Ω'_T is the group generated by the elements

$$u_i = \mathrm{diag}(1, \ldots, 1, \underbrace{\varpi}_{-i}, 1, \ldots, 1, \underbrace{\frac{1}{\varpi}}_{i}, 1 \ldots, 1).$$

We would like to give an explicit splitting of W° into this group by restricting the one

for the usual Iwahori-Weyl group $\widetilde{W}$, following Section 2.4.3. In particular, we consider $(N_G(S)(F) \cap K')/T(F)_1$, which realizes $\sigma \in W^\circ \cong S_n \rtimes (\mathbf{Z}/2\mathbf{Z})^n$ as the usual permutation matrices on $\{-n, \dots, n\}$ but multiplied by the scalar $\mathrm{sign}(\sigma)$ to fix the determinant. We now observe that the Kottwitz homomorphism (which is defined as the determinant modulo ϖ) is trivial on both W° and on u_i – this is why using I' in place of I gives us an ordinary affine Weyl group.

We can identify the affine Weyl group of U with that of a symplectic group G of rank $2n$ over F. Let us use the form $J_{\mathrm{Sp}} = \begin{pmatrix} & A_m \\ -A_m & \end{pmatrix}$, where A_m is as in (2.3), for the symplectic group. Note that $\Omega_G = 1$. The diagonal maximal torus T_{Sp} has $\Omega_{T_{\mathrm{Sp}}}$ a free $\mathbf{Z}$-algebra generated by the matrices

$$u_{i,\mathrm{Sp}} = \mathrm{diag}(1, \dots, 1, \underbrace{\varpi}_{-i}, 1, \dots, 1, \underbrace{\frac{1}{\varpi}}_{i}, 1 \dots, 1),$$

and the isomorphism between W^{aff} for the unitary group and the presentation $\Omega_{T_{\mathrm{Sp}}} \rtimes W^\circ_{\mathrm{Sp}}$ of the symplectic affine Weyl group $W^{\mathrm{aff}}_{\mathrm{Sp}}$ is the obvious one.

One can check (by examining the extra affine root in each case) that the associated Coxeter systems $(W^{\mathrm{aff}}, \Sigma)$ and $(W^{\mathrm{aff}}_{\mathrm{Sp}}, \Sigma_{\mathrm{Sp}})$ are the same, as well as the numbers $d(v)$, which in this case are all 1 for both groups as they are residually split. We summarize the result of the calculation; a more detailed discussion can be found in [81, §1.15], [18], or [53, §2.d.2]. Let S be the maximal split torus consisting of $\mathrm{diag}(s_{-n}, \dots, s_n)$ with $s_{-n}s_n = 1$, $s_0 = 1$, and each $s_i \in F^+$. We define generators

$$a_i(t) = \mathrm{diag}(1, \dots, 1, \underbrace{t}_{-i}, 1, \dots, 1, \underbrace{t^{-1}}_{i}, 1, \dots, 1)$$

of the group $X_*(S)$ and let $\{b_i\}$ be the dual basis to $\{a_i\}$. Then the affine root system of

the unitary group is given by

$$\Phi^{\text{aff}} = \left\{ \pm b_i \pm b_j + \frac{1}{2}\mathbf{Z} \right\}_{i \neq j} \cup \left\{ \pm b_i + \frac{1}{2}\mathbf{Z} \right\} \cup \left\{ \pm 2b_i + \frac{1}{2} + \mathbf{Z} \right\}. \tag{2.12}$$

By rescaling of the first and third set of affine roots by 2 and the second by 4, this has the same set of root hyperplanes as $\{\pm 2b_i \pm 2b_j + \mathbf{Z}\}_{i \neq j} \cup \{\pm 4b_i \pm \mathbf{Z}\}$. But setting Σ to be the root system C_m, this is $2\Sigma + \mathbf{Z}$. We see directly that this identifies the Bernstein presentations by noting that (2.10) depends on Σ and not the rational root system.

Remark 2.4.9. It is interesting to see what is going on at the level of affine Dynkin diagrams. If G is a ramified special unitary group of odd rank $2m + 1 \geq 5$, then the affine Dynkin diagram associated to G is

$$\underset{\alpha_1}{\circ} \Leftarrow \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_m}{\circ} \Leftarrow \underset{\alpha_{m+1}}{\circ} \tag{2.13}$$

and the split form of Sp_{2m} has a similar-looking diagram

$$\underset{\alpha_1}{\circ} \Rightarrow \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_m}{\circ} \Leftarrow \underset{\alpha_{m+1}}{\circ}. \tag{2.14}$$

The rescaling of roots to obtain Σ for the unitary group corresponds exactly to the switching of the orientation of the leftmost arrow; the root on the left in each diagram corresponds to the extra affine root. This can be seen in Tits's calculation of a basis for the root system of G [81, §1.15]. This basis is $\left\{a_1, a_2 - a_1, \ldots, a_m - a_{m-1}, 2a_m + \frac{1}{2}\right\}$, where the ordering aligns with the numbering of diagram (2.13). Then if we halve the aforementioned rescaling of (2.12), this corresponds exactly to rescaling a_1 by a factor of 2, which explains the orientation change.

2.4.5 Ramified quasi-split unitary groups in even dimension and odd special orthogonal groups

We study the case of a ramified quasi-split group in even dimension $2n$. A discussion of this case is already available in [17, §2.1], though our primary goal here is to identify the Bernstein presentation of the Weyl group slightly more explicitly than in that paper, while also giving an exposition of some of the notions discussed in earlier sections. As before, let F/F^+ be a ramified extension of local fields, and let ϖ be a uniformizer of F such that $\varpi^2 \in F^+$. Let U be the unitary group preserving the Hermitian form defined by

$$J = \begin{pmatrix} & & & & \varpi \\ & & & \iddots & \\ & & \varpi & & \\ & -\varpi & & & \\ \iddots & & & & \\ -\varpi & & & & \end{pmatrix} = \begin{pmatrix} & \varpi A_m \\ -\varpi A_m & \end{pmatrix}$$

where A_m is as in (2.3). Note that this is different than the matrix $J_{m,m}$ defined in (2.2); we will explain the reason for this choice shortly. Note that $J = CJ_{m,m}{}^t\overline{C}$ where $C = \mathrm{diag}(\varpi, \dots, \varpi, 1, \dots, 1)$. In particular, an isomorphism from the unitary group $U(J'_{m,m})$ to U is given by the map $u \mapsto C^{-1}uC$. Since C commutes with the diagonal matrices, the maximal torus remains unchanged, and thus the entire discussion of Sections 2.2 and 2.3 as well. Also note that U is also the group stabilizing the anti-Hermitian form

$$J' = \begin{pmatrix} & A_m \\ -A_m & \end{pmatrix}.$$

Then the goal of this section is to prove the following result.

Proposition 2.4.10. *There is an isomorphism between the Iwahori-Hecke algebra of U and the Iwahori-Hecke algebra of the split odd special orthogonal group over F acting on a space of dimension $2m+1$. Moreover, this isomorphism identifies the operator*

$$[Iu_iI] = [I\,\mathrm{diag}(1,\dots,1,\underbrace{\varpi}_{-i},1,\dots,1,\underbrace{-\frac{1}{\varpi}}_{i},1\dots,1)]$$

with

$$[Iu_{i,\mathrm{SO}}I] = [I\,\mathrm{diag}(1,\dots,1,\underbrace{\varpi}_{-i},1,\dots,1,\underbrace{\frac{1}{\varpi}}_{i},1\dots,1)I]$$

for $i \in \{1,\dots,m\}$ and identifies the rational Weyl groups correspondingly.

We will describe the apartment and local Dynkin diagram for U. We write H for the underlying anti-Hermitian space, with ordered basis $e_{-m},\dots,e_{-1},e_1,\dots,e_m$. Then $\langle e_i,e_j\rangle = \mathrm{sign}(j)\delta_{i,-j}$, where $\delta_{\cdot,\cdot}$ denotes the Kronecker δ function and $\mathrm{sign}(j) = 1$ if $j > 0$ and $\mathrm{sign}(j) = -1$ if $j < 0$.

A maximal split torus is given by

$$S(F) = \left\{\mathrm{diag}(s_{-m},\dots,s_{-1},s_1,\dots,s_m) | s_i \in F^+ \text{ and } s_is_{-i} = 1\right\}.$$

(In this case $A = S$ in the notation of Section 2.4.2.) Moreover, the centralizer of S is a maximal torus T, given by the diagonal matrices in U. For $i \in \{1,\dots,m\}$, let $a_i : \mathbf{G}_m \to S$ be defined by

$$a_i(t) = \mathrm{diag}(1,\dots,1,\underbrace{t}_{-i},1,\dots,1,\underbrace{t^{-1}}_{i},1,\dots,1).$$

A basis for $X_*(S)$ is then given by the a_i. We write b_i for the dual basis. Moreover, we write $b_{-i} = -b_i$ for $i \in \{1,\dots,m\}$. Then a relative root system for U is given by

$$\Phi = \{b_i + b_j\}_{i,j\in\{-m,\dots,-1,1,\dots,m\},i\neq -j}.$$

The basis associated to the upper-triangular Borel consists of the roots $b_{-i} + b_{-i+1}$ for $i \in \{1, \ldots, m-1\}$ together with $2b_{-1}$. For each of these roots, we define the associated space in $U(F^+)$ as follows. We write these as a map u_α of either F^+ or F into $U(F^+)$ for $\alpha \in \Phi$. First assume $\alpha = b_i + b_j$ with $i \neq \pm j$. In this case, define $u_\alpha : F \to U(F^+)$ by $u_\alpha(f) = \mathbf{1}_{2m} + (\overline{f})_{-j,i} + (f)_{-i,j}$, where $(\cdot)_{ij}$ denotes the matrix with $\cdot$ in entry ij and 0 elsewhere, using the modified labeling of the basis in terms of e_i, $i \in \{-m, \ldots, -1, 1, \ldots, m\}$ as above. If $\alpha = 2b_i$, then we define $u_\alpha : F^+ \to U(F^+)$ by $u_\alpha(f) = \mathbf{1}_{2m} + (f)_{-i,i}$.

The extended affine Weyl group $\widetilde{W}$ of U is the quotient $N(F)/T(F)_1$, where T is the diagonal maximal torus and $T(F)_1$ is as defined in Section 2.4.2. In fact, $T(F)_1 = T(F)^1$ in this case since T is induced. In particular, the group W' of Section 2.4.2 is also equal to $\widetilde{W}$. We would like to compute the Bernstein presentation of $\widetilde{W}$ explicitly. Before that, we remark that the group K of matrices with integral entries form a special maximal compact subgroup (which is also a special maximal parahoric) and that an Iwahori subgroup I is given by those matrices that are upper triangular modulo ϖ. The reduction modulo ϖ lands in a symplectic group over the residue field κ_F of F; this is easy to see by thinking of U as $U(J')$. This convenient definition is the reason for our choice of the form J' above rather than $J_{m,m}$.

The group Ω_T is a free $\mathbf{Z}$-algebra of rank m. The embedding into $\widetilde{W}$ mentioned in Section 2.4.3 is generated by the matrices

$$u_i = \mathrm{diag}(1, \ldots, 1, \underbrace{\varpi}_{-i}, 1, \ldots, 1, \underbrace{-\frac{1}{\varpi}}_{i}, 1 \ldots, 1)$$

for $i \in \{1, \ldots, m\}$. Corresponding to our choice of K, we obtain a splitting of the Bernstein presentation of $\widetilde{W}$ as in Section 2.4.3. The embedding of W° associated to K is via the usual permutation matrices representing the Weyl group $S_m \rtimes (\mathbf{Z}/2\mathbf{Z})^m$, except with certain signs changed to ensure they are unitary. To be precise, the reflections associated to the roots $b_{-i} +$

b_{-i+1}, $i \in \{1, \ldots, m-1\}$, have positive signs on all entries, while the reflection associated to $2b_{-1}$ has a -1 on the upper off-diagonal entry; the other matrices are determined by these.

Our choice of Iwahori earlier also gives a splitting of $\widetilde{W} \to \Omega_U$. In the Bernstein presentation just described, the image of the nontrivial element of Ω_U under the splitting is given by $(-u_m, w_m) \in \Omega_T \rtimes W^\circ$, where w_m is the element of the Weyl group that switches the sign of u_m and fixes everything else. One can check that this element stabilizes the Iwahori subgroup; in carrying out this calculation, it is important to notice that the lower left corner entry of a matrix in the Iwahori subgroup lies in $F \cap \varpi \mathcal{O}_K = \varpi^2 \mathcal{O}_F$. We can also describe the Kottwitz homomorphism $\Omega_T \rtimes W^\circ \to \Omega_U$, using the chosen splitting of W° to see this as an internal semi-direct product decomposition of $\widetilde{W}$. The Kottwitz homomorphism is simply the determinant modulo ϖ, valued in ± 1. Then every u_i maps to -1, while all of W° maps to 1, as can be seen from the description of the matrix representatives above.

We now identify the Iwahori-Hecke algebra of U with that of the split orthogonal group $G = \mathrm{SO}_{2m+1}/F$; we use the form $J_{\mathrm{SO}} = A_{2m+1}$ to define the latter group. Let T_{SO} be the maximal torus and define

$$u_{i,\mathrm{SO}} = \mathrm{diag}(1, \ldots, 1, \underbrace{\varpi}_{-i}, 1, \ldots, 1, \underbrace{\frac{1}{\varpi}}_{i}, 1 \ldots, 1),$$

where the basis of the underlying orthogonal space is indexed by $e_{-m}, \ldots, e_m$. Then we get an obvious isomorphism between $\Omega_{T_{\mathrm{SO}}} \rtimes W^\circ_{\mathrm{SO}}$ and $\Omega_T \rtimes W^\circ$. We also note that the matrices used for the even unitary group on $e_{-m}, \ldots, e_{-1}, e_1, \ldots, e_m$, but enlarged to act trivially on e_0, are the representatives for the splitting of W° associated to the hyperspecial maximal parahoric K given by the integral matrices. Let I be the preimage of the upper-triangular Borel in $\mathrm{SO}(\kappa_F)$.

In terms of $\Omega_T \rtimes W^\circ$, we calculate the Kottwitz homomorphism and the splitting associated to the Iwahori I. The subgroup W° automatically maps to 1 under the spinor norm,

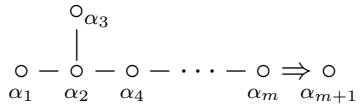
since the Kottwitz homomorphism is trivial on parahoric subgroups. (See Definition 1 of the appendix to [52].) By definition, the spinor homomorphism maps the reflection ω_i through the vector $pe_{-i} - e_i$ to $-p$ for any i. This reflection sends

$$\sum_j f_j e_j \mapsto \sum_j f_j e_j - 2\frac{\left\langle \sum_j f_j e_j, pe_{-i} - e_i \right\rangle}{\langle pe_{-i} - e_i, pe_{-i} - e_i \rangle}(pe_{-i} - e_i)$$
$$= \sum_j f_j e_j + \frac{pf_i - f_{-i}}{p}(pe_{-i} - e_i) = -\frac{1}{p} f_{-i}e_i + pf_i e_{-i} + \sum_{j \neq \pm i} f_j e_j.$$

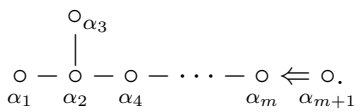
The spinor norm of the reflection through $e_{-i} - e_i$ is -1. Composing, we deduce that the Kottwitz homomorphism sends u_i to -1 for all i, which matches the description for the even unitary group.

Moreover, we can check that the choice of Iwahori I gives the same splitting of the Kottwitz homomorphism. Indeed, if one conjugates by the matrix $(-u_{m,\mathrm{SO}}, w_{m,\mathrm{SO}})$, where $w_{m,\mathrm{SO}}$ switches the sign of $u_{m,\mathrm{SO}}$, one can check that I is sent to itself. The key observation is that for a matrix $(a_{ij}) \in I$ to be orthogonal, $a_{m,m}a_{m,-m} + a_{m,m-1}a_{m,-m+1} + \cdots + a_{m,-m}a_{m,m} = 0$. By the Iwahori condition, we deduce that $2a_{m,m}a_{m,-m} \equiv 0 (\mathrm{mod}\ \pi^2)$. Since $a_{m,m}$ is invertible and p is odd, $\pi^2 | a_{m,-m}$.

As in the odd rank case, after consulting the classification of affine diagrams [81] and checking the remaining simple affine root and its pairings with the other simple roots agree up to the orientation of an edge, one deduces that the Iwahori-Hecke algebras are isomorphic. In particular, at least when $m \geq 3$, the affine diagram for the unitary group is



while SO_{2m+1} has the diagram



We also describe the calculation of Σ for any m. The affine root system of the unitary group is given by

$$\left\{\pm b_i \pm b_j + \frac{1}{2}\mathbf{Z}\right\}_{i\neq j} \cup \{\pm 2b_i + \mathbf{Z}\}.$$

(See [52, §2.d.1].) If we rescale this by multiplying the first set by 2 we obtain

$$\{\pm 2b_i \pm 2b_j + \mathbf{Z}\}_{i\neq j} \cup \{\pm 2b_i + \mathbf{Z}\}.$$

This is precisely $2\Sigma + \mathbf{Z}$, where $\Sigma = B_m$.

2.4.6 Similitude groups and restriction

We would like to understand the relation between the unitary groups considered above and the unitary similitude groups described in Section 2.2.1.

Let $E/\mathbf{Q}_p$ be a ramified quadratic extension, and let $F_i^+/\mathbf{Q}_p$ be an unramified field extension for each i. We set $F_i = F_i^+ E$ and form a Hermitian space H_i of dimension m for each i using the form in Section 2.4.4 or 2.4.5. We then consider the unitary similitude group G of $\bigoplus_i H_i$ over $\mathbf{Q}_p$ with similitude lying in $\mathbf{Q}_p^\times$; an element $g \in G$ is a tuple of matrices g_i that stabilize the form on H_i up to the same similitude $\mu(g)$.

We first compute Ω_G. We observe that Ω_G is the same as Ω_U, where U is the quotient of G by its simply-connected derived subgroup. This subgroup is cut out by the intersection of two conditions, namely that the determinant (which is the natural map $G \mapsto \prod_i \mathrm{Res}^{F_i}_{\mathbf{Q}_p} \mathbf{G}_m$) and similitude are both trivial. Thus U is a subgroup of $\mathbf{G}_m \times \prod_i \mathrm{Res}^{F_i}_{\mathbf{Q}_p} \mathbf{G}_m$. Recall that

by the definition of the similitude group, g_i and $\overline{g}_i$ are related by $g_i J^t \overline{g}_i = \mu(g) J$. It follows that

$$\det g_i \cdot \det \overline{g}_i = \mu(g)^n \tag{2.15}$$

for all i, where n is the dimension of the Hermitian form.

Proposition 2.4.11. *The condition (2.15) defines U.*

Proof. It remains only to check that G surjects onto U. Suppose first that n is even. Given a choice of $\mu(g) \in \mathbf{Q}_p$ and $\det g \in \prod_i \mathrm{Res}^{F_i}_{\mathbf{Q}_p} \mathbf{G}_m$ satisfying (2.15), we observe first that $\mathrm{Nm}^{F_i}_{F_i^+}((\det g_i) \cdot \mu(g)^{-\frac{n}{2}}) = 1$. By Hilbert's Theorem 90 it can be written in the form $\alpha_i^{-1}\overline{\alpha}_i$ for some $\alpha_i \in F_i$. Then the element $g = (g_i)_i$ with

$$g_i = \mathrm{diag}(\mu(g)\alpha_i^{-1}, \underbrace{\mu(g), \ldots, \mu(g)}_{\frac{n}{2}-1}, \underbrace{1, \ldots, 1}_{\frac{n}{2}-1}, \overline{\alpha}_i)$$

lies in G and has determinant $\det g$ and similitude $\mu(g)$.

If n is odd, we instead use the matrix

$$\mathrm{diag}(\underbrace{\mu(g), \ldots, \mu(g)}_{\frac{n-1}{2}}, \mu(g)^{\frac{1-n}{2}} \det(g_i), \underbrace{1, \ldots, 1}_{\frac{n-1}{2}}).$$

□

We define an isomorphism $U \to \mathbf{G}_m \times \prod_i \mathrm{Res}^{F_i}_{\mathbf{Q}_p} U(1)$ by sending the element $(x, (y_i)) \in \mathbf{G}_m \times \prod_i \mathrm{Res}^{F_i}_{\mathbf{Q}_p} \mathbf{G}_m$ to $(x, (y_i x^{-m}))$. It follows that for unitary similitude groups of either odd or even rank, the Kottwitz group Ω_G is of the form $\mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z})^{\{F_i^+\}}$, and the Kottwitz homomorphism is the product of $\mathrm{ord}_p \mu(\cdot)$ with the sign of $\det(\cdot) \cdot \mu(\cdot)^{-m} \bmod \varpi$. These are compatible with the previously calculated Kottwitz homomorphisms for unitary groups, in line with the known functoriality of this construction.

It follows immediately from this and the fact that the affine diagrams only depend on the derived subgroup that if we use the product of the usual Iwahori-Matsumoto presentations of the Hecke algebras of $U(H_i)$, there is a very natural embedding of this product into the Iwahori-Hecke algebra of G corresponding to just looking at the subalgebra generated by elements whose Ω_G component is 0 in the $\mathbf{Z}$ factor. Moreover, if we take the quotient of the Hecke algebra in the odd rank case by all of the $\mathbf{Z}/2\mathbf{Z}$-factors of Ω_G, we obtain a natural embedding for the Hecke algebra corresponding to the replacement I' for the Iwahori chosen in Section 2.4.4. We set up some notation that will be used in subsequent sections.

