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Two instability results in general relativity

Moschidis, Georgios

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TWO INSTABILITY RESULTS IN GENERAL RELATIVITY

GEORGIOS MOSCHIDIS

A DISSERTATION
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Abstract

A fundamental question arising in the study of the Einstein equations in general relativity is that of global stability of solutions under perturbations of their initial state. In this thesis, we will establish two results pertinent to instability phenomena for the Einstein equations, associated to the presence of two distinct geometric structures on the underlying spacetimes.

In the first part of the thesis, we will focus on the class of stationary and asymptotically flat spacetimes; such spacetimes arise naturally as models of the final state of the evolution of isolated self-gravitating systems in the universe. Our first main result in this setting will be a general instability statement for solutions ψ to the scalar wave equation $\square_g \psi = 0$ (a simple linear toy model for the non-linear Einstein equations) on spacetimes $(\mathcal{M}, g)$ possessing a non-empty ergoregion but no event horizon. This statement was conjectured and heuristically supported by Friedman in 1978. As a corollary of this result, we will also obtain an instability statement for the acoustical wave equation on the hydrodynamic vortex, a phenomenon first studied numerically by Oliveira–Cardoso–Crispino.

In the second part of the thesis, we will turn our attention to the class of asymptotically anti-de Sitter spacetimes. These spacetimes arise as solutions of the Einstein equations with a negative cosmological constant and have a prominent role in the high energy physics literature. In this setting, it was conjectured by Dafermos–Holzegel in 2006 that anti-de Sitter spacetime is unstable as a solution of the vacuum Einstein equations when reflecting boundary conditions are imposed on conformal infinity. The second main result of this thesis will be a proof of the instability of AdS for the Einstein–null dust system.

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Chapter 1

Introduction

The general theory of relativity is a geometric theory of gravity, in which space and time are jointly modeled by a 4 dimensional *Lorentzian* manifold $(\mathcal{M}, g)$, so-called *spacetime*. Gravity is perceived as the effect of the warping of the geometry of $(\mathcal{M}, g)$ on the motion of observers in the spacetime, while the geometry of $(\mathcal{M}, g)$ depends on the distribution of matter through Einstein's celebrated field equations; expressed in units where $G = c = 1$, these equations read

$$(1.0.1) \quad Ric_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu},$$

where $Ric_{\mu\nu}$ is the Ricci tensor and R the scalar curvature of g , respectively, while $T_{\mu\nu}$ is the *energy-momentum tensor* associated to the matter present on $\mathcal{M}$.

The system of equations (1.0.1) is not closed; assuming the presence of a collection of matter fields $\{\Psi_i\}$ on $\mathcal{M}$ (which are often modelled by functions or tensor fields defined on $\mathcal{M}$), (1.0.1) must be supplemented by a set of equations governing the behaviour of Ψ_i on $(\mathcal{M}, g)$, provided by the corresponding field theory for the matter fields, as well as a relation defining $T_{\mu\nu}$ in terms of g and $\{\Psi_i\}$. In the special case when matter is absent (i. e. $T_{\mu\nu} = 0$), (1.0.1) reduces to the *vacuum* Einstein equations

$$(1.0.2) \quad Ric_{\mu\nu} = 0.$$

General relativity was the culmination of a long research effort undertaken by Einstein, with the aim of reconciling gravity with the special theory of relativity; the latter had already yielded an axiomatic framework in which an alternative to Newtonian mechanics, consistent with Maxwell's theory of electromagnetism, could be derived (see [58, 119]). Within the special theory of relativity, as reformulated by Minkowski [105], space and time were unified into a 4-dimensional continuum modelled by $\mathbb{R}^{3+1}$ and equipped with the flat Lorentzian metric

$$(1.0.3) \quad \eta = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

In this setting, the motion of observers was assumed to take place along timelike curves with respect to η . After formulating the equivalence principle in 1907, Einstein was soon led to the pursuit of a geometric theory of gravity, in which *free-falling* observers were assumed to follow timelike *geodesics* in the ambient spacetime modelled by a more general Lorentzian manifold $(\mathcal{M}, g)$.¹ Crucially, Newtonian gravity emerged as a natural constraint for any successful formulation of such a theory when passing to the weak-field, non-relativistic limit.² A timelike geodesic γ in $(\mathbb{R}^{3+1}, g)$ had to reduce, in this limit, to the orbit traced out in space and time by a particle following Newton's laws of motion, under the influence of the Newtonian potential V defined by solving

$$(1.0.4) \quad \Delta_{\mathbb{R}^3} V = 4\pi\rho,$$

where, in the standard Cartesian coordinates (x^0, x^1, x^2, x^3) on $\mathbb{R}^{3+1}$, $\rho = T_{00}$ and $\Delta_{\mathbb{R}^3} = \sum_{i=1}^3 \partial_i^2$. After a series of setbacks (and a frantic race with Hilbert), Einstein's effort finally came to fruition in 1915; he presented the field equations (1.0.1) to the Prussian Academy of Sciences on the 25th November of the same year (see [59]).

¹This hypothesis, which is known as the *geodesic hypothesis*, can be verified in the context of general relativity through a field-theoretic approach, by considering the evolution of localized, self-gravitating, non-linear massive scalar fields, governed by (1.0.1) coupled to a non-linear Klein–Gordon equation admitting soliton-type solutions; see [143].

²This is the limit when $\mathcal{M} \simeq \mathbb{R}^{3+1}$, $g_{\mu\nu}$ is close to the flat metric $\eta_{\mu\nu}$ on $\mathbb{R}^{3+1}$, $\rho = T_{00}$ dominates the rest of the components of $T_{\mu\nu}$ (in the standard Cartesian coordinates (x^0, x^1, x^2, x^3) on $\mathbb{R}^{3+1}$), and the geodesics γ under consideration correspond to observers moving at non-relativistic speeds, i. e. $\dot{\gamma}^0 \gg \dot{\gamma}^i$, $i = 1, 2, 3$.

In the first few decades following the conception of the theory, the understanding of the field equations (1.0.1) progressed at a slow pace. Although many seminal explicit solutions of (1.0.1) had already emerged by the late 30s (see, e.g. [132, 65, 114]), paving the ground for striking future advances in the field, general relativity only moved to the forefront of theoretical physics much later. The so-called golden period of general relativity, spanning the 60s and 70s, was characterized by a large number of major breakthroughs in the understanding of (1.0.1), which also led also to a revolution in the fields of astrophysics and cosmology. The concept of a black hole, the incompleteness theorems of Penrose [115] and Hawking–Penrose [78] and the cosmic censorship conjectures of Penrose [116, 117] are part of the legacy of this period. Subsequently, during the late 70s–90s, general relativity further evolved as a major field of interplay between mathematics and physics, with progress driven by the introduction of “heavy” analytical tools from the field of PDEs, resulting in the resolution of the positive mass conjecture by Schoen–Yau [131], the establishment of the non-linear stability of Minkowski spacetime by Christodoulou–Klainerman [32] and the resolution of the weak and strong cosmic censorship conjectures in the special case of spherically symmetric models by Christodoulou [26, 27, 29, 30]. Today, general relativity is an active field of research, lying at the intersection of non-linear analysis, geometry and physics; the recent landmark experimental observation of gravitational waves (one of the most spectacular predictions of (1.0.2)) by the LIGO collaboration [62] serves to illustrate that the bond between the deeper understanding of the mathematical structure of (1.0.1) and the physical consequences of the theory remains as strong as ever.

The Einstein equations with a *cosmological constant* Λ ,

$$(1.0.5) \quad Ric_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu},$$

constitute a variation of (1.0.1) written down in 1917 by Einstein [60], in an attempt to “correct” the apparent incompatibility of (1.0.1) with the existence of a static, spatially compact model of the universe. After the expansion of the universe was discovered by Hubble, Einstein repudiated the introduction of Λ as “the greatest blunder” of his life. However, in recent years, physicists have reintroduced Λ : Astronomical observations indicating an

accelerating expansion of the universe [123, 10] suggest that, for the study of gravitational effects at a cosmological scale, the case $\Lambda > 0$ should be considered; on the other hand, the high energy physics community is primarily interested in the case $\Lambda < 0$, motivated by duality considerations (see Section 1.2.2 for a brief discussion of the latter case).

In this thesis, we will present two results (appearing in [112, 109]) related to the stability properties of (1.0.5) in the cases $\Lambda = 0$ and $\Lambda < 0$. Before introducing these results to the reader, however, we will briefly review the basic notions associated to the initial value problem for the Einstein equations (1.0.5), allowing us to pose questions about the dynamics of (1.0.5). We will also discuss a few seminal results and unanswered conjectures associated to the stability question of (1.0.5) in the cases when $\Lambda = 0$ and $\Lambda < 0$.

1.1 Hyperbolicity and dynamics: The Cauchy problem for the Einstein equations

The analysis of a given system of partial differential equation usually begins by classifying the system into a broader category of equations with similar qualitative characteristics. In the case of second order equations, for instance, standard examples of such categories include the families of elliptic, parabolic and hyperbolic equations. Restricting (for the purposes of this section) to the vacuum Einstein equations, so as to suppress all issues related to the equations associated to matter, the system (1.0.2) can be expressed in any local coordinate chart as

$$(1.1.1) \quad Ric_{\mu\nu}[g] = \frac{1}{2}g^{\alpha\beta}(\partial_\mu\partial_\alpha g_{\beta\nu} + \partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\mu\partial_\nu g_{\alpha\beta} - \partial_\alpha\partial_\beta g_{\mu\nu}) + Q_{\mu\nu}(g; \partial g) = 0,$$

where $Q_{\mu\nu}$ is a smooth function of its variables. Thus, it can be readily verified that, unlike its elliptic Newtonian predecessor (1.0.4), the system (1.1.1) does not fall into any of the traditional standard classes of second order system of PDEs.

Our apparent inability to suitably classify the system (1.1.1) is intimately tied to the geometric nature of (1.0.2), as can be effectively illustrated by the following simple example:

Denoting for any $\rho > 0$

$$\mathcal{B}_\rho = \left\{ \sum_{\mu} (x^\mu)^2 < \rho \right\} \subset \mathbb{R}^{3+1},$$

let g be a given smooth solution of (1.0.2) on $\mathcal{B}_1$. Choosing a non-trivial diffeomorphism $F : \mathcal{B}_1 \rightarrow \mathcal{B}_1$ which restricts to the identity on $\mathcal{B}_1 \setminus \mathcal{B}_{1/2}$,³ let us denote by g' the pull-back F^*g of g through F . Then, it trivially follows that g' also solves the geometric equation (1.1.1). Expressing g, g' in the Cartesian coordinate system (x^0, x^1, x^2, x^3) on $\mathcal{B}_1$, the choice of F implies that

$$(1.1.2) \quad \begin{aligned} g_{\mu\nu}(x) &= g'_{\mu\nu}(x) \text{ for all } x \in \mathcal{B}_1 \setminus \mathcal{B}_{\frac{1}{2}}, \\ g_{\mu\nu}(x) &\neq g'_{\mu\nu}(x) \text{ for some } x \in \mathcal{B}_{\frac{1}{2}}. \end{aligned}$$

Note that (1.1.2) implies that (1.1.1) can not be an elliptic, parabolic or hyperbolic system, since standard uniqueness results for these classes of systems of PDEs would imply that $g_{\mu\nu} = g'_{\mu\nu}$ everywhere on $\mathcal{B}_1$. The above argument, known simply as the *hole argument*, was a major source of confusion for Einstein, who devised it in 1913; in an attempt to circumvent what he perceived as a paradox, he was subsequently led, for a few years, to the pursuit of a non-fully covariant theory of gravity.

Taking into account the freedom in changing the underlying coordinate chart, which constitutes the *gauge freedom* of the system (1.0.2), it was gradually understood that (1.0.2) is of hyperbolic nature. A hint for this fact is provided by the following observation: Restricting the gauge freedom of (1.0.2) by requiring that (1.0.2) is expressed in a local coordinate chart (x^0, x^1, x^2, x^3) on $(\mathcal{M}, g)$ which satisfies the *wave condition* (identified already by de Donder [50] in 1921)

$$(1.1.3) \quad \square_g x^\mu = 0,$$

³Such an F can not be an isometry for g , since in that case the condition $F = id$ on $\mathcal{B}_1 \setminus \mathcal{B}_{1/2}$ would imply that F is trivial.

where $\square_g$ is the wave operator

$$(1.1.4) \quad \square_g \doteq \frac{1}{\sqrt{-\det(g)}} \partial_\mu (g^{\mu\nu} \sqrt{-\det(g)} \partial_\nu),$$

the general expression (1.1.1) for (1.0.2) reduces to the following system of quasilinear wave equations:

$$(1.1.5) \quad \frac{1}{2} \square_g g_{\mu\nu} + Q_{\mu\nu}(g; \partial g) = 0.$$

The hyperbolic nature of the system (1.0.2) implies that the right setting to study its solutions is, therefore, that of the *initial value problem* (also known as the Cauchy problem).

Tightly linked to the initial value problem for hyperbolic equations is the notion of *global hyperbolicity*, introduced by Leray [98]:

Global hyperbolicity. A time-orientable Lorentzian manifold $(\mathcal{M}, g)$ is called globally hyperbolic if it contains a spacelike hypersurface Σ (called a *Cauchy hypersurface*) with the property that every inextendible causal (i.e. timelike or null) curve ζ in $\mathcal{M}$ meets Σ at exactly one point.

The assumption of global hyperbolicity poses a few topological restrictions on the underlying manifold, since a smooth Lorentzian manifold $(\mathcal{M}, g)$ containing a smooth Cauchy hypersurface Σ is necessarily difeomorphic to $\mathbb{R} \times \Sigma$ (see [12]).

A Cauchy hypersurface provides a natural location for setting up initial data for hyperbolic equations: Given a globally hyperbolic background $(\mathcal{M}, g)$ (with fixed time orientation) and a smooth Cauchy hypersurface Σ , the initial value problem for the linear wave equation

$$(1.1.6) \quad \begin{cases} \square_g \psi = 0, \\ \psi|_\Sigma = \psi_0, \\ N(\psi)|_\Sigma = \psi_1, \end{cases}$$

where N is a future directed unit normal to Σ and $\psi_0, \psi_1 : \Sigma \rightarrow \mathbb{R}$ are given smooth functions,

is *well-posed*: It admits a unique solution $\psi : \mathcal{M} \rightarrow \mathbb{R}$ which depends continuously (in the C^∞ topology) on the initial data ψ_0, ψ_1 . More generally, one can similarly formulate the initial value problem for a quasilinear wave equation defined on a background manifold $\mathcal{M}$:

$$(1.1.7) \quad \begin{cases} \square_{g(x;\psi,\partial\psi)} \psi(x) + Q(x; \psi, \partial\psi) = 0, \\ \psi|_\Sigma = \psi_0, \\ N(\psi)|_\Sigma = \psi_1, \end{cases}$$

with $\Sigma \subset \mathcal{M}$ being a two-sided hypersurface which is spacelike with respect to the metric $g(x; \psi(x), \partial\psi(x))|_{x \in \Sigma}$ (with $(\psi(x), \partial\psi(x))|_{x \in \Sigma}$ evaluated in terms of (ψ_0, ψ_1)) and N being the unit normal to Σ with respect to the same metric. In this case, well-posedness consists of establishing the existence of an open neighborhood $\mathcal{U} \subset \mathcal{M}$ of Σ , on which (1.1.7) admits a unique solution ψ , changing continuously under small variations of (ψ_0, ψ_1) ; a necessary condition on $\mathcal{U}$, in this case, is that it is globally hyperbolic with Cauchy hypersurface Σ , when equipped with the metric $g(x; \psi, \partial\psi)$. The analytical tools and techniques for establishing the well-posedness of the initial value problem (1.1.7) were developed in the 30s, through the work of mathematicians such as Schauder, Sobolev and Petrovskii (see e.g. [129, 135, 118])

Returning to the vacuum Einstein equations (1.0.2), in view of the previous discussion and the geometric character of (1.0.2), it appears natural to formulate the initial value problem for (1.0.2) in terms of the induced metric $\bar{g}$ and the second fundamental form k of a Cauchy hypersurface Σ of a solution $(\mathcal{M}, g)$, which can be thought of as the geometric analogues of $\psi|_\Sigma$ and $N(\psi)|_\Sigma$ in (1.1.7). However, $\bar{g}$ and k in this case can not be freely prescribed; they are related through the *vacuum constraint equations* (a consequence of (1.0.2) and the Gauss and Codazzi equations applied to the embedded hypersurface Σ):

$$(1.1.8) \quad \begin{aligned} R_{\bar{g}} + k_i^i k_j^j - k_j^i k_i^j &= 0, \\ \nabla_j^{(\bar{g})} k_i^j - \nabla_i^{(\bar{g})} k_j^j &= 0, \end{aligned}$$

where $R_{\bar{g}}$ and $\nabla^{(\bar{g})}$ are the scalar curvature and covariant derivative of $(\Sigma, \bar{g})$, respectively.

The initial value problem for (1.0.2) is thus formulated as follows:

Cauchy problem for the EVE (1.1.5). Given a smooth, 3-dimensional manifold Σ , a Riemannian metric $\bar{g}$ and a symmetric $(0, 2)$ -tensor k on Σ satisfying the constraint equations (1.1.8), determine a $3 + 1$, globally hyperbolic Lorentzian manifold $(\mathcal{M}, g)$ solving (1.0.2), together with an embedding $i : \Sigma \rightarrow \mathcal{M}$, such that $i(\Sigma)$ is a Cauchy hypersurface of $\mathcal{M}$, with induced metric $\bar{g}$ and second fundamental form k .

In this setting, $(\Sigma, \bar{g}, k)$ is referred to as an *initial data set* for (1.0.2), while a corresponding solution $(\mathcal{M}, g)$ to (1.0.2) is called a *development* of $(\Sigma, \bar{g}, k)$.

The above formulation of the initial value problem for (1.0.2) appeared for the first time in the landmark work [64] by Choquet-Bruhat in 1952; the fact that this formulation is indeed the appropriate one stems from the following fundamental well-posedness result, also obtained in [64]:

Theorem (Choquet-Bruhat, 1952). *For any sufficiently regular initial data set $\mathcal{S} = (\Sigma, \bar{g}, k)$ of (1.0.2), there exists an associated globally hyperbolic development $(\mathcal{M}, g)$ solving (1.0.2). Furthermore, $(\mathcal{M}, g)$ is locally unique in the following sense: For any other such development $(\mathcal{M}', g')$ of $\mathcal{S}$, denoting with i, i' the corresponding embeddings of Σ in $\mathcal{M}, \mathcal{M}'$, there exists an open set $\mathcal{U} \subset \mathcal{M}$ containing $i(\Sigma)$ and an isometric embedding $\mathcal{F} : (\mathcal{U}, g) \rightarrow (\mathcal{M}', g')$ commuting with the corresponding embeddings i, i' of Σ , i. e. $\mathcal{F} \circ i = i'$.*

Let us note that the proof of the above result also yields the continuous dependence of g on $(\bar{g}, k)$ (in a suitable sense), in a high-regularity topology. The uniqueness part of the above theorem was essentially established already in [137].

It is worth mentioning that the well-posedness results of [64] are not a simple corollary of the corresponding results for (1.1.7), a special case of which is the system (1.1.5). The reduction to the well-posedness results for systems of the form (1.1.7) in [64] required two crucial observations for solutions $(\mathcal{M}, g)$ of (1.0.2):

- In any coordinate chart $\{x^\mu\}$ (not necessarily satisfying the gauge condition (1.1.3)), if $(\mathcal{M}, g)$ solves the modified equation (1.1.5), then, as a consequence of the Bianchi

identities for $Ric(g)$, the quantity

$$(1.1.9) \quad \mathcal{D}^\alpha = g^{\beta\gamma} \Gamma_{\beta\gamma}^\alpha$$

(where $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols of g), which vanishes precisely when the coordinates $\{x^\mu\}$ satisfy (1.1.3), is governed by a wave equation with the same principal symbol as (1.1.5), i. e.

$$(1.1.10) \quad \nabla^\mu \nabla_\mu \mathcal{D}^\alpha + g^{\alpha\beta} Ric_{\beta\mu}(g) \mathcal{D}^\mu = 0.$$

Since the systems (1.1.5) and (1.1.5) are equivalent when $\mathcal{D} \equiv 0$, finding a solution of the coupled system (1.1.5) and (1.1.10) (consisting solely of quasilinear wave equations with the same principal symbol) with $(\mathcal{D}, \partial\mathcal{D})|_\Sigma = 0$ yields a solution to the original Einstein equations (1.0.2).

- The initial conditions $\mathcal{D}|_\Sigma = 0$ and $\partial\mathcal{D}|_\Sigma = 0$ for (1.1.10) can be both satisfied exactly when the constraint equations (1.1.8) are satisfied.

While [64] established the aforementioned well-posedness results for initial data with sufficiently high regularity, a lot of work has been recently devoted to obtaining the same results under minimal regularity assumptions; the so-called L^2 curvature conjecture, corresponding to the case when $\bar{g} \in H^2(\Sigma)$ and $k \in H^1(\Sigma)$, was finally settled in [92].

Using a suitable gluing procedure for developments of the same initial data set (which is possible in view of the local uniqueness statement of the above theorem), the results of [64] are sufficient to establish the existence of a unique maximal development of a given initial data set (see [25]):

Theorem (Choquet-Bruhat–Geroch, 1969). *For any sufficiently regular initial data set $\mathcal{S} = (\Sigma, \bar{g}, k)$ of (1.0.2), there exists a unique globally hyperbolic development $(\mathcal{M}, g)$ of $\mathcal{S}$ solving (1.0.2) which is maximal in the following sense: For any other globally hyperbolic development $(\mathcal{M}', g')$ of $\mathcal{S}$, there exists an isometric embedding $\mathcal{F} : (\mathcal{M}', g') \rightarrow (\mathcal{M}, g)$ commuting with the corresponding embeddings of Σ .*

We will refer to the maximal development provided by the above theorem as the *maximal globally hyperbolic development*. The uniqueness of this development is fundamental for any discussion on the dynamics (as well as the predictability) of the Einstein equations. Although the initial proof of the above theorem in [25] made use of Zorn's lemma, this construction has been recently "dezornified" in [128].

Remark. We should notice that a unique maximal globally hyperbolic development does not always exist for scalar, quasilinear wave equations of the form (1.1.7) on a fixed background $\mathcal{M}$: there are examples of initial value problems of the form (1.1.7) admitting two solutions ψ_1, ψ_2 defined, respectively, on two regions $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{M}$, satisfying $\mathcal{U}_1 \not\subset \mathcal{U}_2$ and $\mathcal{U}_2 \not\subset \mathcal{U}_1$, such that $\mathcal{U}_1, \mathcal{U}_2$ are globally hyperbolic with respect to the associated metrics $g(x; \psi_i, \partial\psi_i)$,⁴ but which cannot both be embedded into a larger common development $(\mathcal{U}, \psi)$ (see [61]). One of the special features of the Einstein equations (1.0.2) that enables the existence of a unique maximal globally hyperbolic development is the fact that the underlying manifold $\mathcal{M}$ is constructed as part of the solution, and, therefore, it is not a priori the case that all developments of an initial data set $(\Sigma, \bar{g}, k)$ are defined on subregions of the same background manifold.

While in this section we have restricted ourselves for simplicity to the vacuum Einstein equations (1.0.2) with zero cosmological constant, all the well-posedness results of [64]-[25] extend trivially to include the case of an arbitrary cosmological constant Λ , as well as the general case (1.0.5) for most interesting matter fields for which the corresponding field equations admit a well-posed initial value problem on Cauchy hypersurfaces of the background spacetime. In the latter case, the initial data set $(\Sigma, \bar{g}, k)$ must be supplemented with initial data for the matter fields, while the constraint equations (1.1.8) are modified as follows:

$$(1.1.11) \quad \begin{aligned} R_{\bar{g}} + k_i^i k_j^j - k_j^i k_i^j &= 16\pi T_{\mu\nu} N^\nu N^\mu + 2\Lambda, \\ \nabla_j^{(\bar{g})} k_i^j - \nabla_i^{(\bar{g})} k_j^j &= \Pi_i^\nu T_{\mu\nu} N^\mu, \end{aligned}$$

where N denotes the unit normal to Σ associated to its embedding in $\mathcal{M}$, while

⁴Thus, necessarily, $\psi_1 = \psi_2$ on $\mathcal{U}_1 \cap \mathcal{U}_2$.

$\Pi : T^*\mathcal{M} \rightarrow T^*\Sigma$ denotes the associated projection operator along N ; all quantities in the right hand side of (1.1.11) can in principle be formally computed in terms of $(\bar{g}, k)$ and the initial data $\{\Psi_i^{(0)}\}$ for the matter fields $\{\Psi_i\}$.

1.2 The global stability problem

The well-posedness of the initial value problem for the Einstein equations and the existence of a unique maximal globally hyperbolic development, discussed in the previous section, allow us to pose questions about the global dynamics of certain families of initial data. In particular, a prominent question arising naturally in this context is the following:

How much does the global geometry of the maximal globally hyperbolic development $(\mathcal{M}, g; \{\Psi_i\})$ of an initial data set $\mathcal{S} = (\Sigma, \bar{g}, \bar{k}; \{\Psi_i^{(0)}\})$ for (1.0.5) (and the corresponding equations for matter) change if $\mathcal{S}$ are slightly perturbed?

The answer to this stability question has immediate implications for the physical relevance of $(\mathcal{M}, g; \{\Psi_i\})$ as a model of a gravitational phenomenon that can be observed in the universe.

Related to the stability of a spacetime $(\mathcal{M}, g)$ as a solution to (1.0.5) is the study of the stability properties of solutions ϕ to the scalar wave equation

$$(1.2.1) \quad \square_g \phi = 0$$

on $(\mathcal{M}, g)$, which can be viewed as a simple toy model for (1.0.5).⁵

In this section, we will briefly discuss some of the most significant results and conjectures related to the stability properties of a few prominent explicit solutions $(\mathcal{M}, g; \{\Psi_i\})$ of (1.0.5), as well as the stability properties of (1.2.1) on certain backgrounds $(\mathcal{M}, g)$. We will restrict our discussion to the cases $\Lambda = 0$ and $\Lambda < 0$ in (1.0.5), since the results of this thesis will be relevant only to those cases.

⁵Equation (1.2.1) also appears in other fields of mathematical physics, such as fluid mechanics; later in this thesis, we will state a few results for (1.2.1) on backgrounds arising in the study of the Euler equations for fluids.

1.2.1 The case $\Lambda = 0$: Asymptotically flat spacetimes

The simplest solution of (1.0.1) in the absence of matter is the flat spacetime $(\mathbb{R}^{3+1}, \eta)$, also called *Minkowski spacetime*. Viewed in the context of the initial value problem for (1.0.2), $(\mathbb{R}^{3+1}, \eta)$ is the maximal globally hyperbolic development of the trivial initial data $(\mathbb{R}^3, e, 0)$, where e is the standard Euclidean metric on $\mathbb{R}^3$.

Tied to the asymptotic geometry of $(\mathbb{R}^{3+1}, \eta)$ is the notion of null infinity: *Future null infinity* $\mathcal{I}^+$ can be defined, in a limiting procedure, as the set of the “future endpoints at infinity” of all half-lines of the form $\{t - r, \theta, \varphi = \text{const}\}$, in the standard polar coordinate chart (t, r, θ, φ) on $\mathbb{R}^{3+1}$; equipped with the natural topology associated to its parametrization by $u = t - r$, θ and φ , $\mathcal{I}^+$ is homeomorphic to $\mathbb{R} \times \mathbb{S}^2$. Similarly, *past null infinity* $\mathcal{I}^-$ can be defined as the set of all “past endpoints at infinity” of all half-lines of the form $\{t + r, \theta, \varphi = \text{const}\}$. Alternatively, $\mathcal{I}^+$ and $\mathcal{I}^-$ can be viewed as null hypersurfaces attached to $(\mathbb{R}^{3+1}, \eta)$ at its conformal boundary at infinity, by conformally identifying $(\mathbb{R}^{3+1}, \eta)$ with the region $\{0 \leq \bar{r} < \pi\} \cap \{|\bar{t}| < \pi - \bar{r}\}$ in the Einstein cylinder $(\mathbb{R} \times \mathbb{S}^3, -d\bar{t}^2 + g_{\mathbb{S}^3})$ through the mapping

$$\begin{aligned} (1.2.2) \quad \bar{t}(t, r) &= \tan^{-1}(t + r) + \tan^{-1}(t - r), \\ \bar{r}(t, r) &= \tan^{-1}(t + r) - \tan^{-1}(t - r), \\ (\bar{\theta}, \bar{\varphi}) &= (\theta, \varphi), \end{aligned}$$

where $(\bar{t}, \bar{r}, \theta, \varphi)$ are the standard polar coordinates on $\mathbb{R} \times \mathbb{S}^3$ (with $\bar{r} \in [0, \pi]$).

A significant class of gravitational phenomena in the universe can be observed in the context of isolated, self-gravitating systems; these systems can be modelled by solutions $(\mathcal{M}, g)$ to (1.0.1) which are *asymptotically flat*, i. e. possess an asymptotic region the geometry of which is close to that of $(\mathbb{R}^{3+1}, \eta)$. However, in the language of the initial value problem for (1.0.1), the appropriate way to define classes of solutions $(\mathcal{M}, g; \{\Psi_i\})$ of (1.0.1) is through conditions on the properties of their corresponding initial data sets $(\Sigma, \bar{g}, k; \{\Psi_i^{(0)}\})$; in this setting, the geometric properties of spacetimes $(\mathcal{M}, g)$ in any such class are then derived from the conditions on the initial data through the study of the evolution equations (1.0.1).

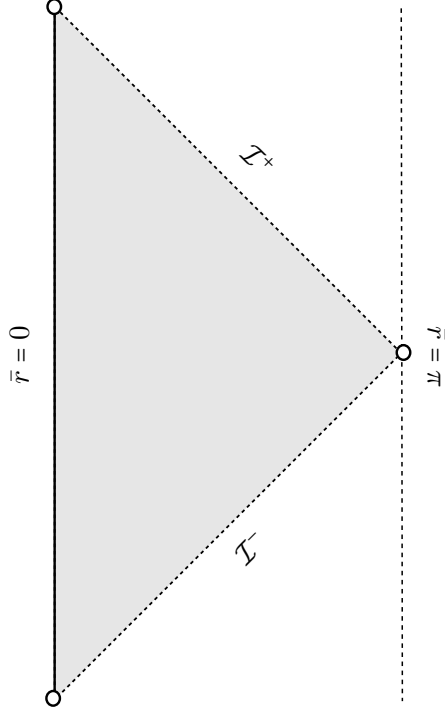


Figure 1: The above figure depicts the image of $(\mathbb{R}^{3+1}, \eta)$ in $\mathbb{R} \times \mathbb{S}^3$, after taking the quotient by the spheres of symmetry $t, r = \text{const}$. The future and past null infinity, $\mathcal{I}^+$ and $\mathcal{I}^-$, respectively, appear naturally in this picture as null hypersurfaces bounding the image of $(\mathbb{R}^{3+1}, \eta)$.

In this context, asymptotic flatness is defined at the level of the initial data as follows:

Definition. An initial data set $(\Sigma, \bar{g}, k; \{\Psi_i^{(0)}\})$ for (1.0.1) is called *strongly asymptotically flat* with m ends if there exists a compact subset $\mathcal{K} \subset \Sigma$, such that $\Sigma \setminus \mathcal{K}$ has m components, each of which is diffeomorphic to $\mathbb{R}^3 \setminus B_0(1)$ (where $B_0(1)$ is the Euclidean unit ball centered at 0), such that, on each component, in the pullback of the Cartesian coordinates $\{x^i\}$ through the associated diffeomorphism, there exists some $M \in \mathbb{R}$ so that:

$$(1.2.3) \quad \bar{g}_{ij} = \left(1 - \frac{2M}{r}\right) e_{ij} + o_2(r^{-1}) \text{ and } k_{ij} = o_1(r^{-2}),$$

where $r = \sqrt{\sum_{i=1}^3 (x^i)^2}$ and $f = o_l(g)$ denotes that $f, r \nabla f, \dots, r^l \nabla^l f$ are of order $o(g)$ as $r \rightarrow +\infty$.

We will now proceed to discuss the global stability problem for (1.0.1) for a special class of strongly asymptotically flat initial data sets.

The non-linear stability of Minkowski spacetime

Among all strongly asymptotically flat initial data sets for (1.0.2), the trivial data $(\mathbb{R}^3, e, 0)$ are of particular importance for the theory, since $(\mathbb{R}^{3+1}, \eta)$ provides, within general relativity, a geometric model for the special theory of relativity. Therefore, the global stability properties of $(\mathbb{R}^{3+1}, \eta)$ against vacuum perturbations of its initial data is of fundamental importance for the pertinence of general relativity as a theory of gravity incorporating the principle of relativity.

Establishing the global stability of Minkowski spacetime remained for a long time a challenging open problem, owing to the borderline stability properties of general quasilinear wave equations of the form (1.1.7) on $\mathbb{R}^{3+1}$, as well as the difficulties associated to the geometric character of the system (1.0.1). Eventually, this problem was successfully addressed in the monumental work of Christodoulou–Klainerman [32] (in which a rich non-linear structure associated to (1.0.1) was uncovered and exploited):

Theorem (Christodoulou–Klainerman, 1993). *For any strongly asymptotically flat initial data set $(\mathbb{R}^3, \bar{g}, k)$ of (1.0.2), such that $(\bar{g}, k)$ are sufficiently close to the trivial ones in a suitable, high-regularity, weighted Sobolev norm, the corresponding maximal globally hyperbolic development $(\mathcal{M}, g)$ is geodesically complete. Furthermore, along any future inextendible, timelike geodesic γ , the geometry of $(\mathcal{M}, g)$ approaches asymptotically that of $(\mathbb{R}^{3+1}, \eta)$ in a quantitative sense. A “complete” future null infinity $\mathcal{I}^+$ can also be attached to $(\mathcal{M}, g)$, such that, in addition, from every point $p \in \mathcal{M}$, there exists a future directed causal curve asymptotically terminating at $\mathcal{I}^+$.*

The original proof of the above result in [32] was of geometric character; an alternative proof, cast in a coordinate system satisfying the wave condition (1.1.3), was obtained in [99]. For a proof of the global stability of Minkowski spacetime in the presence of various forms of matter, see [136, 140, 100, 63].

The results of [32] allow one to infer properties related to asymptotic flatness at the *spacetime* level, assuming strong asymptotic flatness at the level of the initial data. In particular, let $(\mathcal{M}, g)$ be a solution of (1.0.2) arising as a development of a strongly asymptotically flat initial data set $(\Sigma, \bar{g}, k)$, not necessarily close to the trivial one. Denoting

by $\mathcal{S}$ any of the connected components of the asymptotic region $\{r \geq R \gg 1\}$ in Σ and restricting to the domain of dependence $\mathcal{D}(\mathcal{S})$ of $\mathcal{S}$ in $(\mathcal{M}, g)$, the results of [32] imply that a “past complete” piece of future null infinity $\mathcal{I}^+$ can be attached to $\mathcal{D}(\mathcal{S})$. Informally, this is equivalent to stating that the geometry of $\mathcal{D}(\mathcal{S}) \subset \mathcal{M}$ is close to that of the region $\{|t| - r \leq u_0\}$ in $(\mathbb{R}^{3+1}, \eta)$, for some $u_0 \in \mathbb{R}$.

The Schwarzschild and Kerr black hole exteriors

Switching our attention to more complicated asymptotically flat solutions of (1.0.2), a prominent family of explicit solutions of (1.0.2) is the *Schwarzschild* family $(\mathcal{M}_M, g_M)$, $M > 0$, discovered already in 1916 [132]: Defined on $\mathcal{M}_M = \mathbb{R} \times (2M, +\infty) \times \mathbb{S}^2$, the Schwarzschild metric g_M takes the form:

$$(1.2.4) \quad g_M = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

It can be readily verified from the expression (1.2.4) that the hypersurface $\Sigma = \{t = 0\} \subset \mathcal{M}_M$ is a Cauchy hypersurface of $(\mathcal{M}_M, g_M)$ and the induced vacuum initial data set on Σ is strongly asymptotically flat. Furthermore, a complete future and past null infinity, $\mathcal{I}^+$ and $\mathcal{I}^-$, respectively, can be regularly attached to $(\mathcal{M}_M, g_M)$.

Although the expression (1.2.4) becomes singular when $r = 2M$, this singularity is merely a manifestation of a coordinate breakdown; following Kruskal [96], switching to the coordinate system (T, R, θ, φ) defined implicitly by the relations

$$\begin{aligned} T^2 - R^2 &= \left(1 - \frac{r}{2M}\right)e^{r/2M}, \\ \frac{T}{R} &= \tanh\left(\frac{t}{4M}\right), \end{aligned}$$

one can readily verify that $(\mathcal{M}_M, g_M)$ can be smoothly extended to a larger globally hyperbolic spacetime $(\tilde{\mathcal{M}}_M, \tilde{g}_M)$. In this extension, the boundary $\mathcal{H}$ of the original spacetime $(\mathcal{M}_M, g_M)$ consists of the union of two null hypersurfaces $\mathcal{H}^+$ and $\mathcal{H}^-$, called the *future* and *past event horizon*, respectively, intersecting along a spacelike two sphere. For any point p of $(\tilde{\mathcal{M}}_M, \tilde{g}_M)$ lying to the future of $\mathcal{H}^+$, there exists no future directed causal curve

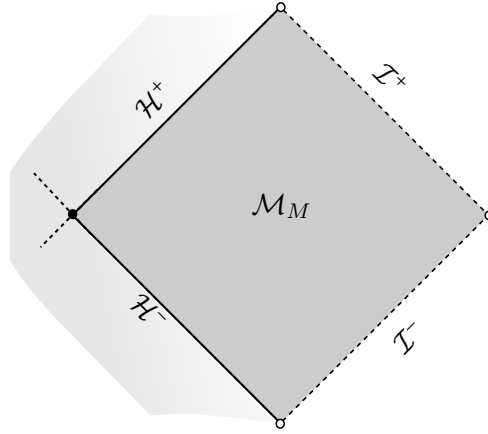


Figure 2: The above figure depicts a compactified image of the Schwarzschild exterior spacetime $(\mathcal{M}_M, g_M)$, after taking the quotients with respect to the spheres of symmetry. In the extension $(\tilde{\mathcal{M}}_M, \tilde{g}_M)$ of $(\mathcal{M}_M, g_M)$, the region lying to the future of $\mathcal{H}^+$ constitutes the black hole region, while the region in the past of $\mathcal{H}^-$ is the white hole region.

emanating from p and reaching the asymptotically flat region $\{r \geq R \gg 2M\}$ of $(\mathcal{M}_M, g_M)$; for this reason, the region lying to the future of $\mathcal{H}^+$ is dubbed the *black hole* region. Similarly, the region of $(\tilde{\mathcal{M}}_M, \tilde{g}_M)$ lying to the past of $\mathcal{H}^-$, called the *white hole* region, cannot communicate with $\{r \geq R \gg 2M\}$ through past directed causal curves. In order to distinguish $(\mathcal{M}_M, g_M)$ from its extension $(\tilde{\mathcal{M}}_M, \tilde{g}_M)$, the spacetime $(\mathcal{M}_M, g_M)$ is often called the Schwarzschild exterior spacetime.

The Schwarzschild family is embedded into a larger, 2-parameter family of explicit, asymptotically flat solutions of (1.0.2): the *Kerr* family $(\mathcal{M}_{M,a}, g_{M,a})$, discovered in 1963 [91]. In the Boyer–Lindquist coordinate chart $(t, r, \vartheta, \varphi)$ on $\mathcal{M}_{M,a} = \mathbb{R} \times (r_+, +\infty) \times \mathbb{S}^2$, the metric $g_{M,a}$ takes the form

$$(1.2.5) \quad g_{M,a} = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar \sin^2 \vartheta}{\rho^2} dt d\varphi + \frac{\varrho^2}{\Delta} dr^2 + \varrho^2 d\vartheta^2 + \sin^2 \vartheta \frac{\Pi}{\rho^2} d\varphi^2,$$

where

$$\begin{aligned}\varrho^2 &= r^2 + a^2 \cos^2 \vartheta, \\ \Delta &= (r - r_+) \cdot (r - r_-), \\ r_{\pm} &= M \pm \sqrt{M^2 - a^2}, \\ \Pi &= (r^2 + a^2)^2 - a^2 \sin^2 \vartheta \Delta.\end{aligned}$$

The Schwarzschild metric (1.2.4) corresponds to (1.2.5) for $a = 0$. When the parameters a , M lie in the range $0 \leq |a| < M$, the metric (1.2.5) is called *subextremal*, with the *extremal* case corresponding to $|a| = M$.

In the subextremal range $0 \leq |a| < M$, similarly as in the case of Schwarzschild exterior, $(\mathcal{M}_{M,a}, g_{M,a})$ can be extended beyond $r = r_+$ to a larger globally hyperbolic spacetime $(\tilde{\mathcal{M}}_{M,a}, \tilde{g}_{M,a})$; $(\mathcal{M}_{M,a}, g_{M,a})$ is also bounded, in this extension, by a pair of intersecting null hypersurfaces $\mathcal{H}^+$, $\mathcal{H}^-$, separating $(\mathcal{M}_{M,a}, g_{M,a})$ from a black hole region and a white hole region, respectively. When $|a| = M$, a smooth extension of $(\mathcal{M}_{M,a}, g_{M,a})$ beyond $r = r_+$ through a pair null hypersurfaces $\mathcal{H}^+$ and $\mathcal{H}^-$ (again bounding a black hole and a white hole region, respectively) also exists. However, in this case the hypersurfaces $\mathcal{H}^+$, $\mathcal{H}^-$ do not intersect; moreover, the extension can not be performed within the class of globally hyperbolic spacetimes.

The metrics (1.2.5) are *stationary*, i.e. possess a Killing vector field T (equal to ∂_t in the (t, r, θ, φ) coordinate chart), such that T is timelike in the asymptotically flat region $\{r \geq R \gg r_+\}$. The significance of the Kerr family in the context of the vacuum equations (1.0.2) lies in the fact that $(\mathcal{M}_{M,a}, g_{M,a})$, for $0 \leq |a| \leq M$, are actually expected to be the *only* stationary black hole exterior solutions $(\mathcal{M}, g)$ of (1.0.2) with a smooth event horizon $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$. This conjecture, also called the *no-hair conjecture*, has been established for some significant classes of spacetimes $(\mathcal{M}, g)$: Under the condition that $\mathcal{H}$ is non-degenerate, the case when $(\mathcal{M}, g)$ is axisymmetric was treated in [22], while the case when $(\mathcal{M}, g)$ is real analytic was shown to reduce to the axisymmetric case by the results of [79, 125, 33]. Beyond the class of real analytic solutions, [2, 3] treated the case when $(\mathcal{M}, g)$ is merely smooth but is assumed to be either close to a member of the Kerr family (1.2.5), or satisfying

a certain smallness condition on the bifurcation sphere $\mathcal{H}^+ \cap \mathcal{H}^-$ (corresponding to $|a| \ll M$ in (1.2.5)).

The stability properties of the Kerr exterior spacetimes

The following stability result regarding the subextremal Kerr exterior spacetimes is widely expected to hold:

Conjecture (Non-linear stability of Kerr). *For any $0 \leq |a_0| < M_0$ and any initial data set $(\Sigma, \bar{g}, k)$ for (1.0.2) which is sufficiently close (in a suitable weighted, high-regularity norm) to those of $(\mathcal{M}_{M_0, a_0}, g_{M_0, a_0})$, there exist parameters a, M , in the subextremal range, close to a_0, M_0 , such that, in the black hole exterior region of the maximal globally hyperbolic development $(\mathcal{M}, g)$ of $(\Sigma, \bar{g}, k)$, g converges to $g_{a, M}$ along inextendible timelike geodesics in a quantitative sense.*

For a more precise formulation of the conjectured stability properties of the subextremal Kerr family, see e.g. [40].

In view of the no-hair conjecture, which implies that the Kerr family is the only vacuum model for non-dynamic black holes, the Kerr stability conjecture is fundamental for the relevance of black holes as realistic astrophysical objects. We should note that the statement of the above conjecture does not imply the stability of the Schwarzschild family (1.2.4) by itself; the Kerr family (1.2.5) for $|a| \ll M$ provides examples of stationary perturbations of the Schwarzschild initial data which, trivially, do not decay to the Schwarzschild metric towards the future.

In recent years, a substantial amount of research has been devoted in the direction of establishing the Kerr stability conjecture. As part of this research effort, the stability properties of the scalar wave equation (1.2.1) on the spacetimes $(\mathcal{M}_{M, a}, g_{M, a})$ have been studied extensively:

- Starting with the simpler case of the Schwarzschild exterior family (1.2.4), boundedness estimates for solutions φ to (1.2.1) was obtained in [90], while polynomial decay estimates for φ were established in [44, 43, 16, 17]. In these works, a wide array of analytical tools were developed in order to treat the novel geometric features of

$(\mathcal{M}_M, g_M)$ (compared to flat backgrounds) affecting the decay properties of (1.2.1), such as the existence of a non-degenerate event horizon and the presence of trapped null geodesics orbiting the black hole.

- In the case of the subextremal Kerr exterior family (1.2.5), the process of establishing estimates for solutions φ to (1.2.1) is complicated by the phenomenon of *superradiance*, which is associated to the fact that the natural conserved energy flux for φ , arising from the stationary Killing field T , is not coercive, as a result of T becoming spacelike in a neighborhood of the horizon. The first non-trivial stability result for the wave equation (1.2.1) on $(\mathcal{M}_{M,a}, g_{M,a})$ with $0 < |a| < M$ was, thus, the mode stability result of [141], establishing the non-existence of solutions φ to (1.2.1) of the form

$$\varphi(t, r, \theta, \varphi) = e^{-i\omega t} \varphi_\omega(r, \theta, \varphi)$$

with $\text{Im}(\omega) > 0$ and finite initial energy (this result was extended to exclude mode solutions with $\text{Im}(\omega) = 0$ in [134]). Following an intense research effort by many authors [46, 47, 45, 139, 6], boundedness and polynomial decay estimates for solutions φ to (1.2.1) on $(\mathcal{M}_{M,a}, g_{M,a})$ in the full subextremal range $0 < |a| < M$ were finally obtained in [48]. For various improvements, see [102, 130, 108].

- In the extremal case $|a| = M$, restricted to axisymmetric solutions φ of (1.2.1), [8, 9] has established a mix of stability and instability results, the latter associated to the degeneracy of the event horizon $\mathcal{H}$ in this case. Beyond axisymmetry, however, little is known in this case; see [101, 76] for a discussion on the stability properties of (1.2.1) and their consequences for the non-linear stability problem for the extremal Kerr exterior.

Moving beyond the toy model provided by the scalar wave equation (1.2.1), polynomial decay estimates for the *linearization* of the vacuum Einstein equations (1.0.2) around the Schwarzschild exterior spacetimes were obtained for the first time by Dafermos–Holzegel–Rodnianski [40]; in this work, the analytical tools developed earlier for the treatment of (1.2.1) were crucial for establishing decay estimates of a novel gauge invariant quantity,

from which estimates for the rest of the unknowns of the linearized system could be inferred. Decay estimates for a similar gauge arising in the linearization of (1.0.2) the Kerr exterior family with $|a| \ll M$ were also obtained by the same authors in [41]. An alternative proof of the linear stability of Schwarzschild, in a gauge obtained from the linearization of (1.1.3), was established in the recent work of Johnson [89].

In the case of the non-linear vacuum Einstein equations (1.0.2), the stability problem of the Schwarzschild exterior spacetime, restricted to perturbations which are axisymmetric and polarized, was addressed by Klainerman–Szeftel [93].

1.2.2 The case $\Lambda < 0$

In the case when the cosmological constant Λ is negative, the simplest solution of the Einstein vacuum equations

$$(1.2.6) \quad Ric_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

is *Anti-de Sitter* spacetime $(\mathcal{M}_{AdS}, g_{AdS})$. In the standard polar coordinate chart (t, r, θ, φ) on $\mathcal{M}_{AdS} \simeq \mathbb{R}^{3+1}$, the AdS metric takes the form (see [78])

$$(1.2.7) \quad g_{AdS} = -\left(1 - \frac{1}{3}\Lambda r^2\right)dt^2 + \left(1 - \frac{1}{3}\Lambda r^2\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

The spacetime $(\mathcal{M}_{AdS}, g_{AdS})$ has constant, negative sectional curvature, and can be viewed as the Lorentzian analogue of the hyperbolic space in Riemannian geometry.

Despite being geodesically complete, $(\mathcal{M}_{AdS}, g_{AdS})$ fails to be globally hyperbolic. In particular, it can be conformally identified with the interior of $(\mathbb{R} \times \mathbb{S}_+^3, g_E)$, where $\mathbb{S}_+^3$ is the closed upper hemisphere of $\mathbb{S}^3$ and g_E is the *Einstein* metric

$$(1.2.8) \quad g_E = -dt^2 + g_{\mathbb{S}^3},$$

with $g_{\mathbb{S}^3}$ being the round metric on $\mathbb{S}^3$. Through this identification, the timelike boundary

$$(1.2.9) \quad \mathcal{I} = \mathbb{R} \times \partial\mathbb{S}_+^3 \simeq \mathbb{R} \times \mathbb{S}^2$$

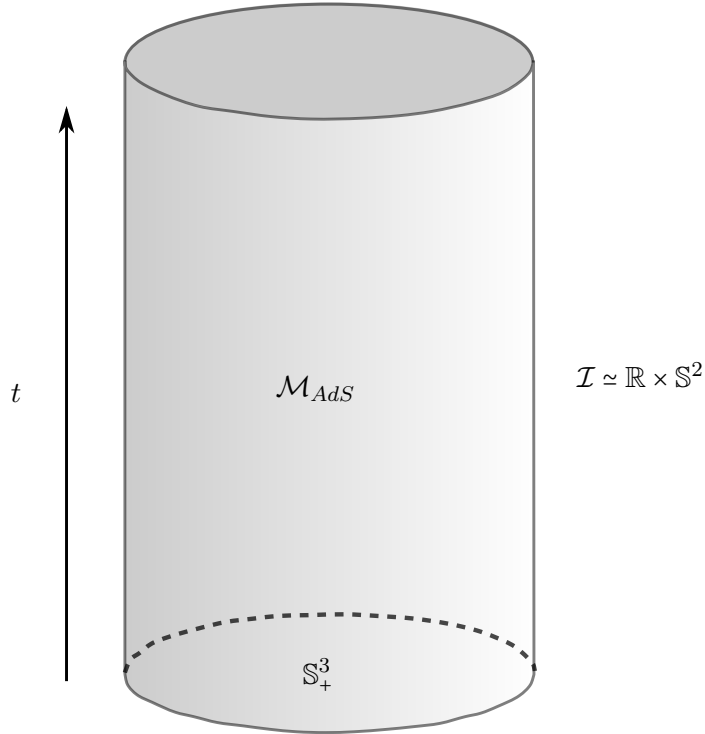


Figure 3: The AdS spacetime $(\mathcal{M}_{AdS}, g_{AdS})$ is conformally equivalent to the interior of $(\mathbb{R} \times \mathbb{S}^3_+, g_E)$. The boundary $\mathcal{I}$ of $(\mathbb{R} \times \mathbb{S}^3_+, g_E)$ can be considered as a conformal boundary at infinity for $(\mathcal{M}_{AdS}, g_{AdS})$.

of $(\mathbb{R} \times \mathbb{S}^3_+, g_E)$ is naturally attached to $(\mathcal{M}_{AdS}, g_{AdS})$ as a “conformal boundary at infinity”.

More generally, the class of *asymptotically AdS* spacetimes consists of those spacetimes $(\mathcal{M}, g)$ possessing an asymptotic region with geometry resembling that of $(\mathcal{M}_{AdS}, g_{AdS})$ in the vicinity of $\mathcal{I}$ (see [70]). Necessarily, an asymptotically AdS spacetime is not globally hyperbolic. Therefore, as we will see in the following section, the class of this spacetimes can not be studied within the framework of the initial value problem for (1.0.5) discussed in Section 1.1, unless additional information is provided along $\mathcal{I}$.

In 1998, Maldacena, Gubser–Klebanov–Polyakov and Witten [103, 75, 142] proposed the *AdS/CFT conjecture*, suggesting a correspondence between certain conformal field theories defined on $\mathcal{I}$ (in the strongly coupled regime) and supergravity on spacetimes asymptotically of the form $(\mathcal{M}_{AdS} \times S^k, g_{AdS} + g_{S^k})$, where (S^k, g_{S^k}) is a suitable compact Riemannian manifold of dimension k (in fact, the conjecture is actually stated for the higher dimensional analogue of $(\mathcal{M}_{AdS}, g_{AdS})$). Following the introduction of this conjecture, asymptotically

AdS spacetimes became a subject of intense study in the high energy physics literature (see e. g. [1, 77, 5] and references therein).

We will now proceed to discuss a few fundamental results and a seminal open problem related to the global dynamics of (1.0.5) in the asymptotically AdS setting.

The initial-boundary value problem and the AdS instability conjecture

The appropriate framework for the study of the dynamics of asymptotically AdS solutions $(\mathcal{M}, g)$ to (1.2.6) is that of an *initial value problem* with suitable *boundary conditions* prescribed asymptotically on $\mathcal{I}$. In the more general case when matter fields are included in the equations (1.0.5), boundary conditions on $\mathcal{I}$ for the matter fields should also be specified.

The issue of the right boundary conditions on $\mathcal{I}$ leading to well posedness for the resulting initial-boundary value problem for (1.2.6) was first addressed by Friedrich in [70]. Working in the conformally compactified picture in a conformally adapted gauge, Friedrich recast the vacuum Einstein equations (1.2.6) into a hyperbolic system of equations (the so-called *conformal field equations*), with unknowns various gauge dependent quantities and the (renormalised) Weyl tensor $W_{\alpha\beta\gamma\delta}$. In this setting, [70] established the following general result:

Theorem (Friedrich, 1995). *Let $(\Sigma, \bar{g}, k)$ be a geodesically complete, asymptotically AdS initial data set for (1.2.6) which admits a smooth conformal compactification $(\Sigma \cup \partial\Sigma, \Omega^2 \bar{g}, \Omega k)$ (where Ω is a boundary defining function of $\partial\Sigma$). Given a conformal Lorentzian structure on $\mathcal{I} \simeq \partial\Sigma \times \mathbb{R}$ (compatible with the initial data on $\partial\Sigma \times \{0\}$ in an “adapted gauge”), there exists a unique asymptotically AdS solution g of (1.2.6) on $\mathcal{M} = \Sigma \times (-\delta, \delta)$, for some $\delta \ll 1$, which is regular (in the conformally compactified picture) up to $\mathcal{I}$ and induces the given initial data on $\Sigma \times \{0\}$ and the given boundary structure on $\mathcal{I}$.*

For a more precise formulation of the above result, see [70].

In the spherically symmetric case, well-posedness for more general boundary conditions and matter fields, allowing *singular* behaviour in the conformally compactified picture, was obtained in [83, 87] (see also [82, 71]). Most physically interesting boundary conditions

on $\mathcal{I}$ leading to a well-posed initial-boundary value problem for (1.2.6) can be classified as either *reflecting*, for which an appropriate “energy flux” for g through $\mathcal{I}$ (corresponding to the flux of the renormalised Weyl tensor $W_{\alpha\beta\gamma\delta}$ in the vacuum case) vanishes, or *dissipative* (allowing for a non-vanishing outgoing “energy flux” for g through $\mathcal{I}$), with substantially different global dynamics associated to each case; see the discussion in [82].

In the framework of the initial-boundary value problem for (1.2.6), a natural question to pose is that of the global stability properties of the trivial solution $(\mathcal{M}_{AdS}, g_{AdS})$ under initial perturbations. Of course, the answer to this question is expected to depend on the choice of the boundary conditions on $\mathcal{I}$. In 2006, Dafermos and Holzegel [38, 37] suggested the following conjecture:

AdS instability conjecture. *There exist arbitrarily small perturbations to the initial data of $(\mathcal{M}_{AdS}, g_{AdS})$ (measured with respect to a suitable initial data norm) which, under evolution by (1.2.6) with a reflecting boundary condition on $\mathcal{I}$, lead to the development of black hole regions. In particular, $(\mathcal{M}_{AdS}, g_{AdS})$ is non-linearly unstable.*

This conjecture was motivated in [38] by the study of asymptotically AdS solutions to (1.2.6) with biaxial Bianchi IX symmetry in $4 + 1$ dimensions, a symmetry class in which the vacuum Einstein equations (1.2.6) reduce to a $1 + 1$ hyperbolic system with non-trivial dynamics. In this setting, it was observed in [38] that perturbations of the initial data of $(\mathcal{M}_{AdS}, g_{AdS})$ (which, if not trivial, necessarily have strictly positive ADM mass M_{ADM} , in view of [72]) cannot settle down to a horizonless static spacetime, since M_{ADM} is conserved along $\mathcal{I}$ under reflecting boundary conditions and no static asymptotically AdS solution of (1.2.6) with $M_{ADM} > 0$ exists (according to [18]).

The AdS instability conjecture stands in contrast to the non-linear stability of Minkowski spacetime $(\mathbb{R}^{3+1}, \eta)$, in the case $\Lambda = 0$ (see Section 1.2.1), or de Sitter space $(\mathcal{M}_{dS}, g_{dS})$, in the case $\Lambda > 0$ (see Friedrich [69]). The proof of the non-linear stability of $(\mathbb{R}^{3+1}, \eta)$ and $(\mathcal{M}_{dS}, g_{dS})$ is based on a stability mechanism related to the fact that linear fields on those spacetimes satisfy sufficiently strong decay rates. The decay rates are, however, borderline in the case $\Lambda = 0$, and thus the stability of $(\mathbb{R}^{3+1}, \eta)$ is a deep fact depending on the precise non-linear structure of the system (1.2.6), whereas, in the case $\Lambda > 0$, the decay

is exponential and stability can be inferred relatively easily. In contrast, on $(\mathcal{M}_{AdS}, g_{AdS})$, it can be shown that linear fields satisfying a reflecting boundary condition on $\mathcal{I}$ remain bounded, but do *not* decay in time. It is precisely the lack of a sufficiently fast decay rate at the linear level which is associated to the possibility of non-linear instability.

We should point out that the prescription of a reflecting boundary condition on $\mathcal{I}$ is essential for the AdS instability conjecture: For maximally dissipative boundary conditions, it is expected that $(\mathcal{M}_{AdS}, g_{AdS})$ is non-linearly *stable*, in view of the superpolynomial decay rates obtained for the linearised vacuum Einstein equations (and other linear fields) around $(\mathcal{M}_{AdS}, g_{AdS})$ by Holzegel–Luk–Smulevici–Warnick in [82].

The numerical study of the AdS instability conjecture has been mainly focused on the simpler setting of the *spherically symmetric* Einstein–scalar field system

$$(1.2.10) \quad \begin{cases} Ric_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}[\varphi], \\ \square_g \varphi = 0. \end{cases}$$

Following the seminal work of Bizon–Rostworowski [15], a vast number of numerical and heuristic works by several authors have been dedicated to the study of the AdS instability conjecture in this setting (see, e. g. [52, 19, 104, 35, 14, 54, 74, 88]). See also the discussion in Section 3.1.3 of Chapter 3.

1.3 Outline of the thesis: The main results

In this section, we will introduce the two main results of this thesis, which should be naturally viewed in the context of the stability problem for the Einstein equations (1.0.5) with $\Lambda = 0$ and $\Lambda < 0$, respectively: A general instability result for the scalar wave equation (1.2.1) on stationary and asymptotically flat spacetimes with a non-empty ergoregion, and a proof of the AdS instability conjecture for the Einstein–null dust system. A more precise formulation of these two results will be presented in Chapters 2 and 3, respectively. All the results of this thesis can be also found in [112, 109].⁶

⁶These results have been also presented at the following conferences: “Workshop: Geometry - In honour of Mario Micallef’s 60th birthday”, University of Warwick, 8-10 May 2017; “Workshop: Advances in

1.3.1 A proof of Friedman’s ergosphere instability for scalar waves on asymptotically flat spacetimes

The asymptotic state of an isolated, self-gravitating system is often modelled, in general relativity, by a globally hyperbolic spacetime $(\mathcal{M}, g)$ which is *asymptotically flat* and *stationary*. Restricting, for simplicity, to the *domain of outer communications* of any such spacetime $(\mathcal{M}, g)$ (i.e. the set of points which can communicate with the asymptotically flat region both with future directed and past directed causal curves), the *ergoregion* (or “ergosphere”) $\mathcal{E}$ of $(\mathcal{M}, g)$ is defined as

$$(1.3.1) \quad \mathcal{E} \doteq \overline{\{p \in \mathcal{M} \mid g(T_p, T_p) > 0\}},$$

where T is the stationary Killing vector field of $(\mathcal{M}, g)$.

A prime example of a stationary and asymptotically flat spacetime possessing a non-empty ergoregion $\mathcal{E}$ is the Kerr exterior spacetime $(\mathcal{M}_{M,a}, g_{M,a})$ with $0 < |a| \leq M$ (see (1.2.5)). The presence of a non-empty ergoregion $\mathcal{E}$ gives rise to the phenomenon of *superradiance* for solutions to (1.2.1) on $(\mathcal{M}, g)$: When $\mathcal{E} \neq \emptyset$, there exist solutions φ to (1.2.1) such that their T -energy flux through future null infinity $\mathcal{I}^+$ is greater than their T -energy flux initially. In general, superradiance poses a serious difficulty in obtaining stability results for equation (1.2.1). However, in the case of $(\mathcal{M}_{M,a}, g_{M,a})$, superradiance does not eventually render equation (1.2.1) unstable, owing, partly, to the presence of the future *event horizon* $\mathcal{H}^+$, allowing for part of the energy of solutions of (1.2.1) to “leave” the black hole exterior (see the discussion in Section 1.2.1 on the stability properties of (1.2.1) on $(\mathcal{M}_{M,a}, g_{M,a})$). Notice, however, that superradiance-related mode instabilities do appear on $(\mathcal{M}_{M,a}, g_{M,a})$ for the Klein–Gordon equation (see [133]), or even for the wave equation with a (well-chosen) short-range non-negative potential (see [110]).

Stationary and asymptotically flat spacetimes $(\mathcal{M}, g)$ possessing a non-empty ergoregion $\mathcal{E}$ but lacking a future event horizon $\mathcal{H}^+$ appear in the literature as models for rapidly rotating self-gravitating objects, for instance, as models of self-gravitating dense rotating

General Relativity”, ESI, Vienna, 28 August–1 September 2017; and “Workshop on General Relativity and AdS/CFT”, Fields Institute, 23–27 October 2017.

fluids (see [20]). In [66], Friedman studied the instability properties of equation (1.2.1) on such spacetimes, making the following observation: There exist smooth solutions φ to (1.2.1) with negative T -energy flux initially, i. e.

$$(1.3.2) \quad \int_{\Sigma} J_{\mu}^T(\varphi) n_{\Sigma}^{\mu} < 0$$

on a Cauchy hypersurface Σ of $(\mathcal{M}, g)$ (see Section 2.2 of Chapter 2 for the relevant notation on vector field currents); thus, in view of the conservation of the T -energy flux, the absence of a future event horizon $\mathcal{H}^+$ and the non-negativity of $J_{\mu}^T(\cdot) n_{\Sigma}^{\mu}$ outside $\mathcal{E}$, any such function φ satisfies for all $\tau \geq 0$:

$$(1.3.3) \quad \int_{\Sigma_{\tau} \cap \mathcal{E}} J_{\mu}^T(\varphi) n_{\Sigma_{\tau}}^{\mu} \leq \int_{\Sigma} J_{\mu}^T(\varphi) n_{\Sigma}^{\mu} < 0$$

(where Σ_{τ} denotes the image of Σ under the flow of T for time τ). Therefore, the local energy of φ can not decay to 0 with time.

Based on the above observation, Friedman provided a heuristic argument suggesting that, under the additional assumption that the spacetime $(\mathcal{M}, g)$ is real analytic, any such solution φ satisfies

$$(1.3.4) \quad \limsup_{\tau \rightarrow +\infty} \int_{\Sigma_{\tau}} J_{\mu}^N(\varphi) n_{\Sigma_{\tau}}^{\mu} = +\infty$$

for a globally timelike T -invariant vector field N , coinciding with T in the asymptotically flat region. In view of the connection between equation (1.2.1) and the Einstein equations (1.0.1) (briefly discussed in Section 1.2), Friedman suggested that such spacetimes can not appear as the final state of the evolution of a self-gravitating system (see [66] for more details).⁷ For a numerical investigation of Friedman's instability, see [34, 144, 21].

In Chapter 2 of this thesis, we will provide a rigorous proof of Friedman's instability

⁷In general, it is expected that rotating, self-gravitating compressible fluids satisfying the Einstein–Euler system suffer (at the linearised level) from the so-called Chandrasekhar–Friedman–Schutz instability, see [23, 68]. In this case, the associated instability mechanism does not depend on the existence of an ergoregion. However, in the presence of an ergoregion, the linearised Einstein–Euler system is also expected to exhibit an instability similar to Friedman's instability for equation (1.2.1), associated to the so-called *w-modes* (see [95, 94, 67]). At the non-linear level, it is believed that the effect of the CFS instability will generally dominate the effect of the w-mode instability in an astrophysical setting (see [94]).

for equation (1.2.1). Our proof will in fact not require that $(\mathcal{M}, g)$ is real analytic, but we will assume, instead, a substantially weaker unique continuation condition for equation (1.2.1) through a subset of the boundary $\partial\mathcal{E}_{ext}$ of the “extended” ergoregion $\mathcal{E}_{ext}$, where we define $\mathcal{E}_{ext}$ to be equal to the union of the ergoregion $\mathcal{E}$ with the connected components of $\mathcal{M} \setminus \mathcal{E}$ which intersect neither $\mathcal{H}^+$ nor the asymptotically flat region of $\mathcal{M}$.⁸ Note that, in the case when $\mathcal{M} \setminus \mathcal{E}$ is connected, $\mathcal{E}_{ext}$ coincides with $\mathcal{E}$. In particular, we will establish the following result:

Theorem 1.1. *Let $(\mathcal{M}^{d+1}, g)$, $d \geq 2$, be a smooth, globally hyperbolic, stationary and asymptotically flat spacetime with a non-empty ergoregion $\mathcal{E}$ and a future event horizon $\mathcal{H}^+$ which is either empty or satisfies $\mathcal{E} \cap \mathcal{H}^+ = \emptyset$. Assume, in addition, that the following unique continuation condition through the boundary $\partial\mathcal{E}_{ext}$ of the “extended” ergoregion $\mathcal{E}_{ext}$ holds:*

Unique continuation condition: *There exists a point $p \in \partial\mathcal{E}_{ext}$ and an open neighborhood $\mathcal{U}$ of p in $\mathcal{M}$ such that, for any solution φ to equation (1.2.1) on $\mathcal{M}$ with $\varphi \equiv 0$ on $\mathcal{M} \setminus \mathcal{E}_{ext}$, we have $\varphi = 0$ also on $\mathcal{E}_{ext} \cap \mathcal{U}$.*

Then, there exists a smooth solution φ to (1.2.1) with compactly supported initial data on a Cauchy hypersurface Σ of $(\mathcal{M}, g)$, such that

$$(1.3.5) \quad \limsup_{\tau \rightarrow +\infty} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu = +\infty,$$

where T is the stationary Killing field of $(\mathcal{M}, g)$, N is a globally timelike and T -invariant vector field on $\mathcal{M}$, coinciding with T in the asymptotically flat region of $\mathcal{M}$, and Σ_τ is the image of Σ under the flow of T for time τ .

Remark. Note that the assumption $\mathcal{E} \cap \mathcal{H}^+ = \emptyset$ excludes the Kerr exterior family with angular momentum $a \neq 0$.

For a more detailed statement of Theorem 1.1 and the assumptions on the spacetimes under consideration, see Section 2.1 of Chapter 2. For a comparison between the heuristics of Friedman in [66] and the methods and results of Chapter 2, see Section 2.8 of Chapter 2.

⁸Notice that $\partial\mathcal{E}_{ext} \subset \partial\mathcal{E}$.

Let us remark that, in the detailed statement of Theorem 1.1, we will introduce an additional restriction on the class of spacetimes $(\mathcal{M}, g)$ under consideration, namely the condition that every connected component of $\mathcal{M} \setminus \mathcal{E}$ intersecting $\mathcal{H}^+$ also intersects the asymptotically flat region of $(\mathcal{M}, g)$. However, our proof of Theorem 1.1 can be adapted to the case when this condition does not hold.

We should also remark that the unique continuation condition through an open subset of $\partial\mathcal{E}_{ext}$, appearing in the statement of Theorem 1.1, is always satisfied in the case when $(\mathcal{M}, g)$ possesses an axisymmetric Killing field Φ such that the span of T, Φ on $\partial\mathcal{E}_{ext}$ contains a timelike direction, or in the case when the spacetime $(\mathcal{M}, g)$ is real analytic in an open subset $\mathcal{U} \subset \mathcal{M}$ such that $\mathcal{U} \cap \partial\mathcal{E}_{ext} \neq \emptyset$; see the discussion in Section 2.1.3. It would be natural to expect that this condition can be completely removed from the statement of Theorem 1.1, but we have not succeeded so far in doing so.

The instability mechanism proposed by Friedman is of interest not only in general relativity, but also in all areas of mathematical physics where stationary and asymptotically flat Lorentzian manifolds $(\mathcal{M}, g)$, and the associated wave equation (1.2.1), arise. For instance, in the field of fluid mechanics, the steady flow of a (locally) irrotational, inviscid and barotropic fluid on an open subset $\mathcal{V}$ of $\mathbb{R}^3$ gives rise to a stationary Lorentzian metric g on $\mathcal{M} = \mathbb{R} \times \mathcal{V}$, the so called *acoustical* metric, and the wave equation (1.2.1) associated to g governs the evolution of small perturbations of the flow. In [113], the authors investigate numerically the Friedman instability for the acoustic wave equation on the hydrodynamic vortex $(\mathbb{R} \times \mathcal{V}_{hyd, \delta}, g_{hyd})$, where $\mathcal{V}_{hyd, \delta} = \mathbb{R}^3 \setminus \{\bar{r} \leq \delta\}$ for some $\delta \ll 1$ (in the cylindrical $(\bar{r}, \vartheta, z)$ coordinate system) and

$$(1.3.6) \quad g_{hyd} = -\left(1 - \frac{C^2}{\bar{r}^2}\right)dt^2 + d\bar{r}^2 - 2Cdt d\vartheta + \bar{r}^2 d\vartheta^2 + dz^2,$$

with suitable boundary conditions imposed for (1.2.1) at $\bar{r} = \delta$. Note that the quotient of $(\mathbb{R} \times \mathcal{V}_{hyd, \delta}, g_{hyd})$ by the group of translations in the z direction is asymptotically flat, possesses a non-empty ergoregion $\mathcal{E} = \{\delta < \bar{r} \leq C\}$ (corresponding to the region where the fluid velocity exceeds the speed of sound) and has no event horizon.

As a straightforward application of Theorem 1.1, we will establish a Friedman-type

instability for the acoustical wave equation on the hydronamic vortex:

Corollary 1.1. *For any $\delta < 1$, there exist smooth and z -invariant solutions φ_D, φ_N to the acoustical wave equation (1.2.1) on $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$, satisfying Dirichlet and Neumann boundary conditions, respectively, on $\{\bar{r} = \delta\}$, with smooth initial data at time $t = 0$ which are compactly supported when restricted on $\{z = 0\}$, such that (in the $(t, \bar{r}, \vartheta, z)$ coordinate chart on $\mathbb{R} \times \mathcal{V}_{hyd,\delta}$):*

$$(1.3.7) \quad \limsup_{\tau \rightarrow +\infty} \int_{\{t=\tau\} \cap \{z=0\} \cap \{\bar{r} \geq \delta\}} (|\partial_t \varphi_D|^2 + |\nabla_{\mathbb{R}^3} \varphi_D|^2) \bar{r} d\bar{r} d\vartheta = +\infty$$

and

$$(1.3.8) \quad \limsup_{\tau \rightarrow +\infty} \int_{\{t=\tau\} \cap \{z=0\} \cap \{\bar{r} \geq \delta\}} (|\partial_t \varphi_N|^2 + |\nabla_{\mathbb{R}^3} \varphi_N|^2) \bar{r} d\bar{r} d\vartheta = +\infty.$$

For a more detailed statement of Corollary 1.1, see Section 2.1.2 of Chapter 2.

Remark. While Corollary 1.1 provides an instability statement for both Dirichlet and Neumann boundary conditions for φ on $\{\bar{r} = \delta\}$, only the Neumann condition is relevant for the case when $\{\bar{r} = \delta\}$ represents a physical boundary for the fluid flow, since in this case the fluid velocity is only allowed to be tangential to $\{\bar{r} = \delta\}$.

1.3.2 A proof of the instability of AdS for the Einstein–null dust system with an inner mirror

The AdS instability conjecture, discussed in Section 1.2.2, was originally stated by [38, 37] in the setting of the Einstein vacuum equations (1.2.6). However, the conjecture is interesting also in the case when matter fields are involved, under the assumption that the matter chosen is “well behaved”; in the context of the global stability problem for (1.0.5), this assumption includes the condition that the trivial solution of (1.0.5) with $\Lambda = 0$ and $\Lambda > 0$ is not rendered unstable in the presence of such matter fields.

The study of the AdS instability conjecture in the presence of matter can also provide a more tractable route to obtaining intuition about the vacuum case: In the setting of the Einstein–scalar field system (1.2.10), for instance, one can restrict to the class of per-

turbations of $(\mathcal{M}_{AdS}, g_{AdS})$ which are spherically symmetric, effectively reducing (1.2.10) to an 1 + 1-dimensional hyperbolic system, while at the same time retaining many of the qualitative dynamical properties of (1.2.6). For this reason, the numerical and heuristic study of the AdS instability conjecture so far has been focused on the system (1.2.10); see [52, 19, 104, 35, 14, 54, 74, 88].

In Chapter 3 of the thesis, we will establish the proof of the AdS instability conjecture in the setting of the simplest possible matter model admitting non-trivial spherically symmetric dynamics, namely the spherically symmetric Einstein–massless Vlasov system further reduced to the case when the Vlasov field f is supported only on radial geodesics. This is a singular reduction; the resulting system is equivalent to the *spherically symmetric Einstein–null dust system*, allowing for both ingoing and outgoing dust (we will also refer to this system as the spherically symmetric Einstein–radial massless Vlasov system). This system has been studied in the $\Lambda = 0$ case by Poisson and Israel [120].

A serious problem with the spherically symmetric Einstein–null dust system is that it suffers from a severe break down when the null dust reaches the centre $r = 0$. In particular, in any reasonable initial data topology, the spherically symmetric Einstein–null dust system is not well posed and $(\mathcal{M}_{AdS}, g_{AdS})$ is not a Cauchy stable solution of it. One way to restore the well posedness of this system (a necessary step for the study of the AdS instability conjecture in this setting) is to place an *inner mirror* at some radius sphere $\{r = r_0\}$ with $r_0 > 0$ and study the evolution of the system in the exterior region $\{r \geq r_0\}$. However, fixing the mirror radius r_0 results in a trivial global stability statement for $(\mathcal{M}_{AdS}, g_{AdS})$, as initial data perturbations with total ADM mass $\tilde{m}_{ADM} < \frac{1}{2}r_0$ cannot form a black hole. Thus, it is necessary to allow the radius r_0 to shrink to 0 as the total ADM mass of the initial data shrinks to 0, in order to address the AdS instability conjecture in this setting. See the discussion in Section 3.1 in Chapter 3 for more details.

A non-technical statement of the main result in Chapter 3 is the following:

Theorem 3.1 (rough version). The AdS spacetime $(\mathcal{M}_{AdS}^{3+1}, g_{AdS})$ is non-linearly unstable under evolution by the spherically symmetric Einstein–null dust system with a reflecting boundary condition on $\mathcal{I}$ and an inner mirror, in the following sense:

There exists a one parameter family of spherically symmetric initial data $\mathcal{S}_\epsilon$, $\epsilon \in (0, 1]$ and a family of inner mirror radii $r = r_{0\epsilon}$ (with $r_{0\epsilon} \xrightarrow{\epsilon \rightarrow 0} 0$) satisfying the following properties:

1. As $\epsilon \rightarrow 0$, $\|\mathcal{S}_\epsilon\|_{CS} \rightarrow 0$, i. e. $\mathcal{S}_\epsilon$ converge to the initial data $\mathcal{S}_0$ of $(\mathcal{M}_{AdS}, g_{AdS})$.
2. For any $\epsilon > 0$, the maximal future development $(\mathcal{M}_\epsilon, g_\epsilon)$ of $\mathcal{S}_\epsilon$ contains a trapped surface and, thus, a black hole region. Moreover, $(\mathcal{M}_\epsilon, g_\epsilon)$ possesses a complete conformal infinity $\mathcal{I}$.

The norm $\|\cdot\|_{CS}$ in 1 measures the concentration of the energy of $\mathcal{S}_\epsilon$ in annuli of width $\sim r_{0\epsilon}$ and has the property that the Einstein–null dust system with a reflecting boundary condition on $\mathcal{I}$ is well-posed and $(\mathcal{M}_{AdS}, g_{AdS})$ is Cauchy stable with respect to $\|\cdot\|_{CS}$ independently of the precise value of $r_{0\epsilon}$.

For progressively more detailed statements of Theorem 3.1, see Sections 3.1.1 and 3.4 in Chapter 3. For further discussion on the need of an inner mirror at $r \sim r_{0\epsilon}$ and its relation to natural dispersive mechanisms appearing in other matter models, see Section 3.1 in Chapter 3.

While we will not address the issue of the end state of the evolution of $\mathcal{S}_\epsilon$, it can be easily inferred from our proof of Theorem 3.1 that the spacetimes $(\mathcal{M}_\epsilon, g_\epsilon)$ settle down to a member of the Schwarzschild–AdS family (see also [111]).

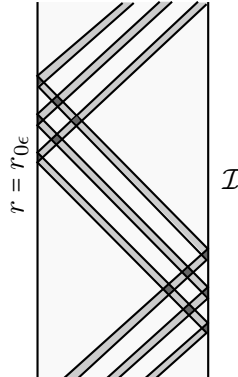


Figure 4: The family of initial data $\mathcal{S}_\epsilon$ that we construct for the proof of Theorem 3.1 give rise to a large number of null dust beams, that are successively reflected off $\mathcal{I}$ and the inner mirror at $r = r_{0\epsilon}$. The Cauchy stability statement for $\|\cdot\|_{CS}$ implies that the number of reflections necessarily goes to $+\infty$ as $\epsilon \rightarrow 0$.

The trivial instability at $r = 0$ occurring for the spherically symmetric Einstein–null dust system is absent in the case of smooth solutions to the general spherically symmetric Einstein–massless Vlasov system (not reduced to the radial case). In particular, the smooth initial value problem for the spherically symmetric Einstein–massless Vlasov system is well-posed, and placing an inner mirror at $r = r_0 > 0$ is not necessary.⁹ For a proof of the AdS instability in this setting, see the forthcoming [106].

1.4 Notational conventions

In this section, we will introduce some notational conventions appear throughout this thesis. We will assume that the reader is familiar with basic notions and the standard notations of Lorentzian geometry, as introduced, for instance, in [124].

1.4.1 Constants and dependence on parameters

We will adopt the following convention for denoting constants appearing in inequalities: We will use capital letters (e.g. C) to denote “large” constants, typically appearing on the right hand side of inequalities. (Such constants can be “freely” replaced by larger ones without rendering the inequality invalid.) Lower case letters (e.g. c) will be used to denote “small” constants (which can similarly be freely replaced by smaller ones). The same characters will be frequently used to denote different constants, even in adjacent lines.

In Chapter 2 of the thesis, we will assume that all non-explicit constants will depend on the specific geometric aspects of $(\mathcal{M}, g)$ and we will not keep track of this dependence, except for some very specific cases. However, since we will introduce a plethora of parameters throughout Sections 2.3–2.5 of Chapter 2, we will always keep track of the dependence of all constants on each of these parameters. Once a parameter is fixed (which will be clearly stated in the text), the dependence of constants on it will be dropped.

⁹In fact, well-posedness for the smooth initial value problem for the Einstein–Vlasov system also holds outside spherical symmetry, see [24]. In the case $\Lambda = 0$, the stability of Minkowski spacetime for the Einstein–massless Vlasov system without any symmetry assumptions was recently established by Taylor [140].

1.4.2 Inequality symbols

We will use the notation $f_1 \lesssim f_2$ for two real functions f_1, f_2 as usual to imply that there exists some $C > 0$, such that $f_1 \leq C f_2$. This constant C might depend on free parameters, and these parameters will be stated clearly in each case. If nothing is stated regarding the dependence of this constant on parameters, it should be assumed that it only depends on the geometry of the spacetime $(\mathcal{M}, g)$ under consideration.

We will denote $f_1 \sim f_2$ when we can bound $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$. The notation $f_1 \ll f_2$ will be equivalent to the statement that $\frac{|f_1|}{|f_2|}$ can be bounded by some sufficiently small positive constant, the magnitude and the dependence of which on variable parameters will be clear in each case from the context. For any function $f : \mathcal{M} \rightarrow [0, +\infty)$ on a background manifold $\mathcal{M}$, $\{f \gg 1\}$ will denote the subset $\{f \geq C\}$ of $\mathcal{M}$ for some constant $C \gg 1$.

For functions $f_1, f_2 : [x_0, +\infty) \rightarrow \mathbb{R}$, the notation $f_1 = o(f_2)$ will mean that $\frac{|f_1|}{|f_2|}$ can be bounded by some continuous function $h : [x_0, +\infty) \rightarrow (0, +\infty)$ such that $h(x) \rightarrow 0$ as $x \rightarrow +\infty$. This bound h might depend on free parameters, and this fact will be clear in each case from the context.

1.4.3 Vector field multipliers and currents

Throughout Chapter 2 of the thesis, we will frequently use the language of Lagrangean currents and vector field multipliers for equation (1.2.1): On any Lorentzian manifold $(\mathcal{M}, g)$, the energy momentum tensor $T_{\mu\nu}$ associated to the wave operator (1.1.4) takes the following form for any smooth function $\psi : \mathcal{M} \rightarrow \mathbb{C}$

$$(1.4.1) \quad T_{\mu\nu}(\psi) = \frac{1}{2} \left(\partial_\mu \psi \cdot \partial_\nu \bar{\psi} + \partial_\mu \bar{\psi} \cdot \partial_\nu \psi \right) - \frac{1}{2} (\partial^\lambda \psi \cdot \partial_\lambda \bar{\psi}) g_{\mu\nu}.$$

For any continuous and piecewise C^1 vector field X on $\mathcal{M}$, the following associated currents can be defined almost everywhere:

$$(1.4.2) \quad J_\mu^X(\psi) = T_{\mu\nu}(\psi) X^\nu,$$

$$(1.4.3) \quad K^X(\psi) = T_{\mu\nu}(\psi) \nabla^\mu X^\nu.$$

The following divergence identity then holds almost everywhere on $\mathcal{M}$:

$$(1.4.4) \quad \nabla^\mu J_\mu^X(\psi) = K^X(\psi) + \operatorname{Re}\left\{(\square_g \psi) \cdot X \bar{\psi}\right\}.$$

Chapter 2

A proof of Friedman’s ergosphere instability for scalar waves on asymptotically flat spacetimes

In this chapter, we will obtain the proof of Friedman’s ergosphere instability for solutions φ to the scalar wave equation

$$(2.0.1) \quad \square_g \varphi = 0$$

on a broad class of stationary and asymptotically flat spacetimes $(\mathcal{M}, g)$ with a non-trivial ergoregion $\mathcal{E}$, establishing, in particular, Theorem 1.1 and Corollary 1.1 (presented in the introduction). The precise assumptions on the spacetimes under consideration and the main results of this part will be presented in Section 2.1; the proof of these results will occupy Sections 2.3–2.7. A brief comparison between the proof of the main theorem and the heuristics of Friedman will be presented in Section 2.8.

2.1 Assumptions on the spacetimes and statement of the main results

In this section, we will outline in detail the assumptions on the spacetimes $(\mathcal{M}, g)$ under consideration, and we will state the main results of this chapter of the thesis. We will also briefly discuss some of the assumptions on $(\mathcal{M}, g)$, as well as some aspects of the proof of the main results.

2.1.1 Assumptions on the spacetime geometry

Let $(\mathcal{M}^{d+1}, g)$, $d \geq 2$, be a smooth, globally hyperbolic Lorentzian manifold with piecewise smooth boundary $\partial\mathcal{M}$ (allowed to be empty). Before stating our main results, we will need to introduce a number of assumptions on the structure of $(\mathcal{M}, g)$. In Section 2.1.4, we will present some explicit examples of spacetimes $(\mathcal{M}, g)$ satisfying all the assumptions that will be introduced in this section.

Assumption G1 (Asymptotic flatness and stationarity).

We will assume that $(\mathcal{M}, g)$ satisfies the following conditions:

- There exists a Killing field T on $(\mathcal{M}, g)$ with complete orbits which is tangential to $\partial\mathcal{M}$, as well as a smooth Cauchy hypersurface $\tilde{\Sigma} \subset \mathcal{M}$, such that $T|_{\tilde{\Sigma}}$ is everywhere transversal to $\tilde{\Sigma} \setminus \partial\mathcal{M}$ and timelike outside a compact subset of $\tilde{\Sigma}$.
- The triad $(\tilde{\Sigma}, g_{\tilde{\Sigma}}, k_{\tilde{\Sigma}})$, where $g_{\tilde{\Sigma}}$ is the induced (Riemannian) metric on $\tilde{\Sigma}$ and $k_{\tilde{\Sigma}}$ its second fundamental form, defines an asymptotically flat Riemannian manifold (possibly with boundary $\tilde{\Sigma} \cap \partial\mathcal{M}$), with a finite number of asymptotically flat ends (possibly more than one); see also the definition in Section 2.1.1 of [107]. Let $\mathcal{I}_{as}$ be the asymptotically flat region of $\mathcal{M}$ (see [107] for the relevant definition). Expressed in a polar coordinate chart of the form (t, r, σ) in each connected component of $\mathcal{I}_{as}$, g has the following form:

$$(2.1.1) \quad g = -\left(1 + O_4(r^{-1})\right)dt^2 + \left(1 + O_4(r^{-1})\right)dr^2 + r^2 \cdot \left(g_{\mathbb{S}^{d-1}} + O_4^{\mathbb{S}^{d-1}}(r^{-1})\right) + O_4(1)dtd\sigma,$$

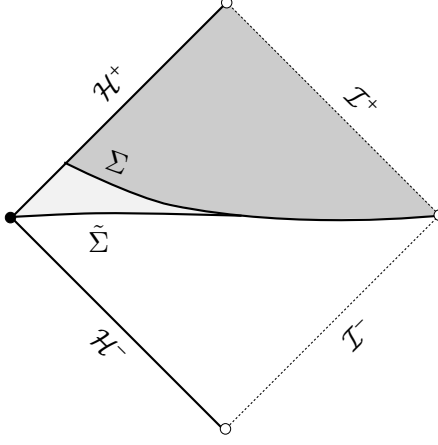


Figure 5: The subextremal Kerr exterior spacetime $(\mathcal{M}_{M,a}, g_{M,a})$ satisfies Assumptions G1 and G2, but not Assumption G3. In the case of $(\mathcal{M}_{M,a}, g_{M,a})$, the intersection of the hypersurfaces $\tilde{\Sigma}, \Sigma$, defined in Assumption G1, with the 1+1 dimensional slice $\{\theta = \pi/2, \phi = 0\} \subset \mathcal{M}_{M,a}$ are schematically as depicted above.

where $O_4^{\mathbb{S}^{d-1}}(\rho^{-1})$ is a symmetric $(0,2)$ -tensor field on the coordinate sphere $\{r = \rho\} \simeq \mathbb{S}^{d-1}$ with $O_4(\rho^{-1})$ asymptotics as $\rho \rightarrow +\infty$. See Section 2.2.5 for the $O_k(\cdot)$, $O_k^{\mathbb{S}^{d-1}}(\cdot)$ notation and Section 2.2.4 for the σ notation on the angular variables of a polar coordinate chart.

