From the holdings of the Department of Special Collections , Princeton University Library, Princeton, NJ

These documents are either in the public domain, or can only be used for educational and research purposes (Fair use) as per U.S. Copyright law (text below). By accessing this file, all users agree that their use falls in the public domain or within fair use as defined by the copyrightlaw. We do not charge any permission or usage fees for the publication of images of material inour collections, including those provided by Princeton University Library, via our website, catalog, or directly from staff.For more information regarding copyright, credit, and citation please review the information onthe Copyright, Credit and Citation Guidelines page of our website.
https://library.princeton.edu/special-collections/policies/copyright-credit-and-citation-guidelines. If you have any questions, including how to order higher-quality images from the collection, contact Special Collections Public Services via our Ask Us! Form. https://library.princeton.edu/special-collections/ask-us

---

Doctoral Dissertations, 2011-2026 Mathematics

2020

On certain families of special cycles on Shimura varieties

Jin, Zhaorong

Princeton, NJ : Princeton University

---

http://arks.princeton.edu/ark:/88435/dsp01d217qs44g

U.S. Copyright law: The copyright law of the United States (Title 17, §108, United States Code) governs the making of photocopies or other reproductions of copyrighted material. Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be used for any purpose other than private study, scholarship, or research. If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of fair use that user may be liable for copyright infringement. This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve a violation of copyright law. https://www.copyright.gov/title17/title17.pdf

On certain families of special cycles on Shimura varieties

Zhaorong Jin

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Christopher McLean Skinner

June 2020

© Copyright by Zhaorong Jin, 2020.

All Rights Reserved

Abstract

In this two-part thesis, we study certain families of special cycles on Shimura varieties, which have interesting arithmetic applications. In the first part, extending the ideas of Darmon and Rotger, we construct a p-adic family of Hirzebruch-Zagier cycles on Shimura threefolds obtained from products of modular curves and Hilbert modular surfaces. This family gives rise to a big cohomology class residing in the Galois cohomology of a certain Λ-adic Galois representation. We then establish a regulator formula for the cycles, which allows us to relate the big cohomology class to a twisted triple product p-adic L-function. As an application, we establish new instances of the equivariant BSD-conjecture in rank 0 and study the arithmetic of rational elliptic curves over quintic fields. In the second part, we follow the ideas of Loeffler, Skinner and Zerbes to set up and conduct local automorphic computations on the algebraic group $\mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, establishing technical results that should lead to the norm relations of an Euler system for $\mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$.

Acknowledgements

First, I would like to express my sincere gratitude to my advisor, Professor Christopher Skinner, for his constant support and encouragement, and years of enlightening mathematical conversations and guidance. I would also like to thank Professor Shou-Wu Zhang for reading the thesis, and Professor Nicholas Katz and Professor Peter Sarnak for serving on my committee.

Special thanks to Michele Fornea for countless fruitful discussions. Work in the first part of this thesis is in collaboration with Michele, and has been presented at the number theory seminars at Princeton University and Harvard University. I am very grateful for his passion and dedication to our joint work.

I would also like to thank Professor David Loeffler and Professor Sarah Zerbes for proposing the $\mathrm{GSp}_4 \times \mathrm{GL}_2$-Euler system and answering my questions, thank Chi-Yun Hsu and Ryotaro Sakamoto for their interest on this project (which is a joint work in progress building on the results in the second part of the thesis), as well as for correcting a key mistake in Proposition 12.3.2, and thank Giada Grossi for sharing her preprint and helping me understand the subtle technical details in [LSZ17].

Many thanks to our wonderful graduate program administrator, Jill LeClair, for answering all my logistic questions and making my life in Fine Hall so smooth and easy. Thanks to all my friends and fellow graduate students, among them Weibo Fu, Shilin Lai, Lue Pan, Congling Qiu, Boya Wen and Ruixiang Zhang, for all the conversations and cheerfulness.

Finally, I owe my deepest gratitude to my parents, Zhijun Jin and Yanxiao Han, and my girl friend Weitao Shuai, for their continuous support and love.

To my family and friends.

Contents

Abstract iii

Acknowledgements iv

I On generalized Hirzebruch-Zagier cycles 2

1 Introduction 3

1.1 History and background 3

1.2 A closer look at the work of Darmon-Rotger 5

1.3 Hirzebruch-Zagier cycles 7

1.4 Notations and conventions 15

2 Preliminaries on Hilbert modular forms 19

2.1 Basic notations 19

2.2 Hilbert modular forms 20

2.3 Relation to classical Hilbert modular forms 22

2.4 Adelic Fourier expansions 23

2.5 Hecke Theory 25

2.6 p-adic modular forms 29

2.7 Hida families 31

2.8 Diagonal restriction 34

2.9 Twists of Hilbert modular forms. 34

3 Automorphic p-adic L-functions 41

3.1 Preparations with **I**-adic forms 41

3.2 The automorphic p-adic L-function 45

3.3 Link to the complex L-value in weight one. 46

4 Hilbert modular varieties and Hirzebruch-Zagier classes 48

4.1 Basic notations 48

4.2 Hecke correspondences and operators 50

4.3 Atkin-Lehner morphism 52

4.4 Hirzebruch-Zagier cycles 62

4.5 Big cohomology classes 67

4.6 A refinement of specialization formula 69

5 Galois representations 73

5.1 Galois representations at finite level 73

5.2 Big Galois representations. 77

5.3 Étale cohomology of towers of Hilbert modular surfaces 81

5.4 Local cohomology class 88

6 Big pairing 96

6.1 Algebra interlude 96

6.2 A weak Λ-adic Eichler-Shimura map 100

6.3 Big pairing. 103

6.4 On Dieudonné modules 111

7 Motivic p-adic L-functions 116

7.1 Perrin-Riou's regulator . . . 116
7.2 The motivic p-adic L-function. . . . 121

8 p-adic Gross-Zagier formulas 123
8.1 P-syntomic cohomology . . . 123
8.2 Syntomic Abel-Jacobi map . . . 125
8.3 Abel-Jacobi map of Hirzebruch-Zagier cycles . . . 128
8.4 Overconvergent Hilbert Modular Forms . . . 133
8.5 Relation to p-adic modular forms. . . . 142
8.6 The formula . . . 150
8.7 An explicit reciprocity law . . . 155

9 On the equivariant BSD-conjecture 157
9.1 Artin representations and Selmer groups . . . 157
9.2 Existence of prime p . . . 159
9.3 Proof of main theorems . . . 165

II Local computations for a $\mathrm{GSp}_4 \times \mathrm{GL}_2$ Euler system 168

10 Introduction 169
10.1 Recent progress in Euler systems . . . 169
10.2 Outline of the proof and main results . . . 171
10.3 Notations and conventions . . . 175

11 Local representation theory 178
11.1 Principal series of $\mathrm{GL}_2(\mathbb{Q}_l)$ . . . 178
11.2 Siegel sections . . . 181
11.3 Principal series of $\mathrm{GSp}_4(\mathbb{Q}_l)$ . . . 182

12 Hecke operators and zeta integrals **186**

12.1 Local zeta integrals 186

12.2 Double coset computations 189

12.3 Hecke operators and local zeta integrals 198

13 Towards norm relations **206**

13.1 A local bilinear form 206

13.2 An application of Frobenius reciprocity 210

13.3 Main results towards vertical norm relations 214

13.4 Main result towards tame norm relations 222

13.5 Divisibility of m_η 224

General notations

Throughout the two parts of the thesis, we will use the following notations.

- For any number field F, we use $\mathcal{O}_F$ to denote its ring of integers. For any place v of F, let F_v denote the corresponding local field, and $\mathcal{O}_{F,v}$ its ring of integers. More generally, for any ideal $\mathfrak{m}$ of $\mathcal{O}_F$, let $F_{\mathfrak{m}} = \prod_{v|\mathfrak{m}} F_v$, and $\mathcal{O}_{F,\mathfrak{m}} = \prod_{v|\mathfrak{m}} \mathcal{O}_{F,v}$.

- $\mathbb{A}_F$ is the ring of adeles, $\mathbb{A}_{F,f}$ is the ring of finite adeles, and $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ is the infinite part.

- For any $y \in \mathbb{A}_F$, we use $y_\infty \in F_\infty$ to denote its infinite part, and $y_f \in \mathbb{A}_{F,f}$ to denote its finite part. For any place v of F, $y_v \in F_v$ is the v-component of y, and for any ideal $\mathfrak{m}$ of $\mathcal{O}_F$, let $y_{\mathfrak{m}} = (y_v)_{v|\mathfrak{m}} \in F_{\mathfrak{m}}$.

Part I

On generalized Hirzebruch-Zagier cycles

Chapter 1

Introduction

1.1 History and background

One of the earliest results towards the Birch and Swinnerton-Dyer conjecture is the seminal work of Coates and Wiles [CW77], in which they established the algebraic rank zero case of the BSD conjecture for elliptic curves with complex multiplications. From a modern perspective, their proof involves interpolating the so-called elliptic units into a p-adic family (a "*big cohomology class*"), and then relating its image under some "big exponential map" to Katz's two-variable p-adic L-function of an imaginary quadratic field (an "*explicit reciprocity law*"), which is done via the so-called p-adic Kronecker limit formula relating the special values of p-adic L-function to p-adic logarithms of elliptic units (a "*p-adic regulator formula*"). In this way, one can obtain special value formulas outside of the range of interpolation, connecting the Hasse-Weil L-value of an CM elliptic curve with a global class in the Selmer group, thereby establishing an instance of the BSD conjecture.

Recently, there have been several works in the spirit of Coates-Wiles, among which [BDP13] and the sequels [DR14], [DR17] are the most prominent. In [BDP13], the

elliptic units are replaced by Heegner points on modular curves (or more generally, generalized Heegner cycles in the product of Kuga-Sato varieties), and Katz's p-adic L-function is replaced by certain p-adic Rankin L-series constructed in op.cit., called the anticyclotomic p-adic L-function, which interpolates the central values of the Rankin-Selberg L-function of a modular form and a Hecke character of an imaginary quadratic field. The main result of op.cit. is a p-adic analogue of the Gross-Zagier formula, relating the images of the generalized Heegner cycles under the *p-adic Abel-Jacobi map* to the special values of anticyclotomic p-adic L-function at critical points that lie outside the range of classical interpolation. This again can be seen as an instance of the p-adic regulator formulas, and has many applications to the study of the BSD conjecture and more generally the Bloch-Kato conjectures, including the recent work of [Ski20] and [CH18].

The work of Darmon and Rotger focuses on the connection between the so-called Gross-Kudla-Schoen diagonal cycles and a triple product p-adic L-function interpolating the (square root of) central critical values of the Garrett-Rankin L-function attached to a triple of modular forms. In [DR14], the authors extended the method of [HT01] to construct a three-variable triple product p-adic L-function, and proved a p-adic Gross-Zagier formula relating the special value of the p-adic L-function to the p-adic Abel-Jacobi image of the diagonal cycles. In [DR17], by modifying the diagonal cycles in triple products of modular curves, they put the p-adic Abel-Jacobi image of the diagonal cycles into a one-variable global cohomology class, and established an explicit reciprocity law, which proved that the image of the global cohomology class under Perrin-Riou's big logarithm map is essentially the restriction of the triple product p-adic L-function constructed earlier to a one variable line in the three variable weight space. By specializing at weights outside the range of "geometric interpolation" defining the global cohomology class, they are able to realize the central critical

value of certain twisted L-function of an elliptic curve as the obstruction to a cohomology class being crystalline at p, thereby establishing instances of the equivariant BSD conjecture.

1.2 A closer look at the work of Darmon-Rotger

As the present work is greatly inspired by and very closely related to [DR17], it is worthwhile to discuss the main features of op.cit. in more detail.

Let f be a newform of weight two, level N_f and trivial character, and $\mathbf{g}$ and $\mathbf{h}$ two ordinary Hida families of tame levels N_g and H_h with inverse tame characters, such that N_f and $H_g N_h$ are coprime, and so are p and $N = \mathrm{lcm}(N_f, N_g, N_h)$. For each $\alpha \geq 1$, we have the diagonally embedded curve inside the threefold

$$X_1(Np^\alpha) \hookrightarrow X_0(Np) \times X_1(Np^\alpha) \times X_1(Np^\alpha),$$

which, after suitably twisting and taking finite quotients, gives rise to a family $\{\Delta_\alpha\}_{\alpha \geq 1}$ of null-homologous cycles compatible under the pushforward of the natural projection down the p-tower as α varies. After taking the inverse limit of the cohomology classes $\mathrm{AJ}_p^{\text{ét}}(\Delta_\alpha)$ obtained under the p-adic étale Abel-Jacobi map and projecting to suitably isotypic quotients of the étale cohomology of the modular threefold, one obtains the global cohomology class

$$\kappa(f, \mathbf{gh}) \in \mathrm{H}^1(\mathbb{Q}, \mathbb{V}_{f\mathbf{gh}}(N)),$$

where $\mathbb{V}_{f\mathbf{gh}}(N)$ is the direct sum of several copies of the tensor product of V_f, $\mathbb{V}_{\mathbf{g}}$, $\mathbb{V}_{\mathbf{h}}$, the (classical and Λ-adic) Galois representations associated to f, $\mathbf{g}$ and $\mathbf{h}$.