Definition 2.4.12. We define $u_{i,j}$ to be the element of G that is equal to the matrix u_j of Section 2.4.4 or Section 2.4.5 for the factor over F_i^+, and for every other place i', $u_{i',j}$ is the identity matrix in that factor.

Recall that ϖ is a uniformizer in E with $\varpi^2 \in \mathbf{Q}_p$, and that ϖ was used in the definition of u_m above. We write $\varpi\mathbf{1}_n$ to denote the element of the unitary similitude group whose F_i factor is $\varpi\mathbf{1}_n$ for *all* i.

Since the unramified representations of the Hecke algebras under consideration are finite-dimensional representations on $\mathbf{C}$-vector spaces, it follows from Schur's lemma that the action of the central subalgebra generated by $\mathbf{Z} \subseteq \Omega_G$ is via a multiplicative character, and thus the restriction of an irreducible representation of Iwahori level of the similitude group to the unitary group remains irreducible, and the exact data lost in this restriction is captured by the scalar action of a Hecke operator corresponding to a generator of $\mathbf{Z} \subseteq \Omega_G$. We can use the operator $I\varpi\mathbf{1}_nI$ for this purpose. Moreover, since $I\varpi\mathbf{1}_nI = I\varpi\mathbf{1}_n$, the action of this Hecke operator is in fact the image of $\varpi\mathbf{1}_n$ under the central character of the representation. Since the transfers described in Section 2.2.2 are compatible with central characters at all places, establishing the local Langlands correspondence at primes over p requires only understanding the restriction to the ordinary unitary group, allowing us to utilize, for instance, Theorem 2.2.8 in this endeavor.

2.5 Weight structure of almost unramified representations of p-adic unitary groups

We study principal series of unitary groups corresponding to the sets of Satake parameters appearing in Theorem 2.3.1. The results of this section are simply calculations of special cases of general theorems of Reeder on the structure of unramified principal series of split groups [58]. In fact, the case we will need later is essentially the most trivial case of Reeder's results. We briefly summarize Reeder's work as it pertains here in Section 2.5.1, and perform the calculations using the classification determined in Section 2.3 in Section 2.5.2.

The main theorem of this section is as follows.

Theorem 2.5.1. *Let π and τ be as in the statement of Theorem 2.2.8. Moreover, assume that π_v is K-spherical for the group K defined either in Section 2.4.4 or 2.4.5. Let $\{\alpha_i\}_{i\in I}$ be the multiset of Satake parameters of π_v, which must be of the form described in Theorem 2.3.1. Assume that $\alpha_i = q_v\alpha_j$ for some $i, j \in I$. In the odd rank $n = 2m+1$ case, let I be the subgroup called I' in Section 2.4.4, and in the even rank $n = 2m$ case, let I be the Iwahori subgroup chosen in Section 2.4.5. Then there exists a vector $\xi \in \pi_v$ that is an eigenvector for the weight space and upon which the action of $[Iu_mI]$ is by multiplication by a number with complex absolute value strictly greater than $q^{\frac{n-1}{2}}$, where u_m is the element of the affine Weyl group defined in Section 2.4.4 or 2.4.5.*

In fact, Reeder's results allow one to give a complete description of all the eigenvectors for the weight space and their eigenvalues.

2.5.1 Weight spaces of unramified representations of split groups

We summarize work of Reeder giving a criterion for non-vanishing of weight spaces for an unramified principal series representation of a split group [58, §1-5]. We warn that while

Reeder uses $\widetilde{\cdot}$ for the p-adic group and ordinary letters for the dual groups, we largely maintain the notation used in earlier sections.

Let F be a non-archimedean field with uniformizer ϖ, ring of integers $\mathfrak{O}_F$, and residue field κ_F of order q. Let G be a split connected reductive group over $\mathfrak{O}_F$ and let $\widehat{G}$ be its dual group over $\mathbf{C}$. Write K for the hyperspecial maximal compact $G(\mathfrak{O}_F)$. Fix an F-split maximal torus S and Borel subgroup B of G, and write N for the unipotent radical of B (all over $\mathfrak{O}_F$). Also write δ for the modulus character of B and I for the Iwahori subgroup given by the preimage of $B(\kappa_F)$ in K. Write W for the Weyl group $N_G(S)/C_G(S)$. We write $S_0 = S(\mathfrak{O}_K)$. Then $\Omega_S = S(F)/S_0$ is called the weight space of G, and we write λ_s for the coset of $s \in S(F)$ in Ω_S. (This notation is consistent with the discussion of the Kottwitz homomorphism in Section 2.4.1.)

Let $\widehat{S} \cong \mathbf{C}^\times \otimes X^*(S)$ be the group of characters $\chi : \Omega_S \to \mathbf{C}^\times$. Write $\widehat{G}$, $\widehat{B}$, and $\widehat{N}$ for the dual group, dual Borel, and unipotent radical of $\widehat{B}$, respectively. We also write $\widehat{\Phi}$ for the root system of $\widehat{G}$ and $\widehat{\Phi}^+$ for the positive roots with respect to $\widehat{B}$. There is a natural isomorphism between $X^*(\widehat{S})$ and the lattice $S(F)/S_0$, given by $\lambda \in S(F)/S_0 \mapsto e_\lambda \in X^*(\widehat{S})$, where

$$e_\lambda(z \otimes \mu) = z^{\mathrm{ord}_\pi \mu(\lambda)}.$$

Via this identification, we realize $S(F)/S_0$ as a lattice inside the dual $\widehat{\mathfrak{s}}^*$ of the Lie algebra $\widehat{\mathfrak{s}} \cong \mathbf{C} \otimes X^*(S)$ of $\widehat{S}$. Also let $\hat{\mathfrak{g}}$ denote the Lie algebra of $\widehat{G}$. We write $\widehat{\mathfrak{s}}^*_{\mathbf{R}} = \mathbf{R} \otimes S(F)/S_0$, which sits inside $\widehat{\mathfrak{s}}^*$.

For $\chi \in \widehat{S}$, which may also be viewed as a character of $S(F)/S_0$, we now define

$$W_\chi = \{w \in W : \chi w = \chi\}, S_\chi = \left\{\alpha \in \Phi : e_\alpha(\chi) = q^{-1}\right\}, \text{ and } \widehat{\Phi}^+_\chi = \left\{\alpha \in \widehat{\Phi}^+ : e_\alpha(\chi) = 1\right\}.$$

We also define

$$R_\chi = \left\{w \in W_\chi : w\widehat{\Phi}^+_\chi = \widehat{\Phi}^+_\chi\right\},$$

which is always 1 if $\widehat{G}$ is simply connected. Each χ defines a principal series representation

$$I(\chi) = \mathrm{Ind}_B^G \chi = \left\{ f \in C^\infty(G) : f(bg) = \chi(p)\delta_B(b)^{\frac{1}{2}} f(g) \text{ for all } b \in B, g \in G \right\},$$

where we've used the normalized induction. We write $\Xi(\chi)$ for the Iwahori invariants of the unique K-spherical irreducible subquotient of $I(\chi)$.

For any χ we consider, we will always require

$$|e_\alpha(\chi)| \leq 1 \text{ for all } \alpha \in \widehat{\Phi}^+. \tag{2.16}$$

This can always be arranged by changing χ within its W-orbit. Already under the weaker requirement $S_\chi \subseteq \widehat{\Phi}^+$, the K-spherical subquotient of $I(\chi)$ is the unique quotient of a certain summand of $I(\chi)$ (which is all of $I(\chi)$ in the case where $R_\chi = 1$). See [58, §1.2] or [57] for more details on this point.

We want to understand the action of a certain subalgebra Θ of the Iwahori-Hecke algebra on $\Xi(\chi)$. For any $u \in \Omega_S$, we may write $u = u_a u_b^{-1}$, where u_a and u_b pair nonnegatively with every positive root (meaning that $\alpha(u_a), \alpha(u_b) \in \mathfrak{O}_F$ for each positive α). Then we can associate to u the element $[Iu_aI][Iu_bI]^{-1}$ in the Iwahori-Hecke algebra of G; this is independent of choices. We define Θ to the $\mathbf{C}$-subalgebra of the Iwahori-Hecke algebra generated by the elements $[Iu_aI][Iu_bI]^{-1}$ for $u \in \Omega_S$. Given a character $\chi : \Omega_S \to \mathbf{C}^\times$, there is a unique extension to a homomorphism $\Theta \to \mathbf{C}$ compatible with the homomorphism $S(F)/S_0 \to \Theta$, which we call χ_Θ. We work out the case of the trivial representation in Example 2.5.3 to illustrate this construction.

There is a natural Θ-module decomposition $\Xi(\chi) = \bigoplus_{w \in W_\chi \backslash W} \Xi(\chi, w)$, where the $\Xi(\chi, w)$ are generalized eigenspaces of $\Xi(\chi)$ with eigenvalue $((\chi w) \cdot \delta_B^{-\frac{1}{2}})_\Theta$ on Θ. Reeder's work [58] proves many interesting properties about this module as well as the Iwahori invariants of other subquotients of $I(\chi)$; we will be only be interested here in identifying elements

$w \in W_\chi \backslash W$ such that $\Xi(\chi, w) \neq 0$. If this holds for a suitable choice of w, then any eigenvector $\xi \in \Xi(\chi, w)$ for Θ will meet the conclusion of Theorem 2.5.1 via the translation from a split group to a ramified unitary group in Section 2.4.

The main geometric object of interest is

$$\mathfrak{q} = \left\{ x \in \widehat{\mathfrak{g}} | \operatorname{ad}(\chi) x = q^{-1} x \right\} \subseteq \widehat{\mathfrak{u}}.$$

Note that the containment in $\widehat{\mathfrak{u}}$ follows from $S_\chi \subseteq \widehat{\Phi}^+$. We will be interested in the action of the group $\widehat{M} = C_{\widehat{G}}(\chi)$ on $\mathfrak{q}$ via the adjoint action. In fact, the action of $\widehat{M}$ on $\mathfrak{q}$ makes $\mathfrak{q}$ into a prehomogeneous vector space with finitely many orbits [59]. We denote the open orbit by $\mathfrak{q}_0$.

We can now state the criterion for whether $\Xi(\chi, w)$ is nonvanishing. We remark that Reeder [58, Corollary 5.13] actually determines a criterion directly for a similar question about the Steinberg subquotient of $I(\chi)$ instead. However, he uses a duality proved by Rodier [61] to deduce a nonvanishing criterion [58, Corollary 5.15] for the spherical representation as well.

Theorem 2.5.2 ([58, Corollary 5.15])**.** *Let $\widehat{\mathfrak{u}}^{\mathrm{op}}$ denote the unipotent radical of the Borel opposite to $\widehat{B}$. We have $\Xi(\chi, w) \neq 0$ if and only if $\mathfrak{q}_0 \cap \operatorname{ad}(w)\widehat{\mathfrak{u}}^{\mathrm{op}} \neq \emptyset$.*

We remark that this theory is related to a comparison result between the Jacquet module and Iwahori invariants of a representation. The general result, is that for a smooth representation V of a split group G as above, with S_0 the integral points of the maximal torus as before, there is a natural isomorphism

$$V^I \to (V_N)^{S_0} \otimes (\delta_B^{-1})_\Theta \tag{2.17}$$

of Θ modules. Here $V_N = V/V(N)$, where $V(N) = \operatorname{span}\{v - nv | n \in N, v \in V\}$ and $(V_N)^{S_0}$

is given the natural action of Θ induced by that of $S(F)/S_0$ as prescribed above.

Example 2.5.3. We give one example to illustrate these result. If $\chi = \delta_B^{-\frac{1}{2}}$, then $I(\chi)$ contains the trivial representation as a subrepresentation. It is trivial to calculate the Jacquet module of this subrepresentation in this case; it is just the trivial representation. Then (2.17) implies that the Iwahori-Hecke algebra should act on the trivial representation by δ_B^{-1}. Since the action of $[IuI]$ in the Iwahori-Hecke algebra on a vector v in the trivial representation is simply multiplication by $[IuI : I]$, (2.17) is essentially the claim that if $u = u_a u_b^{-1}$, where these elements are in the positive Weyl chamber, then the measure of $[Iu_aI]$ divided by the measure of $[Iu_bI]$ is given by $\delta_B^{-1}(u)$. This calculation can be found in [14, Lemma 1.5.1].

Now we apply Theorem 2.5.2 to this setting. We modify χ by the longest Weyl element so that it meets condition 2.16; this yields $\chi = \delta_B^{\frac{1}{2}}$. Then the space $\mathfrak{q}$ is the product of the simple root spaces, and $\mathfrak{q}_0$ consists of points of $\mathfrak{q}$ that are nonzero in each root space. If the intersection $\mathrm{ad}(w)\widehat{\mathfrak{u}}^{\mathrm{op}}$ is nontrivial for an element $w \in W$, then $\mathrm{ad}(w)\widehat{\mathfrak{u}}^{\mathrm{op}}$ must include all of the root spaces for simple positive roots. Since $\widehat{\mathfrak{u}}^{\mathrm{op}}$ is a subalgebra, it is closed under the Lie bracket, and so is $\mathrm{ad}(w)\widehat{\mathfrak{u}}^{\mathrm{op}}$. Therefore $\mathrm{ad}(w)\widehat{\mathfrak{u}}^{\mathrm{op}} = \mathfrak{u}$ and $\chi w = \delta_B^{-\frac{1}{2}}$. In particular, the action of Θ on the unique nonzero weight space is given by $((\chi w) \cdot \delta_B^{-\frac{1}{2}})_\Theta = (\delta_B^{-1})_\Theta$. This is exactly what we calculated above.

2.5.2 Proof of Theorem 2.5.1

Let G be either SO_{2m+1} or Sp_{2m}, and let S be the diagonal maximal torus. Also let B be the upper triangular Borel subgroup. Assume that the character $\chi : \Omega_S \to \mathbf{C}^\times$ is identified with one of the characters of the relevant unitary group classified in Theorem 2.3.1 using Proposition 2.4.7 or 2.4.10. Then possibly after changing χ within its W-orbit, we can assume that the image of u_i is $\gamma_i q_v^{k_i}$ for $k_i \in \frac{1}{2}\mathbf{Z}_{\leq 0}$ and $\gamma_i \in S^1 \subseteq \mathbf{C}$, and moreover that the k_i are non-decreasing as i decreases from n to 1. (Recall that u_1 is the matrix whose

nonidentity diagonal entries are closest to the center, rather than u_m.) Then it is easy to see that the condition (2.16) is met.

Let $w = w_0$ be the longest element of W. We have $\mathrm{ad}(w_0)\widehat{\mathfrak{u}}^{\mathrm{op}} = \widehat{\mathfrak{u}}$. It follows trivially that $\mathfrak{q}_0 \cap \widehat{\mathfrak{u}} \neq \emptyset$. By Theorem 2.5.2, $\Xi(\chi, w_0)$ is nonzero. Let $\xi \in \Xi(\chi, w_0)$ be an eigenvector for Θ. Then since u_m is in the positive Weyl chamber and w_0 acts by inversion on χ, $[Iu_mI]$ acts by $\gamma_n^{-1} q^{-k_m} \delta_B(u_m)^{-\frac{1}{2}}$ on ξ. Some k_i must be nonzero by the hypothesis that $\alpha_i = q_v \alpha_j$ for some i, j. Therefore we have $k_m < 0$, since k_m is the smallest of the k_i. One calculates that $\delta_B(u_m)^{-\frac{1}{2}} = q^{\frac{n-1}{2}}$, where $n = 2m$ or $2m+1$ is the rank of the underlying Hermitian space.

Using the isomorphisms of Iwahori-Hecke algebras given in Section 2.4, we can translate this result immediately into the desired one for unitary groups.

2.6 Crystalline periods in families of Galois representations

We investigate the variation of crystalline periods in families of Galois representations. We will use this to construct a period in the filtered ϕ-module of the Galois representation attached to π. Then certain results on purity, combined with the bound found in Theorem 2.5.1, will give the contradiction needed to show that π_p is tempered and that compatibility with base change holds. We first introduce some of the needed ingredients in Section 2.6.1 and 2.6.2. Then we formalize the properties of the desired p-adic family of Galois representations in Section 2.6.6. In the process, we will connect our discussion to the automorphic context in preparation for Section 2.7, where we describe the construction of these families.

2.6.1 Purity

Fix an isomorphism $\overline{\mathbf{Q}}_p \cong \mathbf{C}$. Let $E \subseteq \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ and let F be a number field. We define an *algebraic* p-adic Galois representation $\rho : G_F \to \mathrm{GL}_n(E)$ to be one that is ramified away from a finite set of places and is de Rham at places above p. The Fontaine-Mazur conjecture predicts that such representations arise from geometry. A consequence is that one expects such a ρ to be *pure* at all places, which is a condition on the interaction between the Frobenius action and monodromy of ρ. For Galois representations attached to regular algebraic conjugate self-dual cuspidal automorphic representations on GL_n over a CM field, this is now known by the deep work of many mathematicians, including Deligne, Clozel, Harris, Taylor, Yoshida, Shin, Barnet-Lamb, Gee, Geraghty, and Caraiani [28, 79, 71, 16, 1, 2, 12, 13]. We will need to utilize their results, so we review what this purity means, closely following the exposition of Taylor and Yoshida [79, §1].

Let L be a finite extension of $\mathbf{Q}_p$ with residue field κ_L. The basic object of study is a Weil-Deligne representation, which is given by a triple (V, r, N). Here V is a vector space over an algebraically closed field of characteristic 0, $r : W_L \to \mathrm{GL}_n(V)$ is a representation of the Weil group with open kernel, and $N \in \mathrm{End}(V)$ satisfies $r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}_L^{-1}(\sigma)|_L N$, where Art denotes the Artin map, and $|\cdot|_L$ is normalized to take uniformizers to $\#\kappa_L^{-1}$. (Recall our requirement that the Artin map take uniformizers to geometric Frobenius lifts.) We say (V, r, N) is Frobenius semi-simple if r is semi-simple.

For $q \in \mathbf{R}_{>0}$, a Weil q-number is an element $\alpha \in \overline{\mathbf{Q}}$ such that the image of α in $\mathbf{C}$ has absolute value $q^{\frac{1}{2}}$ under every embedding. The Frobenius semi-simple representation (V, r, N) is *strictly pure of weight* $w \in \mathbf{R}$ if $N = 0$ and $r(\phi)$ has Weil $(\#\kappa_L)^w$-numbers as eigenvalues for any lift ϕ of the *geometric* Frobenius. We say (V, r, N) is *mixed* if there is an exhaustive, separated increasing filtration $F_i \subseteq V$ such that the i^{th} graded piece $\mathrm{gr}_i = F_i/F_{i+1}$ is strictly pure of weight i. Note that $N(F_i) \subseteq F_{i-2}$. For $w \in \mathbf{R}$, if (V, r, N) is mixed with weights in $w + \mathbf{Z}$ and if for every $i \in \mathbf{Z}_{>0}$ we have $N^i : \mathrm{gr}_{w+i} V \xrightarrow{\sim} \mathrm{gr}_{w-i} V$, we say (V, r, N)

is pure of weight w. We can make the following simple observation.

Lemma 2.6.1. *If (V, r, N) is pure of weight $w \in \mathbf{Z}$, the eigenvalues of a geometric Frobenius element in* $\ker N$ *are at most $(\#\kappa_F)^{\frac{w}{2}}$ in absolute value.*

Proof. If (V, r, N) is pure of weight w, then for any $i \in \mathbf{Z}_{>0}$, the map $N|_{F_{w+i}} : F_{w+i} \to \mathrm{gr}_{w-i} V$ has kernel exactly F_{w+i-1}, so any element of $F_{w+i} \setminus F_{w+i-1}$ cannot be in the kernel of N. Therefore $\ker N \subseteq F_w$. But if we pick a basis of F_w consisting of geometric Frobenius eigenvectors and consider the purity condition, we find that any Frobenius eigenvalue on $\ker N$ must be at most $(\#\kappa_F)^{\frac{w}{2}}$ in absolute value. □

Pure Weil-Deligne representations satisfy a number of interesting properties. For instance, purity is stable under finite extensions of the ground field L, and corresponds to temperedness under the local Langlands correspondence. Also, given (V, r), there is at most one N making (V, r, N) pure.

We are particularly interested in purity for a Galois representation $\rho : G_F \to \mathrm{GL}_n(E)$ as above, but restricted to places above p. At other places, there is a very simple way to associate a Weil-Deligne representation to the restriction. At places above p, one uses p-adic Hodge theory. We describe how this process goes in the special case where ρ is semi-stable, since this is all we will need. See [79, §1] for the general case; we follow the conventions of that paper. Also recall that we have fixed an isomorphism $\iota : \mathbf{C} \to \overline{\mathbf{Q}}_p$. Using this isomorphism it makes sense to talk about the complex absolute values of a number lying in $\overline{\mathbf{Q}}_p$, though of course we will only do this for Weil numbers.