- Let $\mathcal{H} = \partial(J^+(\mathcal{I}_{as}) \cap J^-(\mathcal{I}_{as}))$ be the horizon of $\mathcal{M}$, split as $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$, with $\mathcal{H}^+ = \overline{J^+(\mathcal{I}_{as})} \cap \partial J^-(\mathcal{I}_{as})$ and $\mathcal{H}^- = \overline{J^-(\mathcal{I}_{as})} \cap \partial J^+(\mathcal{I}_{as})$. Then $\mathcal{H}$ coincides with $\partial\mathcal{M}$, and $\mathcal{H}^+$ and $\mathcal{H}^-$ are smooth null hypersurfaces with smooth boundary $\mathcal{H}^+ \cap \mathcal{H}^-$, with $T \neq 0$ on $\mathcal{H} \setminus \mathcal{H}^-$ (the case $\mathcal{H}^+ = \emptyset$ or $\mathcal{H}^- = \emptyset$ is also trivially included in this condition).

See Assumption 1 in Section 2.1.1 of [107] for a detailed statement of these conditions and their related geometric constructions, as well as their implications on the geometry of $\mathcal{M}$. Notice that the domain of outer communications of the asymptotically flat region $\mathcal{I}_{as}$ of $\mathcal{M}$ is the whole of $\mathcal{M} \setminus \mathcal{H}$. In view of the remarks in Section 2.1.1 of [107], $\mathcal{H}^+$ and $\mathcal{H}^-$ are invariant under the flow of the stationary Killing field T .

Let Σ be a spacelike hypersurface intersecting $\mathcal{H}^+$ transversally (if $\mathcal{H}^+ \neq \emptyset$) and satisfying

$\Sigma \cap \mathcal{H}^- = \emptyset$, such that Σ coincides with $\tilde{\Sigma}$ outside a small neighborhood of $\mathcal{H}^+$. In view of the remarks in Section 2.1.1 of [107], our assumption on the structure of $(\mathcal{M}, g)$ implies that $\Sigma \cap \mathcal{H}^+$ is compact. Notice that, in case $\mathcal{H}^+ \neq \emptyset$, Σ will not be a Cauchy hypersurface of $\mathcal{M}$.

We will also fix a smooth spacelike hyperboloidal hypersurface $\mathcal{S} \subset \mathcal{M}$ terminating at future null infinity $\mathcal{I}^+$ as in Section 2.1.1 of [107], such that $\mathcal{S}|_{\{r \leq R_1\}} \equiv \Sigma|_{\{r \leq R_1\}}$ for some fixed constant $R_1 \gg 1$.

Remark. In view of the remarks of Section 2.1.1 of [107], the causal future sets $J^+(\Sigma), J^+(\mathcal{S})$ of $\Sigma, \mathcal{S}$, respectively, in $\mathcal{M}$ coincide with the future domains of dependence $\mathcal{D}^+(\Sigma), \mathcal{D}^+(\mathcal{S})$ of $\Sigma, \mathcal{S}$. Furthermore, the images of $\Sigma, \mathcal{S}$ under the flow of T covers the whole of $\mathcal{M} \setminus \mathcal{H}^-$.

As in Section 2.1.1 of [107], we will extend the polar radius coordinate function $r : \mathcal{I}_{as} \rightarrow (0, +\infty)$ as a non-negative, smooth and T -invariant function on the whole of $\mathcal{M} \setminus \mathcal{H}^-$, such that $r > 0$ on $\mathcal{M} \setminus \mathcal{H}$ and $r|_{\mathcal{H}^+ \setminus \mathcal{H}^-} = 0$, $dr|_{\mathcal{H}^+ \setminus \mathcal{H}^-} \neq 0$. We will also define the function $t : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ by the relations

$$(2.1.2) \quad t|_{\Sigma} = 0 \text{ and } T(t) = 1,$$

as well as the function $\bar{t} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ by

$$(2.1.3) \quad \bar{t}|_{\mathcal{S}} = 0 \text{ and } T(\bar{t}) = 1.$$

Note that $t = \bar{t}$ on $\{r \leq R_1\}$.

We will introduce the reference Riemannian metric

$$(2.1.4) \quad g_{ref} \doteq dt^2 + g_{\Sigma}$$

on $\mathcal{M} \setminus \mathcal{H}^- \simeq \mathbb{R} \times \Sigma$. We will denote the natural extension of g_{ref} on $\oplus_{l_1, l_2 \in \mathbb{N}} (\otimes^{l_1} T(\mathcal{M} \setminus \mathcal{H}^-) \otimes^{l_2} T^*(\mathcal{M} \setminus \mathcal{H}^-))$ also as g_{ref} .

Assumption G2 (Killing horizon with positive surface gravity).

In the case $\mathcal{H}^+ \neq \emptyset$, we will assume that the Killing field T , when restricted to $\mathcal{H}^+ \setminus \mathcal{H}^-$, is parallel to the null generators of $\mathcal{H}^+ \setminus \mathcal{H}^-$. Furthermore, we will assume that there exists a T -invariant strictly timelike vector field N on $J^+(\Sigma)$, which, when restricted on $J^+(\Sigma) \cap \mathcal{H}$, satisfies

$$(2.1.5) \quad K^N(\psi) \geq c J_\mu^N(\psi) N^\mu$$

for some $c > 0$ and any $\psi \in C^1(\mathcal{M})$ (see Section 2.2 for the notation on vector field currents). We will extend N on the whole of $\mathcal{M} \setminus \mathcal{H}^-$ by the condition $[T, N] = 0$.

We will call the vector field N the *red shift* vector field. The reason for this name is that a vector field of that form was shown to exist for a general class of Killing horizons with positive surface gravity by Dafermos and Rodnianski in [44]. However, here we will just assume the existence of such a vector field without specifying its geometric origin.

Note that we can modify the vector field N away from the horizon $\mathcal{H}$, so that in the asymptotically flat region $\{r \gg 1\}$ (i.e. $\mathcal{I}_{as}$) it coincides with T , and still retain the bound (2.1.5) on $J^+(\Sigma) \cap \mathcal{H}$.¹ We will hence assume without loss of generality that N has been chosen so that $N \equiv T$ in the region $\{r \gg 1\}$.

Due to the smoothness of N , there exists an $r_0 > 0$, such that (2.1.5) also holds (possibly with a smaller constant c on the right hand side) in a neighborhood of $\mathcal{H}^+ \setminus (\mathcal{H}^+ \cap \mathcal{H}^-)$ in $\mathcal{M}$ of the form $\{r \leq r_0\}$. For $r \gg 1$, since $N \equiv T$ there, we have $K^T(\psi) \equiv 0$. Hence, due to the T -invariance of N and the compactness of the sets of the form $\{r \leq R\} \cap \Sigma$, there exists a (possibly large) constant $C > 0$ such that

$$(2.1.6) \quad |K^N(\psi)| \leq C \cdot J_\mu^N(\psi) N^\mu$$

everywhere on $\mathcal{M}$ for any $\psi \in C^\infty(\mathcal{M})$.

¹The convexity of the cone of the future timelike vectors over each point of $\mathcal{M}$ is used in this argument.

Without loss of generality, we will also assume that r_0 is sufficiently small so that

$$(2.1.7) \quad dr \neq 0$$

on $\{r \leq 3r_0\}$. This is possible, since $dr|_{\mathcal{H}^+ \setminus \mathcal{H}^+ \cap \mathcal{H}^-} \neq 0$ and $\Sigma \cap \mathcal{H}^+$ is compact.

In the case $\mathcal{H}^+ = \emptyset$, we will fix N to be an arbitrary T -invariant timelike vector field on $\mathcal{M} \setminus \mathcal{H}^-$, such that $N \equiv T$ for $r \gg 1$, and we will set $r_0 = \frac{1}{4} \inf_{\Sigma} r$ (which is possible since $r > 0$ on Σ when $\mathcal{H}^+ = \emptyset$), so that $\{r \leq 3r_0\} = \emptyset$. In this case, (2.1.5), (2.1.6) and (2.1.7) are trivially satisfied.

Assumption G3 (Non-empty ergoregion avoiding the future event horizon).

We will assume that the ergoregion of $(\mathcal{M}, g)$ is non-empty, i. e.

$$(2.1.8) \quad \mathcal{E} \doteq \overline{\{g(T, T) > 0\}} \neq \emptyset,$$

and furthermore

$$(2.1.9) \quad \mathcal{E} \cap \mathcal{H}^+ = \emptyset.$$

Notice that the condition (2.1.9) is trivially satisfied when $\mathcal{H}^+ = \emptyset$. Note also that the subextremal Kerr exterior family with $a \neq 0$ has a non-empty ergoregion, but does not satisfy (2.1.9).

In the case when $\mathcal{H}^+ \neq \emptyset$, we will also assume that every connected component of $\mathcal{M} \setminus \mathcal{E}$ that intersects $\mathcal{H}^+$ also intersects the asymptotically flat region $\mathcal{I}_{as}$ of $(\mathcal{M}, g)$.²

Remark. The assumption that every component of $\mathcal{M} \setminus \mathcal{E}$ intersecting $\mathcal{H}^+$ also intersects $\mathcal{I}_{as}$ is not necessary for the results of this chapter, which can also be established without this condition. The reason for adopting this assumption is that it leads to considerable simplifications in the proof of the Carleman-type estimates in Section 2.5.

²Note that $\mathcal{I}_{as}$ and $\mathcal{H}^+$ might have several components.

We will assume that T is strictly timelike on the complement of $\mathcal{H} \cup \mathcal{E}$, i.e.:

$$(2.1.10) \quad g(T, T) < 0 \text{ on } \mathcal{M} \setminus (\mathcal{E} \cup \mathcal{H}).$$

Furthermore, we will assume that the boundary $\partial\mathcal{E}$ of $\mathcal{E}$ is a smooth hypersurface of $\mathcal{M}$.

The complement $\mathcal{M} \setminus \mathcal{E}$ of $\mathcal{E}$ might consist of more than one components. In view of our assumption that every component of $\mathcal{M} \setminus \mathcal{E}$ intersecting $\mathcal{H}^+$ also intersects $\mathcal{I}_{as}$, the connected components of $\mathcal{M} \setminus \mathcal{E}$ fall into two disjoint categories: The ones that intersect the asymptotically flat region $\mathcal{I}_{as}$ and the future event horizon $\mathcal{H}^+$, and the ones that intersect neither $\mathcal{I}_{as}$ nor $\mathcal{H}^+$. Let us call the union of the components of $\mathcal{M} \setminus \mathcal{E}$ falling into the last category the *enclosed* region of $\mathcal{M}$, and denote it by $\mathcal{M}_{enc}$.³ We will also introduce the notion of the *extended* ergoregion of $\mathcal{M}$ defined by

$$(2.1.11) \quad \mathcal{E}_{ext} \doteq \mathcal{E} \cup \mathcal{M}_{enc}.$$

Note that, since $\mathcal{E}_{ext} \cap \mathcal{H}^+ = \emptyset$, we have $r > 0$ on $\mathcal{E}_{ext}$. Thus, in view of the T -invariance of $\mathcal{E}_{ext}$, we can assume without loss of generality that r_0 has been fixed sufficiently small so that $\{r \leq r_0\} \cap \mathcal{E}_{ext} = \emptyset$. Note also that $\partial\mathcal{E}_{ext} \subseteq \partial\mathcal{E}$.

Assumption A1 (*Unique continuation around $p \in \partial\mathcal{E}_{ext}$*).

We will assume that there exists a point p on the boundary $\partial\mathcal{E}_{ext}$ of $\mathcal{E}_{ext}$ and an open neighborhood $\mathcal{U}$ of p in $\mathcal{M}$, such that for any $\psi \in H_{loc}^1(\mathcal{M} \setminus \mathcal{H}^-)$ solving the wave equation (2.0.1) and satisfying $\psi \equiv 0$ on $\mathcal{M} \setminus \mathcal{E}_{ext}$, we also have $\psi \equiv 0$ on $\mathcal{U}$. Since T is a Killing field of $\mathcal{M}$, the same result also holds on any T -translate of $\mathcal{U}$, and, for this reason, we will assume without loss of generality that $\mathcal{U}$ is T -invariant. Furthermore, since $\partial\mathcal{E}_{ext} \subset \partial\mathcal{E}$, we will assume without loss of generality that $\mathcal{U}$ is small enough so that $\mathcal{U} \cap \mathcal{E}_{ext} \subset \mathcal{E}$.

Remark. Assumption A1 is satisfied in the case when $\mathcal{M}$ is axisymmetric with axisymmetric Killing field Φ , such that $[\Phi, T] = 0$ and the span of $\{\Phi, T\}$ is timelike, or in the case when there exists a point $p \in \partial\mathcal{E}$ such that g is real analytic on an open neighborhood of p in $\mathcal{M}$.

³The reason for calling $\mathcal{M}_{enc}$ enclosed is because $\partial\mathcal{M}_{enc} \subset \partial\mathcal{E}$, i.e. $\mathcal{M}_{enc}$ is enclosed by the ergoregion.

See Section 2.1.3.

2.1.2 The main results

The main result of this chapter is the following (see also [112]):

Theorem 2.1. *Let $(\mathcal{M}^{d+1}, g)$, $d \geq 2$, be a globally hyperbolic Lorentzian manifold satisfying Assumptions G1, G2, G3 and A1, and let the vector field T , N and the spacelike hypersurface Σ be as described in Assumptions G1-G2. Then, there exists a smooth function $\varphi: J^+(\Sigma) \rightarrow \mathbb{C}$ solving the wave equation (2.0.1) on $J^+(\Sigma)$ with compactly supported initial data on Σ , such that*

$$(2.1.12) \quad \limsup_{\tau \rightarrow +\infty} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu = +\infty.$$

The proof of Theorem 2.1 will be presented in Section 2.3.

in Section 2.3, proceeds by contradiction. In particular, assuming that every smooth solution φ of equation (2.0.1) on $(\mathcal{M}, g)$ with compactly supported initial data satisfies

$$(2.1.13) \quad \limsup_{\tau \rightarrow +\infty} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu < +\infty,$$

it is shown that φ decays in time on $\mathcal{M} \setminus \mathcal{E}$. This fact is then shown to lead to a contradiction after a suitable choice of the initial data for φ , combined with the unique continuation assumption of Theorem 1.1. See Section 2.3 for more details. The decay of φ on $\mathcal{M} \setminus \mathcal{E}$ is established through some suitable Carleman-type estimates, derived in Section 2.5. These estimates could have been obtained by methods similar to the ones implemented in [107], but we chose instead to provide an alternative proof, based entirely on the method of first order multipliers for equation (2.0.1). For more details on this, see Section 2.1.5.

Remark. The proof of Theorem 2.1 immediately generalises to the case when the boundary of the spacetime $(\mathcal{M}, g)$ has a smooth, timelike and T -invariant component $\partial_{tim}\mathcal{M}$, such that $\Sigma \cap \partial_{tim}\mathcal{M}$ is compact and $\partial_{tim}\mathcal{M} \cap \mathcal{H} = \emptyset$, assuming that Dirichlet or Neumann boundary conditions are imposed for equation (2.0.1) on $\partial_{tim}\mathcal{M}$. In this case, we have to assume that the double $(\tilde{\mathcal{M}}, \tilde{g})$ of $(\mathcal{M}, g)$ across $\partial_{tim}\mathcal{M}$ is a globally hyperbolic spacetime

satisfying Assumptions G1, G2, G3 and A1 (see Section 2.5.9 for the relevant constructions).

Let us also note that we can readily replace the qualitative instability statement (2.1.12) with the following quantitative statement: For any $C > 0$, there exists a solution φ to equation (2.0.1) as in the statement of Theorem 2.1, such that

$$(2.1.14) \quad \limsup_{\tau \rightarrow +\infty} \left((\log(2 + \tau))^{-C} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu \right) = +\infty.$$

See the remark at the beginning of Section 2.3. However, we do not expect the logarithmic rate of growth in (2.1.14) to be sharp.

As a straightforward application of Theorem 2.1, we will obtain the following instability estimate for solutions to the acoustical wave equation on the hydronamic vortex $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$, where $\mathcal{V}_{hyd,\delta} \subset \mathbb{R}^3$ is the set $\{\bar{r} \geq \delta\}$ (in the cylindrical $(\bar{r}, \vartheta, z)$ coordinate system on $\mathbb{R}^3$) and g_{hyd} is given by the expression (1.3.6):

Corollary 2.1. *For any $\delta < 1$, there exist smooth and z -invariant solutions φ_D, φ_N to (2.0.1) on $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$, satisfying the boundary conditions*

$$(2.1.15) \quad \varphi_D|_{\{\bar{r}=\delta\}} = 0$$

and

$$(2.1.16) \quad \partial_{\bar{r}} \varphi_N|_{\{\bar{r}=\delta\}} = 0$$

and having smooth initial data at time $t = 0$ which are compactly supported when restricted on $\{z = 0\}$, such that (in the $(t, \bar{r}, \vartheta, z)$ coordinate chart on $\mathbb{R} \times \mathcal{V}_{hyd,\delta}$):

$$(2.1.17) \quad \limsup_{\tau \rightarrow +\infty} \int_{\{t=\tau\} \cap \{z=0\} \cap \{\bar{r} \geq \delta\}} (|\partial_t \varphi_D|^2 + |\nabla_{\mathbb{R}^3} \varphi_D|^2) \bar{r} d\bar{r} d\vartheta = +\infty,$$

$$(2.1.18) \quad \limsup_{\tau \rightarrow +\infty} \int_{\{t=\tau\} \cap \{z=0\} \cap \{\bar{r} \geq \delta\}} (|\partial_t \varphi_N|^2 + |\nabla_{\mathbb{R}^3} \varphi_N|^2) \bar{r} d\bar{r} d\vartheta = +\infty.$$

For the proof of Corollary 2.1, see Section 2.7.

2.1.3 Discussion on Assumption A1

There exists a class of natural geometric conditions, such that spacetimes $(\mathcal{M}, g)$ satisfying these conditions (in addition to Assumptions G1–G3) automatically satisfy Assumption A1. Examples of such conditions are the following:

- Assumption A1 is always satisfied on spacetimes $(\mathcal{M}, g)$ having an axisymmetric Killing field Φ , such that $[\Phi, T] = 0$ and the span of Φ, T on $\partial\mathcal{E}_{ext}$ contains a timelike direction. This is a consequence of Lemma 2.1 at the end of this section (choosing $\mathcal{U}$ to be a suitable small neighborhood of a point $p \in \partial\mathcal{E}_{ext} \setminus \{\Phi = 0\}$ and $\mathcal{S} = \partial\mathcal{E}_{ext} \cap \mathcal{U}$ in the statement of Lemma 2.1).
- Assumption A1 is always satisfied on spacetimes $(\mathcal{M}, g)$ on which there exists a point $p \in \partial\mathcal{E}_{ext}$ and an open neighborhood $\mathcal{U}$ of p such that $(\mathcal{U}, g)$ is a real analytic Lorentzian manifold and $\partial\mathcal{E}_{ext} \cap \mathcal{U}$ is a real analytic hypersurface. This is a consequence of Holmgren’s uniqueness theorem (see [80]).

On the other hand, we believe that Assumption A1 does not hold on all spacetimes satisfying Assumptions G1–G3. In particular, by adjusting the arguments of [4], we were able to construct a 3 + 1-dimensional spacetime $(\mathcal{M}, g)$, satisfying Assumptions G1–G3, as well as a suitable T -invariant smooth potential $V : \mathcal{M} \rightarrow \mathbb{C}$, so that Assumption A1 for equation

$$(2.1.19) \quad \square_g \varphi - V \varphi = 0$$

in place of (2.0.1) is not satisfied.⁴ Note that such a construction is non-trivial, in view of the requirement that $T(V) = 0$; for instance, Assumption A1 always holds for equation (2.1.19) on stationary spacetimes without an ergoregion (see [138]). We will not pursue this issue any further here.

⁴In view of the requirement $T(V) = 0$, this construction boils down to the adaptation of the construction of [4] on the quotient $\mathcal{M}/T$ of $\mathcal{M}$ by the orbits of T , with $\square_g$ replaced by $\square_{g/T}$, where $\square_{g/T}$ is obtained from $\square_g$ by dropping the terms involving T derivatives. Note that $\square_{g/T}$ is hyperbolic in $\text{int}(\mathcal{E})/T$ and elliptic on $(\mathcal{M} \setminus \mathcal{E})/T$.

Although we believe that Assumption A1 can be removed from the statement of Theorem 2.1, we were not able to do so.

The following lemma can be used to establish that Assumption A1 always holds in the presence of a second Killing field Φ on $\mathcal{M}$ such that the span of T, Φ is timelike:

Lemma 2.1. *Let $\mathcal{U}$ be an open subset of a smooth spacetime $(\mathcal{M}, g)$ with two Killing fields T, Φ such that $[T, \Phi] = 0$, $\Phi \neq 0$ on $\mathcal{U}$ and the span of T, Φ contains a timelike direction everywhere on $\mathcal{U}$. Let also $\mathcal{S} \subset \mathcal{U}$ be a T, Φ -invariant smooth hypersurface, separating $\mathcal{U}$ into two connected components $\mathcal{U}_1, \mathcal{U}_2$, so that $\mathcal{U}_2$ lies in the domain of dependence of $\mathcal{U}_1$. Then, any $\psi \in H_{loc}^2(\mathcal{U})$ solving (2.0.1) on $(\mathcal{U}, g)$ such that $\psi \equiv 0$ on $\mathcal{U}_1$ must vanish everywhere on $\mathcal{U}$.*

Proof. Since $[T, \Phi] = 0$ and $\Phi \neq 0$ on $\mathcal{U}$, we can assume without loss of generality (by shrinking $\mathcal{U}$ if necessary) that $\mathcal{U}$ is covered by a coordinate chart $(t, \varphi, x^2, \dots, x^d)$ such that:

1. $T(\varphi), T(x^2), \dots, T(x^d) = 0$,
2. $\Phi(t), \Phi(x^2), \dots, \Phi(x^d) = 0$,
3. $T(t) = \Phi(\varphi) = 1$,
4. $\mathcal{S} = \{x^1 = 0\}$.

In view of the fact that the span of Φ, T contains a timelike direction everywhere on $\mathcal{U}$, the wave operator (2.0.1) in the $(t, \varphi, x^2, \dots, x^d)$ coordinate system takes the form (using the shorthand notation $x = (x^2, \dots, x^d)$):

$$(2.1.20) \quad \square_g \psi = \Delta_x \psi + \sum_{j=2}^d \left(a_{tj}^{(2)}(x) \partial_{x^j} \partial_t \psi + a_{\varphi j}^{(2)}(x) \partial_{x^j} \partial_\varphi \psi + a_{tj}^{(1)}(x) \partial_t \psi + a_{\varphi j}^{(1)}(x) \partial_\varphi \psi \right) + \\ + a_{tt}(x) \partial_t^2 \psi + a_{t\varphi}(x) \partial_t \partial_\varphi \psi + a_{\varphi\varphi}(x) \partial_\varphi^2 \psi,$$

where the operator Δ_x in the right hand side of (2.1.20) is a t, φ -invariant second order elliptic operator in the $x^2, \dots, x^d$ variables. Since the coefficients of (2.1.20) are independent of t, φ and Δ_x is elliptic, the proof of the Lemma follows readily by the unique continuation result of Tataru [138]. $\square$

2.1.4 Examples of spacetimes satisfying Assumptions G1–G3 and A1

In this section, we will examine some explicit examples of spacetimes satisfying all of the Assumptions G1–G3 and A1.

An example with $\mathcal{H}^+ = \emptyset$. Our first example will be a simple spacetime with no event horizon. Let $\mathcal{M} = \mathbb{R}^{3+1}$, and let us fix two smooth functions $\chi_{\bar{r}} : [0, +\infty) \rightarrow [0, 1]$ and $\chi_{\vartheta} : [0, \pi] \rightarrow [0, 1]$, satisfying $\chi_{\bar{r}} \equiv 0$ on $[0, 3] \cup [6, +\infty)$, $\chi_{\bar{r}} \equiv 1$ on $[4, 5]$, $\chi_{\vartheta} \equiv 0$ on $[0, \frac{\pi}{6}] \cup [\frac{5\pi}{6}, \pi]$ and $\chi_{\vartheta} \equiv 1$ on $[\frac{\pi}{4}, \frac{3\pi}{4}]$. We will also assume that $\chi_{\bar{r}}, \chi_{\vartheta}$ have been chosen so that the set of zeros of the function

$$(2.1.21) \quad f(\bar{r}, \vartheta) \doteq 1 - (\chi_{\bar{r}}(\bar{r})\chi_{\vartheta}(\vartheta))^2$$

is a smooth curve without self-intersections in the open rectangle $(3, 6) \times (\frac{\pi}{6}, \frac{5\pi}{6})$, and the region $\{f \leq 0\} \subset (3, 6) \times (\frac{\pi}{6}, \frac{5\pi}{6})$ is simply connected.

We will consider the following metric on $\mathcal{M}$ in the usual time-polar coordinate chart $(t, \bar{r}, \vartheta, \varphi)$ on $\mathbb{R}^{3+1}$:

$$(2.1.22) \quad g = -\left(1 - (\chi_{\bar{r}}(\bar{r})\chi_{\vartheta}(\vartheta))^2\right)dt^2 - 1000\chi_{\bar{r}}(\bar{r})\chi_{\vartheta}(\vartheta)dtd\varphi + d\bar{r}^2 + \bar{r}^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2).$$

Note that g is everywhere non-degenerate, and has Lorentzian signature. Furthermore, $(\mathcal{M}, g)$ is a globally hyperbolic spacetime, with Cauchy hypersurface $\{t = 0\}$, satisfying the following properties:

1. The vector field $T = \partial_t$ is a Killing field of $(\mathcal{M}, g)$. Furthermore, $(\mathcal{M}, g)$ is asymptotically flat and satisfies Assumption G1. Notice that $(\mathcal{M}, g)$ has no event horizon, since every point in $\mathcal{M}$ can be connected with the asymptotically flat region $\mathcal{I}_{as} = \{\bar{r} \geq R_0 \gg 1\}$ through both a future directed and a past directed timelike curve, by following the flow of the timelike vector fields $\partial_{\bar{r}} + C(\partial_t + \frac{1}{10}\chi_{\bar{r}}(\bar{r})\chi_{\vartheta}(\vartheta)\partial_{\varphi})$ and $\partial_{\bar{r}} - C(\partial_t + \frac{1}{10}\chi_{\bar{r}}(\bar{r})\chi_{\vartheta}(\vartheta)\partial_{\varphi})$, respectively (for some fixed $C \gg 1$). The function $r : \mathcal{M} \rightarrow [0, +\infty)$, introduced in Assumption G1, can be chosen to be equal to $(1 + \bar{r}^2)^{1/2}$.

2. The spacetime $(\mathcal{M}, g)$ has no event horizon, and, thus, it trivially satisfies Assumption G2.

3. The ergoregion $\mathcal{E} = \{g(T, T) > 0\}$ of $(\mathcal{M}, g)$ is non-empty, and satisfies

$$(2.1.23) \quad \{4 \leq \bar{r} \leq 5\} \cap \left\{ \frac{\pi}{4} \leq \vartheta \leq \frac{3\pi}{4} \right\} \subset \mathcal{E} \subset \{3 \leq \bar{r} \leq 6\} \cap \left\{ \frac{\pi}{6} \leq \vartheta \leq \frac{5\pi}{6} \right\}.$$

Since $\mathcal{H}^+ = \emptyset$, we have $\mathcal{E} \cap \mathcal{H}^+ = \emptyset$. Furthermore, since we assumed that the region $\{f \leq 0\} \subset (3, 6) \times (\frac{\pi}{6}, \frac{5\pi}{6})$ for the function (2.1.21) is simply connected, we can readily infer that that $\mathcal{M} \setminus \mathcal{E}$ is connected and, thus, $\mathcal{E}_{ext} = \mathcal{E}$. Furthermore, the boundary $\partial\mathcal{E}$ of $\mathcal{E}$ is a smooth hypersurface of $\mathcal{M}$ and (2.1.10) is satisfied, in view of our assumption on the set of zeros of the function (2.1.21). Therefore, $(\mathcal{M}, g)$ satisfies Assumption G3.

4. The spacetime $(\mathcal{M}, g)$ possesses an additional Killing field, i.e. $\Phi = \partial_\varphi$. The span of T, Φ contains the everywhere timelike vector field $T + \frac{1}{10}\chi_{\bar{r}}(\bar{r})\chi_\vartheta(\vartheta)\Phi$ and, thus, Lemma 2.1 implies that $(\mathcal{M}, g)$ satisfies Assumption A1. In particular, any point $p \in \partial\mathcal{E}_{ext} \subseteq \partial\mathcal{E}$ and any open neighborhood $\mathcal{U}$ of p in $\mathcal{M}$ satisfy the unique continuation property of Assumption A1.

Therefore, $(\mathcal{M}, g)$ satisfies all of the Assumptions G1–G3 and A1.

Remark. The hydrodynamic vortex $(\mathbb{R} \times \mathcal{V}_{hyd, \delta}, g_{hyd})$ of Corollary 2.1 is not a globally hyperbolic spacetime, since its boundary $\partial(\mathbb{R} \times \mathcal{V}_{hyd, \delta}) = \{\bar{r} = \delta\}$ is a timelike hypersurface. However, as we will show in the proof of Corollary 2.1, the double of $(\mathbb{R} \times \mathcal{V}_{hyd, \delta}, g_{hyd})$ across $\partial(\mathbb{R} \times \mathcal{V}_{hyd, \delta})$ is a globally hyperbolic spacetime without an event horizon, satisfying Assumptions G1–G3 and A1. In addition, the double of $(\mathbb{R} \times \mathcal{V}_{hyd, \delta}, g_{hyd})$ is an example of a spacetime having two asymptotically flat ends, with $(\mathbb{R} \times \mathcal{V}_{hyd, \delta}, g_{hyd}) \setminus \mathcal{E}$ having two connected components.

An example with $\mathcal{H}^+ \neq \emptyset$. We will now proceed to construct a slightly more complicated example of a spacetime satisfying Assumptions G1–G3 and A1, possessing in addition a non-empty event horizon. Note that, as we mentioned in Section 2.1.1, the subextremal Kerr

exterior family $(\mathcal{M}_{M,a}, g_{M,a})$ does not satisfy G3, since the future event horizon $\mathcal{H}^+$ and the ergoregion $\mathcal{E}$ of $(\mathcal{M}_{M,a}, g_{M,a})$ have a non-empty intersection.

For any $M > 0$, let $\mathcal{M}_M$ be diffeomorphic to $\mathbb{R} \times (2M, +\infty) \times \mathbb{S}^2$. Let $\chi_{\bar{r}}, \chi_{\vartheta}$ be as before, assuming, in addition, that they have been chosen so that the set of zeros of the function

$$(2.1.24) \quad f_M(\bar{r}, \vartheta) \doteq \left(1 - \frac{2M}{\bar{r}} - (\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta))^2\right)$$

is a smooth curve without self-intersections in the open rectangle $(3M, 6M) \times (\frac{\pi}{6}, \frac{5\pi}{6})$ and the region $\{f_M \leq 0\} \subset (3, 6) \times (\frac{\pi}{6}, \frac{5\pi}{6})$ is simply connected.

Let us consider the following metric in the $(t, \bar{r}, \vartheta, \varphi)$ coordinate chart on $\mathcal{M}_M$:

$$(2.1.25) \quad g_M = -\left(1 - \frac{2M}{\bar{r}} - (\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta))^2\right)dt^2 - 1000M\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta)dtd\varphi +$$

$$(2.1.26) \quad + \left(1 - \frac{2M}{\bar{r}}\right)^{-1}d\bar{r}^2 + \bar{r}^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2).$$

The metric g_M is everywhere non-degenerate, and has Lorentzian signature.

The spacetime $(\mathcal{M}_M, g_M)$ is isometric to the Schwarzschild exterior spacetime $(\mathcal{M}_{M,Sch}, g_{M,Sch})$ outside the region $\{3M \leq \bar{r} \leq 6M\} \cap \{\frac{\pi}{6} \leq \vartheta \leq \frac{5\pi}{6}\}$ and, thus, it can be extended into a larger globally hyperbolic spacetime $(\tilde{\mathcal{M}}_M, \tilde{g}_M)$.⁵ This extension can be chosen to be the Schwarzschild maximal extension across $\bar{r} = 2M$ (see e. g. Section 2 [47]).⁶ Let us denote

$$(2.1.27) \quad \overline{\mathcal{M}}_M = i(\mathcal{M}_M) \cup \partial\mathcal{M}_M,$$

where $i: \mathcal{M}_M \rightarrow \tilde{\mathcal{M}}_M$ is the natural inclusion of $(\mathcal{M}_M, g_M)$ into its extension and $\partial\mathcal{M}_M$ is the boundary of $i(\mathcal{M}_M)$ inside $\tilde{\mathcal{M}}_M$. Note that, in view of the properties of the maximally extended Schwarzschild spacetime, $(\overline{\mathcal{M}}_M, g_M)$ is a smooth Lorentzian manifold with piecewise smooth boundary $\partial\mathcal{M}_M$, consisting of two intersecting smooth null hypersurfaces. The functions $\bar{r}, \vartheta, \varphi$ can be smoothly extended on $\partial\mathcal{M}_M$, with $\bar{r}|_{\partial\mathcal{M}_M} = 2M$.

The spacetime $(\overline{\mathcal{M}}_M, g_M)$ is globally hyperbolic, with $\Sigma = \overline{\{t = 0\}}$ being a smooth Cauchy

⁵Of course, the coordinate chart $(t, \bar{r}, \vartheta, \varphi)$ will not be regular up to the boundary of $\mathcal{M}_M$ in this extension.

⁶Note that the $\bar{r}$ coordinate function on $\mathcal{M}_M$ corresponds to the usual r coordinate function on Schwarzschild exterior $\mathcal{M}_{M,Sch}$

hypersurface, and satisfies the following properties:

1. The vector field $T = \partial_t$ on $\mathcal{M}_M$ extends smoothly on $\partial\mathcal{M}_M$ and is a Killing vector field of $(\overline{\mathcal{M}}_M, g_M)$. Furthermore, the spacetime $(\overline{\mathcal{M}}_M, g_M)$ is asymptotically flat and satisfies Assumption G1. Note that the event horizon $\mathcal{H}$ of $(\overline{\mathcal{M}}_M, g_M)$ coincides with $\partial\mathcal{M}_M$, since all the points in $\mathcal{M}_M$ can be joined with the asymptotically flat region $\mathcal{I}_{as} = \{\bar{r} \geq R_0 \gg 1\}$ through both a future directed and a past directed timelike curve, by following the flow of the timelike vector fields $\partial_{\bar{r}} + C(\partial_t + \frac{1}{10M}\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta)\partial_{\varphi})$ and $\partial_{\bar{r}} - C(\partial_t + \frac{1}{10M}\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta)\partial_{\varphi})$, respectively (for some fixed $C \gg 1$). The function $r : \overline{\mathcal{M}}_M \rightarrow [0, +\infty)$, introduced in Assumption G1, can be chosen to be equal to $\bar{r} - 2M$.
2. There exists a T -invariant neighborhood $\mathcal{V}$ of $\mathcal{H} = \partial\mathcal{M}_M$ in $\overline{\mathcal{M}}_M$, so that $(\mathcal{V}, g_M)$ is isometric to a neighborhood of the event horizon $\mathcal{H}_{M, Sch}$ of Schwarzschild exterior spacetime. In particular, T is parallel to the null generators of $\mathcal{H}^+ \setminus \mathcal{H}^-$ and there exists a T -invariant timelike vector field N on $\overline{\mathcal{M}}_M$ as in Assumption G2, satisfying (2.1.5) (see [44, 47]). In particular, $(\overline{\mathcal{M}}_M, g_M)$ satisfies Assumption G2.
3. The ergoregion $\mathcal{E} = \{g(T, T) > 0\}$ of $(\overline{\mathcal{M}}_M, g_M)$ is non-empty, and satisfies

$$(2.1.28) \quad \{4M \leq \bar{r} \leq 5M\} \cap \{\frac{\pi}{4} \leq \vartheta \leq \frac{3\pi}{4}\} \subset \mathcal{E} \subset \{3M \leq \bar{r} \leq 6M\} \cap \{\frac{\pi}{6} \leq \vartheta \leq \frac{5\pi}{6}\}.$$

Thus, $\mathcal{E} \cap \mathcal{H}^+ = \emptyset$, since $\bar{r} > 2M$ on $\mathcal{E}$. Furthermore, since the function (2.1.24) was assumed to have the property that the set $\{f_M \leq 0\}$ is simply connected, we can readily infer that $\overline{\mathcal{M}}_M \setminus \mathcal{E}$ is connected. Thus, $\mathcal{E}_{ext} = \mathcal{E}$ and $\mathcal{H}^+$ lies in the same connected component of $\overline{\mathcal{M}}_M \setminus \mathcal{E}$ as the asymptotically flat region $\mathcal{I}_{as}$. The boundary $\partial\mathcal{E}$ of $\mathcal{E}$ is a smooth hypersurface of $\overline{\mathcal{M}}_M$ and (2.1.10) is satisfied, in view of our assumption on the set of zeros of the function (2.1.24). Therefore, $(\overline{\mathcal{M}}_M, g_M)$ satisfies Assumption G3.

4. In view of the fact that $(\mathcal{M}_M, g_M)$ possesses an additional Killing field, namely $\Phi = \partial_{\varphi}$, and the span of T, Φ contains the vector field $T + \frac{1}{10M}\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta)\Phi$ which

is everywhere timelike on $\mathcal{M}_M$, Lemma 2.1 implies that $(\overline{\mathcal{M}}_M, g_M)$ satisfies Assumption A1. In particular, any point $p \in \partial\mathcal{E}_{ext} = \partial\mathcal{E}$ and any open neighborhood $\mathcal{U}$ of p satisfy the unique continuation property of Assumption A1.

Thus, $(\overline{\mathcal{M}}_M, g_M)$ satisfies Assumptions G1–G3 and A1, and, in addition, $(\overline{\mathcal{M}}_M, g_M)$ has a non-empty future event horizon.

2.1.5 A remark on the Carleman-type estimates in the proof of Theorem 2.1

As we discussed in the introduction, a crucial step in the proof of Theorem 2.1 consists of showing that, under the assumption that

$$(2.1.29) \quad \limsup_{\tau \rightarrow +\infty} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu < +\infty$$

holds for every smooth function $\varphi : J^+(\Sigma) \rightarrow \mathbb{C}$ solving the wave equation (2.0.1) on $J^+(\Sigma)$ with compactly supported initial data on Σ , we also have that φ decays on $\mathcal{M} \setminus \mathcal{E}$; see Section 2.3 for more details. This fact is inferred using some suitable Carleman-type estimates on $(\mathcal{M} \setminus \mathcal{E}, g)$ for φ which are particularly useful when φ has localised frequency support in time (see Proposition 2.2 in Section 2.5; for the technical details related to the frequency decomposition of φ , see Section 2.4).

The aforementioned estimates could have been established using the techniques of our previous [107].⁷ However, we chose, instead, to provide an alternative proof, based entirely on the use of first order multipliers for equation (2.0.1). As a consequence, we obtain an alternative proof for the estimates of Section 7.1 of [107], as well as for the Carleman-type estimates established in [126] for the inhomogeneous Helmholtz equation

$$(2.1.30) \quad \Delta_{\bar{g}} u + \omega^2 u - V u = G,$$

$0 < \text{Im}(\omega) \ll 1$, $\text{Re}(\omega) \neq 0$, on an asymptotically conic Riemannian manifold $(\Sigma, \bar{g})$, where

⁷Although the techniques of [107] can be modified to yield the required Carleman-type estimates on the spacetimes under consideration, they can not be used as a black box in the current setting, since the geometry of the spacetime in a neighborhood of $\mathcal{E}$ and the allowed range in the space dimension d (which includes the case $d = 2$) differ from the setting of [107].

the potential $V : \Sigma \rightarrow \mathbb{R}$ satisfies some suitable decay conditions on the asymptotically conic end of Σ . An alternative approach based on the positive commutator method and leading to similar Carleman-type inequalities on asymptotically flat backgrounds with a uniformly timelike Killing field has also been implemented in the recent [?]. For a more detailed statement of these results, see Section 2.5.

2.2 Special notational conventions and Hardy inequalities

In this Section, given a stationary and asymptotically flat spacetime $(\mathcal{M}, g)$ satisfying Assumptions G1–G3, we will introduce some special notational conventions that will be applied on $(\mathcal{M}, g)$, in addition to the ones introduced in the previous sections. These conventions will be irrelevant for Chapter 3 of the thesis. We will also explicitly state some auxiliary Hardy type estimates, that will be used at various points in the proof of Theorem 2.1.

2.2.1 Some special subsets of $\mathcal{M}$

The future event horizon of $\mathcal{M}$ will be denoted by $\mathcal{H}^+$, and the past event horizon by $\mathcal{H}^-$, i. e.

$$\begin{aligned}\mathcal{H}^+ &= \overline{J^+(\mathcal{I}_{as})} \cap \partial J^-(\mathcal{I}_{as}), \\ \mathcal{H}^- &= \overline{J^-(\mathcal{I}_{as})} \cap \partial J^+(\mathcal{I}_{as}).\end{aligned}$$

For any $\tau_1 \leq \tau_2$, we will denote

$$(2.2.1) \quad \mathcal{R}(\tau_1, \tau_2) \doteq \{\tau_1 \leq t \leq \tau_2\} \subset \mathcal{M} \setminus \mathcal{H}^-$$

and

$$(2.2.2) \quad \Sigma_\tau \doteq \{t = \tau\},$$

where the function $t : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ is defined in Assumption G1.

The ergoregion of $\mathcal{M}$, defined by (2.1.8), will be denoted by $\mathcal{E}$. The boundary of $\mathcal{E}$ (which is smooth, according to Assumption G3) will be denoted by $\partial\mathcal{E}$. We will fix a smooth T -invariant spacelike vector field $n_{\partial\mathcal{E}}$ in a small T -invariant neighborhood of $\partial\mathcal{E}$, such that $n_{\partial\mathcal{E}}|_{\partial\mathcal{E}}$ is the unit normal of $\partial\mathcal{E}$. We will denote with $\mathcal{E}_{ext}$ the extended ergoregion of $(\mathcal{M}, g)$, defined by (2.1.11). Notice that $\mathcal{E} \subseteq \mathcal{E}_{ext}$, but $\partial\mathcal{E}_{ext} \subseteq \partial\mathcal{E}$.

For any $\delta > 0$, we will denote

$$(2.2.3) \quad \mathcal{E}_\delta = \{x \in \mathcal{M} \setminus \mathcal{H}^- \mid \text{dist}_{g_{ref}}(x, \mathcal{E}_{ext}) \leq \delta\}.$$

Note that $\mathcal{E}_{ext} \subset \mathcal{E}_\delta$ for any $\delta > 0$, and $\cap_{\delta>0} \mathcal{E}_\delta = \mathcal{E}_{ext}$.

2.2.2 Notations on metrics, connections and integration

For any pseudo-Riemannian manifold $(\mathcal{N}, h_{\mathcal{N}})$ appearing in this chapter, we will denote with $dh_{\mathcal{N}}$ the natural volume form associated with $h_{\mathcal{N}}$. Recall that in any local coordinate chart $(x^1, x^2, \dots, x^k)$ on $\mathcal{N}$, $dh_{\mathcal{N}}$ is expressed as

$$dh_{\mathcal{N}} = \sqrt{|\det(h_{\mathcal{N}})|} dx^1 \dots dx^k.$$

We will also denote with $\nabla_{h_{\mathcal{N}}}$ the natural connection associated to $h_{\mathcal{N}}$. When $(\mathcal{N}, h_{\mathcal{N}}) = (\mathcal{M}, g)$, we will denote $\nabla_{h_{\mathcal{N}}}$ simply as ∇ . If $h_{\mathcal{N}}$ is Riemannian, $|\cdot|_{h_{\mathcal{N}}}$ will denote the associated norm on the tensor bundle of $\mathcal{N}$.

For any integer $l \geq 0$, we will denote with $(\nabla^{h_{\mathcal{N}}})^l$ or $\nabla_{h_{\mathcal{N}}}^l$ the higher order operator

$$(2.2.4) \quad \underbrace{\nabla_{h_{\mathcal{N}}} \dots \nabla_{h_{\mathcal{N}}}}_{l \text{ times}}.$$

Note that the product (2.2.4) is not symmetrised. We will also adopt the convention that we will always use Latin characters to denote such powers of covariant derivative operators. On the other hand, Greek characters will be used for the indices of a tensor in an abstract index notation.

For any smooth and spacelike hypersurface $\mathcal{S} \subset \mathcal{M}$, $g_{\mathcal{S}}$ will denote the induced (Rie-

mannian) metric on $\mathcal{S}$, and $n_{\mathcal{S}}$ the future directed unit normal to $\mathcal{S}$.

Some examples of pseudo-Riemannian manifolds that will appear throughout this chapter are $(\mathcal{M}, g)$, $(\mathcal{M}, g_{ref})$ and $(\Sigma_{\tau}, g_{\Sigma_{\tau}})$, where g_{ref} is the reference Riemannian metric (2.1.4). We will raise and lower indices of tensors on $\mathcal{M}$ only with the use of g .

In some cases, we will omit the volume form dg or $dg_{\Sigma_{\tau}}$ when integrating over domains in $\mathcal{M}$ or the hypersurfaces Σ_{τ} , respectively.

In the case of a smooth null hypersurface $\mathcal{H} \subset \mathcal{M}$, the volume form with which integration will be considered will as usual depend on the choice of a future directed null generator $n_{\mathcal{H}}$ for $\mathcal{H}$. For any such choice of $n_{\mathcal{H}}$, selecting an arbitrary vector field X on $T_{\mathcal{H}}\mathcal{M}$ such that $g(X, n_{\mathcal{H}}) = -1$ enables the construction of a non-degenerate top dimensional form on $\mathcal{H}$: $dvol_{\mathcal{H}} \doteq i_X dg$, which depends on the on the precise choice of $n_{\mathcal{H}}$, but not on the choice for X . In that case, $dvol_{\mathcal{H}}$ (or $dvol_{n_{\mathcal{H}}}$) will be the volume form on $\mathcal{H}$ associated with $n_{\mathcal{H}}$.

2.2.3 Coordinate charts on $\mathcal{M} \setminus \mathcal{H}^-$

Using the function t as a projection, we can identify $\mathcal{M} \setminus \mathcal{H}^-$ with $\mathbb{R} \times \Sigma$. Under this identification, any local coordinate chart $(x^1, \dots, x^d)$ on a subset $\mathcal{V}$ of Σ can be extended to a coordinate chart $(t, x^1, \dots, x^d)$ on $\mathbb{R} \times \mathcal{V} \subset \mathbb{R} \times \Sigma$, and in this chart, we have $\partial_t = T$. We will usually work in such coordinate charts throughout this chapter.

In view of the the flat asymptotics of $(\mathcal{M}, g)$ and the fact that Σ intersects $\mathcal{H}^+$ transversally, the coarea formula yields that in the region $J^+(\Sigma)$, the volume forms dg and $dt \wedge dg_{\Sigma}$ are equivalent, i.e. there exists a $C > 0$, such that for any integrable $\varphi : \mathcal{M} \rightarrow [0, +\infty)$ and any $0 \leq \tau_1 \leq \tau_2$ (identifying $\mathcal{M} \setminus \mathcal{H}^-$ with $\mathbb{R} \times \Sigma$):

$$(2.2.5) \quad C^{-1} \int_{\mathcal{R}(\tau_1, \tau_2)} \varphi dg \leq \int_{\tau_1}^{\tau_2} \left(\int_{\Sigma} \varphi(t, x) dg_{\Sigma} \right) dt \leq C \int_{\mathcal{R}(\tau_1, \tau_2)} \varphi dg.$$

Similarly, for any $\delta > 0$, there exists a $C_{\delta} > 0$ so that for any integrable $\varphi : \mathcal{M} \rightarrow [0, +\infty)$ and any $\tau_1 \leq \tau_2$ (not necessarily non-negative):

$$(2.2.6) \quad C_{\delta}^{-1} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \delta\}} \varphi dg \leq \int_{\tau_1}^{\tau_2} \left(\int_{\Sigma \cap \{r \geq \delta\}} \varphi(t, x) dg_{\Sigma} \right) dt \leq C_{\delta} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \delta\}} \varphi dg.$$

2.2.4 Notations for derivatives on $\mathbb{S}^{d-1}$

In this chapter, we will frequently work in polar coordinates in the asymptotically flat region of $(\mathcal{M}, g)$ or (Σ, g_Σ) . For this reason, we will adopt the same shorthand σ -notation for the angular variables in such a polar coordinate, as we did in [107, 108]. See Section 3.6 of [107] for a detailed statement of this convention.

As an example of this convention, on subset $\mathcal{U}$ of a spacetime $\mathcal{M}$ covered by a polar coordinate chart $(u_1, u_2, \sigma) : \mathcal{U} \rightarrow \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{S}^{d-1}$, for any function $h : \mathcal{U} \rightarrow \mathbb{C}$ and any symmetric $(l, 0)$ -tensor b on $\mathbb{S}^{d-1}$, the following schematic notation for the contraction of the tensor $(\nabla_{g_{\mathbb{S}^{d-1}}})^l h(u_1, u_2, \cdot)$ with b will be frequently used:

$$(2.2.7) \quad b \cdot \partial_\sigma^l h(u_1, u_2, \cdot) \doteq b^{i_1 \dots i_l} (\nabla_{g_{\mathbb{S}^{d-1}}}^l)_{i_1 \dots i_l} h(u_1, u_2, \cdot),$$

where $g_{\mathbb{S}^{d-1}}$ is the standard metric on the unit sphere $\mathbb{S}^{d-1}$. Furthermore, we will also denote in this case

$$(2.2.8) \quad |\partial_\sigma^l h(u_1, u_2, \cdot)| \doteq |\nabla_{g_{\mathbb{S}^{d-1}}}^l h(u_1, u_2, \cdot)|_{g_{\mathbb{S}^{d-1}}}.$$

Notice, also, the following commutation relation holds:

$$(2.2.9) \quad [\mathcal{L}_{\partial_{u_i}}, \nabla^{\mathbb{S}^{d-1}}] = 0,$$

where ∂_{u_i} is the coordinate vector field associated to the coordinate function u_i , $i = 1, 2$. Therefore, we will frequently denote for any function $h : \mathcal{U} \rightarrow \mathbb{C}$:

$$(2.2.10) \quad \mathcal{L}_{\partial_{u_i}} \nabla^{\mathbb{S}^{d-1}} h \doteq \partial_{u_i} \partial_\sigma h,$$

and, in this notation, we will be allowed to commute ∂_{u_i} with ∂_σ , as if ∂_σ was a regular coordinate vector field. See Section 3.6 of [107] for more details.

2.2.5 The $O_k(\cdot)$ notation

For any integer $k \geq 0$ and any $b \in \mathbb{R}$, the notation $h = O_k(r^b)$ for some smooth function $h : \mathcal{M} \rightarrow \mathbb{C}$ will be used to denote that, in the (t, r, σ) polar coordinate chart on each connected component of the region $\{r \gg 1\}$ of $\mathcal{M}$ (see Assumption 1):

$$(2.2.11) \quad \sum_{j=0}^k \sum_{j_1+j_2+j_3=j} r^{j_2+j_3} |\partial_\sigma^{j_1} \partial_t^{j_2} \partial_r^{j_3} h| \leq C \cdot r^b$$

for some constant $C > 0$ depending on k and h . The same notation (omitting the ∂_t derivatives) will also be used for functions on regions of manifolds cover by an (r, σ) polar coordinate chart.

Similarly, the notation $h = O_k^{\mathbb{S}^{d-1}}(r^b)$ will be used to denote a smooth tensor field h on $\mathcal{M}$ such that, in the (t, r, σ) polar coordinate chart on each connected component of the region $\{r \gg 1\}$ of $\mathcal{M}$, h is tangential to the $\{r = \text{const}\}$ coordinate spheres (i. e. h contracted with ∂_r, ∂_t or $dr, d\sigma$, depending on its type, yields zero), and satisfies $|h|_{g_{\mathbb{S}^{d-1}}} = O_k(r^b)$. The type of the tensor h will always be clear from the context.

2.2.6 Hardy-type inequalities

Frequently throughout this chapter, we will need to control the weighted L^2 norm of some function u by some weighted L^2 norm of its derivative ∇u . This will always be accomplished with the use of some variant of the following Hardy-type inequality on $\mathbb{R}^d$ (which is true for $d \geq 1$, although we will only need it for $d \geq 2$):

Lemma 2.2. *For any $a > 0$, there exists some $C_a > 0$ such that for any smooth and compactly supported function $u : \mathbb{R}^d \rightarrow \mathbb{C}$ and any $0 < R_1 < R_2$ we can bound*

$$(2.2.12) \quad \begin{aligned} \int_{\mathbb{R}^d \cap \{R_1 \leq r \leq R_2\}} r^{-d+a} |u|^2 dx + \int_{\{r=R_1\}} R_1^{-(d-1)+a} |u|^2 dg_{\{r=R_1\}} &\leq \\ &\leq C_a \int_{\mathbb{R}^d \cap \{R_1 \leq r \leq R_2\}} r^{-(d-2)+a} |\partial_r u|^2 dx + \int_{\{r=R_2\}} R_2^{-(d-1)+a} |u|^2 dg_{\{r=R_2\}} \end{aligned}$$

and

(2.2.13)

$$\begin{aligned} & \int_{\mathbb{R}^d \cap \{R_1 \leq r \leq R_2\}} r^{-d} |u|^2 dx + \int_{\{r=R_1\}} R_1^{-(d-1)} \log(R_1) |u|^2 dg_{\{r=R_1\}} \leq \\ & \leq C \int_{\mathbb{R}^d \cap \{R_1 \leq r \leq R_2\}} r^{-(d-2)} (\log(r))^2 |\partial_r u|^2 dx + \int_{\{r=R_2\}} R_2^{-(d-1)} \log(R_2) |u|^2 dg_{\{r=R_2\}}. \end{aligned}$$

In the above, r is the polar distance on $\mathbb{R}^d$, dx is the usual volume form on $\mathbb{R}^d$ and $dg_{\{r=R\}}$ is the volume form of the induced metric on the sphere $\{r = R\} \subset \mathbb{R}^d$.

The proof of Lemma 2.2 is straightforward (see also Section 3.9 of [107]).

2.3 Proof of Theorem 2.1

The proof of Theorem 2.1 will proceed by contradiction: We will assume that all smooth solutions φ to (2.0.1) on $\mathcal{D}(\Sigma)$ with compactly supported initial data on Σ satisfy

$$(2.3.1) \quad \mathcal{E}[\varphi] \doteq \sup_{\tau \geq 0} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu < +\infty,$$

and we will reach a contradiction after choosing φ appropriately. To this end, we will need to establish a decay without a rate result outside the extended ergoregion $\mathcal{E}_{ext}$ for solutions φ to (2.0.1), given the bound (2.3.1); see Proposition 2.1 in Section 2.3.2. This result is highly non-trivial and actually lies at the heart of the proof of Theorem 2.1, with Sections 2.4–2.5 being devoted to the development of the necessary technical machinery for the proof of Proposition 2.1. In fact, the proof of Proposition 2.1 will be postponed until Section 2.6.

Remark. Instead of assuming (2.3.1), our proof of Theorem 2.1 also applies under the weaker assumption:

$$(2.3.2) \quad \sup_{\tau \geq 0} \left((\log(2 + \tau))^{-C} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu \right) < +\infty$$

for an arbitrary $C > 0$. Furthermore, as a consequence of the discussion in Section 2.5.9 (see also the remark below Proposition 2.1), the proof of Theorem 2.1 also applies without any significant change in the case when $(\mathcal{M}, g)$ has a T -invariant timelike boundary component

$\partial_{tim}\mathcal{M}$, with $\partial_{tim}\mathcal{M} \cap \Sigma$ compact and $\partial_{tim}\mathcal{M} \cap \mathcal{H} = \emptyset$, and φ is assumed to satisfy either Dirichlet or Neumann boundary conditions on $\partial_{tim}\mathcal{M}$ (see Section 2.5.9 for more details on the assumptions on the geometry of $(\mathcal{M}, g)$ in this case).

In Sections 2.3.1–2.3.3, we will establish some auxiliary results concerning the behaviour of solutions φ to (2.0.1), that will be used in the Section 2.3.4 to complete the proof of Theorem 2.1.

2.3.1 Construction of initial data on Σ with negative higher order energy

In this section, we will establish the following result:

Lemma 2.3. *There exists a smooth initial data set $(\varphi^{(0)}, \varphi^{(1)}) : \Sigma \rightarrow \mathbb{C}^2$ supported in $\Sigma \cap \mathcal{U}$ (where $\mathcal{U} \subset \mathcal{M}$ is the set described in Assumption A1) so that the function $\varphi : \mathcal{D}(\Sigma) \rightarrow \mathbb{C}$, defined by solving*

$$(2.3.3) \quad \begin{cases} \square_g \varphi = 0 & \text{on } \mathcal{D}(\Sigma), \\ (\varphi|_{\Sigma}, T\varphi|_{\Sigma}) = (\varphi^{(0)}, \varphi^{(1)}), \end{cases}$$

satisfies

$$(2.3.4) \quad \int_{\Sigma} J_{\mu}^T(T\varphi)n^{\mu} = -1.$$

Remark. Notice that the initial value problem (2.3.3) is well posed, since the vector field T , although not everywhere timelike, is everywhere transversal to Σ . For the construction of initial data sets $(\varphi^{(0)}, \varphi^{(1)})$ satisfying

$$\int_{\Sigma} J_{\mu}^T(\varphi)n^{\mu} = -1$$

instead of (2.3.4), see [66].

Proof. Since $\mathcal{U}$ is an open subset of $\mathcal{M}$ intersecting $\mathcal{E}$ (according to Assumption A1), in view of the definition (2.1.8) of $\mathcal{E}$ we infer that there exists a point $q \in \mathcal{U} \cap \Sigma$ and a contractible open neighborhood $\mathcal{V}$ of q in $\mathcal{M}$ such that T is strictly spacelike on $\mathcal{V}$. Therefore, provided

$\mathcal{V}$ is sufficiently small, there exists a vector field L on $\mathcal{V}$ satisfying

$$(2.3.5) \quad g(L, L) = 0, \quad g(L, T) > 0 \text{ and } \nabla L = 0.$$

The condition $\nabla L = 0$ on $\mathcal{V}$ implies that there exists a function $w : \mathcal{V} \rightarrow \mathbb{R}$ such that

$$(2.3.6) \quad \nabla w = L.$$

Let us fix a smooth cut-off function $\chi : \mathcal{M} \rightarrow [0, 1]$ supported in $\mathcal{V}$ such that $\chi(q) = 1$. Then, for any $l \gg 1$, the function

$$(2.3.7) \quad \tilde{\varphi}_l \doteq \chi e^{ilw}$$

on $\mathcal{M}$ is supported in $\mathcal{V}$ and satisfies (in view of (2.3.5) and (2.3.6))

$$(2.3.8) \quad \square_g \tilde{\varphi}_l = \chi l^2 \partial_\mu w \partial^\mu w e^{ilw} + O(l) = O(l).$$

Furthermore, we compute:

$$(2.3.9) \quad \begin{aligned} \int_{\Sigma} J^T(T\tilde{\varphi}_l) n^\mu &= \int_{\Sigma} \left(n(T\tilde{\varphi}_l) \cdot T^2 \tilde{\varphi}_l - \frac{1}{2} g(n, T) \partial_\mu T \tilde{\varphi}_l \partial^\mu T \tilde{\varphi}_l \right) dg_{\Sigma} = \\ &= \int_{\Sigma \cap \mathcal{V}} \left((\chi l^2 (nw)(Tw) e^{ilw} + O(l)) (\chi l^2 (Tw)^2 e^{ilw} + O(l)) \right. \\ &\quad \left. - \frac{1}{2} g(n, T) (\chi l^2 (\partial_\mu w)(Tw) e^{ilw} + O(l)) (\chi l^2 (\partial^\mu w)(Tw) e^{ilw} + O(l)) \right) dg_{\Sigma} \\ &= \int_{\Sigma \cap \mathcal{V}} \left(\chi^2 l^4 (g(n, L)(g(T, L))^3 - \frac{1}{2} g(n, T) g(L, L)(g(T, L))^2) + O(l^3) \right) dg_{\Sigma}, \end{aligned}$$

which, in view of (2.3.5) (and the fact that $g(n, L) < 0$), yields:

$$(2.3.10) \quad \int_{\Sigma} J^T(T\tilde{\varphi}_l) n^\mu = -c_0 l^4 + O(l^3)$$

for some $c_0 > 0$.

Let us set

$$(2.3.11) \quad (\varphi^{(0)}, \varphi^{(1)}) \doteq (\tilde{\varphi}_l|_\Sigma, T\tilde{\varphi}_l|_\Sigma).$$

Note that $(\varphi^{(0)}, \varphi^{(1)})$ is supported in $\mathcal{V} \cap \Sigma \subset \mathcal{U} \cap \Sigma$. Then, the function

$$(2.3.12) \quad \check{\varphi} \doteq \varphi - \tilde{\varphi}_l,$$

where φ is defined by (2.3.3), satisfies (in view of (2.3.3), (2.3.8) and (2.3.11)):

$$(2.3.13) \quad \begin{cases} \square_g \check{\varphi} = O(l) & \text{on } \mathcal{D}(\Sigma), \\ (\check{\varphi}|_\Sigma, T\check{\varphi}|_\Sigma) = (0, 0). \end{cases}$$

In view of the fact that $\check{\varphi}|_\Sigma = 0$ and $\nabla \check{\varphi}|_\Sigma = 0$ (implying also that $\nabla_{g_\Sigma}^2 \check{\varphi}|_\Sigma = 0$ and $\nabla_{g_\Sigma} T\check{\varphi}|_\Sigma = 0$), the expression of the wave operator in a coordinate chart of the form (t, x) on $\mathcal{V}$ readily yields

$$(2.3.14) \quad (\square_g \check{\varphi})|_\Sigma = (g^{00} T^2 \check{\varphi})|_\Sigma = (g^{0\mu} \partial_\mu (T\check{\varphi}))|_\Sigma = \left(\frac{1}{g(n, T)} n(T\check{\varphi}) \right)|_\Sigma.$$

Thus, from (2.3.13), (2.3.14) and the fact that $\square_g \check{\varphi}$ is supported in $\mathcal{V}$, we can readily bound

$$(2.3.15) \quad \int_\Sigma J_\mu^N(T\check{\varphi}) n^\mu = O(l^2).$$

From (2.3.12), a Cauchy–Schwarz inequality implies:

$$(2.3.16) \quad \left| \int_\Sigma J_\mu^T(T\varphi) n^\mu - \int_\Sigma J_\mu^T(T\tilde{\varphi}_l) n^\mu \right| \leq C \int_\Sigma J_\mu^N(T\check{\varphi}) n^\mu,$$

and thus, in view also of (2.3.10) and (2.3.15):

$$(2.3.17) \quad \int_\Sigma J_\mu^T(T\varphi) n^\mu = -c_0 l^4 + O(l^3) < 0,$$

provided $l \gg 1$. Multiplying $(\varphi^{(0)}, \varphi^{(1)})$ with a suitable non-zero constant, we can therefore

achieve (2.3.4), and therefore the proof of the Lemma is complete. $\square$

2.3.2 Decay outside the extended ergoregion

The following proposition, establishing decay without a rate outside the ergoregion for solutions to equation (2.0.1), lies at the heart of the proof of Theorem 2.1:

Proposition 2.1. *Let $\varphi: \mathcal{D}(\Sigma) \rightarrow \mathbb{C}$ be a smooth function satisfying (2.0.1) with compactly supported initial data on Σ , and let us set $\psi = T\varphi$. Assume that the energy bound (2.3.1) holds for φ , ψ , $T\psi$ and $T^2\psi$,⁸ i. e.:*

$$(2.3.18) \quad \mathcal{E}[\varphi] + \mathcal{E}[\psi] + \mathcal{E}[T\psi] + \mathcal{E}[T^2\psi] < +\infty.$$

Then for any $0 < \epsilon < 1$, any $\delta_1 > 0$, any $R, \tau_* \gg 1$ and any $\bar{\tau}_0 \gg 1$, there exists a $\tau_{\mathfrak{h}} \geq \bar{\tau}_0 + \tau_*$ depending on $\epsilon, \delta_1, R, \tau_*, \bar{\tau}_0, \mathcal{E}_{\log}[\varphi], \mathcal{E}_{\log}[\psi], \mathcal{E}_{\log}[T\psi]$ and $\mathcal{E}[T^2\psi]$ such that

$$(2.3.19) \quad \sum_{j=0}^1 \int_{(\mathcal{R}(\tau_{\mathfrak{h}} - \tau_*, \tau_{\mathfrak{h}} + \tau_*) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} (J_{\mu}^N(T^j\psi)n^{\mu} + |T^j\psi|^2) < \epsilon$$

(see (2.4.17) for the definition of the quantity $\mathcal{E}_{\log}[\cdot]$)

For the proof of Proposition 2.1, see Section 2.6.

Remark. The proof of Proposition 2.1 also applies when $\mathcal{E} = \emptyset$. Furthermore, in view of the discussion in Section 2.5.9, the proof of Proposition 2.1 in Section 2.6 also applies in the case when $(\mathcal{M}, g)$ has a T -invariant timelike boundary component $\partial_{tim}\mathcal{M}$, with $\partial_{tim}\mathcal{M} \cap \Sigma$ compact and $\partial_{tim}\mathcal{M} \cap \mathcal{H} = \emptyset$, and φ is assumed to satisfy either Dirichlet or Neumann boundary conditions on $\partial_{tim}\mathcal{M}$.⁹ As a consequence, the proof of Theorem 2.1 will also apply in this case as well (all the other steps in the proof of Theorem 2.1 immediately generalise in this case without any change).

⁸Note that, since T is a Killing field of $(\mathcal{M}, g)$, the functions ψ , $T\psi$ and $T^2\psi$ also solve (2.0.1) with compactly supported initial data on Σ .

⁹In this case, we have to assume that the double $(\tilde{\mathcal{M}}, \tilde{g})$ of $(\mathcal{M}, g)$ across $\partial_{tim}\mathcal{M}$ is a globally hyperbolic spacetime satisfying Assumptions G1–G3 (note that Assumption A1 is not necessary for the proof of Proposition 2.1).

2.3.3 Limiting behaviour for solutions of (2.0.1)

We will need the following lemma on the behaviour of ψ asymptotically as $t \rightarrow +\infty$, following essentially from Proposition 2.1:

Lemma 2.4. *Let $\varphi, \psi : \mathcal{D}(\Sigma) \rightarrow \mathbb{C}$ be as in the statement of Proposition 2.1, and let us define, for any $\tau \geq 0$, the function $\psi_\tau : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ as follows:*

$$(2.3.20) \quad \psi_\tau(t, x) \doteq \begin{cases} \psi(t + \tau, x), & t \geq -\tau, \\ 0, & t < -\tau. \end{cases}$$

Then, there exists an increasing sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of non-negative numbers and a function $\tilde{\psi} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ with $\tilde{\psi}, T\tilde{\psi} \in H_{loc}^1(\mathcal{M} \setminus \mathcal{H}^-)$, such that $\tilde{\psi}$ solves (2.0.1) on $\mathcal{M} \setminus \mathcal{H}^-$, satisfying in addition

$$(2.3.21) \quad \int_{-\tau_*}^{+\tau_*} \int_{\Sigma_\tau} (J_\mu^N(\tilde{\psi}) + J_\mu^N(T\tilde{\psi})) n^\mu d\tau < +\infty \text{ for any } \tau_* > 0,$$

$$(2.3.22) \quad \tilde{\psi} \equiv 0 \text{ on } \mathcal{M} \setminus (\mathcal{E}_{ext} \cup \mathcal{H}^-)$$

and $(\psi_{\tau_n}, T\psi_{\tau_n}) \rightarrow (\tilde{\psi}, T\tilde{\psi})$ weakly in $H_{loc}^1(\mathcal{M} \setminus \mathcal{H}^-) \times H_{loc}^1(\mathcal{M} \setminus \mathcal{H}^-)$ and strongly in $H_{loc}^1(\mathcal{M} \setminus (\mathcal{E}_{ext} \cup \mathcal{H}^-)) \times H_{loc}^1(\mathcal{M} \setminus (\mathcal{E}_{ext} \cup \mathcal{H}^-))$ and in $L_{loc}^2(\mathcal{M} \setminus \mathcal{H}^-) \times L_{loc}^2(\mathcal{M} \setminus \mathcal{H}^-)$ in the following sense:

- *For any compactly supported test functions $\{\zeta_j\}_{j=0,1} \in L^2(\mathcal{M} \setminus \mathcal{H}^-)$ and compactly supported vector fields $\{X_j\}_{j=0,1}$ on $\mathcal{M} \setminus \mathcal{H}^-$ such that $|X_j|_{g_{ref}} \in L^2(\mathcal{M} \setminus \mathcal{H}^-)$:*

$$(2.3.23) \quad \lim_{n \rightarrow +\infty} \sum_{j=0}^1 \int_{\mathcal{M} \setminus \mathcal{H}^-} \text{Re} \left\{ g_{ref}(\nabla(T^j \psi_{\tau_n} - T^j \tilde{\psi}), X_j) + (T^j \psi_{\tau_n} - T^j \tilde{\psi}) \zeta_j \right\} dg = 0.$$

- For any compact subset $\mathcal{K} \subset \mathcal{M} \setminus \mathcal{H}^-$ and any $\delta > 0$:

$$(2.3.24) \quad \lim_{n \rightarrow +\infty} \left(\sum_{j=0}^1 \int_{\mathcal{K}} |T^j \psi_{\tau_n} - T^j \tilde{\psi}|^2 dg + \sum_{j=0}^1 \int_{\mathcal{K} \setminus \mathcal{E}_{\delta}} |\nabla(T^j \psi_{\tau_n}) - \nabla(T^j \tilde{\psi})|_{g_{ref}}^2 dg \right) = 0.$$

Proof. Let us fix four sequences of positive numbers $\{\varepsilon_n\}_{n \in \mathbb{N}}$, $\{\delta_n\}_{n \in \mathbb{N}}$, $\{R_n\}_{n \in \mathbb{N}}$ and $\{\tau_n^*\}_{n \in \mathbb{N}}$ such that $\varepsilon_n, \delta_n \rightarrow 0$ and $R_n, \tau_n^* \rightarrow +\infty$ as $n \rightarrow +\infty$. We then define the sequence $\{\tau_n\}_{n \in \mathbb{N}}$ inductively: Setting $\tau_0 = 0$, τ_n is defined for any $n \geq 1$ as the value $\tau_{\natural} > 0$ from Proposition 2.1 for ε_n in place of ε , δ_n in place of δ , R_n in place of R , τ_n^* in place of τ_* and τ_{n-1} in place of $\bar{\tau}_0$ (notice that the last condition guarantees that τ_n is an increasing sequence). Then, Proposition 2.1 applied for the pair $(\psi, T\psi)$ implies that the pair $(\psi_{\tau_n}, T\psi_{\tau_n})$ (which is merely a τ_n -translate of $(\psi, T\psi)$ in the region $\{t \geq -\tau_n\}$) satisfies the following estimate for any $n \in \mathbb{N}$:

$$(2.3.25) \quad \sum_{j=0}^1 \int_{-\tau_n^*}^{+\tau_n^*} \left(\int_{(\Sigma_t \setminus \mathcal{E}_{\delta_n}) \cap \{r \leq R_n\}} (J_{\mu}^N(T^j \psi_{\tau_n}) n^{\mu} + |T^j \psi_{\tau_n}|^2) dt \right) < \varepsilon_n.$$

In view of the bounds (2.3.18) and 2.3.25, as well as the Poincare-type inequality

$$(2.3.26) \quad \int_{\mathcal{R}(\bar{\tau}_1, \bar{\tau}_2) \cap \{r \leq R\}} |\psi_{\tau_n}|^2 \leq CR^2 \int_{\mathcal{R}(\bar{\tau}_1, \bar{\tau}_2) \cap \{r \leq 2R\}} J_{\mu}^N(\psi_{\tau_n}) n^{\mu} + C \int_{\mathcal{R}(\bar{\tau}_1, \bar{\tau}_2) \cap \{R \leq r \leq 2R\}} |\psi_{\tau_n}|^2$$

holding for any $\bar{\tau}_1 \leq \bar{\tau}_2$, we infer that, for any compact subset $\mathcal{K}$ of $\mathcal{M} \setminus \mathcal{H}^-$, setting

$$(2.3.27) \quad n_0(\mathcal{K}) = \min \left\{ n \in \mathbb{N} : \mathcal{K} \text{ is contained in the set } \left\{ \max\{-\tau_n, -\tau_n^*\} < t < \tau_n^* \right\} \cap \{r \leq R_n\} \right\},$$

there exists a $C = C_{\mathcal{K}}$ such that:

$$(2.3.28) \quad \sup_{n \geq n_0(\mathcal{K})} \left(\sum_{j=0}^1 \int_{\mathcal{K}} (J_{\mu}^N(T^j \psi_{\tau_n}) n^{\mu} + |T^j \psi_{\tau_n}|^2) \leq C_{\mathcal{K}}(\mathcal{E}[\psi] + \mathcal{E}[T\psi]) + \sup_{n \geq n_0(\mathcal{K})} \varepsilon_n < +\infty. \right.$$

For any compact $\mathcal{K} \subset \mathcal{M} \setminus \mathcal{H}^-$, Rellich–Kondrachov’s theorem yields that the embedding $H^1(\mathcal{K}) \times H^1(\mathcal{K}) \hookrightarrow L^2(\mathcal{K}) \times L^2(\mathcal{K})$ is compact. Thus, (2.3.28) implies that for any compact $\mathcal{K} \subset \mathcal{M} \setminus \mathcal{H}^-$ and any infinite subset $A \subseteq \mathbb{N}$, there exists an infinite subset $B_{\mathcal{K}, A} \subseteq A$

such that the subsequence $\{(\psi_{\tau_n}, T\psi_{\tau_n})\}_{n \in B_{\mathcal{K}, A}}$ of $\{(\psi_{\tau_n}, T\psi_{\tau_n})\}_{n \in \mathbb{N}}$ converges weakly in the $H^1(\mathcal{K}) \times H^1(\mathcal{K})$ norm and strongly in the $L^2(\mathcal{K}) \times L^2(\mathcal{K})$ to some limit pair $(\tilde{\psi}_{\mathcal{K}}, \dot{\psi}_{\mathcal{K}})$ in $H^1(\mathcal{K}) \times H^1(\mathcal{K})$. Note that in this case, we necessarily have $\dot{\psi}_{\mathcal{K}} = T\tilde{\psi}_{\mathcal{K}}$ in the sense of distributions.

Let $\{\mathcal{K}_m\}_{m \in \mathbb{N}}$ be a sequence of compact subsets of $\mathcal{M} \setminus \mathcal{H}^-$ such that $\mathcal{K}_m \subset \mathcal{K}_{m+1}$ and $\cup_{m \in \mathbb{N}} \mathcal{K}_m = \mathcal{M} \setminus \mathcal{H}^-$. Then, setting $A_{-1} = B_{\mathcal{K}_0, \mathbb{N}}$, $A_m = B_{\mathcal{K}_m, A_{m-1}}$ for $m \in \mathbb{N}$, and defining recursively

$$(2.3.29) \quad A = \cup_{m \in \mathbb{N}} \left\{ \min \left(A_m \setminus \{n : n < m\} \right) \right\},$$

we infer that there exists a pair $(\tilde{\psi}, T\tilde{\psi}) \in H_{loc}^1(\mathcal{M} \setminus \mathcal{H}^-) \times H_{loc}^1(\mathcal{M} \setminus \mathcal{H}^-)$ such that the subsequence $\{\psi_{\tau_n}, T\psi_{\tau_n}\}_{n \in A}$ satisfies (2.3.23) and, for any compact $\mathcal{K} \subset \mathcal{M} \setminus \mathcal{H}^-$ (after permanently renumbering the indices of $\{\psi_{\tau_n}\}_{n \in A}$ through a map $\mathbb{N} \rightarrow A$) :

$$(2.3.30) \quad \lim_{n \rightarrow +\infty} \sum_{j=0}^1 \int_{\mathcal{K}} |T^j \psi_{\tau_n} - T^j \tilde{\psi}|^2 dg = 0.$$

Since the functions ψ_{τ_n} solve (2.0.1) on $\{t > -\tau_n\}$, $\tilde{\psi}$ also solves (2.0.1) on $\mathcal{M} \setminus \mathcal{H}^-$ in the sense of distributions, in view of (2.3.23). Furthermore, in view of (2.3.18), we can bound for any $\tau_* > 0$

$$(2.3.31) \quad \sup_{n \in \mathbb{N}} \left\{ \sum_{j=0}^1 \int_{\max\{-\tau_*, -\tau_n\}}^{\tau_*} \left(\int_{\Sigma_\tau} J_\mu^N(T^j \psi) n^\mu \right) d\tau \right\} < +\infty$$

and, thus, (2.3.21) holds. The identity (2.3.22) follows by letting $n \rightarrow +\infty$ in (2.3.25). Finally, (2.3.24) follows from (2.3.22), (2.3.25) and (2.3.30). $\square$

2.3.4 Finishing the proof

Let us assume, for the sake of contradiction, that any smooth solution φ to (2.0.1) on $\mathcal{D}(\Sigma)$ with compactly supported initial data on Σ satisfies (2.3.1).

Let $\varphi : \mathcal{D}(\Sigma) \rightarrow \mathbb{C}$ be as in the statement of Lemma 2.3, and let us set

$$(2.3.32) \quad \psi = T\varphi.$$

In view of Lemma 2.3, $(\psi, T\psi)|_\Sigma$ is smooth and compactly supported in $\mathcal{U} \cap \Sigma$, and moreover

$$(2.3.33) \quad \int_\Sigma J_\mu^T(\psi) n^\mu = -1.$$

Let $\{\tau_n\}_{n \in \mathbb{N}}$ be the sequence defined by Lemma 2.4, and let $\phi_{\tau_n}, \tilde{\psi} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ be the functions defined by Lemma 2.4.

We will make use of the following identity, appearing also in [66], holding for any acausal, inextendible and piecewise smooth hypersurface $\mathcal{S} \subset \mathcal{M} \setminus \mathcal{H}^-$ such that T is everywhere transversal to $\mathcal{S}$ and any smooth function $\varphi_1 : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ such that $\text{supp}(\varphi_1) \cap \mathcal{S}$ is compact and $\text{supp}(\varphi_1) \cap \mathcal{S} \cap \mathcal{H}^+ = \emptyset$:

$$(2.3.34) \quad \int_\Sigma J_\mu^T(\varphi_1) n_\Sigma^\mu = \int_\Sigma \text{Re}\{n_\Sigma \varphi_1 \cdot T \bar{\varphi}_1 - \varphi_1 \cdot n_\Sigma (T \bar{\varphi}_1)\} dg_\Sigma - \int_\Sigma \text{Re}\{\varphi_1 \square_g \bar{\varphi}_1\} g(n_\Sigma, T) dg_\Sigma,$$

where n_Σ is the future directed unit normal to $\mathcal{S}$.

Proof of (2.3.34). One way to obtain (2.3.34) is the following: Since $\text{supp}(\varphi_1) \cap \mathcal{S}$ is compact and $\text{supp}(\varphi_1) \cap \mathcal{S} \cap \mathcal{H}^+ = \emptyset$, we can assume without loss of generality (by changing φ_1 away from $\mathcal{S}$ if necessary) that φ_1 has compact support in $\mathcal{M} \setminus (\mathcal{H}^+ \cup \mathcal{H}^-)$. Then, integrating the identity

$$(2.3.35) \quad -2 \text{Re}\{T \varphi_1 \square_g \bar{\varphi}_1\} = -\text{Re}\{T \varphi_1 \square_g \bar{\varphi}_1 - \varphi_1 \square_g (T \bar{\varphi}_1) + T(\varphi_1 \square_g \bar{\varphi}_1)\}$$

over $J^-(\mathcal{S})$, we readily obtain:

$$(2.3.36) \quad -2 \int_{J^-(\mathcal{S})} \text{Re}\{T \varphi_1 \square_g \bar{\varphi}_1\} dg = - \int_{J^-(\mathcal{S})} \text{Re}\{T \varphi_1 \square_g \bar{\varphi}_1 - \varphi_1 \square_g (T \bar{\varphi}_1)\} dg + \int_\Sigma \text{Re}\{\varphi_1 \square_g \bar{\varphi}_1\} g(n_\Sigma, T) dg_\Sigma.$$

Using the identities

$$(2.3.37) \quad -2 \int_{J^-(\mathcal{S})} \text{Re}\{T \varphi_1 \square_g \bar{\varphi}_1\} dg = \int_\Sigma J_\mu^T(\varphi) n_\Sigma^\mu$$

and

$$-\int_{J^-(\mathcal{S})} \text{Re}\{T\varphi_1 \square_g \bar{\varphi}_1 - \varphi_1 \square_g (T\bar{\varphi}_1)\} dg = \int_{\mathcal{S}} \text{Re}\{n_{\mathcal{S}}\varphi_1 \cdot T\bar{\varphi}_1 - \varphi_1 \cdot n_{\mathcal{S}}(T\bar{\varphi}_1)\} dg_{\mathcal{S}}$$

(holding because of the assumption that φ_1 has compact support in $\mathcal{M} \setminus (\mathcal{H}^+ \cup \mathcal{H}^-)$), we finally obtain (2.3.34).

We will also introduce the following (indefinite) inner product on the hypersurfaces Σ_{τ} :
For any two functions $\varphi_1, \varphi_2 : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ such that for any $\tau_* > 0$:

$$\sup_{\tau \in [-\tau_*, \tau_*]} \sum_{j=1}^2 \int_{\Sigma_{\tau}} (J_{\mu}^N(\varphi_j) + J_{\mu}^N(T\varphi_j)) n^{\mu} < +\infty$$

and at least one of them has compact support in space (i.e. for any $\tau_* > 0$, its support in $\{-\tau_* \leq t \leq \tau_*\}$ is compact), we will define for any $\tau \in \mathbb{R}$:

$$(2.3.38) \quad \langle \varphi_1, \varphi_2 \rangle_{T, \tau} = \frac{1}{2} \int_{\Sigma_{\tau}} \text{Re}\left\{ (n_{\mathcal{S}}\varphi_1 \cdot T\bar{\varphi}_2 + n_{\mathcal{S}}\varphi_2 \cdot T\bar{\varphi}_1) - (\varphi_1 \cdot n_{\mathcal{S}}(T\bar{\varphi}_2) + \varphi_2 \cdot n_{\mathcal{S}}(T\bar{\varphi}_1)) \right\}.$$

Note that, if both φ_1 and φ_2 solve equation (2.0.1) and at least one of them is supported away from $\mathcal{H}^+$, then for any $\tau_1 \leq \tau_2$ the following identity holds:

$$(2.3.39) \quad \langle \varphi_1, \varphi_2 \rangle_{T, \tau_1} = \langle \varphi_1, \varphi_2 \rangle_{T, \tau_2}.$$

The equality (2.3.39) readily follows after integrating the identity

$$(2.3.40) \quad \frac{1}{2} \text{Re}\left\{ (\square_g \varphi_1 T\bar{\varphi}_2 + \square_g \varphi_2 T\bar{\varphi}_1) - (\varphi_1 \square_g (T\bar{\varphi}_2) + \varphi_2 \square_g (T\bar{\varphi}_1)) \right\} = 0$$

over $\mathcal{R}(\tau_1, \tau_2)$.

Remark. Note that, in the case when φ_1 and φ_2 solve equation (2.0.1) and at least one of them is supported away from $\mathcal{H}^+$, the expression (2.3.38) is the inner product of φ_1, φ_2 associated to the $\int_{\Sigma_{\tau}} J_{\mu}^T(\cdot) n^{\mu}$ “norm”, in view of (2.3.34). Thus, (2.3.39) is a consequence of the conservation of the T -energy flux.