We may specialize the class $\kappa(f, \mathbf{gh})$ at classical points (y, z) of the weight spaces

of the Hida families $\mathbf{g}$ and $\mathbf{h}$ such that y and z have the same weight ℓ and character at p, obtaining a collection of classes $\kappa(f, g_y, h_z) \in \mathrm{H}^1(\mathbb{Q}, V_{fg_yh_z}(N))$ varying p-adically analytically as (g_y, h_z) varies over pairs of specializations of $\mathbf{g}$ and $\mathbf{h}$ with common weights and characters at p. In the case of weight two specializations, as they "arise from geometry", the work of Saito, later refined by Skinner, combined with a result of Nekovář, shows $k(f, g_y, h_z)$ are crystalline after restricting to a decomposition group at p.

On the other hand, although $\kappa(f, \mathbf{gh})$ is constructed by interpolating geometric constructions attached to weight two specializations, it makes sense to consider its weight one specializations $\kappa(f, g_1, h_1)$. These classes, lying outside the range of "geometric interpolation", have no a priori reason to be crystalline at p. In the case where f is a newform associated to an elliptic curve E, and g_y and h_z are the ordinary p-stabilizations of weight one forms associated to two two-dimensional odd Artin representations ρ_1 and ρ_2, it can be shown the classes being non-crystalline at p have very strong implications for the arithmetic of E. For example, the $\rho_1 \otimes \rho_2$-isotypic component of the Mordell-Weil group of E must be trivial.

It is now that the decisive input of a p-adic regulator formula comes into play: using Besser's theory of finite polynomial cohomology, which is an extension of Coleman's theory of p-adic integration, one is able to compute a formula for $\mathrm{AJ}_{\mathrm{syn}}(\Delta_\alpha)$ paired with test vectors, where $\mathrm{AJ}_{\mathrm{syn}}$ is the *syntomic Abel-Jacobi map* and is related to $\mathrm{AJ}_p^{\text{ét}}$ via the Bloch-Kato logarithm. By comparing the values of $\mathrm{AJ}_{\mathrm{syn}}(\Delta_\alpha)$ with the special values of the triple product p-adic L-function at weight two points, one obtains an explicit reciprocity law: if $\mathcal{L}$ is Perrin-Riou's big logarithm interpolating the Bloch-Kato logarithms, then $\mathcal{L}(\kappa(f, \mathbf{gh}))$ is essentially the triple product p-adic L-function. In particular, specializing at weight one expresses the complex L-value $L(E, \rho_1 \otimes \rho_2, 1)$ as the obstruction of $\kappa(f, g_1, h_1)$ being crystalline, which is the last

missing piece in the proof of an instance of the equivariant BSD conjecture, among other applications to the arithmetic of elliptic curves.

1.3 Hirzebruch-Zagier cycles

The present work seeks to explore a similar theme in the setting of the diagonal curve inside the product of a modular curve and a Hilbert modular surface, which is sometimes referred to as a *Hirzebruch-Zagier cycles*. Recent work of [BCF] has constructed the corresponding p-adic L-function in this setting, the twisted triple product p-adic L-function associated to a Hida family of elliptic modular forms and a Hida family of Hilbert modular forms over a real quadratic field L, and established a p-adic Gross-Zagier formula similar to the one in [DR14]. In this part of the thesis, which is a joint work with Michele Fornea, the second author of [BCF], we establish results analogous to [DR17] in the case when the prime p splits in L. In particular, we study the p-adic variation of the Hirzebruch-Zagier cycles and their relations to the twisted triple product p-adic L-function. We interpolate the Abel-Jacobi image of the cycles into a big *Hirzebruch-Zagier class*, and by extending the p-adic regulator formula in [BCF] to the entire p-tower of modular varieties, we prove an explicit reciprocity law. As an application, we establish a new instance of the equivariant BSD conjecture in rank zero and prove arithmetic properties of elliptic curves over S_5-quintic fields.

One can think of the setting of [DR17] as a degenerate case of our work, where L degenerates to $\mathbb{Q} \oplus \mathbb{Q}$. When L is a real quadratic field in which p splits, the local behavior of the automorphic representation of $\mathrm{Res}^L_{\mathbb{Q}}\mathrm{GL}_2(\mathbb{Q})$ is very similar to the tensor product of two automorphic representations of $\mathrm{GL}_2(\mathbb{Q})$ (except for the presence of a rank one group of global units), and correspondingly locally at p the Hilbert

modular surface behaves similar to a product of two modular curves. Nevertheless, the main technical difficulties lie in the complexity of the geometry of Hilbert modular surfaces. Extra work is needed to ensure the cycles are twisted in a way so that they interpolate p-adically.

Main results

Let L be a real quadratic field and $\varrho : G_L \to \mathrm{GL}_2(\mathbb{C})$ a totally odd, irreducible two-dimensional Artin representation of the absolute Galois group of L. The Asai representation

$$\mathrm{As}(\varrho) := \otimes\text{-}\mathrm{Ind}_L^{\mathbb{Q}}(\varrho)$$

is a 4-dimensional complex representation obtained as the tensor induction of ϱ from G_L to $G_{\mathbb{Q}}$. We suppose that ϱ has conductor $\mathfrak{Q}$ and that the tensor induction of the determinant $\det(\varrho)$ is the trivial character so that $\mathrm{As}(\varrho)$ is self-dual. Let $H/\mathbb{Q}$ be the finite extension cut out by the representation $\mathrm{As}(\varrho)$.

For any rational elliptic curve $E_{/\mathbb{Q}}$ of conductor N_E prime to $\mathfrak{Q}$, we define the $\mathrm{As}(\varrho)$-isotypic component of the Mordell-Weil group of E as

$$E(H)^{\mathrm{As}(\varrho)} := \mathrm{Hom}_{\mathrm{Gal}(H/F)}(\mathrm{As}(\varrho), E(H) \otimes \mathbb{C}),$$

and let the algebraic rank of E twisted by $\mathrm{As}(\varrho)$ be defined as

$$r_{\mathrm{alg}}(E, \mathrm{As}(\varrho)) = \dim_{\mathbb{C}} E(H)^{\mathrm{As}(\varrho)}.$$

On the analytic side, there is the Hasse-Weil-Artin L-series $L(E, \mathrm{As}(\varrho), s)$, which is the L-function associated to the Galois representation $\mathrm{As}(\varrho) \otimes V_p(E)$. By the modularity of totally odd Artin representations and rational elliptic curves ([Wil95],[TW95],

[PS16]), this is identified with the twisted triple product L-function attached to the weight two modular form associated to E and the parallel weight one Hilbert modular form associated to ϱ. We define the analytic rank of E twisted by $\mathrm{As}(\varrho)$ to be

$$r_{\mathrm{an}}(E, \mathrm{As}(\varrho)) := \mathrm{ord}_{s=1} L(E, \mathrm{As}(\varrho), s).$$

One of our main results establishes the following new special case of the equivariant BSD conjecture:

Theorem 1.3.1. Suppose that N_E is coprime to $\mathfrak{Q}$ and split in L. If there is a rational prime p such that p is coprime to $N\mathfrak{Q}$, split over L into narrowly principal ideals such that the eigenvalues of Frob_p on $\mathrm{As}(\varrho)$ are all distinct modulo p and there are no totally positive units in L congruent to -1 modulo p, then

$$r_{\mathrm{an}}\big(E, \mathrm{As}(\varrho)\big) = 0 \quad \Longrightarrow \quad r_{\mathrm{alg}}\big(E, \mathrm{As}(\varrho)\big) = 0.$$

We will see later that using some elementary algebraic number theory we can always pick such a prime p. Applying the above theorem to the Artin representation constructed in [For18], we obtain the following result regarding the arithmetic of rational elliptic curves over S_5-quintic fields:

Corollary 1.3.2. Let $K/\mathbb{Q}$ be a non-totally real S_5-quintic extension of positive discriminant whose Galois closure $\widetilde{K}/\mathbb{Q}$ contains a real quadratic field L. Suppose N_E is odd, unramified in $K/\mathbb{Q}$ and split in L, then

$$r_{\mathrm{an}}(E/K) = r_{\mathrm{an}}(E/\mathbb{Q}) \quad \Longrightarrow \quad r_{\mathrm{alg}}(E/K) = r_{\mathrm{alg}}(E/\mathbb{Q}).$$

Overview of the proof

Let $\mathtt{g}_\varrho$ be the Hilbert cuspform of parallel weight one associated to the Artin representation $\varrho : G_L \to \mathrm{GL}_2(\mathbb{C})$, and $\mathtt{f}_\circ$ the elliptic cuspform associated to $E_{/\mathbb{Q}}$.

Assume there exists a rational prime p coprime to the levels of $\mathtt{f}_\circ$ and $\mathtt{g}_\varrho$ such that

- $\mathtt{f}_\circ$ is ordinary and non-Eisenstein at p;
- the eigenvalues of Frob_p on $\mathrm{As}(\varrho)$ are all distinct modulo p;
- $p\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$ splits into the product of two narrowly principal ideals, and
- there are no totally positive units of L congruent to -1 modulo p.

We will see in the last chapter of Part I that for any real quadratic field L the set of prime p satisfying the last two conditions has a positive density. which means we can always choose a p satisfying all the assumptions above.

After choosing a rational prime p and an ordinary p-stabilization $\mathtt{g}_\varrho^{(p)}$, one can find a nearly ordinary Hida family $\mathscr{G}_{\text{n.o.}}$ passing through it. In the present work we focus our attention on the one-variable Hida family $\mathscr{G}$ interpolating parallel weight Hilbert modular forms of certain special level structures (the $K_{\diamond,t}(p^\alpha)$ level subgroups defined below), which can be obtained by partially specializing the three variable Hida family $\mathscr{G}_{\text{n.o.}}$ to one variable. The Hida family is equipped with a big Galois representations, $\mathrm{As}(\mathbf{V}_{\mathscr{G}})$, which is of rank 4 over the field of fractions $\mathbf{Q}_{\mathscr{G}}$ of $\mathbf{I}_{\mathscr{G}}$. Here $\mathbf{I}_{\mathscr{G}}$ is the ring of coefficients of $\mathscr{G}$, which is a finite flat algebra over the standard Iwasawa algebra $\mathbf{\Lambda} = \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]$. Let $\mathcal{A}(\mathbf{I}_{\mathscr{G}})$ be the set of *arithmetic points* inside the weight space $\mathcal{W}_{\mathscr{G}} = \mathrm{Spf}(\mathbf{I}_{\mathscr{G}})^{\mathrm{rig}}$ associated to $\mathbf{I}_{\mathscr{G}}$, and let $\mathrm{P}_\circ \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ be the weight one point corresponding to $\mathtt{g}_\varrho^{(p)}$, so that

$$\mathrm{As}(\mathbf{V}_{\mathscr{G}_{\mathrm{P}_\circ}}) \cong \mathrm{As}(\varrho).$$

On the other hand, there is the Galois representation $\mathrm{V}_{\mathsf{f}_\circ} \cong V_p(E)$ associated to the classical modular form $\mathsf{f}_\circ$. One can show that there exists a twist $\mathbf{V}^\dagger_{\mathscr{G},\mathsf{f}_\circ}$ of

$$\mathbf{V}_{\mathscr{G},\mathsf{f}_\circ} = \mathrm{As}(\mathbf{V}_{\mathscr{G}})(-1) \otimes \mathbf{V}_{\mathsf{f}_\circ}$$

interpolating Kummer self-dual representations, and such that the specialization at $\mathrm{P}_\circ$ is

$$\mathbf{V}^\dagger_{\mathscr{G}_{\mathrm{P}_\circ},\mathsf{f}_\circ} = \mathrm{V}_{\varrho,E}.$$

Refining the construction in [BCF], we construct a rigid analytic function on $\mathcal{W}_{\mathscr{G}}$, the *automorphic p-adic L-function*,

$$\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ) : \mathcal{W}_{\mathscr{G}} \longrightarrow \mathbb{C}_p,$$

which is essentially the restriction of the three variable twisted triple product p-adic L-function of op.cit. The value of $\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)$ at the weight one point $\mathrm{P}_\circ$ is, up to a nonzero constant,

$$\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)(\mathrm{P}_\circ) \dot{\sim} L(E, \mathrm{As}(\varrho), 1).$$

Furthermore, its value at *any* arithmetic point $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight two is explicitly given in terms of p-adic modular forms. We will establish a p-adic regulator formula relating these values to the syntomic Abel-Jacobi images of Hirzebruch-Zagier cycles.

Hirzebruch-Zagier classes.