Let $v|p$ be a place of F and write ρ_v for the restriction of ρ to G_{F_v}. Also write W for the underlying vector space of ρ. Let $F_{v,\mathrm{ur}} \subseteq F_v$ be the maximal subfield of F_v that is unramified over $\mathbf{Q}_p$. Also denote by $\varphi : B_{\mathrm{st}} \to B_{\mathrm{st}}$ the crystalline Frobenius endomorphism, which is semilinear with respect to the action of a geometric Frobenius element. Then for

any embedding $\nu : F_{v,\mathrm{ur}} \to \overline{\mathbf{Q}}_p$ we define

$$D_{\mathrm{st},\nu}(W) = (W \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})^{G_{F_v}} \otimes_{(\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_{v,\mathrm{ur}}),1\otimes\nu} \overline{\mathbf{Q}_p}.$$

In this notation, there is a natural action of φ on $D_{\mathrm{st},\nu}(W)$ induced by its action on the B_{st} factor; although φ is not invertible on B_{st}, it is on $D_{\mathrm{st},\nu}(W)$. We define a representation $r : W_K \to \mathrm{GL}(D_{\mathrm{st},\nu}(W))$ by setting $r(g) = \varphi^{m_g[\kappa_{F_v}:\mathbf{F}_p]}$, where the action of g on the residue field is by $\mathrm{Frob}_p^{m_g}$. (Recall that we use Frob_p for the *geometric* Frobenius.) We also obtain an action of $1 \otimes N$ from the operator N on B_{st}. The triple $(D_{\mathrm{st},\pi}(W), r^{\mathrm{ss}}, 1 \otimes N)$ gives a Weil-Deligne representation (via ι), where r^{ss} denotes the semi-simplification, as one can check. Moreover, it follows from the definition that the kernel of N is exactly

$$D_{\mathrm{cris},\nu}(W) = (W \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})^{G_{K_v}} \otimes_{(\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_{v,\mathrm{ur}}),1\otimes\nu} \overline{\mathbf{Q}_p}.$$

Thus the action of a lift $g \in W_F$ of the geometric Frobenius is given by $\varphi^{[\kappa_{F_v}:\mathbf{F}_p]}$ on $D_{\mathrm{cris},\nu}(W)$. Therefore, by Lemma 2.6.1, if $D_{\mathrm{cris},\nu}(W)$ is pure of weight w, the eigenvalues of $\varphi^{[\kappa_{F_v}:\mathbf{F}_p]}$ are bounded by $(\#\kappa_{F_v})^{\frac{w}{2}}$ in absolute value. In addition, the Weil-Deligne representation is independent of the choice of ν. We summarize this discussion as follows.

Proposition 2.6.2. *Suppose that $F_v/\mathbf{Q}_p$ is a finite extension. The eigenvalues of the $[\kappa_{F_v} : \mathbf{F}_p]^{th}$ power of the crystalline Frobenius on a pure semi-stable p-adic Galois representation of G_{F_v} of weight w are bounded by $(\#\kappa_{F_v})^{\frac{w}{2}}$ in absolute value.*

It is known that tensor products of pure representations are pure. In an earlier version of this work, we needed to show that exterior powers of representations also preserve purity. For future applications, we provide the (trivial) argument here.

Proposition 2.6.3. *Suppose that (V, r, N) is a pure Weil-Deligne representation of W_F of dimension n, where F is an p-adic local field. Then for any Schur functor $c_\lambda \in \mathbf{C}[S_m]$, λ a*

partition of m, $c_\lambda V$ is a pure Weil-Deligne representation.

Proof. Since $V^{\otimes m}$ is pure, it suffices by [79, Lemma 1.4.(5c)] to show that

$$V^{\otimes m} = \bigoplus_\lambda c_\lambda V^{\otimes m}$$

as an internal direct sum of Weil-Deligne subrepresentations. For this, we simply need that each $c_\lambda V^{\otimes m}$ is preserved by W_F and N. But both of these actions commute with the permutation action by S_m and thus the entire group algebra $\mathbf{C}[S_m]$, so they leave $c_\lambda V^{\otimes m}$ invariant.

□

2.6.2 Interpolating crystalline periods

We discuss how to interpolate a single crystalline period, which a technique initiated by Kisin [36]. Nothing in this section is original. The form we will need uses an enhancement by Nakamura [50, §3] to Kisin's results that allows one to work with a representation of G_L for a finite extension $L/\mathbf{Q}_p$ instead of $G_{\mathbf{Q}_p}$. We reword Nakamura's arguments in a form that is similar to that of the proof of [36, Proposition 3.14]. We also mention a lemma due to Luu [46, Lemma 2.10] that will be very useful for applying the main result; this lemma is based on one proved by Kisin [36, Lemma 6.7].

In the approach taken by Kisin and Nakamura, one must bound the amount of variation allowed for the crystalline period over the family.

Definition 2.6.4. If $\mathfrak{R}$ is an E-affinoid algebra and $Y \in \mathfrak{R}^\times$, then $\mathfrak{R}^\times$ is called Y-small if there exists $\lambda \in (\mathfrak{R} \otimes_E E')^\times$ for some finite Galois extension E' of E such that $E'[\lambda] \subseteq \mathfrak{R} \otimes_E E'$ is an étale E' algebra and $Y\lambda^{-1} - 1$ is topologically nilpotent.

Given a point of $\operatorname{Spm} \mathfrak{R}$, we can find an open affinoid neighborhood such that the restric-

tion of Y is Y-small, which will be sufficient for our application.

Remark 2.6.5. Liu [42] has proven a substantially more general version of the crystalline period interpolation used here using finiteness results for cohomology by Kedlaya, Pottharst, and Xiao [35]. Liu allows one to interpolate multiple crystalline periods in a single family with multiple fixed Hodge-Tate-Sen weights, while previous results only allowed one to interpolate a single period. Moreover, he removes the Y-smallness hypothesis. However, we will only need the simpler interpolation result stated below.

Let $L/\mathbf{Q}_p$ and $E/\mathbf{Q}_p$ be finite extensions such that E contains the normal closure of L. Define $\Sigma = \{\sigma : L \to E\}$. Let $\mathfrak{R}$ be an affinoid algebra over E and let $\mathfrak{M}$ be a finite free $\mathfrak{R}$-module equipped with a continuous G_L-action. Fix $Y \in \mathfrak{R}^\times$ such that $\mathfrak{R}$ is Y-small. Let the Sen polynomial with respect to the embedding σ be denoted $P_{\phi,\sigma}(T) \in \mathfrak{R}[T]$. We write $f = [\kappa_L : \mathbf{F}_p]$, where κ_L is the residue field of L. We refer to Nakamura [50, §3.1] for the definition of $B^+_{\max,L}$. We remark that Nakamura makes requirements for all $\sigma \in \Sigma$ and draws conclusions for each σ; however, his proof works on one σ at a time, using the decompositions (from the proof of [50, Lemma 3.11]) $(B^+_{\max,L}\hat{\otimes}_{\mathbf{Q}_p}\mathfrak{R})^{\varphi^f=Y} = \bigoplus_{\sigma\in\Sigma}(B^+_{\max,L}\hat{\otimes}_{L,\sigma}\mathfrak{R})^{\varphi^f=Y}$ and $B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{\mathbf{Q}_p}\mathfrak{R} = \bigoplus_{\sigma\in\Sigma} B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}\mathfrak{R}$. In our application, in any case, we will have a fixed Hodge-Tate weight for each σ.

Theorem 2.6.6 ([50, Proposition 3.14])**.** *Fix $\sigma \in \Sigma$. Assume that there is a factorization $P_{\phi,\sigma}(T) = TQ(T)$, and define $P(k) = \prod_{j=0}^{i-1} Q(-j)$. Suppose that there exists a collection $\{\mathfrak{R}_i\}_{i\in I}$ of affinoid $\mathfrak{R}$-algebras such that for each $k \geq 0$ there exists a subset $I_k \subseteq I$ with the following properties.*

1. *For $i \in I_k$, every $\mathfrak{R}_i$-linear, G_L-equivariant map*

$$\mathfrak{M}^\vee \otimes_{\mathfrak{R}} \mathfrak{R}_i \to B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}\mathfrak{R}_i$$

factors through

$$\mathfrak{M}^\vee \otimes_{\mathfrak{R}} \mathfrak{R}_i \to (B^+_{\max,L}\hat{\otimes}_{L,\sigma}\mathfrak{R}_i)^{\varphi^f=Y}.$$

2. *For each $i \in I_k$, the image of $P(k)$ in $\mathfrak{R}_i$ is a unit.*

3. *The map $\mathfrak{R} \to \prod_{i\in I_k} \mathfrak{R}_i$ is injective.*

Suppose that $F \subseteq \mathbf{C}_p$ is a closed subfield containing E, and $g : \mathfrak{R} \to F$ is a continuous homomorphism. Then we have a nonzero, F-linear, G_L-equivariant homomorphism

$$M_R^\vee \to (B^+_{\max,L} \otimes_{L,\sigma} F)^{\varphi^f=g(Y)}.$$

In particular, we have

$$D^+_{\mathrm{cris}}(\mathfrak{M} \otimes_{\mathfrak{R}} F)^{\varphi^f=g(Y)} \neq 0.$$

We will first need to study the relationship between the crystalline and de Rham period rings. By Proposition 3.7 of [50], for sufficiently large k, there exists a G_L-equivariant short exact sequence of orthonormalizable $\mathfrak{R}$-Banach spaces

$$0 \to (B^+_{\max,L}\hat{\otimes}_{L,\sigma}\mathfrak{R})^{\varphi^f=Y} \xrightarrow{h_k} B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}\mathfrak{R} \xrightarrow{g_k} U_k \to 0. \tag{2.18}$$

Moreover, for any continuous homomorphism $\mathfrak{R} \to \mathfrak{R}'$ of affinoid algebras, the formation of this sequence commutes with taking a completed tensor product with $\mathfrak{R}'$.

Lemma 2.6.7. *For k such that (2.18) holds, any $\mathfrak{R}$-linear G_L-equivariant homomorphism*

$$\psi : \mathfrak{M}^\vee \to B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}\mathfrak{R} \tag{2.19}$$

factors through $(B^+_{\max,L}\hat{\otimes}_{L,\sigma}\mathfrak{R})^{\varphi^f=Y}$.

Proof. Fix an integer k as in the statement of the lemma. We set $h = h_k$, $g = g_k$, and denote by h_i, g_i, and ψ_i the results of applying the functor $\otimes_{\mathfrak{R}}\mathfrak{R}_i$ to the morphisms h, g, and ψ. Orthonormalizability of the $\mathfrak{R}$-Banach spaces in (2.18) shows that the completed tensor product of each term of (2.18) with the homomorphism $\mathfrak{R} \hookrightarrow \prod_{i\in I_k}\mathfrak{R}_i$ is injective. From this and the compatibility of (2.18) with base change we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (B^+_{\max,L}\hat{\otimes}_{L,\sigma}\mathfrak{R})^{\varphi^f=Y} & \xrightarrow{h} & B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}\mathfrak{R} & \xrightarrow{g} & U_k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_{i\in I_k}(B^+_{\max,L}\hat{\otimes}_{L,\sigma}\mathfrak{R}_i)^{\varphi^f=Y} & \xrightarrow{\prod_i h_i} & \prod_{i\in I_k} B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}\mathfrak{R}_i & \xrightarrow{\prod_i g_i} & \prod_{i\in I_k} U_k\hat{\otimes}_{\mathfrak{R}}\mathfrak{R}_i & \longrightarrow & 0 \end{array} \tag{2.20}$$

with exact G_L-equivariant rows. By hypothesis, the morphism $\prod_i \psi_i$ factors through the submodule $\prod_{i\in I_k}(B^+_{\max,L}\hat{\otimes}_{L,\sigma}\mathfrak{R}_i)^{\varphi^f=Y}$, so the composite map

$$\mathfrak{M}^\vee \hookrightarrow \mathfrak{M}^\vee \otimes_{\mathfrak{R}} \prod_i \mathfrak{R}_i \xrightarrow{\prod_i \psi_i} \prod_{i\in I_k} B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}\mathfrak{R}_i \to \prod_{i\in I_k} U_k\hat{\otimes}_{\mathfrak{R}}\mathfrak{R}_i$$

vanishes. Since the rightmost downwards arrow in (2.20) is injective, the image of ψ lands in $(B^+_{\max,L}\hat{\otimes}_{L,\sigma}\mathfrak{R})^{\varphi^f=Y}$ as needed.

□

Define $H \subseteq \mathfrak{R}$ to be the smallest ideal of $\mathfrak{R}$ such that any homomorphism ψ as in Lemma 2.6.7 factors through $B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}H$.

Lemma 2.6.8. *The Zariski-open set* $\mathrm{Spm}(\mathfrak{R}) \setminus \mathrm{Spm}(\mathfrak{R}/H)$ *is scheme-theoretically dense in* $\mathrm{Spm}(R)$.

Proof. We claim that $H\mathfrak{R}_i = \mathfrak{R}_i$. Suppose otherwise. Regard $\mathfrak{R}$ as a L-algebra using the assumption $L^{\mathrm{nor}} \subseteq E$. By [36, Corollary 2.6], setting $E = L$ in Kisin's notation, the map

$$(B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}\mathfrak{M})^{G_L}_{P(k)} \otimes_{\mathfrak{R}} (\mathfrak{R}_i/H\mathfrak{R}_i) \to (B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}(\mathfrak{M} \otimes_{\mathfrak{R}} (\mathfrak{R}_i/H\mathfrak{R}_i)))^{G_L}_{P(k)}$$

is an isomorphism of finite free $(\mathfrak{R}_i/H\mathfrak{R}_i)$-modules of rank one. However, this is the 0 map by definition of H. Thus $\mathrm{Spm}(\mathfrak{R}/H)$ does not meet the image of $\coprod_{i\in I_k}\mathrm{Spm}(\mathfrak{R}_i)$, which is dense by the third condition of the theorem. In particular, $\mathrm{Spm}(\mathfrak{R})\setminus\mathrm{Spm}(\mathfrak{R}/H)$ must be scheme-theoretically dense.

□

We can now use these to prove the main theorem.

Proof of Theorem 2.6.6. Define $\mathfrak{R}_F = \mathfrak{R}\otimes_E F$, $H_F = H\otimes_E F = H\mathfrak{R}_F$, and $\mathfrak{m} = \ker(\mathfrak{R}_F \to F)$. By Lemma (2.6.8), together with stability of scheme-theoretic density under flat base change (see [36, §5.1]), $\mathrm{Spm}(\mathfrak{R}_F)\setminus\mathrm{Spm}(\mathfrak{R}_F/H_F)$ is scheme-theoretically dense in $\mathrm{Spm}(\mathfrak{R}_F)$. In particular, we may choose n such that $H_F \not\subseteq \mathfrak{m}^n$ but $H_F \subseteq \mathfrak{m}^{n-1}$. Let ψ be chosen so that the image of the base extension ψ_F in $B^+_{\mathrm{dR}}/t^kB^+_{\mathrm{dR}}\hat{\otimes}_{L,\sigma}\mathfrak{R}_F$ is not contained in $\mathfrak{m}^n$. (This is possible, as otherwise we could replace H with $H\cap\mathfrak{m}^n$.) From this and our choice of k we obtain a nonzero homomorphism

$$\mathfrak{M}^\vee \xrightarrow{\psi} (B^+_{\max,L}\hat{\otimes}_{L,\sigma}H)^{\varphi=Y} \to (B^+_{\max,L}\hat{\otimes}_{L,\sigma}\mathfrak{m}^{n-1}/\mathfrak{m}^n)^{\varphi=Y}.$$

By a suitable choice of map $\mathfrak{m}^{n-1}/\mathfrak{m}^n \to F$, we obtain a nonzero homomorphism

$$\mathfrak{M}^\vee \to (B^+_{\max,L}\hat{\otimes}_{L,\sigma}\mathfrak{m}^{n-1}/\mathfrak{m}^n)^{\varphi=Y} \to (B^+_{\max,L}\hat{\otimes}_{L,\sigma}F)^{\varphi=Y}$$

as needed.

□

In order to check the first condition in Theorem 2.6.6, we will use the following lemma of Luu, with notation slightly modified to match with that of Nakamura's result. We state the result here with respect to a chosen embedding σ; the proof is identical once one replaces tensor products over $\mathbf{Q}_p$ with tensor products over L with respect to $\sigma : L \to E$. Note

also that Luu uses the opposite of our convention with respect to Hodge-Tate weights; our weights are the negatives of the roots of the Sen polynomial, so that the cyclotomic character has Hodge-Tate weight -1. For this reason the inequalities below are reversed.

Lemma 2.6.9 ([46, Lemma 2.10]). *Let L, E be finite extensions of $\mathbf{Q}_p$, let $\alpha \in E^\times$, let $k \in \mathbf{Z}_{\geq 1}$, and let $\Sigma = \{\sigma : L \to E\}$ as above. Fix $\sigma \in \Sigma$. Let G_L act continuously on a finite dimensional E-vector space V with $D_{\mathrm{cris},\sigma}(V)^{\varphi^{[\kappa_L:\mathbf{F}_p]}=\alpha} \neq 0$. Assume moreover that V is Hodge-Tate with respect to the embedding σ, with weights $\{0, k_{\sigma,1}, \ldots, k_{\sigma,n-1}\}$ such that $0 < k_{\sigma,2} \leq \cdots \leq k_{\sigma,n-1}$ and $k_{\sigma,2} > \max(k, [L : \mathbf{Q}_p]v_p(\alpha))$. Then the natural map*

$$D_{\mathrm{cris},\sigma}(V)^{\varphi^{[\kappa_L:\mathbf{F}_p]}=\alpha} \to (B^+_{\mathrm{dR}}/t^k B^+_{\mathrm{dR}} \widehat{\otimes}_{L,\sigma} V)^{G_L}$$

is an isomorphism.

2.6.3 Hecke algebras and slopes

Let π be a cohomological automorphic representation on a reductive group G over $\mathbf{Q}$ such that $G(\mathbf{Q}_p)$ is quasi-split, and let the dominant algebraic weight λ of G be the weight of π_∞, which we recall means that $\pi_\infty \otimes \mathbf{V}_\lambda^\vee$ has nonvanishing cohomology. (The restriction to $\mathbf{Q}$ is only for convenience; groups over more general fields can be treated by restriction of scalars.) In this section, we discuss the notion of the *slope* of π. When putting π in a sufficiently small (in the rigid geometric sense) p-adic family, the slope will be constant in the family. Everything in this section is taken from Urban [82, §4], and we use similar notation when possible.

We introduce a modified global Hecke algebra of G. We fix a finite set S of finite places of $\mathbf{Q}$ and let $K^p = \prod_{v \neq p} K_v$ be a product of open compact subgroups of $G(\mathbf{Q}_v)$ such that for $v \notin S$, K_v is a hyperspecial maximal compact subgroup. At p, let T be a maximal torus, let B be a Borel with Levi factor T, and let N be the unipotent radical of B. Fix compatible

integral structures for $G(\mathbf{Q}_p), T, N$, and B, and let I be an Iwahori subgroup compatible with the choice of B. Let

$$T^- = \left\{ t \in T(\mathbf{Q}_p) : tnt^{-1} \in N(\mathbf{Z}_p) \text{ for } n \in N(\mathbf{Z}_p) \right\}.$$

Note that the condition $t \in T^-$ is equivalent to asking that t pair nonnegatively with every positive root, which is the notion used in Section 2.5.1. We also write T^{--} for the subset of $t \in T^-$ that pair strictly positively with every positive root; we could also use the condition $\cap_i t^i N(\mathbf{Z}_p) t^{-i} = \{1\}$. We also write $\Delta^- = IT^-I$. We then define $\mathfrak{U}_p = C_c^\infty(I \backslash \Delta^- / I, \mathbf{Q}_p)$. This is an abelian algebra isomorphic to the subalgebra $\mathbf{Z}[T^-/T(\mathbf{Z}_p)]$ of the weight space Θ of Section 2.5.1. We define the commutative algebra $\mathfrak{R}_{S,p} = \mathfrak{U}_p \otimes \prod_{v \notin S} \mathfrak{H}_{K_v}$, where $\mathfrak{H}_{K_v}$ denotes the Hecke algebra $C_c^\infty(K_v \backslash G(\mathbf{Q}_v) / K_v, \mathbf{Q}_p)$. We define $\mathfrak{H}_p(K^p) = \mathfrak{U}_p \otimes \prod_{v \neq p} \mathfrak{H}_{K_v}$, which unlike $\mathfrak{R}_{S,p}$ is not abelian in general. We finally define $\mathfrak{H}_p = C_c^\infty(G(\mathbf{A}_f^p), \mathbf{Q}_p) \otimes \mathfrak{U}_p$; this is the union over all $\mathfrak{H}_{K_v}$.

Remark 2.6.10. In the above, one can replace I with the deeper Iwahori subgroup I_m of elements of $G(\mathbf{Z}_p)$ lifting those in $B(\mathbf{Z}_p / p^m \mathbf{Z}_p)$; however, we will not concern ourselves with this here. For an automorphic representation π with $\pi^{I_m K^p}$ nontrivial for some m, we can always define an associated representation of $\mathfrak{H}_p(K^p)$. However, the representation of $\mathfrak{U}_p$ might send an element of T^- to 0. If this happens, we say that π has *infinite slope* and otherwise we say that π has *finite slope*. In the cases we consider, π will automatically have finite slope by construction. In either case, the reason why we use $\mathfrak{U}_p$ as opposed to the full Iwahori-Hecke algebra is because the operators outside $\mathfrak{U}_p$ cannot act on the larger distribution spaces considered in the p-adic context.

Definition 2.6.11. A *finite slope admissible representation* σ of level K^p is an absolutely irreducible admissible representation of $\mathfrak{H}_p(K^p)$ defined over a p-adic field E satisfying two conditions.

- The action of $\mathfrak{U}_p$ is of finite slope.
- There is a $\mathfrak{O}_E$-lattice in $\sigma^{I_m K^p}$ that is stable by the action of the $\mathbf{Z}_p$-valued Hecke operators in $\mathfrak{H}_p(K^p)$.

Note that it makes sense to consider the trace of any element of $\mathfrak{H}_p$ on σ; we denote this function by J_σ.