For any $\tau \geq 0$, the T -energy identity for ψ in the region $\mathcal{R}(0, \tau)$ combined with (2.3.33)

yields:

$$(2.3.41) \quad \int_{\Sigma_\tau} J_\mu^T(\psi) n^\mu + \int_{\mathcal{H}^+ \cap \mathcal{R}(0, \tau)} J_\mu^T(\psi) n_{\mathcal{H}^+}^\mu = -1.$$

Since T is causal on $\mathcal{M} \setminus \mathcal{E}$, we can bound for any $\tau \geq 0$ and any $\delta > 0$:

$$(2.3.42) \quad \int_{\Sigma_\tau \setminus \mathcal{E}_\delta} J_\mu^T(\psi) n^\mu + \int_{\mathcal{H}^+ \cap \mathcal{R}(0, \tau)} J_\mu^T(\psi) n_{\mathcal{H}^+}^\mu \geq 0.$$

Therefore, (2.3.41) and (2.3.42) imply that for any $\tau \geq 0$, $\delta > 0$:

$$(2.3.43) \quad \int_{\Sigma_\tau \cap \mathcal{E}_\delta} J_\mu^T(\psi) n^\mu \leq -1.$$

Since the functions ψ_{τ_n} satisfy (2.3.20), from (2.3.43) we obtain for any $\delta > 0$, any $\tau > -\tau_n$, and any $n \in \mathbb{N}$:

$$(2.3.44) \quad \int_{\Sigma_\tau \cap \mathcal{E}_\delta} J_\mu^T(\psi_{\tau_n}) n^\mu \leq -1.$$

Let $\chi : \mathcal{M} \setminus \mathcal{H}^- \rightarrow [0, 1]$ be a smooth function of compact support such that $\chi \equiv 1$ on $\mathcal{R}(-1, 2) \cap \mathcal{E}_{\delta_0}$ for some $0 < \delta_0 < 1$ and $\text{supp}(\chi) \cap \mathcal{H}^+ = \emptyset$. Applying the identity (2.3.34) for the function $\chi \psi_{\tau_n}$, and using the fact that ψ_{τ_n} solves (2.0.1), we obtain for any $n \in \mathbb{N}$ and any $0 < \tau_0 \leq 1$:

$$(2.3.45) \quad \begin{aligned} \int_0^{\tau_0} \left(\int_{\Sigma_s} J_\mu^T(\chi \psi_{\tau_n}) n^\mu \right) ds &= \int_0^{\tau_0} \left(\int_{\Sigma_s} \text{Re} \{ n(\chi \psi_{\tau_n}) T(\chi \bar{\psi}_{\tau_n}) - (\chi \psi_{\tau_n}) n_{\mathcal{S}}(T(\chi \bar{\psi}_{\tau_n})) \} dg_\Sigma \right) ds - \\ &\quad - \int_0^{\tau_0} \left(\int_{\Sigma_s} \text{Re} \{ \chi \psi_{\tau_n} (2 \nabla^\mu \chi \nabla_\mu \bar{\psi}_{\tau_n} + (\square_g \chi) \bar{\psi}_{\tau_n}) \} g(n, T) dg_\Sigma \right) ds. \end{aligned}$$

In view of (2.3.44) and the fact that $\chi \equiv 1$ on $\mathcal{R}(-1, 2) \cap \mathcal{E}_{\delta_0}$, (2.3.45) yields:

$$(2.3.46) \quad \begin{aligned} \int_0^{\tau_0} \left(\int_{\Sigma_s \cap \mathcal{E}_{\delta_0}} \text{Re} \{ n \psi_{\tau_n} T \bar{\psi}_{\tau_n} - \psi_{\tau_n} n_{\mathcal{S}}(T \bar{\psi}_{\tau_n}) \} dg_\Sigma \right) ds &\leq \\ &\leq -\tau_0 + C \sum_{j=0}^1 \int_{\text{supp}(\chi) \setminus \mathcal{E}_{\delta_0}} (|\nabla T^j \psi_{\tau_n}|_{g_{ref}}^2 + |T^j \psi_{\tau_n}|^2) dg. \end{aligned}$$

Let us examine the properties of (2.3.46) as $n \rightarrow +\infty$.

1. In view of (2.3.24) and the fact that $\text{supp}(\chi)$ is compact, the right hand side of (2.3.46) converges to $-\tau_0$ as $n \rightarrow +\infty$.
2. For any compact subset $\mathcal{K} \subset \mathcal{M} \setminus \mathcal{H}^+$ and any pair of sequences $(\varphi_n^{(1)}, \varphi_n^{(2)})_{n \in \mathbb{N}} \in L^2(\mathcal{K}) \times L^2(\mathcal{K})$ such that $\sup_n \|\varphi_n^{(1)}\|_{L^2(\mathcal{K})} < +\infty$, $\varphi_n^{(1)} \rightarrow \varphi^{(1)}$ weakly in $L^2(\mathcal{K})$ and $\varphi_n^{(2)} \rightarrow \varphi^{(2)}$ strongly in $L^2(\mathcal{K})$, one readily obtains that

$$(2.3.47) \quad \lim_{n \rightarrow +\infty} \int_{\mathcal{K}} \varphi_n^{(1)} \varphi_n^{(2)} dg = \int_{\mathcal{K}} \varphi^{(1)} \varphi^{(2)} dg.$$

Therefore, (2.3.18), (2.3.23) and (2.3.24) imply that:

$$(2.3.48) \quad \lim_{n \rightarrow +\infty} \int_0^{\tau_0} \left(\int_{\Sigma_s \cap \mathcal{E}_{\delta_0}^c} \text{Re}\{n\psi_{\tau_n} T\bar{\psi}_{\tau_n} - \psi_{\tau_n} n_{\mathcal{S}}(T\bar{\psi}_{\tau_n})\} dg_{\Sigma} \right) ds = \\ \int_0^{\tau_0} \left(\int_{\Sigma_s \cap \mathcal{E}_{\delta_0}^c} \text{Re}\{n\tilde{\psi} T\bar{\tilde{\psi}} - \tilde{\psi} n_{\mathcal{S}}(T\bar{\tilde{\psi}})\} dg_{\Sigma} \right) ds.$$

Thus, taking the limit $n \rightarrow +\infty$ in (2.3.48), we obtain for any $0 < \tau_0 \leq 1$:

$$(2.3.49) \quad \int_0^{\tau_0} \left(\int_{\Sigma_s \cap \mathcal{E}_{\delta_0}^c} \text{Re}\{n\tilde{\psi} T\bar{\tilde{\psi}} - \tilde{\psi} n_{\mathcal{S}}(T\bar{\tilde{\psi}})\} dg_{\Sigma} \right) ds \leq -\tau_0.$$

According to Lemma 2.4, $\tilde{\psi}$ belongs to $H_{loc}^1(\mathcal{M} \setminus \mathcal{H}^-)$ and vanishes outside $\mathcal{E}_{ext}$, and, thus, Assumption A1 implies that

$$(2.3.50) \quad \tilde{\psi} \equiv 0 \text{ on } \mathcal{U}.$$

Since $(\psi, T\psi)|_{\Sigma}$ is compactly supported in $\mathcal{U} \cap \Sigma$ and $\mathcal{U}$ is open, in view of the finite speed of propagation property of equation (2.0.1), there exists some $0 < \tau_0 \leq 1$ (depending on the support of ψ on $\Sigma \cap \mathcal{U}$), such that for all $0 \leq \bar{\tau} \leq \tau_0$:

$$(2.3.51) \quad (\psi, T\psi) = (0, 0) \text{ on } \Sigma_{\bar{\tau}} \setminus \mathcal{U}.$$

In view of the fact that $\mathcal{U}$ is translation invariant, (2.3.38), (2.3.50) and (2.3.51) imply that for any $\tau \in \mathbb{R}$:

$$(2.3.52) \quad \int_0^{\tau_0} \langle \psi, \mathcal{F}_\tau^* \tilde{\psi} \rangle_{T, \bar{\tau}} d\bar{\tau} = 0$$

(the expression (2.3.52) is well defined, in view of (2.3.21)), where

$$(2.3.53) \quad \mathcal{F}_\tau^* \tilde{\psi}(t, x) \doteq \tilde{\psi}(t + \tau, x).$$

In view of Assumption G3, we have $\mathcal{E}_{ext} \cap \mathcal{H}^+ = \emptyset$. Thus, since $\tilde{\psi}$ vanishes outside $\mathcal{E}_{ext}$, we have $\tilde{\psi} \equiv 0$ on $\mathcal{H}^+$. This fact, combined with (2.3.52) and the identity (2.3.39) (applied to a sequence of smooth approximations of $\tilde{\psi}$ in the norm defined by (2.3.21)) yields for any $s, \tau \in \mathbb{R}$:

$$(2.3.54) \quad \int_s^{s+\tau_0} \langle \psi, \mathcal{F}_\tau^* \tilde{\psi} \rangle_{T, \bar{\tau}} d\bar{\tau} = 0.$$

In view of the definitions (2.3.20) and (2.3.53), the identity (2.3.54) for $s = \tau_n$ and $\tau = -s$ yields:

$$(2.3.55) \quad \int_0^{\tau_0} \langle \psi_{\tau_n}, \tilde{\psi} \rangle_{T, \bar{\tau}} d\bar{\tau} = 0.$$

Thus, since $\tilde{\psi}$ is supported in $\mathcal{E}_{ext}$ and $\mathcal{R}(0, 1) \cap \mathcal{E}_{ext}$ is compact, (2.3.23) implies, after letting $n \rightarrow +\infty$ in (2.3.55):

$$(2.3.56) \quad \int_0^{\tau_0} \langle \tilde{\psi}, \tilde{\psi} \rangle_{T, \bar{\tau}} d\bar{\tau} = 0$$

or, in view of (2.3.38):

$$(2.3.57) \quad \int_0^{\tau_0} \left(\int_{\Sigma_s} \operatorname{Re} \{ n \tilde{\psi} T \tilde{\psi} - \tilde{\psi} n_S (T \tilde{\psi}) \} dg_{\Sigma_s} \right) ds = 0.$$

The contradiction now follows after comparing (2.3.57) with (2.3.49) (using also the fact that $\tilde{\psi}$ is supported in $\mathcal{E}_{ext}$). Thus, the proof of Theorem 2.1 is complete. $\square$

2.4 Frequency decomposition

As we remarked in Section 2.3, Sections 2.4–2.5 will be devoted to the development of the technical machinery required for the proof of Proposition 2.1. In particular, in this section, we will assume that we are given a smooth function $\psi : \mathcal{M} \rightarrow \mathbb{C}$ solving the wave equation (2.0.1) on $\mathcal{D}(\Sigma)$ (i.e. the domain of dependence of Σ) with compactly supported initial data on Σ , such that

$$(2.4.1) \quad \mathcal{E}[\psi] \doteq \sup_{\tau \geq 0} \int_{\Sigma_\tau} J^N(\psi) n^\mu < +\infty.$$

We will also introduce the frequency parameters $\omega_+ > 1$ and $0 < \omega_0 < 1$, and we will decompose the function ψ into components with localised frequency support (associated to the t variable). We will always identify $\mathcal{M} \setminus \mathcal{H}^-$ with $\mathbb{R} \times \Sigma$ under the flow of T as explained in Section 2.2.3. The constructions in this section will be similar to the associated constructions in Section 4 of [107].

2.4.1 Weighted energy estimates for ψ

Before proceeding to cut off ψ in the frequency space, we will first derive a few bounds for some suitable weighted energies of ψ .

In view of the finite speed of propagation for solutions to (2.0.1) and the fact that $(\psi, T\psi)|_{\Sigma_0}$ is compactly supported, we infer that $(\psi, T\psi)|_{\Sigma_\tau}$ is also compactly supported for any $\tau \geq 0$. The following lemma is a straightforward application of the finite speed of propagation property of equation (2.0.1):

Lemma 2.5. *For any $a > 0$, any $R \gg 1$ (so that T is timelike in $\{r \geq R\}$), any $\tau_1 \geq 0$ and any $\tau \in \mathbb{R}$:*

$$(2.4.2) \quad \int_{\Sigma_\tau \cap \mathcal{D}(\Sigma_{\tau_1} \cap \{r \geq R\})} (\log(r))^a J_\mu^T(\psi) n^\mu \leq C_a (\log(2 + |\tau - \tau_1|))^{a+1} \int_{\Sigma_{\tau_1} \cap \{r \geq R\}} (\log(r))^a J_\mu^T(\psi) n^\mu$$

and

$$(2.4.3) \quad \int_{\Sigma_{\tau} \cap \mathcal{D}(\Sigma_{\tau_1} \cap \{r \geq R\})} r^a J_{\mu}^T(\psi) n^{\mu} \leq C_a (1 + |\tau - \tau_1|)^a \int_{\Sigma_{\tau_1} \cap \{r \geq R\}} r^a J_{\mu}^T(\psi) n^{\mu},$$

where $\mathcal{D}(\Sigma_{\tau_1} \cap \{r \geq R\}) \subset \{r \geq R\}$ is the domain of dependence of $\Sigma_{\tau} \cap \{r \geq R\}$ and $C_a > 0$ depends only on a and the geometry of $(\mathcal{M}, g)$.

Proof. Let us define for any $k \geq 1$ the sets

$$(2.4.4) \quad \mathcal{A}_k = \{2^k \leq r \leq 2^{k+1}\} \subset \mathcal{M},$$

and let us set

$$(2.4.5) \quad \mathcal{A}_0 = \{r \leq 1\}.$$

Then, in view of the asymptotics (2.1.1) of g in each connected component of the asymptotically flat region $\mathcal{I}_{as}$, there exists a constant $C > 0$ depending on the geometry of $(\mathcal{M}, g)$ such that for any $\tau_1 \geq 0$, $\tau \in \mathbb{R}$ and any $k \in \mathbb{N}$:

$$(2.4.6) \quad \Sigma_{\tau_1} \cap \left(J^+(\mathcal{A}_k \cap \Sigma_{\tau}) \cup J^-(\mathcal{A}_k \cap \Sigma_{\tau}) \right) \subset \bigcup_{n=\max\{0, k - \log_2(|\tau - \tau_1| + 1) - C\}}^{k + \log_2(|\tau - \tau_1| + 1) + C} \mathcal{A}_n \cap \Sigma_{\tau_1}.$$

Applying for any $k \in \mathbb{N}$ the conservation of the T -energy flux in the spacetime region $J^-(\mathcal{A}_k \cap \Sigma_{\tau}) \cap \mathcal{D}^+(\Sigma_{\tau_1} \cap \{r \geq R\})$, in the case $\tau \geq \tau_1$, or the region $J^+(\mathcal{A}_k \cap \Sigma_{\tau}) \cap \mathcal{D}^-(\Sigma_{\tau_1} \cap \{r \geq R\})$,¹⁰ in the case $\tau \leq \tau_1$, we readily obtain in view of (2.4.6) (using also the fact that T is timelike for $r \geq R$):

$$(2.4.7) \quad \int_{\mathcal{A}_k \cap \Sigma_{\tau} \cap \mathcal{D}(\Sigma_{\tau_1} \cap \{r \geq R\})} J_{\mu}^T(\psi) n^{\mu} \leq \sum_{n=\max\{0, k - \log_2(|\tau - \tau_1| + 1) - C\}}^{k + \log_2(|\tau - \tau_1| + 1) + C} \int_{\mathcal{A}_n \cap \Sigma_{\tau_1} \cap \{r \geq R\}} J_{\mu}^T(\psi) n^{\mu}.$$

¹⁰Here, $\mathcal{D}^+(B)$ is the future domain of dependence of the set $B \subset \mathcal{M}$, while $\mathcal{D}^-(B)$ is the past domain of dependence

Multiplying (2.4.7) with k^a and summing over $k \in \mathbb{N}$, we obtain:

(2.4.8)

$$\begin{aligned}
& \sum_{k=0}^{\infty} k^a \int_{\mathcal{A}_k \cap \Sigma_{\tau} \cap \mathcal{D}(\Sigma_{\tau_1} \cap \{r \geq R\})} J_{\mu}^T(\psi) n^{\mu} \\
& \leq \sum_{k=0}^{\infty} \left(k^a \sum_{n=\max\{0, k - \log_2(|\tau - \tau_1| + 1) - C\}}^{k + \log_2(|\tau - \tau_1| + 1) + C} \int_{\mathcal{A}_n \cap \Sigma_{\tau_1} \cap \{r \geq R\}} J_{\mu}^T(\psi) n^{\mu} \right) \leq \\
& \leq C \sum_{k=0}^{\infty} \left(\left(\sum_{j=k}^{k + \log_2(|\tau - \tau_1| + 1) + C} j^a \right) \int_{\mathcal{A}_k \cap \Sigma_{\tau_1} \cap \{r \geq R\}} J_{\mu}^T(\psi) n^{\mu} \right) \leq \\
& \leq C_a \sum_{k=0}^{\infty} \left((k + \log_2(|\tau - \tau_1| + 1) + C)^{a+1} \int_{\mathcal{A}_k \cap \Sigma_{\tau_1} \cap \{r \geq R\}} J_{\mu}^T(\psi) n^{\mu} \right).
\end{aligned}$$

Inequality (2.4.2) follows readily from (2.4.8). Inequality (2.4.3) follows in the same way, after multiplying (2.4.7) with 2^{ka} and summing over $k \in \mathbb{N}$. $\square$

In view of (2.4.1) and the conservation of the T -energy flux in the region $\{t_- \geq 0\} \cap \{t \leq 0\}$, we can bound:

$$(2.4.9) \quad \sup_{\tau \in \mathbb{R}} \int_{\Sigma_{\tau} \cap \{t_- \geq 0\}} J_{\mu}^N(\psi) n^{\mu} \leq \mathcal{E}[\psi]$$

(note that $\Sigma_{\tau} \cap \{t_- \geq 0\} = \Sigma_{\tau}$ when $\tau \geq 0$). Furthermore, in view of (2.4.2) for $\tau_1 = 0$ and the Hardy inequality (2.2.13), we can estimate:

$$(2.4.10) \quad \sup_{\tau \leq 0} \left((\log(2 + |\tau|))^{-3} \int_{\Sigma_{\tau} \cap \{t_- > 0\}} (1 + r)^{-2} |\psi|^2 \right) \leq C \int_{\Sigma_0} (\log(2 + r))^3 J_{\mu}^N(\psi) n^{\mu}.$$

2.4.2 Frequency cut-off

Let us fix a constant $R_1 \gg 1$ large in terms of the geometry of $(\mathcal{M}, g)$, as well as a smooth cut-off function $\chi_1 : [0, +\infty) \rightarrow [0, 1]$ satisfying $\chi_1(r) = 0$ for $r \leq R_1$ and $\chi_1(r) = 1$ for $r \geq R_1 + 1$. As in Section 4 of [107], we will define the following distorted time function on $\mathcal{M} \setminus \mathcal{H}^-$:

$$(2.4.11) \quad t_- = t + \frac{1}{2} \chi_1(r) (r - R_1).$$

Note that $\{t = 0\} \subset J^+(\{t_- = 0\})$.

We will also fix another smooth cut-off function $\chi_2 : \mathbb{R} \rightarrow [0, 1]$, satisfying $\chi_2 \equiv 0$ on $(-\infty, 0]$ and $\chi_2 \equiv 1$ on $[1, +\infty)$, and we will define the function $\psi_c : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ as

$$(2.4.12) \quad \psi_c \doteq \begin{cases} \chi_2(t_-) \cdot \psi, & t_- \geq 0, \\ 0, & t_- \leq 0. \end{cases}$$

Since ψ solves (2.0.1), ψ_c solves

$$(2.4.13) \quad \square_g \psi_c = F,$$

where

$$(2.4.14) \quad F = 2\partial^\mu \chi_2(t_-) \cdot \partial_\mu \psi + \square_g \chi_2(t_-) \cdot \psi$$

is supported in $\{0 \leq t_- \leq 1\}$.

Noting that $r \gtrsim |\tau|$ on $\{t = \tau\} \cap \{0 \leq t_- \leq 1\}$ for $\tau \leq 0$, combining (2.4.9) and (2.4.10) (in each asymptotically flat end of Σ_τ) with the Hardy-type inequality (obtained after averaging (2.2.13) over R_2 , using also a Poincare-type inequality in the near region $\{r \lesssim 1\}$):

$$(2.4.15) \quad \begin{aligned} & \int_{\Sigma_\tau \setminus \mathcal{D}(\Sigma_0 \cap \{r \geq R\})} (1+r)^{-2} |\psi_c|^2 \\ & \leq C \int_{\Sigma_\tau \setminus \mathcal{D}(\Sigma_0 \cap \{r \geq 2R\})} (\log(2+r))^2 J_\mu^N(\psi_c) n^\mu \\ & \quad + C \int_{\Sigma_\tau \cap \mathcal{D}(\Sigma_0 \cap \{R \leq r \leq 2R\})} (1+r)^{-2} \log(r) |\psi_c|^2 \\ & \leq C (\log(2+|\tau|))^2 \int_{\Sigma_\tau} J_\mu^N(\psi_c) n^\mu + C \log(2+|\tau|) \int_{\Sigma_\tau \cap \mathcal{D}(\Sigma_0 \cap \{r \geq R\})} (1+r)^{-2} |\psi_c|^2, \end{aligned}$$

we obtain in view of (2.4.12):

$$(2.4.16) \quad \sup_{\tau \geq 0} \int_{\Sigma_\tau} J^N(\psi_c) n^\mu + \sup_{\tau \leq 0} \left(|\tau|^{-2} (\log(2 + |\tau|))^{-4} \int_{\Sigma_\tau} J_\mu^N(\psi_c) n^\mu \right) + \\ + \sup_{\tau \in \mathbb{R}} \left((\log(2 + |\tau|))^{-4} \int_{\Sigma_\tau} (1 + r)^{-2} |\psi_c|^2 \right) \leq C \mathcal{E}_{\log}[\psi],$$

where

$$(2.4.17) \quad \mathcal{E}_{\log}[\psi] \doteq \mathcal{E}[\psi] + \int_{\Sigma_0} (\log(2 + r))^3 J_\mu^N(\psi) n^\mu.$$

Remark. In dimensions $d \geq 3$, inequality (2.4.16), as well as most of the estimates of this section, holds without the logarithmic loss (since (2.4.10) holds without a logarithmic loss in this case).

We will now proceed to perform a cut-off procedure on ψ_c in the frequency domain. Let $0 < \omega_0 < 1$ be a (small) positive constant, and $\omega_+ \gg \omega_0$ a (large) positive constant, and let us set

$$(2.4.18) \quad n = \lceil \log_2 \frac{\omega_+}{\omega_0} \rceil$$

and, for any integer $1 \leq k \leq n$:

$$(2.4.19) \quad \omega_k = 2^k \omega_0.$$

Fixing a third smooth cut-off function $\chi_3 : \mathbb{R} \rightarrow [0, 1]$ satisfying $\chi_3 \equiv 1$ on $[-1, 1]$ and $\chi_3 \equiv 0$ on $(-\infty, -2] \cup [2, +\infty)$, we will define the following Schwartz functions on $\mathbb{R}$:

$$(2.4.20) \quad \zeta_0(t) = \int_{-\infty}^{+\infty} e^{i\omega t} \chi_3(\omega_0^{-1} \omega) d\omega, \\ \zeta_k(t) = \int_{-\infty}^{+\infty} e^{i\omega t} (\chi_3(\omega_k^{-1} \omega) - \chi_3(\omega_{k-1}^{-1} \omega)) d\omega, \text{ for } 1 \leq k \leq n \\ \zeta_{\leq \omega_+}(t) = \sum_{k=0}^n \zeta_k(t).$$

Notice that the Fourier transform of ζ_k is supported in $\{\omega_{k-1} \leq |\omega| \leq 2\omega_k\}$ (setting $\omega_{-1} = 0$), while the frequency support of $\zeta_{\leq \omega_+}$ is contained in $\{|\omega| \leq 4\omega_+\}$. Furthermore, the following Schwartz bounds hold for any integers $m, m' \in \mathbb{N}$ and $0 \leq k \leq n$:

$$(2.4.21) \quad \sup_{t \in \mathbb{R}} |\omega_k^{-1-m'} (1 + |\omega_k t|^m) \left(\frac{d}{dt}\right)^{m'} \zeta_k(t)| \leq C_m$$

and

$$(2.4.22) \quad \sup_{t \in \mathbb{R}} |\omega_+^{-1-m'} (1 + |\omega_+ t|^m) \left(\frac{d}{dt}\right)^{m'} \zeta_{\leq \omega_+}(t)| \leq C_m.$$

Using $\zeta_k, \zeta_{\leq \omega_+}$, we will define, for $0 \leq k \leq n$, the “frequency decomposed” components $\psi_k, \psi_{\leq \omega_+}, \psi_{\geq \omega_+} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ of ψ through the following relations (identifying $\mathcal{M} \setminus \mathcal{H}^-$ with $\mathbb{R} \times \Sigma$ through the flow of T):

$$(2.4.23) \quad \psi_k(t, \cdot) = \int_{-\infty}^{+\infty} \zeta_k(t-s) \psi_c(s, \cdot) ds,$$

$$(2.4.24) \quad \psi_{\leq \omega_+}(t, \cdot) = \int_{-\infty}^{+\infty} \zeta_{\leq \omega_+}(t-s) \psi_c(s, \cdot) ds$$

and

$$(2.4.25) \quad \psi_{\geq \omega_+}(t, \cdot) = \psi_c(t, \cdot) - \psi_{\leq \omega_+}(t, \cdot).$$

Note that the integrals (2.4.23) and (2.4.24) do not necessarily converge pointwise for all $(t, x) \in \mathbb{R} \times \Sigma$, since the bound (2.4.1) does not suffice to exclude the pointwise exponential growth of ψ in the t variable. Instead, in view of (2.4.16), (2.4.21) and (2.4.22), the restrictions $(\psi_k, T\psi_k)|_{\Sigma_\tau}$, $(\psi_{\leq \omega_+}, T\psi_{\leq \omega_+})|_{\Sigma_\tau}$ and $(\psi_{\geq \omega_+}, T\psi_{\geq \omega_+})|_{\Sigma_\tau}$ are only defined as finite energy functions on Σ_τ for any $\tau \in \mathbb{R}$, satisfying the following bound for any $a > 0$ (derived from

(2.4.16), (2.4.21), (2.4.22) and Young's inequality):

(2.4.26)

$$\begin{aligned} & \sup_{\tau \geq 0} \left((1 + \omega_k^{-2-a})^{-1} \int_{\Sigma} (|N\psi_k(\tau, x)|^2 + |\nabla_{g_{\Sigma}} \psi_k(\tau, x)|_{g_{\Sigma}}^2) dg_{\Sigma} \right) + \\ & + \sup_{\tau \leq 0} \left((1 + \omega_k^{-2-a})^{-1} |\tau|^{-2} (\log(2 + |\tau|))^{-4} \int_{\Sigma} (|N\psi_k(\tau, x)|^2 + |\nabla_{g_{\Sigma}} \psi_k(\tau, x)|_{g_{\Sigma}}^2) dg_{\Sigma} \right) + \\ & + \sup_{\tau \in \mathbb{R}} \left((1 + \omega_k^{-a})^{-1} (\log(2 + |\tau|))^{-4} \int_{\Sigma} (1 + r)^{-2} |\psi_k(\tau, x)|^2 dg_{\Sigma} \right) \leq C_a \mathcal{E}_{log}[\psi], \end{aligned}$$

$$\begin{aligned} (2.4.27) \quad & \sup_{\tau \geq 0} \int_{\Sigma} (|N\psi_{\leq \omega_+}(\tau, x)|^2 + |\nabla_{g_{\Sigma}} \psi_{\leq \omega_+}(\tau, x)|_{g_{\Sigma}}^2) dg_{\Sigma} + \\ & + \sup_{\tau \leq 0} \left(|\tau|^{-2} (\log(2 + |\tau|))^{-4} \int_{\Sigma} (|N\psi_{\leq \omega_+}(\tau, x)|^2 + |\nabla_{g_{\Sigma}} \psi_{\leq \omega_+}(\tau, x)|_{g_{\Sigma}}^2) dg_{\Sigma} \right) + \\ & + \sup_{\tau \in \mathbb{R}} \left((\log(2 + |\tau|))^{-4} \int_{\Sigma} (1 + r)^{-2} |\psi_{\leq \omega_+}(\tau, x)|^2 dg_{\Sigma} \right) \leq C_a \mathcal{E}_{log}[\psi] \end{aligned}$$

and

$$\begin{aligned} (2.4.28) \quad & \sup_{\tau \geq 0} \int_{\Sigma} (|N\psi_{\geq \omega_+}(\tau, x)|^2 + |\nabla_{g_{\Sigma}} \psi_{\geq \omega_+}(\tau, x)|_{g_{\Sigma}}^2) dg_{\Sigma} + \\ & + \sup_{\tau \leq 0} \left(|\tau|^{-2} (\log(2 + |\tau|))^{-4} \int_{\Sigma} (|N\psi_{\geq \omega_+}(\tau, x)|^2 + |\nabla_{g_{\Sigma}} \psi_{\geq \omega_+}(\tau, x)|_{g_{\Sigma}}^2) dg_{\Sigma} \right) + \\ & + \sup_{\tau \in \mathbb{R}} \left((\log(2 + |\tau|))^{-4} \int_{\Sigma} (1 + r)^{-2} |\psi_{\geq \omega_+}(\tau, x)|^2 dg_{\Sigma} \right) \leq C_a \mathcal{E}_{log}[\psi]. \end{aligned}$$

Defining, similarly F_k , $F_{\leq \omega_+}$ and $F_{\geq \omega_+}$ in terms of F as in (2.4.23)–(2.4.25) (replacing ψ_c with F), in view of (2.4.13) we obtain the following relations (for any $0 \leq k \leq n$):

$$(2.4.29) \quad \square_g \psi_k = F_k,$$

$$(2.4.30) \quad \square_g \psi_{\leq \omega_+} = F_{\leq \omega_+}$$

and

$$(2.4.31) \quad \square_g \psi_{\geq \omega_+} = F_{\geq \omega_+}.$$

2.4.3 Bounds for the frequency-decomposed components

In this section, we will establish some useful estimates for the energy of $\psi_k, \psi_{\leq \omega_+}, \psi_{\geq \omega_+}$, as well as for the “error” terms $F_k, F_{\leq \omega_+}, F_{\geq \omega_+}$, in terms of $\mathcal{E}_{log}[\psi]$.

We start with an estimate for weighted spacetime norms of the terms $F_k, F_{\leq \omega_+}, F_{\geq \omega_+}$.

Lemma 2.6. *We can bound for any $0 \leq k \leq n$, any $q, q' \in \mathbb{N}$ and any $0 \leq \tau_1 \leq \tau_2$:*

$$(2.4.32) \quad \int_{\mathcal{R}(\tau_1, \tau_2)} r^q |F_k|^2 \leq C_{qq'} (1 + \omega_k^{-q-2}) \cdot (1 + \omega_k \tau_1)^{-q'} \mathcal{E}_{log}[\psi].$$

The same inequality also holds for $F_{\leq \omega_+}, F_{\geq \omega_+}$ in place of F_k (with ω_+ in place of ω_k).

Proof. In view of (2.4.14) and the fact that

$$(2.4.33) \quad F_k(t, \cdot) = \int_{-\infty}^{+\infty} \zeta_k(t-s) F(s, \cdot) dt,$$

we can estimate (denoting with x the space variable in the splitting $\mathcal{M} \setminus \mathcal{H}^- = \mathbb{R} \times \Sigma$)

$$(2.4.34) \quad \begin{aligned} \int_{\mathcal{R}(\tau_1, \tau_2)} r^q |F_k|^2 &\leq C \int_{\Sigma} \int_{\tau_1}^{\tau_2} r^q \left| \int_{-\infty}^{\infty} \zeta_k(t-s) F(s, x) ds \right|^2 dt dg_{\Sigma} \\ &\leq C \left(\int_{\Sigma} r^q \int_{\tau_1}^{\tau_2} \left| \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \zeta_k(t-s) F(s, x) ds \right|^2 dt dg_{\Sigma} \right). \end{aligned}$$

From the Schwartz bound (2.4.21) for $m = q' + q + 5$ and $m' = 0$ (using also (2.4.14)), we can estimate:

$$(2.4.35) \quad \begin{aligned} \int_{\Sigma} r^q \int_{\tau_1}^{\tau_2} \left| \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \zeta_k(t-s) F(s, x) ds \right|^2 dt dg_{\Sigma} &\leq \\ &\leq C_{qq'} \int_{\Sigma} r^q \int_{\tau_1}^{\tau_2} \left| \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{\omega_k}{(1 + \omega_k |t-s|)^{q'+q+5}} F(s, x) ds \right|^2 dt dg_{\Sigma} \leq \end{aligned}$$

$$\begin{aligned}
&\leq C_{qq'} \int_{\Sigma} r^q \int_{\tau_1}^{\tau_2} \left(\int_{-\infty}^{+\infty} \frac{\omega_k}{(1 + \omega_k |\lambda|)^{q'+q+5}} d\lambda \right) \times \\
&\quad \times \left(\int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{\omega_k}{(1 + \omega_k |t-s|)^{q'+q+5}} |F(s, x)|^2 ds \right) dt dg_{\Sigma} \leq \\
&\leq C_{qq'} \int_{\Sigma} r^q \int_{\tau_1}^{\tau_2} \left(\int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{\omega_k}{(1 + \omega_k |t-s|)^{q'+q+5}} |F(s, x)|^2 ds \right) dt dg_{\Sigma} \leq \\
&\leq C_{qq'} \int_{\Sigma} r^q \left(\int_{\tau_1}^{\tau_2} \frac{\omega_k}{(1 + \omega_k |\tau + \frac{1}{2}\chi_1(r)(r-R_1)|)^{q'+q+5}} d\tau \right) \times \\
&\quad \times \left(\int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} |F(s, x)|^2 ds \right) dg_{\Sigma} \leq \\
&\leq C_{qq'} \frac{1 + \omega_k^{-q}}{(1 + \omega_k \tau_1)^{q'}} \int_{\Sigma} (1 + \omega_k r)^{-4} \left(\int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} |F(t, x)|^2 dt \right) dg_{\Sigma} \leq \\
&\leq C_{qq'} \frac{1 + \omega_k^{-q}}{(1 + \omega_k \tau_1)^{q'}} \int_{\Sigma} (1 + \omega_k r)^{-4} \left(\int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} (J_{\mu}^N(\psi) n^{\mu} + |\psi|^2) dt \right) dg_{\Sigma} \leq \\
&\leq C_{qq'} \frac{1 + \omega_k^{-q-2}}{(1 + \omega_k \tau_1)^{q'}} \int_{-\infty}^0 (1 + \tau)^{-2} \left(\int_{\Sigma_{\tau} \cap \{0 \leq t \leq 1\}} (J_{\mu}^N(\psi) n^{\mu} + (1+r)^{-2} |\psi|^2) \right) d\tau
\end{aligned}$$

(for the last inequality, we used the fact that $t \sim r$ on $\{0 \leq t \leq 1\}$). Therefore, from (2.4.34), (2.4.35), (2.4.9) and (2.4.10), we readily obtain (2.4.32).

The estimate for $F_{\leq \omega_+}$ and $F_{\geq \omega_+}$ follows in exactly the same way. $\square$

We will also need the following qualitative decay statement near spacelike infinity for the functions ψ_k , $\psi_{\leq \omega_+}$ and $\psi_{\geq \omega_+}$:

Lemma 2.7. *For any $q \in \mathbb{N}$, any $\tau \geq 0$ and any $0 \leq k \leq n$:*

$$(2.4.36) \quad \limsup_{R \rightarrow +\infty} \left(R^q \int_{\Sigma_{\tau} \cap \{R \leq r \leq R+1\}} (J_{\mu}^N(\psi_k) n^{\mu} + |\psi_k|^2) \right) = 0.$$

The relation (2.4.36) also holds for $\psi_{\leq \omega_+}$, $\psi_{\geq \omega_+}$ in place of ψ_k .

Proof. The proof of Lemma 2.7 is a straightforward consequence of the compact support of $(\psi, T\psi)|_{\Sigma}$ and the Schwartz bounds (2.4.21), (2.4.22).

Let $R_0(\psi)$ be sufficiently large, so that $(\psi, T\psi)|_{\Sigma}$ is supported in $\{r \leq R_0(\psi) - 1\}$. Then, in view of the finite speed of propagation property of equation (2.0.1), there exists a $C > 0$ (depending only on the geometry of $(\mathcal{M}, g)$), so that the function ψ is supported in

$\{r \leq R_0(\psi) + C|t|\} \subset \mathcal{M}$. Thus,

$$(2.4.37) \quad \psi \equiv 0 \text{ on } \{|t| \geq C^{-1}(r - R_0(\psi))\}.$$

Then, in view of (2.4.12), (2.4.23), (2.4.21) and (2.4.37), we can bound for any $\tau \geq 0$, $R > R_0(\psi) + C\tau$ and $0 \leq k \leq n$:

$$(2.4.38) \quad \begin{aligned} & \int_{\Sigma_\tau \cap \{R \leq r \leq R+1\}} (J_\mu^N(\psi_k) n^\mu + |\psi_k|^2) \\ & \leq C \int_{\Sigma \cap \{R \leq r \leq R+1\}} \sum_{j=0}^1 \left| \int_{-\infty}^{+\infty} \zeta_k(\tau - s) \nabla^j \psi_c(s, x) ds \right|_{g_{ref}}^2 dg_\Sigma \leq \\ & \leq C \left(\int_{-\infty}^{+\infty} |\zeta_k(s)| ds \right) \sum_{j=0}^1 \int_{\Sigma \cap \{R \leq r \leq R+1\}} \int_{-\infty}^{+\infty} |\zeta_k(\tau - s)| |\nabla^j \psi_c(s, x)|_{g_{ref}}^2 ds dg_\Sigma \leq \\ & \leq C_q \sum_{j=0}^1 \int_{\Sigma \cap \{R \leq r \leq R+1\}} \int_{-\infty}^{+\infty} \frac{\omega_k}{1 + (\omega_k |\tau - s|)^{q+4}} |\nabla^j \psi_c(s, x)|_{g_{ref}}^2 ds dg_\Sigma \leq \\ & \leq C_q \sum_{j=0}^1 \int_{\Sigma \cap \{R \leq r \leq R+1\}} \int_{C^{-1}(R - R_0(\psi))}^{+\infty} \frac{\omega_k}{1 + (\omega_k |\tau - s|)^{q+4}} |\nabla^j \psi(s, x)|_{g_{ref}}^2 ds dg_\Sigma. \end{aligned}$$

In view of the bounds (2.4.9) and (2.4.10), inequality (2.4.38) yields:

$$(2.4.39) \quad \begin{aligned} & \int_{\Sigma_\tau \cap \{R \leq r \leq R+1\}} (J_\mu^N(\psi_k) n^\mu + |\psi_k|^2) \\ & \leq C_q R^2 \mathcal{E}_{log}[\psi] \int_{\Sigma \cap \{R \leq r \leq R+1\}} \int_{C^{-1}(R - R_0(\psi))}^{+\infty} \frac{\omega_k}{1 + (\omega_k |\tau - s|)^{q+4}} (\log(2 + |s|))^3 ds dg_\Sigma \leq \\ & \leq C_q \frac{\omega_k^{-1} R^2}{1 + (\omega_k (C^{-1}(R - R_0(\psi)) - \tau))^{q+2}} \mathcal{E}_{log}[\psi]. \end{aligned}$$

Thus, (2.4.36) readily follows from (2.4.39).

The relation (2.4.36) for $\psi_{\leq \omega_+}$, $\psi_{\geq \omega_+}$ in place of ψ_k follows in exactly the same way, using (2.4.22) in place of (2.4.21). $\square$

We will now proceed to obtain local in time estimates of the form $\int_{-\infty}^{\infty} |\partial_t \psi_k|^2 dt \sim \omega_k^2 \int_{-\infty}^{\infty} |\psi_k|^2 dt$. Let us define the following Schwartz functions on $\mathbb{R}$, similar to (2.4.20):

$$\begin{aligned}
(2.4.40) \quad \xi_0(t) &= \int_{-\infty}^{+\infty} e^{i\omega t} \chi_3\left(\frac{1}{2}\omega_0^{-1}\omega\right) d\omega, \\
\xi_k(t) &= \int_{-\infty}^{+\infty} e^{i\omega t} \left(\chi_3\left(\frac{1}{2}\omega_k^{-1}\omega\right) - \chi_3(2\omega_{k-1}^{-1}\omega)\right) d\omega, \text{ for } 1 \leq k \leq n.
\end{aligned}$$

Notice that, for any $0 \leq k \leq n$ (setting $\omega_{-1} = 0$), $\chi_3(\frac{1}{2}\omega_k^{-1}\omega) - \chi_3(2\omega_{k-1}^{-1}\omega) = 1$ for all $\omega \in \mathbb{R}$ such that $\chi_3(\omega_k^{-1}\omega) - \chi_3(\omega_{k-1}^{-1}\omega) \neq 0$, and thus:

$$(2.4.41) \quad \hat{\xi}_k = \hat{\xi}_k \cdot \hat{\xi}_k,$$

where $\hat{\cdot}$ denotes the Fourier transform operator on $\mathbb{R}$. Moreover, the following Schwartz bound holds for any integers $m, m' \in \mathbb{N}$ and $0 \leq k \leq n$:

$$(2.4.42) \quad \sup_{t \in \mathbb{R}} \left| \omega_k^{-1-m'} (1 + |\omega_k t|^m) \left(\frac{d}{dt}\right)^{m'} \xi_k(t) \right| \leq C_{m,m'}.$$

The relation (2.4.41), as well as the definition (2.4.20), implies for any $0 \leq k \leq n$ the following self reproducing formula for ψ_k :

$$(2.4.43) \quad \psi_k(t, \cdot) = \int_{-\infty}^{\infty} \xi_k(t-s) \cdot \psi_k(s, \cdot) ds,$$

where, again, the integral in the right hand side of (2.4.43) converges with respect to the $\int_{\Sigma_t} J_{\mu}^N(\cdot) n^{\mu}$ norm (in view of (2.4.26), (2.4.42) and Young's inequality).

For any $1 \leq k \leq n$, we will also introduce the anti-derivatives of ξ_k , defined as

$$(2.4.44) \quad \tilde{\xi}_k(t) = \int_{-\infty}^{+\infty} \frac{1}{i\omega} e^{i\omega t} \left(\chi_3\left(\frac{1}{2}\omega_k^{-1}\omega\right) - \chi_3(2\omega_{k-1}^{-1}\omega)\right) d\omega,$$

thus satisfying for any $m \in \mathbb{N}$ the Schwartz bound

$$(2.4.45) \quad \sup_{t \in \mathbb{R}} \left| (1 + |\omega_k t|^m) \tilde{\xi}_k(t) \right| \leq C_m,$$

as well as the frequency-domain identity:

$$(2.4.46) \quad \hat{\zeta}_k = \hat{\tilde{\zeta}}_k \cdot i\omega \hat{\zeta}_k.$$

In view of (2.4.46), as well as the definition (2.4.20), we obtain for any $1 \leq k \leq n$:

$$(2.4.47) \quad \psi_k(t, \cdot) = \int_{-\infty}^{\infty} \tilde{\zeta}_k(t-s) \cdot T\psi_k(s, \cdot) ds,$$

where the integral in the right hand side of (2.4.47) converges with respect to the $\int_{\Sigma_t} J_{\mu}^N(\cdot) n^{\mu}$ norm.

We can now establish the following lemma:

Lemma 2.8. *For any $1 \leq k \leq n$, any $0 \leq \tau_1 \leq \tau_2$, any T -invariant L_{loc}^{∞} function $\chi: \mathcal{M} \setminus \mathcal{H}^{-} \rightarrow [0, +\infty)$, any $R \geq 0$ and any $0 < a < 1$, we can bound*

$$(2.4.48) \quad \begin{aligned} c\omega_k^2 \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 - C_a \omega_k^2 (1 + \omega_k^{-5-a}) \sup_{\{r \leq R\}} \chi \cdot \mathcal{E}_{log}[\psi] &\leq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |T\psi_k|^2 \leq \\ &\leq C\omega_k^2 \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 + C_a \omega_k^2 (1 + \omega_k^{-1-a}) (\log(2 + \tau_2))^4 R^2 \sup_{\{r \leq R\}} \chi \cdot \mathcal{E}_{log}[\psi], \end{aligned}$$

and similarly for $k = 0$:

$$(2.4.49) \quad \begin{aligned} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |T\psi_0|^2 &\leq \\ &\leq C\omega_0^2 \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 + C_a \omega_0^2 (1 + \omega_0^{-1-a}) (\log(2 + \tau_2))^4 R^2 \sup_{\{r \leq R\}} \chi \cdot \mathcal{E}_{log}[\psi]. \end{aligned}$$

Remark: Notice that the constant multiplying the error term in the right hand side of (2.4.48) depends on R and τ_2 , while this is not the case in the left hand side.

Proof. For any $0 \leq k \leq n$, from (2.4.43) and (2.4.42) (for $m = 5$, $m' = 1$) we can estimate for

any $\tau \geq 0$:

$$\begin{aligned}
(2.4.50) \quad & \int_{\Sigma \cap \{r \leq R\}} \chi(x) |T\psi_k(\tau, x)|^2 dg_\Sigma = \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left| \int_{-\infty}^{+\infty} \xi'_k(\tau - s) \psi_k(s, x) ds \right|^2 dg_\Sigma \leq \\
& \leq C \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left| \int_{-\infty}^{+\infty} \frac{\omega_k^2}{1 + |\omega_k(\tau - s)|^5} \psi_k(s, x) ds \right|^2 dg_\Sigma \leq \\
& \leq C \omega_k^2 \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} ds \right) \times \\
& \quad \times \left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} |\psi_k(s, x)|^2 ds \right) dg_\Sigma \leq \\
& \leq C \omega_k^2 \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma.
\end{aligned}$$

Thus, integrating (2.4.50) over $\{\tau_1 \leq \tau \leq \tau_2\}$ we obtain:

$$\begin{aligned}
(2.4.51) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |T\psi_k|^2 \leq \\
& \leq C \omega_k^2 \int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma d\tau \leq \\
& \leq C \omega_k^2 \left(\int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma d\tau + \right. \\
& \quad \left. + \int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma d\tau \right) \leq \\
& \leq C \omega_k^2 \left(\left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k \lambda|^5} d\lambda \right) \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 + \right. \\
& \quad + \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{\tau_1} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} d\tau \right) \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma + \\
& \quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_2}^{+\infty} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} d\tau \right) \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma \right) \leq \\
& \leq C \omega_k^2 \left(\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 + \right. \\
& \quad \left. + \sup_{r \leq R} \chi \cdot \int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{1}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} |\psi_k(s, x)|^2 ds dg_\Sigma \right).
\end{aligned}$$

From (2.4.26) we can readily estimate for any $0 < a < 1$:

$$\begin{aligned}
(2.4.52) \quad & \int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{1}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} |\psi_k(s, x)|^2 ds dg_\Sigma \leq \\
& \leq C_a R^2 \left(\int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{1}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} (\log((2 + |s|)))^4 ds \right) (1 + \omega_k^{-a}) \mathcal{E}_{log}[\psi] \leq \\
& \leq C_a R^2 (1 + \omega_k^{-1-2a}) (\log(2 + \tau_2))^4 \mathcal{E}_{log}[\psi].
\end{aligned}$$

Thus, from (2.4.51) and (2.4.52) we readily infer the right “half” of inequality (2.4.48), as well as inequality (2.4.49).

In order to establish the left “half” of inequality (2.4.48), we will work similarly, using formula (2.4.47) in place of (2.4.43). In particular, from (2.4.47) and (2.4.45) (for $m = 5$) we obtain for any $\tau \geq 0$ and any $1 \leq k \leq n$:

$$\begin{aligned}
(2.4.53) \quad & \int_{\Sigma \cap \{r \leq R\}} \chi(x) |\psi_k(\tau, x)|^2 dg_\Sigma = \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left| \int_{-\infty}^{+\infty} \tilde{\xi}_k(\tau - s) T\psi_k(s, x) ds \right|^2 dg_\Sigma \leq \\
& \leq C \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left| \int_{-\infty}^{+\infty} \frac{1}{1 + |\omega_k(\tau - s)|^5} T\psi_k(s, x) ds \right|^2 dg_\Sigma \leq \\
& \leq C \omega_k^{-2} \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} ds \right) \times \\
& \quad \times \left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} |T\psi_k(s, x)|^2 ds \right) dg_\Sigma \leq \\
& \leq C \omega_k^{-2} \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma.
\end{aligned}$$

Integrating (2.4.53) over $\{\tau_1 \leq \tau \leq \tau_2\}$, we obtain:

$$\begin{aligned}
(2.4.54) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 \leq \\
& \leq C \omega_k^{-2} \int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma d\tau \leq
\end{aligned}$$

$$\begin{aligned}
&\leq C\omega_k^{-2} \left(\int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma d\tau + \right. \\
&\quad \left. + \int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma d\tau \right) \leq \\
&\leq C\omega_k^{-2} \left(\left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k \lambda|^5} d\lambda \right) \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |T\psi_k|^2 + \right. \\
&\quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{\tau_1} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} d\tau \right) \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma + \right. \\
&\quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_2}^{+\infty} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} d\tau \right) \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma \right) \leq \\
&\leq C\omega_k^{-2} \left(\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 + \right. \\
&\quad \left. + \sup_{r \leq R} \chi \cdot \int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{1}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} |T\psi_k(s, x)|^2 ds dg_\Sigma \right).
\end{aligned}$$

From (2.4.26) we can estimate:

$$\begin{aligned}
(2.4.55) \quad &\int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{1}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} |T\psi_k(s, x)|^2 ds dg_\Sigma \leq \\
&\leq C_a \left(\int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{(1 + \max\{0, -s\})^2 (\log(2 + \max\{0, -s\}))^4}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} ds \right) (1 + \omega_k^{-2-a}) \mathcal{E}_{\log}[\psi] \leq \\
&\leq C_a (1 + \omega_k^{-5-2a}) \mathcal{E}_{\log}[\psi].
\end{aligned}$$

Thus, the left “half” of inequality (2.4.48) follows from (2.4.54) and (2.4.55). $\square$

We will also need the following estimate in the case when ψ is of the form $T\varphi$, where φ is a smooth solution to the wave equation on $\mathcal{D}(\Sigma)$:

Lemma 2.9. *Let ψ be of the form*

$$(2.4.56) \quad \psi = T\varphi,$$

where $\varphi : \mathcal{D}(\Sigma) \rightarrow \mathbb{C}$ is a smooth function solving (2.0.1) with compactly supported initial data on Σ , such that $\mathcal{E}[\varphi] < +\infty$. Then, for any $0 \leq \tau_1 \leq \tau_2$, any $0 < a < 1$ and any $R \geq 0$ we

can bound:

(2.4.57)

$$\begin{aligned} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} (J_\mu^N(\psi_0) N^\mu + |\psi_0|^2) \leq C \omega_0^2 \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} (J_\mu^N(\varphi) N^\mu + |\varphi|^2) + \\ + C_a (\omega_0^2 (1 + \omega_0^{-1-a}) R^2 (\log(2 + |\tau_2|))^4 + (1 + \omega_0 \tau_1)^{-1} R^2) \mathcal{E}_{log}[\varphi]. \end{aligned}$$

Proof. The bounds (2.4.9), (2.4.2) for φ in place of ψ (combined with the Hardy-type inequalities (2.2.13) and (2.4.15)) imply that

$$(2.4.58) \quad \sup_{\tau \in \mathbb{R}} \int_{\Sigma_\tau \cap \{t_- \geq 0\}} J^N(\varphi) n^\mu + \sup_{\tau \in \mathbb{R}} \left((\log(2 + |\tau|))^4 \int_{\Sigma_\tau \cap \{t_- \geq 0\}} (1 + r)^{-2} |\varphi|^2 \right) \leq C \mathcal{E}_{log}[\varphi].$$

From (2.4.56), (2.4.12) and (2.4.23) we calculate:

(2.4.59)

$$\begin{aligned} \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\psi_0(s, x)|^2 d\tau dg_\Sigma &= \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \left| \int_{-\infty}^{+\infty} \zeta_0(t-s) \psi_c(s, x) ds \right|^2 dtdg_\Sigma = \\ &= \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \left| \int_{-\infty}^{+\infty} \zeta_0(t-s) \chi_2(s + \frac{1}{2} \chi_1(r)(r - R_1)) \partial_t \varphi(s, x) ds \right|^2 dtdg_\Sigma = \\ &= \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \left| - \int_{-\infty}^{+\infty} \frac{d}{ds} (\zeta_0(t-s) \chi_2(s + \frac{1}{2} \chi_1(r)(r - R_1))) \cdot \varphi(s, x) ds \right|^2 dtdg_\Sigma, \end{aligned}$$

noting that the integrating by parts in the last step of (2.4.59) is possible in view of the Schwartz bound (2.4.21) on ζ_0 and (2.4.58).

In view of (2.4.21), the relation (2.4.59) yields:

(2.4.60)

$$\begin{aligned} \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\psi_0(s, x)|^2 d\tau dg_\Sigma &\leq \\ &\leq C \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \left(\left| \int_{-\infty}^{+\infty} \frac{\omega_0^2}{1 + |\omega_0(t-s)|^3} \chi_2(s + \frac{1}{2} \chi_1(r)(r - R_1)) \cdot \varphi(s, x) ds \right|^2 + \right. \\ &\quad \left. + \left| \int_{-\infty}^{+\infty} \frac{\omega_0}{1 + |\omega_0(t-s)|^3} \chi_2'(s + \frac{1}{2} \chi_1(r)(r - R_1)) \cdot \varphi(s, x) ds \right|^2 \right) dtdg_\Sigma \leq \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\int_{-\infty}^{+\infty} \frac{\omega_0}{1 + |\omega_0(t-s)|^3} ds \right) \times \\
&\quad \times \left\{ \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{+\infty} \frac{\omega_0}{1 + |\omega_0(t-s)|^3} |\varphi(s, x)|^2 ds dt dg_{\Sigma} + \right. \\
&\quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{\omega_0}{1 + |\omega_0(t-s)|^3} |\varphi(s, x)|^2 ds dt dg_{\Sigma} \right\} \leq \\
&\leq C \left\{ \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{[\tau_1, \tau_2] \cap [-\frac{1}{2}\chi_1(r)(r-R_1), +\infty)} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_0}{1 + |\omega_0(t-s)|^3} dt \right) |\varphi(s, x)|^2 ds dg_{\Sigma} + \right. \\
&\quad + \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{[-\frac{1}{2}\chi_1(r)(r-R_1), +\infty) \setminus [\tau_1, \tau_2]} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_0}{1 + |\omega_0(t-s)|^3} dt \right) |\varphi(s, x)|^2 ds dg_{\Sigma} + \\
&\quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_0}{1 + |\omega_0(t-s)|^3} dt \right) |\varphi(s, x)|^2 ds dg_{\Sigma} \right\} \leq \\
&\leq C \left\{ \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\varphi(s, x)|^2 ds dg_{\Sigma} + \right. \\
&\quad + \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{[-\frac{1}{2}\chi_1(r)(r-R_1), +\infty) \setminus [\tau_1, \tau_2]} \frac{1}{1 + \min\{|\omega_0(\tau_1-s)|^2, |\omega_0(\tau_2-s)|^2\}} \times \\
&\quad \times |\varphi(s, x)|^2 ds dg_{\Sigma} + \\
&\quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{1}{1 + |\omega_0(\tau_1-s)|^2} |\varphi(s, x)|^2 ds dg_{\Sigma} \right\}.
\end{aligned}$$

From (2.4.58), we can estimate for any $0 < a < 1$:

$$\begin{aligned}
(2.4.61) \quad &\omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{[-\frac{1}{2}\chi_1(r)(r-R_1), +\infty) \setminus [\tau_1, \tau_2]} \frac{1}{1 + \min\{|\omega_0(\tau_1-s)|^2, |\omega_0(\tau_2-s)|^2\}} \times \\
&\quad \times |\varphi(s, x)|^2 ds dg_{\Sigma} \leq \\
&\leq C_a \omega_0^2 (1 + \omega_0^{-1-a}) R^2 (\log(2 + |\tau_2|))^4 \mathcal{E}_{\log}[\varphi]
\end{aligned}$$

and

$$\begin{aligned}
(2.4.62) \quad &\int_{\Sigma \cap \{r \leq R\}} \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{1}{1 + |\omega_0(\tau_1-s)|^2} |\varphi(s, x)|^2 ds dg_{\Sigma} \leq \\
&\leq C_a (1 + \omega_0^{-1-a}) (1 + \omega_0 \tau_1)^{-1} R^2 \mathcal{E}_{\log}[\varphi].
\end{aligned}$$

Thus, from (2.4.60) we obtain for any $0 < a < 1$:

$$(2.4.63) \quad \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\psi_0(s, x)|^2 d\tau dg_\Sigma \leq C\omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\varphi(s, x)|^2 ds dg_\Sigma + \\ + C_a \left(\omega_0^2 (\log(2 + |\tau_2|))^4 + (1 + \omega_0 \tau_1)^{-1} \right) (1 + \omega_0^{-1-a}) R^2 \mathcal{E}_{\log}[\varphi].$$

Repeating the same procedure with $T\psi_0$ and $\nabla_{g_\Sigma}\psi_0$ in place of ψ_0 , we similarly obtain:

$$(2.4.64) \quad \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |T\psi_0(s, x)|^2 d\tau dg_\Sigma \leq C\omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |T\varphi(s, x)|^2 ds dg_\Sigma + \\ + C_a \left(\omega_0^2 (\log(2 + |\tau_2|))^4 + (1 + \omega_0 \tau_1)^{-1} \right) (1 + \omega_0^{-1-a}) R^2 \mathcal{E}_{\log}[\varphi]$$

and

$$(2.4.65) \quad \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\nabla_{g_\Sigma}\psi_0(s, x)|_{g_\Sigma}^2 d\tau dg_\Sigma \leq C\omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\nabla_{g_\Sigma}\varphi(s, x)|_{g_\Sigma}^2 ds dg_\Sigma + \\ + C_a \left(\omega_0^2 (\log(2 + |\tau_2|))^4 + (1 + \omega_0 \tau_1)^{-1} \right) (1 + \omega_0^{-1-a}) R^2 \mathcal{E}_{\log}[\varphi].$$

Inequality (2.4.57) readily follows after adding (2.4.63), (2.4.64) and (2.4.65). $\square$

We will finally establish the following bound for the energy of the high frequency part $\psi_{\geq \omega_+}$ of ψ :

Lemma 2.10. *For any $\tau \geq 0$ and any $m \in \mathbb{N}$ such that*

$$(2.4.66) \quad \sum_{j=0}^m \mathcal{E}[T^j \psi] < +\infty,$$

there exists a constant $C_m > 0$ depending only on m such that:

$$(2.4.67) \quad \int_{\{t=\tau\}} J_\mu^N(\psi_{\geq \omega_+}) n^\mu \leq \frac{C_m}{\omega_+^{2m}} \left(\sum_{j=0}^m \mathcal{E}[T^j \psi] + \mathcal{E}_{\log}[\psi] \right).$$

Proof. We can assume without loss of generality that $m \geq 1$, since the $m = 0$ case is a direct

consequence of (2.4.28). Let us introduce the function $\check{\xi}_m : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{C}$ by the formula

$$(2.4.68) \quad \check{\xi}_m(t) = \int_{-\infty}^{+\infty} (i\omega_+^{-1}\omega)^{-m} e^{i\omega t} (1 - \chi(\omega_+^{-1}\omega)) d\omega.$$

Note that, when $m = 1$, the right hand side of (2.4.68) diverges when $t = 0$. In view of the bound

$$(2.4.69) \quad \left| \int_1^{+\infty} \frac{1}{y^m} e^{i\lambda y} dy \right| \leq \begin{cases} C(|\log(\lambda)| + 1), & m = 1 \\ C, & m > 1, \end{cases}$$

as well as the relation

$$(2.4.70) \quad \begin{aligned} t\check{\xi}_m(t) &= i \int_{-\infty}^{+\infty} e^{i\omega t} \frac{d}{d\omega} ((i\omega)^{-m} (1 - \chi(\omega_+^{-1}\omega))) d\omega = \\ &= -m\omega_+^{-1} \check{\xi}_{m+1}(t) - i\omega_+^{-1} \int_{-\infty}^{+\infty} e^{i\omega t} (i\omega_+^{-1}\omega)^{-m} \chi'(\omega_+^{-1}\omega) d\omega, \end{aligned}$$

from (2.4.68) we infer that for any integer $q \in \mathbb{N}$ and any $t \neq 0$:

$$(2.4.71) \quad |\check{\xi}_m(t)| \leq C_{qm} \omega_+ \frac{|\log(|\omega_+ t|)| + 1}{|\omega_+ t|^q + 1}.$$

Defining the tempered distribution

$$(2.4.72) \quad \zeta_{\geq \omega_+} \doteq \delta_D - \zeta_{\leq \omega_+},$$

where δ_D is Dirac's delta function and $\zeta_{\leq \omega_+}$ is defined by (2.4.20), the Fourier transforms of $\check{\xi}_m$ and $\zeta_{\geq \omega_+}$ satisfy the relation:

$$(2.4.73) \quad \hat{\zeta}_{\geq \omega_+} = \omega_+^{-m} \hat{\check{\xi}}_m \cdot (i\omega)^m \hat{\zeta}_{\geq \omega_+},$$

yielding the following relation for $\psi_{\geq \omega_+}$ in physical space:

$$(2.4.74) \quad \psi_{\geq \omega_+}(t, \cdot) = \omega_+^{-m} \int_{-\infty}^{+\infty} \check{\xi}_m(t-s) T^m \psi_{\geq \omega_+}(s, \cdot) ds,$$

where, again, the integral in the right hand side of (2.4.74) converges in the $\int_{\Sigma_t} J_\mu^N(\cdot) n^\mu$ norm.

From (2.4.74) and (2.4.71) we can estimate for any $\tau \in \mathbb{R}$:

(2.4.75)

$$\begin{aligned}
& \int_{\Sigma} (|T\psi_{\geq\omega_+}(\tau, x)|^2 + |\nabla_{g_{\Sigma}}\psi_{\geq\omega_+}(\tau, x)|_{g_{\Sigma}}^2) dg_{\Sigma} = \\
& = \omega_+^{-2m} \int_{\Sigma} \left(\left| \int_{-\infty}^{+\infty} \check{\xi}_m(\tau-s) T^{m+1}\psi_{\geq\omega_+}(s, x) ds \right|^2 + \right. \\
& \quad \left. + \left| \int_{-\infty}^{+\infty} \check{\xi}_m(\tau-s) \nabla_{g_{\Sigma}} T^m\psi_{\geq\omega_+}(s, x) ds \right|_{g_{\Sigma}}^2 \right) dg_{\Sigma} \leq \\
& \leq C_m \omega_+^{-2m} \left\{ \int_{\Sigma} \left(\left| \int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} T^{m+1}\psi_{\geq\omega_+}(s, x) ds \right|^2 + \right. \right. \\
& \quad \left. \left. + \left| \int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} \nabla_{g_{\Sigma}} T^m\psi_{\geq\omega_+}(s, x) ds \right|_{g_{\Sigma}}^2 \right) dg_{\Sigma} \right\} \leq \\
& \leq C_m \omega_+^{-2m} \left(\int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} ds \right) \times \\
& \quad \times \left(\int_{\Sigma} \int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} (|T^{m+1}\psi_{\geq\omega_+}(s, x)|^2 + \right. \\
& \quad \left. + |\nabla_{g_{\Sigma}} T^m\psi_{\geq\omega_+}(s, x)|_{g_{\Sigma}}^2) ds dg_{\Sigma} \right) \leq \\
& \leq C_m \omega_+^{-2m} \left(\int_{\Sigma} \int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} (|T^{m+1}\psi_{\geq\omega_+}(s, x)|^2 + \right. \\
& \quad \left. + |\nabla_{g_{\Sigma}} T^m\psi_{\geq\omega_+}(s, x)|_{g_{\Sigma}}^2) ds dg_{\Sigma} \right).
\end{aligned}$$

In view of (2.4.25) and the Schwartz bounds (2.4.22), we readily obtain that for any $\tau \in \mathbb{R}$:

$$\begin{aligned}
(2.4.76) \quad & \int_{\Sigma} (|T^{m+1}\psi_{\geq\omega_+}(\tau, x)|^2 + |\nabla_{g_{\Sigma}} T^m\psi_{\geq\omega_+}(\tau, x)|_{g_{\Sigma}}^2) dg_{\Sigma} \leq \\
& \leq C \int_{-\infty}^{+\infty} \frac{\omega_+}{(1+\omega_+|\tau-s|)^4} \left(\int_{\Sigma} (|T^{m+1}\psi_c(s, x)|^2 + |\nabla_{g_{\Sigma}} T^m\psi_c(s, x)|_{g_{\Sigma}}^2) dg_{\Sigma} \right) ds.
\end{aligned}$$

In view of the definition (2.4.12) of ψ_c , (2.4.76) yields

$$\begin{aligned}
(2.4.77) \quad & \int_{\Sigma} (|T^{m+1}\psi_{\geq\omega_+}(\tau, x)|^2 + |\nabla_{g_{\Sigma}} T^m\psi_{\geq\omega_+}(\tau, x)|_{g_{\Sigma}}^2) dg_{\Sigma} \leq \\
& \leq C \sum_{j=0}^m \int_{-\infty}^{+\infty} \frac{\omega_+}{(1+\omega_+|\tau-s|)^4} \left\{ \int_{\Sigma \cap \{t \geq 0\}} (|T^{j+1}\psi(s, x)|^2 + |\nabla_{g_{\Sigma}} T^j\psi(s, x)|_{g_{\Sigma}}^2) dg_{\Sigma} + \right. \\
& \quad \left. + \int_{\Sigma \cap \{0 \leq t \leq 1\}} |\psi(s, x)|^2 dg_{\Sigma} \right\} ds.
\end{aligned}$$

Thus, (2.4.75), (2.4.77), (2.4.10), and (2.4.66) (combined with the conservation of the T -energy flux in the region $\{t_- \geq 0\} \cap \{t \leq 0\}$) imply:

(2.4.78)

$$\begin{aligned} \int_{\Sigma} (|T\psi_{\geq\omega_+}(\tau, x)|^2 + |\nabla_{g_{\Sigma}}\psi_{\geq\omega_+}(\tau, x)|_{g_{\Sigma}}^2) dg_{\Sigma} \leq \\ \leq C_m \omega_+^{-2m} \left(\int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} (1+|s|^2) (\log(2+|s|))^4 ds \right) \times \\ \times \left(\sum_{j=0}^m \mathcal{E}[T^j\psi] + \mathcal{E}_{\log}[\psi] \right), \end{aligned}$$

from which (2.4.67) readily follows. $\square$

2.5 A Carleman-type estimate outside the extended ergoregion

In this section, we will establish the following estimate for solutions φ to the inhomogeneous wave equation

$$(2.5.1) \quad \square_g \varphi = G$$

on $(\mathcal{M}, g)$:

Proposition 2.2. *For any $s, R \gg 1$ sufficiently large in terms of the geometry of $(\mathcal{M}, g)$ and any $0 < \epsilon_0 < 1$, there exists a smooth T -invariant function $f : \mathcal{M} \setminus \mathcal{H}^- \rightarrow (0, +\infty)$ satisfying*

$$(2.5.2) \quad f = \begin{cases} e^{2sw_R} + e^{2s\tilde{w}_R}, & r \leq R, \\ C_s \left(\frac{r}{R} - \frac{9}{10} \log\left(\frac{r}{R}\right) \right) & r \geq R, \end{cases}$$

where the functions $w_R, \tilde{w}_R : \{r \leq R\} \rightarrow \mathbb{R}$ satisfy

1. $w_R \equiv \tilde{w}_R$ on $\{r \leq \frac{1}{4}r_0\} \cup \mathcal{E}_{ext} \cup \{r \geq \frac{1}{2}R_0\}$,
2. $\sup_{\{r \leq R\}} w_R - \inf_{\{r \leq R\}} w_R + \sup_{\{r \leq R\}} \tilde{w}_R - \inf_{\{r \leq R\}} \tilde{w}_R \leq C\epsilon_0^{-1} R^{3\epsilon_0}$ for some absolute

constant $C > 0$,

3. $\inf_{\{\frac{1}{4}r_0 \leq r \leq R\} \setminus \mathcal{E}_{2\delta}} w_R \geq \max_{\mathcal{E}_\delta} w_R + c_\delta R^{-3\epsilon_0}$ and $\inf_{\{\frac{1}{4}r_0 \leq r \leq R\} \setminus \mathcal{E}_{2\delta}} \tilde{w}_R \geq \max_{\mathcal{E}_\delta} \tilde{w}_R + c_\delta R^{-3\epsilon_0}$
for any $0 < \delta \ll 1$,

4. $\sum_{j=1}^4 (|\nabla^j w_R|_{g_{ref}} + |\nabla^j \tilde{w}_R|_{g_{ref}}) \leq C$,

so that the following statement holds: For any $0 < \delta, \epsilon_0 \ll 1$, any $s, R \gg 1$ satisfying $\epsilon_0 s R^{-9\epsilon_0} \gg 1$, any $0 \leq \tau_1 \leq \tau_2$ and any smooth function $\varphi: \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ solving (2.5.1) with compact support on the hypersurfaces $\{t = \tau\}$ for any $\tau_1 \leq \tau \leq \tau_2$, we can estimate:

(2.5.3)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R_0\} \setminus \mathcal{E}_\delta} (f + \inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}} f) \left\{ s R^{-3\epsilon_0} |\nabla_{g_\Sigma} \varphi|_{g_\Sigma}^2 - C_\delta s R^{-3\epsilon_0} |T\varphi|^2 + s^3 R^{-9\epsilon_0} |\varphi|^2 \right\} dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq \frac{1}{2}R\}} f \left\{ s R^{-3\epsilon_0} r^{-\frac{5}{2}} (|\partial_r \varphi|^2 + r^{-2} |\partial_\sigma \varphi|^2) + \right. \\
& \quad \left. + s R^{-3\epsilon_0} r^{-2} |T\varphi|^2 + \epsilon_0 s^3 R^{-9\epsilon_0} r^{-4} |\varphi|^2 \right\} dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} f \left\{ r^{-\frac{5}{2}} (|\partial_r \varphi|^2 + r^{-2} |\partial_\sigma \varphi|^2) + \right. \\
& \quad \left. + R \partial_r w_R (c R^{-2} s |T\varphi|^2 - C R^{-4} s^3 |\varphi|^2) \right\} dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} f(R) \left\{ r^{-\frac{5}{2}} (|\partial_r \varphi|^2 + r^{-2} |\partial_\sigma \varphi|^2) + r^{-2} |T\varphi|^2 - C R^{-1} r^{-3} |\varphi|^2 \right\} dg \leq \\
& \leq C_\delta \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{E}_\delta} f \left\{ s^2 R^{-6\epsilon_0} |\nabla \varphi|_{g_{ref}}^2 + s^4 R^{-12\epsilon_0} |\varphi|^2 \right\} dg + \\
& + C \left| \int_{\mathcal{R}(\tau_1, \tau_2)} G(\nabla^\mu f \nabla_\mu \bar{\varphi} + O(\sum_{j=1}^2 (1+r)^{j-2} |\nabla^j f|_{g_{ref}}) \bar{\varphi}) dg \right| + \\
& + C \sum_{j=1}^2 \int_{\Sigma_{\tau_j}} (|\nabla f|_{g_{ref}} |\nabla \varphi|_{g_{ref}}^2 + (\sum_{j=1}^3 (1+r)^{j-3} |\nabla^j f|_{g_{ref}}) |\varphi|^2) dg_\Sigma.
\end{aligned}$$

The proof of Proposition 2.2 will be given in Section 2.5.7. It will be based on the construction of a suitable multiplier for the inhomogeneous wave equation (2.5.1), which will be presented in Sections 2.5.2–2.5.3, as well as an intricate integration-by-parts procedure, that will be performed in Section 2.5.4.

Remark. In fact, Proposition 2.2 also holds in the case when $\mathcal{E} = \emptyset$. We should also remark that Proposition 2.2 applies in the case when $(\mathcal{M}, g)$ has a T -invariant timelike boundary component $\partial_{tim}\mathcal{M}$, with $\partial_{tim}\mathcal{M} \cap \Sigma$ compact and $\partial_{tim}\mathcal{M} \cap \mathcal{H} = \emptyset$, and φ is assumed to satisfy either Dirichlet or Neumann boundary conditions on $\partial_{tim}\mathcal{M}$ (see Section 2.5.9 for more details).

Furthermore, the proof of Proposition 2.2 applies without any change after replacing equation (2.5.1) with

$$(2.5.4) \quad \square_g \varphi - V\varphi = G,$$

for any smooth and T -invariant function $V : \mathcal{M} \rightarrow \mathbb{R}$ satisfying either $\partial_r V < 0$ and $V \rightarrow 0$ as $r \rightarrow +\infty$ in the asymptotically flat region of $(\mathcal{M}, g)$, or $\sup_{\mathcal{M}}((1+r)^{2+\eta}|V|) < +\infty$ for some $\eta > 0$. In the later case, the constants in the analogue of (2.5.3) can be chosen to depending only on η and $\sup_{\mathcal{M}}((1+r)^{2+\eta}|V|)$. In view of the fact that the proof of Proposition 2.2 also applies for equation (2.5.4), the degeneracy of (2.5.3) in the frequency regime $\{|\omega| \ll 1\}$ can not be fully removed without some additional argument depending on the fact that $V = 0$ in equation (2.5.1), or yielding an estimate depending qualitatively on V (see also the discussion in [126]). However, this degeneracy does not cause any problem in the proof of Theorem 2.1.

Finally, let us remark that the estimate (2.5.3) can be readily used to show that any smooth solution φ to equation (2.0.1) on $\mathcal{M}$ of the form $\varphi = e^{-i\omega t}\varphi_\omega$ with $\omega \in \mathbb{R} \setminus \{0\}$, $T(\varphi_\omega) = 0$ and

$$(2.5.5) \quad \lim_{\rho \rightarrow +\infty} \int_{\{r=\rho\} \cap \{t=0\}} (|\varphi_\omega|^2 + |\nabla \varphi_\omega|^2) = 0$$

vanishes identically on $\mathcal{M} \setminus \mathcal{E}_{ext}$.

As a corollary of Proposition 2.2, given $\omega_+ > 1$ and $0 < \omega_0 < 1$, we will establish the following estimate for the frequency localised components ψ_k of any solution ψ to the wave equation (2.0.1) on $(\mathcal{M}, g)$ satisfying the bound (2.4.1) (see the relevant constructions in Section 2.4):

Corollary 2.2. *For any smooth solution ψ to (2.0.1) satisfying (2.4.1), any integer $1 \leq k \leq n$, any $0 < \delta_1, \delta_2, \epsilon_0 < 1$, any $R_1 \geq 0$, any $0 \leq \tau_1 \leq \tau_2$ we can bound:*

$$\begin{aligned}
(2.5.6) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R_1\} \setminus \mathcal{E}_{2\delta_1}} \left(J_\mu^N(\psi_k) n^\mu + |\psi_k|^2 \right) \leq \\
& \leq \delta_2 \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{E}_{\delta_1}} \left(J_\mu^N(\psi_k) n^\mu + |\psi_k|^2 \right) + \\
& + C_{\epsilon_0 \delta_1 R_1} (1 + \omega_k^{-10}) (\log(2 + \tau_2))^4 \cdot e^{C_{\epsilon_0 \delta_1} \cdot \max\{|\omega_k|, |\omega_k|^{-\epsilon_0}, -\log \delta_2\}} \mathcal{E}_{\log}[\psi],
\end{aligned}$$

where $C_{\epsilon_0 \delta_1 R_1}$ depends only on $\epsilon_0, \delta_1, R_1$ and the geometry of $(\mathcal{M}, g)$, while $C_{\epsilon_0 \delta_1}$ depends only on ϵ_0, δ_1 and the geometry of $(\mathcal{M}, g)$.

The proof of Corollary 2.2 will be presented in Section 2.5.8.

Finally, let us sketch an additional application of Proposition 2.2 in the Riemannian setting. Let $(\Sigma^d, \bar{g})$, $d \geq 3$, be an asymptotically conic Riemannian manifold, with the asymptotics described in [126], and let us consider the unique solution $u \in L^2(\Sigma)$ of the inhomogeneous Helmholtz equation

$$(2.5.7) \quad \Delta_{\bar{g}} u + \omega^2 u - V u = G$$

on $(\Sigma, \bar{g})$ for a suitably decaying source term $G : \Sigma \rightarrow \mathbb{C}$, with $0 < \text{Im}(\omega) \ll 1$, $\text{Re}(\omega) \neq 0$ and a potential $V : \Sigma \rightarrow \mathbb{R}$ satisfying either $\partial_r V < 0$ in the asymptotically conic region of $(\Sigma, \bar{g})$ and $V \rightarrow 0$ as $r \rightarrow +\infty$ (where r is the radial coordinate function in the asymptotically conic region of Σ , extended to a positive function everywhere on Σ), or

$$(2.5.8) \quad \sup_{\Sigma} \left((r^{-2-\eta} + |\omega| r^{-1-\eta}) |V| \right) < +\infty.$$

Then, applying Proposition 2.2 on the product spacetime $(\mathbb{R} \times \Sigma, g = -dt^2 + \bar{g})$ for the function $\varphi = e^{-i\omega t} u$ solving (in view of (2.5.7))

$$(2.5.9) \quad \square_g \varphi - V \varphi = e^{-i\omega t} G,$$

and using the charge estimate

$$(2.5.10) \quad \operatorname{Im}(\omega^2) \int_{\Sigma} |u|^2 d\bar{g} = \int_{\Sigma} \operatorname{Im}(G\bar{u}) d\bar{g}$$

(combined with elliptic estimates for (2.5.7), as is done, for instance, in [126]) one readily obtains the (quantitative in V) global Carleman-type estimates of [126], albeit with a worse dependence on ω as $\operatorname{Re}(\omega) \rightarrow 0$. Thus, the proof of Proposition 2.2 yields a proof of the Carleman-type estimates used in [126, 107] based entirely on the method of first order multipliers.

Remark. A multiplier-based proof of a similar set of Carleman-type estimates for equation (2.5.7) restricted, however, to the high frequency regime $\omega \gg 1$ was obtained previously in [49].

2.5.1 Parameters and cut-off functions in the proof of Proposition 2.2

Let $R_0 \gg 1$ be large in terms of the geometry of $(\mathcal{M}, g)$, such that $\{r \geq \frac{1}{8}R_0\} \subset \mathcal{I}_{as}$ (R_0 will be considered fixed and, thus, we will not use any special notation to denote the dependence of constants on R_0). In addition to the parameters $\delta, \varepsilon_0, s, R$ appearing in the statement of Proposition 2.2, we will introduce the parameters $R \gg R_0$ and $0 < \delta_0, \delta_1, \delta_2 \ll 1$. We will assume without loss of generality that $0 < \varepsilon_0 \ll 1$. These additional parameters will be fixed in the proof of Proposition 2.2.

In the region $\{r \geq \frac{1}{2}R_0\}$, the vector field ∂_r will simply denote the associated coordinate vector field in the (t, r, σ) coordinate chart in each connected component of this region.

Fixing a smooth function $\chi_4 : \mathbb{R} \rightarrow [0, 1]$ satisfying $\chi_4(x) = 0$ for $x \leq \frac{3}{4}$ and $\chi_4(x) = 1$ for $x \geq 1$, we will define the following smooth cut-off functions:

$$(2.5.11) \quad \chi_{\geq R_0}(r) \doteq \chi_4\left(\frac{r}{R_0}\right),$$

$$(2.5.12) \quad \chi_{\leq R}(r) \doteq \chi_4\left(\frac{R}{r}\right).$$

Remark. Note that $\chi_{\leq R} \equiv 1$ for $r \leq R$ and $\chi_{\leq R} \equiv 0$ for $r \geq \frac{4}{3}R$, while $\chi_{\geq R_0} \equiv 1$ for $r \geq R_0$ and $\chi_{\geq R_0} \equiv 0$ for $r \leq \frac{3}{4}R_0$.

2.5.2 Construction of the auxiliary functions $w_R, \tilde{w}_R$

In this section, we will construct the pair of functions $w_R, \tilde{w}_R : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ appearing in the statement of Proposition 2.2, depending on the parameters $\delta_0, \delta_1, \varepsilon_0, s, R$. These functions will be used extensively in the next sections.

First, we will establish the following lemma:

Lemma 2.11. *There exists a smooth and T -invariant function $\bar{w} : \mathcal{M} \setminus \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties:*

1. *The restriction $\bar{w}|_\Sigma$ of $\bar{w}$ on Σ is a Morse function on $\overline{\Sigma \setminus \mathcal{E}_{ext}}$, with no critical points on $\partial \mathcal{E}_{ext}$. Furthermore, none of the (at most finite) critical points $\{x_j\}_{j=1}^k$ of $\bar{w}|_\Sigma$ on $\Sigma \setminus \mathcal{E}_{ext}$ is a point of local maximum of $\bar{w}|_\Sigma$.*
2. *In the region $\{0 < r \leq \frac{1}{8}r_0\}$,¹¹ $\bar{w}$ is a function of r and satisfies*

$$(2.5.13) \quad \nabla^\mu r \nabla_\mu \bar{w} > 0.$$

3. *In the region $\{r \geq \frac{1}{2}R_0\}$, $\bar{w}$ is a function of r , and satisfies (2.5.13).*

4. *On $(\overline{\mathcal{M} \setminus \mathcal{E}_{ext} \cup \mathcal{H}}) \setminus (\mathbb{R} \times \cup_{j=1}^k \{x_j\})$ we have*

$$(2.5.14) \quad \nabla^\mu \bar{w} \nabla_\mu \bar{w} > 0.$$

5. *For any $0 < \delta \ll 1$, we have*

$$(2.5.15) \quad \inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_{2\delta}} \bar{w} > \max_{\mathcal{E}_\delta} \bar{w}.$$

Proof. The proof of Lemma 2.11 will be based on ideas from [126, 107].

Let $R_0 \gg 1$ be a fixed constant large in terms of the geometry of $(\mathcal{M}, g)$. For any $0 \leq \Upsilon < 1$, let $\bar{w}_\Upsilon : (\Sigma \cap \{\frac{1}{6}r_0 \leq r \leq \frac{1}{4}R_0\}) \setminus \mathcal{E}_{ext} \rightarrow \mathbb{R}$ be the (unique) smooth solution of the

¹¹Recall that $\{r \leq \frac{1}{8}r_0\} \cap \Sigma$ is a neighborhood of $\mathcal{H}^+ \cap \Sigma$ in Σ

elliptic boundary value problem:

$$(2.5.16) \quad \begin{cases} \Delta_{g_\Sigma} \bar{w}_\gamma = \gamma & \text{on } (\Sigma \cap \{\frac{1}{6}r_0 < r < \frac{1}{4}R_0\}) \setminus \mathcal{E}_{ext}, \\ \bar{w}_\gamma|_{r=\frac{1}{4}R_0} = 2, \\ \bar{w}_\gamma|_{r=\frac{1}{6}r_0} = 1, \\ \bar{w}_\gamma|_{\partial \mathcal{E}_{ext}} = 1. \end{cases}$$

Let us extend $\bar{w}_\gamma$ on the whole of $\{\frac{1}{6}r_0 < r < \frac{1}{4}R_0\} \setminus \mathcal{E}_{ext} \subset \mathcal{M} \setminus \mathcal{H}^-$ by the requirement that $T\bar{w}|_\gamma = 0$.

Since g_Σ is smooth, $\bar{w}_\gamma$ depends smoothly on γ (see [73]). In view of the fact that every connected component of $\mathcal{M} \setminus \mathcal{E}$ intersecting $\mathcal{H}^+$ also intersects $\mathcal{I}_{as}$ (see Assumption G3), when $\gamma = 0$, the maximum principle and Hopf's lemma (see [73]) imply that for any $\delta > 0$

$$(2.5.17) \quad \inf_{r=\frac{1}{4}R_0} (\nabla^\mu r \nabla_\mu \bar{w}_0), \quad \inf_{r=\frac{1}{6}r_0} (\nabla^\mu r \nabla_\mu \bar{w}_0), \quad \inf_{\partial \mathcal{E}_{ext}} (n_{\partial \mathcal{E}}(\bar{w}_0)), \quad \left(\inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_\delta} \bar{w}_0 - \max_{\partial \mathcal{E}_{ext}} \bar{w}_0 \right) > 0$$

(see Section 2.2.1 for the definition of $n_{\partial \mathcal{E}}$). Therefore, there exists a $\gamma_0 \in (0, 1)$ and a $c_0 > 0$, such that:

$$(2.5.18) \quad \inf_{r=\frac{1}{4}R_0} (\nabla^\mu r \nabla_\mu \bar{w}_{\gamma_0}), \quad \inf_{r=\frac{1}{6}r_0} (\nabla^\mu r \nabla_\mu \bar{w}_{\gamma_0}), \quad \inf_{\mathcal{E}_{c_0} \setminus \mathcal{E}_{ext}} (n_{\partial \mathcal{E}}(\bar{w}_{\gamma_0})) \geq c_0 > 0$$

and, for all $0 < \delta < 1$ (and some fixed $c_1 > 0$):

$$(2.5.19) \quad \inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_\delta} \bar{w}_{\gamma_0} - \max_{\partial \mathcal{E}_{ext}} \bar{w}_{\gamma_0} > c_1 \delta > 0.$$

In view of 2.5.18 and (2.5.19), we can extend $\bar{w}_{\gamma_0}$ as a T -invariant function on the whole

of $\mathcal{M} \setminus \mathcal{H}^-$ in such a way, so that

$$(2.5.20) \quad \inf_{\{\frac{1}{16}r_0 \leq r \leq \frac{1}{6}r_0\} \cup \{\frac{1}{4}R_0 \leq r \leq R_0\}} (\nabla^\mu r \nabla_\mu \bar{w}_{\gamma_0}) \geq \frac{1}{10}c_0 > 0,$$

$$(2.5.21) \quad \inf_{\{\delta \leq r \leq \frac{1}{6}r_0\}} (\nabla^\mu r \nabla_\mu \bar{w}_{\gamma_0}) \geq c(\delta) > 0,$$

$$(2.5.22) \quad \inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_{2\delta}} \bar{w}_{\gamma_0} > \max_{\mathcal{E}_{\delta}} \bar{w}_{\gamma_0} + \frac{1}{2}c_1\delta$$

for all $0 < \delta \ll 1$ (where $c(\delta)$ is a positive function of $\delta > 0$) and, in addition, $\bar{w}_{\gamma_0}$ is a function of r in the region $\{r \leq \frac{1}{8}r_0\} \cup \{r \geq \frac{1}{2}R_0\}$. Notice also that, since $\bar{w}_{\gamma_0}|_{\partial\mathcal{E}_{ext}} = 1$ and $n_{\partial\mathcal{E}}(\bar{w}_{\gamma_0})|_{\partial\mathcal{E}_{ext}} \geq c_0$, we have

$$(2.5.23) \quad (\nabla^\mu \bar{w}_{\gamma_0} \nabla_\mu \bar{w}_{\gamma_0})|_{\partial\mathcal{E}_{ext}} \geq \frac{1}{2}c_0^2.$$

With $\bar{w}_{\gamma_0}$ constructed as above, we can thus readily choose $\bar{w}$ so that it satisfies the following conditions:

1. $\bar{w}|_\Sigma$ and $\bar{w}_{\gamma_0}|_\Sigma$ satisfy

$$(2.5.24) \quad |\bar{w}|_\Sigma - \bar{w}_{\gamma_0}|_\Sigma|_{C^2(\Sigma)} < \frac{1}{100} \min\{\gamma_0, c_0\}.$$

2. $\bar{w}|_\Sigma$ is a Morse function on an open neighborhood of $\overline{(\Sigma \cap \{\frac{1}{6}r_0 \leq r \leq \frac{1}{4}R_0\}) \setminus \mathcal{E}_{ext}}$.