For $\alpha \geq 1$ and compact open $K \subseteq \mathrm{GL}_2(\mathbb{A}_{L,f})$ hyperspecial at p, let

$$K_{\diamond,t}(p^\alpha) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(p^\alpha) \middle| \; a_{\mathfrak{p}_1} d_{\mathfrak{p}_1} \equiv a_{\mathfrak{p}_2} d_{\mathfrak{p}_2},\; d_{\mathfrak{p}_1} d_{\mathfrak{p}_2} \equiv 1 \pmod{p^\alpha} \right\}$$

and denote by $S(K_{\diamond,t}(p^\alpha))$ the corresponding Hilbert modular surface. We also set

$$K_0'(p^\alpha) = K_0(p^\alpha) \cap \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}),$$

and denote by $Y(K_0'(p^\alpha))$ the corresponding open modular curve.

Inspired by ideas in [DR17], for every $\alpha \geq 1$ we produce a null-homologous codimension 2 cycle, called *Hirzebruch-Zagier cycle*,

$$\Delta_\alpha^\circ \in \mathrm{CH}^2\big(Z_\alpha(K)\big)\big(\mathbb{Q}(\zeta_{p^\alpha})\big) \otimes \mathbb{Z}_p$$

on the modular threefold $Z_\alpha(K) = S(K_{\diamond,t}(p^\alpha)) \times Y(K_0'(p))$. Moreover, the action of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q})$ is such that Δ_α° corresponds to a null-homologous rational cycle class

$$\Delta_\alpha^\circ \in \mathrm{CH}^2\big(Z_\alpha^\dagger(K)\big)(\mathbb{Q}) \otimes \mathbb{Z}_p$$

on a twisted threefold $Z_\alpha^\dagger(K)$ with the following property: for every $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight two and level p^α there is a natural Galois equivariant surjection

$$\mathrm{H}^3_{\text{ét}}\big(Z_\alpha^\dagger(K)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(2)\big) \twoheadrightarrow \mathbf{V}^\dagger_{\mathscr{G}_{\mathrm{P}},\mathfrak{f}_\circ}.$$

The ordinary parts of the Abel-Jacobi images

$$\mathrm{AJ}_p^{\text{ét}}(\Delta_\alpha^\circ) \in \mathrm{H}^1\big(\mathbb{Q}, \mathrm{H}^3_{\text{ét}}\big(Z_\alpha^\dagger(K)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2)\big)\big)$$

can be made compatible under the degeneracy maps $\varpi_2 : Z_{\alpha+1}^\dagger(K) \to Z_\alpha^\dagger(K)$ and packaged together to form a global big cohomology class

$$\boldsymbol{\kappa}_{\mathscr{G},\mathfrak{f}_\circ} \in \mathrm{H}^1\big(\mathbb{Q}, \mathbf{V}^\dagger_{\mathscr{G},\mathfrak{f}_\circ}\big).$$

This class retains information about the Abel-Jacobi image of algebraic cycles at arithmetic points of weight two and can be specialized at the arithmetic point $\mathrm{P}_\circ$ of weight one

$$\boldsymbol{\kappa}_{\mathscr{G},\mathsf{f}_\circ}(\mathrm{P}_\circ) \in \mathrm{H}^1\big(\mathbb{Q}, \mathrm{V}_{\varrho,E}\big),$$

mirroring the automorphic p-adic L-function. In order to make apparent the relationship between $\boldsymbol{\kappa}_{\mathscr{G},\mathsf{f}_\circ}$ and the automorphic p-adic L-function, we use Perrin-Riou's machinery to fabricate the *motivic p-adic L-function*.

The motivic p-adic L-function.

This construction is naturally divided into two steps. First, the localization at p of the big cohomology class can be projected to a Galois cohomology group

$$\boldsymbol{\kappa}_p := \mathrm{Im}(\boldsymbol{\kappa}_{\mathscr{G},\mathsf{f}_\circ}) \in \mathrm{H}^1\big(\mathbb{Q}_p, \mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}}(\boldsymbol{\Theta})\big)$$

valued in a subquotient $\mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}}(\boldsymbol{\Theta})$ of the Galois representation $\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ}$ on which $G_{\mathbb{Q}_p}$ acts through explicit characters. Perrin-Riou's big logarithm ([KLZ17], Theorem 8.2.3) valued in the big Dieudonné module $\mathbb{D}(\mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}})$ gives an element

$$\boldsymbol{\mathcal{L}}(\boldsymbol{\kappa}_p) \in \mathbb{D}\big(\mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}}\big)$$

interpolating the Bloch-Kato logarithm of the specialization of the class at arithmetic points of weight ≥ 2, and the Bloch-Kato dual exponential at the weight one point $\mathrm{P}_\circ$. The second step requires the construction of a linear map

$$\big\langle\ , \omega_{\breve{\mathscr{G}}} \otimes \eta'_\circ \big\rangle : \mathbb{D}\big(\mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}}\big) \longrightarrow \boldsymbol{\Pi} \otimes_{\boldsymbol{\Lambda}} \mathbf{I}_{\mathscr{G}},$$

where $\mathbf{\Pi} = \varprojlim_{\alpha} \mathbb{Q}_p[1 + p\mathbb{Z}_p/1 + p^{\alpha}\mathbb{Z}_p]$, producing functions out of elements of the Dieudonné module. As we are dealing with Hilbert modular surfaces and Ohta's Λ-adic Eichler-Shimura isomorphism [Oht95] is not available in this setting, the linear map we construct only takes values in the ring of $\mathbb{C}_p$-valued functions with domain the subset of arithmetic points of weight 2 in $\mathcal{A}(\mathbf{I}_{\mathscr{G}})$. The motivic p-adic L-function is defined as

$$\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}}, \mathsf{f}_\circ) := \left\langle \boldsymbol{\mathcal{L}}(\boldsymbol{\kappa}_p),\ \omega_{\breve{\mathscr{G}}} \otimes \eta'_\circ \right\rangle.$$

By construction, at a weight two point P of level p^{α} it specializes to (essentially)

$$\mathrm{AJ}_{\mathrm{syn}}(\Delta_\alpha^\circ)(\omega_{\breve{\mathscr{G}}_{\mathrm{P}}} \otimes \eta_\circ), \tag{1.1}$$

where $\omega_{\breve{\mathscr{G}}_{\mathrm{P}}}$ and $\eta_\circ$ are certain test vectors associated to $\mathscr{G}$ and $\mathsf{f}_\circ$.

A regulator formula and an explicit reciprocity law

Using Besser's theory of finite polynomial cohomology, we may evaluate (1.1) explicitly (the regulator formula) with a calculation similar in spirit to the ones found in [DR17], [LSZ16] and [BCF], establishing the following specialization formula for *every* arithmetic point $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight two:

$$\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)(\mathrm{P}) \dot{\sim} \left\langle e_{\mathrm{ord}} \zeta^* \big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathscr{G}}_{\mathrm{P}})^{[\mathcal{P}]}), \mathsf{f}_\circ^{(p)} \right\rangle_{\mathrm{Pet}},$$

a Petersson inner product involving (ordinary projections of) p-adic modular forms. This last value is almost exactly the special value of our automorphic p-adic L-function at the same weight two points, which is enough to show that $\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ) \in \mathbf{I}_{\mathscr{G}}$ and $\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}}, \mathsf{f}_\circ) \in \mathbf{\Pi} \otimes \mathbf{I}_{\mathscr{G}}$ coincide (up to a constant in $\mathbf{I}_{\mathscr{G}}$ which never vanishes at arithmetic points). This is our explicit reciprocity law.

As a side remark, this in particular shows that $\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)$ in fact lies in the smaller, much more manageable fraction field of $\mathbf{I}_{\mathscr{G}}^{\times}$. It can also be thought of as saying that the motivic p-adic L-function, which is a priori only defined on arithmetic points of weight two, can be meromorphically continued to the entire weight space (and thus justifies the terminology).

The explicit reciprocity law is a bridge between the automorphic and the algebro-geometric worlds, which can be used to transfer the information about the non-vanishing of a special L-value into the existence of non-trivial annihilators of the Mordell-Weil group.

Theorem 1.3.3. Suppose there is a rational prime p satisfying the assumptions as above. If $\mathsf{g}_\varrho^{(p)}$ is any ordinary p-stabilization of g_ϱ, then

$$L(E, \mathrm{As}(\varrho), 1) \neq 0 \qquad \Longleftrightarrow \qquad \boldsymbol{\kappa}_{\mathscr{G}, \mathsf{f}_\circ}(\mathrm{P}_\circ) \in \mathrm{H}^1\big(\mathbb{Q}, \mathrm{V}_{\varrho, E}\big) \quad \text{not crystalline at } p.$$

By assuming that p splits in L and the eigenvalues of Frob_p on $\mathrm{As}(\varrho)$ are all distinct, the eigenform g_ϱ has four distinct ordinary p-stabilizations. Hence, we obtain four linearly independent classes by repeatedly applying Theorem 1.3.3. As the self-dual representation $\mathrm{As}(\varrho)$ is four dimensional, these annihilators suffice to prove that the relevant part of the Mordell-Weil group is trivial.

1.4 Notations and conventions

Throughout this part of the thesis, we will adopt the following general notations and conventions.

- L is a real quadratic field (except for the preliminary materials of Chapter 3 which applies to all totally real fields L). Let $\mathcal{O}_L$ be its ring of integers, and $\mathfrak{d}_L$

the different ideal.

- $\mathsf{f}_\circ$ is an elliptic modular form of weight 2 and level N with trivial character, and g_0 is a Hilbert modular form over L of parallel weight 1 and level $\mathfrak{Q}$ with character $(\chi_\circ, \mathbb{1})$, such that N and $\mathfrak{Q}$ are coprime, and $\chi_\circ|_{\mathbb{Q}} = 1$. Put $M = N \cdot \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q})$.

- p is a prime number which splits completely over L into narrowly principal ideals: $p\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$. We assume that there are no totally positive units of $\mathcal{O}_L^\times$ congruent to -1 modulo p, that p is coprime to $M\mathfrak{d}_L$, and that p is Eisenstein for $\mathsf{f}_\circ$ (in other words, $\mathsf{f}_\circ$ is not congruent to an Eisenstein series modulo p).

- E is a finite extension of $\mathbb{Q}_p$ containing all Fourier coefficients of f_0 and g_0, and O is its ring of integers.

- Fix a system of compatible p-power roots of unity ζ_{p^α}.

- Fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. For any number field F inside $\bar{\mathbb{Q}}$, let G_F denote the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$. Similarly, fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. For any $E \subset \bar{\mathbb{Q}}_p$, let G_E denote the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E)$.

- Fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ which corresponds to a place of $\bar{\mathbb{Q}}$ above $\mathfrak{p}_1$. In this way we also identify a decomposition group D_p of $G_{\mathbb{Q}}$ at p, which is also a decomposition group $D_{\mathfrak{p}_1}$ of G_L at $\mathfrak{p}_1$, with $G_{\mathbb{Q}_p}$.

- For any function $\mathsf{f} : G(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$ (which in particular includes Hilbert modular forms) and any $A \in G(\mathbb{A}_{\mathbb{Q}})$, the slashing operator is given by

$$\mathsf{f}|_A(x) = \mathsf{f}(xA).$$

  In particular, $\mathsf{f}|_A|_B = \mathsf{f}|_{BA}$.

- The normalization of local class field theory is that the local Artin map sends a uniformizer to an arithmetic Frobenius, so that the p-adic cyclotomic character $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$ is the reciprocal of the map induced by the natural projection $\mathbb{Q}_p^\times \twoheadrightarrow \mathbb{Z}_p^\times$ via the local Artin map. In particular, for $a \in \mathbb{Z}_p^\times$, the element $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q})$ associated to a by class field theory sends ζ_{p^α} to $\zeta_{p^\alpha}^{a^{-1}}$. We will use adelic characters and the corresponding Galois characters interchangeably.

- For any p-adic representation V and character χ of $G_\mathbb{Q}$ or $G_{\mathbb{Q}_p}$, we will use $V(\chi) := V \otimes \chi$ to denote the twisted Galois representation. When χ is the m-th power of the cyclotomic character for some integer m, we simply write $V(m)$ for the Tate twist. To avoid overly cumbersome notation, when χ is the m-th power of the cyclotomic character times another Galois character η, we simply write $V(m+\eta)$ for the twist $V(\chi)$.

- $G = \mathrm{Res}^L_\mathbb{Q}\mathrm{GL}_2$ is the algebraic group over $\mathbb{Q}$ obtained from restriction of scalars.

- In general we will use K with various subscripts to denote open compact subgroups of $G(\mathbb{A}_{\mathbb{Q},f})$, and use a prime (e.g. K') to denote open compact subgroups of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$. We also use S to denote Shimura varieties for G of a certain level (e.g. $S(K)$), and Y to denote Shimura varieties for GL_2 (e.g. $Y(K')$).

- We will use letters in script font to denote Hida families or Λ-adic forms (e.g. $\mathscr{G}$), and use sans serif letters to denote classical modular forms (so we will use g_P to denote the classical specialization of a Hida family $\mathscr{G}$ at an arithmetic point P).

- We will use $\langle , \rangle$ to denote both the Petersson inner product of modular forms and the de Rham Poincaé pairing on modular varieties. In case there is risk

of confusion, we will use subscripts (Pet and dR) to differentiate the two. For the Poincaré pairing, if multiple pairings on different varieties are involved, we will also indicate them using subscripts (e.g. $\langle , \rangle_{\mathrm{dR}, Y_\alpha}$ would mean the de Rham Poincaré pairing on the modular curve Y_α).