Recall that π itself is not directly placed in a p-adic family. Instead, we associate to π a finite slope admissible representation σ_π of the Hecke algebra $\mathfrak{H}_p(K^p)$ as follows. Using the identification ι, we can think of π via the associated representation of the Hecke algebra on a $\overline{\mathbf{Q}}_p$-vector space. Away from $S \cup \{p\}$, the action of $\mathfrak{H}_{K_v}$ is just the natural one on the K_v-spherical vector. At p, if $\pi_p^I \neq 0$, we obtain a representation of $\mathfrak{U}_p$ by making an additional choice of an eigenvector $\xi \in \pi_p^I$ for $\mathfrak{U}_p$. This is all the data we need to produce the finite slope admissible representation attached to π.

For the remainder of this chapter we fix a multiplicative section

$$\upsilon : T(\mathbf{Q}_p)/T(\mathbf{Z}_p) \to T(\mathbf{Q}_p). \tag{2.21}$$

If the group $G(\mathbf{Q}_p)$ is unramified, it is possible to define υ to have the property that $\lambda_{\mathrm{alg}}(\upsilon(t)) = |\lambda_{\mathrm{alg}}(t)|_p^{-1}$ for every algebraic character λ_{alg} on G. (Recall that λ is the algebraic weight associated to π_∞, and so can be evaluated on an element of the chosen maximal torus of $G(\mathbf{Q}_p)$.) In ramified cases such as ours, this is not always possible. To see why it is necessary to make such a choice, see the proof of [82, Lemma 4.6.2]. We define υ to send every $u_{v,m}$ to $u_{v,m}$ and to take $\varpi \mathbf{1}_n$ to itself in the notation of Definition 2.4.12. This choice will be important in the proof of Theorem 2.2.5 given in Section 2.7.3.

Now we return to the situation of π of algebraic weight λ and chosen vector ξ. We twist the $\mathfrak{U}_p$ action on ξ by multiplying it by $\lambda(\upsilon(t))$. This completely determines a character

θ_π of $\mathfrak{R}_{S,p}$. To obtain the finite slope admissible representation σ_π, one simply takes the irreducible $\mathfrak{H}_p(K^p)$-constituent of $\pi_f^{IK^p}$ containing ξ.

An intermediate step in constructing a p-adic family is to piece together various finite slope admissible representations in a natural way.

Definition 2.6.12. Let $\mathfrak{H}'_p$ be the ideal of $\mathfrak{H}_p$ generated by $f \otimes u_t \in C_c^\infty(G(\mathbf{A}_f^p), \mathbf{Q}_p) \otimes \mathfrak{U}_p$ for $t \in T^{--}$. If E is a finite extension of $\mathbf{Q}_p$, an E-valued virtual finite slope character distribution is a $\mathbf{Q}_p$-linear map $J : \mathfrak{H}_p \to E$ such that there exist finite slope admissible representations σ_i and integers m_i with the following two properties.

1. For all $t \in T^{--}$, $s \in \mathbf{Q}$, and K^p, there are only finitely many i with $m_i \neq 0$, $v_p(\theta_{\sigma_i}(u_t)) \leq h$, and $\sigma_i^{K^p} \neq 0$.

2. For any $f \in \mathfrak{H}'_p$, $J(f) = \sum_{i=1}^\infty m_i J_{\sigma_i}(f)$.

We drop the word "virtual" if $m_i \geq 0$ for all i, and sometimes refer to this as *effective*. In that case we can form the direct sum of $\sigma_i^{m_i}$ and complete with respect to the supremum norm to form a p-adic Banach space over $\mathbf{C}_p$. The elements of $\mathfrak{H}'_p$ act completely continuously.

Later we will consider a rigid analytic weight space $\mathfrak{X}$, and a certain ring of rigid analytic functions $\Lambda_{\mathfrak{X},\mathbf{Q}_p} \subseteq \mathfrak{O}_\mathfrak{X}(\mathfrak{X})$ on $\mathfrak{X}$. Then an *analytic family of finite slope character distributions* is a function $J : \mathfrak{H}'_p \to \Lambda_{\mathfrak{X},\mathbf{Q}_p}$ that specializes to a finite slope character distribution at every point of $\mathfrak{X}$.

We finally come to the topic of the *slope* of a representation. Suppose that the homomorphism $\theta : \mathfrak{U}_p \to \overline{\mathbf{Q}}_p^\times$ is of finite slope. Also let F be an extension of $\mathbf{Q}_p$ over which T splits. Then we define the slope μ_θ to be the element of $X^*(T_{/F})^{\mathrm{Gal}(F/\mathbf{Q}_p)} \otimes_\mathbf{Z} \mathbf{Q}$ such that

$$(\mu_\theta, \mu^\vee) = v_p(\theta(u_{\mu^\vee}(p)))$$

for all $\mu^\vee \in X_*(T_{/F})^{+,\mathrm{Gal}(F/\mathbf{Q}_p)}$, where we normalize the valuation v_p by $v_p(p) = 1$. Here $X_*(T_{/F})^{+,\mathrm{Gal}(F/\mathbf{Q}_p)}$ refers to the dual of the cone generated by the positive roots.

If θ takes integral values on $\mathfrak{U}_p$, μ_θ will lie in the subset $X^*(T)_{+,\mathbf{Q}} \subseteq X^*(T) \otimes_{\mathbf{Z}} \mathbf{Q}$ consisting of the $\mathbf{Q}_{\geq 0}$-linear combinations of the simple roots. We define a renormalized action $*$ of the Weyl group on an algebraic character λ by $w * \lambda = w(\lambda + \rho) - \rho$. Now suppose λ is a dominant algebraic weight. Then we say that θ is *noncritical with respect to* λ if $\mu_\theta - \lambda + w * \lambda \notin X^*(T)_{+,\mathbf{Q}}$ for all nontrivial w.

2.6.4 Slopes on ramified unitary similitude groups

Suppose that G is now one of the unitary groups of Section 2.2.1, and we identify $G_{/\mathbf{Q}_p}$ with a similitude group as described in Section 2.4.6.

For illustration purposes, we first check a standard fact about slopes explicitly in this case.

Proposition 2.6.13. *Suppose that μ is any slope lying in $X^*(T)_{+,\mathbf{Q}}$. Then there exists $N \in \mathbf{Z}_{\geq 1}$ such that if λ is any algebraic weight whose pairing with any positive simple coroot is at least N, then μ is non-critical with respect to λ.*

Proof. By definition, μ is non-critical with respect to λ if $\mu - \lambda + w * \lambda$ does not lie in $X^*(T)_{+,\mathbf{Q}}$, which consists of all weights of the form $\sum_j a_{\nu,j}\alpha_{\nu,j}$ where $\alpha_j = t_{\nu,j}/t_{\nu,j+1}$ and $a_i \in \mathbf{Q}_{\geq 0}$. We first note that the multiple of the similitude in μ is 0 since it lies in $X^*(T)_{+,\mathbf{Q}}$, and this similitude is unchanged by $-\lambda + w * \lambda$ since the w action does not change the similitude of λ. Thus we may assume that $c = 0$, where λ corresponds to $\underline{c}$. Similarly, we may translate every $\underline{c}_\nu$ by an element of $\mathbf{Q}$ so that $\sum_{i=1}^n c_{\nu,i} = 0$ without affecting $-\lambda + w * \lambda$; this does not affect the pairings of λ with the positive simple coroots. We may now write $\lambda = \underline{c} = \sum_j a_{\nu,j}\alpha_{\nu,j}$ with every $a_{\nu,j} \geq \frac{N}{2}$. We observe that if w is nontrivial, the usual action of w on λ (rather than the normalized one), which we write $w \cdot \lambda$, produces $w \cdot k\lambda = k \sum_j a^w_{\nu,j}\alpha_{\nu,j}$ where at

least one $a^w_{\nu,j} = -a_{\nu,j}$; fix such a j. The difference between the normalized and usual actions is constant, so for N taken sufficiently large, the multiple of $\alpha_{\nu,j}$ in $-\lambda + w * \lambda$ becomes arbitrarily negative, so there is some N_w such that $\mu - k\lambda + w * k\lambda \notin X^*(T)_{+,\mathbf{Q}}$. Now we just take N to be the largest of the N_w. □

We need slightly more precise information about the interaction between slope and weight. We will calculate two quantities.

1. The first is the slope of the character $u_t \mapsto \lambda(\upsilon(t))$ for λ the algebraic character attached to the data $\underline{c} = (c, (\underline{c}_{\nu_i}))$. This is exactly the renormalization term used in order to produce a finite slope admissible representation from a p-stabilization of an automorphic representation.

2. The second is the slope of a character of $\mathfrak{U}_p$ corresponding to an ordered (for each i) list of Satake parameters $(\beta, (\alpha_{i,-n}, \dots, \alpha_{i,n})_i)$ in $\overline{\mathbf{Q}}_p$ (via ι) obeying the symmetries $\alpha_{i,-n} = \alpha_{i,n}^{-1}$ and $\alpha_{i,0} = 1$ if n is odd. These Satake parameters correspond to the unramified character χ of the torus of G, where $(\alpha_{i,-n}, \dots, \alpha_{i,n})$ are the Satake parameters of the factor corresponding to the field F_i, and β is the image of $\varpi \mathbf{1}_n$ under the central character, where $\varpi \in E$ is a uniformizer with $\varpi^2 \in \mathbf{Q}_p$.

Let $F^{+\prime}$ be the normal closure in $\overline{\mathbf{Q}}_p$ of the compositum of the F_i^+, which by the hypothesis that p is unramified in F^+ is linearly disjoint from E, and let $F' = EF^{+\prime}$. Given any character $\lambda \in X^*(T_{/F'})$, the naturally defined function $\lambda : X_*(T_{/F'})^{\mathrm{Gal}(F'/\mathbf{Q}_p)} \to \mathbf{Q}$ is equal to the function λ_{slp} defined by

$$\lambda_{\mathrm{slp}} = \frac{1}{\#\,\mathrm{Gal}(F'/\mathbf{Q}_p)} \sum_{\sigma \in \mathrm{Gal}(F'/\mathbf{Q}_p)} \sigma(\lambda)$$

since the pairing is Galois equivariant. Moreover, the element $\lambda_{\mathrm{slp}} \in X^*(T_{/F'}) \otimes \mathbf{Q}$ is Galois invariant. By definition of υ we have $\mathrm{ord}_p(\lambda(\upsilon(u_{\mu^\vee(p)}))) = \mathrm{ord}_p(\lambda(u_{\mu^\vee(p)}))$, so since by defi-

nition the slope of $\lambda(v(t))$ is equal to the element of $X^*(T_{/F})^{\mathrm{Gal}(F'/\mathbf{Q}_p)} \otimes \mathbf{Q}$ which calculates $\mathrm{ord}_p(\lambda(u_{\mu^\vee(p)}))$ when paired with $\mu^\vee$, the slope of λ is λ_{slp}.

This makes the first calculation above very direct. Let λ be attached to $\underline{c}$. We want to calculate $\frac{1}{\#\,\mathrm{Gal}(F'/\mathbf{Q}_p)} \sum_{\sigma \in \mathrm{Gal}(F'/\mathbf{Q}_p)} \sigma(\lambda)$. We first average over the action of $\mathrm{Gal}(F^{+\prime}/\mathbf{Q}_p)$. For each orbit of the action of $G_{\mathbf{Q}_p}$ on the embeddings $\nu : F^+ \to \mathbf{R}$, this simply averages over the relevant $\underline{c}_\nu$ with ν in that orbit. The action of $\mathrm{Gal}(E/\mathbf{Q}_p)$ interchanges $c_{\nu,i}$ and $-c_{\nu,n+1-i}$ and sends c to $c + \sum_{\nu,i} c_{\nu,i}$, so the resulting average replaces c with $c + \frac{1}{2}\sum_{\nu,i} c_{\nu,i}$ and the data $(\underline{c}_\nu)$ is proportional to $\sum_{\nu \text{ orbit}} (c_{\nu,i} - c_{\nu,n+1-i})$. Observe that the result of the averaging is not necessarily a weight (for integrality reasons), but still satisfies the inequalities defining strict dominance.

The second calculation requires passing through a series of definitions. First, we write $u_{i,1}, \ldots, u_{i,m}$ for the matrices such that on the F_i factor $u_{i,j}$ is the u_j defined in Section 2.4.4 and 2.4.5, but at every other F_i factor, $u_{i,j}$ is the identity. We pick χ within its conjugacy class so that $\chi(u_{i,j}) = \alpha_{i,-j}$. Now suppose that ξ is a p-stabilization corresponding to the orderings $(\alpha_{i,-n}, \ldots, \alpha_{i,n})$, meaning that it is an eigenvector of Θ acting on the space $\Xi(\chi, 1)$ for the χ just defined, using the notation of Section 2.5.1. Then Θ acts on ξ with eigenvalue $(\chi \cdot \delta_{B'}^{-\frac{1}{2}})_\Theta$, where B' is the Borel on the associated split group as in the proof of Theorem 2.5.1, which means in particular that for any $t \in T^-$, the action of $[ItI]$ is given by $(\chi \cdot \delta_{B'}^{-\frac{1}{2}})(t)$. Recall that $\mathfrak{U}_p$ is the algebra generated by $[ItI]$ for $t \in T^-$, so this completely determines the character of $\mathfrak{U}_p$.

According to Urban's definition, the slope is determined by the pairing of this character with elements $\mu^\vee(p) \in T^{--}$, where $\mu^\vee$ is a cocharacter. One such element is, in the odd case, $\mathrm{diag}(p^{2m}, p^{2m-2}, \ldots, p^2, 1, p^{-2}, \ldots, p^{-2m})$ in the factor corresponding to every F_i, or $\mathrm{diag}(p^{2m}, \ldots, p^2, p^{-2}, \ldots, p^{-2m})$ everywhere in the even case. Let $\mu^\vee_{\mathrm{init}}$ denote the cocharacter such that $\mu^\vee_{\mathrm{init}}(p)$ is equal to this element. If μ_ξ is the slope of the character $(\chi \cdot \delta_{B'}^{-\frac{1}{2}})(t)$, we

have

$$(\mu_\xi, \mu_{\text{init}}^\vee) = \text{ord}_p((\chi\delta_{B'}^{\frac{1}{2}})(\mu_{\text{init}}^\vee(p))). \tag{2.22}$$

Now let $\mu_{i,j}^\vee$ be such that

$$\mu_{i,j}^\vee(p) = \text{diag}(1, \dots, 1, \underbrace{p}_{-j}, 1, \dots, 1, \underbrace{p^{-1}}_{j}, 1, \dots, 1),$$

in the F_i factor and is the identity on every other factor. Then up to a diagonal matrix with unit entries, this is the same matrix as $u_{i,j}^d$ (in the notation of Definition 2.4.12), where $\frac{1}{d} = \text{ord}_p(\varpi)$. (We happen to have $d = 2$ in fact.) Then $\mu_{\text{init}}^\vee + \mu_{i,j}^\vee \in X_*(T_{/F})^{+,\text{Gal}(F/\mathbf{Q}_p)}$ even though $\mu_{i,j}^\vee$ is not, so

$$(\mu_\xi, \mu_{\text{init}}^\vee + \mu_{i,j}^\vee) = \text{ord}_p((\chi\delta_{B'}^{\frac{1}{2}})((\mu_{\text{init}}^\vee + \mu_{i,j}^\vee)(p))).$$

Subtracting (2.22) from this equation, we find $(\mu_\xi, \mu_{i,j}^\vee) = \text{ord}_p((\chi\delta_{B'}^{\frac{1}{2}})(\mu_{i,j}^\vee(p)))$. Moreover, since χ and $\delta_{B'}^{\frac{1}{2}}$ are unramified, we have

$$\begin{aligned}(\mu_\xi, \mu_{i,j}^\vee) = \text{ord}_p((\chi\delta_{B'}^{\frac{1}{2}})(\mu_{i,j}^\vee(p))) &= \text{ord}_p((\chi\delta_{B'}^{\frac{1}{2}})(u_{i,j}^d)) \\ &= \begin{cases} d\,\text{ord}_p(\alpha_{i,-j} q^{\frac{m+j-1}{2}}) & n \text{ even} \\ d\,\text{ord}_p(\alpha_{i,-j} q^{\frac{m+j}{2}}) & n \text{ odd.} \end{cases}\end{aligned}$$

It follows that $\text{ord}_p(\alpha_{i,-j})$ and thus $\text{ord}_p(\alpha_{i,j})$ is determined entirely by its slope μ_ξ. One can also relate β to μ_ξ by using a central cocharacter; the calculation is identical so we omit it.

Proposition 2.6.14. *There exists an $M \in \mathbf{Z}_{\geq 0}$ so that if the absolute value of the pairing of μ_ξ with every positive coroot is at least N, the condition of Theorem 2.2.8 that $\alpha_{i,j} \neq q_i \alpha_{i,j'}$ and $\alpha_{i,j} \neq \alpha_{i,j'}$ for $j \neq j'$ must hold. (Here q_i is the size of the residue field of F_i.)*

Proof. We observe that $\text{ord}_p(\alpha_{i,j} q_i^{-1} \alpha_{i,j'}^{-1})$ and $\text{ord}_p(\alpha_{i,j}\alpha_{i,j'}^{-1})$ are related in a precise way to

$(\mu_\xi, \mu_{i,j}^\vee - \mu_{i,j'}^\vee)$, if neither j nor j' is 0, so a sufficiently large lower bound on $|(\mu_\xi, \mu_{i,j}^\vee - \mu_{i,j'}^\vee)|$ makes this relationship impossible. For the case where n is odd and j or j' is 0, we just use $|(\mu_\xi, \mu_{i,j}^\vee)|$ instead. □

It is easy to see that the hypothesis of Proposition 2.6.14 is satisfied for $\mu_\xi + k\lambda$ as in Proposition 2.6.13 for some lower bound on k in terms of M.

2.6.5 Behavior of finite slope automorphic representations with very regular weight

We have explained how to attach a finite slope admissible representation σ to a p-stabilized classical automorphic representation π. There should be a sense in which σ can be thought of as automorphic. In particular, there should be a class of *finite slope automorphic representations*, generalizing the notion of classical overconvergent modular forms. We refer to Urban [82, §4.3.4] for the definition of this class of representations, but mention only that by definition, they appear as subquotients of the representation of $\mathfrak{H}_p$ on the cohomology of certain distribution spaces $\mathfrak{D}_\lambda$ that generalize the construction of Stevens in the modular symbol case. Here λ is an *analytic* weight, which is the product of an algebraic weight λ_{alg} and a finite order character at p; see [82, §3.2.4] for more details. We will only need to consider weights where this character is trivial, though there are points of the eigenvariety under consideration that have a nontrivial character. The p-stabilization of classical regular cuspidal automorphic representations of non-critical slope, renormalized as described in Section 2.6.3, give examples of finite slope automorphic representations. In this section we consider the converse direction. We will utilize the observations made in Propositions 2.6.13 and 2.6.14 to study the existence and behavior of such forms when they appear in a family.

Suppose that π is a cohomological automorphic representation on one of the unitary groups G of Section 2.2.1 and has Iwahori level at a prime p that ramifies in E. Also let

G_0, H, and H_0 be defined as in Section 2.2.1. Let λ be the algebraic character of π_∞. Write π_0 for a constituent of the restriction of π to G_0; note that the underlying space $\pi_{0,p} = \pi_p$ by the results of Section 2.4.6. We also write τ_0 for the H_0-part of the base change of π; then τ_0 is a weak base change of π_0. Let $\theta : \mathfrak{R}_{S,p} \to \overline{\mathbf{Q}}_p^\times$ and the finite slope representation σ be associated to a p-stabilization ξ of π as in Section 2.6.3. (We do not assume that ξ is the p-stabilization constructed in Theorem 2.5.1.) Write μ_θ for the slope of θ as defined above. We have a ordered list of Satake parameters of π_0, $(\alpha_{-m}, \ldots, \alpha_m)$, associated to ξ. Then $\mu_\theta = \mu_\xi + \lambda_{\mathrm{slp}}$, where μ_ξ is the slope of θ before the twist by $\lambda(\upsilon(\cdot))$.

Now assume σ' is another finite slope admissible representation of the unitary group G, with associated $\mathfrak{R}_{S,p}$ character θ' and of analytic weight λ'. If θ' is p-adically close to θ, then in fact we have $\mu_\theta = \mu_{\theta'}$. This holds for all points sufficiently close to θ in a p-adic family of $\mathfrak{R}_{S,p}$-characters (or of finite slope representations). A natural way to produce from λ_{alg} a large class of very regular weights is to fix a regular dominant weight λ_{reg} and consider $(p-1)k\lambda_{\mathrm{reg}} + \lambda_{\mathrm{alg}}$ for the elements $k \in \mathbf{Z}_{>0} \subseteq \mathbf{Z}_p$ that lie in a small neighborhood of the origin in $\mathbf{Z}_p$. The benefit of this construction is that in "weight space", which we define later, these points all lie on a line. Then one can look at forms above these chosen points.

One consequence of non-criticality (see Theorem 2.8.3) is that an automorphic finite slope representation of non-critical slope that has nontrivial overconvergent Euler-Poincaré characteristic (in a suitable sense – see Section 2.8.1) is classical of the same (usual automorphic) multiplicity. Using that result, the preceding discussion shows that in a suitable family of finite slope forms mapping to a suffiently large weight space, classical points should be dense in a very strong sense.

We are interested not only in knowing whether σ' as above is classical, but, assuming that it is classical and that λ'_{alg} is sufficiently regular, that it satisfies local-global compatibility for the transfer to a general linear group.