3. $\bar{w}|_\Sigma = \bar{w}_{\gamma_0}|_\Sigma$ in the region $\mathcal{E} \cup \{r \leq \frac{1}{6}r_0\} \cup \{r \geq \frac{1}{4}R_0\}$.

4. $T\bar{w} = 0$.

Remark. Note that the compatibility of conditions 1 and 2 follows from the density of the set of Morse functions on $\overline{(\Sigma \cap \{\frac{1}{6}r_0 \leq r \leq \frac{1}{4}R_0\}) \setminus \mathcal{E}_{ext}}$ in $C^2(\overline{(\Sigma \cap \{\frac{1}{6}r_0 \leq r \leq \frac{1}{4}R_0\}) \setminus \mathcal{E}_{ext}})$.

In view of (2.5.16), (2.5.20), (2.5.21) and (2.5.24), $\bar{w}$ satisfies

$$(2.5.25) \quad \Delta_{g_\Sigma} \bar{w}|_\Sigma > 0$$

on $\overline{\{\frac{1}{6}r_0 < r < \frac{1}{4}R_0\} \setminus \mathcal{E}_{ext}}$ and

$$(2.5.26) \quad \inf_{\{\frac{1}{16}r_0 \leq r \leq \frac{1}{6}r_0\} \cup \{r \geq \frac{1}{4}R_0\}} (\nabla^\mu r \nabla_\mu \bar{w}) \geq \frac{1}{10}c_0 > 0,$$

$$(2.5.27) \quad \inf_{\{\delta \leq r \leq \frac{1}{6}r_0\}} (\nabla^\mu r \nabla_\mu \bar{w}) \geq c(\delta) > 0.$$

Therefore, none of the critical points of $\bar{w}|_\Sigma$ on $\overline{\Sigma \setminus \mathcal{E}_{ext}}$ is a point of local maximum. Furthermore, Conditions 2 and 3 imply that $\bar{w}|_\Sigma$ is a Morse function on $\overline{\Sigma \setminus \mathcal{E}_{ext}}$. Since $T(\bar{w}) = 0$ and T is strictly timelike on $\mathcal{M} \setminus (\mathcal{E}_{ext} \cup \mathcal{H})$, in view of (2.5.23) and Condition 1 we have

$$(2.5.28) \quad \nabla^\mu \bar{w} \nabla_\mu \bar{w} > 0 \text{ on } (\overline{\mathcal{M} \setminus \mathcal{E}_{ext} \cup \mathcal{H}}) \setminus (\mathbb{R} \times \cup_{j=1}^k \{x_j\}),$$

where $\{x_j\}_{j=1}^k$ are the (at most finite) critical points of $\bar{w}|_\Sigma$ on $\overline{\Sigma \setminus \mathcal{E}_{ext}}$, none of which lies on $\partial \mathcal{E}_{ext}$ (in view of (2.5.23) and Condition 1). Finally, in view of (2.5.18), (2.5.22) and (2.5.24), inequality (2.5.15) holds for all $0 < \delta \ll 1$. $\square$

Lemma 2.12. *For any $0 < \delta_0 \ll 1$ small in terms of the geometry of $(\mathcal{M}, g)$, there exists a pair of smooth and T -invariant functions $w, \tilde{w} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$, as well as a finite number of points $\{x_j\}_{j=1}^k, \{\tilde{x}_j\}_{j=1}^k \in \Sigma \cap \{r \leq \frac{1}{2}R_0 + 4\delta_0\} \setminus \mathcal{E}_{8\delta_0}$, such that the following statements hold:*

1. *Defining for any $\rho > 0$ the subsets*

$$(2.5.29) \quad \mathcal{B}_{crit}(\rho) = \mathbb{R} \times (\cup_{j=1}^k B_{g_\Sigma}(x_j, \rho)),$$

$$(2.5.30) \quad \tilde{\mathcal{B}}_{crit}(\rho) = \mathbb{R} \times (\cup_{j=1}^k B_{g_\Sigma}(\tilde{x}_j, \rho))$$

of $\mathcal{M} \setminus \mathcal{H}^-$, where $B_{g_\Sigma}(x_j, \rho) \subset (\Sigma, g_\Sigma)$ is the closed Riemannian ball of radius ρ centered at x_j , we have:

$$(2.5.31) \quad \mathcal{B}_{crit}(\delta_0) \subset \tilde{\mathcal{B}}_{crit}(4\delta_0),$$

$$(2.5.32) \quad \tilde{\mathcal{B}}_{crit}(\delta_0) \subset \mathcal{B}_{crit}(4\delta_0),$$

$$(2.5.33) \quad \mathcal{B}_{crit}(\delta_0) \cap \tilde{\mathcal{B}}_{crit}(\delta_0) = \emptyset.$$

2. The functions $w, \tilde{w}$ coincide outside $\mathcal{B}_{crit}(4\delta_0)$:

$$(2.5.34) \quad w \equiv \tilde{w} \text{ on } \mathcal{M} \setminus (\mathcal{B}_{crit}(4\delta_0) \cup \mathcal{H}^-).$$

3. The functions $w, \tilde{w}$ satisfy the following non-degeneracy conditions for some absolute constant $c_0 > 0$ (independent of δ_0):

$$(2.5.35) \quad \inf_{\{r \geq \frac{1}{8}r_0\} \setminus (\mathcal{E}_{ext} \cup \mathcal{B}_{crit}(\delta_0))} \nabla^\mu \nabla^\nu w \nabla_\mu w \nabla_\nu w \geq c_0 > 0,$$

$$(2.5.36) \quad \inf_{\{r \geq \frac{1}{8}r_0\} \setminus (\mathcal{E}_{ext} \cup \tilde{\mathcal{B}}_{crit}(\delta_0))} \nabla^\mu \nabla^\nu \tilde{w} \nabla_\mu \tilde{w} \nabla_\nu \tilde{w} \geq c_0 > 0.$$

4. For any T -invariant vector fields $X, \tilde{X}$ on $\{r \geq \frac{1}{8}r_0\} \setminus (\mathcal{E}_{ext} \cup \mathcal{B}_{crit}(\delta_0))$ such that $X(w) = 0$ and $\tilde{X}(\tilde{w}) = 0$, the following one sided bounds hold:

$$(2.5.37) \quad \nabla_\mu \nabla_\nu w X^\mu X^\nu > -\delta_0 \frac{\nabla^\mu \nabla^\nu w \nabla_\mu w \nabla_\nu w}{g_{ref}(dw, dw)} g_{ref}(X, X),$$

$$(2.5.38) \quad \nabla_\mu \nabla_\nu \tilde{w} \tilde{X}^\mu \tilde{X}^\nu > -\delta_0 \frac{\nabla^\mu \nabla^\nu \tilde{w} \nabla_\mu \tilde{w} \nabla_\nu \tilde{w}}{g_{ref}(d\tilde{w}, d\tilde{w})} g_{ref}(\tilde{X}, \tilde{X}),$$

where g_{ref} is the reference Riemannian metric (2.1.4).

5. The functions $w, \tilde{w}$ satisfy

$$(2.5.39) \quad \max_{\mathcal{B}_{crit}(\delta_0)} w < \min_{\mathcal{B}_{crit}(\delta_0)} \tilde{w},$$

$$(2.5.40) \quad \max_{\tilde{\mathcal{B}}_{crit}(\delta_0)} \tilde{w} < \min_{\tilde{\mathcal{B}}_{crit}(\delta_0)} w.$$

6. For any $0 < \delta \ll 1$:

$$(2.5.41) \quad \inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_{2\delta}} w > \max_{\mathcal{E}_\delta} w,$$

$$(2.5.42) \quad \inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_{2\delta}} \tilde{w} > \max_{\mathcal{E}_\delta} \tilde{w}$$

and

$$(2.5.43) \quad \inf_{\{r \geq \frac{1}{2}r_0\} \setminus \mathcal{E}_{ext}} w > \max_{\{r \leq \frac{1}{4}r_0\}} w,$$

$$(2.5.44) \quad \inf_{\{r \geq \frac{1}{2}r_0\} \setminus \mathcal{E}_{ext}} \tilde{w} > \max_{\{r \leq \frac{1}{4}r_0\}} \tilde{w}$$

Proof. Let $\bar{w} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ be as in the statement of Lemma 2.11, and let $\{x_j\}_{j=1}^k$ be the (at most finite) critical points of $\bar{w}|_\Sigma$ in $\overline{\Sigma \setminus \mathcal{E}_{ext}}$. According to Lemma 2.11, none of these points lies on $\partial \mathcal{E}_{ext}$ or on $\{r \leq \frac{1}{8}r_0\} \cup \{r \geq \frac{1}{2}R_0\}$ and, thus, provided $\delta_0 \ll 1$, we have

$$(2.5.45) \quad \{x_j\}_{j=1}^k \in \Sigma \cap \left\{ \frac{1}{8}r_0 \leq r \leq \frac{1}{2}R_0 \right\} \setminus \mathcal{E}_{16\delta_0}.$$

Let $l > 0$ be large in terms of δ_0 , and let us define the T -invariant function $w : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ as

$$(2.5.46) \quad w = e^{l\bar{w}}.$$

Then, the one sided bounds (2.5.35) and (2.5.37) readily follow from the properties of $\bar{w}$ (see Lemma 2.11), as well as the identities

$$(2.5.47) \quad \nabla_\mu w = l(\nabla_\mu \bar{w})e^{l\bar{w}}$$

and

$$(2.5.48) \quad \nabla_\mu \nabla_\nu w = \left(l^2 (\nabla_\mu \bar{w})(\nabla_\nu \bar{w}) + l \nabla_\mu \nabla_\nu \bar{w} \right) e^{l\bar{w}},$$

provided l is sufficiently large in terms of δ_0 . Inequality (2.5.41) follows readily from (2.5.15).

Since the points $\{x_j\}_{j=1}^k$ satisfy (2.5.45) and none of them is a point of local maximum for $\bar{w}|_\Sigma$ (see Lemma 2.11), for any $0 < \delta_0 \ll 1$, there exists a diffeomorphism $\mathcal{X} : \Sigma \rightarrow \Sigma$ such that $\mathcal{X} = Id$ on $\Sigma \setminus \cup_{j=1}^k B_{g_\Sigma}(x_j, 4\delta_0)$ and for all $1 \leq j \leq k$:

$$(2.5.49) \quad 2\delta_0 < dist_{g_\Sigma}(x_j, \mathcal{X}(x_j)) < 4\delta_0,$$

$$(2.5.50) \quad \mathcal{X}(B_{g_\Sigma}(x_j, \delta_0)) = B_{g_\Sigma}(\mathcal{X}(x_j), \delta_0),$$

$$(2.5.51) \quad \mathcal{X}(B_{g_\Sigma}(\mathcal{X}(x_j), \delta_0)) = B_{g_\Sigma}(x_j, \delta_0),$$

and

$$(2.5.52) \quad \max_{B_{g_\Sigma}(x_j, \delta_0)} \bar{w} < \min_{B_{g_\Sigma}(\mathcal{X}(x_j), \delta_0)} \bar{w}.$$

Setting $\tilde{x}_j \doteq \mathcal{X}(x_j)$ for $j = 1, \dots, k$, provided δ_0 is sufficiently small in terms of the geometry of $(\mathcal{M}, g)$, we have:

$$(2.5.53) \quad \{\tilde{x}_j\}_{j=1}^k \in \Sigma \cap \{r \leq \frac{1}{2}R_0 + 4\delta_0\} \setminus \mathcal{E}_{8\delta_0}.$$

Extending $\mathcal{X}$ on the whole of $\mathcal{M} \setminus \mathcal{H}^-$ by the requirement that it commutes with the flow of T , i. e.:

$$(2.5.54) \quad \mathcal{L}_T \circ \mathcal{X} = \mathcal{X} \circ \mathcal{L}_T,$$

and defining the function $\tilde{w} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ as

$$(2.5.55) \quad \tilde{w} \doteq w \circ \mathcal{X},$$

we infer that, in view of (2.5.47), (2.5.48) and the properties of $\mathcal{X}$, the relations (2.5.34), (2.5.36) and (2.5.38) hold, provided l is sufficiently large in terms of δ and the precise choice of $\mathcal{X}$. Furthermore, in view of (2.5.52), inequalities (2.5.39) and (2.5.40) hold. Finally, inequalities (2.5.41) and (2.5.42) follow trivially from (2.5.15) and the fact that $\mathcal{X} = Id$ on $\mathcal{E}_\delta$ for $\delta \leq 4\delta_0$, while (2.5.43) and (2.5.44) follow from (2.5.13). $\square$

Lemma 2.13. *For any $R_0 \gg 1, 0 < \delta_0 \ll 1, 0 < \delta_1 \ll 1, s \gg 1, 0 < \epsilon_0 \ll 1$ and $R \gg \max\{R_0, \epsilon_0^{-1}\}$, there exists a pair of smooth and T -invariant function $w_R, \tilde{w}_R : \{r \leq R\} \subset \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ satisfying the following properties:*

1. In the region $\{r \leq R_0\}$:

$$(2.5.56) \quad w_R = R^{-3\epsilon_0} w$$

and

$$(2.5.57) \quad \tilde{w}_R = R^{-3\epsilon_0} \tilde{w},$$

where $w, \tilde{w}$ are the functions from Lemma 2.12.

2. In the region $\{R_0 \leq r \leq R^{\epsilon_0}\}$, w_R is a function of r and $\tilde{w}_R = w_R$. The following bounds are also satisfied for some constants depending only on R_0 and δ_0 (and the precise choice of w):

$$(2.5.58) \quad 0 < cR^{-3\epsilon_0} \leq \partial_r w_R \leq C,$$

$$(2.5.59) \quad \partial_r^2 w_R + r^{-1} \partial_r w_R \geq cR^{-3\epsilon_0} + \left| r^{-\frac{1}{2}} \partial_r^2 w_R \right| + \left| r^{-\frac{3}{2}} \partial_r w_R \right|,$$

$$(2.5.60) \quad |\partial_r^2 w_R|, |\partial_r^3 w_R|, |\partial_r^4 w_R| \leq C.$$

3. In the region $\{R^{\epsilon_0} \leq r \leq \frac{1}{2}R\}$:

$$(2.5.61) \quad w_R = \tilde{w}_R = C_1 \epsilon_0^{-1} \left(\frac{r}{R} \right)^{\epsilon_0} + C_2$$

for some constants C_1, C_2 depending only on R_0, δ_0 (and the precise choice of w).

4. In the region $\{\frac{1}{2}R \leq r \leq R\}$:

$$(2.5.62) \quad w_R = \tilde{w}_R = v_s \left(\frac{r}{R} \right) + C_3$$

for some constant C_3 depending on R_0, δ_0, δ_1 (and the precise choice of w), where the function $v_s : [\frac{1}{2}, 1] \rightarrow \mathbb{R}$ depends on $s, \epsilon_0, \delta_0, \delta_1$ and satisfies (for some constants

$c_{\delta_0}, C_{\delta_0} > 0$ depending on δ_0, R_0 and the precise choice of w):

$$(2.5.63) \quad \frac{dv_s}{dx} \geq c_{\delta_0} s^{-1},$$

$$(2.5.64) \quad \left| \frac{d^2 v_s}{dx^2} \right| \leq C_{\delta_0} (\delta_1 s + \delta_1^{-1}) \frac{dv_s}{dx}$$

$$(2.5.65) \quad \left| \frac{d^2 v_s}{dx^2} \right|, \left| \frac{d^3 v_s}{dx^3} \right|, \left| \frac{d^4 v_s}{dx^4} \right| \leq C_{\delta_0} \delta_1^{-1}$$

and, for $x \in [\frac{3}{4}, 1]$:

$$(2.5.66) \quad v_s(x) = \frac{1}{2s} \log \left(x - \frac{9}{10} \log(x) \right).$$

Remark. Notice that we can bound on $\{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}$

$$(2.5.67) \quad \partial_r^2 w_R + r^{-1} \partial_r w_R > c_0 \varepsilon_0 R^{-\varepsilon_0} \cdot r^{-2+\varepsilon_0} + |r^{-\frac{1}{2}} \partial_r^2 w_R| + |r^{-\frac{3}{2}} \partial_r w_R|,$$

$$(2.5.68) \quad \partial_r w_R > c_0 R^{-\varepsilon_0} r^{-1+\varepsilon_0},$$

$$(2.5.69) \quad \sum_{j=1}^4 |r^j \partial_r^j w_R| < C_0 R^{-\varepsilon_0} r^{\varepsilon_0}.$$

Proof. The construction of w_R (and, similarly, $\tilde{w}_R$) can be readily performed in view of the following observations:

- In view of Condition 1 and the properties of the function w , for $r = R_0$ we have:

$$(2.5.70) \quad \partial_r w_R(R_0) \sim_{R_0} R^{-3\varepsilon_0},$$

$$(2.5.71) \quad \partial_r^2 w_R(R_0), r^{-1} \partial_r w_R(R_0) \sim_{R_0} R^{-3\varepsilon_0},$$

while Condition 3 requires that, for $r = R^{\varepsilon_0}$:

$$(2.5.72) \quad \partial_r w_R(R^{\varepsilon_0}) = C_1 R^{-2\varepsilon_0 + \varepsilon_0^2} \gg \partial_r w_R(R_0),$$

$$(2.5.73) \quad \partial_r^2 w_R(R^{\varepsilon_0}) = C_1 (\varepsilon_0 - 1) R^{-3\varepsilon_0 + \varepsilon_0^2}$$

$$(2.5.74) \quad \partial_r^2 w_R(R^{\varepsilon_0}) + r^{-1} \partial_r w_R(R^{\varepsilon_0}) \geq c \varepsilon_0 R^{-3\varepsilon_0 + \varepsilon_0^2} + \left| r^{-\frac{1}{2}} \partial_r^2 w_R(R^{\varepsilon_0}) \right| + \left| r^{-\frac{3}{2}} \partial_r w_R(R^{\varepsilon_0}) \right|.$$

Therefore, we can readily construct the function $\partial_r w_R$ (as a function of r) on the interval $\{R_0 \leq r \leq R^{\varepsilon_0}\}$ (and then integrate in order to obtain w_R and the constant C_2 in (2.5.61)), so that (2.5.58)–(2.5.60) are satisfied. In particular, $\partial_r w_R$ can be constructed as an increasing function of r (i.e. with $\partial_r^2 w_R > 0$) up to $r = R^{\varepsilon_0} - 1$, while for $r \in [R^{\varepsilon_0} - 1, R^{\varepsilon_0}]$, $\partial_r w_R$ is constructed a smooth function of r extending (2.5.61) from $\{r \geq R^{\varepsilon_0}\}$ under the requirement that it satisfies the one sided bound

$$(2.5.75) \quad \partial_r^2 w_R(r) \geq -(1 + \varepsilon_0^2) |\partial_r^2 w_R(R^{\varepsilon_0})|.$$

- Let $\tilde{v} : [\frac{3}{5}, 1] \rightarrow \mathbb{R}$ be a smooth and strictly increasing function such that $\tilde{v}(x) = -(x-1)^2 - 10$ for $x \in [\frac{3}{5}, \frac{7}{10}]$ and $\tilde{v}(x) = \log(x - \frac{9}{10} \log(x))$ for $x \in [\frac{3}{4}, 1]$. Then, provided $s \gg 1$ and $\delta_1 < 1$, it can be readily inferred that there exists a C^5 and piecewise C^6 function $\tilde{v}_s : [\frac{1}{2}, 1] \rightarrow \mathbb{R}$, which is smooth on $[\frac{1}{2}, 1] \setminus \{\frac{3}{5}\}$, satisfying $\frac{d\tilde{v}_s}{dx} \geq \frac{1}{10s} C_1$, $\sum_{j=1}^4 \left| \frac{d^j \tilde{v}_s}{dx^j} \right| \leq 10C_1 \delta_1^{-1}$ and

$$\tilde{v}_s(x) = \begin{cases} C_1 \varepsilon_0^{-1} x^{\varepsilon_0} + C_2 - C_3 & \text{for } x \in [\frac{1}{2}, \frac{11}{20}] \\ -\delta_1 (x - \frac{3}{5})^6 - \frac{1}{2s} ((x-1)^2 + 10) & \text{for } x \in [\frac{23}{40}, \frac{3}{5}) \\ \frac{1}{2s} \tilde{v}(x), & \text{for } x \in [\frac{3}{5}, 1] \end{cases}$$

for a suitable constant $C_3 > 0$ depending on C_1, C_2 . The function v_s is then constructed by mollifying $\tilde{v}_s$ around $x = \frac{3}{5}$.

□

2.5.3 The seed functions $f, \tilde{f}, h$ and $\tilde{h}$

In this section, we will construct (using the auxiliary functions from the previous section) the seed functions for the multipliers that will be used in the proof of Proposition 2.2.

We will assume without loss of generality that $0 < \delta_0, \delta_1 \ll 1$, $s \gg 1$, $0 < \varepsilon_0 \ll 1$ and $R \gg \max\{R_0, \varepsilon_0^{-1}\}$. Let $w_R, \tilde{w}_R : \{r \leq R\} \rightarrow \mathbb{R}$ be the functions from Lemma 2.13 (associated to the parameters $s, R, \varepsilon_0, \delta_0, \delta_1$). We define the smooth and T -invariant functions $f, \tilde{f} :$

$\mathcal{M} \setminus \mathcal{H}^- \rightarrow (0, +\infty)$ as follows:

$$(2.5.76) \quad f = \begin{cases} e^{2sw_R}, & \text{on } \{r \leq R\} \\ C_4^{2s} \cdot \left(\frac{r}{R} - \frac{9}{10} \log\left(\frac{r}{R}\right) \right), & \text{on } \{r \geq R\}, \end{cases}$$

and

$$(2.5.77) \quad \tilde{f} = \begin{cases} e^{2s\tilde{w}_R}, & \text{on } \{r \leq R\} \\ C_4^{2s} \cdot \left(\frac{r}{R} - \frac{9}{10} \log\left(\frac{r}{R}\right) \right), & \text{on } \{r \geq R\}, \end{cases}$$

where $C_4 > 0$ is chosen so that f and $\tilde{f}$ are smooth at $r = R$ (which is possible in view of (2.5.66)).

Let $h : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ be a smooth and T -invariant function satisfying the following conditions (provided $\delta_1, \delta_2 \ll 1$):

1. In the region $\{r \leq \frac{4}{3}R\}$:

$$(2.5.78) \quad h = -\chi_{\leq \frac{1}{2}R_0} s \delta_1 \frac{\nabla_\mu \nabla_\nu w_R \nabla^\mu w_R \nabla^\nu w_R}{g_{ref}(dw_R, dw_R)} e^{2sw_R} + \chi_{\geq R_0} (r^{-1} - r^{-\frac{3}{2}}) \partial_r f,$$

where

$$(2.5.79) \quad \chi_{\leq \frac{1}{2}R_0} \doteq \chi_4\left(\frac{R_0}{2r}\right).$$

2. In the region $\{\frac{4}{3}R \leq r \leq \delta_2^{-1}R\}$, h is a function of r , satisfying

$$(2.5.80) \quad cf(R)r^{-2} < h \leq \min\{(r^{-1} - r^{-\frac{3}{2}})\partial_r f, (1 - r^{-\frac{1}{2}})\partial_r^2 f\}$$

and

$$(2.5.81) \quad -\square_g h \leq CR^{-4}f(R)$$

for some absolute constants $C, c > 0$.

3. In the region $\{r \geq \delta_2^{-1}R\}$:

$$(2.5.82) \quad h = \frac{1}{2} \partial_r^2 f.$$

Notice that h can indeed be defined as above on the interval $\{(1 + 2\delta_2)R \leq r \leq \delta_2^{-1}R\}$ (provided δ_2 is smaller than an absolute constant), in view of the fact that

$$(2.5.83) \quad \min\{\partial_r^2 f, r^{-1} \partial_r f\} \gtrsim f(R) r^{-2},$$

$$(2.5.84) \quad \square_g h = (1 + O(r^{-1})) \partial_r^2 h + ((d-1)r^{-1} + O(r^{-2})) \partial_r h$$

and

$$(2.5.85) \quad \sum_{j=1}^4 r^{j-4} |\partial_r^j f| \leq CR^{-4} f(R)$$

on that interval, while $(r^{-1} - r^{-\frac{3}{2}}) \partial_r f < \partial_r^2 f$ for $R \leq r < \frac{4}{3}R$ (provided $R \gg 1$) and $r^{-1} \partial_r f > \partial_r^2 f$ for $r \geq \delta_2^{-1}R$ (provided $\delta_2 \ll 1$).

We also define $\tilde{h} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ in the same way as h , but with $\tilde{w}_R$ and $\tilde{f}$ in place of w_R and f , respectively.

2.5.4 The integration-by-parts scheme

In this section, we will establish a general identity obtained from equation (2.5.1) and a suitable first order multiplier, after successively integrating by parts over $\mathcal{R}(\tau_1, \tau_2)$. This identity will lie at the core of the proof of Proposition 2.2.

Let f, h be as in Section 2.5.3. We introduce the following multiplier for equation (2.5.1)

$$(2.5.86) \quad 2\nabla^\mu f \cdot \nabla_\mu \varphi + \square_g f \cdot \varphi.$$

Multiplying (2.4.29) with the complex conjugate of (2.5.86) and integrating by parts

over $\mathcal{R}(\tau_1, \tau_2)$, we obtain:

(2.5.87)

$$\begin{aligned}
\int_{\mathcal{R}(\tau_1, \tau_2)} Re\left\{2\nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2\right\} dg = & - \int_{\mathcal{R}(\tau_1, \tau_2)} Re\left\{G\left(2\nabla^\mu f \nabla_\mu \bar{\varphi} + (\square_g f) \bar{\varphi}\right)\right\} dg \\
& - \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} Re\left\{(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f) \varphi \nabla_\nu \bar{\varphi} \right. \\
& \quad \left. - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} - \frac{1}{2} (\nabla_\nu (\square_g f)) |\varphi|^2\right\} n_{\Sigma_{\tau_j}}^\nu dg_{\Sigma_{\tau_j}} \\
& - \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} Re\left\{(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f) \varphi \nabla_\nu \bar{\varphi} \right. \\
& \quad \left. - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} - \frac{1}{2} (\nabla_\nu (\square_g f)) |\varphi|^2\right\} n_{\mathcal{H}^+}^\nu dvol_{\mathcal{H}^+}.
\end{aligned}$$

Let us split the left hand side of (2.5.87) as

$$\begin{aligned}
\int_{\mathcal{R}(\tau_1, \tau_2)} Re\left\{2\nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2\right\} dg = & \\
(2.5.88) \quad & = \int_{\mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} Re\left\{2\nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2\right\} dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} (1 - \chi_{\leq R}) Re\left\{2\nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2\right\} dg.
\end{aligned}$$

Using the identity

$$(2.5.89) \quad \nabla_\mu \varphi \nabla_\nu \bar{\varphi} = f^{-1} \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \frac{1}{2} f^{-1} (\nabla_\mu f \varphi \nabla_\nu \bar{\varphi} + \nabla_\nu f \bar{\varphi} \nabla_\mu \varphi) - \frac{1}{4} f^{-2} \nabla_\mu f \nabla_\nu f |\varphi|^2$$

and integrating by parts in the $\varphi \nabla \varphi$ terms, we have:

$$\begin{aligned}
\int_{\mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} Re\left\{2\nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2\right\} dg = & \\
= \int_{\mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} Re\left\{2f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) + (\nabla_\nu (f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f) - \right. & \\
& \left. - \frac{1}{2} f^{-2} \nabla^\mu \nabla^\nu f \nabla_\mu f \nabla_\nu f - \frac{1}{2} \square_g^2 f) |\varphi|^2\right\} dg + & \\
(2.5.90) \quad & + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\nu \chi_{\leq R} \cdot f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f |\varphi|^2 dg + \\
& + \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \chi_{\leq R} \cdot f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\Sigma_{\tau_j}}^\nu dg_{\Sigma_{\tau_j}} + \\
& + \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} \cdot f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\mathcal{H}^+}^\nu dvol_{\mathcal{H}^+}.
\end{aligned}$$

Thus, in view of (2.5.90), the identity (2.5.88) yields:

(2.5.91)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2 \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2 \right\} dg = \\
& = \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2 \chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} \right\} dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \left(\chi_{\leq R} \nabla_\nu (f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f) - \frac{1}{2} \chi_{\leq R} f^{-2} \nabla^\mu \nabla^\nu f \nabla_\mu f \nabla_\nu f - \frac{1}{2} \square_g^2 f \right) |\varphi|^2 dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} (\nabla_\nu \chi_{\leq R} \cdot f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f) |\varphi|^2 dg + \\
& + \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\Sigma_{\tau_j}}^\nu dg_{\Sigma_{\tau_j}} + \\
& + \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} \cdot f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\mathcal{H}^+}^\nu d\operatorname{vol}_{\mathcal{H}^+}.
\end{aligned}$$

Adding to (2.5.91) the identity

(2.5.92)

$$\begin{aligned}
0 = \int_{\mathcal{R}(\tau_1, \tau_2)} & \left(-2 \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2 \chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 + \right. \\
& \left. + \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-1} (\partial_r f)^2 \partial_r (|\varphi|^2) + \frac{1}{2} \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-2} (\partial_r f)^3 |\varphi|^2 \right) dg
\end{aligned}$$

(recall that ∂_r is the coordinate vector field in the (t, r, σ) coordinate chart in each connected component of the region $\{r \geq \frac{1}{4} R_0\}$) and integrating by parts in the $\partial_r (|\varphi|^2)$ term, we obtain:

(2.5.93)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2 \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2 \right\} dg = \\
& = \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2 \chi_{\leq R} \left(f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right) + \right. \\
& \quad \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2 \chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 + \mathcal{A}_f^{(R)} |\varphi|^2 \right\} dg + \\
& + \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\Sigma_{\tau_j}}^\nu dg_{\Sigma_{\tau_j}} + \\
& + \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} \cdot f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\mathcal{H}^+}^\nu d\operatorname{vol}_{\mathcal{H}^+},
\end{aligned}$$

where

$$\begin{aligned}
(2.5.94) \quad \mathcal{A}_f^{(R)} &\doteq \chi_{\leq R} \nabla_\nu (f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f) - \frac{1}{2} \chi_{\leq R} f^{-2} \nabla^\mu \nabla^\nu f \nabla_\mu f \nabla_\nu f - \frac{1}{2} \square_g^2 f + \\
&- \chi_{\geq R_0} \chi_{\leq R} \partial_r (r^{-1} f^{-1} (\partial_r f)^2) - \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-1} (\partial_r f)^2 \operatorname{div}(\partial_r) + \\
&+ \frac{1}{2} \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-2} (\partial_r f)^3 + \nabla_\nu \chi_{\leq R} \cdot f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f - \\
&- \partial_r (\chi_{\geq R_0} \chi_{\leq R}) r^{-1} f^{-1} (\partial_r f)^2.
\end{aligned}$$

Thus, (2.5.87) and (2.5.93) yield:

$$\begin{aligned}
(2.5.95) \quad &\int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2 \chi_{\leq R} \left(f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right) + \right. \\
&+ 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2 \chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 + \mathcal{A}_f^{(R)} |\varphi|^2 \Big\} dg = \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G(2 \nabla^\mu f \nabla_\mu \bar{\varphi} + (\square_g f) \bar{\varphi}) \right\} dg - \mathcal{B}_f^{(R)}[\varphi; \tau_1, \tau_2],
\end{aligned}$$

where

$$\begin{aligned}
(2.5.96) \quad \mathcal{B}_f^{(R)}[\varphi; \tau_1, \tau_2] &= \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \operatorname{Re} \left\{ \left(2 \nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \right. \right. \\
&+ \left. \left(\chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f - \frac{1}{2} (\nabla_\nu (\square_g f)) \right) |\varphi|^2 \right\} n_{\Sigma_{\tau_j}}^\nu dg_{\Sigma_{\tau_j}} + \\
&+ \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \left(2 \nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \right. \right. \\
&+ \left. \left(\chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f - \frac{1}{2} (\nabla_\nu (\square_g f)) \right) |\varphi|^2 \right\} n_{\mathcal{H}^+}^\nu d\operatorname{vol}_{\mathcal{H}^+}.
\end{aligned}$$

Finally, for $h : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ as in Section 2.5.3, adding to (2.5.95) the Lagrangean identity

$$\begin{aligned}
(2.5.97) \quad &\int_{\mathcal{R}(\tau_1, \tau_2)} \left(-2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + (\square_g h) |\varphi|^2 \right) dg = \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \{ G \cdot 2h \bar{\varphi} \} dg + \\
&+ \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \operatorname{Re} \left\{ \left(2h \nabla_\nu \varphi \bar{\varphi} - \nabla_\nu h |\varphi|^2 \right) n_{\Sigma_{\tau_j}}^\nu \right\} dg_{\Sigma_{\tau_j}} + \\
&+ \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \left(2h \nabla_\nu \varphi \bar{\varphi} - \nabla_\nu h |\varphi|^2 \right) n_{\mathcal{H}^+}^\nu \right\} d\operatorname{vol}_{\mathcal{H}^+},
\end{aligned}$$

we obtain:

(2.5.98)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + \right. \\
& \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - \\
& \quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg = \\
& = - \int_{\mathcal{R}(\tau_1, \tau_2)} Re \left\{ G \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} + (\square_g f - 2h) \bar{\varphi} \right) \right\} dg - \mathcal{B}_{f,h}^{(R)} [\varphi; \tau_1, \tau_2],
\end{aligned}$$

where

$$\begin{aligned}
(2.5.99) \quad \mathcal{A}_{f,h}^{(R)} & \doteq \square_g h + \chi_{\leq R} \nabla_\nu (f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f) - \frac{1}{2} \chi_{\leq R} f^{-2} \nabla^\mu \nabla^\nu f \nabla_\mu f \nabla_\nu f - \frac{1}{2} \square_g^2 f - \\
& - \chi_{\geq R_0} \chi_{\leq R} \partial_r (r^{-1} f^{-1} (\partial_r f)^2) - \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-1} (\partial_r f)^2 \text{div}(\partial_r) + \\
& + \frac{1}{2} \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-2} (\partial_r f)^3 + \nabla_\nu \chi_{\leq R} \cdot f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f - \\
& - \partial_r (\chi_{\geq R_0} \chi_{\leq R}) r^{-1} f^{-1} (\partial_r f)^2
\end{aligned}$$

and

(2.5.100)

$$\begin{aligned}
\mathcal{B}_{f,h}^{(R)} [\varphi; \tau_1, \tau_2] & = \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} Re \left\{ \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f - 2h) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \right. \right. \\
& \quad \left. \left. + (\chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f))) |\varphi|^2 \right) n_{\Sigma_{\tau_j}}^\nu \right\} dg_{\Sigma_{\tau_j}} + \\
& + \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} Re \left\{ \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f - 2h) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \right. \right. \\
& \quad \left. \left. + (\chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f))) |\varphi|^2 \right) n_{\mathcal{H}^+}^\nu \right\} dvol_{\mathcal{H}^+}.
\end{aligned}$$

In the next sections, we will establish a number of estimates for the left hand side of (2.5.98) that will lead to the proof of Proposition 2.2.

2.5.5 Estimates for the zeroth order term

In this section, we will establish some bounds for the coefficient $\mathcal{A}_{f,h}^{(R)}$ of the zeroth order term appearing in the left hand side of (2.5.98).

In view of the choice of the functions f, h in Section 2.5.3, we can readily calculate that on $\{r \leq R\}$ (where $\chi_R \equiv 1$), the quantity $\mathcal{A}_{f,h}^{(R)}$ in (2.5.99) has the form

$$(2.5.101) \quad \mathcal{A}_{f,h}^{(R)} = \left\{ \mathcal{A}_{w_R,3} s^3 + \mathcal{A}_{w_R,2} s^2 + \mathcal{A}_{w_R,1} s \right\} e^{2s w_R},$$

where

$$(2.5.102) \quad \begin{aligned} \mathcal{A}_{w_R,3} = & \left(4 - \chi_{\leq \frac{1}{2}R_0} \delta_1 \frac{\nabla^\alpha w_R \nabla_\alpha w_R}{g_{ref}(dw_R, dw_R)} \right) \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R + \\ & + 4\chi_{\geq R_0} r^{-1} (1 + O(r^{-\frac{1}{2}})) (\partial_r w_R)^3, \end{aligned}$$

$$(2.5.103) \quad \begin{aligned} \mathcal{A}_{w_R,2} = & 4\nabla_\nu \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R - 4\nabla_\nu \nabla^\nu \nabla^\mu w_R \nabla_\mu w_R - \\ & - 4\nabla^\mu (\square_g w_R) \nabla_\mu w_R - 2(\square_g w_R)^2 + \\ & + 4\chi_{\geq R_0} r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^2 w_R \partial_r w_R + \\ & + 4\chi_{\geq R_0} r^{-2} (1 + O(r^{-\frac{1}{2}})) (\partial_r w_R)^2 + \\ & + O(\delta_1) \chi_{\leq \frac{1}{2}R_0} \sum_{j=0}^1 |\nabla^{2+j} w_R|_{g_{ref}} |\nabla^{2-j} w_R|_{g_{ref}} + \\ & + O(|\nabla \chi_{\leq R_0}|_{g_{ref}} + |\nabla \chi_{\leq \frac{1}{2}R_0}|_{g_{ref}}) (|\nabla^2 w_R|_{g_{ref}}^2 + |\nabla w_R|_{g_{ref}}^2), \end{aligned}$$

$$(2.5.104) \quad \begin{aligned} \mathcal{A}_{w_R,1} = & -\square_g^2 w_R + 2\chi_{\geq R_0} r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^3 w + 2(d-3)\chi_{\geq R_0} r^{-2} (1 + O(r^{-\frac{1}{2}})) \partial_r^2 w - \\ & - 2(d-3)\chi_{\geq R_0} r^{-3} (1 + O(r^{-\frac{1}{2}})) \partial_r w + \\ & + O(\delta_1) \chi_{\leq \frac{1}{2}R_0} \sum_{j_1+j_2+j_3=1}^2 \frac{|\nabla^{2+j_1} w_R|_{g_{ref}} |\nabla^{1+j_2} w_R|_{g_{ref}} |\nabla^{1+j_3} w_R|_{g_{ref}}}{|\nabla w_R|_{g_{ref}}^2} + \\ & + 2(\square_g \chi_{\leq R_0}) r^{-1} \partial_r w_R - \delta_1 (\square_g \chi_{\leq \frac{1}{2}R_0}) \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R \frac{\nabla^\alpha w_R \nabla_\alpha w_R}{g_{ref}(dw_R, dw_R)} + \\ & + O\left(\sum_{j=1}^2 (|\nabla^j \chi_{\leq R_0}|_{g_{ref}} + |\nabla^j \chi_{\leq \frac{1}{2}R_0}|_{g_{ref}}) (|\nabla^2 w_R|_{g_{ref}} + |\nabla w_R|_{g_{ref}}) \right) \end{aligned}$$

(with the constants implicit in the $O(\cdot)$ notation depending only on the geometry of $(\mathcal{M}, g)$).

Remark. Notice the cancellation of the $O(s^4)$ terms that were expected to appear in (2.5.101).

Bound on $\{r \leq R_0\}$

In view of (2.5.101)–(2.5.104), the properties of the function w_R (see Lemma 2.13) and the form (2.1.1) of the metric g in the region $r \gg 1$ imply that in the region $\{r \leq R_0\}$:

$$(2.5.105) \quad \mathcal{A}_{f,h}^{(R)} = \left\{ \mathcal{A}_{w,3;R_0} R^{-9\epsilon_0} s^3 + O_{\delta_0}(1) R^{-6\epsilon_0} s^2 + O_{\delta_0}(1) R^{-3\epsilon_0} s \right\} e^{2sw_R},$$

where

$$(2.5.106) \quad \mathcal{A}_{w,3;R_0} = \left(4 - \chi_{\leq \frac{1}{2}R_0} \delta_1 \frac{\nabla^\alpha w \nabla_\alpha w}{g_{ref}(dw, dw)} \right) \nabla^\mu \nabla^\nu w \nabla_\mu w \nabla_\nu w + 4 \chi_{\geq R_0} r^{-1} (1 + O(r^{-\frac{1}{2}})) (\partial_r w)^3,$$

w is the function from Lemma 2.12 and the constants implicit in the $O_{\delta_0}(1)$ notation depend only on R_0, δ_0 .

Bounds on $\{R_0 \leq r \leq R\}$

In the region $\{R_0 \leq r \leq R\}$, the expressions (2.5.102), (2.5.103) and (2.5.104) simplify as follows, in view of (2.1.1) and the fact that w_R is a function of r for $r \geq R_0$:

$$(2.5.107) \quad \mathcal{A}_{w_R,3} = 4(1 + O(r^{-\frac{1}{2}})) \partial_r^2 w_R (\partial_r w_R)^2 + 4r^{-1} (1 + O(r^{-\frac{1}{2}})) (\partial_r w_R)^3,$$

$$(2.5.108) \quad \begin{aligned} \mathcal{A}_{w_R,2} = & -4(1 + O(r^{-\frac{1}{2}})) \partial_r w_R \partial_r^3 w_R - 2(1 + O(r^{-\frac{1}{2}})) (\partial_r^2 w_R)^2 + \\ & + (4 - 8(d-1)) r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^2 w_R \partial_r w_R + \\ & + (4(d-2) - 2(d-1)^2) r^{-2} (1 + O(r^{-\frac{1}{2}})) (\partial_r w_R)^2, \end{aligned}$$

$$(2.5.109) \quad \begin{aligned} \mathcal{A}_{w_R,1} = & - (1 + O(r^{-\frac{1}{2}})) \partial_r^4 w_R - 2(d-1) r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^3 w_R - \\ & - (d-3)^2 r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^2 w_R + (d-3)^2 r^{-3} (1 + O(r^{-\frac{1}{2}})) \partial_r w_R. \end{aligned}$$

Therefore, the properties of the function w_R (see Lemma 2.13) imply the following relations for $\mathcal{A}_{f,h}^{(R)}$ on $\{R_0 \leq r \leq R\}$ (provided R_0 is sufficiently large in terms of the geometry of $(\mathcal{M}, g)$):

1. In the region $\{R_0 \leq r \leq R^{\varepsilon_0}\}$, (2.5.58)–(2.5.60) yield:

$$(2.5.110) \quad \mathcal{A}_{f,h}^{(R)} \geq \left\{ c_{\delta_0} R^{-9\varepsilon_0} s^3 - C_{\delta_0} s^2 - C_{\delta_0} s \right\} e^{2sw_R}$$

for some constants $c_{\delta_0}, C_{\delta_0} > 0$ depending on δ_0, R_0 .

2. In the region $\{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}$, (2.5.67)–(2.5.69) yield:

$$(2.5.111) \quad \mathcal{A}_{f,h}^{(R)} \geq \left\{ c_{\delta_0} \varepsilon_0 R^{-3\varepsilon_0} \cdot r^{-4+3\varepsilon_0} s^3 - C_{\delta_0} R^{-2\varepsilon_0} r^{-4+2\varepsilon_0} s^2 - C_{\delta_0} R^{-\varepsilon_0} r^{-4+\varepsilon_0} s \right\} e^{2sw_R}.$$

3. In the region $\{\frac{1}{2}R \leq r \leq R\}$, (2.5.62)–(2.5.65) yield:

$$(2.5.112) \quad \mathcal{A}_{f,h}^{(R)} \geq -C_{\delta_0} R^{-4} \left\{ v'_s\left(\frac{r}{R}\right) s^3 + s^2 + s \right\} e^{2sw_R},$$

where $C_{\delta_0} > 0$ depends only on δ_0, R_0 .

Bound on $\{r \geq R\}$

In the region $\{R \leq r \leq \delta_2^{-1}R\}$, (2.5.76), (2.5.78) and (2.5.81) yield:

$$(2.5.113) \quad \mathcal{A}_{f,h}^{(R)}(r) \geq -CR^{-4}f(R)$$

for some absolute constant $C > 0$, while for $r \geq \delta_2^{-1}R$ we have (provided $\delta_2 \ll 1$):

$$(2.5.114) \quad \begin{aligned} \mathcal{A}_{f,h}^{(R)} &= -\frac{(d-1)}{2} r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^3 f - \frac{(d-1)(d-3)}{2} r^{-2} (1 + O(r^{-\frac{1}{2}})) \partial_r^2 f + \\ &\quad + \frac{(d-1)(d-3)}{2} r^{-3} (1 + O(r^{-\frac{1}{2}})) \partial_r f = \\ &= \frac{1}{2} (d-1) r^{-4} f(R) \left((d-3) \left(\frac{r + O(r^{\frac{1}{2}})}{R} \right) + \frac{9}{5} + O(r^{-\frac{1}{2}}) \right) \geq \\ &\geq f(R) \left(\frac{1}{2} (d-1)(d-3) R^{-1} r^{-3} (1 + O(r^{-\frac{1}{2}})) + cr^{-4} \right) \end{aligned}$$

for some absolute constant $c > 0$.

2.5.6 Estimates for the first order terms

In this Section, we will establish various bounds for the quantity

$$(2.5.115) \quad \begin{aligned} & Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + \right. \\ & \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} \end{aligned}$$

appearing in the integral in left hand side of (2.5.98). Thus, combined with the bounds of Section 2.5.5 for the zeroth order term $\mathcal{A}_{f,h}^{(R)} |\varphi|^2$ in left hand side of (2.5.98), the results of this section will provide all the necessary estimates leading to the proof of Proposition 2.2.

Let us denote with g^{-1} the natural extension of the metric (2.1.1) on the cotangent bundle $T^*\mathcal{M}$ of $\mathcal{M}$. Since we have identified $\mathcal{M} \setminus \mathcal{H}^-$ with $\mathbb{R} \times \Sigma$ under the flow of T , g^{-1} splits naturally in any local coordinate chart $(t, x^1, \dots, x^d)$ on $\mathbb{R} \times \Sigma$ as

$$(2.5.116) \quad g^{-1} = g^{00} T \otimes T + \frac{1}{2} g^{0i} (T \otimes \partial_{x^i} + \partial_{x^i} \otimes T) + (g^{-1})_\Sigma^{ij} \partial_{x^i} \otimes \partial_{x^j},$$

where $(g^{-1})_\Sigma$ is a symmetric $(2,0)$ -tensor on Σ . In view of Assumption G3, the expression (2.5.116) and the fact that g^{-1} is non-degenerate and has Lorentzian signature imply that $(g^{-1})_\Sigma$ has Riemannian signature on $\Sigma \setminus (\mathcal{E} \cup \mathcal{H}^+)$ and Lorentzian signature on $\Sigma \cap \text{int}(\mathcal{E})$, while $(g^{-1})_\Sigma$ degenerates on $\Sigma \cap (\partial \mathcal{E} \cup \mathcal{H}^+)$. Using the tensor $(g^{-1})_\Sigma$, we can conveniently bound for any $\varphi \in C^1(\mathcal{M} \setminus \mathcal{H}^-)$ (for some constant $C > 0$ depending only on the geometry of $(\mathcal{M}, g)$):

$$(2.5.117) \quad \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \geq (g^{-1})_\Sigma^{ij} \partial_i \varphi \partial_j \bar{\varphi} - C |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} |T\varphi| - C |T\varphi|^2,$$

where the indices i, j in the abstract index notation $(g^{-1})_\Sigma^{ij} \partial_i \varphi \partial_j \bar{\varphi}$ run over the variables $\{x^i\}_{i=1}^d$ in any local coordinate chart on $(\mathcal{M} \setminus \mathcal{H}^-)$ of the form $(t, x^1, \dots, x^d)$.

Bound on $\mathcal{E}_{ext} \cup \{r \leq \frac{1}{4}r_0\}$

For any $0 \leq \tau_1 \leq \tau_2$, we can readily bound from above on $\mathcal{R}(\tau_1, \tau_2) \cap (\mathcal{E}_{ext} \cup \{r \leq \frac{1}{4}r_0\})$ in view of (2.5.76) and (2.5.78):

(2.5.118)

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2) \cap (\mathcal{E}_{ext} \cup \{r \leq \frac{1}{4}r_0\})} Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\ & \quad - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\ & \quad \left. + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \leq \\ & \leq C_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap (\mathcal{E}_{ext} \cup \{r \leq \frac{1}{4}r_0\})} e^{2sw_R} \left((R^{-6\varepsilon_0} s^2 + 1) |\nabla \varphi|_{g_{ref}}^2 + (R^{-12\varepsilon_0} s^4 + 1) |\varphi|^2 \right) dg. \end{aligned}$$

Bound on $\{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}$

In the region $\{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\}$, in view of (2.5.37) and (2.5.76), we can estimate (using a Cauchy–Schwarz inequality):

(2.5.119)

$$\begin{aligned} & Re \left\{ 2f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) \right\} = \\ & = e^{2sw_R} \left\{ 8s^2 |\nabla^\mu w_R \nabla_\mu \varphi|^2 + 16s^3 \nabla^\mu w_R \nabla_\mu w_R Re \{ \varphi \cdot \nabla^\nu w_R \nabla_\nu \bar{\varphi} \} + \right. \\ & \quad + 8s^4 (\nabla^\mu w_R \nabla_\mu w_R)^2 |\varphi|^2 + 4s \nabla^\mu \nabla^\nu w_R \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\ & \quad \left. + 8s \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R Re \{ \varphi \cdot \nabla_\nu \bar{\varphi} \} + 4s^3 \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R |\varphi|^2 \right\} \geq \\ & \geq e^{2sw_R} \left\{ 8s^2 |\nabla^\mu w_R \nabla_\mu \varphi + s^2 \nabla^\mu w_R \nabla_\mu w_R \cdot \varphi|^2 - Cs_{\delta_0} \frac{\nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R}{g_{ref}(dw_R, dw_R)} |\nabla \varphi|_{g_{ref}}^2 + \right. \\ & \quad \left. + 3s^3 \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R |\varphi|^2 \right\} \end{aligned}$$

for some absolute constant $C > 0$. Thus, (2.5.119), (2.5.78) and (2.5.117) imply that we can bound from below on the region $\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}$:

(2.5.120)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} Re \left\{ 2f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} e^{2sw_R} \left\{ \frac{\nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R}{g_{ref}(dw_R, dw_R)} \left(cs \delta_1 (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} + \right. \right. \\
& \quad \left. \left. + cs^3 g_{ref}(dw_R, dw_R) |\varphi|^2 \right) - Cs \left(\sum_{j=1}^2 |\nabla^j w_R|_{g_{ref}} \right) |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} |T\varphi| - \right. \\
& \quad \left. - Cs \left(\sum_{j=1}^2 |\nabla^j w_R|_{g_{ref}} \right) |T\varphi|^2 \right\} dg
\end{aligned}$$

for some absolute constants $c, C > 0$.

Bound on $\{\frac{1}{2}R_0 \leq r \leq R_0\}$

In the region $\{r \geq \frac{1}{2}R_0\}$, the functions f, h depend only on r . In particular, we compute in the (t, r, σ) coordinate system in each connected component of the region $\{r \geq \frac{1}{2}R_0\}$ for any function $\psi \in C^1(\mathcal{M})$:

$$\begin{aligned}
(2.5.121) \quad \nabla^\mu \nabla^\nu f \nabla_\mu \psi \nabla_\nu \bar{\psi} &= ((1 + O(r^{-1})) \partial_r^2 f + O(r^{-2}) \partial_r f) |\partial_r \psi|^2 + \\
&+ r^{-3} (1 + O(r^{-1})) \partial_r f |\partial_\sigma \psi|^2 + O(r^{-2}) \partial_r f |T\psi|^2.
\end{aligned}$$

Therefore, (2.5.76) and (2.5.78) yield the following lower bound:

$$\begin{aligned}
(2.5.122) \quad \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\
\quad \left. - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \right. \\
\quad \left. + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} dg \geq
\end{aligned}$$

$$\begin{aligned}
&\geq c \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} e^{2sw_R} \left\{ s^2 (\partial_r w_R)^2 + O(1)s \sum_{j=1}^2 |\nabla^j w_R|_{g_{ref}} e^{-2sw_R} |\partial_r (e^{sw_R} \varphi)|^2 + \right. \\
&\quad + s(\partial_r w_R) (\chi_{\geq R_0} (r^{-\frac{3}{2}} + O(r^{-2})) + O(\delta_1) |\nabla w_R|_{g_{ref}}^{-1} |\nabla^2 w_R|_{g_{ref}}) |\partial_r \varphi|^2 + \\
&\quad + s(\partial_r w_R) (r^{-3} + O(r^{-4}) + \chi_{\geq R_0} (r^{-\frac{7}{2}} - r^{-3}) + O(\delta_1) |\nabla w_R|_{g_{ref}}^{-1} |\nabla^2 w_R|_{g_{ref}}) |\partial_\sigma \varphi|^2 + \\
&\quad \left. + s(\partial_r w_R) (\chi_{\geq R_0} (r^{-1} + O(r^{-\frac{3}{2}})) + O(\delta_1) |\nabla w_R|_{g_{ref}}^{-1} |\nabla^2 w_R|_{g_{ref}}) |T\varphi|^2 \right\} dg
\end{aligned}$$

for some $c > 0$ depending on the geometry of $(\mathcal{M}, g)$.

Bound on $\{R_0 \leq r \leq R\}$

In the region $\{R_0 \leq r \leq R\}$, in view of (2.5.76), (2.5.78) and (2.5.121), we can readily estimate from below:

$$\begin{aligned}
(2.5.123) \quad &\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq R\}} Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + \right. \\
&\quad \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} dg \geq \\
&\geq c \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq R\}} e^{2sw_R} \left\{ s^2 (1 + O(r^{-1})) (\partial_r w_R)^2 + \right. \\
&\quad + s(\partial_r^2 w_R + O(r^{-2}) \partial_r w_R) e^{-2sw_R} |\partial_r (e^{sw_R} \varphi)|^2 + \\
&\quad + s(\partial_r w_R) (r^{-\frac{3}{2}} + O(r^{-2})) |\partial_r \varphi|^2 + s(\partial_r w_R) (r^{-\frac{7}{2}} + O(r^{-4})) |\partial_\sigma \varphi|^2 + \\
&\quad \left. + cs(\partial_r w_R) (r^{-1} + O(r^{-\frac{3}{2}})) |T\varphi|^2 \right\} dg
\end{aligned}$$

for some $c > 0$ depending on the geometry of $(\mathcal{M}, g)$.

Bound on $\{r \geq R\}$

In the region $\{r \geq R\}$, we can estimate in view of (2.5.121):

(2.5.124)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} \text{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& \quad - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& \quad \left. + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} \left\{ \chi_{\leq R} (2(1 + O(r^{-1})) \partial_r^2 f - 2(r^{-1} - r^{-\frac{3}{2}} + O(r^{-2})) \partial_r f) f^{-1} |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + \right. \\
& \quad + \left(2(1 - \chi_{\leq R}) ((1 + O(r^{-1})) \partial_r^2 f + O(r^{-2}) \partial_r f) - 2(1 + O(r^{-1})) h \right) |\partial_r \varphi|^2 + \\
& \quad + \left(2r^{-3} (1 + O(r^{-1})) \partial_r f - 2(r^{-2} + O(r^{-3})) h \right) |\partial_\sigma \varphi|^2 + \\
& \quad \left. + (2h + O(r^{-2}) \partial_r f) |T\varphi|^2 \right\} dg.
\end{aligned}$$

2.5.7 Proof of Proposition 2.2

1. In view of (2.5.105), (2.5.106), (2.5.120) and Lemma 2.12, we can bound:

(2.5.125)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} \text{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& \quad - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& \quad \left. + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus (\mathcal{E}_{ext} \cup \mathcal{B}_{crit}(\delta_0))} e^{2sw_R} \left\{ c_{\delta_0 \delta_1} s R^{-3\epsilon_0} (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} - \right. \\
& \quad - C s R^{-3\epsilon_0} |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} |T\varphi| - C s R^{-3\epsilon_0} |T\varphi|^2 + \\
& \quad \left. + (c_{\delta_0} s^3 R^{-9\epsilon_0} - C_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) |\varphi|^2 \right\} dg - \\
& - C_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{B}_{crit}(\delta_0)} e^{2sw_R} \left\{ s R^{-3\epsilon_0} |\nabla \varphi|_{g_{ref}}^2 + s^3 R^{-9\epsilon_0} |\varphi|^2 \right\} dg.
\end{aligned}$$

Repeating the same procedure for $\tilde{f}$ in place of f , $\tilde{h}$ in place of h and $\tilde{w}_R$ in place of w_R (see Lemma 2.13), from (the analogues of) (2.5.105), (2.5.106), (2.5.120) for w_R

in place of $\tilde{w}_R$ we obtain:

(2.5.126)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} Re \left\{ 2\chi_{\leq R} \tilde{f}^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (\tilde{f}^{\frac{1}{2}} \varphi) \nabla_\nu (\tilde{f}^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& \quad - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} \tilde{f}^{-1} (\partial_r \tilde{f}) |\partial_r (\tilde{f}^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu \tilde{f} \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& \quad + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r \tilde{f}) |\partial_r \varphi|^2 - 2\tilde{h} \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{\tilde{f}, \tilde{h}}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus (\mathcal{E}_{ext} \cup \tilde{\mathcal{B}}_{crit}(\delta_0))} e^{2s\tilde{w}_R} \left\{ c_{\delta_0 \delta_1} s R^{-3\epsilon_0} (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} - \right. \\
& \quad - C s R^{-3\epsilon_0} |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} |T\varphi| - C s R^{-3\epsilon_0} |T\varphi|^2 + \\
& \quad + (c_{\delta_0} s^3 R^{-9\epsilon_0} - C_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) |\varphi|^2 \Big\} dg - \\
& - C_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \tilde{\mathcal{B}}_{crit}(\delta_0)} e^{2s\tilde{w}_R} \left\{ s R^{-3\epsilon_0} |\nabla \varphi|_{g_{ref}}^2 + s^3 R^{-9\epsilon_0} |\varphi|^2 \right\} dg.
\end{aligned}$$

Adding (2.5.125) and (2.5.126) and using (2.5.39) and (2.5.40), we obtain provided s is large in terms of δ_0 :

(2.5.127)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& \quad - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& \quad + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f, h}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} Re \left\{ 2\chi_{\leq R} \tilde{f}^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (\tilde{f}^{\frac{1}{2}} \varphi) \nabla_\nu (\tilde{f}^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& \quad - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} \tilde{f}^{-1} (\partial_r \tilde{f}) |\partial_r (\tilde{f}^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu \tilde{f} \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& \quad + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r \tilde{f}) |\partial_r \varphi|^2 - 2\tilde{h} \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{\tilde{f}, \tilde{h}}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} (e^{2sw_R} + e^{2s\tilde{w}_R}) \left\{ c_{\delta_0 \delta_1} s R^{-3\epsilon_0} (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} - \right. \\
& \quad - C s R^{-3\epsilon_0} |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} |T\varphi| - C s R^{-3\epsilon_0} |T\varphi|^2 + \\
& \quad + (c_{\delta_0} s^3 R^{-9\epsilon_0} - C_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) |\varphi|^2 \Big\} dg.
\end{aligned}$$

2. In view of (2.5.105), (2.5.106), (2.5.122) and Lemma 2.12, we can estimate

$$\begin{aligned}
(2.5.128) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} e^{2sw_R} \left\{ (c_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) e^{-2sw_R} |\partial_r (e^{sw_R} \varphi)|^2 + \right. \\
& + s R^{-3\epsilon_0} (c_{\delta_0} \chi_{\geq R_0} (r^{-\frac{3}{2}} + O(r^{-2})) - C_{\delta_0} \delta_1) |\partial_r \varphi|^2 + \\
& + s R^{-3\epsilon_0} (c_{\delta_0} (r^{-3} + O(r^{-4})) + \chi_{\geq R_0} (r^{-\frac{7}{2}} - r^{-3})) - C_{\delta_0} \delta_1) |\partial_\sigma \varphi|^2 - \\
& \left. - C_{\delta_0} s R^{-3\epsilon_0} |T\varphi|^2 + (c_{\delta_0} s^3 R^{-9\epsilon_0} - C_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) |\varphi|^2 \right\} dg.
\end{aligned}$$

Provided $sR^{-3\epsilon_0}$ is sufficiently large in terms of δ_0 , we can estimate

$$\begin{aligned}
(2.5.129) \quad & (c_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) e^{-2sw_R} |\partial_r (e^{sw_R} \varphi)|^2 - C_{\delta_0} \delta_1 s R^{-3\epsilon_0} |\partial_r \varphi|^2 \geq \\
& \geq c_{\delta_0} \delta_1 s R^{-3\epsilon_0} |\partial_r \varphi|^2 - C_{\delta_0} \delta_1 s^3 R^{-9\epsilon_0} |\varphi|^2
\end{aligned}$$

and thus (2.5.128) yields:

$$\begin{aligned}
(2.5.130) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} e^{2sw_R} \left\{ c_{\delta_0} \delta_1 s R^{-3\epsilon_0} |\partial_r \varphi|^2 + s R^{-3\epsilon_0} (c_{\delta_0} r^{-\frac{7}{2}} - C_{\delta_0} \delta_1) |\partial_\sigma \varphi|^2 - \right. \\
& \left. - C_{\delta_0} s R^{-3\epsilon_0} |T\varphi|^2 + ((c_{\delta_0} - C_{\delta_0} \delta_1) s^3 R^{-9\epsilon_0} - C_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) |\varphi|^2 \right\} dg.
\end{aligned}$$

3. In view of (2.5.110), (2.5.123) and Lemma 2.13, we can bound:

$$\begin{aligned}
(2.5.131) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq R^{\varepsilon_0}\}} \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) \left| \partial_r (f^{\frac{1}{2}} \varphi) \right|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq R^{\varepsilon_0}\}} e^{2sw_R} \left\{ (c_{\delta_0} s^2 R^{-6\varepsilon_0} - C_{\delta_0} s) e^{-2sw_R} \left| \partial_r (e^{sw_R} \varphi) \right|^2 + \right. \\
& + c_{\delta_0} s R^{-3\varepsilon_0} r^{-\frac{3}{2}} |\partial_r \varphi|^2 + c_{\delta_0} s R^{-3\varepsilon_0} r^{-\frac{7}{2}} |\partial_\sigma \varphi|^2 + \\
& + c_{\delta_0} s R^{-3\varepsilon_0} r^{-1} |T\varphi|^2 + (c_{\delta_0} R^{-9\varepsilon_0} s^3 - C_{\delta_0} s^2 - C_{\delta_0} s) |\varphi|^2 \Big\} dg.
\end{aligned}$$

4. In view of (2.5.111), (2.5.123) and (2.5.67)–(2.5.69), we can bound:

$$\begin{aligned}
(2.5.132) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}} \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) \left| \partial_r (f^{\frac{1}{2}} \varphi) \right|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}} e^{2sw_R} \left\{ r^{-2} (c_{\delta_0} r^{2\varepsilon_0} s^2 R^{-2\varepsilon_0} - \right. \\
& - C_{\delta_0} s (r^{\varepsilon_0} R^{-\varepsilon_0} + r^{2\varepsilon_0} R^{-2\varepsilon_0})) e^{-2sw_R} \left| \partial_r (e^{sw_R} \varphi) \right|^2 + \\
& + c_{\delta_0} r^{-\frac{5}{2} + \varepsilon_0} s R^{-\varepsilon_0} |\partial_r \varphi|^2 + c_{\delta_0} r^{-\frac{9}{2} + \varepsilon_0} s R^{-\varepsilon_0} |\partial_\sigma \varphi|^2 + c_{\delta_0} r^{-2 + \varepsilon_0} s R^{-\varepsilon_0} |T\varphi|^2 + \\
& + r^{-4} (c_{\delta_0} \varepsilon_0 r^{3\varepsilon_0} s^3 R^{-3\varepsilon_0} - C_{\delta_0} r^{2\varepsilon_0} s^2 R^{-2\varepsilon_0} - C_{\delta_0} r^{\varepsilon_0} s R^{-\varepsilon_0}) |\varphi|^2 \Big\} dg.
\end{aligned}$$

5. In view of (2.5.112), (2.5.123) and (2.5.63)–(2.5.65), we can bound:

$$\begin{aligned}
(2.5.133) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} e^{2sw_R} \left\{ R^{-2} (c_{\delta_0} - C_{\delta_0} (\delta_1 + s^{-1} \delta_1^{-1})) e^{-2sw_R} |\partial_r (e^{sw_R} \varphi)|^2 + \right. \\
& + c_{\delta_0} R^{-\frac{5}{2}} |\partial_r \varphi|^2 + c_{\delta_0} R^{-\frac{9}{2}} |\partial_\sigma \varphi|^2 + \\
& + c_{\delta_0} R^{-2} v'_s \left(\frac{r}{R} \right) s |T\varphi|^2 - C_{\delta_0} R^{-4} \left(v'_s \left(\frac{r}{R} \right) s^3 + s^2 + s \right) \Big\} dg.
\end{aligned}$$

6. In view of (2.5.113), (2.5.124), (2.5.76), (2.5.78) and (2.5.80), we can estimate:

$$\begin{aligned}
(2.5.134) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R \leq r \leq \delta_2^{-1} R\}} \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \right. \\
& - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R \leq r \leq \delta_2^{-1} R\}} \left\{ 2\chi_{\leq R} \frac{R^2}{r^2} \left(\frac{9}{5} - \frac{r}{R} \right) R^{-2} f(R) \cdot f^{-1} |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + \right. \\
& + \left((1 - \chi_{\leq R}) f(R) r^{-\frac{5}{2}} - 2\chi_{\leq R} \left(\frac{R^2}{r^2} \left(\frac{r}{R} - \frac{9}{10} \right) (R^{-2} + O(R^{-\frac{5}{2}})) f(R) \right) |\partial_r \varphi|^2 + \right. \\
& + c f(R) r^{-\frac{9}{2}} |\partial_\sigma \varphi|^2 + c f(R) r^{-2} |T\varphi|^2 - C R^{-4} f(R) |\varphi|^2 \Big\} dg.
\end{aligned}$$

Remark. Notice that the positivity of the coefficient of $|\partial_r (f^{\frac{1}{2}} \varphi)|^2$ in the right hand side of (2.5.134) follows from the fact that, in view of (2.5.76), provided R is sufficiently large in terms of the geometry of $(\mathcal{M}, g)$, we can bound for $R \leq r \leq \frac{4}{3}R$ (i.e. on $\operatorname{supp}(\chi_{\leq R}) \cap \{r \geq R\}$):

$$(2.5.135) \quad \partial_r^2 f + r^{-\frac{3}{2}} \partial_r f > (r^{-1} + O(r^{-2})) \partial_r f + O(r^{-1}) \partial_r^2 f + f(R) \left(\frac{9}{5} - \frac{r}{R} \right) R^{-2}.$$

Applying the product rule and a Cauchy–Schwarz inequality on the first term of the

right hand side of (2.5.134), we obtain:

$$\begin{aligned}
(2.5.136) \quad & 2\chi_{\leq R} \frac{R^2}{r^2} \left(\frac{9}{5} - \frac{r}{R} \right) R^{-2} f(R) \cdot f^{-1} |\partial_r(f^{\frac{1}{2}}\varphi)|^2 \geq \\
& \geq 2\chi_{\leq R} \frac{R^2}{r^2} \left(\frac{9}{5} - \frac{r}{R} - \frac{1}{100} \right) R^{-2} f(R) |\partial_r \varphi|^2 - C\chi_{\leq R} R^{-4} f(R) |\varphi|^2
\end{aligned}$$

for some absolute constant $C > 0$. Thus, (2.5.134) yields (provided $R \gg 1$):

$$\begin{aligned}
(2.5.137) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R \leq r \leq \delta_2^{-1} R\}} Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}}\varphi) \nabla_\nu (f^{\frac{1}{2}}\bar{\varphi}) - \right. \\
& - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}}\varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R \leq r \leq \delta_2^{-1} R\}} f(R) \left\{ cr^{-\frac{5}{2}} |\partial_r \varphi|^2 + cr^{-\frac{9}{2}} |\partial_\sigma \varphi|^2 + \right. \\
& \left. + cr^{-2} |T\varphi|^2 - CR^{-4} |\varphi|^2 \right\} dg.
\end{aligned}$$

7. In view of (2.5.114), (2.5.124), (2.5.76) and (2.5.82), we can bound:

$$\begin{aligned}
(2.5.138) \quad & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \delta_2^{-1} R\}} Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}}\varphi) \nabla_\nu (f^{\frac{1}{2}}\bar{\varphi}) - \right. \\
& - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}}\varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \Big\} dg \geq \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \delta_2^{-1} R\}} f(R) \left\{ cr^{-2} |\partial_r \varphi|^2 + cR^{-1} r^{-3} |\partial_\sigma \varphi|^2 + cr^{-2} |T\varphi|^2 - CR^{-1} r^{-3} |\varphi|^2 \right\} dg.
\end{aligned}$$

In view of (2.5.118), (2.5.127), (2.5.130), (2.5.131), (2.5.132), (2.5.137), (2.5.138), as well as the fact that $\tilde{f} \equiv f$ and $\tilde{h} \equiv h$ on $\mathcal{M} \setminus (\mathcal{B}_{crit}(4\delta_0) \cup \mathcal{H}^-)$, we obtain from (2.5.98) (provided $\delta_0, \delta_2 > 0$ are sufficiently small in terms of R_0 and the geometry of $(\mathcal{M}, g)$, $\delta_1 > 0$ is sufficiently small in terms of δ_0 and R_0 , s is sufficiently large in terms of δ_0, δ_1 and $\varepsilon_0 s R^{-9\varepsilon_0}$

is sufficiently large in terms of δ_0, δ_1):

(2.5.139)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} (f + \tilde{f}) \left\{ c_{\delta_0 \delta_1} s R^{-3\varepsilon_0} (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} - C s R^{-3\varepsilon_0} |\nabla_{g_{\Sigma}} \varphi|_{g_{\Sigma}} |T\varphi| - \right. \\
& \quad \left. - C s R^{-3\varepsilon_0} |T\varphi|^2 + c_{\delta_0} s^3 R^{-9\varepsilon_0} |\varphi|^2 \right\} dg + \\
& + c_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \frac{1}{2}R_0\}} (f + \tilde{f}) \left\{ \chi_{\{\frac{1}{2}R_0 \leq r \leq R_0\}} \delta_1 s R^{-3\varepsilon_0} + \chi_{\{R_0 \leq r \leq R^{\varepsilon_0}\}} s R^{-3\varepsilon_0} r^{-\frac{3}{2}} + \right. \\
& \quad + \chi_{\{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}} s R^{-\varepsilon_0} r^{-\frac{5}{2} + \varepsilon_0} + \chi_{\{\frac{1}{2}R \leq r \leq \delta_2^{-1}R\}} r^{-\frac{5}{2}} + \\
& \quad \left. + \chi_{\{r \geq \delta_2^{-1}R\}} r^{-2} \right\} (|\partial_r \varphi|^2 + r^{-2} |\partial_{\sigma} \varphi|^2) dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq \frac{1}{2}R\}} (f + \tilde{f}) \left\{ \chi_{\{\frac{1}{2}R_0 \leq r \leq R_0\}} \left(-C_{\delta_0} s R^{-3\varepsilon_0} |T\varphi|^2 + c_{\delta_0} s^3 R^{-9\varepsilon_0} |\varphi|^2 \right) + \right. \\
& \quad + c_{\delta_0} \chi_{\{R_0 \leq r \leq R^{\varepsilon_0}\}} \left(s R^{-3\varepsilon_0} r^{-1} |T\varphi|^2 + s^3 R^{-9\varepsilon_0} |\varphi|^2 \right) + \\
& \quad \left. + \chi_{\{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}} \left(s R^{-\varepsilon_0} r^{-2 + \varepsilon_0} |T\varphi|^2 + c_{\delta_0} \varepsilon_0 s^3 R^{-3\varepsilon_0} r^{-4 + 3\varepsilon_0} |\varphi|^2 \right) \right\} dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} (f + \tilde{f}) v'_s \left(\frac{r}{R} \right) \left(c_{\delta_0} R^{-2} s |T\varphi|^2 - C_{\delta_0} R^{-4} s^3 |\varphi|^2 \right) dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} (f(R) + \tilde{f}(R)) \left(c r^{-2} |T\varphi|^2 - C R^{-1} r^{-3} |\varphi|^2 \right) dg \leq \\
& \leq C_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap (\mathcal{E}_{ext} \cup \{r \leq \frac{1}{4}r_0\})} (f + \tilde{f}) \left\{ s^2 R^{-6\varepsilon_0} |\nabla \varphi|_{g_{ref}}^2 + s^4 R^{-12\varepsilon_0} |\varphi|^2 \right\} dg - \\
& - \int_{\mathcal{R}(\tau_1, \tau_2)} Re \left\{ G(2(\nabla^{\mu} f + \nabla^{\mu} \tilde{f}) \nabla_{\mu} \bar{\varphi} + (\square_g f + \square_g \tilde{f} - 2h - 2\tilde{h}) \bar{\varphi}) \right\} dg - \\
& - \mathcal{B}_{f,h}^{(R)}[\varphi; \tau_1, \tau_2] - \mathcal{B}_{\tilde{f},h}^{(R)}[\varphi; \tau_1, \tau_2],
\end{aligned}$$

where, for any set $A \subset \mathcal{M}$, we denote with χ_A the characteristic function of A .

In view of the fact that $(g^{-1})_{\Sigma}$ is positive definite on $\mathcal{M} \setminus \mathcal{E} \cup \mathcal{H}$, we can estimate on $\mathcal{M} \setminus \mathcal{E}_{\delta} \cup \{r \geq \frac{1}{4}r_0\}$ for any $\delta > 0$:

(2.5.140)

$$c_{\delta_0 \delta_1} s R^{-3\varepsilon_0} (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} - C s R^{-3\varepsilon_0} |\nabla_{g_{\Sigma}} \varphi|_{g_{\Sigma}} |T\varphi| \geq c_{\delta \delta_0 \delta_1} s R^{-3\varepsilon_0} |\nabla_{g_{\Sigma}} \varphi|_{g_{\Sigma}} - C_{\delta} s R^{-3\varepsilon_0} |T\varphi|^2.$$

Furthermore, if $\chi_{r_0} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow [0, 1]$ is a smooth T -invariant function supported in $\{r \leq r_0\}$

such that $\chi_{r_0} \equiv 1$ on $\{r \leq \frac{1}{2}r_0\}$, then, after integrating by parts in the identity

$$(2.5.141) \quad \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ N(\chi_{r_0} \bar{\varphi}) \square_g (\chi_{r_0} \varphi) \right\} dg = \\ = \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ N(\chi_{r_0} \bar{\varphi}) (\chi_{r_0} G + 2\nabla^\mu \chi_{r_0} \nabla_\mu \varphi + \square_g \chi_{r_0} \varphi) \right\} dg,$$

using also the bounds (2.1.5) and (2.1.6) from Assumption G2 (as well as a Poincare-type inequality), we readily obtain the red-shift-type estimate

$$(2.5.142) \quad c \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} (J_\mu^N(\varphi) n_{\mathcal{H}^+}^\mu + |\varphi|^2) d\operatorname{vol}_{\mathcal{H}^+} + c \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq \frac{1}{2}r_0\}} (|\nabla \varphi|_{g_{ref}}^2 + |\varphi|^2) dg \leq$$

$$(2.5.143) \quad \leq C \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}r_0 \leq r \leq r_0\}} (|\nabla \varphi|_{g_{ref}}^2 + |\varphi|^2) dg + C \int_{\Sigma_{\tau_1} \cap \{r \leq r_0\}} |\nabla \varphi|_{g_{ref}}^2 - \\ - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \chi_{r_0} N(\chi_{r_0} \bar{\varphi}) \cdot G \right\} dg.$$

Therefore, in view of (2.5.140), (2.5.142) and the fact that

$$(2.5.144) \quad \frac{\sup_{\{r \leq \frac{1}{4}r_0\}} (e^{2s\tilde{w}_R} + e^{2s\tilde{w}_R})}{\inf_{\{r \geq \frac{1}{2}r_0\}} (e^{2s\tilde{w}_R} + e^{2s\tilde{w}_R})} > e^{csR^{-3\varepsilon_0}}$$

(following from (2.5.43), (2.5.44), (2.5.76) and (2.5.77)), (2.5.139) yields (provided $sR^{-3\varepsilon_0} \gg 1$):

$$(2.5.145) \quad \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_0} (f + \tilde{f} + \sup_{\{r \leq \frac{1}{4}r_0\}} f) \left\{ c_{\delta_0 \delta_1} sR^{-3\varepsilon_0} |\nabla_{g_\Sigma} \varphi|_{g_\Sigma}^2 - \right. \\ \left. - C_\delta sR^{-3\varepsilon_0} |T\varphi|^2 + c_{\delta_0} s^3 R^{-9\varepsilon_0} |\varphi|^2 \right\} dg + \\ + c_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \frac{1}{2}R_0\}} (f + \tilde{f}) \left\{ \chi_{\{\frac{1}{2}R_0 \leq r \leq R_0\}} \delta_1 sR^{-3\varepsilon_0} + \chi_{\{R_0 \leq r \leq R^{\varepsilon_0}\}} sR^{-3\varepsilon_0} r^{-\frac{3}{2}} + \right. \\ + \chi_{\{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}} sR^{-\varepsilon_0} r^{-\frac{5}{2} + \varepsilon_0} + \chi_{\{\frac{1}{2}R \leq r \leq \delta_2^{-1}R\}} r^{-\frac{5}{2}} + \\ \left. + \chi_{\{r \geq \delta_2^{-1}R\}} r^{-2} \right\} (|\partial_r \varphi|^2 + r^{-2} |\partial_\sigma \varphi|^2) dg +$$

$$\begin{aligned}
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq \frac{1}{2}R\}} (f + \tilde{f}) \left\{ \chi_{\{\frac{1}{2}R_0 \leq r \leq R_0\}} \left(-C_{\delta_0} s R^{-3\epsilon_0} |T\varphi|^2 + c_{\delta_0} s^3 R^{-9\epsilon_0} |\varphi|^2 \right) + \right. \\
& \quad + C_{\delta_0} \chi_{\{R_0 \leq r \leq R^{\epsilon_0}\}} \left(s R^{-3\epsilon_0} r^{-1} |T\varphi|^2 + s^3 R^{-9\epsilon_0} |\varphi|^2 \right) + \\
& \quad \left. + \chi_{\{R^{\epsilon_0} \leq r \leq \frac{1}{2}R\}} \left(s R^{-\epsilon_0} r^{-2+\epsilon_0} |T\varphi|^2 + c_{\delta_0} \epsilon_0 s^3 R^{-3\epsilon_0} r^{-4+3\epsilon_0} |\varphi|^2 \right) \right\} dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} (f + \tilde{f}) v'_s \left(\frac{r}{R} \right) \left(c_{\delta_0} R^{-2} s |T\varphi|^2 - C_{\delta_0} R^{-4} s^3 |\varphi|^2 \right) dg + \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} (f(R) + \tilde{f}(R)) \left(c r^{-2} |T\varphi|^2 - C R^{-1} r^{-3} |\varphi|^2 \right) dg \leq \\
& \leq C_{\delta_0 \delta} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{C}_{\delta_0}} (f + \tilde{f}) \left\{ s^2 R^{-6\epsilon_0} |\nabla \varphi|_{g_{ref}}^2 + s^4 R^{-12\epsilon_0} |\varphi|^2 \right\} dg - \\
& - \int_{\mathcal{R}(\tau_1, \tau_2)} Re \left\{ G \left(2(\nabla^\mu f + \nabla^\mu \tilde{f}) \nabla_\mu \bar{\varphi} + (\square_g f + \square_g \tilde{f} - 2h - 2\tilde{h}) \bar{\varphi} \right) \right\} dg + \\
& + C \bar{\mathcal{B}}_{f,h;\tilde{f},\tilde{h}}^{(R)}[\varphi; \tau_1, \tau_2], -\mathcal{B}_{f,h}^{(R)}[\varphi; \tau_1, \tau_2] - \mathcal{B}_{\tilde{f},\tilde{h}}^{(R)}[\varphi; \tau_1, \tau_2] + \\
& + C \sup_{\{r \leq \frac{1}{4}r_0\}} f \int_{\Sigma_{\tau_1} \cap \{r \leq r_0\}} |\nabla \varphi|_{g_{ref}}^2,
\end{aligned}$$

where

(2.5.146)

$$\begin{aligned}
\bar{\mathcal{B}}_{f,h;\tilde{f},\tilde{h}}^{(R)}[\varphi; \tau_1, \tau_2] & \doteq \sum_{j=1}^2 \left| \int_{\Sigma_{\tau_j}} Re \left\{ \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f - 2h) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \right. \right. \right. \\
& \quad \left. \left. + \left(\chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f)) \right) |\varphi|^2 \right) n_{\Sigma_{\tau_j}}^\nu \right\} dg_{\Sigma_{\tau_j}} \right| + \\
& + \sum_{j=1}^2 \left| \int_{\Sigma_{\tau_j}} Re \left\{ \left(2\nabla^\mu \tilde{f} \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g \tilde{f} - 2\tilde{h}) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu \tilde{f} \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \right. \right. \right. \\
& \quad \left. \left. + \left(\chi_{\leq R} \tilde{f}^{-1} \nabla_\mu \nabla_\nu \tilde{f} \nabla^\mu \tilde{f} + \nabla_\nu \tilde{h} - \frac{1}{2} (\nabla_\nu (\square_g \tilde{f})) \right) |\varphi|^2 \right) n_{\Sigma_{\tau_j}}^\nu \right\} dg_{\Sigma_{\tau_j}} \right| + \\
& + \sup_{\{r \leq \frac{1}{4}r_0\}} f \int_{\Sigma_{\tau_1} \cap \{r \leq r_0\}} |\nabla \varphi|_{g_{ref}}^2
\end{aligned}$$

(note that the boundary terms on $\mathcal{H}^+$ in the right hand side of (2.5.139) were absorbed by the term in the left hand side of (2.5.142)). Inequality (2.5.3) now readily follows from (2.5.145) in view of (2.5.76), (2.5.77). $\square$

2.5.8 Proof of Corollary 2.2

For any $1 \leq k \leq n$ and $0 < \delta_1, \delta_2, \varepsilon_0 \ll 1$, let us choose the parameters R, s to be sufficiently large in terms of $\delta_1, \delta_2, \varepsilon_0$ and the geometry of $(\mathcal{M}, g)$, satisfying in addition:

$$(2.5.147) \quad R \geq C_{\delta_1 \varepsilon_0} \max\{1, \omega_k^{-\frac{1}{1-9\varepsilon_0}}, (-\log \delta_2)^{\frac{1}{1-9\varepsilon_0}}\},$$

$$(2.5.148) \quad C_{\delta_1 \varepsilon_0}^{\frac{1}{3}} \max\{(1 + \omega_k)R^{9\varepsilon_0}, -\log \delta_2\} \leq s \leq C_{\delta_1 \varepsilon_0}^{-\frac{1}{3}} R \omega_k,$$

for some constant $C_{\delta_1 \varepsilon_0} > 1$ large in terms of δ_1, ε_0 and the geometry of $(\mathcal{M}, g)$ (notice that the bound (2.5.147) guarantees that an s satisfying (2.5.148) exists).