Chapter 2

Preliminaries on Hilbert modular forms

In this chapter we summarize basic theory of Hilbert modular forms and Hida families. We follow the account of [Hid91].

2.1 Basic notations

Let L be a totally real number field with ring of integers $\mathcal{O}_L$ and different $\mathfrak{d}_L$. Later on in the paper we will focus on the case of L being a real quadratic field. Let I_L denote the set of embeddings of L into $\bar{\mathbb{Q}}$. A double digit weight, or simply a weight of modular forms is a pair of elements (k, w) in the free module $\mathbb{Z}[I_L]$ generated by the embeddings in I_L. Let $t_L := \sum_{\sigma\in \mathrm{I}_L} \sigma \in \mathbb{Z}[\mathrm{I}_L]$ denote the special "parallel weight one" element. We set $Z_L(1) := \mathbb{A}_L^\times / \overline{L^\times \det V(p^\infty) L_{\infty,+}^\times}$, the p-adic cyclotomic character

$$\varepsilon_L : Z_L(1) \longrightarrow \mathbb{Z}_p^\times, \qquad y \mapsto y_p^{-t_L} |y_f|_{\mathbb{A}_L}^{-1}.$$

It is clear that the composition of the natural diagonal embedding $\mathbb{A}_{\mathbb{Q}}^{\times} \hookrightarrow \mathbb{A}_L^{\times}$ with ε_L is precisely $\varepsilon_{\mathbb{Q}}^{[L:\mathbb{Q}]}$. The canonical isomorphism $\mathbb{Z}_p^{\times} \cong (1+p\mathbb{Z}_p) \times \mu_{p-1}$ induces a factorization of the character

$$\varepsilon_L = \eta_L \cdot \theta_L.$$

Definition 2.1.1. Let D, G and G^* be the following algebraic groups over $\mathbb{Q}$:

$$D = \mathrm{Res}_{\mathbb{Q}}^{L}\mathbb{G}_m, \qquad G = \mathrm{Res}_{\mathbb{Q}}^{L}\mathrm{GL}_2, \qquad G^* = G \times_D \mathbb{G}_m.$$

We now identify $L_\infty := L \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{R}^{I_L}$ and embed L into $\mathbb{R}^{I_L}$ via the diagonal map. Then the identity component $G(\mathbb{R})_+ = \mathrm{GL}_2(L \otimes_{\mathbb{Q}} \mathbb{R})_+$ of $G(\mathbb{R})$ naturally acts on $\mathcal{H}^{I_L}$ where $\mathcal{H}$ denotes the Poincar'e upper half plane, which contains a fixed square root $i = \sqrt{-1} \in \mathbb{C}$ of -1. Let $\mathbf{i} = (i, ..., i) \in \mathcal{H}^{I_L}$.

We will also use the following power convention repeatedly: for $c = (c_\sigma)_{\sigma \in I_L} \in \mathbb{C}^{I_L}$ and $s = \sum_{\sigma \in I_L} s_\sigma \cdot \sigma \in \mathbb{Z}[I_L]$, let $c^s = \Pi_{\sigma \in I_L} c_\sigma^{s_\sigma}$. As a slight extension of this notation, for $a \in L$, we write $a^s = \Pi_{\sigma \in I_L}(a^\sigma)^{s_\sigma}$, where $(a^\sigma)_{\sigma \in I_L}$ is the image of a under the natural diagonal embedding $L \hookrightarrow \mathbb{R}^{I_L}$.

2.2 Hilbert modular forms

We are now ready to recall the notion of (adelic) holomorphic Hilbert modular forms.

Definition 2.2.1. Let $U \subseteq G(\mathbb{A}_{\mathbb{Q},f})$ be an open compact subgroup, and $(k, w) \in \mathbb{Z}[I_L]^2$ be a double digit weight such that $k - 2w = mt_L$ for some $m \in \mathbb{Z}$. The space of holomorphic Hilbert modular forms of weight (k, w) and level U, which we denote by $M_{k,w}(U; \mathbb{C})$, is the space of functions $\mathsf{f} : G(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$ satisfying the following conditions:

- $\mathsf{f}(\alpha x u) = \mathsf{f}(x) j_{k,w}(u_\infty, \mathbf{i})^{-1}$ for $\alpha \in G(\mathbb{Q})$ and $u \in UC_\infty^+$, where C_∞^+ denotes the

stabilizer of $\mathbf{i}$ in $G(\mathbb{R})_+$ and $j_{k,w}$ is the automorphy factor $j_{k,w}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) = (ad-bc)^{-w}(cz+d)^k$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{R})$ and $z \in \mathcal{H}^{I_L}$;

- for every $x \in G(\mathbb{A}_{\mathbb{Q},f})$ the function $\mathsf{f}_x : \mathcal{H}^{I_L} \to \mathbb{C}$ given by $\mathsf{f}_x(z) = \mathsf{f}(xu_\infty)j_{k,w}(u_\infty, \mathbf{i})$ is holomorphic, where for each $z \in \mathcal{H}^{I_L}$ we choose $u_\infty \in G(\mathbb{R})_+$ such that $u_\infty \mathbf{i} = z$.

If in addition to the above f also satisfies

$$\int_{L\backslash\mathbb{A}_L} \mathsf{f}(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x)dx = 0$$

for all $x \in G(\mathbb{A})$ and all additive Haar measure on $L\backslash\mathbb{A}_L$, then we say f is a cuspform, and we denote by $S_{k,w}(U;\mathbb{C})$ the space of Hilbert modular cuspforms of weight (k,w) and level U.

Let dx be the Tamagawa measure on the quotient space $[G(\mathbb{A}_\mathbb{Q})] := \mathbb{A}_L^\times G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})$. Then for any $\mathsf{f}_1, \mathsf{f}_2 \in S_{k,w}(U;\mathbb{C})$ with $k - 2w = mt_L$, we define the Petersson inner product as

$$\langle \mathsf{f}_1, \mathsf{f}_2 \rangle := \int_{[G(\mathbb{A}_\mathbb{Q})]} \mathsf{f}_1(x)\overline{\mathsf{f}_2(x)}|\det(x)|_{\mathbb{A}_L}^m dx.$$

We now introduce the notation for some standard open compact subgroups of $G(\hat{\mathbb{Z}})$.

Definition 2.2.2. For any $\mathcal{O}_L$ ideal $\mathfrak{N}$, write

- $V_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\widehat{\mathbb{Z}}) \middle| \, c \in \mathfrak{N}\widehat{\mathcal{O}_F} \right\}$,

- $V_1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_0(\mathfrak{N}) \middle| \; d \equiv 1 \pmod{\mathfrak{N}\widehat{\mathcal{O}_F}} \right\}$,
- $V^1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_0(\mathfrak{N}) \middle| \; a \equiv 1 \pmod{\mathfrak{N}\widehat{\mathcal{O}_F}} \right\}$,
- $V(\mathfrak{N}) = V_1(\mathfrak{N}) \cap V^1(\mathfrak{N})$,
- $\Gamma(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(\mathfrak{N}) \middle| \; b \in \mathfrak{N}\widehat{\mathcal{O}_F} \right\}$.

From now we will focus the exposition on Hilbert modular forms with level of the form $V(\mathfrak{N})$. Any form of level U with U containing some $V(\mathfrak{N})$ can be viewed as a form of level $V(\mathfrak{N})$ and thus can be treated similarly.

2.3 Relation to classical Hilbert modular forms

Next we recall the relation of the adelic Hilbert modular forms to classical Hilbert modular forms. As above let $\mathfrak{N}$ be any ideal of $\mathcal{O}_L$. Consider the narrow class group $\mathrm{Cl}_L^+(\mathfrak{N}) := L_+^\times \backslash \mathbb{A}_{L,f}^\times / \det V(\mathfrak{N})$ with cardinality $h = h_L^+(\mathfrak{N})$, and fix a set of representatives $\{t_1, ..., t_h\} \subset \mathbb{A}_{L,f}^\times$ of $\mathrm{Cl}_L^+(\mathfrak{N})$ with $t_1 = 1$. We then obtain a decomposition of the adelic points of G

$$G(\mathbb{A}) = \coprod_{i=1}^{h} G(\mathbb{Q}) \begin{pmatrix} t_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} V(\mathfrak{N}) G(\mathbb{R})^+$$

via strong approximation.

Given a Hilbert cuspform $\mathsf{f} \in S_{k,w}(V(\mathfrak{N}); \mathbb{C})$, for each $i = 1, ..., h$, we consider the

following holomorphic function $\mathsf{f}_i : \mathcal{H}^{I_L} \to \mathbb{C}$ given by

$$\mathsf{f}_i(z) = y_\infty^{-w} \mathsf{f}\left(\begin{pmatrix} t_i^{-1} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix}\right)$$

where $z = x_\infty + \mathbf{i} y_\infty$, $x_\infty \in \mathbb{R}^{I_L}$ and $y_\infty \in \mathbb{R}_+^{I_L}$. Each f_i admits a Fourier expansion

$$\mathsf{f}_i(z) = \sum_{\xi \in (t_i \mathfrak{d}_L^{-1})_+} a(\xi, \mathsf{f}_i) e_L(\xi z) \tag{2.1}$$

where $e_L(\xi z) = \exp\big(2\pi i \sum_{\sigma \in I_L} \sigma(\xi) z_\sigma\big)$. The h-tuple of classical Hilbert modular forms $(\mathsf{f}_1, ..., \mathsf{f}_h)$ or their Fourier expansions determine f uniquely.

2.4 Adelic Fourier expansions

Fix $\mathsf{d}_L \in \mathbb{A}_{L,f}^\times$ such that $\mathsf{d}_L \mathcal{O}_L = \mathfrak{d}_L$ is the absolute different ideal of L. Let L^{Gal} be the Galois closure of L in $\bar{\mathbb{Q}}$ and write $\mathcal{V}$ for the ring of integers or a valuation ring of a finite extension L_0 of F^{Gal} such that for every ideal $\mathfrak{a}$ of $\mathcal{O}_L$ and all $\sigma \in I_L$, the ideal $\mathfrak{a}^\sigma \mathcal{V}$ is principal. Choose a generator $\{\mathfrak{q}^\sigma\} \in \mathcal{V}$ of $\mathfrak{q}^\sigma \mathcal{V}$ for each prime ideal $\mathfrak{q}$ of $\mathcal{O}_L$ and by multiplicativity define $\{\mathfrak{a}^v\} \in \mathcal{V}$ for each fractional ideal $\mathfrak{a}$ of L and each $v \in \mathbb{Z}[\mathrm{I}_L]$.

We may package the collection of classical Fourier expansions in (2.1) for each component f_i together into an Adelic Fourier expansion. As an additional piece of notation, let $\mathbb{A}_{L,+}^\times = \mathbb{A}_{L,f}^\times L_{\infty,+}^\times$.

Theorem 2.4.1. ([Hid91], Theorem 1.1) Let $e_L : \mathbb{C}^{I_L} \longrightarrow \mathbb{C}^\times$ be defined as $e_L(z) = \exp\big(2\pi i \sum_{\sigma \in I_L} z_\tau\big)$ and $\chi_L : \mathbb{A}_L / L \longrightarrow \mathbb{C}^\times$ the additive character of the ideles which satisfies $\chi_L(x) = e_L(x_\infty)$. Each cuspform $\mathsf{f} \in S_{k,w}(V(\mathfrak{N}); \mathbb{C})$ has an adelic q-expansion

of the form

$$\mathsf{f}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbb{A}_L} \sum_{\xi \in L_+} \mathsf{a}(\xi y \mathsf{d}_L, \mathsf{f})\{(\xi y \mathsf{d}_L)^{t_L - w}\}(\xi y_\infty)^{w - t_L} e_L(\mathbf{i}\xi y_\infty)\chi_L(\xi x)$$

for $y \in \mathbb{A}_{L,+}^\times$, $x \in \mathbb{A}_L^\times$, where $\mathsf{a}(-, \mathsf{f}) : \mathbb{A}_{L,+}^\times \longrightarrow \mathbb{C}$ vanishes outside $\widehat{\mathcal{O}}_L L_{\infty,+}^\times$ and depends only on the coset $y^\infty \det V(\mathfrak{N})$.