Proposition 2.6.15. *Let G be a unitary group as in Section 2.2.1. Fix p where G_p is*

ramified and quasi-split and an element $\mu \in X^*(T_p)_{\mathbf{Q},+}$. Then for all sufficiently regular weights λ (where the bounds depend only on μ and the dimension of the Hermitian space, and the meaning of sufficiently regular is in terms of its pairing with the positive simple roots as in Proposition 2.6.14), any automorphic finite slope representation σ of Iwahori level at p, weight λ, slope μ, and nontrivial overconvergent Euler-Poincaré characteristic on the unitary group G is attached to a p-stabilization ξ of a classical representation π. Moreover, the base change τ of π to H satisfies local-global compatibility at p and the functorial restriction of π to G_0 is tempered.

Proof. The existence of π for sufficiently regular λ follows from Proposition 2.6.13 and Theorem 2.8.3. Let π_0 be an irreducible component of the restriction of π to G_0 and let τ_0 be the weak base change given by the GL_n component of the representation provided in Theorem 2.2.2. After possibly increasing the regularity bound, Proposition 2.6.14 shows that the hypotheses of Theorem 2.2.8 hold, which then implies that the $\alpha_{i,j}$ are in fact unitary and local-global compatibility at p holds between π_0 and τ_0. Since Theorem 2.2.2 also matches up the central characters, by the results of Section 2.4.6 we get full compatibility at p.

□

We now need to understand the p-adic Galois representation associated to the sufficiently regular finite slope representation σ examined in Proposition 2.6.15. More precisely, since these σ will be the "nice" elements of our p-adic family, we would like to know of the existence of a suitable crystalline period in the associated Galois representation, and that this crystalline period allows us to check the first condition of Theorem 2.6.6.

In the following, $u_{v,m}$ and $\varpi\mathbf{1}_n$ are as defined in Definition 2.4.12.

Proposition 2.6.16. *Let σ satisfy the hypotheses of Proposition 2.6.15, let π be the associated automorphic representation of G, and let $\rho_\pi : G_F \to \mathrm{GL}(V)$ be the Galois representation associated to π by Theorem 2.2.4, where V is an L-vector space of dimension n for a finite*

extension L of $\mathbf{Q}_p$. Also let $v|p$ be a place of F^+ over p, let $w|v$ be the place of F over v, fix an embedding $\nu : F_w \to L$, and let α be the eigenvalue of the Hecke operator $[Iu_{v,m}I]$ on the p-stabilization ξ of π corresponding to σ. (Here we mean the eigenvalue of this operator before the twist by $\lambda(\upsilon(\cdot))$.) Also let β be the image of $\varpi \mathbf{1}_n$ under the central character. Then $D_{\mathrm{cris},\nu}(V)^{\varphi^f=\alpha\beta} \neq 0$, where f is the degree of the residue field extension of $F_w/\mathbf{Q}_p$.

Now assume that we have a set $\{V_j\}_{j\in\mathbf{Z}_{\geq 1}}$ of Galois representations attached to σ_j as in the preceding paragraph, with corresponding eigenvalues α_j and β_j. However, assume the slope of every σ_j is μ. Assume that the weights λ_j have the following properties.

- *The Hodge-Tate weights of V_j with respect to ν are of the form $0 < k_{\nu,2,j} < \cdots < k_{\nu,n,j}$.*
- *The quantity $\lambda_j(\upsilon(\varpi u_{v,m}))$ is independent of j.*
- *The weights λ_j are increasing with j with respect to the usual partial order and the λ_j become arbitrarily regular in the sense that for any N there is some j so that the pairing of λ_j with any positive simple coroot is at least N.*

Then for j sufficiently large, V_j satisfies condition 1 of Theorem 2.6.6 with $Y = \alpha\beta$.

Proof. Showing $D_{\mathrm{cris},\nu}(V)^{\varphi^f=\alpha\beta} \neq 0$ is just a matter of tracing through the definitions to determine what the crystalline eigenvalues of V should be. If α is the eigenvalue of $[Iu_{v,m}I]$ on ξ, then by the results of Section 2.5.1, α is related to a Satake parameter α' of π_v by $\alpha = q^{\frac{m-1}{2}}\alpha'$. Then α' is a Satake parameter of τ_v by local-global compatibility. Now one refers to Theorem 2.2.4 to examine the Weil-Deligne representation associated to $\rho_\pi = \rho_\tau$, which is independent of the choice of ν. It follows from this and the discussion of Section 2.6.1 that ρ_π has crystalline Frobenius eigenvalues $q^{\frac{m-1}{2}}\alpha''\beta$, where α'' ranges over the Satake parameters of τ, including α'.

To see that V_j satisfies condition 1 of Theorem 2.6.6 for sufficiently large j, we apply Lemma 2.6.9, whose hypotheses are satisfied for a sufficiently regular weight λ_{alg} (which is

directly related to the Hodge-Tate weights as explained in the statement of Theorem 2.2.4). We can calculate the p-adic valuation of $\alpha_j\beta_j$ by dividing the eigenvalue of $I\varpi u_{v,m}I$ by the renormalization factor $\lambda_j(v(\varpi u_{v,m}))$. Since this normalization factor is assumed constant and the slope is constant, so is $\mathrm{ord}_p(\alpha_j\beta_j)$. Then sufficient regularity is enough to ensure the bound $k_{\nu,2,j} > \max(k, [F_w : \mathbf{Q}_p]\,\mathrm{ord}_p(\alpha_j\beta_j))$ holds for sufficiently large j.

□

The last part of Proposition 2.6.16 is only useful if it is possible for $\mathrm{ord}_p\,\lambda(u_{v,m}\varpi)$ to be constant while λ becomes very regular. We will see that this is the case for the family considered in Section 2.7.3.

2.6.6 Properties of a suitable family

The results of Section 2.6.5 tell us that for a Zariski-dense set of points in a suitable p-adic family, there is an associated p-stabilized automorphic representation whose Galois representation has a crystalline period whose eigenvalue is related to the action of $[I\varpi u_{v,m}I]$. We will construct a family of suitable Galois representations in Section 2.7.3. In this section, we describe the properties we will need of this family and prove the compatibility as a consequence.

Theorem 2.6.17. *Suppose that $\mathfrak{X}$ is a reduced affinoid rigid analytic space over a finite extension $L/\mathbf{Q}_p$, $\mathfrak{M}$ is a free coherent sheaf over $\mathfrak{X}$ equipped with a continuous action of G_F, and U is a function in $\mathfrak{O}_{\mathfrak{X}}(\mathfrak{X})$. We assume that $\mathfrak{X}$ is U-small. Suppose that E contains the normal closure of L and fix an embedding ν of L into E. Suppose, moreover, that there is a Zariski-dense set Σ of points of $\mathfrak{X}$ such that the reduction $\overline{\mathfrak{M}}_x = \mathfrak{M}_x \otimes_{\mathfrak{O}_{\mathfrak{X},x}} \kappa(x)$ to the residue field at x is a Galois representation ρ_x that is crystalline at the place $v|p$ with $D_{\nu,\mathrm{cris}}(\rho_x|_{G_{F_v}})^{\varphi^f=U(x)} \neq 0$ and that meets Condition 1 of Theorem 2.6.6 with respect to $U(x)$. We assume that there are functions $\kappa_{1,\nu},\ldots,\kappa_{k,\nu} \in \mathfrak{O}_{\mathfrak{X}}(\mathfrak{X})$ with $\kappa_{1,\nu} = 0$ so that at each $x \in$*

Σ, the Hodge-Tate weights with respect to ν are given by $\kappa_{1,\nu}(x) < \kappa_{2,\nu}(x) < \cdots < \kappa_{k,\nu}(x)$. Moreover, assume that Σ contains points that are arbitrarily regular in the sense that each $\kappa_{i,\nu}(x) - \kappa_{i-1,\nu}(x)$ is unbounded on Σ. Then for any point $y \in \mathfrak{X}$, $D_{\nu,\mathrm{crys}}(\overline{\mathfrak{M}}_y^{\mathrm{ss}})^{\varphi=U(y)} \neq 0$.

Proof. We apply Theorem 2.6.6 to show that $D_{\mathrm{crys}}(\overline{\mathfrak{M}}_y)^{\varphi=U(y)} \neq 0$ as follows. The third condition is just Zariski-density since $\mathfrak{X}$ is reduced, the first condition follows by Lemma 2.6.9 for sufficiently regular points, and the second follows from our requirement that the differences between weights are unbounded. Observe that if a G_K-representation V is an extension of V_1 by V_2, then the left exactness of $D_{\nu,\mathrm{crys}}(\cdot)^{\varphi=U(y)}$ implies that if $D_{\nu,\mathrm{crys}}(V)^{\varphi=U(y)} \neq 0$, either $D_{\nu,\mathrm{crys}}(V_1)^{\varphi=U(y)} \neq 0$ or $D_{\nu,\mathrm{crys}}(V_2)^{\varphi=U(y)} \neq 0$. By picking a Jordan-Holder series for $\overline{\mathfrak{M}}_y$, we deduce the result for $\overline{\mathfrak{M}}_y^{\mathrm{ss}}$ as needed.

□

Corollary 2.6.18. *Suppose that π is a regular cuspidal automorphic representation on the unitary group G of Section 2.2.1 and p is ramified in E. Let w be the weight of π. Assume that F_v is unramified for each $v|p$. Also assume that G_p is quasi-split, the central character is unramified at p, and that the restriction $\pi_{0,p}$ to the product of unitary groups is K-spherical for the special maximal compact subgroup chosen in Sections 2.4.4 and 2.4.5. Fix $v|p$ in F and suppose that $U' = [Iu_{v,m}I]$ acts on π_p by multiplication by an element α of complex absolute value greater than $q^{\frac{n-1}{2}}$. Then it is impossible for the p-adic Galois representation ρ_π to be placed in a p-adic family satisfying the hypotheses of Theorem 2.6.17 with respect to the product U of U' with the Hecke operator $\varpi \mathbf{1}_n I$.*

Proof. This follows from Theorem 2.6.17 and Proposition 2.6.2, since the weight of the Weil-Deligne representation attached to $\rho_\pi|_{G_{F_v}}$ for $v|p$ is $-w+n-1$ and β has complex absolute value $q^{-\frac{w}{2}}$.

□

2.7 Families of automorphic forms and Galois representations

In light of Corollary 2.6.18, in order to prove local-global compatibility at p it suffices to put a representation π that fails the hypothesis of Theorem 2.2.8 into a suitable p-adic family in the sense of Theorem 2.6.17. In order to construct such a family, we utilize work of Urban [82] that constructs eigenvarieties for reductive groups satisfying the Harish-Chandra condition. Under a condition, we produce a family passing through the p-stabilization ξ of π constructed in Theorem 2.5.1. Using the existence of Galois representations attached to classical forms, we construct a pseudorepresentation over this family. Results of Wiles [85] and Taylor [78] then allow us to construct an actual family of Galois representations passing through ρ_π.

We begin by setting up the general situation. Let G be a reductive group over $\mathbf{Q}$ satisfying the Harish-Chandra condition and fix a prime p. Let $K^p = \prod_{v \neq p,\infty} K_v$ be a compact open subgroup of $G(\mathbf{A}_{\mathbf{Q},f}^p)$. We write B, T, and N for the $\mathbf{Z}_p$-groups as described in Section 2.6.3, write $I \subseteq G(\mathbf{Q}_p)$ for the Iwahori subgroup, and write $\mathfrak{H}_p(K^p)$ for the Hecke algebra defined there. Also write $\mathfrak{R}_{S,p}$ for the tame part of the Hecke algebra, also as defined in Section 2.6.3.

Definition 2.7.1. A p-adic weight is a continuous group homomorphism $\lambda : T(\mathbf{Z}_p) \to \overline{\mathbf{Q}}_p^\times$. If λ_{alg} is an algebraic weight, then the associated p-adic weight is simply the composite $T(\mathbf{Z}_p) \to T(F) \overset{\lambda_{\mathrm{alg}}}{\to} F^\times \subseteq \overline{\mathbf{Q}}_p^\times$. An arithmetic weight is the product of an algebraic dominant weight with a finite order character.

We define the analytic weight space at p as follows. We write $Z_p(K^p)$ for the p-adic closure of $Z(\mathbf{Q}) \cap K^p T(\mathbf{Z}_p)$. Then the weight space $\mathfrak{W}$ is the space over $\mathbf{Q}_p$ with $\mathfrak{W}(\overline{\mathbf{Q}}_p) = \mathrm{Hom}_{\mathrm{cts}}(T(\mathbf{Z}_p)/Z_p(K_p), \overline{\mathbf{Q}}_p^\times)$. In general its dimension depends on the defect to the truth of the Leopoldt conjecture. (See [82, §3.4, §4.3.2] for more details on this construction.) Every

continuous map $T(\mathbf{Z}_p)/Z_p(K_p) \to \overline{\mathbf{Q}}_p^\times$ is locally analytic for some radius; in our application we will only need to consider maps that are locally analytic with radius 1 and correspond to points of $\mathfrak{W}(\mathbf{Q}_p)$.

In our case, $Z(\mathbf{Q}) = E^\times$, and only the finite group of units in $E^\times$ can possibly lie in $K^pT(\mathbf{Z}_p)$. By shrinking K^p slightly, we can even render $Z_p(K^p)$ trivial. In any case, the Leopoldt conjecture subtleties do not arise for our unitary groups. However, they will arise for Levi subgroups of our unitary groups in Section 2.8.

2.7.1 Urban's eigenvarieties

Suppose that π is a cuspidal automorphic representation of G such that π_∞ is a regular discrete series representation. Then π has an associated arithmetic weight λ in $\mathfrak{X}(L)$ for some finite extension $L/\mathbf{Q}_p$. Fix a p-stabilization of π, and write σ for the associated renormalized finite slope representation. Also write θ for the associated character of $\mathfrak{R}_{S,p}$.

In the following result, we use the notation $m^{\mathrm{cl}}(\sigma', \mathbf{w}(z))$ for an automorphic multiplicity. We will explain this notation in Section 2.8.1. We will not need the definition for the application here; see Remark 2.7.3.

Theorem 2.7.2 ([82, Theorem 5.4.4])**.** *Suppose that the finite slope admissible representation σ on G is associated to a cuspidal automorphic representation π of strictly dominant arithmetic weight λ. If σ is not critical with respect to λ, then we have the following result.*

There exists

- *an affinoid neighborhood $\mathfrak{U} \subseteq \mathfrak{W}$ of the point λ in weight space,*
- *a finite cover $\mathbf{w} : \mathfrak{F} \to \mathfrak{U}$ of $\mathfrak{U}$,*
- *a homomorphism $\theta_{\mathfrak{F}} : \mathfrak{R}_{S,p} \to \mathfrak{O}_{\mathfrak{F}}(\mathfrak{F})$,*
- *a character distribution $I_{\mathfrak{F}} : \mathfrak{H}_p(K^p) \to \mathfrak{O}(\mathfrak{F})$,*

- *a point* $y \in \mathfrak{F}(\overline{\mathbf{Q}}_p)$ *over* λ,
- *a Zariski dense subset* $\Sigma \subseteq \mathfrak{F}(\overline{\mathbf{Q}}_p)$ *such that* $\mathbf{w}(z)$ *is an arithmetic weight for all* $z \in \Sigma$, *and*
- *for each* $z \in \Sigma$, *a finite set* Σ_z *of irreducible finite slope cohomological cuspidal representations of weight* $\mathbf{w}(z)$

satisfying the following properties.

- *The specialization of* $\theta_{\mathfrak{F}}$ *to* y *is* θ, *and the character distribution* I_σ *of* σ *is an irreducible component of the specialization of* $I_{\mathfrak{F}}$ *to* y.
- *For any point* $z \in \Sigma$, *the specialization* θ_z *is a character occuring in the representation of* $\mathfrak{R}_{S,p}$ *on* σ'^{IK^p} *for all* $\sigma' \in \Sigma_z$.
- *For each* $z \in \Sigma$, *the specialization* I_z *satisfies*

$$I_z(f) = \sum_{\sigma' \in \Sigma_z} m^{\mathrm{cl}}(\sigma', \mathbf{w}(z)) \operatorname{Tr}(\sigma'(f)).$$

- *The set* Σ_z *is a singleton for a Zariski dense subset of the* z.

Remark 2.7.3. We make some clarifying comments about this theorem.

1. There is a global construction of an eigenvariety $\mathfrak{E}$ over the entirety of weight space, rather than just over a local piece like the $\mathfrak{U}$ of Theorem 2.7.2. However, a point $x \in \mathfrak{E}$ parametrizes the data of a weight $\mathbf{w}(x) \in \mathfrak{W}$ and character θ_x of $\mathfrak{R}_{S,p}$, whereas a point of $\mathfrak{F}$ recovers a character distribution on $\mathfrak{H}_p(K^p)$. There is a natural finite map from $\mathfrak{F}$ to $\mathfrak{E}$, and each point y' of the fiber over a point $x \in \mathfrak{E}$ corresponds to a linear combination of characters of finite slope representations $\sigma_{y'}$ whose associated character $\theta_{y'}$ of $\mathfrak{R}_{S,p}$ is equal to θ_x. In fact, one must pass to a finite extension in order to split up the

representation $\mathfrak{H}_p(K^p)$ into isotypical components; one of these components contains the initial representation σ one is interested in producing a family around, and the finite covering $\mathfrak{F} \to \mathfrak{E}$ used in the theorem corresponds to this isotypical component.

2. As a consequence of the Čebotarev density theorem, the Galois representation is the same for any two classical finite slope representations with the same associated character of $\mathfrak{R}_{S,p}$, so we can use the point of $\mathfrak{E}$ in attaching a family of Galois representations and pull back to $\mathfrak{F}$ if desired. For this reason, the inclusion of the multiplicity $m^{\mathrm{cl}}(\sigma', \mathbf{w}(z))$ will not affect our study of the Galois representation as long as we limit ourselves to considering the action of Hecke operators in $\mathfrak{R}_{S,p}$.

2.7.2 Pseudorepresentations and p-adic families of Galois representations

Pseudorepresentations were introduced by Wiles [85] in the case of dimension 2 and generalized by Taylor [78, §1] to any dimension.

Definition 2.7.4 ([78, §1.1])**.** Let G be a group and R be a commutative ring. Then a *pseudorepresentation* $t : G \to R$ of dimension d is a map of sets additionally satisfying

1. $t(1) = d$,
2. $t(g_1 g_2) = t(g_2 g_1)$ for $g_1, g_2 \in G$, and
3. $\sum_{\sigma \in S_{d+1}} \mathrm{sign}(\sigma) T_\sigma(g_1, \ldots, g_{d+1}) = 0$ for all $g_1, \ldots, g_{d+1} \in G$, where $T_\sigma : G^{d+1} \to R$ is defined by
$$(g_1, \ldots, g_{d+1}) \mapsto T(g_{i_1^1} \ldots g_{i_{k_1}^1}) \ldots T(g_{i_1^n} \ldots g_{i_{k_n}^n})$$
where $\sigma = (i_1^1 \ldots i_{k_1}^1) \ldots (i_1^n \ldots i_{k_n}^n)$ is the cycle decomposition.

The *kernel* of t is the set of $h \in G$ such that $T(gh) = T(g)$ for all $g \in G$.

By linearity we can instead regard $t : G \to R$ as a map $R[G] \to R$, which we will denote by the same letter. If R' is an R-algebra, we write $t \otimes_R R'$ for the induced map $R'[G] \to R'$.

Taylor proves the following theorem relating pseudorepresentations and representations.

Theorem 2.7.5 ([78, Theorem 1])**.** *We have the following two facts.*

1. *The trace* $\operatorname{Tr}\rho$ *of a representation* $\rho : G \to \mathrm{GL}_d(R)$ *is a pseudorepresentation of dimension* d*. If* R *is a field of characteristic 0, the kernel of* $\operatorname{Tr}\rho$ *is the kernel of the semisimplification of* ρ*.*

2. *If* $t : G \to R$ *is a pseudorepresentation of dimension* d *and* R *is an algebraically closed field of characteristic 0, then there is a true semisimple representation* $\rho : G \to \mathrm{GL}_d(R)$*, unique up to conjugation, with* $\operatorname{Tr}\rho = t$*.*

Moreover, if we give topologies to G *and* R *and add a continuity hypothesis to* t*, then* ρ *is continuous as well.*

In the context of a pseudorepresentation valued in the functions on a rigid space, it is possible to give a slightly more precise version of this result.

Proposition 2.7.6 ([3, Lemma 7.8.11])**.** *Let* $\mathfrak{X}$ *be a reduced rigid analytic space over a finite extension of* $\mathbf{Q}_p$ *and let* $t : G \to \mathfrak{O}_{\mathfrak{X}}(\mathfrak{X})$ *be a continuous* m*-dimensional pseudorepresentation of a topological group* G*. Let* $\mathfrak{U} \subseteq \mathfrak{X}$ *be an open affinoid subdomain of* $\mathfrak{X}$*. Then there is a normal affinoid* $\mathfrak{Y}$*, a finite surjective map* $\pi : \mathfrak{Y} \to \mathfrak{U}$ *that sends each irreducible component of* $\mathfrak{Y}$ *surjectively onto an irreducible component of* $\mathfrak{U}$*, and a coherent torsion free* $\mathfrak{O}_{\mathfrak{Y}}(\mathfrak{Y})$*-module* $\mathfrak{M}$ *of generic rank* m *equipped with a continuous representation* $\rho : G \to \mathrm{Aut}_{\mathfrak{O}_{\mathfrak{Y}}(\mathfrak{Y})}(\mathfrak{M})$ *of generic trace* t*.*

Generically, ρ *is semi-simple and equal to the direct sum of absolutely irreducible representations. For* y *in the complement of a closed analytic subvariety,* $\mathfrak{M}_y$ *is free of rank*

m over $\mathfrak{O}_{\mathfrak{Y},y}$ and the representation $\mathfrak{M}_y \otimes_{\mathfrak{O}_{\mathfrak{Y},y}} \overline{\kappa(y)}$, is semisimple and isomorphic to the representation associated to $t \otimes_{\mathfrak{O}_{\mathfrak{Y}}(\mathfrak{Y})} \overline{\kappa(y)}$.