By approximating the functions ψ_k , $1 \leq k \leq n$, by smooth solutions to (2.0.1) with compact support in space and using Lemma 2.7 on the decay of ψ_k as $r \rightarrow +\infty$, we infer that Proposition 2.2 also applies for the functions ψ_k . Therefore, using the values of s, R chosen above, we obtain for any $0 \leq \tau_1 \leq \tau_2$ and any $1 \leq k \leq n$:

$$(2.5.149) \quad \begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R_0\} \setminus \mathcal{E}_{\delta_1}} \left(f + \inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}} f \right) \left\{ sR^{-3\varepsilon_0} |\nabla_{g_\Sigma} \psi_k|_{g_\Sigma}^2 - C_{\delta_1} s R^{-3\varepsilon_0} |T\psi_k|^2 + s^3 R^{-9\varepsilon_0} |\psi_k|^2 \right\} dg + \\ & + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq \frac{1}{2}R\}} f \left\{ sR^{-3\varepsilon_0} r^{-\frac{5}{2}} (|\partial_r \psi_k|^2 + r^{-2} |\partial_\sigma \psi_k|^2) + \right. \\ & \quad \left. + sR^{-3\varepsilon_0} r^{-2} |T\psi_k|^2 + \varepsilon_0 s^3 R^{-9\varepsilon_0} r^{-4} |\psi_k|^2 \right\} dg + \\ & + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} f \left\{ r^{-\frac{5}{2}} (|\partial_r \psi_k|^2 + r^{-2} |\partial_\sigma \psi_k|^2) + \right. \\ & \quad \left. + R \partial_r w_R (cR^{-2} s |T\psi_k|^2 - CR^{-4} s^3 |\psi_k|^2) \right\} dg + \\ & + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} f(R) \left\{ r^{-\frac{5}{2}} (|\partial_r \psi_k|^2 + r^{-2} |\partial_\sigma \psi_k|^2) + \right. \\ & \quad \left. + r^{-2} |T\psi_k|^2 - CR^{-1} r^{-3} |\psi_k|^2 \right\} dg \leq \end{aligned}$$

$$\begin{aligned}
&\leq C_{\delta_1} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{O}_{\delta_1}} f \left\{ s^2 R^{-6\epsilon_0} |\nabla \psi_k|_{g_{ref}}^2 + s^4 R^{-12\epsilon_0} |\psi_k|^2 \right\} dg + \\
&\quad + C \left| \int_{\mathcal{R}(\tau_1, \tau_2)} F_k \left(\nabla^\mu f \nabla_\mu \bar{\psi}_k + O \left(\sum_{j=1}^2 (1+r)^{j-2} |\nabla^j f|_{g_{ref}} \right) \bar{\psi}_k \right) dg \right| + \\
&\quad + C \sum_{j=1}^2 \int_{\Sigma_{\tau_j}} \left(|\nabla f|_{g_{ref}} |\nabla \psi_k|_{g_{ref}}^2 + \left(\sum_{j=1}^3 (1+r)^{j-3} |\nabla^j f|_{g_{ref}} \right) |\psi_k|^2 \right) dg_\Sigma.
\end{aligned}$$

In view Lemma 2.8, the bound (2.5.149) implies:

(2.5.150)

$$\begin{aligned}
&\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R_0\} \setminus \mathcal{O}_{\delta_1}} \left(f + \inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{O}} f \right) \left\{ sR^{-3\epsilon_0} |\nabla_{g_\Sigma} \psi_k|_{g_\Sigma}^2 + sR^{-3\epsilon_0} |T\psi_k|^2 + \right. \\
&\quad \left. + (s^3 R^{-9\epsilon_0} - C_{\delta_1} \omega_k^2 s R^{-3\epsilon_0}) |\psi_k|^2 \right\} dg + \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq \frac{1}{2}R\}} f \left\{ sR^{-3\epsilon_0} r^{-\frac{5}{2}} (|\partial_r \psi_k|^2 + r^{-2} |\partial_\sigma \psi_k|^2) + \right. \\
&\quad \left. + sR^{-3\epsilon_0} r^{-2} |T\psi_k|^2 + \epsilon_0 s^3 R^{-9\epsilon_0} r^{-4} |\psi_k|^2 \right\} dg + \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} f \left\{ r^{-\frac{5}{2}} (|\partial_r \psi_k|^2 + r^{-2} |\partial_\sigma \psi_k|^2) + R^{-1} s \partial_r w_R |T\psi_k|^2 + \right. \\
&\quad \left. + R^{-1} s \partial_r w_R (c\omega_k^2 - CR^{-2} s^2) |\psi_k|^2 \right\} dg + \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} f(R) \left\{ r^{-\frac{5}{2}} (|\partial_r \psi_k|^2 + r^{-2} |\partial_\sigma \psi_k|^2) + cr^{-2} |T\psi_k|^2 + \right. \\
&\quad \left. + (c\omega_k^2 - CR^{-2}) r^{-2} |\psi_k|^2 \right\} dg \leq \\
&\leq C_{\delta_1} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{O}_{\delta_1}} f \left\{ s^2 R^{-6\epsilon_0} |\nabla \psi_k|_{g_{ref}}^2 + s^4 R^{-12\epsilon_0} |\psi_k|^2 \right\} dg + \\
&\quad + C \left| \int_{\mathcal{R}(\tau_1, \tau_2)} F_k \left(\nabla^\mu f \nabla_\mu \bar{\psi}_k + O \left(\sum_{j=1}^2 (1+r)^{j-2} |\nabla^j f|_{g_{ref}} \right) \bar{\psi}_k \right) dg \right| + \\
&\quad + C \sum_{j=1}^2 \int_{\Sigma_{\tau_j}} \left(|\nabla f|_{g_{ref}} |\nabla \psi_k|_{g_{ref}}^2 + \left(\sum_{j=1}^3 (1+r)^{j-3} |\nabla^j f|_{g_{ref}} \right) |\psi_k|^2 \right) dg_\Sigma +
\end{aligned}$$

$$\begin{aligned}
& + C \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{H}^+} \left(|\nabla f|_{g_{ref}} J_{\mu}^N(\psi_k) n_{\mathcal{H}^+}^{\mu} + \left(\sum_{j=1}^3 (1+r)^{j-3} |\nabla^j f|_{g_{ref}} \right) |\psi_k|^2 \right) dvol_{\mathcal{H}^+} + \\
& + C \omega_k^2 (1 + \omega_k^{-2}) (\log(2 + \tau_2))^4 R_0^2 \sup_{\{r \leq R_0\}} f \cdot \mathcal{E}_{log}[\psi] + \\
& + C \omega_k^2 (1 + \omega_k^{-6}) \sup_{\{r \leq R\}} f \cdot \mathcal{E}_{log}[\psi].
\end{aligned}$$

In view of the bound (2.5.148) for the parameters R, s , as well as the properties of the function (2.5.2), inequality (2.5.150) yields (using also (2.4.26) and Lemma 2.6, combined with a Cauchy–Schwarz inequality, to estimate the second and third terms in the right hand side of (2.5.150)):

(2.5.151)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \setminus \mathcal{E}_{2\delta_1}} \left((1+r)^{-\frac{5}{2}} |\nabla \psi_k|_{g_{ref}}^2 + (\omega_k^2 r^{-2} + r^{-4}) |\psi_k|^2 \right) dg \leq \\
& \leq \sum_{j=1}^4 \left(|\nabla^j w_R|_{g_{ref}} + |\nabla^j \tilde{w}_R|_{g_{ref}} \right) \left\{ C_{\delta_1} \frac{\sup_{\mathcal{E}_{\delta_1}} (e^{2sw_R} + e^{2s\tilde{w}_R})}{\inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_{2\delta_1}} (e^{2sw_R} + e^{2s\tilde{w}_R})} \times \right. \\
& \quad \times (sR^{-3\epsilon_0}) \int_{\mathcal{R}(\tau_1, \tau_2) \setminus \mathcal{E}_{\delta_1}} \left(|\nabla \psi_k|_{g_{ref}}^2 + |\psi_k|^2 \right) dg + \\
& \quad \left. + C_{\delta_1} (1 + \omega_k^{-10}) (\log(2 + \tau_2))^4 \frac{\sup_{\{r \leq R\}} (e^{2sw_R} + e^{2s\tilde{w}_R})}{\inf_{\{r \leq R\}} (e^{2sw_R} + e^{2s\tilde{w}_R})} \mathcal{E}_{log}[\psi] \right\}.
\end{aligned}$$

In view of the properties of the function 2.5.2, we can estimate

$$(2.5.152) \quad \sup_{\{r \leq R\}} w_R - \inf_{\{r \leq R\}} w_R + \sup_{\{r \leq R\}} \tilde{w}_R - \inf_{\{r \leq R\}} \tilde{w}_R \leq C \epsilon_0^{-1} R^{3\epsilon_0},$$

$$(2.5.153) \quad \inf_{\{\frac{1}{4}r_0 \leq R\} \setminus \mathcal{E}_{2\delta_1}} w_R \geq \max_{\mathcal{E}_{\delta_1}} w_R + c_{\delta_1} R^{-3\epsilon_0},$$

$$(2.5.154) \quad \inf_{\{\frac{1}{4}r_0 \leq r \leq R\} \setminus \mathcal{E}_{2\delta_1}} \tilde{w}_R \geq \max_{\mathcal{E}_{\delta_1}} \tilde{w}_R + c_{\delta_1} R^{-3\epsilon_0}$$

and

$$(2.5.155) \quad \sum_{j=1}^4 \left(|\nabla^j w_R|_{g_{ref}} + |\nabla^j \tilde{w}_R|_{g_{ref}} \right) \leq C.$$

Therefore, inequality (2.5.6) readily follows from 2.5.151, provided $C_{\delta_1 \varepsilon_0}$ in (2.5.148) is sufficiently large in terms of δ_1, ε_0 . $\square$

2.5.9 Proof of Proposition 2.2 in the case of Dirichlet or Neumann boundary conditions

In this section, we will briefly sketch how the proof of Proposition 2.2 can be applied to the case when the boundary $\partial\mathcal{M}$ of $(\mathcal{M}, g)$ is allowed to have a non-trivial timelike component $\partial_{tim}\mathcal{M}$ and equation (2.5.1) is supplemented with Dirichlet or Neumann boundary conditions for φ on $\partial_{tim}\mathcal{M}$.

We will first describe the class of Lorentzian manifolds with such a boundary component on which Proposition 2.2 will apply. Let $(\mathcal{M}^{d+1}, g)$, $d \geq 2$, be a smooth Lorentzian manifold with piecewise smooth boundary $\partial\mathcal{M}$ splitting as

$$(2.5.156) \quad \partial\mathcal{M} = \partial_{hor}\mathcal{M} \cup \partial_{tim}\mathcal{M},$$

where $\partial_{hor}\mathcal{M}$ has the structure of a piecewise smooth null hypersurface and $\partial_{tim}\mathcal{M}$ is a smooth timelike hypersurface, with $\partial_{hor}\mathcal{M} \cap \partial_{tim}\mathcal{M} = \emptyset$. For the discussion of this section, we will assume that $\partial_{tim}\mathcal{M} \neq \emptyset$, but $\partial_{hor}\mathcal{M}$ will be allowed to be empty. Let $(\tilde{\mathcal{M}}, \tilde{g})$ be the double of $(\mathcal{M}, g)$ across $\partial_{tim}\mathcal{M}$, which is defined as the disjoint union of two copies of $(\mathcal{M}, g)$ glued along $\partial_{tim}\mathcal{M}$ (for the relevant definitions, see e. g. [97]). Let $i_1, i_2 : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ be the two natural isometric embeddings of $(\mathcal{M}, g)$ into $(\tilde{\mathcal{M}}, \tilde{g})$. Note that $\tilde{\mathcal{M}} = i_1(\mathcal{M}) \cup i_2(\mathcal{M})$ and $i_1(\partial_{tim}\mathcal{M}) = i_2(\partial_{tim}\mathcal{M})$. Furthermore, $\tilde{\mathcal{M}}$ is a smooth manifold, and the metric $\tilde{g}$ is continuous and piecewise smooth on $\tilde{\mathcal{M}}$ and smooth on $\tilde{\mathcal{M}} \setminus i_1(\partial_{tim}\mathcal{M})$.¹² We will always identify $\mathcal{M}$ with $i_1(\mathcal{M}) \subset \tilde{\mathcal{M}}$.

We will assume that $(\tilde{\mathcal{M}}, \tilde{g})$ is a globally hyperbolic Lorentzian manifold (with the regularity of $\tilde{g}$ as described before), satisfying Assumptions G1, G2 and G3 of Section 2.1 (for the discussion of this Section, we can also allow the case $\mathcal{E} = \emptyset$). Additionally, we will assume that the stationary Killing field T of $\tilde{\mathcal{M}}$ (defined by Assumption G1) is tangent to $i_1(\partial_{tim}\mathcal{M})$. Let also $\Sigma_{\tilde{\mathcal{M}}}, \mathcal{S}_{\tilde{\mathcal{M}}}, \mathcal{H}_{\tilde{\mathcal{M}}}^\pm \subset \tilde{\mathcal{M}}$, $t_{\tilde{\mathcal{M}}} : \tilde{\mathcal{M}} \setminus \mathcal{H}_{\tilde{\mathcal{M}}}^- \rightarrow \mathbb{R}$ and $r_{\tilde{\mathcal{M}}} : \tilde{\mathcal{M}} \setminus \mathcal{H}_{\tilde{\mathcal{M}}}^- \rightarrow [0, +\infty)$ be

¹²The metric $\tilde{g}$ is continuous across $i_1(\partial_{tim}\mathcal{M})$, but fails to be C^1 at all the points of $i_1(\partial_{tim}\mathcal{M})$ on which the second fundamental form of $i_1(\partial_{tim}\mathcal{M})$ is non-zero.

as defined under Assumption G1. We will assume without loss of generality that $\Sigma_{\tilde{\mathcal{M}}}, \mathcal{S}_{\tilde{\mathcal{M}}}$ intersect $i_1(\partial_{tim}\mathcal{M})$ transversally, and that $\Sigma_{\tilde{\mathcal{M}}} \cap i_1(\partial_{tim}\mathcal{M}), \mathcal{S}_{\tilde{\mathcal{M}}} \cap i_1(\partial_{tim}\mathcal{M})$ are compact. Note that the restriction of $\mathcal{H}_{\tilde{\mathcal{M}}}$ on $\mathcal{M}$ coincides with $\partial_{hor}\mathcal{M}$.

Remark. We will use the notation $\Sigma, \mathcal{S}, \mathcal{H}^\pm, t$ and r for the restriction of the hypersurfaces $\Sigma_{\tilde{\mathcal{M}}}, \mathcal{S}_{\tilde{\mathcal{M}}}, \mathcal{H}_{\tilde{\mathcal{M}}}^\pm$ and the functions $t_{\tilde{\mathcal{M}}}, r_{\tilde{\mathcal{M}}}$ on $\mathcal{M} \simeq i_1(\mathcal{M})$.

For any $F \in C^\infty(\mathcal{M})$ and any $(\varphi_0, \varphi_1) \in C^\infty(\Sigma) \times C^\infty(\Sigma)$, the initial-boundary value problem

$$(2.5.157) \quad \begin{cases} \square_g \varphi = G & \text{on } \{t \geq 0\} \\ (\varphi, T\varphi) = (\varphi_0, \varphi_1) & \text{on } \{t = 0\} \\ \varphi = 0 & \text{on } \partial_{tim}\mathcal{M} \end{cases}$$

is well posed on $\{t \geq 0\} \subset \mathcal{M}$. This follows from the assumption that $(\tilde{\mathcal{M}}, \tilde{g})$ is globally hyperbolic. The Dirichlet boundary condition $\varphi|_{\partial_{tim}\mathcal{M}} = 0$ in (2.5.157) can also be replaced by the Neumann boundary condition

$$(2.5.158) \quad n_{\partial_{tim}\mathcal{M}}(\varphi)|_{\partial_{tim}\mathcal{M}} = 0,$$

where $n_{\partial_{tim}\mathcal{M}}$ is the unit normal vector field on $\partial_{tim}\mathcal{M}$, pointing towards the interior of $\mathcal{M}$.

On a spacetime $(\mathcal{M}, g)$ as above, we will extend Proposition 2.2 as follows:

Proposition 2.3. *Let $(\mathcal{M}, g)$ be a Lorentzian manifold with boundary as above. For any $s, R \gg 1$ sufficiently large in terms of the geometry of $(\mathcal{M}, g)$ and any $0 < \epsilon_0 < 1$, there exists a smooth T -invariant function $f : \mathcal{M} \setminus \mathcal{H}^- \rightarrow (0, +\infty)$ as in Proposition 2.2, so that (provided $\epsilon_0 s R^{-9\epsilon_0} \gg 1$), for any $0 < \delta \ll 1$, any $0 \leq \tau_1 \leq \tau_2$ and any smooth function $\varphi : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ with compact support on the hypersurfaces $\{t = \text{const}\}$ solving (2.5.1) and satisfying on $\partial_{tim}\mathcal{M}$ either the Dirichlet condition $\varphi = 0$ or the Neumann condition $n_{\partial_{tim}\mathcal{M}}(\varphi) = 0$, the estimate (2.5.3) holds.*

Proof. The proof of Proposition 2.3 follows in almost exactly the same way as the proof of Proposition 2.2, the only difference being the following: When using the multiplier (2.5.86)

for equation (2.5.1) as in Section 2.5.4 and after performing the same integration-by-parts procedure, one obtains instead of (2.5.98) the following relation:

$$\begin{aligned}
(2.5.159) \quad & \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + \right. \\
& \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg = \\
& = - \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ G(2\nabla^\mu f \nabla_\mu \bar{\varphi} + (\square_g f - 2h)\bar{\varphi}) \right\} dg - \mathcal{B}_{f,h}^{(R)}[\varphi; \tau_1, \tau_2] - \mathcal{B}_{f,h}^{(b)}[\varphi; \tau_1, \tau_2],
\end{aligned}$$

where

$$\begin{aligned}
(2.5.160) \quad & \mathcal{B}_{f,h}^{(b)}[\varphi; \tau_1, \tau_2] = \int_{\partial_{tim} \mathcal{M} \cap \mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f - 2h) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \right. \right. \\
& \left. \left. + \left(f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f)) \right) |\varphi|^2 \right) n_{\partial_{tim} \mathcal{M}}^\nu \right\} dg_{\partial_{tim} \mathcal{M}},
\end{aligned}$$

$g_{\partial_{tim} \mathcal{M}}$ being the induced (Lorentzian) metric on $\partial_{tim} \mathcal{M}$. Notice that (2.5.159) differs from (2.5.98) only by the term (2.5.160) in the right hand side.

Let us assume, without loss of generality, that the function $\bar{w}$ of Lemma 2.11 has been chosen so that it additionally satisfies $n_{\partial_{tim} \mathcal{M}}(\bar{w}) > 0$, with $\bar{w}$ being constant on $\partial_{tim} \mathcal{M}$ (it can be readily checked that Lemma 2.11 can be established under this additional assumption). In the case when φ satisfies the Dirichlet boundary condition $\varphi|_{\partial_{tim} \mathcal{M}} = 0$, it is straightforward to check that this choice of $\bar{w}$ implies (in view of the choice of the functions f, h in Section 2.5.3) that the term (2.5.160) is non-negative, and in particular

$$\begin{aligned}
(2.5.161) \quad & \int_{\partial_{tim} \mathcal{M} \cap \mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f - 2h) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \right. \right. \\
& \left. \left. + \left(f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f)) \right) |\varphi|^2 \right) n_{\partial_{tim} \mathcal{M}}^\nu \right\} dg_{\partial_{tim} \mathcal{M}} \geq \\
& \geq c \int_{\partial_{tim} \mathcal{M} \cap \mathcal{R}(\tau_1, \tau_2)} n_{\partial_{tim} \mathcal{M}}(f) |n_{\partial_{tim} \mathcal{M}}(\varphi)|^2 dg_{\partial_{tim} \mathcal{M}} \geq 0
\end{aligned}$$

for some $c > 0$. Thus, the term (2.5.160) can be dropped from the right hand side of (2.5.159) (thus yielding (2.5.98)) and one can proceed as before to establish (2.5.3).

In the case when φ satisfies the Neumann boundary condition $n_{\partial_{tim}\mathcal{M}}(\varphi)|_{\partial_{tim}\mathcal{M}} = 0$, (2.5.160) is not necessarily non-negative, since the term

$$n_{\partial_{tim}\mathcal{M}}(f)\nabla^\mu\varphi\nabla_\mu\bar{\varphi}$$

in (2.5.160) does not necessarily have a sign (as is the case when $\varphi|_{\partial_{tim}\mathcal{M}} = 0$). In order to absorb this term, we proceed as follows: Let $\mathcal{U} \subset \mathcal{M}$ be a (small) T -invariant tubular neighborhood of $\partial_{tim}\mathcal{M}$ (so that $\mathcal{U} \cap (\mathcal{H} \cup \mathcal{B}_{crit}(8\delta_0)) = \emptyset$), split as $\mathcal{U} \simeq [0, 1) \times \partial_{tim}\mathcal{M}$, where the projection onto the factor $[0, 1)$ is given by a smooth function $\bar{r} : \mathcal{U} \rightarrow [0, 1)$ such that $\nabla^\mu\bar{r}|_{\partial_{tim}\mathcal{M}} = n_{\partial_{tim}\mathcal{M}}^\mu$, and the projection onto $\partial_{tim}\mathcal{M}$ is given by a smooth map $\bar{\sigma} : \mathcal{U} \rightarrow \partial_{tim}\mathcal{M}$. We will extend $n_{\partial_{tim}\mathcal{M}}$ on the whole of $\mathcal{U}$ by the relation

$$(2.5.162) \quad n_{\partial_{tim}\mathcal{M}}^\mu = \nabla^\mu\bar{r}.$$

Let $\chi_c : [0, 1) \rightarrow [0, 1]$ be a smooth function satisfying $\chi_c \equiv 1$ on $[0, \frac{1}{4}]$ and $\chi_c \equiv 0$ on $[\frac{1}{2}, 1)$, and let us define the function $\check{f} : \mathcal{M} \rightarrow \mathbb{R}$ by the relation

$$\check{f}(\bar{r}, \bar{\sigma}) \doteq \chi_c(\bar{r}) \cdot (n_{\partial_{tim}\mathcal{M}}(f))|_{\partial_{tim}\mathcal{M}}(\bar{\sigma}) \text{ on } \mathcal{U} \simeq [0, 1) \times \partial_{tim}\mathcal{M}$$

(where $(n_{\partial_{tim}\mathcal{M}}(f))|_{\partial_{tim}\mathcal{M}}$ is the value of $n_{\partial_{tim}\mathcal{M}}(f)$ on $\{\bar{r} = 0\}$) and

$$\check{f} \equiv 0 \text{ on } \mathcal{M} \setminus \mathcal{U}.$$

Adding to (2.5.159) the identity

(2.5.163)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2\nabla^\mu (\check{f} n_{\partial_{tim}\mathcal{M}}^\vee) \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu (\check{f} n_{\partial_{tim}\mathcal{M}}^\vee) \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} dg = \\
& = - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G \cdot 2\check{f} n_{\partial_{tim}\mathcal{M}}(\bar{\varphi}) \right\} dg - \\
& - \int_{\partial_{tim}\mathcal{M} \cap \mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ (2\check{f} n_{\partial_{tim}\mathcal{M}}(\bar{\varphi}) \nabla_\nu \varphi - \check{f} (n_{\partial_{tim}\mathcal{M}})_\nu \nabla^\mu \varphi \nabla_\mu \bar{\varphi}) n_{\partial_{tim}\mathcal{M}}^\vee \right\} dg_{\partial_{tim}\mathcal{M}}
\end{aligned}$$

and using the Neumann condition $n_{\partial_{tim}\mathcal{M}}(\varphi)|_{\partial_{tim}\mathcal{M}} = 0$, we thus infer:

(2.5.164)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) + 2\nabla^\mu (\check{f} n_{\partial_{tim}\mathcal{M}}^\vee) \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \right. \\
& - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + \\
& + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - (2h + \nabla_\nu (\check{f} n_{\partial_{tim}\mathcal{M}}^\vee)) \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \Big\} dg = \\
& = - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G(2\nabla^\mu f \nabla_\mu \bar{\varphi} + (\square_g f - 2h)\bar{\varphi}) \right\} dg - \mathcal{B}_{f,h}^{(R)}[\varphi; \tau_1, \tau_2] - \check{\mathcal{B}}_{f,h}^{(b)}[\varphi; \tau_1, \tau_2],
\end{aligned}$$

where

(2.5.165)

$$\check{\mathcal{B}}_{f,h}^{(b)}[\varphi; \tau_1, \tau_2] = \int_{\partial_{tim}\mathcal{M} \cap \mathcal{R}(\tau_1, \tau_2)} \left(f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f)) \right) |\varphi|^2 n_{\partial_{tim}\mathcal{M}}^\vee dg_{\partial_{tim}\mathcal{M}}.$$

Notice that, if $sR^{-3\varepsilon_0} \gg 1$, the term (2.5.165) is non-negative (in view of the properties of the functions f, h , see Section 2.5.3), and thus it can be dropped from the right hand side of (2.5.164). Furthermore, if $l \gg 1$ in Lemma 2.11 and $sR^{-3\varepsilon_0} \gg 1$, the terms $2\nabla^\mu (\check{f} n_{\partial_{tim}\mathcal{M}}^\vee) \nabla_\mu \varphi \nabla_\nu \bar{\varphi}$ and $\nabla_\nu (\check{f} n_{\partial_{tim}\mathcal{M}}^\vee) \nabla^\mu \varphi \nabla_\mu \bar{\varphi}$ in the left hand side of (2.5.164) (restricted to the complement of $\mathcal{E}_{ext}$) can be absorbed into the right hand side of 2.5.120. Thus, following exactly the same steps as we did in order to obtain (2.5.3) from (2.5.98) in the case $\partial_{tim}\mathcal{M} = \emptyset$, we can also obtain (2.5.3) from (2.5.164) in the case when $\partial_{tim}\mathcal{M} \neq \emptyset$ and $n_{\partial_{tim}\mathcal{M}}(\varphi)|_{\partial_{tim}\mathcal{M}} = 0$. $\square$

2.6 Proof of Proposition 2.1

Let us introduce the parameters $0 < \omega_0 \ll 1$, $\omega_+ \gg 1$ and $\tau_1 \geq \bar{\tau}_0 + \varepsilon^{-2}\tau_*$ depending on $\varepsilon, \delta_1, R, \tau_*, \bar{\tau}_0, \mathcal{E}_{log}[\varphi], \mathcal{E}_{log}[\psi], \mathcal{E}_{log}[T\psi]$ and $\mathcal{E}[T^2\psi]$ in the statement of Proposition 2.1 (we will fix ω_0 , ω_+ and τ_1 later), and, for $n = \lceil \log_2(\frac{\omega_+}{\omega_0}) \rceil$, let us decompose ψ and $T\psi$ into their frequency localised components $\{\psi_k\}_{k=0}^n$, $\psi_{\geq \omega_+}$ and $\{(T\psi)_k\}_{k=0}^n$, $(T\psi)_{\geq \omega_+}$, respectively, as in Section 2.4.2 (notice that (2.4.1) is satisfied in view of (2.3.18)).

In view of Lemma 2.9, (2.3.1) (and (2.3.18)), as well as a Hardy-type inequality (of the form (2.4.15)), we obtain for any $\tau \geq 2\tau_*$ and any $0 < a \ll 1$:

(2.6.1)

$$\begin{aligned}
\int_{\mathcal{R}(\tau-\tau_*, \tau+\tau_*) \cap \{r \leq R\}} (J_\mu^N(\psi_0)n^\mu + |\psi_0|^2) &\leq \\
&\leq C\omega_0^2 \int_{\mathcal{R}(\tau-\tau_*, \tau+\tau_*) \cap \{r \leq R\}} (J_\mu^N(\varphi)n^\mu + |\varphi|^2) + \\
&\quad + C_a \left(\omega_0^2 (\log(2+\tau))^4 + (1+\omega_0\tau)^{-1} \right) (1+\omega_0^{-1-a}) R^2 \mathcal{E}_{log}[\varphi] \leq \\
&\leq C_a \left\{ \omega_0^2 R^2 \tau_* (\log(2+\tau))^4 + \omega_0^2 (\log(2+\tau))^4 (1+\omega_0^{-1-a}) R^2 + \right. \\
&\quad \left. + (1+\omega_0\tau)^{-1} (1+\omega_0^{-1-a}) R^2 \right\} \mathcal{E}_{log}[\varphi].
\end{aligned}$$

From Lemmas 2.8 and 2.10, we obtain for any $\tau > 0$ and any $\delta > 0$:

$$\begin{aligned}
(2.6.2) \quad &\sum_{j=0}^{\lfloor \frac{\delta^{-1}-1}{2} \rfloor} \int_{\mathcal{R}(\tau+2j\tau_*, \tau+2(j+1)\tau_*) \cap \{r \leq R\}} (J_\mu^N(\psi_{\geq \omega_+})n^\mu + |\psi_{\geq \omega_+}|^2) \leq \\
&\leq \int_{\mathcal{R}(\tau, \tau+\delta^{-1}\tau_*) \cap \{r \leq R\}} (J_\mu^N(\psi_{\geq \omega_+})n^\mu + |\psi_{\geq \omega_+}|^2) \leq \\
&\leq C\delta^{-1}\tau_*\omega_+^{-2} (\mathcal{E}_{log}[\psi] + \mathcal{E}[T\psi]) + C\mathcal{E}_{log}[\psi].
\end{aligned}$$

We will assume that ω_+ is sufficiently large in terms of $\varepsilon, \tau_*, \mathcal{E}_{log}[\psi], \mathcal{E}[T\psi]$ so that

$$(2.6.3) \quad \tau_*\omega_+^{-2} (\mathcal{E}_{log}[\psi] + \mathcal{E}[T\psi]) \ll \varepsilon.$$

Let us also use the ansatz

$$(2.6.4) \quad \omega_0 = \frac{\bar{\omega}_0}{(\log(2 + \tau_1))^8},$$

and let us assume that $\bar{\omega}_0$ is sufficiently small in terms of $\varepsilon, R, \tau_*, \mathcal{E}_{\log}[\varphi]$, and τ_1 is sufficiently large in terms of $\varepsilon, R, \mathcal{E}_{\log}[\varphi], \bar{\omega}_0$, so that for any $\tau_1 \leq \tau \leq 100\tau_1$ (having fixed an $a \in (0, 1)$):

$$(2.6.5) \quad \begin{aligned} & (\omega_0^2 R^2 \tau_* (\log(2 + \tau))^4 + \omega_0^2 (\log(2 + \tau))^4 (1 + \omega_0^{-1-a}) R^2 + \\ & + (1 + \omega_0 \tau)^{-1} (1 + \omega_0^{-1-a}) R^2) \mathcal{E}_{\log}[\varphi] \ll \varepsilon \end{aligned}$$

(later, we will also need to assume that τ_1 is also sufficiently large in terms of ω_+). Then, (2.6.1), (2.6.2), (2.6.5) and (2.6.3) imply that for any $\tau \geq 0$:

$$(2.6.6) \quad \begin{aligned} & \sum_{l=0}^{\lfloor \frac{\delta^{-1}-1}{2} \rfloor} \int_{\mathcal{R}(\tau+2l\tau_*, \tau+2(l+1)\tau_*) \cap \{r \leq R\}} (J_\mu^N(\psi_0) n^\mu + J_\mu^N(\psi_{\geq \omega_+}) n^\mu + |\psi_0|^2 + |\psi_{\geq \omega_+}|^2) \leq \\ & \leq \frac{1}{20} \varepsilon \delta^{-1} + C \mathcal{E}_{\log}[\psi]. \end{aligned}$$

Repeating the same procedure for $T\psi$ in place of ψ and adding the result to (2.6.6), we obtain for any $\delta > 0$ (provided $\bar{\omega}_0$ is fixed sufficiently small in terms of $\varepsilon, R, \tau_*, \mathcal{E}_{\log}[\varphi], \mathcal{E}_{\log}[\psi]$, τ_1 is fixed sufficiently large in terms of $\varepsilon, R, \mathcal{E}_{\log}[\varphi], \mathcal{E}_{\log}[\psi] \bar{\omega}_0$ and ω_+ is fixed sufficiently large in terms of $\varepsilon, \tau_*, \mathcal{E}_{\log}[\psi], \mathcal{E}_{\log}[T\psi], \mathcal{E}[T^2\psi]$):

$$(2.6.7) \quad \begin{aligned} & \sum_{l=0}^{\lfloor \frac{\delta^{-1}-1}{2} \rfloor} \sum_{j=0}^1 \int_{\mathcal{R}(\tau+2l\tau_*, \tau+2(l+1)\tau_*) \cap \{r \leq R\}} (J_\mu^N((T^j\psi)_0) n^\mu + J_\mu^N((T^j\psi)_{\geq \omega_+}) n^\mu + \\ & + |(T^j\psi)_0|^2 + |(T^j\psi)_{\geq \omega_+}|^2) \leq \frac{1}{10} \varepsilon \delta^{-1} + C \sum_{j=0}^1 \mathcal{E}_{\log}[T^j\psi]. \end{aligned}$$

In view of Corollary 2.2, for any $1 \leq k \leq n$, any $0 < \delta_1, \delta_2 < 1$, any $0 < \varepsilon_0 < 1$ and any

$\bar{\tau} \geq \tau_1$, we can bound:

$$\begin{aligned}
(2.6.8) \quad & \int_{(\mathcal{R}(\tau_1, \bar{\tau}) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} \left(J_{\mu}^N(\psi_k) n^{\mu} + |\psi_k|^2 \right) \leq \\
& \leq C_R \delta_2 \int_{\mathcal{R}(\tau_1, \bar{\tau}) \cap \mathcal{E}_{\delta_1/2}} \left(J_{\mu}^N(\psi_k) n^{\mu} + |\psi_k|^2 \right) + \\
& + C_{\varepsilon_0 \delta_1 R} \omega_0^{-10} \left(\log(2 + \bar{\tau}) \right)^4 e^{C_{\varepsilon_0 \delta_1 \omega_+} \max\{\omega_0^{-\varepsilon_0}, -\log \delta_2\}} \mathcal{E}_{\log}[\psi].
\end{aligned}$$

Let us set

$$(2.6.9) \quad \delta_2 = \omega_0^3 \omega_+^{-1} \bar{\delta}_2,$$

where $\bar{\delta}_2$ is sufficiently small in terms of $\varepsilon, \varepsilon_0, R, \tau_*, \mathcal{E}_{\log}[\psi]$. Assuming also that $\bar{\omega}_0$ in (2.6.4) has been fixed sufficiently small in terms of $\varepsilon, \varepsilon_0, R, \tau_*, \mathcal{E}_{\log}[\psi]$, from (2.6.8), (2.4.26), (2.3.18) and the Poincare inequality

$$(2.6.10) \quad \int_{\mathcal{R}(\tau_1, \bar{\tau}) \cap \mathcal{E}_{\delta_1}} |\psi_k|^2 \leq C \int_{\mathcal{R}(\tau_1, \bar{\tau}) \cap \{r \leq R\}} J_{\mu}^N(\psi_k) n^{\mu} + C \int_{(\mathcal{R}(\tau_1, \bar{\tau}) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} |\psi_k|^2,$$

we obtain after summing over all $k \in \{1, \dots, n\}$ provided $\bar{\delta}_2$ is sufficiently small in terms of $\varepsilon, \varepsilon_0, R, \tau_*, \mathcal{E}_{\log}[\psi]$ (recall that $n \sim \log(\omega_0^{-1} \omega_+)$):

$$\begin{aligned}
(2.6.11) \quad & \sum_{k=1}^n \int_{(\mathcal{R}(\tau_1, \bar{\tau}) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} \left(J_{\mu}^N(\psi_k) n^{\mu} + |\psi_k|^2 \right) \leq \\
& \leq C_{\varepsilon_0 R} (\bar{\tau} - \tau_1) \delta_2 \omega_0^{-2} \log(\omega_0^{-1} \omega_+) \mathcal{E}_{\log}[\psi] + \\
& + C_{\varepsilon_0 \delta_1 R} \omega_0^{-10} \left(\log(2 + \bar{\tau}) \right)^4 e^{C_{\varepsilon_0 \delta_1 \omega_+} \max\{\omega_0^{-\varepsilon_0}, -\log \delta_2\}} \mathcal{E}_{\log}[\psi] \leq \\
& \leq \frac{1}{40 \tau_*} \varepsilon \bar{\tau} + C_1 \left(\log(2 + \bar{\tau}) \right)^{14} e^{C_1 \left(\log(2 + \tau_1) \right)^{8\varepsilon_0}},
\end{aligned}$$

where C_1 depends on $\varepsilon, \varepsilon_0, \delta_1, R, \tau_*, \mathcal{E}_{\log}[\psi], \omega_+$.

Repeating the same procedure for $T\psi$ in place of ψ , we obtain the following analogue of (2.6.11):

$$(2.6.12) \quad \sum_{k=1}^n \int_{(\mathcal{R}(\tau_1, \bar{\tau}) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} \left(J_{\mu}^N((T\psi)_k) n^{\mu} + |(T\psi)_k|^2 \right) \leq \frac{1}{40 \tau_*} \varepsilon \bar{\tau} + C_2 \left(\log(2 + \bar{\tau}) \right)^{14} e^{C_2 \left(\log(2 + \tau_1) \right)^{8\varepsilon_0}}$$

where C_2 depends on $\varepsilon, \varepsilon_0, \delta_1, R, \tau_*, \mathcal{E}_{\log}[T\psi], \omega_+$.

From (2.6.11) and (2.6.12) we obtain for any $\delta > 0$ (setting $\bar{\tau} = \tau_1 + \delta^{-1}\tau_*$)

$$(2.6.13) \quad \sum_{l=0}^{\lfloor \frac{\delta^{-1}-1}{2} \rfloor} \left\{ \sum_{k=1}^n \sum_{j=0}^1 \int_{\mathcal{R}(\tau_1+2l\tau_*, \tau_1+2(l+1)\tau_*) \setminus \mathcal{E}_{\delta_1} \cap \{r \leq R\}} \left(J_{\mu}^N((T^j\psi)_k) n^{\mu} + |(T^j\psi)_k|^2 \right) \right\} \leq \\ \leq \frac{1}{20\tau_*} \varepsilon(\tau_1 + \delta^{-1}\tau_*) + C_3 \left(\log(\tau_1 + \delta^{-1}\tau_*) \right)^{14} e^{C_3 \left(\log(2+\tau_1) \right)^{8\varepsilon_0}},$$

where $C_3 = C_1 + C_2$. Adding (2.6.7) (for $\tau = \tau_1$) and (2.6.13), we therefore obtain for any $\delta > 0$:

$$(2.6.14) \quad \sum_{l=0}^{\lfloor \frac{\delta^{-1}-1}{2} \rfloor} \left\{ \sum_{j=0}^1 \int_{\mathcal{R}(\tau_1+2l\tau_*, \tau_1+2(l+1)\tau_*) \setminus \mathcal{E}_{\delta_1} \cap \{r \leq R\}} \left(J_{\mu}^N(T^j\psi) n^{\mu} + |T^j\psi|^2 \right) \right\} \leq \\ \leq \frac{1}{10} \varepsilon \delta^{-1} + C \sum_{j=0}^1 \mathcal{E}_{\log}[T^j\psi] + \frac{1}{20\tau_*} \varepsilon(\tau_1 + \delta^{-1}\tau_*) + C_3 \left(\log(\tau_1 + \delta^{-1}\tau_*) \right)^{14} e^{C_3 \left(\log(2+\tau_1) \right)^{8\varepsilon_0}}.$$

Applying the pidgeonhole principle on (2.6.14) (assuming that $\delta \ll 1$), we infer that there exists some $l_0 \in \{0, \dots, \lfloor \frac{\delta^{-1}-1}{2} \rfloor\}$ such that

$$(2.6.15) \quad \sum_{j=0}^1 \int_{\mathcal{R}(\tau_1+2l_0\tau_*, \tau_1+2(l_0+1)\tau_*) \setminus \mathcal{E}_{\delta_1} \cap \{r \leq R\}} \left(J_{\mu}^N(T^j\psi) n^{\mu} + |T^j\psi|^2 \right) \leq \\ \leq \lfloor \frac{\delta^{-1}-1}{2} \rfloor^{-1} \left(\frac{1}{5} \varepsilon \delta^{-1} + \frac{1}{20} \varepsilon \tau_*^{-1} \tau_1 + C \sum_{j=0}^1 \mathcal{E}_{\log}[T^j\psi] + \right. \\ \left. + C_3 \left(\log(\tau_1 + \delta^{-1}\tau_*) \right)^{14} e^{C_3 \left(\log(2+\tau_1) \right)^{8\varepsilon_0}} \right) \leq \\ \leq \frac{\varepsilon}{2} (1 + \tau_*^{-1} \delta \tau_1) + C \delta \sum_{j=0}^1 \mathcal{E}_{\log}[T^j\psi] + C_3 \delta \left(\log(\tau_1 + \delta^{-1}\tau_*) \right)^{14} e^{C_3 \left(\log(2+\tau_1) \right)^{8\varepsilon_0}}.$$

Thus, provided

$$(2.6.16) \quad \delta = \frac{\bar{\delta}}{\tau_1},$$

where $\bar{\delta}$ is small in terms of $\varepsilon, \tau_*, \mathcal{E}_{\log}[\psi], \mathcal{E}_{\log}[T\psi]$ and the precise choice of the constants C_1, C_2 , and that τ_1 is chosen sufficiently large in terms of $\varepsilon, \varepsilon_0, \tau_*$ and the precise choice

of C_1, C_2 (assuming also that ε_0 has been fixed so that $0 < \varepsilon_0 < \frac{1}{8}$), from (2.6.15) we infer:

$$(2.6.17) \quad \sum_{j=0}^1 \int_{\mathcal{R}(\tau_1+2l_0\tau_*, \tau_1+2(l_0+1)\tau_*) \setminus \mathcal{E}_{\delta_1} \cap \{r \leq R\}} \left(J_{\mu}^N(T^j\psi)n^{\mu} + |T^j\psi|^2 \right) < \varepsilon.$$

Setting $\tau_{\natural} = \tau_1 + (2l_0 + 1)\tau_*$ (and thus $\tau_1 + 2l_0\tau_* = \tau_{\natural} - \tau_*$ and $\tau_1 + 2(l_0 + 1)\tau_* = \tau_{\natural} + \tau_*$), (2.6.17) yields (2.3.19). $\square$

2.7 Proof of Corollary 2.1

The proof of Corollary 2.1 follows immediately from Theorem 2.1 applied to the quotient of $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$ by the translations in the z -direction, i. e. the $2+1$ dimensional spacetime $(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}, \bar{g}_{hyd})$, where $\bar{\mathcal{V}}_{hyd,\delta} = \mathbb{R}^2 \setminus \{\bar{r} \leq \delta\}$ (in the polar $(\bar{r}, \vartheta)$ coordinate system) and

$$(2.7.1) \quad \bar{g}_{hyd} = -\left(1 - \frac{C^2}{\bar{r}^2}\right)dt^2 + d\bar{r}^2 - 2Cdt d\vartheta + \bar{r}^2 d\vartheta^2$$

(see also the remark below Theorem 2.1, as well as Section 2.5.9, regarding the Dirichlet or Neumann boundary conditions on $\{\bar{r} = \delta\}$).

In particular, in the language of Section 2.5.9, $(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}, \bar{g}_{hyd})$ is a smooth Lorentzian manifold with smooth timelike boundary

$$(2.7.2) \quad \partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}) = \{\bar{r} = \delta\}.$$

The double $(\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}, \tilde{g}_{hyd})$ of $(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}, \bar{g}_{hyd})$ across the boundary $\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta})$ is diffeomorphic to $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^1$, with the metric $\tilde{g}_{hyd}$ in the $(t, \bar{r}, \vartheta)$ coordinate chart of $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^1$ having the form:

$$(2.7.3) \quad \tilde{g}_{hyd} = -\left(1 - \frac{C^2}{(|\bar{r} - \delta| + \delta)^2}\right)dt^2 + d\bar{r}^2 - 2Cdt d\vartheta + (|\bar{r} - \delta| + \delta)^2 d\vartheta^2$$

Notice that $(\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}, \tilde{g}_{hyd})$ is a globally hyperbolic spacetime without boundary, with Cauchy hypersurface $\{t = 0\}$. Let $i_1, i_2 : (\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}, \bar{g}_{hyd}) \rightarrow (\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}, \tilde{g}_{hyd})$ be the two natural inclusions (see Section 2.5.9). Then, in the coordinate charts $(t, \bar{r}, \vartheta)$ on $\mathbb{R} \times$

$[\delta, +\infty) \times \mathbb{S}^1 \simeq \mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}$ and $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^1 \simeq \widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}$, we have $i_1((t, \bar{r}, \vartheta)) = (t, \bar{r}, \vartheta)$ and $i_2((t, \bar{r}, \vartheta)) = (t, \delta - \bar{r}, \vartheta)$.

Note that $\tilde{g}_{hyd}$ is smooth everywhere except on $i_1(\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta})) = \{\bar{r} = \delta\}$. Notice also that $(\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}, \tilde{g}_{hyd})$ has no event horizon $\mathcal{H}$ (and thus, trivially, $\mathcal{H} \cap i_1(\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta})) = \emptyset$), and $i_1(\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta})) \cap \{t = 0\}$ is compact. Thus, in view of the remark below Theorem 2.1 on spacetimes with timelike boundary, it only remains to verify that $(\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}, \tilde{g}_{hyd})$ satisfies Assumptions G1–G3 and A1, and that $i_1(\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}))$ is invariant with respect to the stationary Killing field of $(\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}, \tilde{g}_{hyd})$.

1. The vector field ∂_t (in the $(t, \bar{r}, \vartheta)$ coordinate system for $(\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}, \tilde{g}_{hyd})$) is Killing, and the metric (2.7.1) is asymptotically flat (with the asymptotically flat region $\tilde{\mathcal{I}}_{as} = \{\bar{r} \geq R_0 \gg 1\}$ consisting of two connected components) and satisfies Assumption G1. Furthermore, $\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta})$ is ∂_t -invariant.
2. The spacetime $(\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}, \tilde{g}_{hyd})$ has no event horizon $\mathcal{H}$, and thus Assumption G2 is trivially satisfied.
3. The spacetime $(\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}, \tilde{g}_{hyd})$ has a non empty ergoregion $\tilde{\mathcal{E}} = \{2\delta - C < \bar{r} \leq C\}$. The boundary $\partial\tilde{\mathcal{E}} = \{\bar{r} = 2\delta - C\} \cup \{\bar{r} = C\}$ of $\tilde{\mathcal{E}}$ is a smooth hypersurface of $\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}$, and $\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}} \setminus \tilde{\mathcal{E}}$ consists of two connected components, each containing one asymptotically flat end of $\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}$ (and, thus, $\tilde{\mathcal{E}}_{ext} = \tilde{\mathcal{E}}$). In particular, Assumption G3 is satisfied.
4. Assumption A1 is readily satisfied in view of the fact that $(\widetilde{\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}}, \tilde{g}_{hyd})$ is also axisymmetric, with axisymmetric Killing field ∂_ϑ such that $[\partial_\vartheta, \partial_t] = 0$ and the span of $\partial_\vartheta, \partial_t$ contains a timelike direction (see the discussion in Section 2.1.3).

Thus, the proof of Corollary 2.1 is complete.

2.8 Aside: Discussion on Friedman's heuristic argument

In this Section, we will briefly sketch the heuristic arguments developed by Friedman in [66], and we will discuss their connections with the methods used in this chapter.

2.8.1 Friedman's argument

As we already explained in the introduction, on any globally hyperbolic, stationary and asymptotically flat spacetime $(\mathcal{M}, g)$ with a non-empty ergoregion $\mathcal{E}$ and no future event horizon $\mathcal{H}^+$, Friedman constructed, in [66], a class of smooth solutions ψ to the wave equation (2.0.1) satisfying

$$(2.8.1) \quad \int_{\Sigma} J_{\mu}^T(\psi) n^{\mu} = -1,$$

where Σ is a Cauchy hypersurface of $(\mathcal{M}, g)$, T is the stationary Killing field of $(\mathcal{M}, g)$ and n is the future directed unit normal to Σ . In view of the conservation of the T -energy flux for solutions to (2.0.1) on $(\mathcal{M}, g)$ and the fact that $J_{\mu}^T(\psi) n^{\mu} \geq 0$ on $\mathcal{M} \setminus \mathcal{E}$, from (2.8.1) Friedman inferred that for any $\tau \geq 0$:

$$(2.8.2) \quad \int_{\Sigma_{\tau} \cap \mathcal{E}} J_{\mu}^T(\psi) n^{\mu} \leq -1,$$

where Σ_{τ} is defined as in Section 2.2 (i. e. the image of Σ under the flow of T for time τ).

Proceeding to study the consequences of the bound (2.8.2) on the (in)stability properties of equation (2.0.1), Friedman first noted the following dichotomy for the energy flux through the future null infinity $\mathcal{I}^+$ of any solution ψ to (2.0.1), satisfying (2.8.1):¹³ Either

$$(2.8.3) \quad \int_{\mathcal{I}^+} J_{\mu}^T(\psi) n_{\mathcal{I}^+}^{\mu} = +\infty,$$

in which case (in view of (2.8.1) and the conservation of the J^T -flux) there exists a sequence of hyperboloidal hypersurfaces $\mathcal{S}_{\tau_n}$ terminating at $\mathcal{I}^+$ such that

$$(2.8.4) \quad \limsup_{n \rightarrow +\infty} \int_{\mathcal{S}_{\tau_n}} J_{\mu}^T(\psi) n_{\mathcal{S}_{\tau_n}}^{\mu} = +\infty,$$

¹³See [108] for the definition of the Friedlander radiation field and the energy flux of φ through $\mathcal{I}^+$ on general asymptotically flat spacetimes.

or

$$(2.8.5) \quad \int_{\mathcal{I}^+} J_\mu^T(\psi) n_{\mathcal{I}^+}^\mu < +\infty.$$

In case the first scenario (2.8.4) holds, one immediately obtains an energy instability statement for equation (2.0.1). In case the second scenario (2.8.5), Friedman argued (see [66]) that ψ “settles down” to a “non-radiative state” $\tilde{\psi}$, which is to be interpreted as a solution to (2.0.1) such that

$$(2.8.6) \quad \int_{\mathcal{I}^+} J_\mu^T(\tilde{\psi}) n_{\mathcal{I}^+}^\mu = 0.$$

Furthermore, in view of (2.8.2), Friedman argued that $\tilde{\psi}$ should also satisfy for all $\tau \geq 0$:

$$(2.8.7) \quad \int_{\Sigma_\tau \cap \mathcal{E}} J_\mu^T(\tilde{\psi}) n^\mu \leq -1.$$

Assuming that $(\mathcal{M}, g)$ is globally real analytic and that the metric g has a proper asymptotic expansion in powers of r^{-1} in a neighborhood of $\mathcal{I}^+$, Friedman inferred from (2.8.6) (using an adaptation of Holmgren’s uniqueness theorem for analytic linear partial differential equations, see [80]) that

$$(2.8.8) \quad \tilde{\psi} \equiv 0$$

on $(\mathcal{M}, g)$. Thus, (2.8.7) and (2.8.8) yield a contradiction, implying that the scenario (2.8.5) should not occur on such spacetimes.

2.8.2 Comparison with the proof of Theorem 2.1

In general terms, the proof of Theorem 2.1 (see Section 2.3) follows the roadmap of the heuristic arguments of Friedman. In particular, our proof proceeds by contradiction, assuming the energy bound (2.3.1) on the $\{t = \tau\}$ hypersurfaces, which is a slightly stronger assumption than the energy bound (2.8.5) on $\mathcal{I}^+$ in the second scenario considered by Friedman.

In Lemma 2.4, we show that, under the assumption (2.3.1), a function ψ solving (2.0.1) with compactly supported initial data indeed “settles down” to a function $\tilde{\psi}$ (in a well defined way), such that $\tilde{\psi}$ vanishes identically outside the extended ergoregion $\mathcal{E}_{ext}$. This result makes use (through Proposition 2.1) of the Carleman-type estimates of Section 2.5, as well as the bound (2.3.1). Here, assuming merely the bound (2.8.5) on $\mathcal{I}^+$ would not be enough. Note that, in the argument of [66], no justification is provided (even at the heuristic level) of why a function ψ solving (2.0.1) and satisfying (2.8.5) is expected to “settle down” to a non-radiating solution $\tilde{\psi}$ of (2.0.1).

The fact that $\tilde{\psi}$ vanishes outside $\mathcal{E}_{ext}$ follows from the estimates of Section 2.5, without any need to impose a real analyticity assumption on $(\mathcal{M}, g)$ or a complete asymptotic expansion for g on $\mathcal{I}^+$. In general, however, it can not be inferred that $\tilde{\psi}$ vanishes also on $\mathcal{E}$.¹⁴ Thus, a contradiction can not be reached following the argument of Friedman in this setting. Instead, after restricting ourselves to spacetimes $(\mathcal{M}, g)$ satisfying the unique continuation assumption A1, which guarantees that $\tilde{\psi}$ vanishes on $(\mathcal{M} \setminus \mathcal{E}_{ext}) \cup \mathcal{U}$, we reach the desired contradiction by exploiting our freedom to choose the initial data for ψ appropriately: We choose $(\psi, T\psi)|_{\Sigma}$ to be supported in $\Sigma \cap \mathcal{U}$, so that the support of $(\psi, T\psi)|_{\Sigma}$ will be disjoint from $\mathcal{M} \setminus \mathcal{U}$, where the support of all the time translates of $(\tilde{\psi}, T\tilde{\psi})|_{\Sigma}$ is contained. Therefore, ψ and all the time translates of $\tilde{\psi}$ are orthogonal with respect to the (indefinite) T -inner product (2.3.38). This fact leads to relation (2.3.57), from which a contradiction follows readily in view of (2.3.49).

¹⁴We can in fact construct spacetimes $(\mathcal{M}^{d+1}, g)$, $d \geq 3$, with a smooth solution $\tilde{\psi}$ to an equation of the form $\square_g \tilde{\psi} + V\tilde{\psi} = 0$, such that $T(V) = 0$, $\tilde{\psi} \equiv 0$ on $\mathcal{M} \setminus \mathcal{E}$ and $\tilde{\psi}$ not identically 0 in $\mathcal{E}$.

Chapter 3

A proof of the instability of AdS for the Einstein–null dust system with an inner mirror

In this chapter, we will establish Theorem 3.1 (presented in the introduction) on the instability of AdS for the Einstein–null dust system. First, in Section 3.1, we will give an overview of the properties of the matter model under consideration (illustrating the necessity for an introduction of an inner mirror); we will also present a more precise formulation of Theorem 3.1 and we will discuss earlier numerical and heuristic works on the AdS instability conjecture. In Sections 3.2–3.3, we will introduce in detail the Einstein–null dust system with an inner mirror and state the main well-posedness result which are necessary for the study of the stability problem in this setting (established in [111]). The final, detailed statement of Theorem 3.1 will be given in Section 3.4; its proof will then occupy Sections 3.5–3.6.

3.1 Introduction: The Einstein–null dust system with an inner mirror and the instability of AdS

Establishing the AdS instability conjecture for the vacuum Einstein equations (1.2.6) requires presenting an unstable family of initial data sets for (1.2.6) converging to the trivial

data. For this reason, it is tempting to consider simplifying the system (1.2.6) under suitable symmetry assumptions, in order to render the conjecture more tractable. In 3+1 dimensions, the only symmetry class in which (1.2.6) is reduced to a 1+1 hyperbolic system which is, at the same time, consistent with the AdS asymptotics, is the class of spherically symmetric spacetimes. Unfortunately, all solutions to (1.2.6) in this class are locally isometric to a member of the Schwarzschild–AdS family (see [57]). Thus, any attempt to study the AdS instability conjecture for (1.2.6) in 3+1 dimensions can not be reduced to a problem for a 1+1 hyperbolic system.¹

A system closely related to (1.2.6), admitting non-trivial spherically symmetric dynamics in 3+1 dimensions is the Einstein scalar field system

$$(3.1.1) \quad \begin{cases} Ric_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}[\varphi], \\ \square_g \varphi = 0, \\ T_{\mu\nu}[\varphi] \doteq \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2}g_{\mu\nu} \partial^\alpha \varphi \partial_\alpha \varphi \end{cases}$$

The study of the system (3.1.1) reduced under spherical symmetry in the case $\Lambda = 0$ was pioneered by Christodoulou [31]. Spherically symmetric solutions of (3.1.1) share many qualitative properties with general solutions of (1.2.6); for this reason, instead of (1.2.6), numerical and heuristic works on the AdS instability have so far mainly focused on (3.1.1) (see Section (3.1.3) for a more detailed discussion of these works).

In spherical symmetry, expressed in a double null gauge (u, v) , i. e. a gauge where

$$(3.1.2) \quad g = -\Omega^2 du dv + r^2 g_{\mathbb{S}^2},$$

¹This problem is circumvented in 4+1 dimensions by the biaxial Bianchi IX symmetry class referred to earlier (see [13]).

the system (3.1.1) takes the form

$$(3.1.3) \quad \begin{cases} \partial_u \partial_v (r^2) &= -\frac{1}{2}(1 - \Lambda r^2)\Omega^2, \\ \partial_u \partial_v \log(\Omega^2) &= \frac{\Omega^2}{2r^2}(1 + 4\Omega^{-2}\partial_u r \partial_v r) - 8\pi \partial_u \varphi \partial_v \varphi, \\ \partial_v (\Omega^{-2} \partial_v r) &= -4\pi r \Omega^{-2} (\partial_v \varphi)^2, \\ \partial_u (\Omega^{-2} \partial_u r) &= -4\pi r \Omega^{-2} (\partial_u \varphi)^2, \\ \partial_u \partial_v (r\varphi) &= -\frac{\Omega^2 - 4\partial_u r \partial_v r}{4r^2} \cdot r\varphi. \end{cases}$$

The well-posedness of the asymptotically AdS initial-boundary value problem for the system (3.1.3) with reflecting boundary conditions on $\mathcal{I}$ was established by Holzegel and Smulevici in [83].

A spherically symmetric model for (1.2.6) which is even simpler than (3.1.1) is the *Einstein–massless Vlasov* system (see [7, 121]). The case where the Vlasov field is supported only on radial geodesics is a singular reduction of this system which is equivalent to the *Einstein–null dust* system, allowing for both ingoing and outgoing dust (see [122]). This system was studied by Poisson and Israel in their seminal work on mass inflation [120]. In $3+1$ dimensions, it takes the form (in double null coordinates (u, v)):

$$(3.1.4) \quad \begin{cases} \partial_u \partial_v (r^2) &= -\frac{1}{2}(1 - \Lambda r^2)\Omega^2, \\ \partial_u \partial_v \log(\Omega^2) &= \frac{\Omega^2}{2r^2}(1 + 4\Omega^{-2}\partial_u r \partial_v r), \\ \partial_v (\Omega^{-2} \partial_v r) &= -4\pi r^{-1} \Omega^{-2} \bar{\tau}, \\ \partial_u (\Omega^{-2} \partial_u r) &= -4\pi r^{-1} \Omega^{-2} \tau, \\ \partial_u \bar{\tau} &= 0, \\ \partial_v \tau &= 0. \end{cases}$$

In certain cases, the Einstein–null dust system (3.1.4) can be formally viewed as a high frequency limit of the Einstein–scalar field system (3.1.3) (as was already discussed in [120]): Setting

$$\tau \doteq r^2 (\partial_u \varphi)^2, \quad \bar{\tau} \doteq r^2 (\partial_v \varphi)^2$$

in (3.1.1) and dropping all lower order terms from the wave equation for φ , one formally obtains (3.1.4) in the region where $\partial_u \varphi \partial_v \varphi$ is negligible, i.e. outside the intersection of the supports of $\tau, \bar{\tau}$. While this formal limiting procedure can be rigorously justified away from $r = 0$, the dynamical similarities between (3.1.3) and (3.1.4) break down close to $r = 0$. A fundamental difference between these systems is the fact that, while small data asymptotically AdS solutions to (3.1.3) satisfying a reflecting boundary condition at $\mathcal{I}$ remain regular (and “small”) for large times, all non-trivial solutions to the system (3.1.4) break down once the support of $\bar{\tau}$ reaches the axis γ (i.e. the timelike portion of $\{r = 0\}$), independently of the boundary conditions imposed at $\mathcal{I}$. This is an ill-posedness statement for (3.1.4), which needs to be addressed before any attempt to study the AdS instability conjecture in the setting of (3.1.4).

We will now proceed to discuss this difference of (3.1.3) and (3.1.4) in more detail.

Cauchy stability for the Einstein–scalar field system

The following Cauchy stability result holds for the system (3.1.3):

Proposition 3.1 (Cauchy stability for (3.1.3); see [83]). *For a suitable initial data norm $\|\cdot\|_{initial}$, $(\mathcal{M}_{AdS}, g_{AdS})$ is Cauchy stable as a solution of the system (3.1.3) with reflecting boundary conditions on $\mathcal{I}$. That is to say, for all fixed times $T_* > 0$, any perturbation of the initial data of $(\mathcal{M}_{AdS}, g_{AdS})$ which is small enough (when measured in terms of $\|\cdot\|_{initial}$) with respect to T_* gives rise to a solution of (3.1.3) which is regular and close to $(\mathcal{M}_{AdS}, g_{AdS})$ for times up to T_* .*

Remark. In the statement of Proposition 3.1, Cauchy stability of $(\mathcal{M}_{AdS}, g_{AdS})$ refers to stability over fixed compact subsets of the *conformal compactification* of $(\mathcal{M}_{AdS}, g_{AdS})$, such as subsets of the form $\{0 \leq t \leq T_*\}$ in the $(t, r, \vartheta, \varphi)$ coordinate chart. Any such subset contains, in particular, a compact subset of the timelike boundary $\mathcal{I}$.

The initial data norm $\|\cdot\|_{initial}$, for which the Cauchy stability of $(\mathcal{M}_{AdS}, g_{AdS})$ follows from [83], is a higher order, suitably weighted C^k norm. However, this is not the only norm for which $(\mathcal{M}_{AdS}, g_{AdS})$ can be shown to be Cauchy stable: An additional, highly non-trivial example of such a norm is the bounded variation norm of Christodoulou [28]

(modified with suitable r -weights near $r = \infty$). Similar low-regularity norms will also play an important role in this chapter (see (3.1.11)).

Assuming, for simplicity, that initial data are prescribed on the outgoing null hypersurface corresponding to $u = 0$, for $0 \leq v \leq v_*$, a necessary condition for Cauchy stability of $(\mathcal{M}_{AdS}, g_{AdS})$ for the system (3.1.3) with respect to an initial data norm $\|\cdot\|_{initial}$ is that, for any given $R_0 > 0$, $\|\cdot\|_{initial}$ controls the quantity

$$(3.1.5) \quad \mathcal{M} \doteq \sup_{\substack{0 \leq v_1 < v_2 \leq v_*, \\ \frac{r(0, v_2)}{r(0, v_1)} < \frac{3}{2}, \quad r(0, v_2) \leq R_0}} \frac{\tilde{m}(0, v_2) - \tilde{m}(0, v_1)}{(r(0, v_2) - r(0, v_1)) \left| \log \left(\frac{r(0, v_2)}{r(0, v_1)} - 1 \right) \right|},$$

where $\tilde{m}$ is the *renormalised Hawking mass*, defined in terms of the Hawking mass m ,

$$(3.1.6) \quad m \doteq \frac{r}{2} \left(1 - 4\Omega^{-2} \partial_u r \partial_v r \right),$$

by the relation

$$(3.1.7) \quad \tilde{m} \doteq m - \frac{1}{6} \Lambda r^3.$$

This is a consequence of the fact that, when $\mathcal{M}$ exceeds a certain threshold (depending on R_0), there exists a point $p = (u_{\dagger}, v_{\dagger})$ in the development of the initial data, with $0 < u_{\dagger} < v_*$, such that

$$(3.1.8) \quad \frac{2m}{r}(u_{\dagger}, v_{\dagger}) > 1,$$

a result proven by Christodoulou in [26].² The bound (3.1.8) implies that

$$(3.1.9) \quad \partial_u r(u_{\dagger}, v_{\dagger}) < 0, \quad \partial_v r(u_{\dagger}, v_{\dagger}) < 0,$$

i. e. that the symmetry sphere associated to $(u_{\dagger}, v_{\dagger})$ is a *trapped surface*. In particular, $(u_{\dagger}, v_{\dagger})$ is contained in a black hole.³

²The result of [26] was restricted to the case $\Lambda = 0$, but the proof can be readily modified to include the case $\Lambda < 0$.

³We should remark that (3.1.9) follows from (3.1.8) under the assumption that $\partial_u r < 0$ (which always

As a corollary, it follows that the total ADM mass of the initial data, though expressible as a coercive functional on the space of initial data of (3.1.3), does not yield a norm for which $(\mathcal{M}_{AdS}, g_{AdS})$ is Cauchy stable for (3.1.3), since the ADM mass manifestly fails to control (3.1.5).

Break down at $r = 0$ and “trivial” Cauchy instability for the Einstein–null dust system

The following instability result holds for the system (3.1.4) (see [111]):

Proposition 3.2 (Cauchy instability for (3.1.4)). *Any globally hyperbolic spherically symmetric solution $(\mathcal{M}, g; \tau, \bar{\tau})$ of (3.1.4) with non-empty axis γ “breaks down” at the first point when a radial geodesic in the support of $\bar{\tau}$ reaches γ : Beyond that point, $(\mathcal{M}, g; \tau, \bar{\tau})$ is C^0 inextendible as a spherically symmetric solution to (3.1.4). As a result, $(\mathcal{M}_{AdS}, g_{AdS})$ is not a Cauchy stable solution of (3.1.4) for any “reasonable” initial data topology.*

For the precise definition of the notion of C^0 inextendibility as a spherically symmetric solution to (3.1.4), see [111]. Note that this a *stronger* statement than $(\mathcal{M}, g; \tau, \bar{\tau})$ breaking down as a smooth solution of (3.1.4). We should also remark the following regarding Proposition 3.2:

- Proposition 3.2 holds independently of the value of the cosmological constant Λ . In particular, Minkowski spacetime $(\mathbb{R}^{3+1}, \eta)$ is not Cauchy stable for (3.1.4) with $\Lambda = 0$ for any “reasonable” initial data topology.
- Proposition 3.2 yields a uniform upper bound on the time of existence of solutions $(\mathcal{M}, g)$ to (3.1.4) for any initial data set for which $\bar{\tau}$ is not identically equal to 0, depending only on the distance of the initial support of $\bar{\tau}$ from the axis and, thus, independently of the proximity of the initial data to the trivial data (in any reasonable initial data norm). We should also highlight that the instability of Proposition 3.2 has nothing to do with trapped surface formation: Up to the first retarded time when a radial geodesic in the support of $\bar{\tau}$ reaches γ , any solution $(\mathcal{M}, g)$ to (3.1.4) arising from

holds provided, initially, $\partial_u r|_{u=0} < 0$; see [28]).

smooth initial data close to $(\mathcal{M}_{AdS}, g_{AdS})$ remains smooth and close to $(\mathcal{M}_{AdS}, g_{AdS})$, and $(\mathcal{M}, g)$ *contains no trapped surface*. In fact, in this case, despite being C^0 inextendible as a globally hyperbolic spherically symmetric solution to (3.1.4), $(\mathcal{M}, g)$ is globally C^∞ -extendible as a spherically symmetric Lorentzian manifold; see [111].

- The Cauchy stability statement for $(\mathcal{M}_{AdS}, g_{AdS})$ for the system (3.1.3) stated in Proposition 3.1 can be informally interpreted as the result of a natural dispersive mechanism close to the axis γ displayed by the system (3.1.3), which does not allow the energy of φ to concentrate on scales smaller than $\tilde{m}$ in $O(1)$ time, provided a suitable initial norm of φ (controlling at least (3.1.5)) is small enough. No such mechanism is present for the system (3.1.4), as is illustrated by Proposition 3.2.

Resolution of the “trivial” instability of (3.1.4) through an inner mirror

In order to turn the spherically symmetric Einstein–null dust system (3.1.4) into a well-posed, Cauchy-stable system (a necessary step for converting (3.1.4) into an effective model of the vacuum Einstein equations (1.2.6)), it is necessary to explicitly add to (3.1.4) a mechanism that prevents the break down at $r = 0$ described by Proposition 3.2, so that, moreover, an analogue of Proposition 3.1 holds for (3.1.4). This can be achieved by by placing an *inner mirror* at $r = r_0 > 0$, i. e. by restricting (3.1.4) on $\{r \geq r_0\}$, for some $r_0 > 0$, and imposing a reflecting boundary condition on the portion γ_0 of the set $\{r = r_0\}$ which is timelike.

Remark. The reflecting boundary condition on γ_0 can be motivated by the fact that, for smooth spherically symmetric solutions $(\mathcal{M}, g; \varphi)$ to (3.1.1), the function φ , viewed as a function on the quotient of $(\mathcal{M}, g)$ by the spheres of symmetry, satisfies a reflecting boundary condition on the axis.

The well-posedness and the properties of the maximal development for the system (3.1.4) with reflecting boundary conditions on $\mathcal{I}$ and γ_0 are addressed in [111]. The following result is established in [111]:

Theorem 2 (Well posedness for (3.1.4) with an inner mirror). For any $r_0 > 0$ and any smooth asymptotically AdS initial data set $(r, \Omega^2, \tau, \bar{\tau})|_{u=0}$ on $u = 0$, there exists a unique

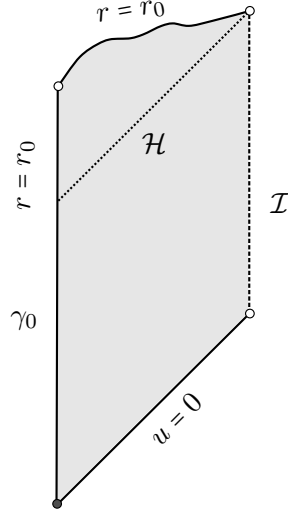


Figure 6: Schematic depiction of the domain on which the maximal future development $(r, \Omega^2, \tau, \bar{\tau})$ of a smooth initial data set on $u = 0$ (with reflecting boundary conditions on $\mathcal{I}$ and γ_0) is defined. A gauge condition ensures that $\mathcal{I}$ and γ_0 are straight vertical lines. Conformal infinity $\mathcal{I}$ is always complete in this setting. In the case when the future event horizon $\mathcal{H}^+$ is non-empty, it is smooth and has infinite affine length. In this case, apart from the mirror γ_0 , the boundary of the domain has a spacelike portion on which $\{r = r_0\}$.

smooth maximal future development $(r, \Omega^2, \tau, \bar{\tau})$ on $\{r \geq r_0\}$, solving (3.1.4) with reflecting boundary conditions on $\mathcal{I}$ and γ_0 , where $r|_{\gamma_0} = r_0$ and γ_0 coincides with the portion of the curve $\{r = r_0\}$ which is timelike (fixing the gauge freedom by imposing a reflecting gauge condition on both $\mathcal{I}$ and γ_0). For this development, $\mathcal{I}$ is complete and $\{r = r_0\}$ is timelike in the past of $\mathcal{I}$ (see Figure 6).

In the case when the future event horizon $\mathcal{H}^+$ is non-empty, it is smooth and future complete. A necessary condition for $\mathcal{H}^+$ to be non-empty is the existence of a point $(u_{\dagger}, v_{\dagger})$ where (3.1.8) holds. If the total mass $\tilde{m}|_{\mathcal{I}}$ and the mirror radius r_0 satisfy

$$(3.1.10) \quad \frac{2\tilde{m}|_{\mathcal{I}}}{r_0} \leq 1 - \frac{1}{3}\Lambda r_0^2,$$

then necessarily $\mathcal{H}^+ = \emptyset$.

For a more detailed statement of Theorem 2, see Section 3.3 and [111].

In view of the fact that $\mathcal{H}^+ = \emptyset$ in the case when the total mass $\tilde{m}|_{\mathcal{I}}$ and the mirror radius r_0 satisfy (3.1.10), in order to address the AdS instability conjecture for the system

(3.1.4) with reflecting boundary conditions on $\mathcal{I}$ and γ_0 , it is necessary to allow r_0 to shrink to 0 with the size of the data. Thus, addressing the AdS instability conjecture in this setting requires establishing a Cauchy stability statement for $(\mathcal{M}_{AdS}, g_{AdS})$ which is *independent of the precise value of the mirror radius r_0* . This is the statement of the following result, proved in [111]:

Theorem 3 (Cauchy stability for (3.1.4) uniformly in r_0). Given $\varepsilon > 0$, $u_* > 0$, there exists a $\delta > 0$ such that the following statement holds: For any $r_0 > 0$ and any initial data set $(r, \Omega^2, \tau, \bar{\tau})|_{u=0}$ satisfying

$$(3.1.11) \quad \|(r, \Omega^2, \tau, \bar{\tau})\|_{u=0} \doteq \sup_{\bar{v}} \int_{u=0} \frac{\bar{\tau}(0, v)}{|\varphi(0, v) - \varphi(0, \bar{v})| + \tan^{-1}(\sqrt{-\Lambda}r_0)} \frac{\sqrt{-\Lambda}dv}{\partial_v \varphi(0, v)} + \\ + \sup_{u=0} \left(\left| \left(1 - \frac{2\tilde{m}}{r}\right)^{-1} - 1 \right| + \sqrt{-\Lambda}\tilde{m} \right) \leq \varepsilon,$$

where

$$(3.1.12) \quad \varphi(0, v) \doteq \tan^{-1}(\sqrt{-\Lambda}r)(0, v),$$

the corresponding solution $(r, \Omega^2, \tau, \bar{\tau})$ to (3.1.4) with reflecting boundary conditions on $\mathcal{I}$ and γ_0 will satisfy

$$(3.1.13) \quad \sup_{0 \leq \bar{u} \leq u_*} \|(r, \Omega^2, \tau, \bar{\tau})\|_{u=\bar{u}} \leq \delta.$$

For a more detailed statement of Theorem 3, see Section 3.3 and [111].

Notice that the norm (3.1.11) vanishes only for the trivial initial data $(r, \Omega^2, 0, 0)$. Informally, Theorem 3 implies that, if the energy of the initial data concentrated on scales proportional to the mirror radius r_0 is small enough, then the energy of the solution to (3.1.4) (with reflecting boundary conditions on $\mathcal{I}$ and γ_0) will remain similarly dispersed for times less than any given constant. In particular, no trapped surface can form in this timescale if δ is chosen sufficiently small.

In Section 3.3, we will also present a Cauchy stability statement for general solutions of (3.1.4) with reflecting boundary conditions on $\mathcal{I}$ and γ_0 , which will be used in the proof of

Theorem 3.1 (see Theorem 3.2).

3.1.1 Statement of the main result: the non-linear instability of AdS

According to Theorem 3, a Cauchy stability statement holds for $(\mathcal{M}_{AdS}, g_{AdS})$ for time intervals which are independent of the precise value of the mirror radius r_0 , depending only on the smallness of the initial data norm (3.1.11). As a result, it is possible to study the AdS instability conjecture for the system (3.1.4) with reflecting boundary conditions on $\mathcal{I}$ and γ_0 , for perturbations which are small with respect to (3.1.11), *allowing the mirror radius r_0 to shrink with the size of the data*. In fact, we will prove the following result:

Theorem 3.1 (more precise version). There exists a family of positive numbers $r_{0\varepsilon}$ (satisfying $r_{0\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0$) and smooth initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ for the system (3.1.4) satisfying the following properties:

1. In the norm $\|\cdot\|_{u=0}$ defined by (3.1.11):

$$(3.1.14) \quad \|(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}\|_{u=0} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

2. For any $\varepsilon > 0$, the maximal development $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$ of $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ for the system (3.1.4) with reflecting boundary conditions on $\mathcal{I}$ and γ_0 , $r|_{\gamma_0} = r_{0\varepsilon}$, contains a trapped sphere, i. e. there exists a point $(u_\varepsilon, v_\varepsilon)$ such that:

$$(3.1.15) \quad \frac{2m^{(\varepsilon)}}{r^{(\varepsilon)}}(u_\varepsilon, v_\varepsilon) > 1.$$

Thus, in view of Theorem 2, $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$ contains a non-empty, smooth and future complete event horizon $\mathcal{H}^+$ and a complete conformal infinity $\mathcal{I}$.

For the definitive statement of Theorem 3.1, see Section 3.4. The following remarks should be made concerning Theorem 3.1:

- In view of the Cauchy stability of $(\mathcal{M}_{AdS}, g_{AdS})$ with respect to (3.1.11) (see Theorem 3), the time⁴ required to elapse before $(1 - \frac{2m^{(\varepsilon)}}{r^{(\varepsilon)}})$ becomes negative necessarily tends

⁴where time is measured with respect to the (dimensionless) coordinate function $\bar{t} = \sqrt{-\Lambda}(u + v)$.

to $+\infty$ as $\varepsilon \rightarrow 0$. However, our proof yields a quantitative upper bound (increasing to $+\infty$ as $\varepsilon \rightarrow 0$) on the time of trapped sphere formation for the development of $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ (see (3.4.6)).

- In view of the fact that $\mathcal{H}^+ = \emptyset$ when (3.1.10) holds, in order for $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$ to satisfy both (3.1.14) and (3.1.15), it is necessary that $r_{0\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$, at a rate which is at least as fast as that of $2\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$,⁵ i. e.:

$$(3.1.16) \quad r_{0\varepsilon} \leq 2\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}.$$

In fact, we choose the family $r_{0\varepsilon}$, $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ of Theorem 3.1 to saturate the bound (3.1.16) in the limit $\varepsilon \rightarrow 0$, i. e.

$$(3.1.17) \quad \lim_{\varepsilon \rightarrow 0} \frac{r_{0\varepsilon}}{2\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}} = 1.$$

For the proof of Theorem 3.1, (3.1.17) is not essential and can be relaxed; however, it is fundamental for our proof that $r_{0\varepsilon}$ is bounded from below by some small multiple of $\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$.

- It follows from the proof of Theorem 2 that, in the case $\Lambda = 0$, Minkowski spacetime $(\mathbb{R}^{3+1}, \eta)$ is *globally stable* (for the system (3.1.4) with reflecting boundary conditions on the inner mirror $\{r = r_0\}$) to initial data perturbations which are small with respect to the norm (3.1.11), independently of the precise choice of r_0 . This fact further justifies the choice of the matter model and the norm (3.1.11) as a setting for establishing the AdS instability conjecture.
- The proof of Theorem 3.1 also applies in the case $\Lambda = 0$ when placing an *outer mirror* at $r = R_0 \gg r_0$ (in addition to the inner mirror at $r = r_0$), i. e. restricting the solutions of (3.1.4) in the region $\{r_0 \leq r \leq R_0\}$ and imposing reflecting boundary conditions on both $\{r = r_0\}$ and $\{r = R_0\}$. This is in accordance with the numerical results of [19] for the system (3.1.1).

⁵Note that the renormalised Hawking mass $\tilde{m}^{(\varepsilon)}$ is constant on $\mathcal{I}$ when imposing a reflecting boundary condition.

- It can be readily inferred by Cauchy stability (see Theorem 3.2 in Section 3.3.4) that, for any $y_\varepsilon \doteq (r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ in the family of initial data of Theorem 3.1, there exists an open neighborhood $\mathcal{W}_\varepsilon$ of initial data around y_ε such that, for all $y \in \mathcal{W}_\varepsilon$, the maximal future development of y also contains a trapped surface. In particular, the set of initial data leading to trapped surface formation is open. An even stronger genericity statement would be the existence of an *open instability corner* in the space of initial data around $(\mathcal{M}_{AdS}, g_{AdS})$ (see [54]), i.e. the existence of a $c_1 > 0$ such that $\{y : \text{dist}(y, y_\varepsilon) \leq c_1 \|y_\varepsilon\|_{u=0}\} \subset \mathcal{W}_\varepsilon$ for all $\varepsilon > 0$ (with $\text{dist}(\cdot, \cdot)$ being the distance function associated to (3.1.11) for $r_0 = r_{0\varepsilon}$). While we have not addressed the issue of genericity of the unstable initial data here, we expect that the proof of Theorem 3.1 can be adapted to yield the existence of an instability corner around $(\mathcal{M}_{AdS}, g_{AdS})$.
- A plethora of numerical works (see, e.g. [15, 19, 14, 54]) suggest that, in the case of the Einstein–scalar field system (3.1.1), for families of initial data $(\varphi_\varepsilon^{(0)}, \varphi_\varepsilon^{(1)})$ for the scalar field φ of the form

$$(3.1.18) \quad (\varphi_\varepsilon^{(0)}, \varphi_\varepsilon^{(1)}) = (\varepsilon \varphi^{(0)}, \varepsilon \varphi^{(1)})$$

(where $(\varphi^{(0)}, \varphi^{(1)})$ is a fixed initial profile), trapped surface formation occurs at time $\sim \varepsilon^{-2}$. However, any rigorous formulation of this statement for general families of initial data, not necessarily of the form (3.1.18), requires fixing an initial data norm for which the initial data size is measured, with different choices of (scale invariant) norms possibly leading to different timescales of trapped surface formation for initial data of size $\sim \varepsilon$. For this reason, given that the initial data $(\tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ in Theorem 3.1 can not be viewed as a rescaling of a fixed profile of the form (3.1.18), we have not tried to optimize the time required for trapped surface formation in Theorem 3.1 in terms of the initial norm (3.1.11).

3.1.2 Sketch of the proof and remarks on Theorem 3.1

We will now proceed to sketch the main arguments involved in the proof of Theorem 3.1.

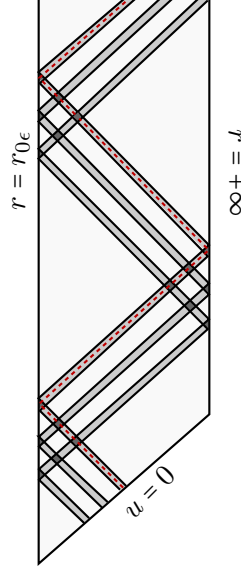


Figure 7: The initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\epsilon)}|_{u=0}$ give rise to a bundle of ingoing beams which are successively reflected off $\{r = r_{0\epsilon}\}$ and $\mathcal{I} = \{r = +\infty\}$. While the number of beams goes to infinity as $\epsilon \rightarrow 0$, for simplicity, we only depict here a bundle of three beams. As long as the total width of the bundle of beams remains small, the interaction set naturally splits into a part which lies close to $\{r = r_{0\epsilon}\}$ and a part near $\mathcal{I}$. We have also marked with a red dashed line the beam lying (initially) to the future of the rest.

Construction of the initial data

The family of initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\epsilon)}|_{u=0}$ in Theorem 3.1 is chosen so that its total ADM mass $\tilde{m}^{(\epsilon)}|_{\mathcal{I}}$ and the mirror radius $r_{0\epsilon}$ satisfy (for $\epsilon \ll 1$)

$$(3.1.19) \quad r_{0\epsilon}, \tilde{m}^{(\epsilon)}|_{\mathcal{I}} \sim \epsilon(-\Lambda)^{-\frac{1}{2}}.$$

In particular, fixing a function $h(\epsilon)$ in terms of ϵ such that

$$\epsilon \ll h(\epsilon) \ll 1,$$

the initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\epsilon)}|_{u=0}$ are constructed so that the null dust initially forms a bundle of narrow ingoing beams emanating from the region $r \sim 1$; see Figure 7. The number of the beams is chosen to be large, i. e. of order $\sim (h(\epsilon))^{-1}$, and the beams are initially separated by gaps of r -width $\sim (h(\epsilon))^{-1}\epsilon(-\Lambda)^{-\frac{1}{2}}$. The large number of beams and their initial separation

are chosen so that

$$(3.1.20) \quad \|(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}\|_{u=0} \sim h(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Remarks on the configuration of the null dust beams

As the solution $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$ arising from the initial data set $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ evolves according to equations (3.1.4), the null dust beams are reflected successively off $\gamma_0 = \{r = r_{0\varepsilon}\}$ and $\mathcal{I}$, as depicted in Figure 7. The beams separate the spacetime into vacuum regions (the larger rectangular regions between the beams in Figure 7), where the renormalised Hawking mass $\tilde{m}^{(\varepsilon)}$ is constant (recall the definition of the Hawking mass and the renormalised Hawking mass by (3.1.6) and (3.1.7), respectively). The *interaction set* of the beams consists of all the points in the spacetime where two different beams intersect (depicted in Figure 7 as the union of all the smaller dark rectangles, lying in the intersection of any two beams). As long as the total width of the bundle of beams remains small, the interaction set can be split into two sets, one consisting of the intersections occurring close to the mirror γ_0 and one consisting of the intersections near $\mathcal{I}$ (see Figure 7).

Every beam is separated by the interaction set into several components. To each such component, we can associate the *mass difference* $\mathfrak{D}\tilde{m}$ between the two vacuum regions which are themselves separated by that beam component. The mass difference $\mathfrak{D}\tilde{m}$ measures the energy content of each beam component and, in view of the non-linearity of the system (3.1.4), it is not necessarily conserved along the beam after an intersection with another beam. Precisely determining the resulting change in the mass difference after the interaction of two beams will be the crux of the proof of Theorem 3.1.

Beam interactions and change in mass difference

In Figure 8, the region around the intersection of an incoming null dust beam ζ_{in} and an outgoing null dust beam ζ_{out} is depicted. This region is separated by the beams into 4 vacuum subregions $\mathcal{R}_1, \dots, \mathcal{R}_4$ with associated renormalised Hawking masses $\tilde{m}_1, \dots, \tilde{m}_4$ (see Figure 8). Before the intersection of the two beams, the mass difference of the incoming

beam ζ_{in} is

$$(3.1.21) \quad \overline{\mathfrak{D}}_- \tilde{m} = \tilde{m}_3 - \tilde{m}_4,$$

while the mass difference of the outgoing beam ζ_{out} is

$$(3.1.22) \quad \mathfrak{D}_- \tilde{m} = \tilde{m}_4 - \tilde{m}_2.$$

After the intersection of the beams, the mass differences associated to ζ_{in} and ζ_{out} become

$$(3.1.23) \quad \overline{\mathfrak{D}}_+ \tilde{m} = \tilde{m}_1 - \tilde{m}_2$$

and

$$(3.1.24) \quad \mathfrak{D}_+ \tilde{m} = \tilde{m}_3 - \tilde{m}_1,$$

respectively.

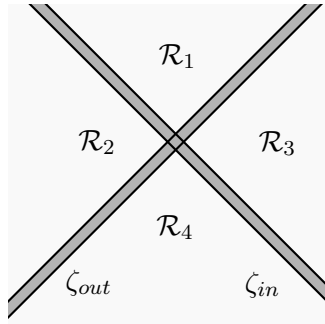


Figure 8: The region in the (u, v) plane around the intersection of an incoming beam ζ_{in} and an outgoing beam ζ_{out} . The regions $\mathcal{R}_i$, $i = 1, \dots, 4$ are vacuum and the renormalised Hawking mass $\tilde{m}$ is constant (and equal to $\tilde{m}_i$) on each of the $\mathcal{R}_i$'s.

Assuming that

$$(3.1.25) \quad \frac{2m}{r} < 1$$

and

$$(3.1.26) \quad \partial_u r < 0 < \partial_v r,$$

we can readily obtain the following differential relations for r and $\tilde{m}$ from (3.1.4):

$$(3.1.27) \quad \begin{aligned} \partial_u \log \left(\frac{\partial_v r}{1 - \frac{2m}{r}} \right) &= - \frac{4\pi}{r} \frac{\tau}{-\partial_u r}, \\ \partial_v \log \left(\frac{-\partial_u r}{1 - \frac{2m}{r}} \right) &= \frac{4\pi}{r} \frac{\bar{\tau}}{\partial_v r} \end{aligned}$$

and

$$(3.1.28) \quad \begin{aligned} \partial_u \tilde{m} &= -2\pi \left(\frac{1 - \frac{2m}{r}}{-\partial_u r} \right) \tau, \\ \partial_v \tilde{m} &= 2\pi \left(\frac{1 - \frac{2m}{r}}{\partial_v r} \right) \bar{\tau}. \end{aligned}$$

We will also assume that:

- The null dust beams ζ_{in} and ζ_{out} are sufficiently narrow so that, on their intersection $\zeta_{in} \cap \zeta_{out}$, r can be considered nearly constant:⁶

$$(3.1.29) \quad \sup_{\zeta_{in} \cap \zeta_{out}} r - \inf_{\zeta_{in} \cap \zeta_{out}} r \ll \varepsilon (-\Lambda)^{-\frac{1}{2}},$$

- $\bar{\mathfrak{D}}_+ \tilde{m} - \bar{\mathfrak{D}}_- \tilde{m}$ and $\mathfrak{D}_+ \tilde{m} - \mathfrak{D}_- \tilde{m}$ are relatively small.⁷

Then, equations (3.1.27)–(3.1.28), combined with the conservation laws

$$\partial_u \bar{\tau} = 0,$$

$$\partial_v \tau = 0,$$

⁶This is possible in view of the fact that, for solutions $(r, \Omega^2, \tau, \bar{\tau})$ to (3.1.4), r remains uniformly continuous in the limit when $\tau, \bar{\tau}$ tend to δ -functions in the u, v variables, respectively.

⁷Note that, necessarily, $\bar{\mathfrak{D}}_+ \tilde{m} - \bar{\mathfrak{D}}_- \tilde{m} = -(\mathfrak{D}_+ \tilde{m} - \mathfrak{D}_- \tilde{m})$.

yield the following relations for the change in the mass difference associated to ζ_{in} and ζ_{out} after their intersection:

$$(3.1.30) \quad \overline{\mathfrak{D}}_+ \tilde{m} = \overline{\mathfrak{D}}_- \tilde{m} \cdot \exp \left(\frac{2}{r} \frac{\overline{\mathfrak{D}}_- \tilde{m}}{1 - \frac{2m_2}{r}} + \mathfrak{Err}_{in} \right)$$

and

$$(3.1.31) \quad \mathfrak{D}_+ \tilde{m} = \mathfrak{D}_- \tilde{m} \cdot \exp \left(- \frac{2}{r} \frac{\overline{\mathfrak{D}}_- \tilde{m}}{1 - \frac{2m_2}{r}} + \mathfrak{Err}_{out} \right),$$

where the error terms $\mathfrak{Err}_{in}, \mathfrak{Err}_{out}$ are negligible compared to the other terms in (3.1.30), (3.1.31) (see also the relations (3.6.51) and (3.6.52) in Section 3.6.1). In particular, whenever an ingoing and an outgoing null dust beam intersect, *the mass difference of the ingoing beam increases, while that of the outgoing beam decreases.*

Remark. Notice that, according to (3.1.30) and (3.1.31), the change in the mass difference of each of the beams ζ_{in}, ζ_{out} after their intersection can be estimated in terms of the mass difference of the other beam and the value of r and $\inf(1 - \frac{2m}{r})$ in the region of intersection. A relation for the change of the mass difference of two infinitely thin, intersecting null dust beams was also obtained in [120].