Concretely, the function $\mathsf{a}(-, \mathsf{f})$ is defined as follows: for every idele $y \in \mathbb{A}_{L,+}^\times$ can be written as $y = \xi t_i^{-1} \mathsf{d} u$ for some $1 \leq i \leq h$, $\xi \in L_+^\times$ and $u \in \det V(\mathfrak{N}) L_{\infty,+}^\times$, then

$$\mathsf{a}(y, \mathsf{f}) := a(\xi, \mathsf{f}_i)\{y^{w - t_L}\}\xi^{t_L - w}|t_i|_{\mathbb{A}_L}.$$

Furthermore, if all $\mathsf{a}(\xi, \mathsf{f}_i)$ belong to $\bar{\mathbb{Q}}$, then we define the p-adic Fourier coefficients $\mathsf{a}_p(-, \mathsf{f}) : \mathbb{A}_{L,+}^\times \longrightarrow \bar{\mathbb{Q}}_p$ by

$$\mathsf{a}_p(y, \mathsf{f}) := a(\xi, \mathsf{f}_i) y_p^{w - t_L} \xi^{t_L - w} \varepsilon_L(a_i)^{-1}.$$

Here $\varepsilon_L : Z_L(1) := \mathbb{A}_L^\times / \overline{L^\times \det V(p^\infty) L_{\infty,+}^\times} \to \bar{\mathbb{Q}}_p^\times$ is the p-adic cyclotomic character given by $y \mapsto y_p^{-t_F} |y^\infty|_{\mathbb{A}_L}^{-1}$.

Using the Fourier expansions we may define rational and integral structures on the space of Hilbert modular forms. For any $\mathcal{V}$-algebra A in $\mathbb{C}$, define

$$S_{k,w}(U; A) = \{\mathsf{f} \in S_{k,w}(U; \mathbb{C}) \mid \mathsf{a}(y, f) \in A\}.$$

2.5 Hecke Theory

Let $U \subset G(\mathbb{A}_{\mathbb{Q},f})$ be an open compact subgroup satisfying $V_0(\mathfrak{N}) \subset V(\mathfrak{N})$. Suppose $\mathcal{V}$ is the valuation ring corresponding to the fixed embedding $\iota_p : L^{\mathrm{Gal}} \hookrightarrow \bar{\mathbb{Q}}_p$, so that we may assume $\{y^{t_L - w}\} = 1$ whenever the ideal $y\mathcal{O}_L$ generated by y is prime to $p\mathcal{O}_L$.

For every $g \in G(\mathbb{A}_{\mathbb{Q}})$, we have the following double coset operator $[UgU]$. Decomposing the double coset into a disjoint union

$$UgU = \coprod_i \gamma_i U,$$

we define

$$\mathsf{f}|[UgU](x) = \sum_i \mathsf{f}(x\gamma_i).$$

Remark. This is the convention used in [Hid91] (see the discussion immediately following equation (2.1)). However, this differs slightly from the double coset operators used in [Hid88], [Hid89b] and [Hid04]. However, to keep the exposition consistent with our main reference [Hid91], we stick with the notation there.

Definition 2.5.1. For every prime ideal $\mathfrak{q}$ of $\mathcal{O}_L$ and a uniformizer $\varpi_\mathfrak{q}$ of $\mathcal{O}_{L,\mathfrak{q}}$, we define the Hecke operator given by the following double coset operator

$$T(\varpi_\mathfrak{q}) = \left[V(\mathfrak{N}) \begin{pmatrix} \varpi_\mathfrak{q} & 0 \\ 0 & 1 \end{pmatrix} V(\mathfrak{N})\right].$$

We will also write

$$T_0(\varpi_\mathfrak{q}) = \{\varpi_\mathfrak{q}^{w-t_L}\} T(\varpi_\mathfrak{q}).$$

Furthermore, for every $a \in \mathcal{O}_{L,\mathfrak{N}}^{\times} = \Pi_{\mathfrak{q}|\mathfrak{N}} \mathcal{O}_{L,\mathfrak{q}}^{\times}$, define the double coset operator

$$T(a,1) = \big[V(\mathfrak{N}) \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} V(\mathfrak{N})\big].$$

Finally, for $z \in Z_G(\mathbb{A}_{\mathbb{Q},f})$ in the center of $G(\mathbb{A}_{\mathbb{Q},f})$, we have the diamond operator $\langle z \rangle$ defined by

$$\langle z \rangle \mathsf{f}(x) = \mathsf{f}(xz).$$

If $\mathfrak{q}$ is coprime to the level $\mathfrak{N}$, then $T(\varpi_{\mathfrak{q}})$ (respectively, $T_0(\varpi_{\mathfrak{q}})$) is independent of the particular choice of the uniformizer $\varpi_{\mathfrak{q}}$, and thus we simply denote it by $T(\mathfrak{q})$ (respectively, $T_0(\mathfrak{q})$). Similarly, the diamond operator $\langle \varpi_{\mathfrak{q}} \rangle$ also only depends on the ideal $\mathfrak{q}$ and we denote it by $\langle \mathfrak{q} \rangle$. In general, for the $V(\mathfrak{N})$ level structure, if $\mathfrak{q}|\mathfrak{N}$, then $T(\varpi_{\mathfrak{q}})$ (respectively, $T_0(\varpi_{\mathfrak{q}})$) does depend on $\varpi_{\mathfrak{q}}$, and we will write it as $U(\varpi_{\mathfrak{q}})$ (respectively, $U_0(\varpi_{\mathfrak{q}})$).

The effects of the Hecke operators on the adelic Fourier coefficients are given by

$$\mathsf{a}_p(y, T(\mathfrak{q})\mathsf{f}) = \mathsf{a}_p(y\varpi_{\mathfrak{q}}, \mathsf{f})\varpi_{\mathfrak{q},p}^{t_L - w} + \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{q})\mathsf{a}_p(y\varpi_{\mathfrak{q}}^{-1}, \langle \mathfrak{q} \rangle \mathsf{f})\varpi_{\mathfrak{q},p}^{w - t_L} \tag{2.2}$$

and

$$\mathsf{a}(y, T_0(\mathfrak{q})\mathsf{f}) = \mathsf{a}_p(y\varpi_{\mathfrak{q}}, \mathsf{f}) + \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{q})\{\mathfrak{q}^{2(w-t_L)}\}\mathsf{a}_p(y\varpi_{\mathfrak{q}}^{-1}, \langle \mathfrak{q} \rangle \mathsf{f}), \tag{2.3}$$

if $\mathfrak{q} \nmid \mathfrak{N}$, and

$$\mathsf{a}_p(y, U(\varpi_{\mathfrak{q}})\mathsf{f}) = \mathsf{a}_p(y\varpi_{\mathfrak{q}}, \mathsf{f})\varpi_{\mathfrak{q},p}^{t_L - w} \tag{2.4}$$

and

$$\mathsf{a}(y, U_0(\varpi_{\mathfrak{q}})\mathsf{f}) = \mathsf{a}(y\varpi_{\mathfrak{q}}, \mathsf{f}) \tag{2.5}$$

if $\mathfrak{q}|\mathfrak{N}$. Here $\varpi_{\mathfrak{q},p}$ denotes the p-component of $\varpi_{\mathfrak{q}}$ (in articular it is always trivial

unless $\mathfrak{q}|p\mathcal{O}_L$). Furthermore, for $a \in \mathcal{O}_{L,\mathfrak{N}}^{\times}$,

$$\mathsf{a}_p(y, T(a,1)\mathsf{f}) = \mathsf{a}_p(ya, \mathsf{f})a_p^{t_L - w}.$$

Definition 2.5.2. The Hecke algebra $\mathbf{h}_{k,w}(U;\mathcal{V})$ is defined as the $\mathcal{V}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_{k,w}(U;\mathbb{C}))$ generated by the Hecke operators $T_0(\mathfrak{q})$ for $\mathfrak{q}$ outside $\mathfrak{N}$, $U_0(\varpi_{\mathfrak{q}})$ for $\mathfrak{q}|\mathfrak{N}$, and $T(a,b)$ for $a, b \in \mathcal{O}_{L,\mathfrak{N}}^{\times}$. For each $\mathcal{V}$-algebra A in $\mathbb{C}$, define

$$\mathbf{h}_{k,w}(U;A) = \mathbf{h}_{k,w}(U;\mathcal{V}) \otimes_{\mathcal{V}} A.$$

Theorem 2.5.3. ([Hid91], Theorem 2.2) Let F/L^{Gal} be any finite field extension and A any $\mathcal{V}$-subalgebra of F. Then there is a natural isomorphism

$$S_{k,w}(U;F) \cong S_{k,w}(U;A) \otimes_A F.$$

Moreover, if A is an integrally closed domain containing $\mathcal{V}$, finite flat over either $\mathcal{V}$ or $\mathbb{Z}_p$, then $S_{k,w}(U;A)$ is stable under the action of $\mathbf{h}_{k,w}(U;A)$, and the pairing

$$(,) : S_{k,w}(U;A) \times \mathbf{h}_{k,w}(U;A) \longrightarrow A, \qquad (f,h) = \mathsf{a}(1, h \cdot f)$$

induces isomorphisms of A-modules

$$\mathbf{h}_{k,w}(U;A) \cong S_{k,w}(U;A)^*, \qquad S_{k,w}(U;A) \cong \mathbf{h}_{k,w}(U;A)^*,$$

where $(-)^*$ denotes the A-linear dual $\mathrm{Hom}_A(-,A)$.

As an extra piece of notation, for any $y \in \hat{\mathcal{O}}_L^{\times}$, we can factor it as $y = au\Pi_{\mathfrak{q}}\varpi_{\mathfrak{q}}^{e(\mathfrak{q})}$ with $a \in \mathcal{O}_{L,\mathfrak{N}}^{\times}$ and $u \in \det V(\mathfrak{N})$. Write $\mathfrak{n}$ for the ideal $(\Pi_{\mathfrak{q}|\mathfrak{N}}\varpi_{\mathfrak{q}}^{e(\mathfrak{q})})\mathcal{O}_L$, then we define

the Hecke operator

$$T(y) = T(a,1)T(\mathfrak{n})\Pi_{\mathfrak{q}|\mathfrak{N}}U(\varpi_{\mathfrak{q}}^{e(\mathfrak{q})}),$$

which only depends on the idele y. We also put $T_0(y) = \{y^{w-t_L}\}T(y)$.

A cuspform f is said to be an eigenform if it is an eigenvector for all the Hecke operators, and it is normalized if $\mathsf{a}(1,\mathsf{f}) = 1$. Note that for a normalized eigenform f the $T_0(y)$-eigenvalue is $\mathsf{a}(y,\mathsf{f})$ for every idele y. By a result of Shimura ([Shi78], Proposition 2.2), the Hecke eigenvalues of an eigenform are all algebraic numbers. In particular, every normalized eigenform f is defined over $\bar{\mathbb{Q}}$.

We next take a closer look at the behaviors at p.

Definition 2.5.4. Let $\mathfrak{p}|p$ be a prime ideal of $\mathcal{O}_L$ and $\varpi_{\mathfrak{p}} \in \mathcal{O}_{L,\mathfrak{p}}$ a uniformizer. A normalized eigenform $\mathsf{f} \in S_{k,w}(U;\bar{\mathbb{Q}})$ is said to be nearly ordinary at $\mathfrak{p}$ if the $U_0(\varpi_{\mathfrak{p}})$ (or $T_0(\mathfrak{p})$ if $\mathfrak{p}$ is away from the level) eigenvalue is a p-adic unit with respect to the fixed embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Note that this notion is independent of the choice of the uniformizer. If f is nearly ordinary at all $\mathfrak{p}|p$, we say f is p-nearly ordinary.

We also have the following V operator:

Definition 2.5.5. For every finite idele $b \in \mathbb{A}_{L,f}^{\times}$ there is an operator $V(b)$ on the space of cuspforms defined by

$$(V(b)\mathsf{f})(x) = \mathrm{N}_{L/\mathbb{Q}}(b\mathcal{O}_L)\mathsf{f}(x\begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix}).$$

Remark. In [Hid91] Section 7B this operator is denoted as $[b]$. The effect on Fourier coefficients is

$$\mathsf{a}_p(y, V(b)\mathsf{f}) = b_p^{w-t_L}\mathsf{a}_p(yb^{-1},\mathsf{f}).$$

In particular, combined with equation (2.4) it shows

$$U(\varpi_{\mathfrak{q}})V(\varpi_{\mathfrak{q}}) = 1$$

for all $\varpi_{\mathfrak{q}}$ a uniformizer of a prime ideal dividing the level $\mathfrak{N}$.

2.6 p-adic modular forms

Now fix a valuation ring O in $\bar{\mathbb{Q}}_p$ finite flat over $\mathbb{Z}_p$ containing $\iota_p(\mathcal{V})$. We also fix an ideal $\mathfrak{N}$ of $\mathcal{O}_L$ which is prime to p and let K be a subgroup of $V_0(\mathfrak{N})$ containing $V_1(\mathfrak{N})$. Put $K(p^\alpha) = K \cap V(p^\alpha)$ and $K(p^\infty) = \cap_\alpha K(p^\alpha)$.