2.7.3 Proof of Theorem 2.2.5 in the non-critical case

Suppose that there is some place $v|p$ of F^+ so that there are Satake parameters $\alpha_{v,i} = q_v\alpha_{v,j}$; otherwise Theorem 2.2.8 gives compatibility. Let ξ be the p-stabilization constructed in Theorem 2.5.1 for the unitary group over F_v^+ and chosen arbitrarily at places other than v. Write $[Iu_{v,m}\varpi I]$ for Hecke operator associated to the element $(\omega\mathbf{1}_n)\cdot u_{v,m}$ as defined in Definition 2.4.12. Assuming that the finite slope representation σ attached to ξ meets the non-criticality hypothesis of Theorem 2.7.2, we construct an associated p-adic family of Galois representations and prove the non-critical case of Theorem 2.2.5.

We now apply Theorem 2.7.2; retain the notation used there. We will carefully pick a strictly dominant integral weight λ_{dom} on G. Recall that λ_{dom} is attached to a datum $\underline{c} = (c, \underline{c}_\nu)_{\nu:F^+\to\mathbf{R}}$. Write $n = 2m+1$ or $2m$ if n is odd or even, respectively. We use the datum given by $c = -2m$ and

$$\underline{c}_\nu = \begin{cases} (2m, 2m-2, \ldots, 2, 0, -2, \ldots, -2m) & n \text{ odd} \\ (2m, 2m-2, \ldots, 2, -2, \ldots, -2m) & n \text{ even} \end{cases}$$

for all ν. We observe that this datum satisfies

$$2c + \left(\sum_{i,\nu} c_{\nu,i}\right) + 2c_{\nu,1} - 2c_{\nu,n} = 0, -c - c_{\nu,1} = 0, \text{ and } c + \left(\sum_{\nu,i} c_{\nu,i}\right) - c_{\nu,n} = 0, \quad (2.23)$$

and is strictly dominant. The first relation in (2.23), combined with our choice of the section υ (see (2.21)) and the factor of 2 multiplying all the c_i, will translate into analyticity for the *classical* action of the Hecke operator $[Iu_{v,m}\varpi I]$ on a Zariski dense subset of classical points

in the family we consider. More precisely, we will use the relation $\lambda_{\mathrm{dom}}(u_{v,m}\varpi) = 1$. The second and third relations translate into fixing the smallest Hodge-Tate weight under each embedding of F into $\mathbf{C}$.

Then the space of points of $\mathfrak{W}$ that may be written in the form $\lambda+\alpha\lambda_{\mathrm{dom}}$ for some $\alpha \in \mathbf{Q}_p$ is a line that intersects $\mathfrak{U}$ nontrivially; let this intersection be denoted by $\mathfrak{Z}$. Since $\mathfrak{F} \to \mathfrak{U}$ is a surjective cover by an equidimensional variety of dimension $\dim \mathfrak{U}$ and $\mathfrak{Z}$ is cut out by $\dim \mathfrak{F} - 1$ functions, every irreducible component of the fiber product $\mathfrak{F} \times_{\mathfrak{U}} \mathfrak{Z}$ has dimension at least 1. Moreover, since the base change of a finite morphism is finite and $\mathfrak{F}$ surjects onto $\mathfrak{U}$, each irreducible component has dimension exactly 1. Write $\mathfrak{F}'$ for the irreducible component containing the point y.

Let C be a positive integer large enough that $\lambda + \alpha p(p-1)\lambda_{\mathrm{dom}}$ meets the conditions of Propositions 2.6.15 and 2.6.16 for $\alpha \in \mathbf{Z}_{>C}$. Then for all $\alpha \in \mathbf{Z}_{>C}$ such that $\lambda + \alpha p(p-1)\lambda_{\mathrm{dom}} \in \mathfrak{U}$, every preimage in $\mathfrak{F}'$ of $\lambda + \alpha p(p-1)\lambda_{\mathrm{dom}}$ is a classical point whose associated Galois representation is crystalline. These points are Zariski-dense in $\mathfrak{Z}$, so since $\mathfrak{F}'$ is an irreducible finite cover, the preimage of this set is also Zariski-dense in $\mathfrak{F}'$. The trace of the action by a Frobenius element at a place of F over a split place of E is given by a fixed Hecke operator, which corresponds to a rigid analytic function on the family. Since these are dense in the Galois group the trace of any element is given by a rigid analytic function. It follows that we obtain a pseudorepresentation $t : G_F \to \mathfrak{O}_{\mathfrak{F}'}(\mathfrak{F}')$ of dimension m.

Note that $\mathfrak{F}'$ is already affinoid since it is a finite cover of an affinoid. We now apply Proposition 2.7.6 to $\mathfrak{F}'$, which gives a normal finite cover $\mathfrak{F}''$ of $\mathfrak{F}'$ and a torsion-free module $\mathfrak{M}$ over $\mathfrak{F}''$ of generic rank m with an action $\rho : G_F \to \mathrm{Aut}_{\mathfrak{O}_{\mathfrak{F}'}(\mathfrak{F}')}(\mathfrak{M})$. However, the affinoid algebra of each irreducible component of $\mathfrak{F}''$ is a Dedekind domain, so $\mathfrak{M}$ is locally free of rank m. Fix a preimage y' of y in $\mathfrak{F}''$ and an irreducible affinoid neighborhood $\mathfrak{G}$ of y' such that $\mathfrak{M}$ is free and every point $z \in \mathfrak{G}$ over a weight $\lambda + \alpha p(p-1)\lambda_{\mathrm{dom}}$ with $\alpha \in \mathbf{Z}_{>C}$ has $\mathfrak{M}_z \otimes_{\mathfrak{O}_{\mathfrak{G},z}} \overline{\kappa(z)}$ isomorphic to the representation attached to $t \otimes_{\mathfrak{O}_{\mathfrak{G}}(\mathfrak{G})} \overline{\kappa(z)}$, where we have

used $\mathfrak{M}$ again for the restriction to $\mathfrak{G}$ and t again for the pseudo-representation valued in $\mathfrak{O}_{\mathfrak{G}}$ given by pullback along the projections. We write Σ (ignoring the one used in Theorem 2.7.2) for the preimages of the $\lambda + \alpha p(p-1)\lambda_{\mathrm{dom}}$ with $\alpha \in \mathbf{Z}_{>C}$.

Remark 2.7.7. We note that $y \notin \Sigma$, and so we only have an isomorphism of $(\mathfrak{M}_y \otimes \overline{\kappa(y)})^{\mathrm{ss}}$ with the Galois representation attached to our original π.

The Hodge-Tate weights of the Galois representations with respect to each embedding of F in $\mathbf{C}$ at points of Σ are specializations of rigid analytic functions $\kappa_{i,\nu}$ given by the recipe of Theorem 2.2.4 applied to $\lambda + \alpha p(p-1)\lambda_{\mathrm{dom}}$. It follows from Zariski-density of these points that the Sen polynomial of $\mathfrak{M}$ is given by $\prod_i (X + \kappa_{i,\nu})$. We twist the entire family by $\mathbf{C}(\kappa_{1,\nu}(y'))$ so that $0 = \kappa_{1,\nu} < \kappa_{2,\nu} < \cdots < \kappa_{n,\nu}$ for every point in Σ. (The constancy of the function $\kappa_{1,\nu}$ follows from (2.23)).) This twisting has the effect of changing the crystalline eigenvalues, but after applying the crystalline interpolation result, we can untwist, so there is no net effect.

The Hecke operator $[Iu_{v,m}\varpi I]$ has image in $\mathfrak{O}_{\mathfrak{G}}(\mathfrak{G})$ via pullback from $\mathfrak{F}$. Moreover, due to (2.23), the renormalization of the action of the action of the classical Hecke operator $[Iu_{v,m}\varpi I]$, which at $z \in \Sigma$ of weight λ_z is by multiplication by $\lambda_z(\upsilon(u_{v,m}\varpi))$, is the same at every classical point of our family. Thus the action of the true, unnormalized Hecke operator $[Iu_{v,m}\varpi I]$ on the classical automorphic representations associated to a point of Σ is also given by a rigid analytic function, related to the image of $[Iu_{v,m}\varpi I]$ in $\mathfrak{O}_{\mathfrak{G}}(\mathfrak{G})$ by division by the number $\lambda(\upsilon(u_{v,m}\varpi))$. We write U for this function. Finally, we shrink $\mathfrak{G}$ again to a smaller affinoid neighborhood of y' so that it is small (in the sense of Section 2.6.2) with respect to the action of U. The resulting family meets all the conditions of Theorem 2.6.17, as needed. As explained in Corollary 2.6.18, since the family both meets the conditions of Theorem 2.6.17 and U acts (due to our choice of p-stabiliation) by an eigenvalue whose complex absolute value violates purity, our original assumption that $\alpha_i = q_v\alpha_j$ for some pair of Satake parameters of π_p must be incorrect, and by Theorem 2.2.8 and the compatibility

of central characters, we conclude local-global compatibility for π at p.

2.8 Classical and overconvergent automorphic multiplicities

So far in this paper we have used the work of Urban [82] as a black box. However, in order to improve our main theorem so that we no longer require the nebulous hypothesis that a certain p-stabilization of π is non-critical, we will need to be more precise about what Urban proves. The key new ingredient is that Urban's eigenvarieties contain a number of non-classical points over which the Galois representation is still classical. The reason for this is that the data of σ at p is a merger of algebraic information from π_∞ and p-adic information from π_p. There are natural operators (coming from the theory of the BGG resolution) that change the algebraic aspect of σ, rendering it non-classical, but which do not modify places away from p. It follows from the construction of the Galois representation that the pseudorepresentation is unaffected by these operators. We have been careful to write Theorem 2.6.17 so it applies equally as well to the images of σ under such operators once we understand how to place these images into a family. The net effect will be a weakening of hypotheses in our Theorem 2.2.5.

2.8.1 Automorphic cohomology

It would take us too far afield to present all of the relevant definitions involved in the overconvergent automorphic cohomology constructed by Urban [82, §3, §4]. Instead, we present the facts that will be necessary for our application. Nothing in this section is original; it is all taken from Urban's paper.

Definition 2.8.1 ([82, Proposition 4.3.5])**.** There exists a distribution $\mathfrak{D}_\lambda(E)$ for each an-

alytic weight λ, where the coefficient field E is a finite extension of $\mathbf{Q}_p$. Then one can examine its cohomology in degree q, which Urban writes $H^q_{\mathrm{fs}}(\widetilde{S}_G, \mathfrak{D}_\lambda(E))$. (Here $\widetilde{S}_G$ is a limit of symmetric spaces of G over varying tame levels and $\mathfrak{D}_\lambda(E)$ is a distribution space.) This cohomology has a natural representation of $\mathfrak{H}_p$ whose trace decomposes as

$$\mathrm{Tr}(f : H^q_{\mathrm{fs}}(\widetilde{S}_G, \mathfrak{D}_\lambda(E))) = \sum_\sigma m^q(\sigma, \lambda) J_\sigma(f)$$

for some finite slope admissible representations σ and multiplicities $m^q(\sigma, \lambda)$, which are defined by this property. We define σ to be *automorphic of weight* λ if $m^q(\sigma, \lambda) \neq 0$ for some q. We define the virtual finite slope character distribution $I^\dagger_G(f, \lambda)$ to be the alternating sum

$$I^\dagger_G(f, \lambda) = \sum_q (-1)^q \,\mathrm{Tr}(f : H^q_{\mathrm{fs}}(\widetilde{S}_G, \mathfrak{D}_\lambda(E))) = \sum_\sigma \sum_q (-1)^q m^q(\sigma, \lambda) J_\sigma(f).$$

Then one defines the overconvergent Euler-Poincaré multiplicity by

$$m^\dagger_G(\sigma, \lambda) = \sum_q (-1)^q m^q(\sigma, \lambda).$$

We have similar definitions for classical multiplicities. Namely, if λ is now arithmetic, one sets

$$I^{\mathrm{cl}}_G(f, \lambda) = \sum_q (-1)^q \,\mathrm{Tr}(f : H^q_{\mathrm{fs}}(\widetilde{S}_G, \mathbf{V}^\vee_\lambda(E))) = \sum_\sigma m^{\mathrm{cl}}(\sigma, \lambda) J_\sigma(f),$$

where $\mathbf{V}^\vee_\lambda(E)$ is now a space of locally algebraic functions transforming by λ and $m^{\mathrm{cl}}(\sigma, \lambda)$ is again an Euler-Poincaré multiplicity. This quantity is related in a precise way to the usual Euler-Poincare multiplicity

$$m_{\mathrm{EP}}(\pi_f, \lambda) = \sum_q (-1)^q \dim_{\mathbf{C}} \mathrm{Hom}_{G_f}(\pi_f, H^q(\widetilde{S}_G, \mathbf{V}^\vee_\lambda(\mathbf{C})))$$

of π, which is nonzero since λ is regular. If σ is a p-stabilized representation attached to π_f, we have

$$m^{\mathrm{cl}}(\sigma,\lambda) = m_{\mathrm{EP}}(\pi_f,\lambda) \times \dim \mathrm{Hom}_{\mathfrak{H}_p}(\sigma, (\pi_f|_{\mathfrak{H}_p})^{\mathrm{ss}}).$$

In particular, this quantity is also nonzero.

We next describe some of the ideas behind Urban's classicality result. The main idea is to compare overconvergent multiplicities with classical ones. For each simple root α of the (absolute) root system of G, there is an operator $\Theta_\alpha^* : \mathfrak{D}_{s_\alpha * \lambda}(L) \to \mathfrak{D}_\lambda(L)$, which induces a morphism on cohomology. Now let λ be an arithmetic weight. Urban calculates an exact sequence built from the sum of Θ_α^* (which is part of a locally analytic version of the BGG resolution)

$$\bigoplus_\alpha \mathfrak{D}_{s_\alpha * \lambda}(L) \to \mathfrak{D}_\lambda(L) \to V_\lambda^\vee(L) \to 0,$$

where $V_\lambda^\vee(L)$ is a space of locally analytic functions that for the purposes of *finite slope* cohomology, calculate the same cohomology as the algebraic representation of weight λ.

One now observes that if σ is non-critical, the operators Θ_α^* take σ to something of slope not in $X^*(T)_{\mathbf{Q},+}$, which one shows cannot appear in cohomology. One can deduce a classicality result from this observation. However, we would like to have a more precise formula involving the automorphic multiplicities. Urban achieves this using the whole BGG resolution.

Definition 2.8.2. We write $l(w)$ for the length of an element of the Weyl group and define a modified operation of the Weyl group on an arithmetic weight λ as follows. We set $w * \lambda$ to have the same finite order character as λ, and algebraic part $w(\lambda_{\mathrm{alg}} + \rho) - \rho$, where ρ is the half-sum of positive roots.

We recall from (2.21) the multiplicative section $\upsilon : T(\mathbf{Q}_p)/T(\mathbf{Z}_p) \to T(\mathbf{Q}_p)$. For an arithmetic weight λ, Urban defines a twisting operation on the Hecke algebra $\mathfrak{H}_p$ by $f \mapsto f^{w,\lambda}$, where $f = f^p \otimes u_t$ maps to $f^{w,\lambda} = (w * \lambda_{\mathrm{alg}} - \lambda_{\mathrm{alg}})(\upsilon(t)) f^p \otimes u_t$. This gives a twisting operation

$\sigma \mapsto \sigma^{w,\lambda}$ on finite slope admissible representations σ. The slope of this representation is given by $\mu_{\sigma^{w,\lambda}} = \mu_\sigma - \lambda_{\mathrm{alg}} + w * \lambda_{\mathrm{alg}}$. Notice that if σ came from a classical form π of weight λ_{alg}, this is the slope it would have if we pretended that its weight is $w * \lambda_{\mathrm{alg}}$. Also notice that σ is unchanged by this twisting at all places away from p. This twisting operation is exactly defined to align with the Θ_α^* actions on cohomology.

Using this notion and a spectral sequence arising from the BGG resolution, Urban derives the following consequence.

Theorem 2.8.3 ([82, Corollary 4.3.12])**.** *Suppose that σ is an automorphic finite slope representation of arithmetic weight λ. Then*

$$m_G^{\mathrm{cl}}(\sigma, \lambda) = \sum_{w \in W} (-1)^{l(w)} m_G^\dagger(\sigma^{w,\lambda}, w * \lambda). \tag{2.24}$$

If σ is non-critical with respect to λ, then all but one of the terms on the right vanish and

$$m_G^{\mathrm{cl}}(\sigma, \lambda) = m_G^\dagger(\sigma, \lambda).$$

Suppose that π is an automorphic representation and σ is a finite slope representation given by a choice of p-stabilization. If $\sigma^{w,\lambda}$ appears in a p-adic family with a dense set of classical points, the pseudorepresentation construction would eventually associate to it a Galois representation whose semi-simplification would have to be ρ_π since its trace is the same. As a reality check, we show that in our unitary group setting, this is consistent with the action $w * \lambda$ on the weight space, which in turn gives the Hodge-Tate weights of ρ_π.

Proposition 2.8.4. *The recipe of Theorem 2.2.4 for producing the Hodge-Tate weights attached to λ gives the same weights for λ and $w * \lambda$.*

Proof. If λ corresponds to $\underline{c}$ as in Section 2.2.2, the Hodge-Tate weights are given by $\{c + j - 1 - c_{\nu,j}\}$ and $\{c' + j - 1 + c_{\nu,n+1-j}\}$ for $c' = c + \sum_\nu \sum_j c_{\nu,j}$. The weight $\rho = \underline{d}$

is attached to the (possibly only half-integral) data $d = 0$ and $d_{\nu,j} = \frac{m+1-2j}{2}$ for all ν. It suffices to check the result for w a transposition $(j, j+1)$, and we can assume $c = 0$. Moreover, by symmetry we need only examine $\{c + j - 1 - c_{\nu,j}\}$. Then $w\lambda$ contributes $j - 1 - c_{\nu,j+1}$ and $j - c_{\nu,j}$ in place of $j - 1 - c_{\nu,j}$ and $j - c_{\nu,j+1}$ to the calculation of Hodge-Tate weights. Also, $w\rho$ contributes $j - 1 - \frac{m+1-2(j+1)}{2} = 2j - \frac{m+1}{2}$ and $j - \frac{m+1-2j}{2} = 2j - \frac{m+1}{2}$ in place of $j - 1 - \frac{m+1-2j}{2} = 2j - \frac{m+3}{2}$ and $j - \frac{m+1-2(j+1)}{2} = 2j - \frac{m-1}{2}$. Thus the contribution of $w(\rho) - \rho$ to the Hodge-Tate weight indexed by j in $\{c + j - 1 - c_{\nu,j}\}$ is addition by 1 and the contribution to the weight indexed by $j+1$ is subtraction by 1. So the resulting Hodge-Tate weights attached to $w(\lambda_{\mathrm{alg}} + \rho) - \rho$ are $j - 1 - c_{\nu,j+1} + 1$ and $j - c_{\nu,j} - 1$, which are the position $j + 1$ and j Hodge-Tate weights attached to λ_{alg} as needed. □

Our strategy for constructing families through finite slope automorphic representations of critical slope is to place these non-arithmetic weight forms into families. There is one additional obstruction beyond understanding the various $m_G^\dagger(\sigma^{w,\lambda}, w * \lambda)$, however. Namely, the eigenvariety can only be attached to a family of *effective* finite slope character distributions, while $I^\dagger(f, \lambda)$ is only virtual. To rectify this, we need to consider only the *cuspidal* part of these character distributions. To do this, we need to make another definition.

Definition 2.8.5 ([82, §4.1.8])**.** Let G be a reductive group over $\mathbf{Q}$ that is quasi-split at p. Fix a minimal parabolic P_0 of G, a Borel subgroup B of $G_{/\mathbf{Q}_p}$ with $B_0 \subseteq P(\mathbf{Q}_p)$, and a maximal torus $T \subseteq B$. Suppose that P is a standard (i.e. containing P_0) parabolic of the unitary group G, where P is defined over $\mathbf{Q}$, so that $P(\mathbf{Q}_p)$ also contains B. Then we can write $P = MN$ where $M(\mathbf{Q}_p)$ contains T. We refer to such an M as a standard Levi subgroup of G, and let L_G be the set of them. Since M is reductive, there are Hecke algebras $\mathfrak{H}_p(M)$ and $\mathfrak{U}_p(M)$. For such an M, let W^M be the set of elements of the Weyl group W of G such that $w^{-1}(\alpha) > 0$ for every α that is a positive root with respect to the Borel $B \cap M$ and torus T.

For each $M \in L_G$ and $w \in W^M$ we define a map $\mathfrak{H}_p(G) \to \mathfrak{H}_p(M)$, which will be denoted $f \mapsto f^{\mathrm{reg}}_{M,w}$. Before giving this definition, we give a classical variant. Namely, for $f \in C_c^\infty(G(\mathbf{A}_f))$ and $M \in L_G$, we write $f_M \in C_c^\infty(M(\mathbf{A}_f))$ for the operator defined by

$$f_M(m) = \int_{K_{\max} \times N(\mathbf{A}_f)} f(k^{-1}mnk)dndk,$$

where $K_{\max} \subseteq C_c^\infty(G(\mathbf{A}_f))$ is a fixed maximal open compact subgroup taken to have measure 1. Then we have $\mathrm{Tr}(f : \mathrm{Ind}_{P(\mathbf{A}_f)}^{G(\mathbf{A}_f)} \sigma) = \mathrm{Tr}(f_M : \sigma)$ for σ an admissible representation of $M(\mathbf{A}_f)$, where here we mean the non-normalized induction.