The instability mechanism

Let us now consider, among the null dust beams arising from the initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\epsilon)}|_{u=0}$, the beam ζ_0 which initially lies to the future of the rest (this is the beam marked with a red dashed line in Figure 7). Denoting

$$(3.1.32) \quad \mathcal{E}_{\zeta_0}[t_*] \doteq \text{mass difference associated to } \zeta_0 \text{ at } \zeta_0 \cap \{u + v = t_*\},$$

we will examine how $\mathcal{E}_{\zeta_0}$ changes along ζ_0 , after each successive intersection of ζ_0 with the rest of the beams:

1. Starting from $u = 0$ up to the first reflection of ζ_0 off the inner mirror γ_0 , the beam ζ_0 is ingoing and intersects all the other beams *after* they are reflected off γ_0 . Thus,

applying (3.1.30) successively at each intersection of ζ_0 with an outgoing beam, we infer that $\mathcal{E}_{\zeta_0}$ increases at this step by a multiplicative factor

$$(3.1.33) \quad A_{in} \geq \exp \left(\frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0})}{r_{\gamma_0}} (1 - \varepsilon) \right),$$

where r_{γ_0} is the value of r at the region of intersection of ζ_0 with the first beam which is reflected off $\{r = r_{0\varepsilon}\}$ (note that r_{γ_0} is also the r -width of the bundle of beams when ζ_0 first reaches the mirror γ_0). In obtaining (3.1.33), we have assumed that $r_{0\varepsilon} \ll r_{\gamma_0} \ll (-\Lambda)^{-\frac{1}{2}}$, $\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} \sim r_{0\varepsilon}$ and $\mathcal{E}_{\zeta_0}|_{u=0} \ll \tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$ (which holds in view of the way the initial data were chosen).

2. The mass difference $\mathcal{E}_{\zeta_0}$ right before and right after the reflection of ζ_0 off γ_0 is the same, in view of the reflecting boundary conditions on γ_0 .
3. From its first reflection off γ_0 up to its first reflection off $\mathcal{I}$, the beam ζ_0 is outgoing and intersects (again) the rest of the beams in the region close to $\mathcal{I}$ (*after* these beams are reflected off $\mathcal{I}$). Applying (3.1.31) successively at each intersection, we infer that $\mathcal{E}_{\zeta_0}$ decreases at this step, being multiplied by a factor

$$(3.1.34) \quad 1 > A_{out} \geq \exp \left(- \frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0})}{r_{\mathcal{I}}} \left(\frac{1}{(1 - \varepsilon - \frac{1}{3}\Lambda r_{\mathcal{I}}^2)} + \varepsilon \right) \right),$$

where $r_{\mathcal{I}}$ is the value of r at the region of intersection of ζ_0 with the first beam which is reflected off $\mathcal{I}$. In obtaining (3.1.34), we have assumed that $r_{\mathcal{I}} \gg (-\Lambda)^{-\frac{1}{2}}$ (which holds in view of the way the initial data were chosen).

4. The mass difference $\mathcal{E}_{\zeta_0}$ right before and right after the reflection of ζ_0 off $\mathcal{I}$ is the same, in view of the reflecting boundary conditions on $\mathcal{I}$.

Therefore, provided $r_{\gamma_0} \ll (-\Lambda)^{-\frac{1}{2}} \ll r_{\mathcal{I}}$, we infer that, after the first reflection of ζ_0 off

γ_0 and $\mathcal{I}$, the mass difference $\mathcal{E}_{\zeta_0}$ increases by a factor

(3.1.35)

$$\begin{aligned} A_{tot} = A_{in} \cdot A_{out} &\geq \exp\left(\frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0})}{r_{\gamma_0}}(1 - \varepsilon) - \frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0})}{r_{\mathcal{I}}}\left(\frac{1}{(1 - \varepsilon - \frac{1}{3}\Lambda r_{\mathcal{I}}^2)} + \varepsilon\right)\right) \\ &\geq \exp\left(\frac{\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0}}{r_{\gamma_0}}\right). \end{aligned}$$

The steps 1–4 in the above procedure can then be repeated for each successive reflection of ζ_0 off γ_0 and $\mathcal{I}$, as long as

$$(3.1.36) \quad r_{0\varepsilon} \ll r_{\gamma_0;n} \ll (-\Lambda)^{-\frac{1}{2}} \ll r_{\mathcal{I};n},$$

where $r_{\gamma_0;n}, r_{\mathcal{I};n}$ are the values of $r_{\gamma_0}, r_{\mathcal{I};n}$ after the n -th reflection of ζ_0 on γ_0 and $\mathcal{I}$ (note that $r_{\gamma_0;n}$ is also the r -width of the bundle of beams at the n -th reflection of ζ_0 off γ_0). Thus, as long as (3.1.36) holds, denoting with $\mathcal{E}_{\zeta_0;n}$ the value of $\mathcal{E}_{\zeta_0}$ at the n -th reflection of ζ_0 off γ_0 , the following inductive bound holds:

$$(3.1.37) \quad \mathcal{E}_{\zeta_0;n} \geq A_{tot;n} \cdot \mathcal{E}_{\zeta_0;n-1},$$

where the multiplicative factor

$$(3.1.38) \quad A_{tot;n} \doteq \exp\left(\frac{\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0;n}}{r_{\gamma_0;n}}\right)$$

is always greater than 1, since $\mathcal{E}_{\zeta_0;n} < \tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$ (see also the relation (3.6.13) in Section 3.6.1).

This is the main mechanism driving the instability, and the proof of Theorem 3.1 is aimed at showing that, for some large enough $n(\varepsilon)$ depending on ε ,

$$(3.1.39) \quad \prod_{n=0}^{n(\varepsilon)} A_{tot;n} > \frac{r_{0\varepsilon}}{2\mathcal{E}_{\zeta_0}|_{u=0}}.$$

Inequality (3.1.39) implies (in view of (3.1.37)) that

$$(3.1.40) \quad \frac{2\mathcal{E}_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}} > 1,$$

i. e. that, after the $n(\varepsilon)$ 'th successive reflection of ζ_0 on γ_0 and $\mathcal{I}$, the mass difference $\mathcal{E}_{\zeta_0}$ has become so large that a trapped surface (in particular, a point where $\frac{2m}{r} > 1$) necessarily forms before ζ_0 reaches the mirror $\gamma_0 = \{r = r_{0\varepsilon}\}$ for the $n(\varepsilon) + 1$ -th time (provided ζ_0 was initially chosen sufficiently “narrow”).⁸

Control of $r_{\gamma_0;n}$ and the final step before trapped surface formation

The main obstacle to establishing (3.1.39) (and, thus, Theorem 3.1) is the following: Once $\mathcal{E}_{\zeta_0}$ exceeds $c \cdot r_{0\varepsilon}$ for some fixed (small) $c > 0$, the total r -width of the bundle of beams close to γ_0 , i. e. $r_{\gamma_0;n}$ in (3.1.38), increases after each successive reflection off γ_0 and $\mathcal{I}$. Thus, the multiplicative factor (3.1.38) decreases as n grows. The increase in $r_{\gamma_0;n}$ is more dramatic when the spacetime is close to having a trapped surface, i. e. when $\frac{2m}{r}$ is close to 1.⁹

Controlling the growth of $r_{\gamma_0;n}$ is achieved by establishing an inductive bound of the following form:

$$(3.1.41) \quad r_{\gamma_0;n} \leq r_{\gamma_0;n-1} \cdot \left(1 + C_0 \frac{r_{0\varepsilon}}{r_{\gamma_0;n-1}} \left(\left| \log \left(1 - \frac{2\mathcal{E}_{\zeta_0;n-1}}{r_{0\varepsilon}} \right) \right| + 1 \right) \right)$$

(see also the relation (3.6.14) in Section 3.6.1). Obtaining the bound (3.1.41) is one of the most demanding parts in the proof of Theorem 3.1 and requires controlling the r -distance $r_{\gamma_0;n}^{(1)}$ of ζ_0 from the second-to-top beam ζ_1 at the n -th reflection off γ_0 for all $n \leq n(\varepsilon)$, i. e. establish a bound of the form

$$(3.1.42) \quad \frac{r_{\gamma_0;n}^{(1)}}{r_{0\varepsilon}} \geq 1 + c_0 (\mathcal{E}_{\zeta_0;0}/r_{0\varepsilon}).$$

(see (3.6.12) in Section 3.6.1). The bound (3.1.42) is in turn obtained by establishing an

⁸We should remark that, once a trapped surface $\mathcal{S}$ has formed, $\{r = r_{0\varepsilon}\} \cap J^+(\mathcal{S})$ (where $J^+(\mathcal{S})$ is the future of $\mathcal{S}$) will be spacelike and we will not study the evolution of the spacetime beyond $\{r = r_{0\varepsilon}\} \cap J^+(\mathcal{S})$. In particular, no more reflections of the beams will occur in the future of $\mathcal{S}$. See Theorem 2.

⁹The example of two outgoing null rays in the exterior of Schwarzschild–AdS, with mass $M \ll (-\Lambda)^{-\frac{1}{2}}$, serves to illustrate this phenomenon: The r -separation of two rays emanating from the region close to the future event horizon $\mathcal{H}^+$, where $\frac{2m}{r} \sim 1$, increases dramatically by the time they reach the region $r \sim (-\Lambda)^{-\frac{1}{2}}$. This is, of course, nothing other than the celebrated *red-shift* effect.

inductive bound of the form

$$(3.1.43) \quad \log \left(\frac{r_{\gamma_0;n-1}^{(1)}}{r_{\gamma_0;n}^{(1)}} \right) \leq C_0 \log \left(\frac{\mathcal{E}_{\zeta_0;n}}{\mathcal{E}_{\zeta_0;n-1}} \right),$$

estimating the decrease of $r_{\gamma_0;n}^{(1)}$ by the increase of $\mathcal{E}_{\zeta_0;n}$ at each reflection (see (3.6.129) in Section 3.6.1). The bound (3.1.42) is inferred from (3.1.43), in view of the fact that $\mathcal{E}_{\zeta_0;n} \geq \mathcal{E}_{\zeta_0;n-1}$ and

$$(3.1.44) \quad \sum_{n=1}^{n(\varepsilon)} \log \left(\frac{\mathcal{E}_{\zeta_0;n}}{\mathcal{E}_{\zeta_0;n-1}} \right) = \log \left(\frac{\mathcal{E}_{\zeta_0;n(\varepsilon)}}{\mathcal{E}_{\zeta_0;0}} \right) \leq \log \left(\frac{r_{0\varepsilon}}{2\mathcal{E}_{\zeta_0;0}} \right).$$

At the level of the initial data, obtaining (3.1.41) and (3.1.43) requires introducing a certain hierarchy for the scales of the r -distances and mass differences associated to the beams initially (see (3.5.2) and (3.5.3) in Section 3.5.1).

Combining (3.1.37) and (3.1.41), we can show that there exists a large $n(\varepsilon)$ such that, after $n(\varepsilon)$ reflections of ζ_0 off γ_0 (but not earlier!), we have

$$(3.1.45) \quad \frac{2\mathcal{E}_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}} > 1 - c(\varepsilon),$$

where $c(\varepsilon) \ll h(\varepsilon)$ is a fixed function of ε . Note that, compared to (3.1.40), (3.1.45) is a slightly weaker bound, which just stops short of implying that a trapped surface is formed. In order to complete the proof of Theorem 3.1, we therefore have to consider two different scenarios for $\mathcal{E}_{\zeta_0;n(\varepsilon)}$:

Case 1. In the case when (3.1.40) holds, the proof of Theorem 3.1 follows readily, since (3.1.40) implies that, before ζ_0 reaches $\{r = r_{0\varepsilon}\}$ for the $n(\varepsilon) + 1$ -th time, a point arises where $\frac{2m}{r} > 1$.

Case 2. In the case when (3.1.45) holds but (3.1.40) is violated, we can bound

$$(3.1.46) \quad 1 - c(\varepsilon) < \frac{2\mathcal{E}_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}} \leq 1.$$

In this case, ζ_0 reaches $\{r = r_{0\varepsilon}\}$ for the $n(\varepsilon) + 1$ -th time before a trapped

surface has formed. One would be tempted to repeat the above procedure for one more reflection, in an attempt to establish that a trapped surface has formed before the $n(\varepsilon) + 2$ -th reflection of ζ_0 off γ_0 . However, the bound (3.1.46) implies that most of the bootstrap assumptions needed for the proof of Theorem 3.1 (which we have suppressed in this sketch for the sake of simplicity) are violated beyond the $n(\varepsilon) + 1$ -th reflection and, thus, the above procedure can not be repeated. For this reason, we choose a different path: Applying a Cauchy stability statement backwards in time (see Theorem 3.2), we show that there exists a small perturbation $(r', (\Omega')^2, \tau', \bar{\tau}')^{(\varepsilon)}|_{u=0}$ of the initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ (satisfying (3.1.20)), such that the perturbed solution $(r', (\Omega')^2, \tau', \bar{\tau}')^{(\varepsilon)}$ to (3.1.4) satisfies (3.1.40) and, furthermore,

$$(3.1.47) \quad \frac{2\mathcal{E}'_{\zeta_0; n(\varepsilon)}}{r_{0\varepsilon}} > 1$$

(where $\mathcal{E}'_{\zeta_0}$ is similarly defined by the relation 3.1.32 for $(r', (\Omega')^2, \tau', \bar{\tau}')^{(\varepsilon)}$ in place of $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$). Thus, we end up in the scenario of Case 1, and the proof of Theorem 3.1 follows readily.

Further remarks on the proof of Theorem 3.1

The proof of Theorem 3.1 involves many technical issues related to the final step of the evolution before a trapped surface is formed. Most of these technical issues simplify considerably in the case when one restricts to showing a weaker instability statement for $(\mathcal{M}_{AdS}, g_{AdS})$, e. g. by replacing (3.1.15) with

$$(3.1.48) \quad \left(1 - \frac{2m}{r}\right)^{(\varepsilon)}|_{(u_\varepsilon, v_\varepsilon)} < \frac{1}{2}.$$

See Sections 3.4 and 3.6.1 for more details.

The mechanism leading to trapped surface formation in the proof of Theorem 3.1 only made use of the fact that we chose the initial data $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ so that the matter was supported in narrow null beams, successively reflected off γ_0 and $\mathcal{I}$, while the matter model

satisfied the condition

$$(3.1.49) \quad T_{uv} = \Omega^2 g^{AB} T_{AB} = 0.$$

Thus, we expect that the same mechanism can be adapted to the case of more general matter fields, which allow for matter to be arranged into narrow and sufficiently localised null beams, satisfying (in a region around the set of intersection of the beams)

$$(3.1.50) \quad T_{uv}, |\Omega^2 g^{AB} T_{AB}| \ll T_{uu} + T_{vv},$$

with such a configuration arising moreover from initial data which are small in a norm for which $(\mathcal{M}_{AdS}, g_{AdS})$ is Cauchy stable. For an application of this mechanism in the case of the spherically symmetric Einstein–massless Vlasov system (without reducing to the radial case and without an inner mirror), see our forthcoming [106].

Finally, let us remark that the general mechanism of instability suggested by the proof of Theorem 3.1 can be summarized as follows: In a configuration consisting of a relatively narrow bundle of nearly-null beams of matter that are successively reflected on $\mathcal{I}$ and $r = 0$ (on an approximately $(\mathcal{M}_{AdS}, g_{AdS})$ background), the energy content of the “top” beam will increase after each pair of reflections. A similar physical space mechanism was described for the Einstein–scalar field system (3.1.3) in [54], where it was suggested that, on a nearly-null scalar field beam successively reflected off $\mathcal{I}$ and the center $r = 0$, the energy density on the top part of the beam tends to increase.

3.1.3 Aside: Earlier numerical and heuristic works, and related problems

Numerical results in the direction of establishing the AdS instability conjecture were first obtained in 2011 by Bizon and Rostworowski in [15], who studied the evolution of spherically symmetric perturbations of $(\mathcal{M}_{AdS}, g_{AdS})$ for the Einstein–scalar field system (3.1.1) in Schwarzschild-type coordinates. More precisely, [15] numerically simulated the evolution of initial data for (3.1.1) with φ initially arranged into small amplitude wave packets. It was found that, for certain families of initial arrangements of this form (of “size” ϵ), after a finite

number of reflections on $\mathcal{I}$ (proportional to ε^{-2}), the energy of the wave packets becomes substantially concentrated, leading to a break down of the coordinate system associated with the threshold of trapped surface formation.

Following [15], a vast amount of numerical and heuristic works have been dedicated to the understanding of the global dynamics of perturbations of $(\mathcal{M}_{AdS}, g_{AdS})$ for the system (3.1.1) (see, e. g., [52, 19, 51, 104, 11, 35, 36, 14, 54, 74, 88, 56, 55, ?]). In these works, the picture that arises regarding the long time dynamics of *generic* spherically symmetric perturbations is rather complicated: Apart from perturbations that lead to instability and trapped surface formation ([52, 19]), it appears that there exist certain types of perturbations (dubbed “islands of stability”) which remain close to $(\mathcal{M}_{AdS}, g_{AdS})$ for long times; see [51, 104, 11, 56]. Perturbations of the latter type might in fact occupy an open set in the moduli space of spherically symmetric initial data for (3.1.1) (see [11, 56]). The question of existence of open “corners” of initial data around $(\mathcal{M}_{AdS}, g_{AdS})$ leading to trapped surface formation has also been studied (see, e. g. [54]).

Another interesting problem in this context is the characterization of the possible end states of the evolution of unstable perturbations of $(\mathcal{M}_{AdS}, g_{AdS})$. In [85], Holzegel–Smulevici established that the Schwarzschild–AdS spacetime $(\mathcal{M}_{Sch}, g_{Sch})$ is an asymptotically stable solution of the system (3.1.1) in spherical symmetry, with perturbations decaying at an exponential rate.¹⁰ This result supports the expectation that all spherically symmetric perturbations of $(\mathcal{M}_{AdS}, g_{AdS})$ for the system (3.1.1) leading to the formation of a trapped surface eventually settle down to a member of the Schwarzschild–AdS family (see [38, 39]). However, beyond spherical symmetry, Holzegel–Smulevici [84, 86] showed that solutions to the linear scalar wave equation

$$(3.1.51) \quad \square_{g_{Sch}} \varphi = 0$$

on $(\mathcal{M}_{Sch}, g_{Sch})$ (and, more generally, on Kerr–AdS) decay at a slow (logarithmic) rate, which is insufficient in itself to yield the non-linear stability of $(\mathcal{M}_{Sch}, g_{Sch})$ (cf. our re-

¹⁰A similar result can presumably also be deduced for the vacuum Einstein equations (1.2.6) reduced under the biaxial Bianchi IX symmetry in $4 + 1$ dimensions, following by an amalgamation of the proofs of [81] and [85].

mark below the statement of the AdS instability conjecture). Thus, [86] conjectured that $(\mathcal{M}_{Sch}, g_{Sch})$ is non-linearly unstable. On the other hand, based on a detailed analysis of quasinormal modes on $(\mathcal{M}_{Sch}, g_{Sch})$, Dias–Horowitz–Marolf–Santos [51] suggested that sufficiently regular, non-linear perturbations of $(\mathcal{M}_{Sch}, g_{Sch})$ still remain small, at least for long times. As a result, the picture regarding the end state of the evolution of generic perturbations of $(\mathcal{M}_{AdS}, g_{AdS})$ outside spherical symmetry remains unclear (see also [88, 53, 127]).

Following [15], the bulk of heuristic works have implemented a frequency space analysis in the study of the AdS instability conjecture. A notable exception is the work [54] of Dimitrakopoulos–Freivogel–Lippert–Yang, where a physical space mechanism possibly leading to instability for the system (3.1.3) is suggested. We will revisit the mechanism of [54] and compare it with the results of this chapter at the end of Section 3.1.2.

3.1.4 Outline of the chapter

This chapter is organised as follows:

In Section 3.2, we will introduce the spherically symmetric Einstein–radial massless Vlasov system in double null coordinates. We will also formulate the notion of reflecting boundary conditions for this system on $\mathcal{I}$ and on timelike hypersurfaces of the form $\{r = r_0\}$.

In Section 3.3, we will formulate the asymptotically AdS characteristic initial-boundary value problem for the spherically symmetric Einstein–radial massless Vlasov system. We will then recall the main results established in [111] regarding the structure of the maximal development and the Cauchy stability properties for this system.

In Section 3.4, we will provide a technical statement of the main result, namely the instability of AdS for the Einstein–radial massless Vlasov system with reflecting boundary conditions on $\{r = r_0\}$ and $\mathcal{I}$. The proof of this result will occupy Sections 3.5 and 3.6.

3.2 The Einstein–massless Vlasov system in spherical symmetry

In this Section, we will review the basic properties of the spherically symmetric Einstein–massless Vlasov system in $3 + 1$ dimensions, expressed in double null coordinates,

following the conventions introduced in [42]. We will also introduce the notion of the reflecting boundary condition on timelike hypersurfaces for the radial massless Vlasov equation. To this end, we will follow the conventions adopted in [111].

3.2.1 Spherically symmetric spacetimes in double null coordinates

Let $(\mathcal{M}^{3+1}, g)$ be a smooth Lorentzian manifold, such that $\mathcal{M}$ is of the form

$$(3.2.1) \quad \mathcal{M} \simeq \mathcal{U} \times \mathbb{S}^2$$

where $\mathcal{U}$ is an open domain of $\mathbb{R}^2$ with piecewise Lipschitz boundary $\partial\mathcal{U}$ and, in the standard (u, v) coordinates on $\mathcal{U}$, g takes the form

$$(3.2.2) \quad g = -\Omega^2(u, v)dudv + r^2(u, v)g_{\mathbb{S}^2},$$

where $g_{\mathbb{S}^2}$ is the standard round metric on $\mathbb{S}^2$ and $\Omega, r : \mathcal{U} \rightarrow (0, +\infty)$ are smooth functions. In addition, we will assume that

$$(3.2.3) \quad \inf_{\mathcal{U}} r > 0.$$

We will also fix a time orientation on $\mathcal{M}$ by requiring that the timelike vector field $N = \partial_u + \partial_v$ is future directed.

Remark. Notice that the action of $SO(3)$ on $(\mathcal{M}, g)$ through rotations of the $\mathbb{S}^2$ factor of (3.2.1) is an isometric action.

We will also define the *Hawking mass* $m : \mathcal{M} \rightarrow \mathbb{R}$ by the expression

$$(3.2.4) \quad m = \frac{r}{2} \left(1 - g(\nabla r, \nabla r) \right).$$

Viewed as a function on $\mathcal{U}$, m takes the form:

$$(3.2.5) \quad m = \frac{r}{2} \left(1 + 4\Omega^{-2} \partial_u r \partial_v r \right).$$

Equivalently, we have

$$(3.2.6) \quad \Omega^2 = 4 \frac{(-\partial_u r) \partial_v r}{1 - \frac{2m}{r}}.$$

In any local coordinate chart (y^1, y^2) on $\mathbb{S}^2$, the non-zero Christoffel symbols of (3.2.2) in the (u, v, y^1, y^2) local coordinate chart on $\mathcal{M}$ are computed as follows:

$$(3.2.7) \quad \begin{aligned} \Gamma_{uu}^u &= \partial_u \log(\Omega^2), \quad \Gamma_{vv}^v = \partial_v \log(\Omega^2), \\ \Gamma_{AB}^u &= \Omega^{-2} \partial_v (r^2) (g_{\mathbb{S}^2})_{AB}, \quad \Gamma_{AB}^v = \Omega^{-2} \partial_u (r^2) (g_{\mathbb{S}^2})_{AB}, \\ \Gamma_{uB}^A &= r^{-1} \partial_u r \delta_B^A, \quad \Gamma_{vB}^A = r^{-1} \partial_v r \delta_B^A, \\ \Gamma_{BC}^A &= (\Gamma_{\mathbb{S}^2})_{BC}^A, \end{aligned}$$

where the latin indices A, B, C are associated to the spherical coordinates y^1, y^2 , δ_B^A is Kronecker delta and $\Gamma_{\mathbb{S}^2}$ are the Christoffel symbols of the round sphere in the (y^1, y^2) coordinate chart.

For any pair of smooth functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ with $f_1', f_2' \neq 0$, the coordinate transformation

$$(3.2.8) \quad (\bar{u}, \bar{v}) = (f_1(u), f_2(v)),$$

mapping $\mathcal{U}$ to $\bar{\mathcal{U}} \subset \mathbb{R}^2$, can be used to diffeomorphically identify $\mathcal{M}$ with $\bar{\mathcal{U}} \times \mathbb{S}^2$. In these new coordinates, the metric g takes the form

$$(3.2.9) \quad g = -\bar{\Omega}^2(\bar{u}, \bar{v}) d\bar{u} d\bar{v} + r^2(\bar{u}, \bar{v}) g_{\mathbb{S}^2},$$

where

$$(3.2.10) \quad \bar{\Omega}^2(\bar{u}, \bar{v}) = \frac{1}{f_1' f_2'} \Omega^2(f_1^{-1}(\bar{u}), f_2^{-1}(\bar{v})),$$

$$(3.2.11) \quad r(\bar{u}, \bar{v}) = r(f_1^{-1}(\bar{u}), f_2^{-1}(\bar{v})).$$

We will frequently make use of such coordinate transformations, without renaming the

coordinates each time.

Note that m is invariant under coordinate transformations of the form $(u, v) \rightarrow (f_1(u), f_2(v))$, i. e.

$$(3.2.12) \quad m(\bar{u}, \bar{v}) = m(f_1^{-1}(\bar{u}), f_2^{-1}(\bar{v})).$$

3.2.2 The radial massless Vlasov equation

Let $(\mathcal{M}, g)$ be as in Section 3.2.1. Let $f \geq 0$ be a measure on $T\mathcal{M}$ which is constant along the geodesic flow, that is to say, in any local coordinate chart (x^0, x^1, x^2, x^3) on $\mathcal{M}$ with associated momentum coordinates (p^0, p^1, p^2, p^3) on the fibers of $T\mathcal{M}$, f satisfies (as a distribution) the first order equation

$$(3.2.13) \quad p^\alpha \partial_{x^\alpha} f - \Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma \partial_{p^\alpha} f = 0,$$

where $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols of g in the chart (x^0, x^1, x^2, x^3) . We will call f a *massless Vlasov field* if it is supported on the set $P \subset T\mathcal{M}$ of null vectors, i. e. on the set

$$(3.2.14) \quad g_{\alpha\beta}(x) p^\alpha p^\beta = 0.$$

Associated to f is a symmetric $(0, 2)$ -form on $\mathcal{M}$ (possibly defined only in the sense of distributions), the *energy momentum* tensor of f , given by the expression

$$(3.2.15) \quad T_{\alpha\beta}(x) = \int_{\pi^{-1}(x)} p_\alpha p_\beta f,$$

where $\pi^{-1}(x)$ denotes the fiber of $T\mathcal{M}$ over $x \in \mathcal{M}$ and the indices of the momentum coordinates are lowered with the use of the metric g , i. e.

$$(3.2.16) \quad p_\gamma = g_{\gamma\delta}(x) p^\delta.$$

Remark. We will only consider distributions f for which the expression (3.2.15) is finite for all $x \in \mathcal{M}$ and depends smoothly on $x \in \mathcal{M}$.

We will consider only distributions f which are spherically symmetric, i.e. invariant under the action of $SO(3)$ on $\mathcal{M}$. In that case, in any (u, v, y^1, y^2) local coordinate chart as in Section 3.2.1, the energy-momentum tensor T is of the form

$$(3.2.17) \quad T = T_{uu}(u, v)du^2 + 2T_{uv}(u, v)dudv + T_{vv}(u, v)dv^2 + T_{AB}(u, v)dy^A dy^B.$$

Furthermore, we will restrict to *radial* Vlasov fields f , i.e. fields supported only on radial null vectors which are normal to the orbits of the action of $SO(3)$ on $\mathcal{M}$. In any (u, v, y^1, y^2) local coordinate chart as in Section 3.2.1 (with associated momentum coordinates (p^u, p^v, p^1, p^2)), a spherically symmetric, radial massless Vlasov field f has the form

$$(3.2.18) \quad f(u, v, y^1, y^2; p^u, p^v, p^1, p^2) = (\bar{f}_{in}(u, v; p^u) + \bar{f}_{out}(u, v; p^v))\delta(\sqrt{(g_{S^2})_{AB}p^A p^B})\delta(\Omega^2 p^u p^v),$$

where $\bar{f}_{in}, \bar{f}_{out} \geq 0$ and δ is the Dirac delta function on $\mathbb{R}$. In this case, the only non-zero components of the energy momentum tensor (3.2.15) are the T_{uu} and T_{vv} components. In particular, in terms of $\bar{f}_{in}, \bar{f}_{out}$, we (formally) compute that

$$(3.2.19) \quad T_{uu}(u, v) = \int_0^{+\infty} \Omega^4(p^v)^2 \bar{f}_{out}(u, v; p^v) r^2 \frac{dp^v}{p^v},$$

$$(3.2.20) \quad T_{vv}(u, v) = \int_0^{+\infty} \Omega^4(p^u)^2 \bar{f}_{in}(u, v; p^u) r^2 \frac{dp^u}{p^u}.$$

Remark. We will only consider the case when $\bar{f}_{in}, \bar{f}_{out}$ are smooth and compactly supported in the p^u, p^v variables, respectively.

In the case when f is of the form (3.2.18), equation (3.2.13) is equivalent to the following system for $\bar{f}_{in}$ and $\bar{f}_{out}$:

$$(3.2.21) \quad \partial_u(\Omega^4 r^4 p^u \bar{f}_{in}) + p^u \partial_{p^u}(\Omega^4 r^4 p^u \bar{f}_{in}) = 0,$$

$$(3.2.22) \quad \partial_v(\Omega^4 r^4 p^v \bar{f}_{out}) + p^v \partial_{p^v}(\Omega^4 r^4 p^v \bar{f}_{out}) = 0.$$

The equations (3.2.21)–(3.2.22) readily yield the following transport equations for T_{uu} , T_{vv} :

$$(3.2.23) \quad \partial_v(r^2 T_{uu}) = 0,$$

$$(3.2.24) \quad \partial_u(r^2 T_{vv}) = 0.$$

Remark. Under a coordinate transformation of the form (3.2.8), $\bar{f}_{in}, \bar{f}_{out}$ transform as

$$(3.2.25) \quad \bar{f}_{in}^{(new)}(f_1(u), f_2(v); f'_1(u)p) = \bar{f}_{in}(u, v; p)$$

and

$$(3.2.26) \quad \bar{f}_{out}^{(new)}(f_1(u), f_2(v); f'_2(v)p) = \bar{f}_{out}(u, v; p).$$

3.2.3 The spherically symmetric Einstein–radial massless Vlasov system

Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold and let $\Lambda < 0$. Let also f be a non-negative measure on $T\mathcal{M}$. The *Einstein–Vlasov* system for $(\mathcal{M}, g; f)$ with cosmological constant Λ is

$$(3.2.27) \quad \begin{cases} Ric_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \\ p^\alpha \partial_{x^\alpha} f - \Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma \partial_{p^\alpha} f = 0, \end{cases}$$

where $T_{\mu\nu}$ is expressed in terms of f by (3.2.15).

Restricting to the case where $(\mathcal{M}, g)$ is a spherically symmetric spacetime as in Section 3.2.1 and f is a radial massless Vlasov field (i. e. has the form (3.2.18)), the system (3.2.27)

is equivalent to the following system for $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$:

$$(3.2.28) \quad \partial_u \partial_v (r^2) = -\frac{1}{2}(1 - \Lambda r^2) \Omega^2,$$

$$(3.2.29) \quad \partial_u \partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2} (1 + 4\Omega^{-2} \partial_u r \partial_v r),$$

$$(3.2.30) \quad \partial_v (\Omega^{-2} \partial_v r) = -4\pi r T_{vv} \Omega^{-2},$$

$$(3.2.31) \quad \partial_u (\Omega^{-2} \partial_u r) = -4\pi r T_{uu} \Omega^{-2},$$

$$(3.2.32) \quad \partial_u (\Omega^4 r^4 p^u \bar{f}_{in}) = -p^u \partial_{p^u} (\Omega^4 r^4 p^u \bar{f}_{in}),$$

$$(3.2.33) \quad \partial_v (\Omega^4 r^4 p^v \bar{f}_{out}) = -p^v \partial_{p^v} (\Omega^4 r^4 p^v \bar{f}_{out}),$$

where T_{uu}, T_{vv} are expressed in terms of $\bar{f}_{out}, \bar{f}_{in}$ by (3.2.19), (3.2.20), respectively. Notice that the system (3.2.28)–(3.2.33) reduces to the following system for $(r, \Omega^2, T_{uu}, T_{vv})$:

$$(3.2.34) \quad \partial_u \partial_v (r^2) = -\frac{1}{2}(1 - \Lambda r^2) \Omega^2,$$

$$(3.2.35) \quad \partial_u \partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2} (1 + 4\Omega^{-2} \partial_u r \partial_v r),$$

$$(3.2.36) \quad \partial_v (\Omega^{-2} \partial_v r) = -4\pi r T_{vv} \Omega^{-2},$$

$$(3.2.37) \quad \partial_u (\Omega^{-2} \partial_u r) = -4\pi r T_{uu} \Omega^{-2},$$

$$(3.2.38) \quad \partial_u (r^2 T_{vv}) = 0,$$

$$(3.2.39) \quad \partial_v (r^2 T_{uu}) = 0.$$

Remark. The system (3.2.34)–(3.2.39) is the Einstein–null dust system with both ingoing and outgoing dust (used as a model for self-gravitating radiation already in [120]). In the notation of Section 3.1,

$$(3.2.40) \quad r^2 T_{vv} = \bar{\tau}$$

and

$$(3.2.41) \quad r^2 T_{uu} = \tau.$$

Defining the renormalised Hawking mass as

$$(3.2.42) \quad \tilde{m} \doteq m - \frac{1}{6}\Lambda r^3,$$

and using the relation (3.2.5), equations (3.2.34)–(3.2.39) formally give rise to the following system for $(r, \tilde{m}, T_{uu}, T_{vv})$ (valid in the region of $\mathcal{U}$ where $\partial_v r > 0$, $\partial_u r < 0$ and $1 - \frac{2m}{r} > 0$):

$$(3.2.43) \quad \partial_u \log \left(\frac{\partial_v r}{1 - \frac{2m}{r}} \right) = -4\pi r^{-1} \frac{r^2 T_{uu}}{-\partial_u r},$$

$$(3.2.44) \quad \partial_v \log \left(\frac{-\partial_u r}{1 - \frac{2m}{r}} \right) = 4\pi r^{-1} \frac{r^2 T_{vv}}{\partial_v r},$$

$$(3.2.45) \quad \partial_u \partial_v r = -\frac{2\tilde{m} - \frac{2}{3}\Lambda r^3}{r^2} \frac{(-\partial_u r)\partial_v r}{1 - \frac{2m}{r}},$$

$$(3.2.46) \quad \partial_u \tilde{m} = -2\pi \frac{\left(1 - \frac{2m}{r}\right)}{-\partial_u r} r^2 T_{uu},$$

$$(3.2.47) \quad \partial_v \tilde{m} = 2\pi \frac{\left(1 - \frac{2m}{r}\right)}{\partial_v r} r^2 T_{vv}$$

$$(3.2.48) \quad \partial_u (r^2 T_{vv}) = 0,$$

$$(3.2.49) \quad \partial_v (r^2 T_{uu}) = 0.$$

3.2.4 The reflecting boundary condition

Let $(\mathcal{M}, g)$ be as in Section 3.2.1. Recall that $\mathcal{M}$ splits topologically as the product

$$\mathcal{M} \simeq \mathcal{U} \times \mathbb{S}^2.$$

Let $\partial_{tim}\mathcal{U}$ be the subset of the boundary $\partial\mathcal{U}$ of $\mathcal{U} \subset \mathbb{R}^2$ consisting of a union of connected, timelike Lipschitz curves with respect to the comparison metric

$$(3.2.50) \quad g_{comp} = -dudv$$

on $\mathbb{R}^2$. Recall that a connected Lipschitz curve γ in $\mathbb{R}^2$ is said to be timelike with respect to (3.2.50) if, for every point $p = (u_*, v_*) \in \gamma$, we have

$$(3.2.51) \quad \gamma \setminus p \subset I^+(p) \cup I^-(p) \doteq (\{u > u_*\} \cap \{v > v_*\}) \cup (\{u < u_*\} \cap \{v < v_*\}).$$

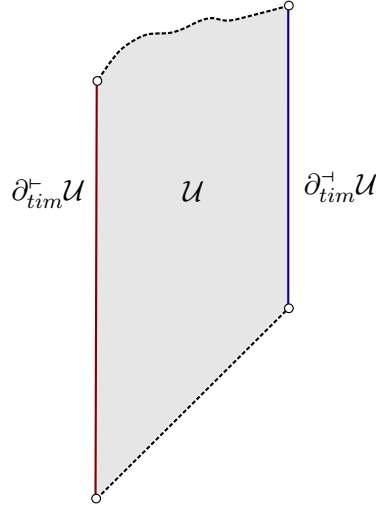


Figure 9: For a domain $\mathcal{U} \subset \mathbb{R}^2$ as depicted above, the timelike portion $\partial_{tim}\mathcal{U}$ of the boundary $\partial\mathcal{U}$ splits as the union of a "left" component $\partial_{tim}^+\mathcal{U}$ and a "right" component $\partial_{tim}^-\mathcal{U}$. In general, $\partial_{tim}^+\mathcal{U}$ and $\partial_{tim}^-\mathcal{U}$ need not necessarily be straight line segments as depicted above. However, in the following sections, we will impose a gauge condition on the domains under consideration that will indeed fix $\partial_{tim}^+\mathcal{U}$ and $\partial_{tim}^-\mathcal{U}$ to be vertical line segments (see Definitions 3.2 and 3.3).

Let us fix $w : \mathcal{U} \cup \partial_{tim}\mathcal{U} \rightarrow \mathbb{R}$ to be a smooth boundary defining function of $\partial_{tim}\mathcal{U}$, i. e.

$$w|_{\partial_{tim}\mathcal{U}} = 0,$$

$$dw|_{\partial_{tim}\mathcal{U}} \neq 0$$

and

$$w|_{\mathcal{U}} > 0.$$

We can split $\partial_{tim}\mathcal{U}$ into its "left" and "right" components as

$$(3.2.52) \quad \partial_{tim}\mathcal{U} = \partial_{tim}^+\mathcal{U} \cup \partial_{tim}^-\mathcal{U},$$

where

$$\partial_{tim}^+ \mathcal{U} = \{(u_0, v_0) \in \partial_{tim} \mathcal{U} : \partial_v w(u_0, v_0) > 0\},$$

$$\partial_{tim}^- \mathcal{U} = \{(u_0, v_0) \in \partial_{tim} \mathcal{U} : \partial_v w(u_0, v_0) < 0\}.$$

Remark. Notice that any future directed radial null geodesic of $\mathcal{M} = \mathcal{U} \times \mathbb{S}^2$ with a future limiting point on $\partial_{tim}^+ \mathcal{U} \times \mathbb{S}^2$ (in the ambient $\mathbb{R}^2 \times \mathbb{S}^2$ topology of $\bar{\mathcal{U}} \times \mathbb{S}^2$) is necessarily ingoing. Similarly, future directed radial null geodesics “terminating” at $\partial_{tim}^- \mathcal{U} \times \mathbb{S}^2$ are necessarily outgoing.

In the next sections, we will only consider the reflection of radial null geodesics on parts of $\partial_{tim} \mathcal{U}$ for which either $r - r_0$ (for some constant $r_0 > 0$) or $1/r$ is a boundary defining function.

Following [111], we will define the reflecting boundary condition on $\partial_{tim} \mathcal{U}$ for the radial massless Vlasov equation as follows:

Definition. A radial massless Vlasov field f on $T\mathcal{M}$ will be said to satisfy the *reflecting boundary condition* on $\partial_{tim} \mathcal{U} \times \mathbb{S}^2$ if and only if

- For any $(u_0, v_0) \in \partial_{tim}^+ \mathcal{U}$ and any $p > 0$:

$$(3.2.53) \quad \lim_{h \rightarrow 0^+} \left(\frac{\bar{f}_{out}(u_0, v_0 + h; \frac{-\partial_u w}{\partial_v w}(u_0, v_0) \cdot \Omega^{-2}(u_0, v_0 + h) \cdot p)}{\bar{f}_{in}(u_0 - h, v_0; \Omega^{-2}(u_0 - h, v_0) \cdot p)} \right) = 1.$$

- For any $(u_1, v_1) \in \partial_{tim}^- \mathcal{U}$ and any $p > 0$:

$$(3.2.54) \quad \lim_{h \rightarrow 0^+} \left(\frac{\bar{f}_{in}(u_1 + h, v_1; \frac{-\partial_v w}{\partial_u w}(u_1, v_1) \cdot \Omega^{-2}(u_1 + h, v_1) \cdot p)}{\bar{f}_{out}(u_1, v_1 - h; \Omega^{-2}(u_1, v_1 - h) \cdot p)} \right) = 1.$$

Note that the relations (3.2.53) and (3.2.54) for $\bar{f}_{in}, \bar{f}_{out}$ imply the following boundary relations for the components (3.2.19)–(3.2.20) of the energy momentum tensor T :

- For any $(u_0, v_0) \in \partial_{tim}^+ \mathcal{U}$:

$$(3.2.55) \quad \lim_{h \rightarrow 0^+} \frac{r^2 T_{uu}(u_0, v_0 + h)}{r^2 T_{vv}(u_0 - h, v_0)} = \left(\frac{-\partial_u w}{\partial_v w}(u_0, v_0) \right)^2.$$

- For any $(u_1, v_1) \in \partial_{tim}^- \mathcal{U}$:

$$(3.2.56) \quad \lim_{h \rightarrow 0^+} \frac{r^2 T_{vv}(u_1 + h, v_1)}{r^2 T_{uu}(u_1, v_1 - h)} = \left(\frac{-\partial_v w}{\partial_u w}(u_0, v_0) \right)^2.$$

3.3 Well-posedness for the boundary–characteristic initial value problem

In this Section, we will formulate the asymptotically AdS initial value problem for the system (3.2.28)–(3.2.33) with reflecting boundary conditions on $\{r = r_0\}$ and $\mathcal{I}$, for some $r_0 > 0$. We will then recall the main results established in [111], regarding the well-posedness and the structure of the maximal development for this system.

3.3.1 Asymptotically AdS characteristic initial data

The following definition was introduced in [111]:

Definition 3.1 (Definition 3.1 in [111]). For any $v_1 < v_2$ and any $r_0 > 0$, let $r_f : [v_1, v_2) \rightarrow [r_0, +\infty)$, $\Omega_f : [v_1, v_2) \rightarrow (0, +\infty)$ and $\bar{f}_{in/f}, \bar{f}_{out/f} : [v_1, v_2) \times (0, +\infty) \rightarrow [0, +\infty)$ be C^∞ functions, such that

$$(3.3.1) \quad r_f(v_1) = r_0$$

and

$$(3.3.2) \quad \lim_{v \rightarrow v_2} r_f(v) = +\infty.$$

Let us define $(\partial_u r)_I : [v_1, v_2) \rightarrow (-\infty, 0)$ by the relation

$$(3.3.3) \quad (\partial_u r)_I(v) = \frac{1}{r_I(v)} \left(-r_I \partial_v r_I(v_1) - \frac{1}{4} \int_{v_1}^v (1 - \Lambda r_I^2(\bar{v})) \Omega_I^2(\bar{v}) d\bar{v} \right).$$

We will call $(r_I, \Omega_I^2, \bar{f}_{in/I}, \bar{f}_{out/I})$ an *asymptotically AdS boundary-characteristic initial data set* on $[v_1, v_2)$ for the system (3.2.28)–(3.2.33) satisfying the reflecting gauge condition at $r = r_0, +\infty$ if:

- (r_I, Ω_I) satisfies the constraint equation

$$(3.3.4) \quad \partial_v (\Omega_I^{-2} \partial_v r_I) = -4\pi r_I (T_{vv})_I \Omega_I^{-2},$$

where

$$(3.3.5) \quad (T_{vv})_I(v) \doteq \int_0^{+\infty} \Omega_I^4(v) (p^u)^2 \bar{f}_{in/I}(v; p^u) r_I^2(v) \frac{dp^u}{p^u}.$$

- $\bar{f}_{out/I}$ solves the massless radial Vlasov equation

$$(3.3.6) \quad \partial_v (\Omega_I^4(v) r_I^4(v) p^v \bar{f}_{out/I}(v, p^v)) + p^v \partial_{p^v} (\Omega_I^4(v) r_I^4(v) p^v \bar{f}_{out/I}(v, p^v)) = 0.$$

- $(\partial_u r)_I$ satisfies

$$(3.3.7) \quad \lim_{v \rightarrow v_2^-} \frac{(\partial_u r)_I}{\partial_v r_I} = 1.$$

- $\bar{f}_{out/I}, \bar{f}_{in/I}$ satisfy the following compatibility conditions at $v = v_1, v_2$ for any $p > 0$:

$$(3.3.8) \quad \frac{\bar{f}_{out/I}(v_1; \frac{-(\partial_u r)_I}{\partial_v r_I}(v_1) \cdot \Omega_I^{-2}(v_1) \cdot p)}{\bar{f}_{in/I}(v_1; \Omega_I^{-2}(v_1) \cdot p)} = 1$$

and

$$(3.3.9) \quad \lim_{h \rightarrow 0^+} \left(\frac{\bar{f}_{in/}(v_2 - h; \frac{\partial_v r_{/}}{-(\partial_u r)_{/}}(v_2 - h) \cdot \Omega_{/}^{-2}(v_2 - h) \cdot p)}{\bar{f}_{out/}(v_2 - h; \Omega_{/}^{-2}(v_2 - h) \cdot p)} \right) = 1.$$

Remark. Notice that the constraint equation (3.3.4) implies

$$(3.3.10) \quad \partial_v(\Omega_{/}^{-2} \partial_v r_{/}) \leq 0.$$

Thus, (3.3.2) yields

$$(3.3.11) \quad \partial_v r_{/} > 0$$

everywhere on $[v_1, v_2)$.

Given any asymptotically AdS boundary-characteristic initial data set $(r_{/}, \Omega_{/}^2, \bar{f}_{in/}, \bar{f}_{out/})$ on $[v_1, v_2)$ with reflecting gauge conditions at $r = r_0, +\infty$, we will also define the initial Hawking mass $m_{/}$ and initial renormalised Hawking mass $\tilde{m}_{/}$ on $[v_1, v_2)$ by the relations

$$(3.3.12) \quad m_{/} \doteq \frac{r_{/}}{2} (1 - 4\Omega_{/}^{-2} (\partial_u r)_{/} \partial_v r_{/}),$$

and

$$(3.3.13) \quad \tilde{m}_{/} \doteq m_{/} - \frac{1}{6} \Lambda r_{/}^3,$$

in accordance with (3.2.6), (3.2.42).

3.3.2 Developments with reflecting boundary conditions on $r = r_0, +\infty$

We will only consider solutions $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ to (3.2.28)–(3.2.33) satisfying a reflecting gauge condition on $\partial_{tim} \mathcal{U}$, which fixes $\partial_{tim} \mathcal{U}$ to be a union of vertical straight lines in the (u, v) -plane. This motivates defining the following class of domains $\mathcal{U}$ in the plane (see [111]):

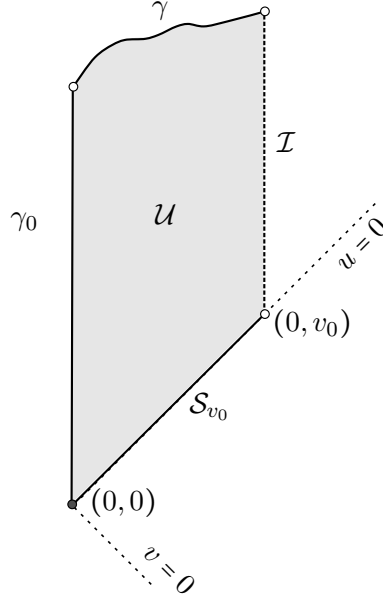


Figure 10: A typical domain $\mathcal{U} \in \mathcal{U}_{v_0}$ would be as depicted above. In the case when the boundary set γ is empty, it is necessary that both γ_0 and $\mathcal{I}$ are unbounded (i. e. extend all the way to $u + v = \infty$).

Definition 3.2 (Definition 3.3 in [111]). For any $v_0 > 0$, let $\mathcal{U}_{v_0}$ be the set of all connected open domains $\mathcal{U}$ of the (u, v) -plane with piecewise Lipschitz boundary $\partial\mathcal{U}$, with the property that $\partial\mathcal{U}$ splits as the union

$$(3.3.14) \quad \partial\mathcal{U} = \gamma_0 \cup \mathcal{I} \cup \mathcal{S}_{v_0} \cup \text{clos}(\gamma),$$

where, for some $0 < u_{\gamma_0}, u_{\mathcal{I}} \leq +\infty$,

$$(3.3.15) \quad \gamma_0 = \{u = v\} \cap \{0 \leq u < u_{\gamma_0}\},$$

$$(3.3.16) \quad \mathcal{I} = \{u = v - v_0\} \cap \{0 \leq u < u_{\mathcal{I}}\},$$

$$(3.3.17) \quad \mathcal{S}_{v_0} = \{0\} \times [0, v_0],$$

and $\gamma : (x_1, x_2) \rightarrow \mathbb{R}^2$ is a Lipschitz, achronal (with respect to the reference Lorentzian

metric (3.2.50)) curve, which is allowed to be empty (the closure $clos(\gamma)$ of γ in (3.3.14) is considered with respect to the standard topology of $\mathbb{R}^2$).

Remark. It follows readily from Definition 3.2 that $\mathcal{U}$ is necessarily contained in the future domain of dependence of $\mathcal{S}_{v_0} \cup \gamma_0 \cup \mathcal{I}$ (with respect to the comparison metric (3.2.50)). In the case when $\gamma = \emptyset$ in (3.3.14), it is necessary that both γ_0 and $\mathcal{I}$ extend all the way to $u + v = +\infty$.

A development of an asymptotically AdS boundary-characteristic initial data set for the system (3.2.28)–(3.2.33) with reflecting boundary conditions on $r = r_0, +\infty$ is defined as follows (see [111]):

Definition 3.3 (Definition 3.4 in [111]). For any $v_0 > 0$ and $r_0 > 0$, let $(r_/, \Omega_/, \bar{f}_{in/}, \bar{f}_{out/})$ be a smooth asymptotically AdS boundary-characteristic initial data set on $[0, v_0)$ for the system (3.2.28)–(3.2.33) satisfying the reflecting gauge condition at $r = r_0, +\infty$, according to Definition 3.1. A future development of $(r_/, \Omega_/, \bar{f}_{in/}, \bar{f}_{out/})$ will consist of an open set $\mathcal{U} \in \mathcal{U}_{v_0}$ (see Definition 3.2) and smooth functions $r : \mathcal{U} \rightarrow (r_0, +\infty)$, $\Omega^2 : \mathcal{U} \rightarrow (0, +\infty)$ and $\bar{f}_{in}, \bar{f}_{out} : \mathcal{U} \times (0, +\infty) \rightarrow [0, +\infty)$ satisfying the following properties:

1. The functions $r, \Omega^2, \bar{f}_{in}, \bar{f}_{out}$ solve the system (3.2.28)–(3.2.33) on $\mathcal{U}$.
2. The functions $r, \Omega^2, \bar{f}_{in}, \bar{f}_{out}$ satisfy the given initial conditions on $\mathcal{S}_{v_0} = \{0\} \times [0, v_0)$, i. e.:

$$(3.3.18) \quad (r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})|_{\mathcal{S}_{v_0}} = (r_/, \Omega_/, \bar{f}_{in/}, \bar{f}_{out/}).$$

3. The functions $(r, \bar{f}_{in}, \bar{f}_{out})$ satisfy on γ_0 the boundary conditions

$$(3.3.19) \quad r|_{\gamma_0} = r_0$$

and

$$(3.3.20) \quad \bar{f}_{out}(u_*, v_*; p) = \bar{f}_{in}(u_*, v_*; p),$$

for all $(u_*, v_*) \in \gamma_0$ and $p > 0$, and on $\mathcal{I}$ the boundary conditions

$$(3.3.21) \quad (1/r)|_{\mathcal{I}} = 0$$

and

$$(3.3.22) \quad \lim_{h \rightarrow 0^+} \left(\frac{\bar{f}_{in}(u_* + h, v_*; \Omega^{-2}(u_* + h, v_*) \cdot p)}{\bar{f}_{out}(u_*, v_* - h; \Omega^{-2}(u_*, v_* - h) \cdot p)} \right) = 1,$$

for all $(u_*, v_*) \in \mathcal{I}$ and $p > 0$.

4. The following are satisfied on γ_0 and $\mathcal{I}$:

$$(3.3.23) \quad \partial_u r|_{\gamma_0} = -\partial_v r|_{\gamma_0}$$

and

$$(3.3.24) \quad \partial_u (1/r)|_{\mathcal{I}} = -\partial_v (1/r)|_{\mathcal{I}}.$$

Remark. Notice that the boundary conditions (3.3.19) and (3.3.21), combined with the form (3.3.15) and (3.3.16) of γ_0 and $\mathcal{I}$, respectively, imply the relations (3.3.23) and (3.3.24). However, the relations (3.3.23) and (3.3.24) should be viewed as *gauge conditions* fixing, in conjunction with (3.3.19) and (3.3.21), the form (3.3.15) and (3.3.16) of γ_0 and $\mathcal{I}$.

If $\mathcal{D} = (\mathcal{U}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ and $\mathcal{D}' = (\mathcal{U}'; r', (\Omega')^2, \bar{f}'_{in}, \bar{f}'_{out})$ are two future developments of the same initial data $(r_I, \Omega_I^2, \bar{f}_{in/I}, \bar{f}_{out/I})$, we will say that $\mathcal{D}'$ is an extension of $\mathcal{D}$, writing $\mathcal{D} \subseteq \mathcal{D}'$, if $\mathcal{U} \subseteq \mathcal{U}'$ and the restriction of $(r', (\Omega')^2, \bar{f}'_{in}, \bar{f}'_{out})$ on $\mathcal{U}$ coincides with $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$.

Remark. If $\mathcal{D} = (\mathcal{U}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ and $\mathcal{D}' = (\mathcal{U}'; r', (\Omega')^2, \bar{f}'_{in}, \bar{f}'_{out})$ are two future developments of the same initial data $(r_I, \Omega_I^2, \bar{f}_{in/I}, \bar{f}_{out/I})$, then

$$(3.3.25) \quad (r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})|_{\mathcal{U} \cap \mathcal{U}'} = (r', (\Omega')^2, \bar{f}'_{in}, \bar{f}'_{out})|_{\mathcal{U} \cap \mathcal{U}'}$$

(see [111]).

3.3.3 The maximal development

The following result was established in [111]:

Theorem 3.3 (Theorem 1 in [111]). For any $v_0 > 0$ and $r_0 > 0$, let $(r_/, \Omega_/, \bar{f}_{in/}, \bar{f}_{out/})$ be a smooth asymptotically AdS boundary-characteristic initial data set on $[0, v_0)$ for the system (3.2.28)–(3.2.33) satisfying the reflecting gauge condition at $r = r_0, +\infty$, according to Definition 3.1, such that the quantities $\frac{\Omega_//^2}{1 - \frac{1}{3}\Lambda r_//^2}, r_//^2(T_{vv})_/$ and $\tan^{-1} r_/$ extend smoothly on $v = v_0$. Then, there exists a unique, smooth future development $(\mathcal{U}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ of $(r_/, \Omega_//^2, \bar{f}_{in/}, \bar{f}_{out/})$ which is maximal, i. e. any other future development $(\mathcal{U}'; r', (\Omega')^2, \bar{f}'_{in}, \bar{f}'_{out})$ of $(r_/, \Omega_//^2, \bar{f}_{in/}, \bar{f}_{out/})$ with $r' \geq r_0$ everywhere on $\mathcal{U}'$ satisfies $\mathcal{U}' \subseteq \mathcal{U}$ and $r', (\Omega')^2, \bar{f}'_{in}, \bar{f}'_{out}$ are the restrictions of $r, \Omega^2, \bar{f}_{in}, \bar{f}_{out}$ on $\mathcal{U}'$.

The maximal future development $(\mathcal{U}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ satisfies the following properties (for the definition of the curves $\gamma_0, \mathcal{I}, \gamma$, see Definition 3.2):

1. The renormalised Hawking mass $\tilde{m}$ is conserved on γ_0 and $\mathcal{I}$, i. e.:

$$(3.3.26) \quad \tilde{m}|_{\gamma_0} = \tilde{m}|_{\gamma_0 \cap \{u=0\}}$$

and

$$(3.3.27) \quad \tilde{m}|_{\mathcal{I}} = \tilde{m}|_{\mathcal{I} \cap \{u=0\}}.$$

2. The curve $\mathcal{I}$ is conformally complete, i. e. $\Omega^2/(1 - \frac{1}{3}\Lambda r^2)$ has a finite limit on $\mathcal{I}$ and:

$$(3.3.28) \quad \int_{\mathcal{I}} \sqrt{\frac{\Omega^2}{1 - \frac{1}{3}\Lambda r^2}} \Big|_{\mathcal{I}} du = +\infty.$$

3. We have

$$(3.3.29) \quad \partial_u r < 0,$$

$$(3.3.30) \quad \left(1 - \frac{2m}{r}\right) \Big|_{J^-(\mathcal{I}) \cup J^-(\gamma_0)} > 0$$

and

$$(3.3.31) \quad \partial_v r|_{J^-(\mathcal{I}) \cup J^-(\gamma_0)} > 0,$$

where

$$(3.3.32) \quad J^-(\mathcal{I}) = \left\{ 0 \leq u < \sup_{\mathcal{I}} u \right\} \cap \mathcal{U}$$

is the causal past of $\mathcal{I}$ and

$$(3.3.33) \quad J^-(\gamma_0) = \left\{ 0 \leq v < \sup_{\gamma_0} v \right\} \cap \mathcal{U}$$

is the causal past of γ_0 (with respect to the reference Lorentzian metric (3.2.50)).

4. In the case $\mathcal{U} \setminus J^-(\mathcal{I}) \neq \emptyset$, the future event horizon

$$(3.3.34) \quad \mathcal{H}^+ \doteq \mathcal{U} \cap \partial J^-(\mathcal{I}) = \left\{ u = \sup_{\mathcal{I}} u \right\} \cap \mathcal{U}$$

has the following properties:

(a) $\mathcal{H}^+$ has infinite affine length, i. e.:

$$(3.3.35) \quad \int_{\mathcal{H}^+} \Omega^2 dv = +\infty.$$

(b) We have

$$(3.3.36) \quad \sup_{\mathcal{H}^+} r = r_S$$

and

$$(3.3.37) \quad \inf_{\mathcal{H}^+} \left(1 - \frac{2m}{r} \right) = 0,$$

where r_S defined by the relation

$$(3.3.38) \quad 1 - 2 \frac{\lim_{v \rightarrow v_0^-} \tilde{m}_I(v)}{r_S} - \frac{1}{3} \Lambda r_S^2 = 0.$$

5. In the case $\mathcal{H}^+ \neq \emptyset$, the curve γ_0 is bounded and satisfies

$$(3.3.39) \quad \gamma_0 \not\subset J^-(\mathcal{I}),$$

i. e. γ_0 contains points lying to the future of $\mathcal{H}^+$.

6. In the case $\mathcal{H}^+ \neq \emptyset$, the curve γ is non-empty, piecewise smooth and r extends continuously on γ with $r|_{\gamma_0} = r_0$. Furthermore, for any point $(u_1, v_1) \in \gamma$, the line $\{v = v_1\}$ intersects $\mathcal{I}$.¹¹

Remark. In the case when $\mathcal{U} \setminus J^-(\mathcal{I}) \neq \emptyset$ (and thus $\mathcal{H}^+ \neq \emptyset$), in view of (3.3.36), (3.3.38) and the fact that $r > r_0$ on $\mathcal{U}$, it is necessary that

$$(3.3.40) \quad 2 \frac{\lim_{v \rightarrow v_0^-} \tilde{m}_I(v)}{r_0} > 1 - \frac{1}{3} \Lambda r_0^2.$$

In a similar way, we can uniquely define the *maximal past development* $(\mathcal{U}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ of $(r_I, \Omega_I^2, \bar{f}_{in/I}, \bar{f}_{out/I})$, satisfying the properties outlined by Theorem 3.3 after performing a “time reversal” transformation $(u, v) \rightarrow (-v, -u)$. Notice that such a coordinate transformation turns an asymptotically AdS boundary-characteristic initial data set on $u = 0$ into an asymptotically AdS boundary-characteristic initial data set on $v = 0$. However, Theorem 3.3 also holds (with exactly the same proof) for such initial data sets.

¹¹In other words, there is no point in γ which lies on the curve $\{v = v_{\mathcal{I}}\}$, where $(u_{\mathcal{I}}, v_{\mathcal{I}})$ is the future limit point of $\mathcal{I}$.

3.3.4 Cauchy stability in a rough norm, uniformly in r_0

In [111], the following “norm” was introduced for smooth asymptotically AdS boundary-characteristic initial data sets $(r_/, \Omega_{}^2, \bar{f}_{in/}, \bar{f}_{out/})$ on $[0, v_0)$ for the system (3.2.28)–(3.2.33):

$$(3.3.41) \quad \|(r_/, \Omega_{}^2, \bar{f}_{in/}, \bar{f}_{out/})\|_{\mathcal{CS}} \doteq (-\Lambda) \sup_{0 \leq v < v_0} \int_0^{v_0} \frac{1}{\varrho_/(v) - \varrho_/(\bar{v}) + \varrho_/(0)} \left(\frac{r_{}^2(T_{vv})_/}{\partial_v \varrho_/} \right) (\bar{v}) d\bar{v} + \\ + \sup_{0 \leq v < v_0} \max \left\{ \frac{2m_}{r_}, 0 \right\} + \sqrt{-\Lambda} \sup_{0 \leq v < v_0} |\tilde{m}_/(v)|,$$

where

$$(3.3.42) \quad \varrho_/(v) \doteq \tan^{-1}(\sqrt{-\Lambda} r_/(v)).$$

Remark. Note that, in (3.3.41),

$$\varrho_/(0) = \tan^{-1}(\sqrt{-\Lambda} r_0).$$

The expression (3.3.41) is invariant under gauge transformations, as well as scale transformations of the form $(u, v) \rightarrow (\lambda u, \lambda v)$, $(r, \tilde{m}, \Lambda) \rightarrow (\lambda r, \lambda \tilde{m}, \lambda^{-2} \Lambda)$, $r_0 \rightarrow \lambda r_0$, $(\bar{f}_{in}, \bar{f}_{out}) \rightarrow (\lambda^{-4} \bar{f}_{in}, \lambda^{-4} \bar{f}_{out})$. Moreover, $\|(r_/, \Omega_{}^2, \bar{f}_{in/}, \bar{f}_{out/})\|_{\mathcal{CS}} = 0$ if and only if $(r_/, \Omega_{}^2, \bar{f}_{in/}, \bar{f}_{out/})$ are the initial data for pure AdS spacetime on $\{r \geq r_0\}$, i. e. if $\bar{f}_{in/} = \bar{f}_{out/} = 0$ and $\tilde{m} = 0$.

The following Cauchy stability result for the trivial initial data was established in [111]:

Proposition 3.3 (Corollary 1 in [111]). *For any (possibly large) $l_* > 0$, there exists a (small) $\epsilon_0 > 0$ and a constant $C_{l_*} > 0$ depending only on l_* , so that the following statement holds: For any $v_0 > 0$ and $0 < r_0 < (-\Lambda)^{-\frac{1}{2}}$, if $(r_/, \Omega_{}^2, \bar{f}_{in/}, \bar{f}_{out/})$ is a smooth asymptotically AdS boundary-characteristic initial data set on $[0, v_0)$ for the system (3.2.28)–(3.2.33) satisfying the reflecting gauge condition at $r = r_0, +\infty$, according to Definition 3.1, such that the quantities $\frac{\Omega_{}^2}{1 - \frac{1}{3}\Lambda r_{}^2}, r_{}^2(T_{vv})_/$ and $\tan^{-1} r_/$ extend smoothly on $v = v_0$ and moreover*

$$(3.3.43) \quad \|(r_/, \Omega_{}^2, \bar{f}_{in/}, \bar{f}_{out/})\|_{\mathcal{CS}} < \epsilon$$

for some $0 < \epsilon \leq \epsilon_0$, then the maximal development $(\mathcal{U}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ satisfies

$$(3.3.44) \quad \mathcal{W}_{l_*} \doteq \{0 < u \leq l_* v_0\} \cap \{u < v < u + v_0\} \subset \mathcal{U}$$

and

$$(3.3.45) \quad \sqrt{-\Lambda} \sup_{\mathcal{W}_{l_*}} |\tilde{m}| + \sup_{\mathcal{W}_{l_*}} \log \left(\frac{1 - \frac{1}{3} \Lambda r^2}{1 - \max\{\frac{2m}{r}, 0\}} \right) + \\ + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{l_*}} \frac{r T_{vv}}{\partial_v r} dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{l_*}} \frac{r T_{uu}}{(-\partial_u r)} du < C_{l_*} \epsilon.$$

Remark. Proposition 3.3 should be interpreted as a Cauchy stability statement for the pure AdS initial data set with respect to the topology defined by (3.3.41) which is independent of the radius r_0 of the reflecting boundary.

Considering the spherically symmetric Einstein–scalar field system (3.1.3) with an inner mirror placed at $\{r = r_0\}$, the analogue of the initial data norm (3.3.41) (obtained using the substitution $(T_{vv})_l \rightarrow (\partial_v \varphi)|_{u=0}$) is rougher compared to the bounded variation norm of Christodoulou (see [28]). It is not known whether (3.1.3), restricted to the exterior of an inner mirror at $\{r = r_0\}$, satisfies a Cauchy stability estimate with respect to the analogue of the initial data norm (3.3.41) which is independent of r_0 (although local existence and uniqueness follow trivially in this case for fixed r_0).

In fact, Proposition 3.3 is a special case of the following Cauchy stability estimate established in [111]:

Theorem 3.2 (Theorem 2 in [111]). For any $v_1 < v_2$ and $0 < r_0 < (-\Lambda)^{-1/2}$, let $(r_{/i}, \Omega_{/i}^2, \bar{f}_{in/i}, \bar{f}_{out/i})$, $i = 1, 2$, be two smooth asymptotically AdS boundary-characteristic initial data sets on $[v_1, v_2)$ for the system (3.2.28)–(3.2.33) satisfying the reflective gauge condition at $r = r_0, +\infty$, according to Definition 3.1, such that the quantities $\frac{\Omega_{/i}^2}{1 - \frac{1}{3} \Lambda r_{/i}^2}$, $r_{/i}^2 (T_{vv})_{/i}$ and $\tan^{-1} r_{/i}$ extend smoothly on $v = v_2$. Assume, also, the following conditions:

1. For some $u_0 > 0$, the maximal future development $(\mathcal{U}_1; r_1, \Omega_1^2, \bar{f}_{in1}, \bar{f}_{out1})$ of

$(r_{/1}, \Omega_{/1}^2, \bar{f}_{in/1}, \bar{f}_{out/1})$ satisfies

$$(3.3.46) \quad \mathcal{W}_{u_0} \doteq \{0 < u < u_0\} \cap \{u + v_1 < v < u + v_2\} \subset \mathcal{U}_1$$

and

$$(3.3.47) \quad \sup_{\mathcal{W}_{u_0}} \left\{ \left| \log \left(\frac{\Omega_1^2}{1 - \frac{1}{3}\Lambda r_1^2} \right) \right| + \left| \log \left(\frac{2\partial_v r_1}{1 - \frac{2m_1}{r_1}} \right) \right| + \left| \log \left(\frac{1 - \frac{2m_1}{r_1}}{1 - \frac{1}{3}\Lambda r_1^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m}_1| \right\} + \\ + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_0}} r_1 \frac{(T_{vv})_1}{\partial_v r_1} dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_0}} r_1 \frac{(T_{uu})_1}{-\partial_u r_1} du = C_0 < +\infty.$$

2. The $(r_{/i}, \Omega_{/i}^2, \bar{f}_{in/i}, \bar{f}_{out/i})$, $i = 1, 2$, are δ -close in the following sense:

$$(3.3.48) \quad \sup_{v \in [v_1, v_2]} \left\{ \left| \log \left(\frac{\Omega_{/1}^2}{1 - \frac{1}{3}\Lambda r_{/1}^2} \right) - \log \left(\frac{\Omega_{/2}^2}{1 - \frac{1}{3}\Lambda r_{/2}^2} \right) \right| + \left| \log \left(\frac{2\partial_v r_{/1}}{1 - \frac{2m_{/1}}{r_{/1}}} \right) - \log \left(\frac{2\partial_v r_{/2}}{1 - \frac{2m_{/2}}{r_{/2}}} \right) \right| + \right. \\ \left. + \left| \log \left(\frac{1 - \frac{2m_{/1}}{r_{/1}}}{1 - \frac{1}{3}\Lambda r_{/1}^2} \right) - \log \left(\frac{1 - \frac{2m_{/2}}{r_{/2}}}{1 - \frac{1}{3}\Lambda r_{/2}^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m}_{/1} - \tilde{m}_{/2}| \right\} (v) \leq \delta$$

and

$$(3.3.49) \quad \sup_{v \in [v_1, v_2]} (-\Lambda) \int_{v_1}^{v_2} \frac{|r_{/1}^2 \frac{(T_{vv})_{/1}}{\partial_v \varrho_1}(\bar{v}) - r_{/2}^2 \frac{(T_{vv})_{/2}}{\partial_v \varrho_2}(\bar{v})|}{|\varrho_{/}(v) - \varrho_{/}(\bar{v})| + \varrho_{/}(v_1)} d\bar{v} \leq \delta,$$

where C_1 is a large fixed absolute constant, δ satisfies

$$(3.3.50) \quad 0 \leq \delta \leq \delta_0 \doteq \exp \left(- \exp \left(C_1 (1 + C_0) \frac{u_0}{v_2 - v_1} \right) \right)$$

and $\varrho_{/}$ is defined by the relation

$$(3.3.51) \quad \varrho_{/}(v) \doteq \tan^{-1} \left(\sqrt{-\frac{\Lambda}{3}} r_{/}(v) \right).$$

Then, the maximal development $(\mathcal{U}_2; r_2, \Omega_2^2, \bar{f}_{in2}, \bar{f}_{out2})$ of $(r_{/2}, \Omega_{/2}^2, \bar{f}_{in/2}, \bar{f}_{out/2})$ satisfies

$$(3.3.52) \quad \mathcal{W}_{u_0} \subset \mathcal{U}_2$$

and

$$(3.3.53) \quad \sup_{\mathcal{W}_{u_0}} \left\{ \left| \log \left(\frac{\Omega_1^2}{1 - \frac{1}{3}\Lambda r_1^2} \right) - \log \left(\frac{\Omega_2^2}{1 - \frac{1}{3}\Lambda r_2^2} \right) \right| + \left| \log \left(\frac{2\partial_v r_1}{1 - \frac{2m_1}{r_1}} \right) - \log \left(\frac{2\partial_v r_2}{1 - \frac{2m_2}{r_2}} \right) \right| + \right. \\ \left. + \left| \log \left(\frac{1 - \frac{2m_1}{r_1}}{1 - \frac{1}{3}\Lambda r_1^2} \right) - \log \left(\frac{1 - \frac{2m_2}{r_2}}{1 - \frac{1}{3}\Lambda r_2^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m}_1 - \tilde{m}_2| \right\} + \\ + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_0}} |r_1(T_{vv})_1 - r_2(T_{vv})_2| dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_0}} |r_1(T_{uu})_1 - r_2(T_{uu})_2| du \\ \leq \exp \left(\exp \left(C_1(1 + C_0) \right) \frac{u_0}{v_2 - v_1} \right) \delta.$$

Remark. By repeating the proof of Theorem 3.2, the Cauchy stability estimate (3.3.53) also holds in the case when $(\mathcal{U}_i; r_i, \Omega_i^2, \bar{f}_{in;i}, \bar{f}_{out;i})$, $i = 1, 2$, are the maximal past developments of $(r_{/i}, \Omega_{/i}^2, \bar{f}_{in/i}, \bar{f}_{out/i})$, i. e. when $\mathcal{W}_{u_0}$ is replaced by

$$(3.3.54) \quad \mathcal{W}_{u_0}^{(-)} \doteq \{-u_0 \leq u < 0\} \cap \{u + v_1 < v < u + v_2\}$$

and (3.3.47) holds on $\mathcal{W}_{u_0}^{(-)}$ in place of $\mathcal{W}_{u_0}$.

3.4 Final statement of Theorem 3.1: the non-linear instability of AdS

The main result that will be established in this chapter of the thesis is the following:

Theorem 3.1 (final version). For any $\varepsilon \in (0, 1]$, there exist $r_{0\varepsilon}, v_{0\varepsilon}$ depending smoothly on ε such that

$$(3.4.1) \quad r_{0\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0$$

and

$$(3.4.2) \quad \sqrt{-\Lambda} v_{0\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \frac{\pi}{\sqrt{3}},$$

as well as a family $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{in/}^{(\varepsilon)}, \bar{f}_{out/}^{(\varepsilon)})$ of smooth asymptotically AdS boundary-characteristic initial data sets for the system (3.2.28)–(3.2.33) satisfying the reflecting gauge condition at $r = r_0, +\infty$, such that the following hold:

1. The family $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{in/}^{(\varepsilon)}, \bar{f}_{out/}^{(\varepsilon)})$ satisfies

$$(3.4.3) \quad \|(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{in/}^{(\varepsilon)}, \bar{f}_{out/}^{(\varepsilon)})\|_{CS} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where $\|\cdot\|_{CS}$ is the norm defined by (3.3.41).

2. There exists a trapped sphere, i. e. point $(u_{\dagger}, v_{\dagger})$ in the maximal future development $(\mathcal{U}_{\varepsilon}; r_{\varepsilon}, \Omega_{\varepsilon}^2, \bar{f}_{in\varepsilon}, \bar{f}_{out\varepsilon})$ of $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{in/}^{(\varepsilon)}, \bar{f}_{out/}^{(\varepsilon)})$ such that

$$(3.4.4) \quad \frac{2m}{r}(u_{\dagger}, v_{\dagger}) > 1.$$

In particular, in view of Theorem 3.3, $(\mathcal{U}_{\varepsilon}; r_{\varepsilon}, \Omega_{\varepsilon}^2, \bar{f}_{in\varepsilon}, \bar{f}_{out\varepsilon})$ has a non-empty future event horizon $\mathcal{H}^+$ (defined by (3.5.27)), satisfying the properties 4.a and 4.b of Theorem 3.3, and a complete conformal infinity $\mathcal{I}$ (satisfying (3.3.28)).

Remark. If

$$(3.4.5) \quad \delta(\varepsilon) \doteq \|(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{in/}^{(\varepsilon)}, \bar{f}_{out/}^{(\varepsilon)})\|_{CS},$$

the point $(u_{\dagger}, v_{\dagger})$ satisfies the upper bound

$$(3.4.6) \quad u_{\dagger} \leq \exp\left(\exp(\delta^6(\varepsilon))\right)v_0.$$

On the other hand, in view of Proposition 3.3, we necessarily have

$$(3.4.7) \quad u_{\dagger} \xrightarrow{\varepsilon \rightarrow 0} +\infty.$$

In the simpler case when one is interested in a weaker instability statement, such as the existence of a point $(u_{\dagger}, v_{\dagger})$ where

$$(3.4.8) \quad \left. \frac{2m}{r} \right|_{(u_{\dagger}, v_{\dagger})} > \frac{1}{2}$$

(instead of the stronger bound (3.4.4)), the proof of Theorem 3.1 can be substantially simplified. In the case of (3.4.8), the upper bound (3.4.6) can be improved into a polynomial bound

$$(3.4.9) \quad u_{\dagger} \leq (\delta(\varepsilon))^{-C_1} v_0,$$

for some fixed $C_1 > 0$.

3.5 Construction of the initial data and notation

As described already in Section 3.1.2 of the introduction, the initial data family in Theorem 3.1 will be such that their development consists of a large number of initially ingoing Vlasov beams. In this section, we will construct such a family $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ of asymptotically AdS boundary-characteristic initial data for (3.2.28)–(3.2.33). The family $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{in/}^{(\varepsilon)}, \bar{f}_{out/}^{(\varepsilon)})$ in the statement of Theorem 3.1 will be eventually obtained from $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ after possibly adding a suitable perturbation (see Section 3.6).

This section is organised as follows: In Section 3.5.1, we will introduce a certain hierarchy of parameters that will be necessary for the construction of $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ in Section 3.5.2. In Section 3.5.3, we will introduce some basic notation related to the maximal future development $(\mathcal{U}_{\varepsilon}; r_{\varepsilon}, \Omega_{\varepsilon}^2, \bar{f}_{in\varepsilon}, \bar{f}_{out\varepsilon})$ of $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$. Finally, in Section 3.5.4, we will perform some basic geometric constructions on $(\mathcal{U}_{\varepsilon}; r_{\varepsilon}, \Omega_{\varepsilon}^2, \bar{f}_{in\varepsilon}, \bar{f}_{out\varepsilon})$, related to the separation of $\mathcal{U}_{\varepsilon}$ into various subregions by the Vlasov beams arising from the initial data.

3.5.1 Parameters and auxiliary functions

Let us fix some smooth and strictly increasing functions $h_0, h_1, h_2 : (0, 1) \rightarrow (0, 1)$, so that

$$(3.5.1) \quad \lim_{\varepsilon \rightarrow 0^+} h_0(\varepsilon) = \lim_{\varepsilon \rightarrow 0^+} h_1(\varepsilon) = \lim_{\varepsilon \rightarrow 0^+} h_2(\varepsilon) = 0,$$

$$(3.5.2) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon \cdot \exp\left(\frac{1}{(h_1(\varepsilon))^6}\right) = \lim_{\varepsilon \rightarrow 0^+} h_1(\varepsilon) \cdot \exp\left(\exp\left(\frac{1}{(h_0(\varepsilon))^6}\right)\right) = 0$$

and

$$(3.5.3) \quad \lim_{\varepsilon \rightarrow 0^+} h_2(\varepsilon) \cdot \exp(\varepsilon^{-2}) = 0.$$

In particular, the following relations hold for $\varepsilon \ll 1$:

$$(3.5.4) \quad h_2(\varepsilon) \ll \varepsilon \ll h_1(\varepsilon) \ll h_0(\varepsilon) \ll 1.$$

Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth cut-off function, satisfying $\chi|_{[-1, 1]} = 1$, $\chi|_{\mathbb{R} \setminus [-2, 2]} = 0$ and

$$(3.5.5) \quad \chi|_{(-2, 2)} > 0,$$

and let $\varepsilon_0 \ll 1$ be a small enough absolute constant. For any $0 < \varepsilon < \varepsilon_0$, any $r_0 > 0$ satisfying

$$(3.5.6) \quad 1 - \exp\left(-2(h_0(\varepsilon))^{-4}\right) < \frac{r_0}{\frac{2}{\sqrt{-\Lambda}}\varepsilon - \frac{1}{3}\Lambda r_0^3} < 1 - \frac{1}{2}\exp\left(-2(h_0(\varepsilon))^{-4}\right)$$

(note that (3.5.6) implies that $\frac{r_0\sqrt{-\Lambda}}{2\varepsilon} = 1 + O\left(\exp\left(-2(h_0(\varepsilon))^{-4}\right)\right)$ as $\varepsilon \rightarrow 0$), we will define the following function on $[0, +\infty) \times (0, +\infty)$:

$$(3.5.7) \quad \bar{f}_\varepsilon(v, p^u) \doteq C_{\varepsilon r_0} \sum_{j=0}^{\lceil 1/h_1(\varepsilon) \rceil} \chi(p^u - 3) \cdot \frac{1}{h_2(\varepsilon)} \chi\left(\frac{(v - v^{(j)})\sqrt{-\Lambda} - 2h_2(\varepsilon)}{h_2(\varepsilon)}\right) \cdot h_{(j)}(\varepsilon) \cdot \varepsilon,$$

for some constant $C_{\varepsilon r_0}$ to be specified in terms of ε, r_0 later, where $\lceil 1/h_1(\varepsilon) \rceil$ denotes the

least integer greater than or equal to $1/h_1(\varepsilon)$,

$$(3.5.8) \quad v^{(j)} = \frac{\pi}{\sqrt{-\Lambda}} - j \frac{\varepsilon}{h_1(\varepsilon)\sqrt{-\Lambda}}$$

for any $0 \leq j \leq \lceil 1/h_1(\varepsilon) \rceil$,

$$(3.5.9) \quad h_{(0)} = h_0,$$

and

$$(3.5.10) \quad h_{(j)} = h_1$$

for all $1 \leq j \leq \lceil 1/h_1(\varepsilon) \rceil$.

3.5.2 Construction of the initial data family

For any $0 < \varepsilon < \varepsilon_0$, any r_0 satisfying (3.5.6), we will define the following asymptotically AdS boundary-characteristic initial data set according to Definition 3.1:

Definition 3.4. For any $0 < \varepsilon < \varepsilon_0$, any r_0 satisfying (3.5.6), we define $v_0 = v_0(r_0, \varepsilon) > 0$ and the set of smooth functions $r_{/\varepsilon} : [0, v_0) \rightarrow [r_0, +\infty)$, $\Omega_{/\varepsilon}^2 : [0, v_0) \rightarrow (0, +\infty)$, $\bar{f}_{in/\varepsilon} : [0, v_0) \times (0, +\infty) \rightarrow [0, +\infty)$ and $\bar{f}_{out/\varepsilon} : [0, v_0) \times (0, +\infty) \rightarrow [0, +\infty)$ by the requirement that $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ is an asymptotically AdS boundary-characteristic initial data set on $[0, v_0)$ for the system (3.2.28)–(3.2.33) satisfying the reflecting gauge condition at $r = r_0, +\infty$ so that

$$(3.5.11) \quad \frac{\partial_v r_{/\varepsilon}}{1 - \frac{2m_{/\varepsilon}}{r_{/\varepsilon}}} = \frac{1}{2}$$

(where $m_{/\varepsilon}$ is defined in terms of $r_{/\varepsilon}, \Omega_{/\varepsilon}^2$ by (3.3.12)),

$$(3.5.12) \quad \bar{f}_{out/\varepsilon} = 0$$

and

$$(3.5.13) \quad \bar{f}_{in/\varepsilon}(v, p^u) = \bar{f}_\varepsilon(v, p^u)$$

for all $0 \leq v \leq v_0$ and $p^u > 0$. The constant $C_{\varepsilon r_0}$ in (3.5.7) is fixed in terms of ε, r_0 by the requirement that

$$(3.5.14) \quad \lim_{v \rightarrow v_0} \tilde{m}_{/\varepsilon} = \frac{\varepsilon}{\sqrt{-\Lambda}}$$

(in particular, there exists some fixed (large) $C_0 > 1$, independent of ε, r_0 , so that $C_{\varepsilon r_0} \in [C_0^{-1}, C_0]$ for any $0 < \varepsilon < \varepsilon_0$, any r_0 satisfying (3.5.6)).¹²

Remark. The conditions (3.5.11)–(3.5.13) determine v_0 and $r_{/\varepsilon}, \Omega_{/\varepsilon}^2$ uniquely in terms of ε, r_0 . While $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ depend on both ε and r_0 , we will only use the subscript ε in their notation, since most of the estimates that we will later establish for their maximal development will depend only on ε .

The initial data $(r_{/\varepsilon}^{(\varepsilon)}, (\Omega_{/\varepsilon}^{(\varepsilon)})^2, \bar{f}_{in/\varepsilon}^{(\varepsilon)}, \bar{f}_{out/\varepsilon}^{(\varepsilon)})$ in the statement of Theorem 3.1 will eventually be chosen to be small perturbations of $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ (see Section 3.6).

3.5.3 Notational conventions and basic computations

For any $0 < \varepsilon < \varepsilon_0$ and any r_0 satisfying (3.5.6), let $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ be the initial data set defined by Definition 3.4. Assuming that ε_0 is fixed small enough, for any $0 < \varepsilon < \varepsilon_0$ and any r_0 satisfying (3.5.6), the initial data set $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ satisfies the following estimate depending only on ε :

$$(3.5.15) \quad \|(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})\|_{CS} \leq Ch_0(\varepsilon),$$

where $\|\cdot\|_{CS}$ is defined by (3.3.41) and $C > 0$ is a fixed constant.

Let $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{in\varepsilon}, \bar{f}_{out\varepsilon})$ be the maximal future development of $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ (see Theorem 3.3). In view of Proposition 3.3, the bound (3.5.15) implies that, for any

¹²In fact, it suffices to choose $C_0 = 50$.