Consider the limit

$$S_{k,w}(K(p^\infty); O) = \varinjlim_{\alpha} S_{k,w}(K(p^\alpha); O)$$

on which the universal p-adic Hecke algebra

$$\mathbf{h}_{k,w}(K(p^\infty); O) = \varprojlim_{\alpha} \mathbf{h}_{k,w}(K(p^\alpha); O)$$

naturally acts. Inside $\mathbf{h}_{k,w}(K(p^\infty); O)$ we have the operators

$$\mathbf{T}(y) = \varprojlim_{\alpha} T(y) y_p^{w-t_L}.$$

There is a p-adic norm on $S_{k,w}(K(p^\infty); O)$ given by $|\mathsf{f}|_p = \sup_y(|\mathsf{a}_p(y, \mathsf{f})|)$. We denote the completion of $S_{k,w}(K(p^\infty); O)$ under this norm by $\bar{\mathbf{S}}_{k,w}(K(p^\infty); O)$. The p-adic Fourier coefficients can be regarded as a continuous function

$$\mathscr{J} \to O, \qquad y \mapsto \mathsf{a}_p(y, \mathsf{f}),$$

where $\mathscr{J} = (\widehat{\mathcal{O}}_L \cap \mathbb{A}_{L,f}^\times)/(K(p^\infty) \cap \mathbb{A}_{L,f}^\times) = \varprojlim_\alpha (\widehat{\mathcal{O}}_L \cap \mathbb{A}_{L,f}^\times)/(K(p^\alpha) \cap \mathbb{A}_{L,f}^\times)$ is endowed with profinite topology and is isomorphic to $\mathcal{O}_p^\times \times \mathscr{I}_L$, where $\mathscr{I}_L$ denotes the semigroup of all integral ideals of L. Writing $\mathscr{C}(X, A)$ for the space of continuous functions on a topological space X with values in A, we see that we have an embedding of $\bar{\mathbf{S}}_{k,w}(K(p^\infty); O)$ into $\mathscr{C}(\mathscr{J}, O)$.

Theorem 2.6.1. The following statements hold regarding the universal p-adic Hecke algebra:

1. ([Hid89b], Theorem 2.3) for all weights $(k, w) \in \mathbb{Z}[I_L]^2$ satisfying $k - 2w = mt_L$ for some $m \in \mathbb{Z}$, there is a canonical algebra isomorphism

$$\mathbf{h}_{k,w}(K(p^\infty); O) \cong \mathbf{h}_{2t_L, t_L}(K(p^\infty); O)$$

   sending $\mathbf{T}(y)$ to $\mathbf{T}(y)$;

2. ([Hid91], Theorem 3.1) the pairing

$$(,) : \ \mathbf{h}_{k,w}(K(p^\infty); O) \times \bar{\mathbf{S}}_{k,w}(K(p^\infty); O) \longrightarrow O, \qquad (\mathsf{h}, \mathsf{f}) = \mathsf{a}_p(1, \mathsf{h} \cdot \mathsf{f})$$

   induces isomorphisms

$$\mathbf{h}_{k,w}(K(p^\infty); O)^* \cong \bar{\mathbf{S}}_{k,w}(K(p^\infty); O) \text{ and } \bar{\mathbf{S}}_{k,w}(K(p^\infty); O)^* \cong \mathbf{h}_{k,w}(K(p^\infty); O).$$

As an immediate corollary, the image of $\bar{\mathbf{S}}_{k,w}(K(p^\infty); O)$ inside $\mathscr{C}(\mathscr{J}, O)$ is independent of the weight (k, w), and we will simply denote it by $\bar{\mathbf{S}}_L(K; O)$. Similarly, we will use $\mathbf{h}_L(K; O)$ to denote $\mathbf{h}_{k,w}(K(p^\infty); O)$. Moreover, if $K = V_1(\mathfrak{N})$, we will simply write $\bar{\mathbf{S}}_L(\mathfrak{N}; O)$ and $\mathbf{h}_L(\mathfrak{N}; O)$. $\bar{\mathbf{S}}_L(K; O)$ is called the space of p-adic cuspforms.

Let $Z_L(K) = \mathbb{A}_L^\times / \overline{L^\times (\mathbb{A}_{L,f} \cap K(p^\infty)) L_{\infty,+}^\times}$. In the case $K = V_1(\mathfrak{N})$, $Z_L(K)$ can

be identified with the strict ray class group of L of modulus $\mathfrak{N}p^\infty$. Put $\mathbb{G}_L(K) = Z_L(K) \times \mathcal{O}_{L,p}^\times$. Then $(z,a) \in \mathbb{G}_L(K)$ acts on $S_{k,w}(K(p^\alpha);O)$ via the operator $\langle z,a\rangle := T(a^{-1},1)\langle z\rangle$, which we will also call diamond operators. In this way, we have a natural embedding of $\mathbb{G}_L(K)$ and hence $O[\![\mathbb{G}_L(K)]\!]$ into $\mathbf{h}_L(\mathfrak{N};O)$, allowing us to view $\mathbf{h}_L(\mathfrak{N};O)$ as an $O[\![\mathbb{G}_L(K)]\!]$-algebra.

Now we normalize this new operator by setting $\langle z,a\rangle_{k,w} = \varepsilon_L(z)^{k-2w}\langle z,a\rangle$, which factors through the finite quotient $\mathrm{Cl}_L(\mathfrak{N}p^\alpha) \times (\mathcal{O}_L/p^\alpha)^\times$ of $\mathbb{G}_L(K)$.

Definition 2.6.2. For characters $\psi : \mathrm{Cl}(\mathfrak{N}p^\alpha) \to O^\times$ and $\psi' : (\mathcal{O}_L/p^\alpha)^\times \to O^\times$, let

$$S_{k,w}(K(p^\alpha),\psi,\psi';O) = \{\mathsf{f} \in S_{k,w}(K(p^\alpha);O) : \langle z,a\rangle\mathsf{f} = \psi(z)\psi'(a)\mathsf{f}, \forall (z,a) \in \mathbb{G}_L(K)\}.$$

2.7 Hida families

Since $\mathbf{h}_L(K;O)$ is a compact ring, we can decompose it as a direct sum of algebras

$$\mathbf{h}_L(K;O) = \mathbf{h}_L^{\text{n.o.}}(K;O) \oplus \mathbf{h}_L^{\text{ss}}(K;O)$$

in such a way that $\mathbf{T}(p)$ is a unit in $\mathbf{h}_L^{\text{n.o.}}(K;O)$ and is topologically nilpotent in $\mathbf{h}_L^{\text{ss}}(K;O)$. We denote by

$$e_{\text{n.o.}} = \lim_{n\to\infty} \mathbf{T}(p)^{n!}$$

the idempotent of the nearly ordinary part. Let $\overline{\mathbf{S}}_L^{\text{n.o.}}(K;O) = e_{\text{n.o.}}\overline{\mathbf{S}}_L(K;O)$ be the space of nearly ordinary p-adic cuspforms. The ring $\mathbf{h}_L^{\text{n.o.}}(K;O)$ is called the universal nearly ordinary Hecke algebra.

We have a short exact sequence

$$1 \longrightarrow \overline{\mathcal{E}}_{\mathfrak{N}p}^+ \backslash \big(1 + p\mathcal{O}_{L,p}\big) \longrightarrow Z_L(K) \longrightarrow \mathrm{Cl}_L^+(\mathfrak{N}p) \longrightarrow 1,$$

where $\mathrm{Cl}_L^+(\mathfrak{N}p)$ is the strict ray class group of modulus $\mathfrak{N}p$ and $\overline{\mathcal{E}}_{\mathfrak{N}p}^+$ is the closure of the totally positive global units of $\mathcal{O}_L$ congruent to 1 modulo $\mathfrak{N}p$. When p is large enough, the group $\mathrm{Cl}_L^+(\mathfrak{N}p)$ has order prime to p, in which case there is a canonical decomposition

$$\begin{aligned} Z_L(K) &\cong \overline{\mathcal{E}}_{\mathfrak{N}p}^+ \backslash \big(1 + p\mathcal{O}_{L,p}\big) \times \mathrm{Cl}_L^+(\mathfrak{N}p) \\ z &\mapsto \big(\xi_z, \bar{z}\big). \end{aligned} \tag{2.6}$$

Putting $\mathfrak{I}_L = \overline{\mathcal{E}}_{\mathfrak{N}p}^+ \backslash \big(1 + p\mathcal{O}_{L,p}\big) \times (1 + p\mathcal{O}_{L,p})$, we obtain a short exact sequence

$$1 \longrightarrow \mathfrak{I}_L \longrightarrow \mathbb{G}_L(K) \longrightarrow \mathrm{Cl}_L^+(\mathfrak{N}p) \times (\mathcal{O}_L/p)^\times \longrightarrow 1$$

which splits canonically. The group $\mathfrak{I}_L$ is a finitely generated $\mathbb{Z}_p$-module of $\mathbb{Z}_p$-rank $r = [L : \mathbb{Q}] + 1 + \delta$, where δ is Leopoldt's defect for L.

Now fix a decomposition of $\mathfrak{I}_L = \mathbf{W} \times \mathfrak{I}_{L,\mathrm{tor}}$, where $\mathfrak{I}_{L,\mathrm{tor}}$ is the torsion subgroup of $\mathfrak{I}_L$ and $\mathbf{W}$ is a free $\mathbb{Z}_p$-module of rank $[L : \mathbb{Q}] + 1 + \delta$. Put $\mathbf{\Lambda}_L = O[\![\mathbf{W}]\!]$.

Theorem 2.7.1. ([Hid89b], Theorem 2.4) The universal nearly ordinary Hecke algebra $\mathbf{h}_L^{\mathrm{n.o.}}(K; O)$ is finite and torsion-free over $\mathbf{\Lambda}_L$.

For each pair of characters $\psi : \mathrm{Cl}(\mathfrak{N}p^\alpha) \to O^\times$ and $\psi' : (\mathcal{O}_L/p^\alpha)^\times \to O^\times$ and a weight $(k, w) \in \mathbb{Z}[I_L]$ satisfying $k - 2w = mt_L$ for some integer m, let $\mathrm{P}_{k,w,\psi,\psi'}$ be the O-algebra homomorphism $O[\![\mathbb{G}_L(K)]\!] \to O$ induced by the character $\mathbb{G}_L(K) \ni (z, a) \mapsto \psi(z)\psi'(a)\varepsilon_L^m(z)a^{t_L - w}$.

Let $\mathbf{F}$ be the fraction field of $\mathbf{\Lambda}_L$, and fix an algebraic closure $\bar{\mathbf{F}}$ of $\mathbf{F}$. Consider a $\mathbf{\Lambda}$-linear map $\lambda : \mathbf{h}_L^{\mathrm{n.o.}}(K; O) \to \bar{\mathbf{F}}$. Since $\mathbf{h}_L^{\mathrm{n.o.}}(K; O)$ is finite over $\mathbf{\Lambda}_L$, the image of λ is contained in the integral closure $\mathbf{I}$ of $\mathbf{\Lambda}_L$ in a finite extension $\mathbf{K}$ of $\mathbf{F}$. If $\mathrm{P} : \mathbf{I} \to O$ is an O-algebra homomorphism which coincides with some $\mathrm{P}_{k,w,\psi,\psi'}$ on $\mathbf{\Lambda}_L$,

then by Theorem 2.4 of [Hid89b], the composition $\lambda_{\mathrm{P}} = \mathrm{P} \circ \lambda$ induces an O-linear map $\lambda_{\mathrm{P}} : \mathbf{h}^{\mathrm{n.o.}}_{k,w}(K(p^\alpha); \bar{\mathbb{Q}}_p) \to \bar{\mathbb{Q}}_p$ for a suitable $\alpha > 0$, and hence by duality (Theorem 2.5.3) we obtain a Hilbert cuspform $\mathsf{f}_{\mathrm{P}} \in S_{k,w}(K(p^\alpha); \bar{\mathbb{Q}}_p)$ such that $\mathsf{a}_p(y, \mathsf{f}_{\mathrm{P}}) = \lambda_{\mathrm{P}}(\mathbf{T}(y))$ for all integral idele y. If λ is a $\mathbf{\Lambda}_L$-algebra homomorphism then f_{P} is an eigenform. This justifies the following definition:

Definition 2.7.2. The space of nearly ordinary $\mathbf{I}$-adic cuspforms of tame level K is

$$\bar{\mathbf{S}}^{\mathrm{n.o.}}_L(K; \mathbf{I}) := \mathrm{Hom}_{\mathbf{\Lambda}_L}(\mathbf{h}^{\mathrm{n.o.}}_L(K(p^\infty); O), \mathbf{I}).$$

If an $\mathbf{I}$-adic cuspforms is also a $\mathbf{\Lambda}_L$-algebra homomorphism, then we say it is a Hida family.