We let $T_M^- = \{t \in T(\mathbf{Q}_p) | t(N(\mathbf{Z}_p) \cap M(\mathbf{Z}_p))t^{-1} \subseteq N(\mathbf{Z}_p)\}$. We have $wtw^{-1} \in T_M^-$ for $t \in T^-$ and $w \in W^M$. We define $\epsilon_{\upsilon,w}(t) = \upsilon(t)^{w(\rho_P)+\rho_P} |t^{w^{-1}(\rho_P)+\rho_P}|_p$ where $\rho_P(m) = \det(\mathrm{ad}\, m : \mathfrak{n})^{\frac{1}{2}}$ is the the modulus function of P and υ is the chosen splitting $T(\mathbf{Q}_p)/T(\mathbf{Z}_p) \to T(\mathbf{Q}_p)$. Here $\mathfrak{n}$ denotes the Lie algebra of N.

We may now define $f^{\mathrm{reg}}_{M,w}$ for $f^p \otimes u_t$ by $\epsilon_{\upsilon,w}(t) f^p_M \otimes u_{wtw^{-1},M}$, where f^p_M is the away-from-p part of the classical f_M and $u_{wtw^{-1},M}$ is the u-operator attached to $wtw^{-1} \in T_M^-$. We extend to $\mathfrak{H}_p$ by linearity.

Now suppose that σ is an irreducible finite slope representation of $\mathfrak{H}_p(M)$. Then we can write $I^G_{M,w}(\sigma)$ for the tensor product of $\mathrm{Ind}_{P(\mathbf{A}_f^p)}^{G(\mathbf{A}_f^p)} \sigma^p$ with the character $\theta_{\sigma,M,w}$ of $\mathfrak{U}_p(G)$ defined by $u_t \mapsto \theta_\sigma(u_{wtw^{-1}}, M)$. By definition, it then follows that $\mathrm{Tr}(f : I^G_{M,w}(\sigma)) = \mathrm{Tr}(f^{\mathrm{reg}}_{M,w}, \sigma)$.

With this definition in hand, we may now describe the Eisenstein part of overconvergent cohomology as follows. First, we define $I^\dagger_{G,0}(f,\lambda) = I^\dagger_{G,G}(f,\lambda) = I^\dagger_G(f,\lambda)$ for any group of rank 0 (so $L_G = \{G\}$). Then inductively, it makes sense to define $I^\dagger_{G,M,w}$, $I^\dagger_{G,M}$, and $I^\dagger_{G,0}$ by

setting

$$I^{\dagger}_{G,M,w}(f,\lambda) = I^{\dagger}_{M,0}(f^{\mathrm{reg}}_{M,w}, w*\lambda + 2\rho_P),$$
$$I^{\dagger}_{G,M}(f,\lambda) = \sum_{w\in W^M_{\mathrm{Eis}}} (-1)^{l(w)+\dim \mathfrak{n}_M} I^{\dagger}_{G,M,w}(f,\lambda)$$
$$\text{and } I^{\dagger}_{G,0}(f,\lambda) = I^{\dagger}_{G}(f,\lambda) - \sum_{M\in L_G\backslash\{G\}} I^{\dagger}_{G,M}(f,\lambda).$$

Here W^M_{Eis} is a certain subset of the elements of W^M [82, §1.4.1]. We also make the obvious definitions of $m^{\dagger}_{G,M,w}(\sigma,\lambda)$, $m^{\dagger}_{G,M}(\sigma,\lambda)$, and $m^{\dagger}_{G,0}(\sigma,\lambda)$. If $m^{\dagger}_{G,0}(\sigma,\lambda)\neq 0$, we say that σ is a finite slope *cuspidal* automorphic representation of weight λ. The main result concerning this definition is as follows.

Proposition 2.8.6 ([82, Corollary 4.7.4])**.** *Let M have discrete series and let d_M be the half the real dimension of the symmetric space of M. Then the product $(-1)^{d_M} I^{\dagger}_{G,M,w}$ is an effective finite slope character distribution. Moreover, $m^{\dagger}_{G,M,w}(\sigma,\lambda) = 0$ always if either*

- *M does not have discrete series or*
- $\dim \mathfrak{X}_{K^p\cap M} < \dim \mathfrak{X}_{K^p}$.

We now state Urban's eigenvariety machine.

Theorem 2.8.7 ([82, Proposition 5.3.10])**.** *Let J be a family of effective finite slope character distributions over the weight space $\mathfrak{W}$. Suppose that σ is a finite slope admissible representation of level K^p such that $m_J(\sigma,\lambda) > 0$ for some weight λ, where $m_J(\sigma,\lambda)$ denotes the multiplicity of σ in the specialization of J at λ. Let θ be the associated character of $\mathfrak{R}_{S,p}$. Write $\mathfrak{R}^{\vee}$ for the p-adic space whose L-points are given by* $\mathrm{Hom}_{\mathrm{cts,alg}}(\mathfrak{R}_{S,p}[u_t^{-1}]^{\wedge}_{t\in T^-}, L)$. *Then there exists an equidimensional rigid analytic space $\mathfrak{E}_{J,K^p}\subseteq \mathfrak{W}\times\mathfrak{R}^{\vee}$ whose L-points correspond exactly to the pairs (λ',θ') of a weight $\lambda'\in\mathfrak{W}(L)$ and character $\theta'\in\mathfrak{R}^{\vee}(L)$ that appear with nontrivial multiplicity in J. Moreover the dimension of $\mathfrak{E}_{J,K^p}$ is equal to* $\dim\mathfrak{W}$.

Moreover, there exists

- *a finite flat covering* $\mathfrak{F}$ *of an open affinoid subdomain* $\mathfrak{X} \subseteq \mathfrak{E}_{J,K^p}$ *containing* (λ, θ)*, such that the composition with the projection to weight space is a finite and generically flat map to an open subset* $\mathfrak{U} \subseteq \mathfrak{W}$*,*
- *a homomorphism* $\theta_{\mathfrak{F}} : \mathfrak{R}_{S,p} \to \mathfrak{O}_{\mathfrak{F}}(\mathfrak{F})$*,*
- *a point* x *above* (λ, θ)*,*
- *for each* $y \in \mathfrak{F}(\overline{\mathbf{Q}}_p)$*, a nonempty finite set* Π_y *of irreducible finite slope representations of* $\mathfrak{H}_p(K^p)$ *such that for each* $\sigma' \in \Pi_y$*,* $\theta_{\sigma'}$ *is the character of* $\mathfrak{R}_{S,p}$ *given by the projection to the factor* $\mathfrak{R}^\vee$ *of the eigenvariety, and*
- *a nontrivial linear map* $I_{\mathfrak{F}} : \mathfrak{H}_p(K^p) \to \mathfrak{O}(\mathfrak{F})$

satisfying the following properties.

- *The specialization* I_y *of* $I_{\mathfrak{F}}$ *to* $y \in \mathfrak{F}(\overline{\mathbf{Q}}_p)$ *is equal to* $\sum_{\sigma \in \Pi_y} m_y(\sigma) J_\sigma$*, where* $m_y(\sigma) > 0$ *if and only if* $m_J(\lambda_y, \sigma) > 0$*.*
- *There exists a Zariski dense subset where* Π_y *is a singleton and the multiplicty* $m_y(\sigma)$ *is constant.*
- *The point* σ *lies in* Π_x*.*
- *If we write* $\theta_{\mathfrak{X}}$ *for the* $\mathfrak{O}_{\mathfrak{X}}(\mathfrak{X})$*-valued character of* $\mathfrak{R}_{S,p}$*, we have* $I_{\mathfrak{F}}(f f') = \theta_{\mathfrak{X}}(f) I_{\mathfrak{F}}(f')$ *for all* $f \in \mathfrak{R}_{S,p}$ *and* $f' \in \mathfrak{H}_p(K^p)$*.*

2.8.2 Critical slopes

In this section, we prove cases of local-global compatibility when the p-stabilization σ of π constructed earlier has critical slope. The main theorem is as follows.

Theorem 2.8.8. *Let π be a regular automorphic representation on the unitary similitude group G constructed in Section 2.2.1. Assume that p ramifies in E and that each place F_v^+ of F^+ over p is unramified. Moreover, assume that G_p is quasi-split, and that π_p is K-spherical for the K chosen in Section 2.4.4 or Section 2.4.5. Suppose that the relation $\alpha_i = q_v\alpha_j$ holds for some pair of Satake parameters of some unitary group factor G'_v of the restriction $\pi_{0,p}$ to the product of unitary groups for each F_v. Let ξ be the p-stabilization constructed for the representation of G'_v under consideration. Pick any p-stabilization for the other factors of $\pi_{0,p}$. Then let σ be the associated finite slope representation. Then for every $w \in W$, we must have $m_0^\dagger(\sigma^{w,\lambda}, w * \lambda) = 0$.*

Proof. Note that by the hypothesis on π_p the p-adic weight λ is algebraic. Assuming that $m_0^\dagger(\sigma^{w,\lambda}, w * \lambda) \neq 0$, we construct a family passing through σ and satisfying the conditions of Corollary 2.6.18, which is a contradiction. To do this, we apply Theorem 2.8.7 to $J = (-1)^{d_G} I_{G,0}$, where d_G is the dimension of the symmetric space of G. (In fact, Theorem 2.7.2 is proved by applying Theorem 2.8.7 to this J.) By Proposition 2.8.6, J is effective, and by hypothesis, $m_0^\dagger(\sigma^{w,\lambda}, w * \lambda) > 0$ (positivity is forced by effectivity of J), so Theorem 2.8.7 applies.

We now apply word-for-word the proof of the non-critical case of Theorem 2.2.5 as stated in Section 2.7.3, substituting $\sigma^{w,\lambda}$ for σ and $w * \lambda$ for λ, though we now need to check two additional things. One is that the Galois representation of ρ_π is actually given by the semisimplification of the specialization of $\mathfrak{M}$ to the point $(\sigma^{w,\lambda}, w * \lambda)$, but this follows from the Čebotarev density theorem and the fact that the restriction of the action of the Hecke algebra to split primes away from p where π is of hyperspecial level is the same. The other is that the Hecke operator $[Iu_{v,m}\varpi I]$ has the correct specialization at our point to give a suitable crystalline period. For this, we need to recall the twisted action of the Hecke algebra on $\sigma^{w,\lambda}$. This is given by $(w * \lambda - \lambda)(\upsilon(t))$ multiplied by the action on σ. Moreover, the action on σ is given by $\lambda(\upsilon(t))$ multiplied by the classical action, so the action on $\sigma^{w,\lambda}$ is

$(w * \lambda)(\upsilon(t))$ multiplied by the classical action.

In the proof of Theorem 2.2.5 given in Section 2.7.3, the crystalline eigenvalue U interpolated is the image of $[Iu_{v,m}\varpi I]$ in $\mathfrak{O}_{\mathfrak{G}}(\mathfrak{G})$ divided by $\lambda(\upsilon(u_{v,m}\varpi))$, where λ is the weight of the original σ. Since we are substituting $\sigma^{w,\lambda}$ for σ and $w * \lambda$ for λ in that argument, if we combine this division with the normalization just calculated, we see that the action of U on $\sigma^{w,\lambda}$ is precisely the classical action of the Hecke operator $[Iu_{v,m}\varpi I]$ on the chosen p-stabilization ξ of π. The remainder of the argument is identical.

□

Recall that we are trying to disallow the possibility that the relation $\alpha_i = q_v\alpha_j$ holds for some i, j. If for π as in Theorem 2.8.8, we can show that some $m^{\dagger}_{G,0}(\sigma^{w,\lambda}, w * \lambda) \neq 0$, we can deduce compatibility from this contradiction. Thus it is useful to have a formula relating the various $m^{\dagger}_{G,0}(\sigma^{w,\lambda}, w * \lambda)$ with the multiplicity $m^{\mathrm{cl}}_G(\sigma, \lambda)$, which is known to be nonzero. Then if every term other than these is zero, we can use this to find some nonzero $m^{\dagger}_{G,0}(\sigma^{w,\lambda}, w * \lambda)$. To produce the needed formula, we simply combine the definitions given in the preceding section with (2.24) to calculate

$$\begin{aligned} m^{\mathrm{cl}}_G(\sigma, \lambda) &= \sum_{w \in W} (-1)^{l(w)} m^{\dagger}_G(\sigma^{w,\lambda}, w * \lambda) \\ &= \sum_{w \in W} (-1)^{l(w)} \left(m^{\dagger}_{G,0}(\sigma^{w,\lambda}, w * \lambda) + \sum_{M \in L_G \setminus \{G\}} m^{\dagger}_{G,M}(\sigma^{w,\lambda}, w * \lambda) \right) \\ &= \sum_{w \in W} (-1)^{l(w)} \Bigg(m^{\dagger}_{G,0}(\sigma^{w,\lambda}, w * \lambda) + \\ &\qquad \sum_{M \in L_G \setminus \{G\}} \left(\sum_{w_0 \in W^M_{\mathrm{Eis}}} (-1)^{l(w) + \dim \mathfrak{n}_M} m^{\dagger}_{G,M,w_0}(\sigma^{w,\lambda}, w * \lambda) \right) \Bigg) . \end{aligned} \tag{2.25}$$

We can use this to prove the following, which allows us to apply Theorem 2.8.8.

Proposition 2.8.9. *Suppose that π is as in Theorem 2.8.8 and either that G is anisotropic*

over $\mathbf{Q}$ *or* $[F^+ : \mathbf{Q}] \geq 2$. *Then some* $m_0^\dagger(\sigma^{w,\lambda}, w * \lambda) \neq 0$.

Proof. As explained in Definition 2.8.1, the left hand side of (2.25) is nonzero since λ is regular and σ is a p-stabilization of π. If G is anisotropic, the set L_G is empty, so the result follows from looking at the right hand side of (2.25).

Now assume that $[F^+ : \mathbf{Q}] \geq 2$. The multiplicity $m_{G,M,w_0}^\dagger(\sigma^{w,\lambda}, w * \lambda)$ vanishes if M does not have discrete series by Proposition 2.8.6. The parabolic subgroups of unitary groups are the stabilizers of self-dual flags [48, §3.2.3], and have Levi factors given by the product of a unitary group with some number of factors of the form $\mathrm{GL}_{d/F}$. We need $d = 1$ for all factors for the group to possess discrete series. We change basis over $\mathbf{Q}$ so that our hermitian space (h, J) over F has the form $\begin{pmatrix} & & A_e \\ & J' & \\ A_e & & \end{pmatrix}$ for some integer e, where (h', J') has an anisotropic unitary group and A_e is defined as in (2.3). Then there are exactly e proper standard parabolic subgroups whose Levi factors have discrete series. These have the form $M_i = \mathrm{GU}(J_i) \oplus (\mathrm{Res}_{\mathbf{Q}}^F \mathrm{GL}_1)^i$ for $i = 1, \ldots, e$, where $J_i = \begin{pmatrix} & & A_{e-i} \\ & J' & \\ A_{e-i} & & \end{pmatrix}$.

Recall that $\mathfrak{X}_{K^p \cap M} = \mathrm{Hom}_{\mathrm{cont}}(T(\mathbf{Z}_p)/Z_p(K_M^p), L)$. The dimension of $\mathfrak{X}_{K^p \cap M}$ is independent of K^p. For the group $H = \mathrm{Res}_{\mathbf{Q}}^F \mathrm{GL}_{1/F}$ and K^p in $H(\mathbf{A}_f)$ taken to be maximal, $Z_p(K_H^p) = Z_H(\mathbf{Q}) \cap K^p$ is the closure of the units of F in $\mathfrak{O}_{F_p}$, where $F_p = F \otimes_{\mathbf{Q}} \mathbf{Q}_p$. The units of F form an infinite set if $[F^+ : \mathbf{Q}] \geq 2$, so this closure has nonzero dimension. For a unitary similitude group G, $Z_p(K^p)$ is given by the p-adic closure of $Z(\mathbf{Q}) \cap K^p T(\mathbf{Z}_p)$, or the integral diagonal matrices with similitude in $\mathbf{Q}^\times$. The only integral matrices with similitude in $\mathbf{Q}^\times$ actually have to have similitude in ± 1, so $Z(\mathbf{Q}) \cap K^p T(\mathbf{Z}_p)$ consists of elements $f \in F$ with $\mathrm{Nm}_{F/F^+} f \in \{\pm 1\}$. This set is finite since the unit groups of these fields have the same rank. By this and the discussion of the last paragraph, $\mathfrak{X}_{K^p}$ has larger dimension than $\mathfrak{X}_{K^p \cap M}$ for any proper Levi factor M that has discrete series. □

Combined with Theorem 2.8.8 and Theorem 2.2.8, Proposition 2.8.9 implies the main result Theorem 2.2.5 under the additional hypothesis that either $[F^+ : \mathbf{Q}] \geq 2$ or G is anisotropic.

Remark 2.8.10. We can also easily show that some $m_0^\dagger(\sigma^{w,\lambda}, w * \lambda) \neq 0$ if either

- π is supercuspidal at any place or
- ρ_π is irreducible (which holds if, for instance, π is discrete series at a finite place).

We only sketch the argument here because it is similar to ones already given and we do not need this for the proof of Theorem 2.2.5. If some term $m_{G,M,w_0}^\dagger(\sigma^{w,\lambda}, w * \lambda) \neq 0$ with M proper, then π has to be a subquotient of an induction from M at all finite places, which is impossible if π is supercuspidal somewhere. We can then apply (2.25) to find that some $m_0^\dagger(\sigma^{w,\lambda}, w * \lambda) \neq 0$.

Now suppose that ρ_π is irreducible and some $m_{G,M,w_0}^\dagger(\sigma^{w,\lambda}, w * \lambda) \neq 0$. Then there is some finite slope automorphic representation σ' on M of weight $w_0 w * \lambda + 2\rho_P$ such that $\sigma^{w,\lambda}$ is a subquotient of the parabolic induction of σ' to G. It is possible to attach a Galois representation to any classical automorphic representation on an M with discrete series. (One must do this in a way that is compatible with the way ρ_π is attached to π; we skip the details.) Then by considering the pseudorepresentation over the eigenvariety of M and moving into classical weight, one attaches a second Galois representation to π by specializing the family of Galois representations at the point of the eigenvariety of M corresponding to σ' and semisimplifying; this Galois representation will have the same trace of Frobenius at unramified split places, and so will be equal to ρ_π. On the other hand, by definition the entire family of Galois representations will be reducible at every classical point. Since reducibility is an open condition, this contradicts the irreducibility of ρ_π.

2.8.3 Finite slope Eisenstein cohomology

We would like to have compatibility even if $F^+ = \mathbf{Q}$. To simplify notation, we assume that $F^+ = \mathbf{Q}$ in this section, although it is not needed. We obtain compatibility by a closer examination of what it means for a term $m^\dagger_{G,M,w_0}(\sigma^{w,\lambda}, w * \lambda)$ to fail to vanish. By definition we have

$$(-1)^{d_M} I^\dagger_{G,M,w_0}(f, \lambda) = (-1)^{d_M} I^\dagger_{M,0}(f^{\mathrm{reg}}_{M,w_0}, w_0 * \lambda + 2\rho_P).$$

In particular, since both sides are effective finite slope chararacter distributions by Proposition 2.8.6, and by the defining property of f^{reg}_{M,w_0}, $I^\dagger_{G,M,w_0}(f, \lambda)$ is the sum, taken with multiplicity, of the traces of all subquotients of the twisted (at p) non-normalized parabolic inductions from M to G of weight $w_0 * \lambda + 2\rho_P$ finite slope cuspidal automorphic representations of M. (Here we mean automorphic in the sense of Definition 2.8.1, so these representations are not necessarily attached to classical automorphic representations.) If we assume that $m^\dagger_{G,M,w_0}(\sigma^{w,\lambda}, w * \lambda) \neq 0$, it must be the case that the finite slope representation σ is, up to some twist of the action at p, a subquotient of a non-normalized parabolic induction $\mathrm{Ind}_{M(\mathbf{A}_f)}^{G(\mathbf{A}_f)} \sigma'$ for some finite slope cuspidal automorphic representation σ' on M of weight $w_0 w * \lambda + 2\rho_P$. We must have $(-1)^{d_M} m^\dagger_{M,0}(\sigma', w_0 w * \lambda + 2\rho_P) > 0$ for this σ'. The remainder of this section is concerned with the consequences of the existence of such a σ'.

We first show that σ' gives rise to representations on $\mathrm{Res}^E_{\mathbf{Q}} \mathrm{GL}_{1/E}$ that have nonvanishing overconvergent automorphic multiplicity.