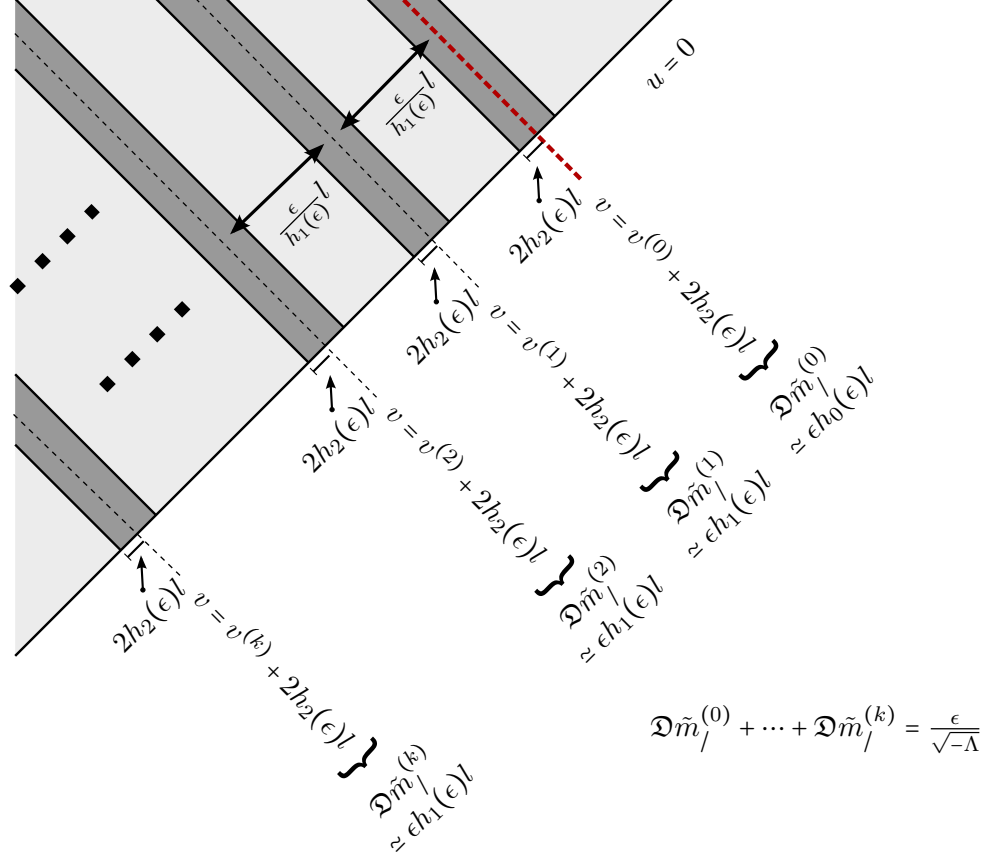


Figure 11: The initial data $(r_{\epsilon/l}, \Omega_{\epsilon/l}^2, \bar{f}_{in\epsilon/l}, \bar{f}_{out\epsilon/l})$ give rise to $k+1$ Vlasov beams, initially arranged as depicted above (using, for convenience, the abbreviation $k = \lceil 1/h_1(\epsilon) \rceil$ and $l = (-\Lambda)^{-1/2}$). For any $0 \leq j \leq k$, we can associate to the beam centered around $v = v^{(j)} + 2h_2(\epsilon)l$ the initial mass difference $\mathfrak{D}\tilde{m}_l^{(j)}$, defined as the renormalised Hawking mass difference between the two vacuum regions surrounding the beam. The top beam (centered initially around $v = v^0 + 2h_2(\epsilon)l$) has a greater initial mass difference than the rest of the beams.

fixed $u_* > 0$ and any $\delta > 0$, there exists an $\varepsilon_{\delta u_*} > 0$ sufficiently small depending only on δ and u_* so that, for any $0 \leq \varepsilon < \varepsilon_{\delta u_*}$:

$$(3.5.16) \quad \mathcal{W}_{u_*} \doteq \{0 < u < u_*\} \cap \{u < v < u + v_0\} \subset \mathcal{U}$$

and

$$(3.5.17) \quad \begin{aligned} & \sqrt{-\Lambda} \sup_{\mathcal{W}_{u_*}} |\tilde{m}_\varepsilon| + \sup_{\mathcal{W}_{u_*}} \log \left(\frac{1 - \frac{1}{3} \Lambda r_\varepsilon^2}{1 - \max\{\frac{2m_\varepsilon}{r_\varepsilon}, 0\}} \right) + \\ & + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_*}} \frac{r_\varepsilon(T_{vv})_\varepsilon}{\partial_v r_\varepsilon} dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_*}} \frac{r_\varepsilon(T_{uu})_\varepsilon}{(-\partial_u r_\varepsilon)} du < \delta, \end{aligned}$$

where $m_\varepsilon, \tilde{m}_\varepsilon, (T_{uu})_\varepsilon, (T_{vv})_\varepsilon$ are defined in terms of $r_\varepsilon, \Omega_\varepsilon^2 \bar{f}_{in\varepsilon}, \bar{f}_{out\varepsilon}$ by (3.2.5), (3.2.42), (3.2.19) and (3.2.20). In particular, if

$$(3.5.18) \quad g_{AdS} = -\Omega_{AdS, r_0, v_0}^2 dudv + r_{AdS, r_0, v_0} g_{\mathbb{S}^2}$$

is the pure AdS metric in a spherically symmetric coordinate chart (u, v) such that $r_{AdS, r_0, v_0} = r_0$ on $\{u = v\}$ and $r_{AdS, r_0, v_0} = +\infty$ on $\{u = v - v_0\}$,¹³ then $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{in\varepsilon}, \bar{f}_{out\varepsilon})$, when restricted on $\mathcal{W}_{u_*}$, is δ -close to $(\mathcal{W}_{u_*}; r_{AdS, r_0, v_0}, \Omega_{AdS, r_0, v_0}^2, 0, 0)$ with respect to the (gauge invariant) distance defined by (3.3.41). Notice also that (3.5.17) implies that, provided δ is small enough, the spacetime $(\mathcal{W}_{u_*} \times \mathbb{S}^2, g_\varepsilon)$ does not contain any trapped surface, where

$$(3.5.19) \quad g_\varepsilon = -\Omega_\varepsilon^2 dudv + r_\varepsilon^2 g_{\mathbb{S}^2}.$$

Notice that, in view of the conservation of $\tilde{m}$ on γ_0 and $\mathcal{I}$ (see (3.3.26) and (3.3.27)), we have:

$$(3.5.20) \quad \tilde{m}_\varepsilon|_{\gamma_0} = 0$$

and

$$(3.5.21) \quad \tilde{m}_\varepsilon|_{\mathcal{I}} = \lim_{v \rightarrow v_0} \tilde{m}_{/\varepsilon}(v) = \frac{\varepsilon}{\sqrt{-\Lambda}}.$$

For each $0 \leq j \leq \lceil 1/h_1(\varepsilon) \rceil$, we can associate to the beam centered at $v = v^{(j)} + \frac{2}{\sqrt{-\Lambda}}h_2(\varepsilon)$ the mass difference

$$(3.5.22) \quad \mathcal{D}\tilde{m}_j^{(j)} \doteq \tilde{m}_{/\varepsilon}\left(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right) - \tilde{m}_{/\varepsilon}(v^{(j)}).$$

Notice that

$$(3.5.23) \quad \mathcal{D}\tilde{m}_j^{(0)} \simeq \frac{h_0(\varepsilon)\varepsilon}{\sqrt{-\Lambda}}$$

¹³Note that such a coordinate chart is not unique.

and, for all $1 \leq j \leq \lceil 1/h_1(\varepsilon) \rceil$:

$$(3.5.24) \quad \mathcal{D}\tilde{m}_j^{(j)} \simeq \frac{h_1(\varepsilon)\varepsilon}{\sqrt{-\Lambda}}.$$

Furthermore:

$$(3.5.25) \quad \sum_{j=0}^{\lceil 1/h_1(\varepsilon) \rceil} \mathcal{D}\tilde{m}_j^{(j)} = \lim_{v \rightarrow v_0} \tilde{m}_{/ \varepsilon}(v) = \frac{\varepsilon}{\sqrt{-\Lambda}}.$$

3.5.4 Some geometric constructions on $\mathcal{U}_\varepsilon$

For any $0 < \varepsilon < \varepsilon_0$ and any r_0 satisfying (3.5.6), we will define some special subsets of the domain $\mathcal{U}_\varepsilon$ of the maximal future development $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{in\varepsilon}, \bar{f}_{out\varepsilon})$ of the initial data set $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$.

Remark. In the rest of this section, we will adopt the convention that the boundary ∂A of a subset $A \subseteq \mathcal{U}_\varepsilon$ is the boundary of A as a subset of $\mathbb{R}^2$ (with respect to the ambient topology of $\mathbb{R}^2$)

Let us define the domain of outer communications $\mathcal{D}_\varepsilon$ of $\mathcal{U}_\varepsilon$ as

$$(3.5.26) \quad \mathcal{D}_\varepsilon \doteq J^-(\mathcal{I}) \cap \mathcal{U}_\varepsilon,$$

where $J^-(\mathcal{I})$ is the causal past of $\mathcal{I}$ with respect to the reference metric (3.2.50) (see (3.3.32)). In accordance with Theorem 3.3, we will also define the future event horizon $\mathcal{H}_\varepsilon^+$ of $\mathcal{U}_\varepsilon$ as

$$(3.5.27) \quad \mathcal{H}_\varepsilon^+ \doteq \partial \mathcal{D}_\varepsilon \cap \mathcal{U}_\varepsilon.$$

Note that we allow $\mathcal{H}_\varepsilon^+$ to be empty. In view of Theorem 3.3, in the case when $\mathcal{H}_\varepsilon^+$ is non-empty, it is necessarily of the form

$$(3.5.28) \quad \mathcal{H}_\varepsilon^+ = \{u = u_{\mathcal{H}_\varepsilon^+}\} \cap \mathcal{U}_\varepsilon$$

and has infinite affine length.

We will also define

$$(3.5.29) \quad \mathcal{J}_\varepsilon \doteq J^-(\gamma_0) \cap \mathcal{U}_\varepsilon.$$

Notice that, as a consequence of Theorem 3.3, on $\mathcal{J}_\varepsilon \cup \mathcal{D}_\varepsilon$ we have

$$(3.5.30) \quad 1 - \frac{2m}{r} > 0,$$

i.e. trapped spheres can only appear in the region $\mathcal{U}_\varepsilon \setminus (\mathcal{J}_\varepsilon \cup \mathcal{D}_\varepsilon)$. In the case $\mathcal{H}_\varepsilon^+ \neq \emptyset$, Theorem 3.3 also implies that $\mathcal{J}_\varepsilon \setminus \mathcal{D}_\varepsilon \neq \emptyset$.

For any $v_* \in [0, v_0]$ and any integer $n \geq 1$, we will define

$$(3.5.31) \quad U_n(v_*) \doteq v_* + (n-1)v_0$$

and

$$(3.5.32) \quad V_n(v_*) = v_* + nv_0.$$

We will also set

$$(3.5.33) \quad V_0(v_*) \doteq v_*.$$

Notice that the segment $\{u = U_n(v_*)\} \cap \mathcal{U}_\varepsilon$ is the image of the ingoing null geodesic of $\mathcal{U}_\varepsilon$ emanating from the point $(0, v_*)$ after n reflections off γ_0 and $n-1$ reflections off $\mathcal{I}$, while the segment $\{v = V_n(v_*)\} \cap \mathcal{U}_\varepsilon$ is the image of the same null geodesic after n reflections off γ_0 and n reflections off $\mathcal{I}$.

Let us define the domains $\mathcal{R}_{\varepsilon n}^{(i,j)} \subset \mathcal{U}_\varepsilon$ for any $n \in \mathbb{N}$, $0 \leq i \leq \lceil 1/h_1(\varepsilon) \rceil$ and $i \leq j \leq \lceil 1/h_1(\varepsilon) \rceil + i + 1$ by the relation

$$(3.5.34) \quad \begin{aligned} \mathcal{R}_{\varepsilon n}^{(i,j)} = & \left\{ U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) < u < U_n(v^{(i-1)}) \right\} \cap \\ & \cap \left\{ V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) < v < V_n(v^{(j-1)}) \right\} \cap \mathcal{U}_\varepsilon, \end{aligned}$$

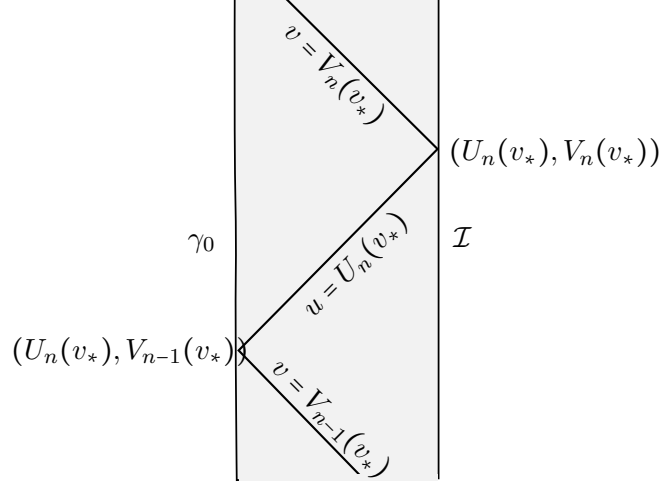


Figure 12: Schematic depiction of the lines $v = V_{n-1}(v_*)$, $u = U_n(v_*)$ and $v = V_n(v_*)$.

where we have used the following conventions in the expression (3.5.34):

1. $U_n(v^{(-1)}) \doteq U_{n+1}(v^{(\lceil 1/h_1(\varepsilon) \rceil)})$.
2. $V_n(v^{(\lceil 1/h_1(\varepsilon) \rceil + l)} + c) \doteq V_{n-1}(v^{(l-1)} + c)$ for any integer $1 \leq l \leq \lceil 1/h_1(\varepsilon) \rceil$ and any $c \geq 0$.

Remark. The boundary of the domains $\mathcal{R}_{\varepsilon n}^{(i,i)}$, $0 \leq i \leq \lceil 1/h_1(\varepsilon) \rceil$, contains a segment $\mathcal{I}$, while the boundary of the domains $\mathcal{R}_{\varepsilon n}^{(i, \lceil 1/h_1(\varepsilon) \rceil + 1 + i)}$, $0 \leq i \leq \lceil 1/h_1(\varepsilon) \rceil$, contains a segment in γ_0 .

Notice that $T_{uu} = T_{vv} = 0$ in $\mathcal{R}_{\varepsilon n}^{(i,j)}$. In particular, all the domains $(\mathcal{R}_{\varepsilon n}^{(i,j)} \times \mathbb{S}^2, g_\varepsilon)$ are isometric to a region of a member of the Schwarzschild-AdS family (or to a region of pure AdS spacetime), and the renormalised mass function $\tilde{m}_\varepsilon$ is constant on them. We will define for any $0 \leq i \leq \lceil 1/h_1(\varepsilon) \rceil$, $i \leq j \leq \lceil 1/h_1(\varepsilon) \rceil + i + 1$ and $n \in \mathbb{N}$ such that $\mathcal{R}_{\varepsilon n}^{(i,j)} \neq \emptyset$:

$$(3.5.35) \quad \tilde{m}_{\varepsilon n}^{(i,j)} \doteq \tilde{m}_\varepsilon|_{\mathcal{R}_{\varepsilon n}^{(i,j)}}.$$

In view of (3.5.20) and (3.5.21), we immediately calculate that for all $0 \leq i \leq \lceil 1/h_1(\varepsilon) \rceil$ and all $n \in \mathbb{N}$ such that $\mathcal{R}_{\varepsilon n}^{(i,i)}, \mathcal{R}_{\varepsilon n}^{(i, \lceil 1/h_1(\varepsilon) \rceil + 1 + i)} \neq \emptyset$:

$$(3.5.36) \quad \tilde{m}_{\varepsilon n}^{(i, \lceil 1/h_1(\varepsilon) \rceil + 1 + i)} = 0$$

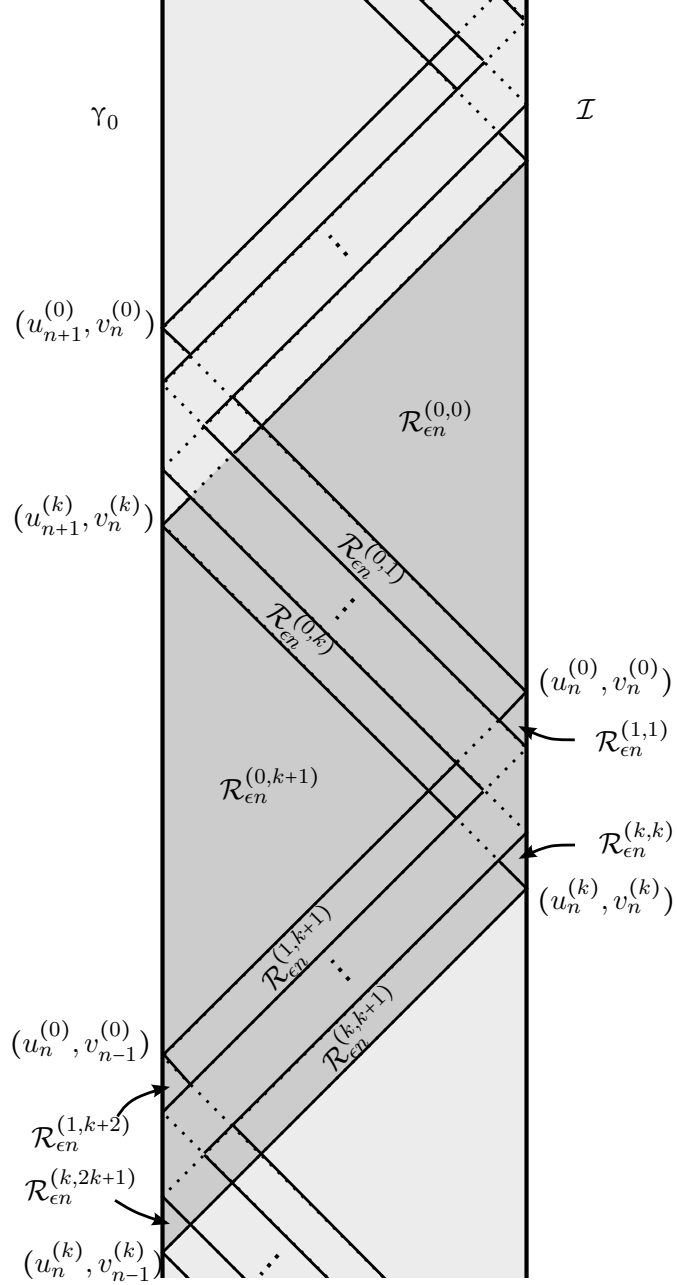


Figure 13: Schematic depiction of the domains $\mathcal{R}_{\epsilon n}^{(i,j)}$ for $0 \leq i \leq k$ and $i \leq j \leq k+i+1$ (where $k = \lceil 1/h_1(\epsilon) \rceil$). We have used the shorthand notation $u_n^{(j)} = U_n(v^{(j)})$ and $v_n^{(j)} = V_n(v^{(j)})$. Having assumed that $h_2(\epsilon) \ll \epsilon$, all the beams of the form $\{U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\epsilon))\}$ and $\{V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\epsilon))\}$, which separate the domains $\mathcal{R}_{\epsilon n}^{(i,j)}$, are depicted as straight line segments.

and

$$(3.5.37) \quad \tilde{m}_{\epsilon n}^{(i,i)} = \tilde{m}_{\epsilon 0}^{(i,i)} = \frac{\epsilon}{\sqrt{-\Lambda}}.$$

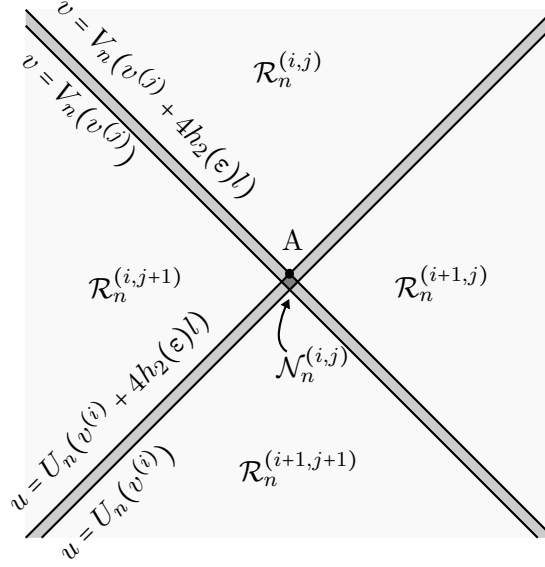


Figure 14: Typical arrangement of neighboring vacuum domains not intersecting γ_0 or $\mathcal{I}$. The point A at the lower corner of $\mathcal{R}_n^{(i,j)}$ satisfies $r(A) = r_n^{(i,j)}$. For simplicity, we have used the shorthand notation $l = (-\Lambda)^{-1/2}$.

For any $n \in \mathbb{N}$, $0 \leq i \leq \lceil 1/h_1(\epsilon) \rceil$ and $i+1 \leq j \leq \lceil 1/h_1(\epsilon) \rceil + i$, we will define the interaction regions:

(3.5.38)

$$\mathcal{N}_{\epsilon n}^{(i,j)} \doteq \left\{ U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\epsilon)) \right\} \cap \left\{ V_n(v^{(j)}) \leq u \leq V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\epsilon)) \right\} \cap \mathcal{U}_\epsilon,$$

where the conventions stated below (3.5.34) hold regarding indices smaller than 0 or larger than $\lceil 1/h_1(\epsilon) \rceil$.

Let us define for any $0 \leq i \leq \lceil 1/h_1(\epsilon) \rceil$, $i \leq j \leq \lceil 1/h_1(\epsilon) \rceil + i + 1$ and $n \in \mathbb{N}$ such that $\mathcal{R}_{\epsilon n}^{(i,j)} \neq \emptyset$:

$$(3.5.39) \quad r_{\epsilon n}^{(i,j)} \doteq r_\epsilon \left(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\epsilon)), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\epsilon)) \right).$$

Note that $r_{\epsilon n}^{(i,i)} = +\infty$ and $r_{\epsilon n}^{(i, \lceil 1/h_1(\epsilon) \rceil + i + 1)} = r_{0\epsilon}$.

Finally, let us remark that, in view of property (3.6.204) of the cut-off used in the construction of the initial data and equations (3.2.48)–(3.2.49), for any $1 \leq n \leq n_f$, $0 \leq i \leq$

$\lceil 1/h_1(\varepsilon) \rceil$, $i \leq j \leq \lceil 1/h_1(\varepsilon) \rceil + i + 1$, we have

$$(3.5.40) \quad T_{uu} > 0 \text{ on } \left\{ U_n(v^{(i)}) < u < U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right\}$$

and

$$(3.5.41) \quad T_{vv} > 0 \text{ on } \left\{ V_n(v^{(j)}) < u < V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right\}.$$

3.6 Proof of Theorem 3.1

In this Section, we will prove Theorem 3.1. In order to simplify our notation, from now on, we will often drop the subscripts ε in notations related to the maximal future development $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{in\varepsilon}, \bar{f}_{out\varepsilon})$ of the initial data $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ (see Definition 3.4).

For any $0 < \varepsilon < \varepsilon_0$ (provided ε_0 is fixed sufficiently small), any $r_0 > 0$ satisfying (3.5.6), let $(\mathcal{U}_\varepsilon; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ be the maximal future development of $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$, and let us define

$$(3.6.1) \quad u_+ \doteq \sup \left\{ u_* > 0 : 1 - \frac{2m}{r} > h_3(\varepsilon) \text{ on } \mathcal{U}_\varepsilon \cap \{u < u_*\} \right\}$$

and

$$(3.6.2) \quad \mathcal{U}_\varepsilon^+ = \mathcal{U}_\varepsilon \cap \left\{ u < \min\{u_+, (h_1(\varepsilon))^{-2} v_{0\varepsilon}\} \right\},$$

where

$$(3.6.3) \quad h_3(\varepsilon) = \exp \left\{ - \exp \left((h_1(\varepsilon))^{-5} \exp \left(- 2(h_0(\varepsilon))^{-4} \right) \right) \right\}.$$

Let us also set

$$(3.6.4) \quad k \doteq \lceil 1/h_1(\varepsilon) \rceil$$

and

$$(3.6.5) \quad n_f \doteq \lfloor (u_+ - v^{(0)})/v_0 \rfloor,$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x , while $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .

The proof of Theorem 3.1 will follow in two steps: First, in Section 3.6.2, we will show that:

$$(3.6.6) \quad \sup_{\mathcal{U}_\epsilon^+} \left(1 - \frac{2m}{r}\right) = h_3(\epsilon),$$

i. e. that $\mathcal{U}_\epsilon^+$ contains a nearly-trapped sphere. Then, in Section 3.6.3, we will show that, at the final step of the evolution, either a trapped sphere is formed, or there exists a small perturbation of the initial data $(r/\epsilon, \Omega_{/\epsilon}^2, \bar{f}_{in/\epsilon}, \bar{f}_{out/\epsilon})$ giving rise to a trapped sphere.

Before proving (3.6.6), we will need to establish some necessary bounds for the evolution of $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ in the region $\mathcal{U}_\epsilon^+$. These bounds, which will be obtained in Section 3.6.1, will be used both in Section 3.6.2 and in Section 3.6.3.

3.6.1 Inductive bounds for the evolution in the region $\mathcal{U}_\epsilon^+$

In this Section, we will establish a number of useful bounds for $(\mathcal{U}_\epsilon^+; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$. These bounds will include a number of inductive bounds for the quantities $\tilde{m}_n^{(1,k+1)}$, $r_n^{(k,k+1)}$ and $r_n^{(1,k+1)}$ (with k defined by (3.6.4)), that will be of fundamental significance in the proof of Theorem 3.1.

In particular, we will prove the following result:

Proposition 3.4. *For any $0 < \epsilon < \epsilon_0$, the following bounds hold for $(\mathcal{U}_\epsilon^+; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$:*

1. On $\mathcal{U}_\epsilon^+$, we can estimate:

$$(3.6.7) \quad \left| \log \left(\frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right) \right| + \left| \log \left(\frac{\partial_v r}{1 - \frac{2m}{r}} \right) \right| \leq (h_1(\epsilon))^{-4} \log((h_3(\epsilon))^{-1}).$$

2. For any $1 \leq n \leq n_f$:

$$(3.6.8) \quad r_n^{(0,k)} \geq \frac{\epsilon^{-\frac{1}{2}}}{\sqrt{-\Lambda}},$$

$$(3.6.9) \quad r_n^{(k,k+1)} \leq \frac{\epsilon^{\frac{1}{2}}}{\sqrt{-\Lambda}},$$

$$(3.6.10) \quad \frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)})}{r_0} \geq \exp\left(-2(h_0(\epsilon))^{-4}\right),$$

$$(3.6.11) \quad \frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)})}{r_0} \leq 1 - \frac{1}{C_0} h_0(\epsilon)$$

and

$$(3.6.12) \quad \frac{r_n^{(1,k+1)}}{r_0} - 1 \geq \exp\left(-(h_0(\epsilon))^{-4}\right),$$

where $C_0 > 1$ is a large fixed constant (independent of all the parameters).

3. For any $2 \leq n \leq n_f$:

$$(3.6.13) \quad \frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \geq 1 + \frac{1}{4} \exp\left(-2(h_0(\epsilon))^{-4}\right) \frac{r_0}{r_n^{(k,k+1)}}$$

and

$$(3.6.14) \quad \frac{r_n^{(k,k+1)} - r_0}{r_{n-1}^{(k,k+1)} - r_0} \leq 1 + 2C_0 \frac{r_0}{r_{n-1}^{(k,k+1)}} \left(\left| \log \left(1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r_0} \right) \right| + (h_0(\epsilon))^{-4} \right).$$

Before presenting the proof of Proposition 3.4 (in Section 3.6.1), we will briefly comment on the nature of the bounds (3.6.7)–(3.6.14) and their relation with the specific choice of the parameters (3.5.2)–(3.5.3).

Remarks on Proposition 3.4

The bounds (3.6.7)–(3.6.14) in Proposition 3.4 lie at the heart of the proof of Theorem 3.1. The precise form of the initial data (3.5.7), the range (3.5.6) for the mirror radius r_0 and the asymptotic bounds (3.5.2)–(3.5.3) on the parameters h_0, h_1, h_2 were carefully chosen so that (3.6.7)–(3.6.14) can be obtained. We will now proceed to briefly comment on the role of the bounds (3.6.7)–(3.6.14) in the proof of Theorem 3.1. The reader is advised to review first the sketch of the proof in Section 3.1.2 of the introduction. Let us remark that, in the notation of Section 3.1.2,

$$(3.6.15) \quad \mathcal{E}_{\zeta_0;n} = \tilde{m}_n^{(1,k+1)},$$

$$(3.6.16) \quad r_{\gamma_0;n} = r_n^{(k,k+1)}$$

and

$$(3.6.17) \quad r_{\gamma_0;n}^{(1)} = r_n^{(1,k+1)}.$$

The bound (3.6.7) is a “trivial” bound controlling quantities related to the chosen gauge. The right hand side of (3.6.7), upon integration across any specific beam (in a direction transversal to the beam), will yield a small quantity, in view of the fact that the width of the null beams emanating from $u = 0$, $v \sim v^{(j)}$ was chosen to be $\sim h_2(\varepsilon)$ and, moreover, $h_2(\varepsilon)$ was chosen in (3.5.3) to be small compared to the right hand side of (3.6.7). This fact will prove convenient for the proof of Proposition 3.4 and Theorem 3.1, as it will enable us to “ignore” the variation of certain quantities across the width of any specific beam. That is to say, the bound (3.6.7) will enable us to frequently treat the null beams as line segments having negligible width.

The bounds (3.6.8)–(3.6.9) are quantitative expressions of the fact that the set of interactions of the beams splits into two portions, one close to $r = r_0$ and one close to $\mathcal{I}$.

The lower bound (3.6.10) is necessary in order to establish (3.6.13). In order to obtain (3.6.10), it is necessary that r_0 satisfies the upper bound of (3.5.6).

The upper bound (3.6.11) implies that a trapped sphere (i.e. a sphere where $\frac{2m}{r} > 1$) can not be formed at $\mathcal{R}_{\varepsilon n}^{(i,j)}$ for any $j > k+1$, since one can also show that $\tilde{m} \leq \tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)}$ in those regions. In order to obtain (3.6.11), it is necessary that the mirror radius r_0 satisfies the lower bound of (3.5.6).

In the language of Section 3.1.2 of the introduction, the bound (3.6.12) states that, when ζ_0 reaches $\{r = r_0\}$ for the n -th time, the r -distance of the top beam ζ_0 from the second-to-top beam ζ_1 , i.e. $r_n^{(1,k+1)} - r_0$, can be bounded from below by a small multiple of r_0 which is large compared to $h_1(\varepsilon)r_0$. As a consequence of (3.6.12) and the bound (3.5.6) for r_0 , for any $i \neq 1$, $\mathcal{R}_{\varepsilon n}^{(i,k+1)}$ does not contain a trapped sphere. As a result, combining (3.6.11), (3.6.8) and (3.6.12), we infer that, among all regions $\mathcal{R}_{\varepsilon n}^{(i,j)}$, a trapped sphere can only appear for $i = 1$, $j = k+1$. This fact serves to simplify the proof of Theorem 3.1, by avoiding considering multiple scenarios of trapped surface formation. Furthermore, it is crucial in obtaining (3.6.14).

Establishing (3.6.12) is the most demanding part of the proof of Proposition 3.4. It requires obtaining a lower bound in the rate of decrease of $r_n^{(1,k+1)}$ in terms of the rate of increase of $\tilde{m}_n^{(1,k+1)}$, using also the fact that $\tilde{m}_n^{(1,k+1)} \lesssim r_0$ before a trapped sphere is formed (see the relations (3.6.129) and (3.6.154) in the next section).

The bound (3.6.13) is a technical version of the bound (3.1.37), and its proof follows from the ideas outlined in Section 3.1.2. In obtaining (3.6.13), the lower bound of (3.6.10) is necessary.

Finally, the bound (3.6.14) is a technical version of the bound (3.1.41) in Section 3.1.2 and provides an estimate for the decrease of the multiplicative factor in the right hand side of (3.6.13). In obtaining (3.6.14) when $\frac{2m}{r} \simeq 1$, the fact that $\frac{2m}{r}$ is bounded away from 1 everywhere but on $\mathcal{R}_{\varepsilon n}^{(i,j)}$ is crucially used (in particular, the bound (3.6.12) is necessary for (3.6.14)).

Remark. As is evident from the above discussion, most of the technical difficulties in the proof of Proposition 3.4 are associated to issues related with the near-trapped regime

$\frac{2m}{r} \simeq 1$. In the case when, instead of the stronger bound (3.4.4), one is merely interested in establishing the weaker instability estimate (3.4.8), the proof of Proposition 3.4 simplifies substantially: In that case, it is not necessary to demand that the worst instability scenario takes place in $\mathcal{R}_{\varepsilon n}^{(1,k+1)}$. In particular, the bounds (3.6.11) and (3.6.12) can be omitted from the proof. Moreover, the lower bound for r_0 in (3.5.6) can be relaxed, and the exponentials in the relations (3.5.2) between $h_0(\varepsilon), h_1(\varepsilon)$ can be replaced by polynomial functions.

Proof of Proposition 3.4

In this section, we will make use of the $O(\cdot)$ convention: For any pair of functions $\mathcal{F}, \mathcal{G}$ defined on the same domain, with $\mathcal{G} \geq 0$, the notation

$$\mathcal{F} = O(\mathcal{G})$$

will imply that

$$|\mathcal{F}| \leq C \cdot \mathcal{G}$$

for some universal constant $C > 0$ which is independent of all the parameters in the statement of Theorem 3.1. We should also remark that, throughout this proof, we will adopt the convention on the indices stated under (3.5.34), i. e.:

1. $U_n(v^{(-1)}) \doteq U_{n+1}(v^{(k)})$.
2. $V_n(v^{(k+l)} + c) \doteq V_{n-1}(v^{(l-1)} + c)$ for any integer $1 \leq l \leq k$ and any $c \geq 0$.

In view of (3.6.1), on $\mathcal{U}_\varepsilon^+$ we have

$$(3.6.18) \quad \partial_u r < 0 < \partial_v r$$

and

$$(3.6.19) \quad \partial_u \tilde{m} \leq 0 \leq \partial_v \tilde{m}.$$

We will split the proof of Theorem 3.1 into two parts: In the first (and shortest) part,

we will establish the bound (3.6.7) through a standard continuity argument. The proof of (3.6.7) will also yield (3.6.8) and (3.6.9). In the second (and more extended) part, we will establish the bounds (3.6.10)–(3.6.14) by induction on n .

Part I: Proof of (3.6.7)–(3.6.9)

Let $u_* > 0$ be such, so that on

$$(3.6.20) \quad \mathcal{U}_\varepsilon^* \doteq \mathcal{U}_\varepsilon^+ \cap \{u < u_*\},$$

we can bound

$$(3.6.21) \quad \left| \log \left(\frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right) \right| + \left| \log \left(\frac{\partial_v r}{1 - \frac{2m}{r}} \right) \right| \leq 2(h_1(\varepsilon))^{-4} \log((h_3(\varepsilon))^{-1}).$$

By showing that (3.6.7) holds on $\mathcal{U}_\varepsilon^*$, it will follow (by applying a standard continuity argument) that (3.6.7) holds on the whole of $\mathcal{U}_\varepsilon^+$.

Inductive formulas for $\partial_u r$ and $\partial_v r$ and proof of (3.6.7). From equation (3.2.45), we can readily derive the following renormalised equation:

$$(3.6.22) \quad \partial_v \partial_u \left\{ \sqrt{-\frac{3}{\Lambda}} \tan^{-1} \left(\sqrt{-\frac{\Lambda}{3}} r \right) \right\} = -2 \frac{\tilde{m}}{r^2} \frac{(1 - \Lambda r^2)}{(1 - \frac{1}{3}\Lambda r^2)} \left(\frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right) \left(\frac{\partial_v r}{1 - \frac{2m}{r}} \right)$$

Let $n \geq 1$, $0 \leq i \leq k$, $i \leq j \leq k + i$, $\bar{u} < u_*$ and v_b be such that

$$U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \leq \bar{u} \leq U_n(v^{(i-1)})$$

and

$$V_n(v^{(j)}) \leq v_b \leq V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)).$$

Integrating equation (3.6.22) for $u = \bar{u}$ from $v = V_n(v^{(j)})$ up to $v = v_b$, using also the fact that

$$\partial_u \left\{ \sqrt{-\frac{3}{\Lambda}} \tan^{-1} \left(\sqrt{-\frac{\Lambda}{3}} r \right) \right\} = \frac{\partial_u r}{1 - \frac{1}{3}\Lambda r^2},$$

we obtain:

$$(3.6.23) \quad \left. \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(\bar{u}, V_n(v^{(j)}))} = \left. \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(\bar{u}, v_b)} + \\ + O\left(\sup_{\{u=\bar{u}\} \cap \{V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq v \leq V_n(v^{(j-1)})\}} \left| \frac{\tilde{m}}{r^2} \left(\frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right) \left(\frac{\partial_v r}{1 - \frac{2m}{r}} \right) \right| h_2(\varepsilon) \right).$$

Using the bootstrap bound (3.6.21), combined with the trivial bounds

$$(3.6.24) \quad \tilde{m} \leq \tilde{m}|_{\mathcal{I}} = \frac{\varepsilon}{\sqrt{-\Lambda}}$$

and

$$(3.6.25) \quad r \geq r_0$$

(following from (3.2.46), (3.2.47) and (3.6.18)), as well as the relation (3.5.3) for $h_2(\varepsilon)$, the relation (3.6.23) yields:

$$(3.6.26) \quad \left. \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(\bar{u}, V_n(v^{(j)}))} = \left. \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(\bar{u}, v_b)} + O((h_2(\varepsilon))^{1/2}).$$

Similarly, integrating (3.6.22) for $v = \bar{v}$ from $u = U_n(v^{(i)})$ up to $u = u_b$ for any $V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq \bar{v} \leq V_n(v^{(j-1)})$ and any $U_n(v^{(i)}) \leq u_b \leq U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))$ (assuming that $u_b < u_*$), we infer:

$$(3.6.27) \quad \left. \frac{\partial_v r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(U_n(v^{(i)}), \bar{v})} = \left. \frac{\partial_v r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(u_b, \bar{v})} + O((h_2(\varepsilon))^{1/2}).$$

By multiplying and dividing each factor with $1 - \frac{2m}{r} = 1 - \frac{2\tilde{m}}{r} - \frac{1}{3}\Lambda r^2$, the relations (3.6.26) and (3.6.27) are equivalent to

$$(3.6.28) \quad \left. \frac{-\partial_u r}{1 - \frac{2m}{r}} \right|_{(\bar{u}, V_n(v^{(j)}))} = \left. \frac{-\partial_u r}{1 - \frac{2m}{r}} \right|_{(\bar{u}, v_b)} \cdot \left(\frac{1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)}}{r(1 - \frac{1}{3}\Lambda r^2)} \Big|_{(\bar{u}, v_b)} + O((h_2(\varepsilon))^{1/2}) \right)$$

and

$$(3.6.29) \quad \frac{\partial_v r}{1 - \frac{2\tilde{m}}{r}} \Big|_{(U_n(v^{(i)}), \bar{v})} = \frac{\partial_v r}{1 - \frac{2\tilde{m}}{r}} \Big|_{(u_b, \bar{v})} \cdot \left(\frac{1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)} \Big|_{(u_b, \bar{v})}}{1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)} \Big|_{(U_n(v^{(i)}), \bar{v})}} + O((h_2(\varepsilon))^{1/2}) \right).$$

Remark. In the vacuum case, where $\tilde{m}$ is constant, the factors in the right hand side of (3.6.28) and (3.6.29) become identically 1. In our case, however, where matter is present, by relaxing our definition of h_2 and considering the limit $h_2 \rightarrow 0$ for fixed ε , the dominant terms in the factors in the right hand side of (3.6.28) and (3.6.29), i. e. the first summands, do *not* converge to 1. This is because, in this limit, while the function r remains C^1 , the renormalised Hawking mass $\tilde{m}$ has a jump discontinuity across the beam.

Since $T_{uu} = T_{vv} = 0$ on $\mathcal{R}_{\varepsilon n}^{(i,j)}$ for any $n \geq 1$, any $0 \leq i \leq k$ and any $i \leq j \leq k + i + 1$, the relations (3.2.43)–(3.2.44) imply that:

$$(3.6.30) \quad \partial_v \left(\frac{-\partial_u r}{1 - \frac{2\tilde{m}}{r}} \right) \Big|_{\mathcal{R}_{\varepsilon n}^{(i,j)}} = \partial_u \left(\frac{\partial_v r}{1 - \frac{2\tilde{m}}{r}} \right) \Big|_{\mathcal{R}_{\varepsilon n}^{(i,j)}} = 0.$$

In particular, along lines of the form $\{u = \bar{u}\}$, the quantity $\frac{-\partial_u r}{1 - \frac{2\tilde{m}}{r}}$ remains constant on $\{u = \bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}$ for any $\bar{u} < u_*$ such that $\{u = \bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}$ is non-trivial. In view of (3.6.28), the quantities $\frac{-\partial_u r}{1 - \frac{2\tilde{m}}{r}} \Big|_{\{u=\bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j+1)}}$ and $\frac{-\partial_u r}{1 - \frac{2\tilde{m}}{r}} \Big|_{\{u=\bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}}$ (for any $\bar{u} < u_*$ such that $\{u = \bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}$ is non-trivial) are related by

$$(3.6.31) \quad \frac{-\partial_u r}{1 - \frac{2\tilde{m}}{r}} \Big|_{\{u=\bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}} = \frac{-\partial_u r}{1 - \frac{2\tilde{m}}{r}} \Big|_{\{u=\bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j+1)}} \times \left(\frac{1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)} \Big|_{(\bar{u}, V_n(v^{(j)}))}}{1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)} \Big|_{(\bar{u}, V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))}} + O((h_2(\varepsilon))^{1/2}) \right).$$

Similarly, the quantity $\frac{\partial_v r}{1 - \frac{2\tilde{m}}{r}}$ remains constant along segments of the form $\{v = \bar{v}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}$,

and $\left. \frac{\partial_v r}{1 - \frac{2m}{r}} \right|_{\{v=\bar{v}\} \cap \mathcal{R}_{\varepsilon n}^{(i+1,j)}}$ and $\left. \frac{\partial_v r}{1 - \frac{2m}{r}} \right|_{\{v=\bar{v}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}}$ are related (in view of (3.6.29)) by

$$(3.6.32) \quad \left. \frac{\partial_v r}{1 - \frac{2m}{r}} \right|_{\{v=\bar{v}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}} = \left. \frac{\partial_v r}{1 - \frac{2m}{r}} \right|_{\{v=\bar{v}\} \cap \mathcal{R}_{\varepsilon n}^{(i+1,j)}} \times \\ \times \left(\frac{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(U_n(v^{(i)}), \bar{v})}}{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), \bar{v})}} + O((h_2(\varepsilon))^{1/2}) \right).$$

We infer, therefore, that for any point $(\bar{u}, \bar{v}) \in \mathcal{R}_{\varepsilon, n}^{(i,j)}$ for some $n \geq 2$, $0 \leq i \leq k$ and $i \leq j \leq k+i+1$ such that $\bar{u} < u_*$, the following relations hold between $(\bar{u}, \bar{v})$ and $(\bar{u}-v_0, \bar{v}-v_0) \in \mathcal{R}_{\varepsilon, n-1}^{(i,j)}$:

$$(3.6.33) \quad \left. \frac{-\partial_u r}{1 - \frac{2m}{r}} \right|_{(\bar{u}, \bar{v})} = \left. \frac{-\partial_u r}{1 - \frac{2m}{r}} \right|_{(\bar{u}-v_0, \bar{v}-v_0)} \times \prod_{\bar{j}=j}^{k+i} \left(\frac{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(\bar{u}, V_n(v^{(\bar{j})})}}{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(\bar{u}, V_n(v^{(\bar{j})} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))}} + O((h_2(\varepsilon))^{1/2}) \right) \times \\ \times \prod_{\bar{i}=i}^{k+i} \left(\frac{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(U_n(v^{(\bar{i})}), \bar{u})}}{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(U_n(v^{(\bar{i})} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), \bar{u})}} + O((h_2(\varepsilon))^{1/2}) \right) \times \\ \times \prod_{\bar{j}=k+i+1}^{k+j} \left(\frac{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(\bar{u}-v_0, V_n(v^{(\bar{j})})}}{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(\bar{u}-v_0, V_n(v^{(\bar{j})} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))}} + O((h_2(\varepsilon))^{1/2}) \right)$$

and

$$(3.6.34) \quad \left. \frac{\partial_v r}{1 - \frac{2m}{r}} \right|_{(\bar{u}, \bar{v})} = \left. \frac{\partial_v r}{1 - \frac{2m}{r}} \right|_{(\bar{u}-v_0, \bar{v}-v_0)} \times \prod_{\bar{i}=i}^{j-1} \left(\frac{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(U_n(v^{(\bar{i})}), \bar{v})}}{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(U_n(v^{(\bar{i})} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), \bar{v})}} + O((h_2(\varepsilon))^{1/2}) \right) \times \\ \times \prod_{\bar{j}=j}^{k+j} \left(\frac{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(\bar{u}-v_0, V_n(v^{(\bar{j})})}}{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(\bar{u}-v_0, V_n(v^{(\bar{j})} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))}} + O((h_2(\varepsilon))^{1/2}) \right) \times \\ \times \prod_{\bar{i}=j}^{k+i} \left(\frac{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(U_n(v^{(\bar{i})}), \bar{v}-v_0)}}{1 - \frac{2\tilde{m}}{r(1-\frac{1}{3}\Lambda r^2)} \Big|_{(U_n(v^{(\bar{i})} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), \bar{v}-v_0)}} + O((h_2(\varepsilon))^{1/2}) \right).$$

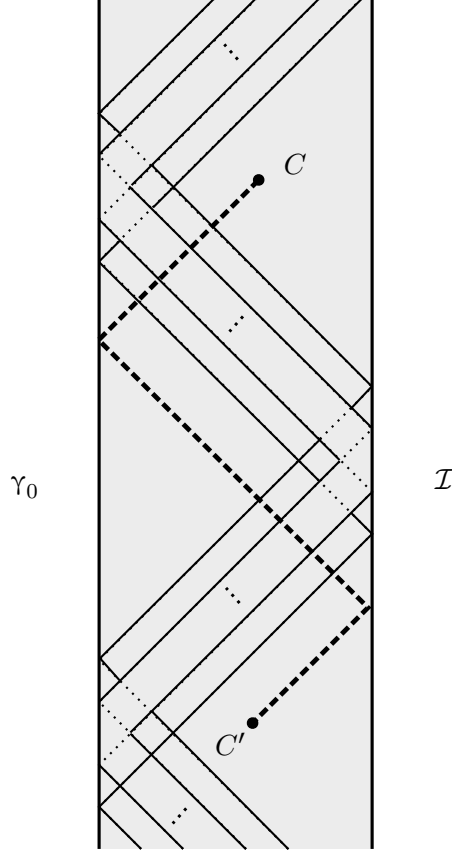


Figure 15: In order to obtain the formula (3.6.33) relating $\frac{-\partial_u r}{1-\frac{2m}{r}}$ at the point $C = (\bar{u}, \bar{v})$ with the same quantity at the point $C' = (\bar{u} - v_0, \bar{v} - v_0)$, we apply the relations (3.6.31) and (3.6.32) along the dashed path depicted above, using also the reflecting gauge condition on γ_0 and $\mathcal{I}$.

The relation (3.6.33) is obtained as follows (see also Figure 15): First, (3.6.31) determines the evolution of $\frac{-\partial_u r}{1-\frac{2m}{r}}$ (according to (3.6.31)) along the line $\{u = \bar{u}\}$ in the past direction, from $(\bar{u}, \bar{v})$ up to γ_0 . Then, using the boundary relation

$$(3.6.35) \quad \left. \frac{-\partial_u r}{1 - \frac{2m}{r}} \right|_{\gamma_0} = \left. \frac{\partial_v r}{1 - \frac{2m}{r}} \right|_{\gamma_0},$$

one repeats the same procedure for $\frac{\partial_v r}{1-\frac{2m}{r}}$ along $\{v = \bar{u}\}$ from γ_0 up to $\mathcal{I}$. Finally, using

$$(3.6.36) \quad \left. \frac{-\partial_u r}{1 - \frac{2m}{r}} \right|_{\mathcal{I}} = \left. \frac{\partial_v r}{1 - \frac{2m}{r}} \right|_{\mathcal{I}},$$

and following the evolution of $\frac{-\partial_u r}{1-\frac{2m}{r}}$ along $\{u = \bar{u} - v_0\}$ from $\mathcal{I}$ up to $(\bar{u} - v_0, \bar{v} - v_0)$, one arrives at (3.6.33). The relation (3.6.34) is similarly obtained by following the same procedure along

the lines $\{v = \bar{v}\}$ (up to $\mathcal{I}$), $\{u = \bar{v} - v_0\}$ (from $\mathcal{I}$ up to γ_0) and $\{v = \bar{v} - v_0\}$ (from $\mathcal{I}$ up to $(\bar{u} - v_0, \bar{v} - v_0)$).

In view of the bound

$$(3.6.37) \quad 1 - \frac{2m}{r} \geq h_3(\varepsilon)$$

on $\mathcal{U}_\varepsilon^+$ (see (3.6.1)), we can estimate in the region $\{r \leq \varepsilon^{1/2}(-\Lambda)^{1/2}\} \cap \mathcal{U}_\varepsilon^+$:

$$(3.6.38) \quad 1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)} = \frac{1 - \frac{2m}{r}}{1 - \frac{1}{3}\Lambda r^2} \geq \frac{1}{2}h_1(\varepsilon).$$

On the other hand, in the region $\{r \geq \varepsilon^{1/2}(-\Lambda)^{1/2}\} \cap \mathcal{U}_\varepsilon^+$, using (3.5.21) to bound $\tilde{m}$ we can trivially estimate (in view also of (3.5.2)):

$$(3.6.39) \quad 1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)} \geq 1 - \frac{2\varepsilon}{\varepsilon^{\frac{1}{2}}} \geq h_1(\varepsilon).$$

Combining (3.6.38) and (3.6.39), using also the fact that $\tilde{m} \geq \tilde{m}|_{\gamma_0} = 0$ on $\mathcal{U}_\varepsilon^+$, we can bound $1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)}$ from above and below everywhere on $\mathcal{U}_\varepsilon^+$ as:

$$(3.6.40) \quad \frac{1}{2}h_3(\varepsilon) \leq 1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)} \leq 1.$$

Thus, by considering the logarithm of the relations (3.6.33)–(3.6.34) and noting that the resulting right hand side contains $\sim k = \lceil 1/h_1(\varepsilon) \rceil$ summands, each controlled with the help of (3.6.40), we readily obtain for any $n \geq 2$, $0 \leq i \leq k$ and $i \leq j \leq k + i + 1$ and any point $(\bar{u}, \bar{v}) \in \mathcal{R}_{\varepsilon n}^{(i,j)}$ with $\bar{u} < u_*$:

$$(3.6.41) \quad \left| \log\left(\frac{-\partial_u r}{1 - \frac{2m}{r}}\right)\Big|_{(\bar{u}, \bar{v})} - \log\left(\frac{-\partial_u r}{1 - \frac{2m}{r}}\right)\Big|_{(\bar{u}-v_0, \bar{v}-v_0)} \right| \leq \frac{C}{h_1(\varepsilon)} \log((h_3(\varepsilon))^{-1})$$

and

$$(3.6.42) \quad \left| \log\left(\frac{\partial_v r}{1 - \frac{2m}{r}}\right)\Big|_{(\bar{u}, \bar{v})} - \log\left(\frac{\partial_v r}{1 - \frac{2m}{r}}\right)\Big|_{(\bar{u}-v_0, \bar{v}-v_0)} \right| \leq \frac{C}{h_1(\varepsilon)} \log((h_3(\varepsilon))^{-1}).$$

In view of (3.6.28)–(3.6.29), the bounds (3.6.41) and (3.6.42) (stated in the case when $(\bar{u}, \bar{v})$

belongs to a vacuum region $\mathcal{R}_{\varepsilon n}^{(i,j)}$) also hold when $(\bar{u}, \bar{v})$ belongs to a beam, i.e. when $U_n(v^{(i)}) \leq \bar{u} \leq U_n(v^{(i-1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))$ or $V_n(v^{(j)}) \leq \bar{v} \leq V_n(v^{(i-1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))$ for some $n \geq 2$, $0 \leq i \leq k$ and $i \leq j \leq k + i + 1$. Therefore, for any $n \geq 2$, the bounds (3.6.28)–(3.6.29) hold on the whole of

$$(3.6.43) \quad \mathcal{U}_{\varepsilon;n}^* \doteq \{U_n(v^{(k)}) \leq u \leq U_{n+1}(v^{(k)})\} \cap \mathcal{U}_{\varepsilon}^*.$$

From (3.6.2) and the definition (3.6.5), it follows that

$$(3.6.44) \quad n_f \leq (h_1(\varepsilon))^{-2}.$$

Since $n \leq n_f$ (because $\mathcal{U}_{\varepsilon}^* \subset \mathcal{U}_{\varepsilon}^+$), by substituting $(\bar{u}, \bar{v}) \rightarrow (\bar{u} - v_0, \bar{v} - v_0)$ in (3.6.41)–(3.6.42) $n - 2$ times and using (3.6.44), (3.5.11), (3.6.40) as well as the Cauchy stability estimate of Proposition 3.3 for the region $\{0 \leq u \leq 2v_0\}$, we readily obtain

$$(3.6.45) \quad \sup_{\mathcal{U}_{\varepsilon}^*} \left\{ \left| \log \left(\frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right) \right| + \left| \log \left(\frac{\partial_v r}{1 - \frac{2m}{r}} \right) \right| \right\} \leq \frac{C}{(h_1(\varepsilon))^3} \log((h_3(\varepsilon))^{-1}).$$

Thus, (3.6.7) holds on $\mathcal{U}_{\varepsilon}^*$ in view of the relation (3.5.2) for the parameter $h_1(\varepsilon)$ (provided ε_0 is small enough). Therefore (as explained in the beginning of the proof), a standard continuity argument yields that (3.6.7) actually holds on the whole of $\mathcal{U}_{\varepsilon}^+$.

Proof of (3.6.8) and (3.6.9). For any $1 \leq n \leq n_f$, we can bound in view of the definition (3.5.8) of $v^{(j)}$ and the bound (3.6.7):

$$\begin{aligned}
(3.6.46) \quad & \left| \tan^{-1} \left(\sqrt{-\frac{\Lambda}{3}} r \right) \right|_{\left(U_n(v^{(0)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), V_n(v^{(k)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right)^-} \\
& \quad - \tan^{-1} \left(\sqrt{-\frac{\Lambda}{3}} r \right) \Big|_{\mathcal{I} \cap \{v = V_n(v^{(k)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))\}} \Big| \\
& = \sqrt{-\frac{\Lambda}{3}} \int_{U_n(v^{(k)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))}^{U_n(v^{(0)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))} \frac{-\partial_u r}{1 - \frac{1}{3} \Lambda r^2} \Big|_{\left(u, V_n(v^{(k)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right)} du \\
& \leq \frac{C \sqrt{-\Lambda}}{(h_1(\varepsilon))^4} \log((h_3(\varepsilon))^{-1}) |v^{(k)} - v^{(0)}| \\
& \leq \frac{C \varepsilon}{(h_1(\varepsilon))^6} \log((h_3(\varepsilon))^{-1})
\end{aligned}$$

and

$$\begin{aligned}
(3.6.47) \quad & \left| \tan^{-1} \left(\sqrt{-\frac{\Lambda}{3}} r \right) \right|_{\left(U_n(v^{(k)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), V_n(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right)^-} \\
& \quad - \tan^{-1} \left(\sqrt{-\frac{\Lambda}{3}} r \right) \Big|_{\gamma_0 \cap \{v = V_n(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))\}} \Big| \\
& = \sqrt{-\frac{\Lambda}{3}} \int_{U_n(v^{(k)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))}^{U_n(v^{(0)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))} \frac{-\partial_u r}{1 - \frac{1}{3} \Lambda r^2} \Big|_{\left(u, V_n(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right)} du \\
& \leq \frac{C \sqrt{-\Lambda}}{(h_1(\varepsilon))^4} \log((h_3(\varepsilon))^{-1}) |v^{(k)} - v^{(0)}| \\
& \leq \frac{C \varepsilon}{(h_1(\varepsilon))^6} \log((h_3(\varepsilon))^{-1}).
\end{aligned}$$

From (3.6.46) and (3.6.47) we readily obtain (3.6.8) and (3.6.9), respectively, in view of the relations (3.5.2) and (3.6.3) for h_1, h_3 , respectively, and the fact that $r|_{\gamma_0} = r_0$, $r|_{\mathcal{I}} = +\infty$.

Part II: Proof of (3.6.10)–(3.6.14)

We will now proceed to establish the bounds (3.6.10)–(3.6.14). To this end, we will first derive some useful estimates for the differences of the renormalised masses $\tilde{m}_n^{(i,j)}$ associated to the vacuum regions around each interaction region $\mathcal{N}_{\varepsilon_n}^{(i,j)}$.¹⁴

¹⁴A relation for the change the mass differences of two intersecting, infinitely thin null dust beams was also obtained in [120].

Relations for the change in the mass difference of the beams. Let us introduce the notion of the mass difference for the beams $\{U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\}$ and $\{V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\}$ around their interaction region $\mathcal{N}_{\varepsilon n}^{(i,j)}$: For any $1 \leq n \leq n_f$, $0 \leq i \leq k$ and $i+1 \leq j \leq k+i$, we define the initial mass differences

$$(3.6.48) \quad \begin{aligned} (\mathfrak{D}_- \tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i+1,j+1)} - \tilde{m}_n^{(i,j+1)} \\ (\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i+1,j)} - \tilde{m}_n^{(i+1,j+1)} \end{aligned}$$

and the final mass differences

$$(3.6.49) \quad \begin{aligned} (\mathfrak{D}_+ \tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i+1,j)} - \tilde{m}_n^{(i,j)} \\ (\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)}. \end{aligned}$$

Note that $(\mathfrak{D}_- \tilde{m})_n^{(i,j)}$ and $(\mathfrak{D}_+ \tilde{m})_n^{(i,j)}$ are the mass differences around the outgoing beam $\{U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\}$ before and after crossing the region $\mathcal{N}_{\varepsilon n}^{(i,j)}$, respectively, while $(\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}$ and $(\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}$ are the mass differences around the ingoing beam $\{V_n(v^{(j)}) \leq v \leq U_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\}$ before and after crossing the region $\mathcal{N}_{\varepsilon n}^{(i,j)}$. Note the trivial identity

$$(3.6.50) \quad (\mathfrak{D}_- \tilde{m})_n^{(i,j)} + (\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,j)} = (\mathfrak{D}_+ \tilde{m})_n^{(i,j)} + (\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}.$$

We will establish the following bounds for any $1 \leq n \leq n_f$, $1 \leq i \leq k$ and $i+1 \leq j \leq k+i$:

$$(3.6.51) \quad (\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)} = (\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,j)} \cdot \exp \left(\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\mathfrak{D}_- \tilde{m})_n^{(i,j)}}{1 - \frac{2\tilde{m}_n^{(i+1,j)}}{\bar{r}_n^{(i,j)}} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,j)})(1 - \mathfrak{Err}_n^{(i,j)}) \right)$$

and:

$$(3.6.52) \quad (\mathfrak{D}_+ \tilde{m})_n^{(i,j)} = (\mathfrak{D}_- \tilde{m})_n^{(i,j)} \cdot \exp \left(- \frac{2}{\bar{r}_n^{(i,j)}} \frac{(\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}}{1 - \frac{2\tilde{m}_n^{(i+1,j)}}{\bar{r}_n^{(i,j)}} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,j)})(1 - \mathfrak{Err}_n^{(i,j)}) \right),$$

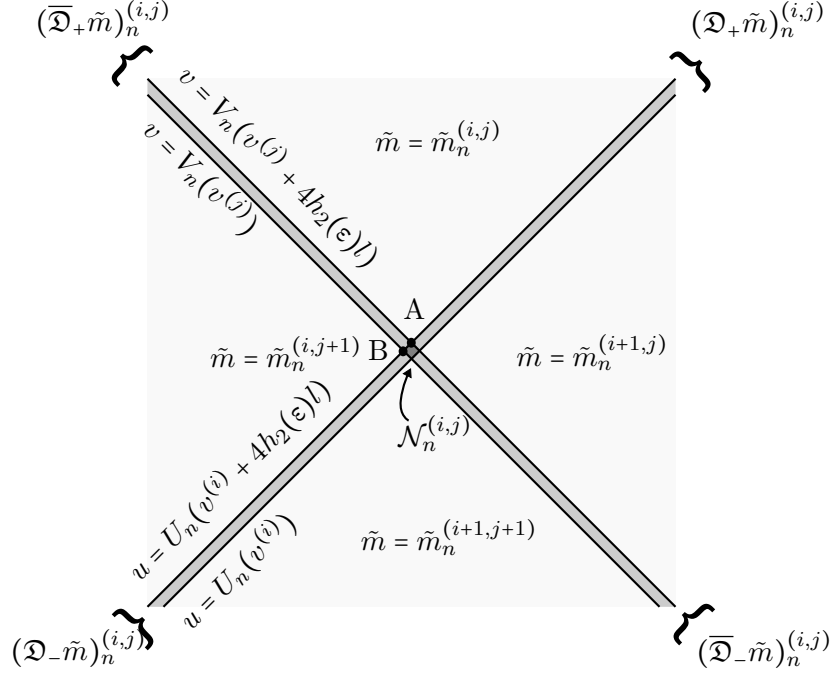


Figure 16: Schematic depiction of two intersecting beams, with associated incoming and outgoing mass differences $(\mathfrak{D}_- \tilde{m})_n^{(i,j)}$, $(\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}$ and $(\mathfrak{D}_+ \tilde{m})_n^{(i,j)}$, $(\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}$, respectively. The point A satisfies $r(A) = r_n^{(i,j)}$, while the point B satisfies $r(B) = \bar{r}_n^{(i,j)}$. For simplicity, we have used the shorthand notation $l = (-\Lambda)^{-1/2}$.

where the terms $\mathfrak{Err}_{1,n}^{(i,j)}$ in (3.6.51) and (3.6.52) are allowed to be different from each other, but they both satisfy the bound

$$(3.6.53) \quad 0 \leq \mathfrak{Err}_{1,n}^{(i,j)} \leq 1 - \frac{\bar{r}_n^{(i,j)} - 2\tilde{m}_n^{(i+1,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^3}{r \left| \left(U_n(v^{(i)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\epsilon)) \right) - 2\tilde{m}_n^{(i,j+1)} - \frac{1}{3}\Lambda r^3 \right| \left(U_n(v^{(i)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\epsilon)) \right)}$$

and $\mathfrak{Err}_{\setminus n}^{(i,j)}$, $\mathfrak{Err}_{/n}^{(i,j)}$ satisfy the bounds

$$(3.6.54) \quad 0 \leq \mathfrak{Err}_{\setminus n}^{(i,j)} \leq 1 - \frac{(\mathfrak{D}_+ \tilde{m})_n^{(i,j)}}{(\mathfrak{D}_- \tilde{m})_n^{(i,j)}}$$

and

$$(3.6.55) \quad 0 \leq \mathfrak{Err}_{/n}^{(i,j)} \leq 1 - \frac{(\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}}{(\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}}.$$

Moreover, the following estimate will be useful in the proof of (3.6.12): For any $1 \leq n \leq n_f$, $1 \leq i \leq k$ and $k+1 \leq j \leq k+i$,

$$(3.6.56) \quad (\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)} \geq (\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,j)} \cdot \exp\left(\frac{1}{5C_0} \frac{(\mathfrak{D}_- \tilde{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}}\right).$$

Remark. Notice that, as a consequence of (3.6.51) and (3.6.52), during the interaction of the two beams at $\mathcal{N}_{\varepsilon n}^{(i,j)}$, the mass difference $\overline{\mathfrak{D}}\tilde{m}$ of the ingoing beam increases, while the mass difference $\mathfrak{D}\tilde{m}$ of the outgoing beam decreases.

Proof of (3.6.51) and (3.6.52). By differentiating (3.2.47) in u and using (3.2.43) and (3.2.48), we readily obtain the following wave-type equation for $\tilde{m}$:

$$(3.6.57) \quad \partial_u \partial_v \tilde{m} = -F(r, \tilde{m}) \partial_u \tilde{m} \partial_v \tilde{m},$$

where

$$(3.6.58) \quad F(r, \tilde{m}) \doteq \frac{2}{r - 2\tilde{m} - \frac{1}{3}\Lambda r^3}.$$

Note that, formally, equation (3.6.57) can be rewritten as

$$(3.6.59) \quad \partial_v \log(-\partial_u \tilde{m}) = -F(r, \tilde{m}) \partial_v \tilde{m}$$

or

$$(3.6.60) \quad \partial_u \log(\partial_v \tilde{m}) = F(r, \tilde{m})(-\partial_u \tilde{m})$$

(note, however, that $\log(-\partial_u \tilde{m})$, $\log(\partial_v \tilde{m})$ will not be well defined when $\partial_u \tilde{m} = 0$ or $\partial_v \tilde{m} = 0$).

For any $1 \leq n \leq n_f$, $1 \leq i \leq k$ and $i+1 \leq j \leq k+i$, integrating equation (3.6.57) first in u , for $U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))$, and then in v , for $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))$,

we obtain:

(3.6.61)

$$\begin{aligned} \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)} &= \int_{V_n(v^{(j)})}^{V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \partial_v \tilde{m}|_{(U_n(v^{(i)}), v)} \times \\ &\quad \times \exp\left(2 \int_{U_n(v^{(i)})}^{U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \frac{-\partial_u \tilde{m}}{r - 2m}|_{(u,v)} du\right) dv. \end{aligned}$$

Remark. Note that, at the formal level, the derivation of (3.6.61) is easiest seen by integrating equation (3.6.60) first in u , then exponentiating, and then integrating in v . This procedure can actually be done rigorously, since $\partial_u \tilde{m} < 0 < \partial_v \tilde{m}$ in the interior of $\mathcal{N}_{\varepsilon n}^{(i,j)}$, in view of (3.2.46), (3.2.47) and (3.5.40)–(3.5.41).

In view of (3.6.18)–(3.6.19), we can bound for any $U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))$ and any $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))$:

$$\begin{aligned} (3.6.62) \quad \bar{r}_n^{(i,j)} - 2\tilde{m}_n^{(i+1,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^3 &\leq (r - 2m)|_{(u,v)} \leq \\ &\leq r|_{(U_n(v^{(i)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)))} - 2\tilde{m}_n^{(i,j+1)} - \frac{1}{3}\Lambda r^3|_{(U_n(v^{(i)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)))}, \end{aligned}$$

where

$$\bar{r}_n^{(i,j)} \doteq r|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), V_n(v^{(j)}))}.$$

Therefore, using (3.6.62) to estimate $\frac{1}{r-2m}$, from (3.6.61) we readily infer that:

(3.6.63)

$$\begin{aligned} \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)} &= \int_{V_n(v^{(j)})}^{V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \partial_v \tilde{m}|_{(U_n(v^{(i)}), v)} \times \\ &\quad \times \exp\left(\frac{2}{\bar{r}_n^{(i,j)}} \frac{\tilde{m}|_{(U_n(v^{(i)}), v)} - \tilde{m}|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), v)}}{1 - \frac{2\tilde{m}_n^{(i+1,j)}}{\bar{r}_n^{(i,j)}} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,j)}(v))\right) dv, \end{aligned}$$

where, for any $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))$, $\mathfrak{Err}_{1,n}^{(i,j)}(v)$ satisfies the bound (3.6.53).

Equations (3.2.46), (3.2.44) and (3.2.23) imply that, for any

$$V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)),$$

$$(3.6.64) \quad \partial_v \left(\tilde{m}|_{(U_n(v^{(i)}), v)} - \tilde{m}|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v)} \right) \leq 0$$

and, therefore, for any $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))$:

$$(3.6.65) \quad (\mathfrak{D}_+ \tilde{m})_n^{(i,j)} \leq \tilde{m}|_{(U_n(v^{(i)}), v)} - \tilde{m}|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v)} \leq (\mathfrak{D}_- \tilde{m})_n^{(i,j)}.$$

The bound (3.6.65) implies that (3.6.63) can be expressed as

$$(3.6.66)$$

$$\begin{aligned} \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)} &= (\tilde{m}_n^{(i+1,j)} - \tilde{m}_n^{(i+1,j+1)}) \times \\ &\times \exp \left(\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\mathfrak{D}_- \tilde{m})_n^{(i,j)}}{1 - \frac{2\tilde{m}_n^{(i+1,j)}}{\bar{r}_n^{(i,j)}} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,j)})(1 - \mathfrak{Err}_n^{(i,j)}) \right) \end{aligned}$$

where $\mathfrak{Err}_{1,n}^{(i,j)}$ satisfies the bound (3.6.53) and $\mathfrak{Err}_n^{(i,j)}$ satisfies the bound (3.6.54). In view of (3.6.48) and (3.6.49), (3.6.66) is equivalent to (3.6.51).

Similarly, integrating equation (3.6.57) first in v , for $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))$, and then in u , for $U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))$ (see also (3.6.59)), we obtain (3.6.52).

Proof of (3.6.56). Recall F defined by (3.6.58) and let us define the function $\bar{F} : \mathcal{D}_{\bar{F}} \rightarrow (0, +\infty)$, where

$$(3.6.67) \quad \mathcal{D}_{\bar{F}} = \left\{ (x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } x - y - \frac{2}{3}\Lambda x^2 > 0 \right\},$$

by the relation

$$(3.6.68) \quad \bar{F}(x, y) \doteq \frac{2}{x - y - \frac{2}{3}\Lambda x^2}.$$

Note that, in view of (3.6.7), (3.6.9), (3.5.6) and (3.5.3), for any $\mu \geq 0$ for which

$$(3.6.69) \quad \inf_{(u,v) \in \mathcal{N}_{\varepsilon n}^{(i,j)}} \left\{ r(u, v) - 2\mu - \frac{1}{3}\Lambda r^2(u, v) \right\} > h_3(\varepsilon),$$

we can readily bound:

$$(3.6.70) \quad \max_{(u,v) \in \mathcal{N}_{\varepsilon n}^{(i,j)}} \bar{F}(r(u,v), \mu) < \min_{(u,v) \in \mathcal{N}_{\varepsilon n}^{(i,j)}} F(r(u,v), \mu)$$

and

$$(3.6.71) \quad \partial_u \bar{F}(r(u,v), \mu), \partial_u F(r(u,v), \mu) > 0$$

(note that $F(r|_{\mathcal{N}_{\varepsilon n}^{(i,j)}}, \mu)$ and $\bar{F}(r|_{\mathcal{N}_{\varepsilon n}^{(i,j)}}, \mu)$ are well-defined and positive under the condition (3.6.69)).

For any $1 \leq n \leq n_f$, $1 \leq i \leq k$ and $k+1 \leq j \leq k+i$, let us consider the following characteristic initial value problem on $\mathcal{N}_{\varepsilon n}^{(i,j)}$:

$$(3.6.72) \quad \begin{cases} \partial_u \partial_v \bar{m} = -\bar{F}(r, \bar{m}) \partial_u \bar{m} \partial_v \bar{m} & \text{on } \mathcal{N}_{\varepsilon n}^{(i,j)}, \\ \bar{m} = \tilde{m} & \text{on } [U_n(v^{(i)}), U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))] \times \{V_n(v^{(j)})\} \cup \\ & \cup \{U_n(v^{(i)})\} \times [V_n(v^{(j)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))]. \end{cases}$$

Note that $\tilde{m}$ satisfies the same characteristic initial value problem with $F(r, \tilde{m})$ in place of $\bar{F}(r, \bar{m})$. Notice also that, in view of (3.2.46)–(3.2.47) and (3.5.40)–(3.5.41), the initial data for $\tilde{m}$ and $\bar{m}$ satisfy:

$$(3.6.73) \quad \partial_u \tilde{m} < 0 \text{ on } \left\{ U_n(v^{(i)}) < u < U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right\}$$

and

$$(3.6.74) \quad \partial_v \tilde{m} > 0 \text{ on } \left\{ V_n(v^{(j)}) < u < V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right\}.$$

Therefore, in view of (3.6.70), (3.6.71) and (3.6.73)–(3.6.74), an application of Lemma 3.1 (see Section 3.6.4) with $\tilde{m}, \bar{m}$ in place of z_2, z_1 , respectively, yields the following a priori

bounds for a solution $\bar{m}$ of (3.6.72):

$$(3.6.75) \quad \bar{m} \leq \tilde{m} \text{ on } \mathcal{N}_{\varepsilon n}^{(i,j)}$$

and

$$(3.6.76) \quad \partial_u \bar{m} < 0 < \partial_v \bar{m} \text{ in the interior of } \mathcal{N}_{\varepsilon n}^{(i,j)}.$$

Notice that the a priori bound (3.6.75) and the initial data in (3.6.72) imply that $\bar{m} \geq 0$ and that (3.6.69) holds for $\mu = \tilde{m}$ and $\mu = \bar{m}$; in particular, $\bar{F}(r, \bar{m})$ is well defined and positive on $\mathcal{N}_{\varepsilon n}^{(i,j)}$. Thus, it readily follows (using standard arguments) that (3.6.72) indeed has a unique smooth solution $\bar{m}$ satisfying (3.6.75).

With $\bar{m}$ defined on $\mathcal{N}_{\varepsilon n}^{(i,j)}$ as above for any $1 \leq n \leq n_f$, $1 \leq i \leq k$ and $k+1 \leq j \leq k+i$, we will define the following modified versions of (3.6.48) and (3.6.49):

$$(3.6.77) \quad \begin{aligned} (\mathfrak{D}_- \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)}))} - \bar{m}|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), V_n(v^{(j)}))} \\ (\overline{\mathfrak{D}}_- \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)))} - \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)}))} \end{aligned}$$

and

$$(3.6.78) \quad \begin{aligned} (\mathfrak{D}_+ \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)))} - \bar{m}|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)))} \\ (\overline{\mathfrak{D}}_+ \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)))} - \bar{m}|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), V_n(v^{(j)}))}. \end{aligned}$$

Note that, in view of the initial data for (3.6.72):

$$(3.6.79) \quad \begin{aligned} (\mathfrak{D}_- \bar{m})_n^{(i,j)} &= (\mathfrak{D}_- \tilde{m})_n^{(i,j)} \\ (\overline{\mathfrak{D}}_- \bar{m})_n^{(i,j)} &= (\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}, \end{aligned}$$

while, in view of the bound (3.6.75) (and the initial data for (3.6.72)):

$$(3.6.80) \quad \begin{aligned} (\mathfrak{D}_+ \bar{m})_n^{(i,j)} &\geq (\mathfrak{D}_+ \tilde{m})_n^{(i,j)} \\ (\overline{\mathfrak{D}_+ \bar{m}})_n^{(i,j)} &\leq (\overline{\mathfrak{D}_+ \tilde{m}})_n^{(i,j)}. \end{aligned}$$

By repeating exactly the same steps that led to (3.6.51) and (3.6.52) but using (3.6.72) instead of (3.6.57), we obtain for any $1 \leq n \leq n_f$, $1 \leq i \leq k$ and $k+1 \leq j \leq k+i$:

$$(3.6.81) \quad (\overline{\mathfrak{D}_+ \bar{m}})_n^{(i,j)} = (\overline{\mathfrak{D}_- \bar{m}})_n^{(i,j)} \cdot \exp \left(\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\mathfrak{D}_- \bar{m})_n^{(i,j)}}{1 - \frac{\tilde{m}_n^{(i+1,j)}}{\bar{r}_n^{(i,j)}} - \frac{2}{3} \Lambda (\bar{r}_n^{(i,j)})^2} (1 - \overline{\mathfrak{Err}}_{1,n}^{(i,j)}) (1 - \overline{\mathfrak{Err}}_n^{(i,j)}) \right)$$

and

$$(3.6.82) \quad (\mathfrak{D}_+ \bar{m})_n^{(i,j)} = (\mathfrak{D}_- \bar{m})_n^{(i,j)} \cdot \exp \left(- \frac{2}{\bar{r}_n^{(i,j)}} \frac{(\overline{\mathfrak{D}_+ \bar{m}})_n^{(i,j)}}{1 - \frac{\tilde{m}_n^{(i+1,j)}}{\bar{r}_n^{(i,j)}} - \frac{2}{3} \Lambda (\bar{r}_n^{(i,j)})^2} (1 - \overline{\mathfrak{Err}}_{1,n}^{(i,j)}) (1 - \overline{\mathfrak{Err}}_n^{(i,j)}) \right),$$

where

$$(3.6.83) \quad 0 \leq \overline{\mathfrak{Err}}_{1,n}^{(i,j)} \leq 1 - \frac{\bar{r}_n^{(i,j)} - \tilde{m}_n^{(i+1,j)} - \frac{2}{3} \Lambda (\bar{r}_n^{(i,j)})^3}{r \left| \left(U_n(v^{(i)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\epsilon)) \right) - \tilde{m}_n^{(i,j+1)} - \frac{2}{3} \Lambda r^3 \right| \left(U_n(v^{(i)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\epsilon)) \right)},$$

$$(3.6.84) \quad 0 \leq \overline{\mathfrak{Err}}_n^{(i,j)} \leq 1 - \frac{(\mathfrak{D}_+ \bar{m})_n^{(i,j)}}{(\mathfrak{D}_- \bar{m})_n^{(i,j)}}$$

and

$$(3.6.85) \quad 0 \leq \overline{\mathfrak{Err}}_n^{(i,j)} \leq 1 - \frac{(\overline{\mathfrak{D}_- \bar{m}})_n^{(i,j)}}{(\overline{\mathfrak{D}_+ \bar{m}})_n^{(i,j)}}$$

(and, as before, we allow the terms $\overline{\mathfrak{Err}}_{1,n}^{(i,j)}$ in (3.6.81) and (3.6.82) to be different).

Because

$$(3.6.86) \quad \tilde{m}_n^{(i+1,j)} \leq \tilde{m}|_X \leq \frac{2}{3} r_0 \leq \frac{2}{3} \bar{r}_n^{(i,j)}$$

(in view of (3.5.21), (3.5.6), (3.6.18) and (3.6.19)), from (3.6.82) (using also (3.6.9) and the fact that $\overline{\mathfrak{Err}}_{1,n}^{(i,j)}, \overline{\mathfrak{Err}}_{/n}^{(i,j)} \geq 0$) we can estimate for any $1 \leq n \leq n_f$, $1 \leq i \leq k$ and $k+1 \leq j \leq k+i$:

(3.6.87)

$$\begin{aligned} (\mathfrak{D}_+ \bar{m})_n^{(i,j)} &= (\mathfrak{D}_- \bar{m})_n^{(i,j)} \cdot \exp \left(- \frac{2}{\bar{r}_n^{(i,j)}} \frac{(\overline{\mathfrak{D}_+ \bar{m}})_n^{(i,j)}}{1 - \frac{\bar{m}_n^{(i+1,j)}}{\bar{r}_n^{(i,j)}} - \frac{2}{3} \Lambda(\bar{r}_n^{(i,j)})^2} (1 - \overline{\mathfrak{Err}}_{1,n}^{(i,j)}) (1 - \overline{\mathfrak{Err}}_{/n}^{(i,j)}) \right) \\ &\geq (\mathfrak{D}_- \bar{m})_n^{(i,j)} \cdot \exp \left(- \frac{8(\overline{\mathfrak{D}_+ \bar{m}})_n^{(i,j)}}{\bar{r}_n^{(i,j)}} \right) \end{aligned}$$

In view of the fact that

$$(\overline{\mathfrak{D}_+ \bar{m}})_n^{(i,j)} \leq (\overline{\mathfrak{D}_+ \bar{m}})_n^{(i,j)} = \bar{m}_n^{(i,j)} - \bar{m}_n^{(i,j+1)} \leq \bar{m}|_{\mathcal{I}} - 0 \leq \frac{2}{3} r_0 \leq \frac{2}{3} \bar{r}_n^{(i,j)}$$

(following from (3.6.80)), (3.6.87) yields

$$(3.6.88) \quad (\mathfrak{D}_+ \bar{m})_n^{(i,j)} \geq e^{-\frac{16}{3}} (\mathfrak{D}_- \bar{m})_n^{(i,j)}.$$

In view of (3.6.84), (3.6.88) implies that

$$(3.6.89) \quad 1 - \overline{\mathfrak{Err}}_{/n}^{(i,j)} \geq \frac{1}{C_0}.$$

Using (3.6.89) in (3.6.81), we obtain:

$$(3.6.90) \quad (\overline{\mathfrak{D}_+ \bar{m}})_n^{(i,j)} \geq (\overline{\mathfrak{D}_- \bar{m}})_n^{(i,j)} \cdot \exp \left(\frac{2}{C_0 \bar{r}_n^{(i,j)}} \frac{(\mathfrak{D}_- \bar{m})_n^{(i,j)}}{1 - \frac{\bar{m}_n^{(i+1,j)}}{\bar{r}_n^{(i,j)}} - \frac{2}{3} \Lambda(\bar{r}_n^{(i,j)})^2} (1 - \overline{\mathfrak{Err}}_{1,n}^{(i,j)}) \right).$$

In view of (3.6.86) and (3.6.9), we can also estimate

$$(3.6.91) \quad \frac{1 - \overline{\mathfrak{Err}}_{1,n}^{(i,j)}}{1 - \frac{\bar{m}_n^{(i+1,j)}}{\bar{r}_n^{(i,j)}} - \frac{2}{3} \Lambda(\bar{r}_n^{(i,j)})^2} \geq \frac{1}{10}$$

and, thus, (3.6.90) yields:

$$(3.6.92) \quad (\overline{\mathfrak{D}_+ \bar{m}})_n^{(i,j)} \geq (\overline{\mathfrak{D}_- \bar{m}})_n^{(i,j)} \cdot \exp \left(\frac{1}{5C_0} \frac{(\mathfrak{D}_- \bar{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}} \right).$$

From (3.6.92) and the relations (3.6.79) and (3.6.80), we readily obtain (3.6.56). $\square$

Proof of (3.6.11). For any $1 \leq n \leq n_f$, from (3.6.51) we readily obtain that, for any $1 \leq i \leq k$:

$$(3.6.93) \quad (\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)} \geq (\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,k+1)}.$$

Applying (3.6.93) successively for $i = 1, 2, \dots, k$, using also the identity

$$(3.6.94) \quad (\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,j)} = (\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i+1,j)}$$

(which follows from the fact that $\tilde{m}$ is constant over each $\mathcal{R}_{\varepsilon n}^{(i,j)}$), we thus infer that, for any $1 \leq i \leq k$:

$$(3.6.95) \quad (\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)} \geq (\overline{\mathfrak{D}}_- \tilde{m})_n^{(k,k+1)} = (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}).$$

Since

$$(3.6.96) \quad \tilde{m}_{n-1}^{(0,0)} = \tilde{m}_{n-1}^{(1,1)} = \tilde{m}|_{\mathcal{I}}$$

and

$$(3.6.97) \quad (\mathfrak{D}_+ \tilde{m})_{n-1}^{(0,1)} = \tilde{m}_{n-1}^{(1,1)} - \tilde{m}_{n-1}^{(0,1)} = (\overline{\mathfrak{D}}_- \tilde{m})_n^{(k,k+1)},$$

from (3.6.95) we infer that, for any $1 \leq i \leq k$:

$$(3.6.98) \quad (\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)} \geq (\mathfrak{D}_+ \tilde{m})_{n-1}^{(0,1)}.$$

Similarly as for the derivation of (3.6.93), applying the relation (3.6.52) successively for

$i = 0$ and $j = 1, 2, \dots, k$ (with $n - 1$ in place of n), we infer:

(3.6.99)

$$(\mathfrak{D}_+ \tilde{m})_{n-1}^{(0,1)} = (\mathfrak{D}_- \tilde{m})_{n-1}^{(0,k)} \cdot \exp \left(- \sum_{j=1}^k \frac{2}{\bar{r}_{n-1}^{(0,j)}} \frac{(\overline{\mathfrak{D}_- \tilde{m}})_{n-1}^{(0,j)}}{1 - \frac{2\tilde{m}_{n-1}^{(0,j)}}{\bar{r}_{n-1}^{(0,j)}} - \frac{1}{3}\Lambda(\bar{r}_{n-1}^{(0,j)})^2} (1 - \mathfrak{Err}_{1,n-1}^{(0,j)})(1 - \mathfrak{Err}_{/n-1}^{(0,j)}) \right).$$

In view of the bound (3.5.21) for the total mass $\tilde{m}|_{\mathcal{I}}$, the lower bound (3.6.8) for $r_n^{(0,k)}$ and the fact that

$$\bar{r}_{n-1}^{(0,j)} \leq r_{n-1}^{(0,k)} \leq \bar{r}_{n-1}^{(0,j)} (1 + (h_2(\epsilon))^{1/2})$$

for $1 \leq j \leq k$ (following from (3.6.7), and (3.5.3)), we can estimate

$$(3.6.100) \quad \sum_{j=1}^k \frac{2}{\bar{r}_{n-1}^{(0,j)}} \frac{(\overline{\mathfrak{D}_- \tilde{m}})_{n-1}^{(0,j)}}{1 - \frac{2\tilde{m}_{n-1}^{(0,j)}}{\bar{r}_{n-1}^{(0,j)}} - \frac{1}{3}\Lambda(\bar{r}_{n-1}^{(0,j)})^2} (1 - \mathfrak{Err}_{1,n-1}^{(0,j)})(1 - \mathfrak{Err}_{/n-1}^{(0,j)}) \leq \epsilon^{\frac{3}{2}}.$$

Therefore, (3.6.99) yields:

$$(3.6.101) \quad \log \frac{(\mathfrak{D}_+ \tilde{m})_{n-1}^{(0,1)}}{(\mathfrak{D}_- \tilde{m})_{n-1}^{(0,k)}} \geq -\epsilon^{3/2}.$$

From (3.6.98) and (3.6.101) we thus infer that, for any $1 \leq i \leq k$ and any $2 \leq n \leq n_f$:

$$(3.6.102) \quad \log \frac{(\overline{\mathfrak{D}_+ \tilde{m}})_n^{(i,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \geq -\epsilon^{3/2}.$$

From (3.6.102) for $i = 1$ and the fact that, for any $1 \leq n \leq n_f$:

$$(3.6.103) \quad (\overline{\mathfrak{D}_+ \tilde{m}})_n^{(1,k+1)} = (\mathfrak{D}_- \tilde{m})_n^{(0,k)} = \tilde{m}_n^{(1,k+1)},$$

we thus infer that, for all $2 \leq n \leq n_f$:

$$(3.6.104) \quad \log \frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \geq -\epsilon^{3/2}.$$

Applying (3.6.104) successively $n - 1$ times, we thus infer for any $2 \leq n \leq n_f$:

$$(3.6.105) \quad \log \frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_1^{(1,k+1)}} \geq -\varepsilon^{3/2}(n-1).$$

The bound (3.5.6) for r_0 and the form (3.5.7) of the initial data imply that

$$(3.6.106) \quad \frac{2(\tilde{m}_j(v^{(0)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) - \tilde{m}_j(v^{(0)}))}{r_0} \geq \frac{4}{C_0}h_0(\varepsilon).$$

Therefore, from (3.6.95) and (3.6.106) we infer that, for all $1 \leq i \leq 1$:

$$(3.6.107) \quad \frac{2(\overline{\mathfrak{D}}_+ \tilde{m})_1^{(i,k+1)}}{r_0} \geq \frac{4}{C_0}h_0(\varepsilon).$$

From (3.6.105) and (3.6.107) for $i = 1$ (when $(\overline{\mathfrak{D}}_+ \tilde{m})_1^{(1,k+1)} = \tilde{m}_1^{(1,k+1)}$), using also the fact that $n_f \leq (h_1(\varepsilon))^{-2}$, we thus deduce that, for all $1 \leq n \leq n_f$:

$$(3.6.108) \quad \frac{2\tilde{m}_n^{(1,k+1)}}{r_0} \geq \frac{2}{C_0}h_0(\varepsilon).$$

The relations (3.5.21) and (3.6.108) readily yield (3.6.11).