We can write $O[\![\mathbb{G}_L(\mathfrak{N})]\!] = \bigoplus_\chi \mathbf{\Lambda}_{L,\chi}$ as a direct sum ranging over all the characters of the torsion subgroup $\mathbb{G}_L(\mathfrak{N})_{\mathrm{tor}} = \mathfrak{I}_{L,\mathrm{tor}} \times \mathrm{Cl}^+_L(\mathfrak{N}p) \times (\mathcal{O}_L/p)^\times$, where each $\mathbf{\Lambda}_{L,\chi}$ is isomorphic to $\mathbf{\Lambda}_L$. Similarly there is a decomposition of the universal nearly ordinary Hecke algebra $\mathbf{h}^{\mathrm{n.o.}}_L(K; O) = \bigoplus_\chi \mathbf{h}^{\mathrm{n.o.}}_L(K; O)_\chi$. Then we may refine the above definition as follows:

Definition 2.7.3. Let $\chi : \mathbb{G}_L(K)_{\mathrm{tor}} \to O^\times$ be a character and $\mathbf{I}$ a $\mathbf{\Lambda}_{L,\chi}$-algebra. The space of nearly ordinary $\mathbf{I}$-adic cuspforms of tame level K and character χ is

$$\bar{\mathbf{S}}^{\mathrm{n.o.}}_L(K, \chi; \mathbf{I}) := \mathrm{Hom}_{\mathbf{\Lambda}_\chi}(\mathbf{h}^{\mathrm{n.o.}}_L(K(p^\infty)_\chi; O), \mathbf{I}).$$

Definition 2.7.4. Let $\mathbf{I}$ be a $\mathbf{\Lambda}_{L,\chi}$-algebra. The set of *arithmetic points* $\mathcal{A}_{L,\chi}(\mathbf{I})$ is the subset of $\mathrm{Hom}_{O\text{-alg}}(\mathbf{I}, \bar{\mathbb{Q}}_p)$ consisting of homomorphisms that coincide with some $\mathrm{P}_{k,w,\psi,\psi'}$ on $\mathbf{\Lambda}_{L,\chi}$.

2.8 Diagonal restriction

If L/F is an extension of totally real fields, there is a restriction map $\mathrm{I}_L \to \mathrm{I}_F$ which induces a group homomorphism $\mathbb{Z}[\mathrm{I}_L] \to \mathbb{Z}[\mathrm{I}_F]$ denoted by $\ell \mapsto \ell_{|F}$ and satisfies $(t_L)_{|F} = [L : F] \cdot t_F$. Let $\mathfrak{N}$ be an ideal of $\mathcal{O}_F$, then the natural inclusion $\zeta : \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \hookrightarrow \mathrm{GL}_2(\mathbb{A}_L)$ defines by composition a *diagonal restriction* map

$$\zeta^* : S_{\ell,x}(V(\mathfrak{N}\mathcal{O}_L); L; \mathbb{C}) \to S_{\ell_{|F},x_{|F}}(V(\mathfrak{N}); F; \mathbb{C}).$$

Lemma 2.8.1. Let $\mathsf{g} \in S_{\ell,x}(V(\mathfrak{N}\mathcal{O}_L); L; \bar{\mathbb{Q}})$, then for $y \in \hat{\mathcal{O}}_F F_{\infty,+}^{\times}$ written as $y = \xi a_i^{-1} \mathsf{d}_F u$ where $\xi \in F_+^{\times}$ and $u \in \det V(\mathfrak{N}) F_{\infty,+}^{\times}$, we have

$$\mathsf{a}_p(y, \zeta^*\mathsf{g}) = y_p^{x_{|F}-t_F} \xi^{t_F - x_{|F}} \varepsilon_F(a_i)^{-1} \sum_{\mathrm{Tr}_{L/F}(\eta)=\xi} \mathsf{a}_p(y_\eta, \mathsf{g})(y_\eta)_p^{t_L - x} \eta^{x - t_L}$$

where $\eta \in L_+^{\times}$ and $y_\eta = \eta a_i^{-1} \mathsf{d}_L u$.

Proof. A direct computation. □

Remark. When p is unramified in $L/\mathbb{Q}$ one sees that $y_p = (\xi u)_p$, $(y_\eta)_p = (\eta u)_p$ and the formula becomes

$$\mathsf{a}_p(y, \zeta^*\mathsf{g}) = \varepsilon_F(a_i)^{-1} \sum_{\mathrm{Tr}_{L/F}(\eta)=\xi} \mathsf{a}_p(y_\eta, \mathsf{g})$$

2.9 Twists of Hilbert modular forms.

Hida defined two kinds of twists for Hilbert modular forms in [Hid91], Section 7F. We recall these constructions and then modify one of them to get a new one.

Let $\Psi : \mathbb{A}_L^{\times}/L^{\times} \to \mathbb{C}^{\times}$ be a Hecke character of conductor $C(\Psi)$ and infinity type

$m \cdot t_L$, $m \in \mathbb{Z}$. Since Ψ has algebraic values on finite ideles, one has the map

$$- \otimes \Psi : S_{k,w}\big(\mathfrak{N}p^{\alpha}, \psi, \psi'; O\big) \to S_{k,w+m\cdot t_L}\big(C(\Psi)\mathfrak{N}p^{\alpha}, \psi\Psi^2, \psi'\Psi_p^{-1}; O\big)$$
$$\mathsf{f} \mapsto \mathsf{f} \otimes \Psi$$

where $\mathsf{f} \otimes \Psi$ has adelic Fourier coefficients

$$\mathsf{a}_p(y, \mathsf{f} \otimes \Psi) = \Psi(y^{\infty})\mathsf{a}_p(y, \mathsf{f})y_p^{m\cdot t_L}.$$

In the special case when $\Psi = |-|_{\mathbb{A}_L}^m$, one finds that

$$\mathsf{f} \otimes |-|_{\mathbb{A}_L}^m \in S_{k,w+m\cdot t_L}\big(\mathfrak{N}p^{\alpha}, \psi, \psi'; O\big)$$

and

$$\mathsf{a}_p(y, \mathsf{f} \otimes |-|_{\mathbb{A}_L}^m) = \varepsilon_F(y)^{-m}\mathsf{a}_p(y, \mathsf{f}).$$

Since Ψ is trivial on the principal ideles, it follows immediately from Lemma 2.8.1 we have the following

Lemma 2.9.1. Let L/F be a finite extension of totally real fields as above, and let $\Psi_F = \Psi|_{\mathbb{A}_F^{\times}}$ be the restriction of the Hecke character Ψ of L to F. Then for any Hilbert modular form g over L,

$$\zeta^*(\mathsf{g} \otimes \Psi) = \zeta^*(\mathsf{g}) \otimes \Psi_F.$$

Another consequence is that this kind of twisting does not affect the classical Fourier expansion on the identity component of the Hilbert modular surface, given that we have fixed a set of representatives $\{t_1 = 1, t_2, ..., t_h\}$ of $\mathrm{cl}_L^+(\mathfrak{N})$ in Section 2.3. More precisely, we have the following

Lemma 2.9.2. Let $\mathsf{g} \in S_{k,w}\big(\mathfrak{N}p^{\alpha}, \psi, \psi'; O\big)$ and g_1 be the first component of the corresponding tuple of classical Hilbert modular forms. Then for any Hecke character $\Psi : \mathbb{A}_L^{\times}/L^{\times} \to \mathbb{C}^{\times}$, we have

$$(\mathsf{g} \otimes \Psi)_1 = \psi(d_L)\mathsf{g}_1.$$

Proof. Since we have chosen $t_1 = 1$ and the idele generator d_L of the different $\mathfrak{d}_L$ to have trivial p-component, by definition the classical Fourier coefficients $\mathsf{a}(\xi, \mathsf{g}_1)$ is related to the p-adic Fourier coefficients via (1.3 a of [Hid91])

$$\mathsf{a}_p(\xi d_L, \mathsf{g}) = \mathsf{a}(\xi, \mathsf{g}_1)$$

where $\xi \in L_+$ and ξd_L is understood as an adele.

Suppose Ψ has infinity type $m \cdot t_L$. By construction of the twist we have

$$\mathsf{a}_p(\xi d_L, \mathsf{g} \otimes \Psi) = \mathsf{a}_p(\xi d_L, \mathsf{g})\Psi((\xi d_L)^{\infty})\xi^{m \cdot t_L} = \mathsf{a}_p(\xi d_L, \mathsf{g})\Psi(d_L)\Psi(\xi) = \mathsf{a}_p(\xi d_L, \mathsf{g})\Psi(d_L)$$

where we regard ξ as an adele via diagonal embedding. This implies

$$\mathsf{a}(\xi, (\mathsf{g} \otimes \Psi)_1) = \Psi(d_L)\mathsf{a}(\xi, \mathsf{g}_1)$$

which finishes the proof. □

Now let $\mathfrak{N}, \mathfrak{A}$ be integral $\mathcal{O}_L$-ideals prime to p and let $\chi : \mathrm{Cl}_L^{+}(\mathfrak{A}p^{\alpha}) \to O^{\times}$ be a finite order Hecke character. Hida defined a function

$$\begin{aligned} -_{|\chi} : S_{k,w}\big(\mathfrak{N}p^{\alpha}, \psi, \psi'; O\big) &\to S_{k,w}\big(\mathfrak{N}p^{\alpha}\mathfrak{A}^2, \psi\chi^2, \psi'; O\big) \\ \mathsf{f} &\mapsto \mathsf{f}_{|\chi} \end{aligned}$$

where $\mathsf{f}_{|\chi}$ is a cuspform whose adelic Fourier coefficients are given by

$$\mathsf{a}_p(y, \mathsf{f}_{|\chi}) = \begin{cases} \chi(y)\mathsf{a}_p(y, \mathsf{f}) & \text{if } y_{\mathfrak{A}_p} \in \mathcal{O}_{L,\mathfrak{A}_p}^{\times} \\ 0 & \text{otherwise} \end{cases}$$

Finally, we define a new kind of twist by slightly modifying Hida's work. Let $\mathfrak{p} \mid p$ be a $\mathcal{O}_L$-prime ideal and $\chi : \mathcal{O}_{L,\mathfrak{p}}^{\times} \to O^{\times}$ a finite order character of conductor $\mathfrak{p}^{c(\chi)}$.

Proposition 2.9.3. There is a function

$$\begin{gathered} - \star \chi : S_{k,w}\big(\mathfrak{N}p^{\alpha}, \psi, \psi'; O\big) \to S_{k,w}\big(\mathfrak{N}p^{\max\{\alpha, c(\chi)\}}, \psi, \psi'\chi^{-1}; O\big) \\ \mathsf{f} \mapsto \mathsf{f} \star \chi \end{gathered}$$

where the adelic Fourier coefficients of $\mathsf{f} \star \chi$ are given by

$$\mathsf{a}_p(y, \mathsf{f} \star \chi) = \begin{cases} \chi(y_{\mathfrak{p}})\mathsf{a}_p(y, \mathsf{f}) & \text{if } y_{\mathfrak{p}} \in \mathcal{O}_{L,\mathfrak{p}}^{\times}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Define

$$\mathsf{h}(x) := \sum_{u \in \left(\mathcal{O}_{F,\mathfrak{p}}/\mathfrak{p}^{c(\chi)}\right)^{\times}} \chi^{-1}(u)\mathsf{f}\left(x \begin{pmatrix} 1 & u\varpi_{\mathfrak{p}}^{-c(\chi)}\mathsf{d}_L \\ 0 & 1 \end{pmatrix}\right).$$

Looking at the adelic q-expansion we see that

$$\mathsf{a}_p(y, \mathsf{h}) = \left(\sum_{u \in \left(\mathcal{O}_{L,\mathfrak{p}}/\mathfrak{p}^{c(\chi)}\right)^{\times}} \chi^{-1}(u)\chi_L\big(yu\varpi_{\mathfrak{p}}^{-c(\chi)}\big)\right)\mathsf{a}_p(y, \mathsf{f}).$$

Then one notices that

$$\sum_{u\in\left(\mathcal{O}_{L,\mathfrak{p}}/\mathfrak{p}^{c(\chi)}\right)^{\times}}\chi^{-1}(u)\chi_L\left(yu\varpi_{\mathfrak{p}}^{-c(\chi)}\right)=\begin{cases}\chi(y_{\mathfrak{p}})G(\chi) & \text{if } y_{\mathfrak{p}}\in\mathcal{O}_{L,\mathfrak{p}}^{\times};\\ 0 & \text{otherwise;}\end{cases}$$

where

$$G(\chi)=\sum_{u}\chi^{-1}(u)\chi_L\left(u\varpi_{\mathfrak{p}}^{-c(\chi)}\right) \tag{2.7}$$

is a Gauss sum. Hence, we define

$$\mathsf{f}\star\chi:=G(\chi)^{-1}\mathsf{h}.$$

Finally, to understand the action of the operators $T(a^{-1},1)$ we directly compute that

$$\mathsf{a}_p\left(y,(\mathsf{f}\star\chi)_{|T(a^{-1},1)}\right)=\chi^{-1}(a_{\mathfrak{p}})\psi'(a)\mathsf{a}_p(y,\mathsf{f}\star\chi).$$

□

Let $\mathsf{f}\in S_{k,w}(\mathfrak{N}p^{\alpha},\psi,\psi';O)$ be an eigencuspform. For $\mathfrak{p}$ a prime ideal of $\mathcal{O}_L$, let $\tau_{\mathfrak{p}^{\alpha}}\in\mathrm{GL}_2(\mathbb{A}_L)$ be defined by

$$(\tau_{\mathfrak{p}^{\alpha}})_{\mathfrak{p}}=\begin{pmatrix}0 & -1\\ \varpi_{\mathfrak{p}}^{\alpha} & 0\end{pmatrix},\qquad(\tau_{\mathfrak{p}^{\alpha}})_v=\mathbb{I}_2\qquad\text{for}\qquad v\neq\mathfrak{p}.$$

Then we have the following operator

$$\mathsf{f}|\tau_{\mathfrak{p}^{\alpha}}^{-1}(x):=\mathsf{f}(x\tau_{\mathfrak{p}^{\alpha}}^{-1}).$$

Lemma 2.9.4. Let $\mathsf{f} \in S_{2t_F,t_F}(\mathfrak{N}p^\alpha, \psi, \psi'; O)$ be an eigenform, then

$$\mathsf{a}_p(\varpi_\mathfrak{p}, \mathsf{f})^\alpha \cdot \big(\mathsf{f}|\tau_{\mathfrak{p}^\alpha}^{-1}\big)^{[\mathfrak{p}]} = G(\psi_\mathfrak{p}(\psi'_\mathfrak{p})^2) \cdot \big(\mathsf{f} \star \psi_\mathfrak{p}(\psi'_\mathfrak{p})^2\big)$$

the equality taking place in $S_{2t_F,t_F}\big(\mathfrak{N}p^\alpha, \psi, \psi' \cdot \psi_\mathfrak{p}^{-1}(\psi'_\mathfrak{p})^{-2}; O\big)$.