Proposition 2.8.11. *Suppose that $H = H_1 \times H_2$ as algebraic groups over $\mathbf{Q}$ and σ_i, $i \in \{1, 2\}$ is a finite slope admissible representation of H_i. Then $m^\dagger_{H,0}(\sigma_1 \otimes \sigma_2, \lambda_1 \lambda_2) = m^\dagger_{H_1,0}(\sigma_1, \lambda_1) m^\dagger_{H_2,0}(\sigma_2, \lambda_2)$.*

Proof. The equality $m^\dagger_H(\sigma_1 \otimes \sigma_2, \lambda_1 \lambda_2) = m^\dagger_{H_1}(\sigma_1, \lambda_1) m^\dagger_{H_2}(\sigma_2, \lambda_2)$ follows formally using the Künneth formula, since $\widetilde{S}_H = \widetilde{S}_{H_1} \times \widetilde{S}_{H_2}$ and $\mathfrak{D}_{\lambda_1 \lambda_2}$ splits up similarly. Then the cuspidal version follows from the definition of $m^\dagger_{H,0}$ since a parabolic on H is the product of parabolics

on H_1 and H_2. $\square$

In particular, if M_i is as in the proof of Proposition 2.8.9, we obtain from σ' finite slope admissible characters ψ_j of $\mathrm{Res}^E_{\mathbf{Q}} \mathrm{GL}_{1/E}$ for $j = 1, \ldots, i$ with $m^{\dagger}_{\mathrm{Res}^E_{\mathbf{Q}} \mathrm{GL}_{1/E}, 0}(\psi_j, \lambda_j) \neq 0$. (We remark that the subscript 0 is unnecessary here.) In other words, ψ_j is a finite slope automorphic representation of weight λ_j. Here λ_j is an algebraic weight defined in terms of M_i, w, w_0, and λ.

Proposition 2.8.12. *A finite slope automorphic representation on* $\mathrm{Res}^E_{\mathbf{Q}} \mathrm{GL}_{1/E}$ *of algebraic weight* λ *is a Hecke character of weight* λ.

Proof. Let $H = \mathrm{Res}^E_{\mathbf{Q}} \mathrm{GL}_{1/E}$. By definition, any cohomology class in $H^0_{\mathrm{fs}}(\widetilde{S}_H, \mathbf{V}^{\vee}_{\lambda})$ is a Hecke character of weight λ. We need to show that $H^0_{\mathrm{fs}}(\widetilde{S}_H, \mathbf{V}^{\vee}_{\lambda}) = H^0_{\mathrm{fs}}(\widetilde{S}_H, \mathfrak{D}_{\lambda})$. In the notation of [82, §3.2.6, 3.2.9], we have $\mathfrak{D}_{\lambda} = V^{\vee}_{\lambda}$ since G has no roots; this is the degenerate case of [82, Theorem 3.3.10]. Then the result follows from [82, Lemma 4.3.8]. $\square$

To calculate what the weight λ_j is, we need to make the identification of M_i with a subgroup of G precise. The map $M_i = \mathrm{GU}(J_i) \oplus (\mathrm{Res}^E_{\mathbf{Q}} \mathrm{GL}_1)^i \to G$ is defined by

$$(g, (t_j)_j) \mapsto \begin{pmatrix} \mathrm{diag}(t_1, \ldots, t_i) & & \\ & g & \\ & & \mu(g)\,\mathrm{diag}(\overline{t}_i^{-1}, \ldots, \overline{t}_1^{-1}) \end{pmatrix}.$$

Now let $\lambda' = w_0 w * \lambda + 2\rho_P$ be attached to the datum $\underline{d} = (d_0, (d_1, \ldots, d_n))$. (Since $F^+ = \mathbf{Q}$ we no longer need to consider the embeddings of F^+ into $\mathbf{R}$.) In terms of $\underline{d}$, we can identify the weight of σ' as follows. The induced map on $\mathrm{GU}(J_i) \oplus (\mathrm{Res}^E_{\mathbf{Q}} \mathrm{GL}_1)^i$ is via the identification of $\mathrm{GU}(J)_{/E}$ with $\mathbf{G}_m \times \mathrm{GL}_n$ given in Section 2.2.1. If we use the notation $(s_0, \mathrm{diag}(s_1, \ldots, s_n))$ to denote an element of the maximal torus of $\mathbf{G}_m \times \mathrm{GL}_n$, the image of $(t, (t_j)_j)$ is $(\mu(t), \mathrm{diag}(t_1, \ldots, t_i, t, \mu(g)\overline{t}_i^{-1}, \ldots, \mu(g)\overline{t}_1^{-1})$. Applying λ', we obtain

$\mu(t)^{d_0}\lambda_{J_i}(t)\prod_j t_j^{d_j}\bar{t}_j^{-d_{n+1-j}}$, where λ_{J_i} is λ' restricted to the middle $n-2i$ entries of the diagonal of GL_n. In particular, the composite of λ' with the embedding of the j^{th} factor of $\mathrm{Res}^E_{\mathbf{Q}}\,\mathrm{GL}_1$ has algebraic weight $(d_j, -d_{n+1-j})$.

We now calculate the motivic weight of ψ_j in a different way. Let ℓ be a split prime of E where σ and σ' are unramified. Then $\mathrm{GU}(J)(\mathbf{Q}_\ell)$ may be identified with $\mathbf{G}_m \times \mathrm{GL}_n$, so that the diagonal maximal torus is $\mathbf{G}_m \times \mathbf{G}_m^n$, and $\pi_\ell = \sigma_\ell$ is an unramified representation given by the normalized induction of a character χ_ℓ of $\mathbf{G}_m \times \mathbf{G}_m^n$ that is unitary when restricted to the $\mathbf{G}_m^n$ factor. We are using here that τ_0 is tempered at every finite place and the compatibility between π_0 and τ_0 at split places, which is part of Theorem 2.2.2. Moreover, this normalized induction is irreducible by the Bernstein-Zelevinsky classification. Using Ind to denote non-normalized induction, we have $\pi_\ell = \mathrm{Ind}_{B_\ell}^{G_\ell}\delta_{B_\ell}^{\frac{1}{2}}\chi_\ell$.

On the other hand, σ_ℓ is also the non-normalized induction from the $\mathbf{Q}_\ell$-component of the parabolic P_i with Levi factor M_i of the unramified representation σ'_ℓ. Let χ'_ℓ be a character of the maximal torus so that σ'_ℓ is $\mathrm{Ind}_{B_{i,\ell}}^{M_{i,\ell}}\delta_{B_{i,\ell}}^{\frac{1}{2}}\chi'_\ell$, where $B_{i,\ell}$ is the upper triangular Borel subgroup of $M_{i,\ell}$. Then we have an isomorphism

$$\mathrm{Ind}_{P_{i,\ell}}^{G_\ell}\mathrm{Ind}_{B_{i,\ell}}^{M_{i,\ell}}\delta_{B_{i,\ell}}^{\frac{1}{2}}\chi'_\ell = \mathrm{Ind}_{B_\ell}^{G_\ell}\delta_{B_{i,\ell}}^{\frac{1}{2}}\chi'_\ell \cong \mathrm{Ind}_{B_\ell}^{G_\ell}\delta_{B_\ell}^{\frac{1}{2}}\chi_\ell.$$

Thus there exists w so that $\delta_{B_\ell}^{\frac{1}{2}}(w\chi_\ell) = \delta_{B_{i,\ell}}^{\frac{1}{2}}\chi'_\ell$; we replace χ_ℓ with $w\chi_\ell$ (which is still unitary) so that $\chi'_\ell\delta_{B_{i,\ell}}^{\frac{1}{2}} = \delta_{B_\ell}^{\frac{1}{2}}\chi_\ell$ or $\chi'_\ell = \delta_{P_i}^{\frac{1}{2}}\chi_\ell$. Write $u_j = \mathrm{diag}(1,\dots,1,\underbrace{p}_{j},1,\dots,1)$. Then for $j \in \{1,\dots,i\}$, $\chi'_\ell(u_j)$ and $\chi'_\ell(u_{n+1-j}^{-1})$ have complex absolute value $p^{\frac{2j-n-1}{2}}$.

Using purity for Hecke characters, we can translate this calculation into one of the weight of the Hecke character ψ_j as follows. We can write $E_\ell = \mathbf{Q}_\ell \times \mathbf{Q}_\ell$, where the first factor corresponds to the composite of chosen embedding $E \subseteq \mathbf{C}$ with the fixed identification $\mathbf{C} \cong \overline{\mathbf{Q}}_\ell$. Then $\psi_j|_{E_\ell^\times}(p,1) = \chi'(u_j)$ and $\psi_j|_{E_\ell^\times}(1,p) = \chi'(u_{n+1-j}^{-1})$. As expected, these numbers have the same complex absolute value, and it follows that the motivic weight of ψ_j

is $n+1-2j$. Thus

$$d_j - d_{n+1-j} = n+1-2j. \tag{2.26}$$

We now calculate the algebraic weight $\lambda_j = (d_j, -d_{n+1-j})$ in terms of $w_0 w$ and λ. Let $\lambda = (c_0, (c_1, \ldots, c_n))$ and write $w' = w_0 w$. Then $\underline{d}$ is attached to $\lambda' = w' * \lambda + 2\rho_P = w'\lambda + w'\rho_B - \rho_B + 2\rho_P$. Suppose that w' takes the weights u_j and u_{n+1-j} to u_k and $u_{k'}$, respectively. We then have

$$\begin{aligned}\lambda_j =& (d_j, -d_{n+1-j}) \\ =& \Bigg(c_k + \frac{n+1-2k}{2} - \frac{n+1-2j}{2} + (n+1-2j)\ , \\ & -c_{k'} - \frac{n+1-2k'}{2} + \frac{2j-n-1}{2} - (2j-n-1) \Bigg).\end{aligned}$$

Using this and (2.26) we find

$$d_j - d_{n+1-j} = c_k - c_{k'} - k + k' + n + 1 - 2j \underbrace{=}_{(2.26)} n+1-2j.$$

It follows that $c_k - c_{k'} = k - k'$. Since $\underline{c}$ is dominant (even regular), $c_k - c_{k'}$ and $k - k'$ have opposite signs, which is a contradiction. Thus all of the Eisenstein multiplicities in (2.25) vanish and thus some $m_G^\dagger(\sigma^{w,\lambda}, w * \lambda)$ must be nonzero. Using this nonvanishing in conjunction with Theorem 2.8.8 and Theorem 2.2.8, we deduce Theorem 2.2.5 in full generality.

Bibliography

[1] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor. Local-global compatibility for $l = p$, I. *Ann. Fac. Sci. Toulouse Math. (6)*, 21(1):57–92, 2012.

[2] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor. Local-global compatibility for $l = p$, II. *Ann. Sci. de l'ENS*, 47:161–175, 2014.

[3] Joël Bellaïche and Gaëtan Chenevier. Families of Galois representations and Selmer groups. *Astérisque*, (324):xii+314, 2009.

[4] Rebecca Bellovin. p-adic Hodge theory in rigid analytic families. *PhD diss.*, 2013.

[5] Laurent Berger and Pierre Colmez. Familles de représentations de de Rham et monodromie p-adique. *Astérisque*, (319):303–337, 2008.

[6] I. N. Bernšteĭn and A. V. Zelevinskiĭ. Representations of the group $\mathrm{GL}(n, F)$, where F is a local non-Archimedean field. *Uspehi Mat. Nauk*, 31(3(189)):5–70, 1976.

[7] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive p-adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977.

[8] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1984.

[9] Nicolas Bourbaki. *Elements of mathematics. Commutative algebra.* Hermann, Paris, 1972. Translated from the French.

[10] Eliot Brenner. Stability of the local gamma factor in the unitary case. *J. Number Theory*, 128(5):1358–1375, 2008.

[11] Olivier Brinon and Brian Conrad. CMI summer school notes on p-adic Hodge theory. *preprint*, 2009.

[12] Ana Caraiani. Local-global compatibility and the action of monodromy on nearby cycles. *Duke Math. J.*, 161(12):2311–2413, 2012.

[13] Ana Caraiani. Monodromy and local-global compatibility for $l = p$. *preprint*, 2012.

[14] William Casselman. Introduction to the theory of admissible representations of p-adic reductive groups. *preprint*, 1995.

[15] Gaëtan Chenevier and Michael Harris. Construction of automorphic Galois representations II. *Cambridge Math. Journal*, 1, 2013.

[16] Laurent Clozel. Purity reigns supreme. *Int. Math. Res. Not. IMRN*, (2):328–346, 2013.

[17] Laurent Clozel and Jack Thorne. Level-raising and symmetric power functoriality, ii. *to appear, Annals of Math.*, 2014.

[18] Laurent Clozel and Jack Thorne. Level-raising and symmetric power functoriality, iii. *preprint*, 2014.

[19] James W. Cogdell and Ilya Piatetski-Shapiro. Derivatives and l-functions for GL(n). *The Heritage of B. Moishezon. Instituto Mexicano de Contadores Publicos (forthcoming)*, 2010.

[20] Brian Conrad. Irreducible components of rigid spaces. *Ann. Inst. Fourier (Grenoble)*, 49(2):473–541, 1999.

[21] Brian Conrad. Relative ampleness in rigid geometry. *Ann. Inst. Fourier (Grenoble)*, 56(4):1049–1126, 2006.

[22] Samuel Eilenberg and Saunders MacLane. Cohomology theory in abstract groups. I. *Ann. of Math. (2)*, 48:51–78, 1947.

[23] Matthew Emerton. Locally analytic vectors in representations of locally p-adic analytic groups. *to appear, Memoirs of the AMS*, 2011.

[24] Stephen Gelbart, Ilya Piatetski-Shapiro, and Stephen Rallis. *Explicit constructions of automorphic L-functions*, volume 1254 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[25] Roger Godement and Hervé Jacquet. *Zeta functions of simple algebras*. Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin, 1972.

[26] Thomas J. Haines and Sean Rostami. The Satake isomorphism for special maximal parahoric Hecke algebras. *Represent. Theory*, 14:264–284, 2010.

[27] Michael Harris, Kai-Wen Lan, Richard Taylor, and Jack Thorne. On the rigid cohomology of certain Shimura varieties. *in preparation*.

[28] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[29] Eugen Hellmann. Families of trianguline representations and finite slope spaces. *ArXiv e-prints*, February 2012.

[30] Adrian Iovita and Alexandru Zaharescu. Galois theory of B_{dR}^{+}. *Compositio Math.*, 117(1):1–31, 1999.

[31] Adrian Iovita and Alexandru Zaharescu. Generating elements for B_{dR}^{+}. *J. Math. Kyoto Univ.*, 39(2):233–248, 1999.

[32] Hervé Jacquet and Joseph Shalika. A lemma on highly ramified ϵ-factors. *Math. Ann.*, 271(3):319–332, 1985.

[33] Andrei Jorza. p-adic families and Galois representations for $\mathrm{GS_p}(4)$ and GL(2). *Math. Res. Lett.*, 19(5):987–996, 2012.

[34] Kiran Kedlaya and Ruochuan Liu. On families of ϕ, Γ-modules. *Algebra Number Theory*, 4(7):943–967, 2010.

[35] Kiran S. Kedlaya, Jonathan Pottharst, and Liang Xiao. Cohomology of arithmetic families of (φ, Γ)-modules. *ArXiv e-prints*, March 2012.

[36] Mark Kisin. Overconvergent modular forms and the Fontaine-Mazur conjecture. *Invent. Math.*, 153(2):373–454, 2003.

[37] Robert E. Kottwitz. Isocrystals with additional structure. II. *Compositio Math.*, 109(3):255–339, 1997.

[38] J.-P. Labesse and R. P. Langlands. L-indistinguishability for SL(2). *Canad. J. Math.*, 31(4):726–785, 1979.

[39] Jean-Pierre Labesse. Cohomologie, stabilisation et changement de base. *Astérisque*, (257):vi+161, 1999. Appendix A by Laurent Clozel and Labesse, and Appendix B by Lawrence Breen.

[40] Erez M. Lapid and Stephen Rallis. On the local factors of representations of classical groups. In *Automorphic representations, L-functions and applications: progress and prospects*, volume 11 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 309–359. de Gruyter, Berlin, 2005.

[41] Ruochuan Liu. Triangulation of refined families. *ArXiv e-prints*, February 2012.

[42] Ruochuan Liu. Semi-stable periods of finite slope families. *preprint*, May 2013.

[43] George Lusztig. Affine Hecke algebras and their graded version. *J. Amer. Math. Soc.*, 2(3):599–635, 1989.

[44] George Lusztig. Classification of unipotent representations of simple p-adic groups. *Internat. Math. Res. Notices*, (11):517–589, 1995.

[45] George Lusztig. Classification of unipotent representations of simple p-adic groups. II. *Represent. Theory*, 6:243–289, 2002.

[46] Martin Luu. Deformation theory and local-global compatibility of Langlands correspondences. *to appear, Memoirs of the AMS*, 2014.

[47] Barry Mazur and Andrew Wiles. On p-adic analytic families of Galois representations. *Compositio Math.*, 59(2):231–264, 1986.

[48] Alberto Mínguez. Unramified representations of unitary groups. In *On the stabilization of the trace formula*, volume 1 of *Stab. Trace Formula Shimura Var. Arith. Appl.*, pages 389–410. Int. Press, Somerville, MA, 2011.

[49] Sophie Morel. *On the cohomology of certain noncompact Shimura varieties*, volume 173 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2010. With an appendix by Robert Kottwitz.

[50] Kentaro Nakamura. Deformations of trianguline B-pairs and Zariski density of two dimensional crystalline representations. *ArXiv e-prints*, June 2010.

[51] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 2008.

[52] G. Pappas and M. Rapoport. Twisted loop groups and their affine flag varieties. *Adv. Math.*, 219(1):118–198, 2008. With an appendix by T. Haines and Rapoport.

[53] G. Pappas and M. Rapoport. Local models in the ramified case. III. Unitary groups. *J. Inst. Math. Jussieu*, 8(3):507–564, 2009.

[54] Stefan Patrikis. Variations on a theorem of Tate. *ArXiv e-prints*, July 2012.

[55] Cristina Pérez-García and Wilhelmus H. Schikhof. *Locally convex spaces over non-Archimedean valued fields*, volume 119 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[56] Ilya Piatetski-Shapiro and Stephen Rallis. ϵ factor of representations of classical groups. *Proc. Nat. Acad. Sci. U.S.A.*, 83(13):4589–4593, 1986.

[57] Mark Reeder. p-adic Whittaker functions and vector bundles on flag manifolds. *Compositio Math.*, 85(1):9–36, 1993.

[58] Mark Reeder. Whittaker functions, prehomogeneous vector spaces and standard representations of p-adic groups. *J. Reine Angew. Math.*, 450:83–121, 1994.

[59] R. W. Richardson. On orbits of algebraic groups and Lie groups. *Bull. Austral. Math. Soc.*, 25(1):1–28, 1982.

[60] T. Richarz. On the Iwahori-Weyl group. *ArXiv e-prints*, October 2013.

[61] François Rodier. Les représentations de $\mathrm{GSp}(4, k)$ où k est un corps local. *C. R. Acad. Sci. Paris Sér. A-B*, 283(7):Ai, A429–A431, 1976.

[62] S. Rostami. The Bernstein presentation for general connected reductive groups. *ArXiv e-prints*, December 2013.

[63] Peter Schneider. *Nonarchimedean functional analysis*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.

[64] Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *preprint*, 2013.

[65] Hans Schoutens. Blowing up in rigid analytic geometry. *Bull. Belg. Math. Soc. Simon Stevin*, 2(4):399–417, 1995.

[66] Shankar Sen. Continuous cohomology and p-adic Galois representations. *Invent. Math.*, 62(1):89–116, 1980.

[67] Shankar Sen. The analytic variation of p-adic Hodge structure. *Ann. of Math. (2)*, 127(3):647–661, 1988.

[68] Shankar Sen. An infinite-dimensional Hodge-Tate theory. *Bull. Soc. Math. France*, 121(1):13–34, 1993.

[69] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.

[70] Sug Woo Shin. On the cohomological base change for unitary similitude groups. *preprint*.

[71] Sug Woo Shin. Galois representations arising from some compact Shimura varieties. *Ann. of Math. (2)*, 173(3):1645–1741, 2011.

[72] Sug Woo Shin and Nicolas Templier. On fields of rationality for automorphic representations. *preprint*, 2013.

[73] Christopher Skinner. Galois representations for automorphic forms on unitary groups over $\mathbf{Q}$. *preprint*.

[74] Christopher Skinner. A note on the p-adic Galois representations attached to Hilbert modular forms. *Doc. Math.*, 14:241–258, 2009.

[75] Christopher Skinner and Eric Urban. Vanishing of L-functions and ranks of Selmer groups. In *International Congress of Mathematicians. Vol. II*, pages 473–500. Eur. Math. Soc., Zürich, 2006.

[76] Christopher Skinner and Eric Urban. The Iwasawa Main Conjectures for GL_2. *Inventiones mathematicae*, pages 1–277, 2013.

[77] John Tate. p-divisible groups. In *Proc. Conf. Local Fields (Driebergen, 1966)*, pages 158–183. Springer, Berlin, 1967.

[78] Richard Taylor. Galois representations associated to Siegel modular forms of low weight. *Duke Math. J.*, 63(2):281–332, 1991.

[79] Richard Taylor and Teruyoshi Yoshida. Compatibility of local and global Langlands correspondences. *J. Amer. Math. Soc.*, 20(2):467–493, 2007.

[80] Michael Temkin. Functorial desingularization of quasi-excellent schemes in characteristic zero: the nonembedded case. *Duke Math. J.*, 161(11):2207–2254, 2012.

[81] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979.

[82] Eric Urban. Eigenvarieties for reductive groups. *Ann. of Math. (2)*, 174(3):1685–1784, 2011.

[83] Eric Urban. On the rank of Selmer groups for elliptic curves over $\mathbf{Q}$. *to appear, Proceedings of the International Colloquium TIFR*, 2013.

[84] Ila Varma. *PhD diss.*, 2014.

[85] A. Wiles. On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988.

[86] Jiu-Kang Yu. On the local Langlands correspondence for tori. In *Ottawa lectures on admissible representations of reductive p-adic groups*, volume 26 of *Fields Inst. Monogr.*, pages 177–183. Amer. Math. Soc., Providence, RI, 2009.