Proof of (3.6.10). In view of the bound (3.6.37), we infer that, for any $1 \leq n \leq n_f$:

$$(3.6.109) \quad 1 - \frac{2\tilde{m}_n^{(1,k+1)}}{r|_{\left(U_n(v^{(1)}), V_n(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\right)}} - \frac{1}{3}\Lambda r^2|_{\left(U_n(v^{(1)}), V_n(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\right)} \geq h_3(\varepsilon).$$

Using the bounds

$$(3.6.110) \quad \frac{r|_{\left(U_n(v^{(1)}), V_n(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\right)}}{r_0} \leq 1 + (h_2(\varepsilon))^{1/2}$$

(derived from (3.6.7), (3.6.9) and (3.5.3)) and

$$(3.6.111) \quad \frac{r_0}{\frac{2}{\sqrt{-\Lambda}}\varepsilon - \frac{1}{3}\Lambda r_0^3} < 1 - \frac{1}{2}\exp\left(-2(h_0(\varepsilon))^{-4}\right)$$

(from (3.5.6)), as well as the relation (3.5.21) for $\tilde{m}|_{\mathcal{I}}$, we can readily derive from (3.6.109) that:

$$(3.6.112) \quad \frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)})}{r_0} \geq 2(1 + (h_2(\varepsilon))^{1/2})^{-1} \exp(-2(h_0(\varepsilon))^{-4}) + \frac{1}{3} \Lambda r_0^2 (1 + (h_2(\varepsilon))^{1/2}).$$

The bound (3.6.10) follows readily from (3.6.111) and (3.6.112).

Proof of (3.6.13). For any $2 \leq n \leq n_f$, applying the relation (3.6.51) successively for $j = k + 1$ and $i = 1, 2, \dots, k$, using also the identity (3.6.94) and the trivial bound

$$(3.6.113) \quad (\mathfrak{D}_- \tilde{m})_n^{(i,j)} (1 - \mathfrak{Err}_n^{(i,k+1)}) \geq (\mathfrak{D}_+ \tilde{m})_n^{(i,k+1)}$$

(following directly from (3.6.54)), we obtain:

$$(3.6.114) \quad \begin{aligned} (\overline{\mathfrak{D}_+ \tilde{m}})_n^{(1,k+1)} &= (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \times \\ &\quad \times \exp \left(\sum_{i=1}^k \frac{2}{\bar{r}_n^{(i,k+1)}} \frac{(\mathfrak{D}_- \tilde{m})_n^{(i,k+1)}}{1 - \frac{2\tilde{m}_n^{(i+1,k+1)}}{\bar{r}_n^{(i,j)}} - \frac{1}{3} \Lambda (\bar{r}_n^{(i,k+1)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,k+1)}) (1 - \mathfrak{Err}_n^{(i,k+1)}) \right) \\ &\geq (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \cdot \exp \left(\sum_{i=1}^k \frac{2}{\bar{r}_n^{(i,k+1)}} \frac{(\mathfrak{D}_+ \tilde{m})_n^{(i,k+1)}}{1 - \frac{2\tilde{m}_n^{(i+1,k+1)}}{\bar{r}_n^{(i,j)}} - \frac{1}{3} \Lambda (\bar{r}_n^{(i,k+1)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,k+1)}) \right). \end{aligned}$$

In view of (3.6.7), (3.5.3), (3.6.53), (3.6.9) and the fact that

$$r_0 \leq \bar{r}_n^{(i,k+1)} \leq \bar{r}_n^{(k,k+1)},$$

for $1 \leq i \leq k$ (following from (3.6.18)), we can bound for any $1 \leq i \leq k$:

(3.6.115)

$$\begin{aligned}
& \frac{2}{\bar{r}_n^{(i,k+1)}} \frac{1}{1 - \frac{2\tilde{m}_n^{(i+1,k+1)}}{\bar{r}_n^{(i,j)}} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,k+1)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,k+1)}) \geq \\
& \geq 2 \min \left\{ \frac{1}{r \left| \left(U_n(v^{(i)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right) - 2\tilde{m}_n^{(i,j+1)} - \frac{1}{3}\Lambda r^3 \right| \left(U_n(v^{(i)}), V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right)} \right. \\
& \quad \left. \frac{1}{\bar{r}_n^{(i,j)} - 2\tilde{m}_n^{(i+1,k+1)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,k+1)})^3} \right\} \geq \\
& \geq \frac{2 - O(\varepsilon)}{r_n^{(k,k+1)}}.
\end{aligned}$$

Furthermore,

$$(3.6.116) \quad \sum_{i=1}^k (\mathfrak{D}_+ \tilde{m})_n^{(i,k+1)} = \sum_{i=1}^k (\tilde{m}_n^{(i+1,k+1)} - \tilde{m}_n^{(i,k+1)}) = \tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)}.$$

Therefore, in view of (3.6.10), (3.6.115), (3.6.116) and the fact that

$$(3.6.117) \quad (\overline{\mathfrak{D}_+ \tilde{m}})_n^{(1,k+1)} = \tilde{m}_n^{(1,k+1)} - \tilde{m}_n^{(1,k+2)} = \tilde{m}_n^{(1,k+1)},$$

the bound (3.6.114) yields

$$\begin{aligned}
(3.6.118) \quad \tilde{m}_n^{(1,k+1)} & \geq (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \cdot \exp \left(\frac{2 - O(\varepsilon)}{r_n^{(k,k+1)}} (\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)}) \right) \\
& \geq (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \cdot \exp \left(\frac{r_0}{2r_n^{(k,k+1)}} \exp(-2(h_0(\varepsilon))^{-4}) \right).
\end{aligned}$$

Using the bound (3.6.101) and the fact that

$$(\mathfrak{D}_+ \tilde{m})_{n-1}^{(0,1)} = \tilde{m}_{n-1}^{(1,1)} - \tilde{m}_{n-1}^{(0,1)} = \tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}$$

and

$$(\mathfrak{D}_- \tilde{m})_{n-1}^{(0,k)} = \tilde{m}_{n-1}^{(1,k+1)},$$

we can estimate:

$$(3.6.119) \quad (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \geq e^{-\varepsilon^{3/2}} \tilde{m}_{n-1}^{(1,k+1)}.$$

From (3.6.118) and (3.6.119) we thus obtain (in view also of (3.6.9) and the properties (3.5.2) of $h_0(\varepsilon)$):

$$(3.6.120) \quad \tilde{m}_n^{(1,k+1)} \geq \tilde{m}_{n-1}^{(1,k+1)} \exp\left(\frac{r_0}{4r_n^{(k,k+1)}} \exp(-2(h_0(\varepsilon))^{-4})\right).$$

In particular, (3.6.13) holds for all $2 \leq n \leq n_f$.

Proof of (3.6.12). Combining (3.6.102) and (3.6.11) (using also (3.6.107) in the case $n = 1$, as well as (3.5.21) and (3.5.6) for $\tilde{m}|_{\mathcal{I}}, r_0$), we can readily estimate for any $1 \leq n \leq n_f$ and any $1 \leq i \leq k$:

$$(3.6.121) \quad \frac{2(\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)}}{r_0} \geq \frac{1}{C_0} h_0(\varepsilon).$$

Similarly, in view of (3.6.94), (3.6.97) and (3.6.101), we can bound for any $1 \leq n \leq n_f$ and any $1 \leq i \leq k$:

$$(3.6.122) \quad \frac{2(\overline{\mathfrak{D}}_- \tilde{m})_n^{(i,k+1)}}{r_0} \geq \frac{1}{C_0} h_0(\varepsilon).$$

Using the relation

$$(3.6.123) \quad (\overline{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)} = \tilde{m}_n^{(i,k+1)} - \tilde{m}_n^{(i,k+2)}$$

and the trivial bounds

$$(3.6.124) \quad \tilde{m}_n^{(i,k+1)} \leq \tilde{m}|_{\mathcal{I}}$$

and

$$(3.6.125) \quad \max_{k+2 \leq j \leq k+i+1} \tilde{m}_n^{(i,j)} = \tilde{m}_n^{(i,k+2)}$$

(following from the monotonicity properties (3.6.19) of $\tilde{m}$), from (3.6.121) we obtain for any $1 \leq n \leq n_f$:

$$(3.6.126) \quad \min_{\substack{1 \leq i \leq k, \\ k+2 \leq j \leq k+i+1}} \frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(i,j)})}{r_0} \geq \frac{1}{C_0} h_0(\varepsilon).$$

In view of (3.5.21) and (3.5.6) and the properties (3.5.2) of $h_0(\varepsilon)$, from (3.6.126) we infer that, for any $1 \leq n \leq n_f$:

$$(3.6.127) \quad \max_{\substack{1 \leq i \leq k, \\ k+2 \leq j \leq k+i+1}} \frac{2\tilde{m}_n^{(i,j)}}{r_0} \leq 1 - \frac{1}{3C_0} h_0(\varepsilon).$$

In particular, for any $1 \leq n \leq n_f$, (3.6.127) implies that:

$$(3.6.128) \quad \sup_{\{u \leq v \leq V_n(v^{(k+1)})\} \cap \{u \geq U_n(v^{(k)})\}} \frac{1}{1 - \frac{2m}{r}} \leq 4C_0(h_0(\varepsilon))^{-1}.$$

The main estimate that will be used in the proof of (3.6.12) is the following bound: For any $1 \leq n_1 < n_2 \leq n_f$ and any $V_{n_2}(v^{(k+2)}) + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \leq v \leq V_{n_2}(v^{(k+1)})$:

$$(3.6.129)$$

$$\begin{aligned} \partial_v r \Big|_{(U_{n_2}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v)} &\geq \partial_v r \Big|_{(U_{n_1-1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v - (n_2 - n_1 - 1)v_0)}^\times \\ &\times \exp \left(-C_0^5(h_0(\varepsilon))^{-3} \max_{n_1 \leq n \leq n_2} \left\{ \frac{r_n^{(1,k+1)}}{r_0} \right\} \log \left(\frac{\tilde{m}_{n_2}^{(1,k+1)}}{\tilde{m}_{n_1}^{(1,k+1)}} \right) - 2\varepsilon^{1/2} \right). \end{aligned}$$

Proof of (3.6.129). For any $1 \leq n \leq n_f$, $1 \leq i \leq k$ and $k+1 \leq j \leq k+i$, integrating (3.2.43) from $u = U_n(v^{(i)})$ up to $U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))$ (and using (3.2.46)), we infer that, for all

$$V_n(v^{(j+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq \bar{v} \leq V_n(v^{(j)}):$$

$$\begin{aligned}
(3.6.130) \quad & \log\left(\frac{\partial_v r}{1 - \frac{2m}{r}}\right)\Big|_{(U_n(v^{(i)}), \bar{v})} - \log\left(\frac{\partial_v r}{1 - \frac{2m}{r}}\right)\Big|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), \bar{v})} \\
&= 4\pi \int_{U_n(v^{(i)})}^{U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \frac{rT_{uu}}{-\partial_u r}\Big|_{(u, \bar{v})} du \\
&= 2 \int_{U_n(v^{(i)})}^{U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \frac{-\partial_u \tilde{m}}{r(1 - \frac{2m}{r})}\Big|_{(u, \bar{v})} du.
\end{aligned}$$

In view of the monotonicity properties (3.6.18), (3.6.19) of r and $\tilde{m}$, we can estimate:

$$\begin{aligned}
(3.6.131) \quad & 2 \int_{U_n(v^{(i)})}^{U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \frac{-\partial_u \tilde{m}}{r(1 - \frac{2m}{r})}\Big|_{(u, \bar{v})} du \leq \\
& \leq \sup_{U_n(v^{(i)}) \leq \bar{u} \leq U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \left(\frac{2}{r(1 - \frac{2m}{r})}\right)\Big|_{(u, \bar{v})} \int_{U_n(v^{(i)})}^{U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} (-\partial_u \tilde{m})\Big|_{(u, \bar{v})} du \leq \\
& \leq \frac{2}{r(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), \bar{v})} \sup_{U_n(v^{(i)}) \leq \bar{u} \leq U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \left(\frac{1}{1 - \frac{2m}{r}}\right)\Big|_{(u, \bar{v})} (\mathfrak{D}_- \tilde{m})_n^{(i, j)}.
\end{aligned}$$

From (3.6.130), (3.6.131) and the bound (3.6.128) for $1 - \frac{2m}{r}$, we infer that

$$\begin{aligned}
(3.6.132) \quad & \log\left(\frac{\partial_v r}{1 - \frac{2m}{r}}\right)\Big|_{(U_n(v^{(i)}), \bar{v})} - \log\left(\frac{\partial_v r}{1 - \frac{2m}{r}}\right)\Big|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), \bar{v})} \leq \\
& \leq 8C_0(h_0(\varepsilon))^{-1} \frac{\bar{r}_n^{(i, j)}}{r(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), \bar{v})} \cdot \frac{(\mathfrak{D}_- \tilde{m})_n^{(i, j)}}{\bar{r}_n^{(i, j)}}.
\end{aligned}$$

Notice that, in view of the bound (3.6.56), we can estimate:

$$(3.6.133) \quad \frac{(\mathfrak{D}_- \tilde{m})_n^{(i, j)}}{\bar{r}_n^{(i, j)}} \leq 5C_0 \log\left(\frac{(\overline{\mathfrak{D}_+ \tilde{m}})_n^{(i, j)}}{(\overline{\mathfrak{D}_- \tilde{m}})_n^{(i, j)}}\right).$$

Thus, from (3.6.132) and (3.6.133), we deduce that, for any $1 \leq n \leq n_f$, $1 \leq i \leq k$ and $k+1 \leq j \leq k+i$ and any $V_n(v^{(j+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq \bar{v} \leq V_n(v^{(j)})$:

(3.6.134)

$$\begin{aligned} \log\left(\frac{\partial_v r}{1 - \frac{2m}{r}}\right)\Big|_{(U_n(v^{(i)}), \bar{v})} - \log\left(\frac{\partial_v r}{1 - \frac{2m}{r}}\right)\Big|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), \bar{v})} \leq \\ \leq 40C_0^2(h_0(\varepsilon))^{-1} \frac{\bar{r}_n^{(i,j)}}{r(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), \bar{v})} \log\left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}}{(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}}\right). \end{aligned}$$

Applying the relation (3.6.134) successively for $i = 1, \dots, k$ and $\bar{v} = v$, using also the fact that $\partial_u\left(\frac{\partial_v r}{1 - \frac{2m}{r}}\right) = 0$ on each $\mathcal{R}_{\varepsilon n}^{(i,j)}$, we obtain:

(3.6.135)

$$\begin{aligned} \frac{\partial_v r}{1 - \frac{2m}{r}}\Big|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), \bar{v})} \geq \frac{\partial_v r}{1 - \frac{2m}{r}}\Big|_{(U_{n-1}(v^{(0)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), \bar{v})} \times \\ \times \exp\left(-40C_0^2(h_0(\varepsilon))^{-1} \sum_{i=1}^k \frac{\bar{r}_n^{(i,k+1)}}{r(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), \bar{v})} \log\left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)}}{(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,k+1)}}\right)\right). \end{aligned}$$

For any $i = 1, \dots, k$, integrating (3.2.43) in u from $u = U_n(v^{(i)})$ up to $U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))$ for $v = v_* \in [\bar{v}, V_n(v^{(j)})]$ and using (3.2.46) and the fact that $\partial_u \tilde{m} = 0$ on $\mathcal{R}_{\varepsilon n}^{(i,j)}$, we infer:

$$\begin{aligned} (3.6.136) \quad \frac{\partial_v r}{1 - \frac{2m}{r}}\Big|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), v_*)} = \frac{\partial_v r}{1 - \frac{2m}{r}}\Big|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), v_*)} \times \\ \times \exp\left(-2 \int_{U_n(v^{(i)})}^{U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \frac{-\partial_u \tilde{m}}{r - 2\tilde{m} - \frac{1}{3}\Lambda r^3}\Big|_{(u, v_*)} du\right). \end{aligned}$$

Using the fact that $r \geq r_0$, from (3.6.136) we infer (in view of the monotonicity property

(3.6.19) for $\tilde{m}$) that

(3.6.137)

$$\begin{aligned}
\frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v_*)} &\geq \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v_*)} \times \\
&\times \exp \left(-2 \int_{U_n(v^{(i)})}^{U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))} \frac{-\partial_u \tilde{m}|_{(u, v_*)}}{r_0 - 2\tilde{m}|_{(u, v_*)} - \frac{1}{3} \Lambda r_0^3} du \right) \\
&= \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v_*)} \times \\
&\times \exp \left(- \int_{U_n(v^{(i)})}^{U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon))} \partial_u \left(\log \left(1 - \frac{2\tilde{m}|_{(u, v_*)}}{r_0} - \frac{1}{3} \Lambda r_0^2 \right) \right) du \right) \\
&= \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v_*)} \cdot \left(\frac{1 - \frac{2\tilde{m}|_{(U_n(v^{(i)}), v_*)}}{r_0} - \frac{1}{3} \Lambda r_0^2}{1 - \frac{2\tilde{m}|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v_*)}}{r_0} - \frac{1}{3} \Lambda r_0^2} \right).
\end{aligned}$$

In view of the bounds (3.6.9) and (3.6.127), (3.6.137) yields:

$$(3.6.138) \quad \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v_*)} \geq \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v_*)} \frac{h_0(\varepsilon)}{4C_0}.$$

Integrating (3.6.138) in $v_* \in [\bar{v}, V_n(v^{(j)})]$ and using (3.6.128) (and (3.6.9)) for the $\frac{1}{1 - \frac{2m}{r}}$ factors, we thus obtain:

$$(3.6.139) \quad \bar{r}_n^{(1, k+1)} - r_0 \geq \left(\bar{r}_n^{(i, k+1)} - r(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), \bar{v}) \right) \frac{(h_0(\varepsilon))^2}{16C_0^2}$$

and, thus (in view of (3.6.7), (3.5.3) and the fact that $r_0 \leq \min\{r_n^{(1, k+1)}, r(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), \bar{v})\}$):

$$(3.6.140) \quad \frac{\bar{r}_n^{(i, k+1)}}{r(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), \bar{v})} \leq 16C_0^2 (h_0(\varepsilon))^{-2} \frac{r_n^{(1, k+1)}}{r_0}.$$

From (3.6.140) and (3.6.94), it follows that:

$$\begin{aligned}
(3.6.141) \quad & \sum_{i=1}^k \frac{\bar{r}_n^{(i,k+1)}}{r(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), v)} \log \left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)}}{(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,k+1)}} \right) \\
& \leq 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \sum_{i=1}^k \log \left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)}}{(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,k+1)}} \right) \\
& = 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \left\{ \sum_{i=1}^{k-1} \log \left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)}}{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i+1,k+1)}} \right) + \log \left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(k,k+1)}}{\tilde{m}|_{\mathcal{I}} - \tilde{m}_{n-1}^{(0,1)}} \right) \right\} \\
& = 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \log \left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(1,k+1)}}{\tilde{m}|_{\mathcal{I}} - \tilde{m}_{n-1}^{(0,1)}} \right) \\
& = 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \log \left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(1,k+1)}}{(\bar{\mathfrak{D}}_+ \tilde{m})_{n-1}^{(0,1)}} \right) \\
& \leq 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \left\{ \log \left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(1,k+1)}}{(\bar{\mathfrak{D}}_- \tilde{m})_{n-1}^{(0,k)}} \right) + \varepsilon^{3/2} \right\} \\
& = 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \left\{ \log \left(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \right) + \varepsilon^{3/2} \right\}
\end{aligned}$$

(where the inequality at the sixth line of (3.6.141) follows from (3.6.101)). Therefore,

(3.6.135) and (3.6.141) yield for any $1 \leq n \leq n_f$ and any $V_n(v^{(k+2)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq v \leq V_n(v^{(k+1)})$:

$$\begin{aligned}
(3.6.142) \quad & \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), v)} \geq \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_{n-1}(v^{(0)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), v)} \times \\
& \times \exp \left(-C_0^5(h_0(\varepsilon))^{-3} \frac{r_n^{(1,k+1)}}{r_0} \left\{ \log \left(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \right) + \varepsilon^{3/2} \right\} \right).
\end{aligned}$$

In view of (3.2.44), we can bound for any $U_{n-1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq u \leq U_{n-1}(v^{(0)})$:

$$(3.6.143) \quad \frac{-\partial_u r}{1 - \frac{2m}{r}} \Big|_{(u, V_{n-1}(v^{(1)}))} \geq \frac{-\partial_u r}{1 - \frac{2m}{r}} \Big|_{(u, V_{n-1}(v^{(k+1)}))}.$$

Hence, using the fact that

- From (3.5.36), (3.6.9):

$$(3.6.144) \quad \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v)} = \frac{\partial_v r}{1 - \frac{1}{3} \Lambda r^2} \Big|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v)} = \\ = (1 + O(\varepsilon)) \partial_v r \Big|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v)},$$

- From (3.2.45), (3.5.21), (3.6.8) and (3.6.7)

$$(3.6.145) \quad \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_{n-1}(v^{(0)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v)} = \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{(U_{n-1}(v^{(0)}), v)} (1 + O(\varepsilon)),$$

- From (3.2.44), (3.2.43) and the gauge condition (3.3.24):

$$(3.6.146) \quad \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{\mathcal{R}_{n-1}^{(1,1)} \cap \{v=\bar{v}\}} = \frac{-\partial_u r}{1 - \frac{2m}{r}} \Big|_{\mathcal{R}_{n-1}^{(1,1)} \cap \{u=\bar{v}-v_0\}},$$

the bounds (3.6.142) and (3.6.143) yield for any $V_n(v^{(k+2)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \leq v \leq V_n(v^{(k+1)})$:

$$(3.6.147) \quad \partial_v r \Big|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)), v)} \geq \frac{-\partial_u r}{1 - \frac{2m}{r}} \Big|_{(v-v_0, V_{n-1}(v^{(k+1)}))} \times \\ \times \exp \left(-C_0^5(h_0(\varepsilon))^{-3} \frac{r_n^{(1,k+1)}}{r_0} \log \left(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \right) - \varepsilon^{1/2} \right).$$

In view of (3.5.36), (3.6.9) and the fact that

$$(3.6.148) \quad \frac{-\partial_u r}{1 - \frac{2m}{r}} \Big|_{\mathcal{R}_{n-1}^{(1,k+2)} \cap \{u=\bar{u}\}} = \frac{\partial_v r}{1 - \frac{2m}{r}} \Big|_{\mathcal{R}_{n-1}^{(1,k+2)} \cap \{v=\bar{u}\}}$$

(following from the gauge condition (3.3.23)), from (3.6.147) we infer for any

$$V_n(v^{(k+2)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq v \leq V_n(v^{(k+1)}):$$

$$(3.6.149) \quad \partial_v r|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), v)} \geq \partial_v r|_{(U_{n-1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), v-v_0)} \times \\ \times \exp\left(-C_0^5(h_0(\varepsilon))^{-3} \frac{r_n^{(1,k+1)}}{r_0} \log\left(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}}\right) - 2\varepsilon^{1/2}\right).$$

Iterating (3.6.149) for $n_1 < n \leq n_2$, we thus obtain (3.6.129).

Let $2 \leq n_1 \leq n_f$ be such so that

$$(3.6.150) \quad r_{n_1}^{(1,k+1)} \leq 2r_0$$

and

$$(3.6.151) \quad r_{n_1-1}^{(1,k+1)} > 2r_0.$$

Note that if no such n_1 exists, then (3.6.12) is automatically true ((3.6.151) holds for $r_1^{(1,k+1)}$ as a corollary of the Cauchy stability estimates of Proposition 3.3 and the choice of the initial data).

Let us also define

$$(3.6.152) \quad n_2 = \max\{n_1 \leq n \leq n_f : r_l^{(1,k+1)} \leq 2r_0 \text{ for all } n_1 \leq l \leq n\}.$$

In order to establish (3.6.12), it suffices to establish that, for all $n_1 \leq n \leq n_2$:

$$(3.6.153) \quad \frac{r_n^{(1,k+1)}}{r_0} - 1 \geq \exp\left(-C_0^7(h_0(\varepsilon))^{-3} \log((h_0(\varepsilon))^{-1})\right).$$

In view of the fact that

$$(3.6.154) \quad \sup_{1 \leq l_1 < l_2 \leq n_*} \frac{\tilde{m}_{l_2}^{(1,k+1)}}{\tilde{m}_{l_1}^{(1,k+1)}} \leq C_0(h_0(\varepsilon))^{-1}$$

(following from (3.5.7), (3.6.86) and the fact that the sequence $\tilde{m}_n^{(1,k+1)}$ is increasing in n as a consequence of (3.6.120)), from (3.6.129) we infer that, for any $n_1 \leq n \leq n_2$ and any

$$V_{n_2}(v^{(k+2)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq v \leq V_{n_2}(v^{(k+1)}):$$

$$(3.6.155) \quad \partial_v r \Big|_{(U_n(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), v)} \geq \partial_v r \Big|_{(U_{n_1-1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), v - (n - n_1)v_0)} \times \\ \times \exp \left(-C_0^6(h_0(\varepsilon))^{-3} \log((h_0(\varepsilon))^{-1}) \right).$$

Thus, integrating (3.6.155) from $v = V_n(v^{(k+2)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))$ up to $v = V_n(v^{(k+1)})$ and using (3.6.151), we immediately infer (3.6.153).

Proof of (3.6.14). In view of (3.5.3), (3.6.32), (3.6.31), as well as the boundary condition (3.3.24) and the bounds (3.6.8) and (3.6.9), the following one-sided bound holds for all $2 \leq n \leq n_f$:

$$(3.6.156) \quad r_n^{(k,k+1)} - r_0 = \int_{V_n(v^{(2k)})}^{V_n(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \partial_v r \Big|_{(U_n(v^{(k)}), v)} dv \\ \leq \int_{V_n(v^{(2k)})}^{V_n(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \frac{\partial_v r}{1 - \frac{2\tilde{m}}{r}} \Big|_{(U_n(v^{(k)}), v)} \left(1 + O(\varepsilon)\right) dv \\ = \int_{U_{n-1}(v^{(k)})}^{U_{n-1}(v^{(0)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \frac{-\partial_u r}{1 - \frac{2\tilde{m}}{r}} \Big|_{(u, V_{n-1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)))} \left(1 + O(\varepsilon)\right) du.$$

We can readily compute (using also (3.5.3), (3.6.9) and (3.6.45)):

$$(3.6.157) \quad \int_{U_{n-1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))}^{U_{n-1}(v^{(0)})} \frac{-\partial_u r}{1 - \frac{2\tilde{m}}{r}} \Big|_{(u, V_{n-1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)))} du \\ = \int_{U_{n-1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))}^{U_{n-1}(v^{(0)})} \left(1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r \Big|_{(u, V_{n-1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)))}} - \frac{1}{3} \Lambda r^2 \Big|_{(u, V_{n-1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)))}\right)^{-1} \times \\ \times (-\partial_u r) \Big|_{(u, V_{n-1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)))} du \\ = \int_{r_0 + O((h_2(\varepsilon))^{1/2})}^{r_{n-1}^{(1,k+1)}} \left(1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r} - \frac{1}{3} \Lambda r^2\right)^{-1} dr \\ \leq r_{n-1}^{(1,k+1)} - r_0 + C_0 \tilde{m}_{n-1}^{(1,k+1)} \left| \log \left(1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r_0}\right) \right|.$$

From (3.5.6) and (3.6.12) we can similarly estimate:

$$\begin{aligned}
(3.6.158) \quad & \int_{U_{n-1}(v^{(k)})}^{U_{n-1}(v^{(1)})} \left| \frac{-\partial_u r}{1 - \frac{2\tilde{m}}{r}} \right|_{(u, V_{n-1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)))} du \leq \\
& \leq \int_{r_{n-1}^{(1,k+1)} + O((h_2(\varepsilon))^{1/2})}^{r_{n-1}^{(k,k+1)}} \left(1 - \frac{2\tilde{m}|_{\mathcal{I}}}{r} + O(\varepsilon) \right)^{-1} dr \leq \\
& \leq r_{n-1}^{(k,k+1)} + C_0 \tilde{m}|_{\mathcal{I}} \left| \log \left(\exp(h_0(\varepsilon))^{-4} + 1 \right) \right| \leq \\
& \leq r_{n-1}^{(k,k+1)} - r_{n-1}^{(1,k+1)} + C_0 \tilde{m}|_{\mathcal{I}} (h_0(\varepsilon))^{-4}.
\end{aligned}$$

From (3.5.3), (3.6.45), (3.6.156), (3.6.157) and (3.6.158) one readily obtains the bound (3.6.14). $\square$

3.6.2 Formation of a nearly-trapped sphere

In this section, we will establish (3.6.6), using the bounds (3.6.7)–(3.6.14) of Proposition 3.4.

Let us set

$$(3.6.159) \quad n_{max} \doteq \max \{ n_* \in \mathbb{N} : \mathcal{R}_n^{(1,k+1)} \subset \mathcal{U}_\varepsilon^+ \text{ for all } n \leq n_* \}.$$

Note that, in view of (3.6.2) and (3.6.5), n_{max} satisfies

$$(3.6.160) \quad n_f \leq n_{max} \leq n_f + 1.$$

Thus, (3.6.1) implies that

$$(3.6.161) \quad n_{max} \leq (h_1(\varepsilon))^{-2}.$$

Notice that (3.6.160) and the definition (3.6.2) imply that, if

$$(3.6.162) \quad n_{max} < \frac{1}{2} (h_1(\varepsilon))^{-2},$$

then, necessarily, (3.6.6) holds. Thus, in order to establish (3.6.6), it suffices to show

(3.6.162).

We will show (3.6.162) by applying Lemma 3.2 (see Section 3.6.4). In particular, setting for any $1 \leq n \leq n_{max} + 1$

$$(3.6.163) \quad \mu_n \doteq \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r_0},$$

$$(3.6.164) \quad \varrho_n \doteq \frac{r_{n-1}^{(k,k+1)}}{r_0},$$

the inductive bounds (3.6.13) and (3.6.14) imply that μ_n, ϱ_n satisfy

$$(3.6.165) \quad \begin{aligned} \varrho_{n+1} &\leq \varrho_n + C_1 \log \left((1 - \mu_n)^{-1} + 1 \right), \\ \mu_{n+1} &\geq \mu_n \exp \left(\frac{c_1}{\varrho_{n+1}} \right), \end{aligned}$$

for any $1 \leq n \leq n_{max} + 1$, with

$$(3.6.166) \quad C_1 = (h_0(\varepsilon))^{-4}$$

and

$$(3.6.167) \quad c_1 = \frac{1}{16} \exp \left(-2(h_0(\varepsilon))^{-4} \right).$$

Furthermore, setting

$$(3.6.168) \quad \delta = h_3(\varepsilon)$$

(where $h_3(\cdot)$ is defined by (3.6.3)), the definition (3.6.159) immediately implies that

$$(3.6.169) \quad \max_{0 \leq n \leq n_{max}} \mu_n < 1 - \delta.$$

Note that, in view of Definition 3.4 and Proposition 3.3, we have

$$(3.6.170) \quad \mu_0 \sim h_0(\varepsilon)$$

and

$$(3.6.171) \quad \varrho_0 \sim (h_1(\varepsilon))^{-1}.$$

Hence, as a consequence of (3.5.2) and (3.6.3),

$$(3.6.172) \quad \left(\frac{C_1}{c_1}\right)^8 \ll \frac{\varrho_0}{\mu_0}$$

and

$$(3.6.173) \quad \delta < \left(\frac{\mu_0}{\varrho_0}\right)^{(C_1/c_1)^4}.$$

The relations (3.6.165)–(3.6.173) allow us to apply Lemma 3.2 (see Section 3.6.4) with $n_* = n_{max} + 1$ for the sequence ϱ_n, μ_n . Thus, in view of Lemma 3.2, we obtain the following upper bound for n_{max} :

$$(3.6.174) \quad n_{max} + 1 \leq \exp \left(\exp \left(2(h_0(\varepsilon))^{-4} \right) \right) (h_1(\varepsilon))^{-1}.$$

In particular, (3.6.162) (and, thus, (3.6.6)) holds.

3.6.3 The final step of the evolution

In this section, we will complete the proof of Theorem 3.1, using the near-trapping bound (3.6.6), the bounds (3.6.7)–(3.6.14) of Proposition 3.4, as well a backwards-in-time Cauchy stability estimate (see Lemma 3.3 in Section 3.6.4).

The bound (3.6.6), combined with the estimates (3.6.11) and (3.6.12) of Proposition 3.4, imply that, necessarily (in view also of (3.5.2), (3.5.3), (3.6.7), (3.6.13) and (3.6.159)):

$$(3.6.175) \quad \frac{2\tilde{m}_{n_{max}+1}^{(1,k+1)}}{r_0} \geq 1 - 2h_3(\varepsilon).$$

Therefore, applying again Lemma 3.2 for μ_n, ϱ_n (defined by (3.6.163), (3.6.164)) and

$n_* = n_{max} + 1$ yields, in view of (3.6.175), that, either

$$(3.6.176) \quad \mu_{n_{max}+1} > 1 + h_3(\varepsilon),$$

or

$$(3.6.177) \quad 1 - 2h_3(\varepsilon) \leq \mu_{n_{max}+1} \leq 1 + h_3(\varepsilon)$$

and

$$(3.6.178) \quad \mu_{n_{max}} \leq 1 - \exp\left(-\exp\left(2(h_0(\varepsilon))^{-4}\right)\right)(h_1(\varepsilon))^2$$

$$(3.6.179) \quad \max\{\rho_{n_{max}+1}, \rho_{n_{max}}\} \leq \exp\left(\exp\left(2(h_0(\varepsilon))^{-4}\right)\right)(h_1(\varepsilon))^{-1} \log\left((h_1(\varepsilon))^{-1}\right).$$

Let us set

$$(3.6.180) \quad \bar{v}_* \doteq V_{n_{max}+1}\left(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right)$$

(recall that (3.6.180) equals $V_{n_{max}}\left(v^{(0)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right)$, in view of our conventions on the indices). The proof of Theorem 3.1 will follow by showing that

- Either

$$(3.6.181) \quad \inf_{\mathcal{U}_\varepsilon \cap \{v=\bar{v}_*\}} \left(1 - \frac{2m}{r}\right) < 0$$

(in which case $(r'_j, (\Omega'_j)^2, \bar{f}'_{in/j}, \bar{f}'_{out/j}) = (r_{/j}, \Omega_{/j}^2, \bar{f}_{in/j}, \bar{f}_{out/j})$ in the statement of Theorem 3.1).

- Or

$$(3.6.182) \quad \inf_{\mathcal{U}_\varepsilon \cap \{v=\bar{v}_*\}} \left(1 - \frac{2m'}{r'}\right) < 0,$$

where $(r', (\Omega')^2, \bar{f}'_{in}, \bar{f}'_{out})$ is a (possibly different) smooth solution to the system (3.2.28)–(3.2.33) arising as a future development of an asymptotically AdS boundary-

characteristic initial data set $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{in/\varepsilon}, \bar{f}'_{out/\varepsilon})$ on $\{u = 0\} \cap \{0 \leq v \leq v_{0\varepsilon}\}$ (satisfying the reflecting gauge condition at $r = r_0, +\infty$) which is $(h_1(\varepsilon))^2$ close to $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ with respect to the norm (3.3.41), i.e. satisfies, in particular, (3.6.279) and (3.6.280) (in which case $(r_{/\varepsilon}^{(\varepsilon)}, (\Omega_{/\varepsilon}^{(\varepsilon)})^2, \bar{f}_{in/\varepsilon}^{(\varepsilon)}, \bar{f}_{out/\varepsilon}^{(\varepsilon)})$ equals $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{in/\varepsilon}, \bar{f}'_{out/\varepsilon})$ in the statement of Theorem 3.1).

Notice that, in both cases, (3.4.3) follows readily from (3.5.15) and (3.5.2). To this end, we will proceed to treat the cases (3.6.176) and (3.6.177) separately.

Case I. Assume that (3.6.176) holds. Then, we will show that (3.6.181) also holds. We will argue by contradiction, assuming that

$$(3.6.183) \quad \inf_{\mathcal{U}_\varepsilon \cap \{v=\bar{v}_*\}} \left(1 - \frac{2m}{r}\right) \geq 0.$$

Let us set

$$(3.6.184) \quad \mathcal{C}_* \doteq \{U_{n_{max}+1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq u < U_{n_{max}+1}(v^{(0)})\} \cap \{v = \bar{v}_*\} \cap \mathcal{U}_\varepsilon.$$

The renormalised mass $\tilde{m}$ is constant on $\mathcal{C}_*$, satisfying in particular

$$(3.6.185) \quad \tilde{m}|_{\mathcal{C}_*} = \tilde{m}_{n_{max}+1}^{(1,k+1)}.$$

Since $\partial_u r < 0$ on $\mathcal{U}_\varepsilon$ (see (3.3.29)), from (3.6.183) and the fact that $\mathcal{C}_*$ does not contain its future endpoint, we infer the following stronger bound:

$$(3.6.186) \quad 1 - \frac{2m}{r}|_{\mathcal{C}_*} > 0.$$

Thus, we also have

$$(3.6.187) \quad \partial_v r|_{\mathcal{C}_*} > 0.$$

We will now show that the future endpoint of $\mathcal{C}_*$ is exactly $(U_{n_{max}+1}(v^{(0)}), \bar{v}_*)$. If there existed some $(u_b, \bar{v}_*) \in (\partial\mathcal{U}_\varepsilon \setminus \mathcal{I})$ such that $U_{n_{max}+1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq u_b < U_{n_{max}+1}(v^{(0)})$,

then Theorem 3.3 on the structure of the maximal future development would imply that r extends continuously on $(u_b, \bar{v}_*)$ with

$$(3.6.188) \quad r(u_b, \bar{v}_*) = r_{0\varepsilon}.$$

However, in that case, (3.6.176), (3.6.185) and (3.6.188) would imply that, for some u_{b*} close enough to u_b

$$(3.6.189) \quad 1 - \frac{2m}{r} \Big|_{(u_{b*}, \bar{v}_*)} < 0,$$

which is a contradiction in view of (3.6.183). Therefore,

$$\{U_{n_{max}+1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq u < U_{n_{max}+1}(v^{(0)})\} \cap \{v = \bar{v}_*\} \cap (\partial\mathcal{U}_\varepsilon \setminus \mathcal{I}) = \emptyset,$$

and, thus

$$(3.6.190) \quad \mathcal{C}_* = \{U_{n_{max}+1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq u < U_{n_{max}+1}(v^{(0)})\} \cap \{v = \bar{v}_*\}.$$

In order to complete the proof in the case when (3.6.176) holds, it suffices to establish that

$$(3.6.191) \quad \limsup_{\bar{u} \rightarrow U_{n_{max}+1}(v^{(0)})} \frac{r|_{(\bar{u}, \bar{v}_*)}}{r_0} \leq 1 + O((h_2(\varepsilon))^{1/2}).$$

Assuming that (3.6.191) holds, from (3.6.176), (3.6.185) and (3.6.191) (in view also of (3.5.3), (3.6.3)) we readily obtain

$$(3.6.192) \quad \liminf_{\bar{u} \rightarrow U_{n_{max}+1}(v^{(0)})} \left(1 - \frac{2m}{r}\right) \Big|_{(\bar{u}, \bar{v}_*)} < -\frac{1}{2}h_3(\varepsilon) < 0,$$

which is a contradiction in view of (3.6.183).

Let us set

$$(3.6.193) \quad \mathcal{B}_* \doteq \{U_{n_{max}+1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq u < U_{n_{max}+1}(v^{(0)})\} \cap \{V_{n_{max}+1}(v^{(k)}) \leq v \leq \bar{v}_*\}.$$

From (3.6.190) and the structure of the maximal future development of general initial data sets for (3.2.28)–(3.2.33) (see Theorem 3.3), we infer that

$$\mathcal{B}_* \subset \mathcal{U}_\varepsilon.$$

Furthermore, in view of (3.2.30) and (3.6.187), we infer that

$$(3.6.194) \quad \partial_v r|_{\mathcal{B}_*} > 0$$

and, thus (in view of (3.3.29)):

$$(3.6.195) \quad 1 - \frac{2m}{r}|_{\mathcal{B}_*} > 0.$$

In view of (3.6.159) and the bounds (3.6.11) and (3.6.12), we have

$$(3.6.196) \quad \{u \leq U_{n_{max}+1}(v^{(1)})\} \cap \mathcal{U}_\varepsilon \subset \mathcal{U}_\varepsilon^+.$$

Therefore, as a consequence of (3.6.7), we can estimate

$$(3.6.197) \quad \log \left(\frac{\partial_v r}{1 - \frac{2m}{r}} \right) \Big|_{\{u=U_{n_{max}+1}(v^{(1)})\} \cap \mathcal{U}_\varepsilon} \leq (h_1(\varepsilon))^{-4} \log((h_3(\varepsilon))^{-1}).$$

Since (3.2.43) implies that

$$(3.6.198) \quad \partial_u \log \left(\frac{\partial_v r}{1 - \frac{2m}{r}} \right) \leq 0,$$

from (3.6.197) (and (3.6.9)) we infer the one sided bound:

$$(3.6.199) \quad \partial_v r|_{\mathcal{B}_*} \leq 2(h_1(\varepsilon))^{-4} \log((h_3(\varepsilon))^{-1}).$$

Integrating (3.6.199) from $v = V_{n_{max}+1}(v^{(k)})$ up to $V_{n_{max}+1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))$ using (3.5.3), we finally obtain (3.6.191). Thus, the proof in the case when (3.6.176) holds is complete.

Case II. Assume that (3.6.177) holds. Then, (3.6.178) and (3.6.179) also hold.

As a consequence of (3.6.11), (3.6.12) and (3.6.13), the bound (3.6.178) implies that

$$(3.6.200) \quad \inf_{\{u \leq U_{n_{max}}(v^{(0)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\} \cap \mathcal{U}_\varepsilon} \left(1 - \frac{2\tilde{m}}{r}\right) \geq \frac{1}{2} \exp\left(-\exp\left(2(h_0(\varepsilon))^{-4}\right)\right)(h_1(\varepsilon))^2.$$

Therefore, using (3.6.11), (3.6.12) and (3.6.13) to estimate $(1 - \frac{2\tilde{m}}{r})$ in the region

$$\{U_{n_{max}}(v^{(0)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq u \leq U_{n_{max}+1}(v^{(0)})\} \setminus \mathcal{R}_{n_{max}+1}^{(1,k+1)},$$

we infer that

$$(3.6.201) \quad \inf_{\{u \leq U_{n_{max}+1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\} \cap \mathcal{U}_\varepsilon} \left(1 - \frac{2\tilde{m}}{r}\right) \geq \frac{1}{2} \exp\left(-\exp\left(2(h_0(\varepsilon))^{-4}\right)\right)(h_1(\varepsilon))^2.$$

Remark. Notice that, while $1 - \frac{2\tilde{m}}{r}$ becomes $\sim h_3(\varepsilon)$ in $\{u \leq U_{n_{max}+1}(v^{(0)})\} \cap \mathcal{U}_\varepsilon$ (in view of (3.6.177)), when restricting to the subregion $\{u \leq U_{n_{max}+1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))\} \cap \mathcal{U}_\varepsilon$, the improved bound (3.6.201) holds.

Let us set

$$(3.6.202) \quad u_* \doteq U_{n_{max}+1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)),$$

noticing that

$$(3.6.203) \quad \text{supp}(r^2 T_{vv}) \cap \{u = u_*\} \subset \{r \leq \varepsilon^{1/2}\}$$

as a consequence of (3.6.8). Let us also fix a smooth cut-off function $\chi_\varepsilon : [u_*, u_* + v_{0\varepsilon}) \rightarrow [0, 1]$

such that

$$(3.6.204) \quad \chi_\varepsilon(v) = 1 \text{ for } v \in [V_{n_{max}+1}(v^{(k+1)}), V_{n_{max}+1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))]$$

and

$$\chi_\varepsilon(v) = 0 \text{ for}$$

$$v \in [u_*, u_* + v_{0\varepsilon}) \setminus [V_{n_{max}+1}(v^{(k+1)} - \frac{1}{\sqrt{-\Lambda}}h_2(\varepsilon)), V_{n_{max}+1}(v^{(k+1)} + \frac{5}{\sqrt{-\Lambda}}h_2(\varepsilon))].$$

We will then define the function $\tilde{T}_{vv} : [u_*, u_* + v_{0\varepsilon}) \rightarrow \mathbb{R}$ by the relation

$$(3.6.205) \quad \tilde{T}_{vv}(v) \doteq \exp\left(-2C_\varepsilon^2 \frac{u_*}{v_0}\right)(h_1(\varepsilon))^2 \chi_\varepsilon(v) T_{vv}(u_*, v),$$

where C_ε is defined by (3.6.273). Notice that, since

$$(3.6.206) \quad 2\pi \int_{V_{n_{max}+1}(v^{(k+1)})}^{V_{n_{max}+1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \frac{(1 - \frac{2m}{r})}{\partial_v r} r^2 T_{vv} \Big|_{(u_*, v)} dv = \tilde{m}_{n_{max}+1}^{(1,k+1)} - \tilde{m}_{n_{max}+1}^{(1,k+2)} = \tilde{m}_{n_{max}+1}^{(1,k+1)},$$

we can readily bound in view of (3.6.205), (3.6.12), (3.6.177) and (3.6.206):

$$(3.6.207) \quad \sup_{u_* \leq \bar{v} \leq u_* + v_{0\varepsilon}} (-\Lambda) \int_{u_*}^{u_* + v_{0\varepsilon}} \frac{r^2(u_*, v) \frac{|\tilde{T}_{vv}(v)|}{\partial_v \rho(u_*, v)}}{|\rho(u_*, v) - \rho(u_*, \bar{v})| + \rho(u_*, u_*)} dv \leq \exp\left(-C_\varepsilon^2 \frac{u_*}{v_{0\varepsilon}}\right)(h_1(\varepsilon))^2,$$

where ρ is defined in terms of r by the relation

$$(3.6.208) \quad \rho \doteq \tan^{-1}\left(\sqrt{-\frac{\Lambda}{3}}r\right).$$

Applying the backwards-in-time Cauchy stability lemma 3.3 (see Section 3.6.4), for u_* given by (3.6.202) and $\tilde{T}_{vv}$ given by (3.6.205) (in view of (3.6.201), (3.6.203) and (3.6.207)), we infer that there exists an smooth asymptotically AdS boundary-characteristic initial data set $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{in/\varepsilon}, \bar{f}'_{out/\varepsilon})$ on $\{u = 0\}$ for the system (3.2.28)–(3.2.33) satisfying the reflecting gauge condition at $r = r_{0\varepsilon}, +\infty$ with the following properties:

1. The initial data sets $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ and $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{in/\varepsilon}, \bar{f}'_{out/\varepsilon})$ satisfy (3.6.279) and (3.6.280).
2. The maximal development $(\mathcal{U}'_\varepsilon; r', (\Omega')^2, \bar{f}'_{in}, \bar{f}'_{out})$ of $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{in/\varepsilon}, \bar{f}_{out/\varepsilon})$ satisfies (3.6.281), (3.6.282) and (3.6.283).

Using primes to denote quantities associated to $(r', (\Omega')^2, \bar{f}'_{in}, \bar{f}'_{out})$, we can readily estimate in view of (3.6.281), (3.6.282), (3.6.283) and (3.6.205):

$$\begin{aligned}
(3.6.209) \quad \tilde{m}'|_{(u_*, \bar{v}_*)} &= \int_{V_{n_{max}+1}(v^{(k+1)})}^{V_{n_{max}+1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \frac{(1 - \frac{2m'}{r'})}{\partial_v r'} (r')^2 T'_{vv} \Big|_{(u_*, v)} dv \\
&= \int_{V_{n_{max}+1}(v^{(k+1)})}^{V_{n_{max}+1}(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon))} \frac{(1 - \frac{2m'}{r})}{\partial_v r} r^2 (T_{vv} + \tilde{T}_{vv}) \Big|_{(u_*, v)} dv \\
&\geq (1 + \exp(-2C_\varepsilon^2 \frac{u_*}{v_0})(h_1(\varepsilon))^2) \tilde{m}|_{(u_*, \bar{v}_*)}.
\end{aligned}$$

Therefore, since $\tilde{m}|_{(u_*, \bar{v}_*)} = \tilde{m}_{n_{max}+1}^{(1, k+1)}$, the bound (3.6.177) (in view also of (3.6.3) and (3.6.273)) implies that

$$(3.6.210) \quad \frac{2\tilde{m}'|_{(u_*, \bar{v}_*)}}{r_0} \geq (1 + \exp(-2C_\varepsilon^2 \frac{u_*}{v_0})(h_1(\varepsilon))^2)(1 - 2h_3(\varepsilon)) \geq 1 + h_3(\varepsilon).$$

Since $\tilde{m}|_{(u_*, \bar{v}_*)}$ is constant on

$$\mathcal{C}'_* \doteq \{U_{n_{max}+1}(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)) \leq u < U_{n_{max}+1}(v^{(0)})\} \cap \{v = \bar{v}_*\} \cap \mathcal{U}'_\varepsilon$$

and satisfies (3.6.210), we can now repeat the same arguments as in Case I (i.e. the case when (3.6.176) holds) in order to infer that (3.6.182) holds.

Thus, the proof of Theorem 3.1 is complete. $\square$

3.6.4 Some auxiliary lemmas

In this section, we will prove some lemmas necessary for the proof of Proposition 3.4 and Theorem 3.1.

A maximum principle for 1 + 1 wave-type equations

The following lemma provides a comparison inequality for certain 1 + 1 equations of wave type, and is used in the proof of Proposition 3.4.

Lemma 3.1. *For any $u_0 < u_1$, $v_0 < v_1$ and $a \in \mathbb{R}$, let*

$F_1, F_2 : [u_0, u_1] \times [v_0, v_1] \times (-\infty, a] \rightarrow (0, +\infty)$ be smooth functions so that

$$(3.6.211) \quad \max_{(u,v) \in [u_0, u_1] \times [v_0, v_1]} F_1(u, v, z) < \min_{(u,v) \in [u_0, u_1] \times [v_0, v_1]} F_2(u, v, z)$$

for any $z \in (-\infty, a]$ and

$$(3.6.212) \quad \partial_z F_1(u, v, z), \partial_z F_2(u, v, z) \geq 0$$

for any $(u, v, z) \in [u_0, u_1] \times [v_0, v_1] \times (-\infty, a]$. Suppose also that

$z_1, z_2 : [u_0, u_1] \times [v_0, v_1] \rightarrow (-\infty, a]$ are smooth solutions to the equations

$$(3.6.213) \quad \partial_v \partial_u z_1 = -F_1(u, v, z_1) \partial_u z_1 \partial_v z_1$$

and

$$(3.6.214) \quad \partial_v \partial_u z_2 = -F_2(u, v, z_2) \partial_u z_2 \partial_v z_2,$$

satisfying the same characteristic initial data

$$(3.6.215) \quad z_1(u, v_0) = z_2(u, v_0) = z_\backslash(u),$$

$$(3.6.216) \quad z_1(u_0, v) = z_2(u_0, v) = z_/(v).$$

where $z_/: [v_0, v_1] \rightarrow (-\infty, a)$ and $z_\backslash : [u_0, u_1] \rightarrow (-\infty, a)$ are smooth functions so that

$$(3.6.217) \quad z_/(v_0) = z_\backslash(1),$$

$$(3.6.218) \quad \partial_v z_/|_{(v_0, v_1)} > 0$$

and

$$(3.6.219) \quad \partial_u z|_{(u_0, u_1)} < 0.$$

Then, the functions z_1, z_2 satisfy

$$(3.6.220) \quad \partial_u z_i < 0 < \partial_v z_i, \quad i = 1, 2$$

in $(u_0, u_1) \times (v_0, v_1)$ and

$$(3.6.221) \quad z_1 \leq z_2$$

everywhere on $[u_0, u_1] \times [v_0, v_1]$.

Proof. We will first establish (3.6.220). By applying a standard continuity argument, rewriting equation (3.6.213) as

$$(3.6.222) \quad \partial_v \log(-\partial_u z_1) = -\partial_v z_1 F_1(u, v, z_1)$$

and integrating in v , using also the property (3.6.219) of the initial data, we obtain that

$$(3.6.223) \quad \partial_u z_1 < 0$$

everywhere on $(u_0, u_1) \times (v_0, v_1)$. Similarly, rewriting (3.6.213) as

$$(3.6.224) \quad \partial_u \log(\partial_v z_1) = -\partial_u z_1 F_1(u, v, z_1)$$

and integrating in u , using (3.6.219), and then repeating the same procedure for z_2 , we finally obtain (3.6.220).

In order to establish (3.6.221), we will argue by continuity: Let $u_* \in [u_0, u_1]$ be such that (3.6.221),

$$(3.6.225) \quad \partial_v z_1 \leq \partial_v z_2$$

and

$$(3.6.226) \quad \partial_u z_1 \leq \partial_u z_2$$

hold on $[u_0, u_*] \times [v_0, v_1]$. Note that $u_* = u_0$ satisfies this condition: In this case, (3.6.221) and (3.6.225) follow directly from (3.6.216), while (3.6.226) follows by integrating (3.6.222) (and its analogue for z_2) and using (3.6.211). We will show that there exists a $\delta > 0$, such that (3.6.221), (3.6.225) and (3.6.226) hold on $[u_0, u_* + \delta] \times [v_0, v_1]$.

For any $\bar{v} \in (v_0, v_1]$, integrating (3.6.213) and (3.6.214) in v along $\{u_*\} \times [v_0, \bar{v}]$, we obtain:

$$(3.6.227) \quad \log(-\partial_u z_1)(u_*, \bar{v}) = - \int_{v_0}^{\bar{v}} F_1(u_*, v, z_1) \partial_v z_1 dv + \log(-\partial_u z_1)(u_*)$$

and

$$(3.6.228) \quad \log(-\partial_u z_2)(u_*, \bar{v}) = - \int_{v_0}^{\bar{v}} F_2(u_*, v, z_2) \partial_v z_2 dv + \log(-\partial_u z_2)(u_*).$$

Let us define the auxiliary functions $F_{1;u_*\bar{v}}, F_{2;u_*\bar{v}} : (-\infty, a] \rightarrow (0, +\infty)$ by the relations

$$(3.6.229) \quad F_{1;u_*\bar{v}}(z) = \max_{v \in [v_0, \bar{v}]} F_1(u_*, v, z)$$

and

$$(3.6.230) \quad F_{2;u_*\bar{v}}(z) = \min_{v \in [v_0, \bar{v}]} F_2(u_*, v, z).$$

In view of (3.6.211), (3.6.212) and the fact that (3.6.221) holds on $\{u_*\} \times [v_0, \bar{v}]$,¹⁵ we can bound for any $v \in [v_0, \bar{v}]$:

$$(3.6.231) \quad F_{1;u_*\bar{v}}(z_1(u_*, v)) < F_{2;u_*\bar{v}}(z_1(u_*, v)) \leq F_{2;u_*\bar{v}}(z_2(u_*, v)).$$

Thus, subtracting (3.6.227) and (3.6.228) and using (3.6.231) and (3.6.225) (and the fact

¹⁵Note that we can immediately restrict from $[u_0, u_1] \times [v_0, v_1]$ to $\{u_*\} \times [v_0, \bar{v}]$ in (3.6.211).

that $\partial_v z_2 > 0$, $\bar{v} > v_0$), we readily infer that

(3.6.232)

$$\begin{aligned} & \log(-\partial_u z_1)(u_*, \bar{v}) - \log(-\partial_u z_2)(u_*, \bar{v}) \\ & \geq \int_{v_0}^{\bar{v}} F_{2;u_*\bar{v}}(z_2(u_*, v)) \partial_v z_2(u_*, v) dv - \int_{v_0}^{\bar{v}} F_{1;u_*\bar{v}}(z_1(u_*, v)) \partial_v z_1(u_*, v) dv \\ & > 0. \end{aligned}$$

From (3.6.232) we thus infer that, for any $v_0 < \bar{v} \leq v_1$:

$$(3.6.233) \quad \partial_u z_1(u_*, \bar{v}) < \partial_u z_2(u_*, \bar{v}).$$

Therefore, since z_1, z_2 are smooth, there exists a continuous function $\delta_u : [v_0, v_1] \rightarrow [0, 1)$ with $\delta_u|_{(v_0, v_1]} > 0$, such that

$$(3.6.234) \quad \partial_u z_1(u, v) \leq \partial_u z_2(u, v) \text{ for } \{v_0 \leq v \leq v_1\} \cap \{u_* \leq u \leq u_* + \delta_u(v)\}.$$

Similarly, by integrating equations (3.6.213) and (3.6.214) in u along $[u_0, u_1] \times \{v_0\}$ and repeating a similar procedure (using (3.6.215)), we also obtain that there exists a continuous function $\delta_v : [u_0, u_1] \rightarrow [0, 1)$ with $\delta_v|_{(u_0, u_1]} > 0$, such that

$$(3.6.235) \quad \partial_v z_1(\bar{u}, v_0) \leq \partial_v z_2(\bar{u}, v_0) \text{ for } \{u_0 \leq u \leq u_1\} \cap \{v_0 \leq v \leq v_0 + \delta_v(u)\}.$$

From (3.6.233) and (3.6.235), we infer that there exists some $\delta > 0$, such that

$$(3.6.236) \quad z_1 \leq z_2 \text{ on } (u_*, u_* + \delta) \times [v_0, v_1].$$

In particular, (3.6.221) holds on $[u_0, u_* + \delta) \times [v_0, v_1]$. Furthermore, for any $\bar{u} \in (u_*, u_* + \delta)$ and any $\bar{v} \in (v_0, v_0 + \delta_v(\bar{u}))$, repeating the procedure leading to (3.6.232) with $\bar{u}$ in place of u_* and using (3.6.235) and (3.6.236) in place of (3.6.225) and (3.6.221), respectively, we

infer that:

$$(3.6.237) \quad \partial_u z_1(\bar{u}, \bar{v}) \leq \partial_u z_2(\bar{u}, \bar{v}).$$

Thus, combining (3.6.234) and (3.6.237), we infer that (3.6.226) holds on $[u_0, u_* + \delta'] \times [v_0, v_1]$, for some $0 < \delta' \leq \delta$. Finally, the bound (3.6.225) on $[u_0, u_* + \delta'] \times [v_0, v_1]$ follows in a similar way as the proof of (3.6.232), by integrating equations (3.6.213) and (3.6.214) in $u \in [u_0, u_* + \delta']$ for any $\bar{v} \in (v_0, v_1)$ and using (3.6.211), (3.6.221) and (3.6.226) (which we have shown that they hold on $[u_0, u_* + \delta'] \times [v_0, v_1]$). We will omit the details. $\square$

A lemma for a system of inductive inequalities

The following lemma is used to show that the inductive bounds (3.6.13) and (3.6.14) for $\tilde{m}_n^{(1,k+1)}$ and $r_n^{(k,k+1)}$ indeed lead to the formation of an almost-trapped surface.

Lemma 3.2. *Let $0 < c_1 \ll 1 \ll C_1$, and $0 < \mu_0 \ll 1 \ll \rho_0$, $0 < \delta \ll 1$ be given variables, such that*

$$(3.6.238) \quad \left(\frac{C_1}{c_1}\right)^8 \ll \frac{\rho_0}{\mu_0}$$

and

$$(3.6.239) \quad \delta < \left(\frac{\mu_0}{\rho_0}\right)^{(C_1/c_1)^4}.$$

Let also $\mu_n, \rho_n > 0$, be sequences of positive numbers, with μ_n increasing in n , such that for $0 \leq n \leq n_*$ they satisfy

$$(3.6.240) \quad \rho_{n+1} \leq \rho_n + C_1 \log((1 - \mu_n)^{-1} + 1),$$

$$(3.6.241) \quad \mu_{n+1} \geq \mu_n \exp\left(\frac{c_1}{\rho_{n+1}}\right),$$

and

$$(3.6.242) \quad \max_{0 \leq n \leq n_* - 1} \mu_n < 1 - \delta.$$

Then,

$$(3.6.243) \quad n_* \leq \left(\frac{C_1}{c_1}\right)^3 \rho_0 \mu_0^{-(C_1/c_1)^2}.$$

Furthermore, if $1 - \delta \leq \mu_{n_*} \leq 1 + \delta$, we can bound:

$$(3.6.244) \quad \mu_{n_*-1} \leq 1 - \left(\frac{c_1}{C_1}\right)^3 \rho_0^{-2} \mu_0^{2(C_1/c_1)^2}$$

and

$$(3.6.245) \quad \max\{\rho_{n_*}, \rho_{n_*-1}\} \leq \left(\frac{C_1}{c_1}\right)^4 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\rho_0}{\mu_0}\right).$$

Remark. Notice that the right hand side of (3.6.243) is independent of δ .

Proof. Let us define for any integer $k \geq 1$

$$(3.6.246) \quad n_k = \max \left\{ 0 \leq n \leq n_* : \mu_l \leq 1 - \frac{1}{2^k} \text{ for all } 0 \leq l \leq n \right\},$$

using the convention

$$(3.6.247) \quad n_0 = 0.$$

Notice that, in view of the fact that the sequence μ_n is increasing, for all $k \geq 1$ and all $n_{k-1} < n \leq n_k$ we can estimate:

$$(3.6.248) \quad 1 - \frac{1}{2^{k-1}} \leq \mu_n \leq 1 - \frac{1}{2^k}$$

(note that, in the case $n_{k-1} = n_k$, there is no n satisfying $n_{k-1} < n \leq n_k$ and (3.6.248)).

Using (3.6.248), from (3.6.240) we can bound for any $k \geq 1$ such that $n_{k-1} < n_k$ and any $n_{k-1} < n \leq n_k$:

$$(3.6.249) \quad \varrho_n \leq \varrho_{n_{k-1}} + 2C_1(\log 2)k(n - n_{k-1})$$

and, therefore, for any $0 \leq n \leq n_k$ we have:

$$(3.6.250) \quad \varrho_n \leq 2C_1(\log 2) \left(\sum_{l=1}^{k-1} l(n_l - n_{l-1}) + k(n - n_{k-1}) \right) + \varrho_0$$

(note that (3.6.250) holds for all $0 \leq n \leq n_k$, while the bounds (3.6.248) and (3.6.249) are non-trivial only for those values of k for which $n_k > n_{k-1}$).

Let us set

$$(3.6.251) \quad k_1 \doteq 32 \lceil \log \frac{C_1}{c_1} \rceil.$$

Then, (3.6.249) implies that, for all $0 \leq n \leq n_{k_1}$

$$(3.6.252) \quad \varrho_n \leq \varrho_0 + 2C_1(\log 2)k_1 n$$

and, thus, (3.6.241) implies that

$$(3.6.253) \quad \log \left(\frac{\mu_{n_{k_1}}}{\mu_0} \right) \geq c_1 \sum_{n=1}^{n_{k_1}} \varrho_n^{-1} \geq c_1 \sum_{n=1}^{n_{k_1}} \frac{1}{\varrho_0 + 2C_1(\log 2)k_1 n} \geq \frac{c_1 \log \left(\frac{\varrho_0 + 2C_1(\log 2)k_1 n_{k_1}}{\varrho_0 + 2C_1(\log 2)k_1} \right)}{4C_1(\log 2)k_1}.$$

From (3.6.246) and (3.6.253) we readily infer that

$$(3.6.254) \quad \frac{c_1 \log \left(\frac{\varrho_0 + 2C_1(\log 2)k_1 n_{k_1}}{\varrho_0 + 2C_1(\log 2)k_1} \right)}{4C_1(\log 2)k_1} \leq -\log(\mu_0)$$

and, therefore (using also (3.6.238)):

$$(3.6.255) \quad n_{k_1} \leq \frac{\varrho_0}{\mu_0^{(C_1/c_1)^2}}.$$

For any $k \geq 2$ such that $n_k > n_{k-1} + 1$, from (3.6.241), (3.6.246), (3.6.248) and (3.6.250) we readily infer:

$$(3.6.256) \quad \frac{1}{2^{k-2}} \geq \log \frac{\mu_{n_k}}{\mu_{n_{k-1}+1}} \geq c_1 \sum_{n=n_{k-1}+2}^{n_k} \varrho_n^{-1} \geq \frac{c_1}{4C_1(\log 2)} \frac{1}{(k-1) + \sum_{l=2}^{k-1} (l-1) \frac{(n_l - n_{l-1})}{n_k - n_{k-1} - 1}}$$

and, hence:

$$\begin{aligned}
 (3.6.257) \quad n_k - n_{k-1} - 1 &\leq \frac{\sum_{l=2}^{k-1} (l-1)(n_l - n_{l-1})}{\frac{c_1}{4C_1(\log 2)} 2^{k-2} - (k-1)} \\
 &\leq \frac{k(k-1)}{2} \frac{\max_{2 \leq l \leq k-1} (n_l - n_{l-1})}{\frac{c_1}{4C_1(\log 2)} 2^{k-2} - (k-1)} \\
 &\leq \frac{k(k-1)}{2} \frac{1}{\frac{c_1}{4C_1(\log 2)} 2^{k-2} - (k-1)} n_{k-1}.
 \end{aligned}$$

The relation (3.6.257) also holds (trivially) when $n_k \leq n_{k-1} + 1$. Thus, for any $k \geq k_1$, the bound (3.6.257) yields:

$$(3.6.258) \quad n_k \leq \left(1 + \frac{C_1}{c_1} 2^{-\frac{k-2}{4}}\right) n_{k-1} + 1$$

and, therefore, for any $k \geq 2$:

$$(3.6.259) \quad n_k \leq 16 \frac{C_1}{c_1} (n_{k_1} + \max\{k - k_1, 0\}).$$

In view of (3.6.255), we thus obtain for any $k \geq 2$:

$$(3.6.260) \quad n_k \leq 16 \frac{C_1}{c_1} \left(\frac{\varrho_0}{\mu_0^{(C_1/c_1)^2}} + \max\{k - k_1, 0\} \right).$$

Let us set

$$(3.6.261) \quad k_2 \doteq 4k_1 + 2 \frac{\log(\varrho_0 / \mu_0^{(C_1/c_1)^2})}{\log 2}.$$

Note that (3.6.239), (3.6.242) and (3.6.246) implies that

$$(3.6.262) \quad n_{k_2} \leq n_* - 1.$$

In view of (3.6.260), we have

$$(3.6.263) \quad n_{k_2} \leq \left(\frac{C_1}{c_1}\right)^3 \frac{\varrho_0}{\mu_0^{(C_1/c_1)^2}}$$

and, for all $k \geq k_2$ (in view of (3.6.257) and (3.6.260)):

$$(3.6.264) \quad n_k - n_{k-1} \leq 1.$$

Furthermore, (3.6.250) implies (in view of (3.6.261) and (3.6.263)) that

$$(3.6.265) \quad \max\{\varrho_{n_{k_2}+1}, \varrho_{n_{k_2}}\} \leq \left(\frac{C_1}{c_1}\right)^3 \frac{\varrho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\varrho_0}{\mu_0}\right).$$

In view of (3.6.246), we have

$$(3.6.266) \quad \mu_{n_{k_2}} \leq 1 - 2^{-k_2} < \mu_{n_{k_2}+1}.$$

We will consider two cases, depending on whether $\mu_{n_{k_2}+1}$ is larger than $1 - \delta$ or not.

1. In the case $\mu_{n_{k_2}+1} \geq 1 - \delta$, (3.6.242) implies that $n_{k_2} + 1 = n_*$. Thus, (3.6.243) follows from (3.6.263). Furthermore, (3.6.245) follows from (3.6.265), while (3.6.244) follows from (3.6.266).
2. In the case $\mu_{n_{k_2}+1} < 1 - \delta$, we can assume without loss of generality that $n_{k_2} \leq n_* - 2$ (otherwise, (3.6.243) follows from (3.6.263)). From (3.6.240), (3.6.265) and (3.6.266), we thus infer that

$$(3.6.267) \quad \varrho_{n_{k_2}+2} \leq \left(\frac{C_1}{c_1}\right)^3 \frac{\varrho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\varrho_0}{\mu_0}\right) + C_1 \log\left((1 - \mu_{n_{k_2}+1})^{-1}\right)$$

Hence, setting

$$(3.6.268) \quad M \doteq \left(\frac{C_1}{c_1}\right)^3 \frac{\varrho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\varrho_0}{\mu_0}\right),$$

from (3.6.241) and (3.6.266) we calculate:

$$(3.6.269) \quad \mu_{n_{k_2}+2} \geq \mu_{n_{k_2}+1} \exp\left(\frac{c_1}{\varrho_{n_{k_2}+2}}\right) \\ \geq \begin{cases} (1 - 2^{-k_2})e^{\frac{c_1}{2M}}, & \text{if } \log((1 - \mu_{n_{k_2}+1})^{-1}) \leq \frac{M}{C_1} \\ \mu_{n_{k_2}+1} \left(1 + \frac{c_1}{C_1 \log((1 - \mu_{n_{k_2}+1})^{-1} + 1)}\right), & \text{if } \log((1 - \mu_{n_{k_2}+1})^{-1}) > \frac{M}{C_1}. \end{cases}$$

If $\log((1 - \mu_{n_{k_2}+1})^{-1}) \leq \frac{M}{C_1}$, in view of (3.6.261) and (3.6.268) we can bound (using also (3.6.238))

$$(3.6.270) \quad (1 - 2^{-k_2})e^{\frac{c_1}{2M}} \geq \left(1 - \frac{\mu_0^{2(C_1/c_1)^2}}{\varrho_0^2}\right) \left(1 + \frac{c_1}{2} \left(\frac{c_1}{C_1}\right)^3 \frac{\mu_0^{(C_1/c_1)^2}}{\varrho_0 \log\left(\frac{\varrho_0}{\mu_0}\right)}\right) > 1 + \delta.$$

If $\log((1 - \mu_{n_{k_2}+1})^{-1}) > \frac{M}{C_1} \gg (C_1/c_1)^3$, we can also estimate:

$$(3.6.271) \quad \mu_{n_{k_2}+1} \left(1 + \frac{c_1}{C_1 \log((1 - \mu_{n_{k_2}+1})^{-1} + 1)}\right) \geq \\ \geq \left(1 - e^{-\log((1 - \mu_{n_{k_2}+1})^{-1})}\right) \left(1 + \frac{c_1}{C_1 \log((1 - \mu_{n_{k_2}+1})^{-1} + 1)}\right) > 1 + \delta.$$

Therefore, (3.6.269) implies that

$$(3.6.272) \quad \mu_{n_{k_2}+2} > 1 + \delta$$

and, hence, $n_{k_2+2} = n_*$. Thus, (3.6.243) follows again from (3.6.263).

□

Cauchy stability backwards in time

The following lemma, which is essentially a backwards-in-time Cauchy stability estimate for late time perturbations of $(\mathcal{U}_\varepsilon; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$, is an easy corollary of Theorem 3.2.

Lemma 3.3. *For any $0 < \epsilon < \epsilon_0$ (provided ϵ_0 is sufficiently small), any $r_0 > 0$ satisfy-*

ing (3.5.6), let $(\mathcal{U}_\epsilon; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ be the maximal future development of $(r|_\epsilon, \Omega_{|_\epsilon}^2, \bar{f}_{in|_\epsilon}, \bar{f}_{out|_\epsilon})$, and let us set

$$(3.6.273) \quad C_\epsilon \doteq \exp\left(\exp\left(-2(h_0(\epsilon))^{-4}\right)(h_1(\epsilon))^{-4}\right).$$

Then, for any $0 \leq u_* \leq (h_1(\epsilon))^{-2}v_{0\epsilon}$ such that

$$(3.6.274) \quad \mathcal{W}_{u_*} \doteq \{0 < u \leq u_*\} \cap \{u < v < u + v_{0\epsilon}\} \subset \mathcal{U}_\epsilon,$$

$$(3.6.275) \quad \sup_{\mathcal{W}_{u_*}} \left(1 - \frac{2\tilde{m}}{r}\right)^{-1} \leq C_\epsilon$$

and

$$(3.6.276) \quad u_* + v_{0\epsilon} \notin \text{supp}(r^2 T_{vv}(u_*, \cdot)),$$

and for any $\tilde{T}_{vv} : (u_*, u_* + v_{0\epsilon}) \rightarrow \mathbb{R}$ smooth and compactly supported satisfying $\tilde{T}_{vv}(\cdot) \geq -T_{vv}(u_*, \cdot)$ and

$$(3.6.277) \quad \sup_{u_* \leq \bar{v} \leq u_* + v_{0\epsilon}} (-\Lambda) \int_{u_*}^{u_* + v_{0\epsilon}} \frac{r^2(u_*, v) \frac{|\tilde{T}_{vv}(v)|}{\partial_v \rho(u_*, v)}}{|\rho(u_*, v) - \rho(u_*, \bar{v})| + \rho(u_*, u_*)} dv \leq \exp\left(-C_\epsilon^2 \frac{u_*}{v_{0\epsilon}}\right) (h_1(\epsilon))^2$$

with

$$(3.6.278) \quad \rho(u, v) \doteq \tan^{-1}\left(\sqrt{-\frac{\Lambda}{3}}r\right),$$

the following statement holds: There exists a smooth asymptotically AdS boundary characteristic initial data set $(r'_{|\epsilon}, (\Omega'_{|\epsilon})^2, \bar{f}'_{in|_\epsilon}, \bar{f}'_{out|_\epsilon})$ on $\{u = 0\}$ for the system (3.2.28)–(3.2.33) satisfying the reflecting gauge condition at $r = r_{0\epsilon}, +\infty$ with the following properties:

1. The initial data sets $(r|_\epsilon, \Omega_{|\epsilon}^2, \bar{f}_{in|_\epsilon}, \bar{f}_{out|_\epsilon})$ and $(r'_{|\epsilon}, (\Omega'_{|\epsilon})^2, \bar{f}'_{in|_\epsilon}, \bar{f}'_{out|_\epsilon})$ are $(h_1(\epsilon))^2$

close in the (3.3.41) norm, and in particular:

(3.6.279)

$$\sup_{v \in [0, v_{0\epsilon})} \left\{ \left| \log \left(\frac{\Omega_{/\epsilon}^2}{1 - \frac{1}{3} \Lambda r_{/\epsilon}^2} \right) - \log \left(\frac{(\Omega'_{/\epsilon})^2}{1 - \frac{1}{3} \Lambda (r'_{/\epsilon})^2} \right) \right| + \left| \log \left(\frac{2\partial_v r_{/\epsilon}}{1 - \frac{2m_{/\epsilon}}{r_{/\epsilon}}} \right) - \log \left(\frac{2\partial_v r'_{/\epsilon}}{1 - \frac{2m'_{/\epsilon}}{r'_{/\epsilon}}} \right) \right| + \left| \log \left(\frac{1 - \frac{2m_{/\epsilon}}{r_{/\epsilon}}}{1 - \frac{1}{3} \Lambda r_{/\epsilon}^2} \right) - \log \left(\frac{1 - \frac{2m'_{/\epsilon}}{r'_{/\epsilon}}}{1 - \frac{1}{3} \Lambda (r'_{/\epsilon})^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m}_{/\epsilon} - \tilde{m}'_{/\epsilon}| \right\} (v) \leq (h_1(\epsilon))^2$$

and

$$(3.6.280) \quad \sup_{\bar{v} \in [0, v_{0\epsilon}]} \int_0^{v_{0\epsilon}} \frac{\left| r_{/\epsilon}^2 \frac{(T_{vv})_{/\epsilon}}{\partial_v \rho_{/\epsilon}}(v) - (r'_{/\epsilon})^2 \frac{(T_{vv})'_{/\epsilon}}{\partial_v \rho'_{/\epsilon}}(v) \right|}{|\rho_{/\epsilon}(\bar{v}) - \rho_{/\epsilon}(v)| + \rho_{/\epsilon}(0)} dv \leq (h_1(\epsilon))^2.$$

2. The maximal future development $(\mathcal{U}'_\epsilon; r', (\Omega')^2, \bar{f}'_{in}, \bar{f}'_{out})$ of $(r'_{/\epsilon}, (\Omega'_{/\epsilon})^2, \bar{f}'_{in/\epsilon}, \bar{f}'_{out/\epsilon})$ satisfies

$$(3.6.281) \quad \mathcal{W}_{u_*} \subset \mathcal{U}'_\epsilon,$$

$$(3.6.282) \quad r'|_{\{u=u_*\} \cap \text{supp}(T_{vv})} = r|_{\{u=u_*\} \cap \text{supp}(T_{vv})}$$

and

$$(3.6.283) \quad T'_{vv}|_{\{u=u_*\}} = T_{vv}|_{\{u=u_*\}} + \tilde{T}_{vv}.$$

Proof. In view of (3.5.6), (3.5.21), (3.6.11) and (3.6.12), we can readily estimate

$$(3.6.284) \quad \sup_{\mathcal{W}_{u_*} \setminus \cup_n \mathcal{R}_n^{(1,k+1)}} \left(1 - \frac{2\tilde{m}}{r} \right)^{-1} \leq 2 \exp \left((h_0(\epsilon))^{-4} \right).$$

Therefore, using (3.6.275) for $\cup_n \mathcal{R}_n^{(1,k+1)}$ and (3.6.284) for $\mathcal{W}_{u_*} \setminus \cup_n \mathcal{R}_n^{(1,k+1)}$, the relations (3.6.33) and (3.6.34) imply (in view of (3.5.3), (3.6.8) and the fact that $u_* \leq (h_1(\epsilon))^{-2} v_{0\epsilon}$)

that

$$(3.6.285) \quad \sup_{\mathcal{W}_{u_*}} \left(\left| \log \left(\frac{-\partial_u r}{1 - \frac{2m}{r}} \right) \right| + \left| \log \left(\frac{\partial_v r}{1 - \frac{2m}{r}} \right) \right| \right) \leq (h_1(\varepsilon))^{-3} \exp((h_0(\varepsilon))^{-4}).$$

Similarly, equations (3.2.43) and (3.2.44), in view of the relations (3.6.33), (3.6.34) (using again the bounds (3.6.275) (3.6.284)) imply that

$$(3.6.286) \quad \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_*}} r T_{vv} dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_*}} r T_{uu} du \leq (h_1(\varepsilon))^{-1} \exp((h_0(\varepsilon))^{-4}).$$

Let us fix a set of smooth functions $r_j^*, (\Omega_j^*)^2 : [u_*, u_* + v_{0\varepsilon}) \rightarrow (0, +\infty)$ and $\bar{f}_{in/j}^*, \bar{f}_{out/j}^* : [u_*, u_* + v_{0\varepsilon}) \times (0, +\infty) \rightarrow [0, +\infty)$ satisfying the following requirements:

1. $(r_j^*, (\Omega_j^*)^2, \bar{f}_{in/j}^*, \bar{f}_{out/j}^*)$ is a smooth asymptotically AdS boundary-characteristic initial data set for the system (3.2.28)–(3.2.33) on $\{u_*\} \times [u_*, u_* + v_{0\varepsilon})$ satisfying the reflecting gauge condition at $r^* = r_{0\varepsilon}, +\infty$.

2. The function r_j^* satisfies for any v such that $(u_*, v) \in \text{supp}(T_{vv})$

$$(3.6.287) \quad r_j^*(v) = r|_{\text{supp}(T_{vv}) \cap \{u=u_*\}}.$$

3. The function $\bar{f}_{in/j}^*$ satisfies for all $v \in [u_*, u_* + v_{0\varepsilon})$:

$$(3.6.288) \quad \int_0^{+\infty} ((\Omega_j^*)^2(v) p^v)^2 \bar{f}_{in/j}^*(v; p^v) (r_j^*)^2(v) \frac{dp^v}{p^v} = T_{vv}(u_*, v) + \tilde{T}_{vv}(v).$$

4. The function $\bar{f}_{out/j}^*$ satisfies for all $(v, p^v) \in [u_*, u_* + v_{0\varepsilon}) \times (0, +\infty)$:

$$(3.6.289) \quad \bar{f}_{out/j}^*(v; p^v) = \bar{f}_{out}(u_*, v; p^v).$$

5. The initial data sets $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})|_{u=u_*}$ and $(r_{\not{I}}^*, (\Omega_{\not{I}}^*)^2, \bar{f}_{in/\not{I}}^*, \bar{f}_{out/\not{I}}^*)$ satisfy

(3.6.290)

$$\begin{aligned} & \sup_{v \in [u_*, u_* + v_{0\varepsilon})} \left\{ \left| \log \left(\frac{\Omega^2}{1 - \frac{1}{3}\Lambda r^2} \right) \right|_{u=u_*} - \log \left(\frac{(\Omega_{\not{I}}^*)^2}{1 - \frac{1}{3}\Lambda (r_{\not{I}}^*)^2} \right) \right| + \\ & + \left| \log \left(\frac{2\partial_v r}{1 - \frac{2m}{r}} \right) \right|_{u=u_*} - \log \left(\frac{2\partial_v r_{\not{I}}^*}{1 - \frac{2m_{\not{I}}^*}{r_{\not{I}}^*}} \right) \right| + \left| \log \left(\frac{1 - \frac{2m}{r}}{1 - \frac{1}{3}\Lambda r^2} \right) \right|_{u=u_*} - \log \left(\frac{1 - \frac{2m_{\not{I}}^*}{r_{\not{I}}^*}}{1 - \frac{1}{3}\Lambda (r_{\not{I}}^*)^2} \right) \right| + \\ & + \sqrt{-\Lambda} |\tilde{m}|_{u=u_*} - \tilde{m}_{\not{I}}^* \Big| \Big\} (v) \leq \exp \left(-\frac{1}{2} C_\varepsilon^2 \frac{u_*}{v_0} \right) (h_1(\varepsilon))^2 \end{aligned}$$

and

$$(3.6.291) \quad \sup_{v \in [v_1, v_2]} \int_{v_1}^{v_2} \frac{|r^2 T_{vv}(u_*, \bar{v}) - (r_{\not{I}}^*)^2 (T_{vv})_{\not{I}}^*(\bar{v})|}{|\varrho_{\not{I}}(v) - \varrho_{\not{I}}(\bar{v})| + \varrho_{\not{I}}(v_1)} d\bar{v} \leq \exp \left(-\frac{1}{2} C_\varepsilon^2 \frac{u_*}{v_0} \right) (h_1(\varepsilon))^2.$$

Remark. As a consequence of (3.6.276), by suitably deforming $r_{\not{I}}^*$ near $v = u_* + v_{0\varepsilon}$, we can always arrange that (3.3.7) and (3.6.287) are satisfied simultaneously. Furthermore, since $\tilde{T}_{vv}$ is compactly supported in $(u_*, u_* + v_{0\varepsilon})$, we can always choose $\bar{f}_{in/\not{I}}^* = f_{in}|_{\{u=u_*\}}$ in a neighborhood of $v = u_*, u_* + v_{0\varepsilon}$, so that (3.3.8) and (3.3.9) are satisfied. Finally, $(r_{\not{I}}^*, (\Omega_{\not{I}}^*)^2, \bar{f}_{in/\not{I}}^*, \bar{f}_{out/\not{I}}^*)$ can be chosen so that (3.6.290) and (3.6.291) are satisfied because of (3.6.277) and the relations (3.2.6), (3.2.44) and (3.2.47).

Let us now consider the two sets of initial data $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})|_{u=u_*}$ and $(r_{\not{I}}^*, (\Omega_{\not{I}}^*)^2, \bar{f}_{in/\not{I}}^*, \bar{f}_{out/\not{I}}^*)$ on $\{u = u_*\} \cap \{u_* \leq v < u_* + v_{0\varepsilon}\}$. The maximal past development of $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})|_{u=u_*}$ (see the remark below Theorem 3.3) coincides with $(\mathcal{W}_{u_*}; r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})$ when restricted on $\{u \geq 0\}$ and, in view of (3.6.285) and (3.6.286), satisfies

(3.6.292)

$$\begin{aligned} & \sup_{\mathcal{W}_{u_*}} \left\{ \left| \log \left(\frac{\Omega^2}{1 - \frac{1}{3}\Lambda r^2} \right) \right| + \left| \log \left(\frac{2\partial_v r}{1 - \frac{2m}{r}} \right) \right| + \left| \log \left(\frac{1 - \frac{2m}{r}}{1 - \frac{1}{3}\Lambda r^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m}| \right\} + \\ & + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_*}} r T_{vv} dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_*}} r T_{uu} du \leq 4(h_1(\varepsilon))^{-3} \exp \left(-(h_0(\varepsilon))^{-4} \right). \end{aligned}$$

Therefore, in view of (3.6.292), (3.6.290) and (3.6.291), Theorem 3.2 applied for the past

development of $(r, \Omega^2, \bar{f}_{in}, \bar{f}_{out})|_{u=u_*}$ on $\mathcal{W}_{u_*}$ (see the remark below Theorem 3.2) implies that the maximal past development $(\mathcal{U}^*; r^*, (\Omega^*)^2, \bar{f}_{in}^*, \bar{f}_{out}^*)$ of $(r', (\Omega')^2, \bar{f}_{in}', \bar{f}_{out}')$ satisfies

$$(3.6.293) \quad \mathcal{W}_{u_*} \subset \mathcal{U}^*$$

and

$$(3.6.294) \quad \begin{aligned} & \sup_{\mathcal{W}_{u_*}} \left\{ \left| \log \left(\frac{\Omega^2}{1 - \frac{1}{3}\Lambda r^2} \right) - \log \left(\frac{(\Omega^*)^2}{1 - \frac{1}{3}\Lambda (r^*)^2} \right) \right| + \left| \log \left(\frac{2\partial_v r}{1 - \frac{2m}{r}} \right) - \log \left(\frac{2\partial_v r^*}{1 - \frac{2m^*}{r^*}} \right) \right| + \right. \\ & \quad \left. + \left| \log \left(\frac{1 - \frac{2m}{r}}{1 - \frac{1}{3}\Lambda r^2} \right) - \log \left(\frac{1 - \frac{2m^*}{r^*}}{1 - \frac{1}{3}\Lambda (r^*)^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m} - \tilde{m}^*| \right\} + \\ & + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_*}} |rT_{vv} - r^*(T_{vv})^*| dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_*}} |rT_{uu} - r^*(T_{uu})^*| du \leq (h_1(\varepsilon))^3. \end{aligned}$$

Thus, the proof of the lemma concludes by setting

$$(3.6.295) \quad (r', (\Omega')^2, \bar{f}_{in}', \bar{f}_{out}') \doteq (r^*, (\Omega^*)^2, \bar{f}_{in}^*, \bar{f}_{out}^*)|_{u=0}.$$

□

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