Proof. For $u, v \in \mathcal{O}_{L,\mathfrak{p}}^\times$ such that $uv \equiv -1 \pmod{\mathfrak{p}^\alpha}$ we have

$$\begin{pmatrix} 0 & -1 \\ \varpi_\mathfrak{p}^\alpha & 0 \end{pmatrix}^{-1} \begin{pmatrix} \varpi_\mathfrak{p}^\alpha & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v\varpi_\mathfrak{p}^{-\alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v & (1+uv)\varpi_\mathfrak{p}^{-\alpha} \\ -\varpi_\mathfrak{p}^\alpha & -u \end{pmatrix}.$$

To simplify notation let $\gamma_{u,\alpha} = \begin{pmatrix} \varpi_\mathfrak{p}^\alpha & u \\ 0 & 1 \end{pmatrix}$ and $\delta_v = \begin{pmatrix} 1 & v\varpi_\mathfrak{p}^{-\alpha} \\ 0 & 1 \end{pmatrix}$. Note that right translation by the matrix $\begin{pmatrix} v & (1+uv)\varpi_\mathfrak{p}^{-\alpha} \\ -\varpi_\mathfrak{p}^\alpha & -u \end{pmatrix}$ gives the action of $\langle v^{-1}, v^{-2}\rangle$. Thus we have

$$\mathsf{f}|\gamma_{u,\alpha}|\tau_{\mathfrak{p}^\alpha}^{-1} = (\langle v^{-1}, v^{-2}\rangle \mathsf{f})|\delta_v = \psi(v^{-1})\psi'(v^{-2}) \cdot \mathsf{f}|\delta_v. \tag{2.8}$$

On one hand, summing the right hand side of (2.8) over $v \in (\mathcal{O}_F/\mathfrak{p}^\alpha)^\times$ gives

$$\sum_{v \in (\mathcal{O}_F/\mathfrak{p}^\alpha)^\times} (\psi(\psi')^2)^{-1}(v)\mathsf{f}|\delta_v = G(\psi_\mathfrak{p}(\psi'_\mathfrak{p})^2) \cdot \big(\mathsf{f} \star \psi_\mathfrak{p}(\psi'_\mathfrak{p})^2\big).$$

On the other hand, summing the left hand side of (2.8) over $u \in (\mathcal{O}_F/\mathfrak{p}^\alpha)^\times$ gives

$$\begin{aligned} \sum_{u \in (\mathcal{O}_F/\mathfrak{p}^\alpha)^\times} \mathsf{f}|\gamma_{u,\alpha}|\tau_{\mathfrak{p}^\alpha}^{-1} &= \Big(\sum_{u \in (\mathcal{O}_F/\mathfrak{p}^\alpha)} \mathsf{f}|\gamma_{u,\alpha}\Big)|\tau_{\mathfrak{p}^\alpha}^{-1} - \Big(\sum_{u' \in (\mathfrak{p}\mathcal{O}_F/\mathfrak{p}^\alpha)} \mathsf{f}|\gamma_{u',\alpha-1}|\tau_{\mathfrak{p}^\alpha}^{-1}\Big)|\begin{pmatrix} 1 & \\ & \varpi_\mathfrak{p} \end{pmatrix} \\ &= (U_\mathfrak{p}^\alpha \mathsf{f})|\tau_{\mathfrak{p}^\alpha}^{-1} - (U_\mathfrak{p}^{\alpha-1}\mathsf{f})|\tau_{\mathfrak{p}^\alpha}^{-1}|\begin{pmatrix} 1 & \\ & \varpi_\mathfrak{p} \end{pmatrix}, \end{aligned}$$

where in the first equality we used the following identity for $u \in \mathfrak{p}(\mathcal{O}_F/\mathfrak{p}^\alpha)$:

$$\begin{pmatrix} 0 & -1 \\ \varpi_\mathfrak{p}^\alpha & 0 \end{pmatrix}^{-1} \begin{pmatrix} \varpi_\mathfrak{p}^\alpha & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_\mathfrak{p} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \varpi_\mathfrak{p}^\alpha & 0 \end{pmatrix}^{-1} \begin{pmatrix} \varpi_\mathfrak{p}^{\alpha-1} & u/\varpi_\mathfrak{p} \\ 0 & 1 \end{pmatrix}.$$

Finally, taking the $\mathfrak{p}$-depletion we obtain

$$\mathsf{a}_p(\varpi_\mathfrak{p}, \mathsf{f})^\alpha \cdot \big(\mathsf{f}|\tau_{\mathfrak{p}^\alpha}\big)^{[\mathfrak{p}]} = G(\psi_\mathfrak{p}(\psi'_\mathfrak{p})^2) \cdot \big(\mathsf{f} \star \psi_\mathfrak{p}(\psi'_\mathfrak{p})^2\big)$$

because

$$\big((U_\mathfrak{p}^{\alpha-1}\mathsf{f})|\tau_{\mathfrak{p}^\alpha}^{-1}|\begin{pmatrix} 1 & \\ & \varpi_\mathfrak{p} \end{pmatrix}\big)^{[\mathfrak{p}]} = 0 \qquad \text{and} \qquad (\mathsf{f} \star \psi_\mathfrak{p}(\psi'_\mathfrak{p})^2)^{[\mathfrak{p}]} = \mathsf{f} \star \psi_\mathfrak{p}(\psi'_\mathfrak{p})^2.$$

□

Chapter 3

Automorphic p-adic L-functions

3.1 Preparations with I-adic forms

From now on we will assume L is a real quadratic field and write $\mathrm{I}_L = \{\mu, \mu'\}$. Let $\chi_\circ : \mathrm{Cl}_L^+(\mathfrak{Q}) \to O^\times$ be a finite order character satisfying $\chi_{\circ|\mathbb{Q}} = 1$ and suppose we are given a primitive Hilbert cuspform $\mathsf{g}_\circ \in S_{t_L,t_L}(\mathfrak{Q}; \chi_\circ; \bar{\mathbb{Q}})$ over L of level $\mathfrak{Q}$, an ideal of $\mathcal{O}_L$, and a primitive elliptic cuspform $\mathsf{f}_\circ \in S_{2,1}(N; \mathbb{1}; \bar{\mathbb{Q}})$ of level $N \in \mathbb{Z}^+$ and trivial character, such that $\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q})$ and N are coprime, and we put $M = \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q}) \cdot N$. Let p be a rational prime which splits completely in L and is coprime to M, and assume further that $\mathsf{g}_\circ$, $\mathsf{f}_\circ$ are p-ordinary. We write $\mathcal{P} = \{\mathfrak{p}, \mathfrak{p}'\}$ for the set of prime $\mathcal{O}_L$-ideals dividing p.

Definition 3.1.1. Let Γ denote the p-adic group $1 + p\mathbb{Z}_p$ and $\mathbf{\Lambda} = O[\![\Gamma]\!]$ the usual Iwasawa algebra. For any $\mathbf{\Lambda}$-algebra $\mathbf{I}$, let $\boldsymbol{\mathcal{W}}(\mathbf{I})$ be the weight space $\boldsymbol{\mathcal{W}}(\mathbf{I}) = \mathrm{Hom}_{O-\mathrm{alg}}(\mathbf{I}, \bar{\mathbb{Q}}_p)$, and the subset of arithmetic points $\mathcal{A}(\mathbf{I})$ consists of those coinciding with continuous homomorphisms

$$\mathsf{w}_{\ell,\chi_\diamond} : \mathbf{\Lambda} \longrightarrow \bar{\mathbb{Q}}_p, \qquad [u] \mapsto \chi_\diamond(u) u^{\ell-2}$$

on $\mathbf{\Lambda}$, for $\ell \in \mathbb{Z}_{\geq 1}$ and $\chi_\diamond : \Gamma \to \bar{\mathbb{Q}}_p^\times$ a finite order character. We say such an arithmetic point is of weight ℓ and character $\chi_\diamond$.

Let $\boldsymbol{\chi}$ be the character

$$\boldsymbol{\chi} : \mathrm{Cl}_L^+(\mathfrak{Q}p) \longrightarrow O^\times, \qquad z \mapsto \chi_\circ(z)\theta_L^{-1}(\bar{z})$$

AND Consider the surjection

$$\phi_{\boldsymbol{\chi}} : O[\![\mathbb{G}_L(\mathfrak{Q})]\!] \twoheadrightarrow O[\![\Gamma]\!], \qquad [(z,a)] \mapsto \boldsymbol{\chi}(z)[\xi_z^{-t_L}]. \tag{3.1}$$

Then for any arithmetic point $\mathrm{P} : O[\![\mathbb{G}_L(\mathfrak{Q})]\!] \to \bar{\mathbb{Q}}_p$ of weight $(\ell t_L, t_L)$ and character $(\chi_\circ\theta_L^{1-\ell}\chi^{-1}, \mathbb{1})$, where $\chi : Z_L(\mathfrak{Q}) \to O^\times$ is the p-power order character factoring through the norm with $\chi = \chi_\diamond \circ \mathrm{N}_{L/\mathbb{Q}}$, we have

$$\mathrm{P}_{\ell t_L, t_L, \chi_\circ\theta_L^{1-\ell}\chi^{-1}, \mathbb{1}} = \mathsf{w}_{\ell,\chi_\diamond} \circ \phi_{\boldsymbol{\chi}} \tag{3.2}$$

Definition 3.1.2. Let $\mathscr{G}_{\mathrm{n.o.}} \in \overline{\mathbf{S}}_L^{\mathrm{n.o.}}(\mathfrak{Q}, \boldsymbol{\chi}; \mathbf{I}_{\mathscr{G}_{\mathrm{n.o.}}})$ be the nearly ordinary Hida family passing through an ordinary p-stabilization $\mathsf{g}_\circ^{(p)}$ of $\mathsf{g}_\circ$, i.e. there exists an arithmetic point $\mathrm{P}_\circ \in \mathcal{A}_{L,\boldsymbol{\chi}}(\mathbf{I}_{\mathscr{G}_{\mathrm{n.o.}}})$ of weight (t_L, t_L) and character $(\chi_\circ, \mathbb{1})$ such that $\mathscr{G}_{\mathrm{n.o.}}(\mathrm{P}_\circ) = \mathsf{g}_\circ^{(p)}$. If we set

$$\mathbf{I}_{\mathscr{G}} := \mathbf{I}_{\mathscr{G}_{\mathrm{n.o.}}} \otimes_{\phi_{\boldsymbol{\chi}}} \mathbf{\Lambda} \qquad \text{and} \qquad \mathscr{G} := (1 \otimes \phi_{\boldsymbol{\chi}}) \circ \mathscr{G}_{\mathrm{n.o.}},$$

then $\mathscr{G} \in \overline{\mathbf{S}}_L^{\mathrm{ord}}(\mathfrak{Q}, \boldsymbol{\chi}; \mathbf{I}_{\mathscr{G}})$ is the ordinary Hida family passing through $\mathsf{g}_\circ^{(p)}$ ([Wil88], Theorem 3).

Remark. By (3.2), $\phi_{\boldsymbol{\chi}}$ induces an injection $\mathcal{A}(\mathbf{I}_{\mathscr{G}}) \hookrightarrow \mathcal{A}_{L,\boldsymbol{\chi}}(\mathbf{I}_{\mathscr{G}_{\mathrm{n.o.}}})$, and the image consists of those arithmetic points of weight $(\ell t_L, t_L)$ and character $(\chi_\circ\theta_L^{1-\ell}\chi^{-1}, \mathbb{1})$,

where $\chi = \chi_\diamond \circ \mathrm{N}_{L/\mathbb{Q}}$ factors through the norm. In particular, the arithmetic point $\mathrm{P}_\circ \in \mathcal{A}_{L,\boldsymbol{\chi}}(\mathbf{I}_{\mathscr{G}_{\mathrm{n.o.}}})$ of weight (t_L, t_L) and character $(\chi_\circ, \mathbb{1})$ in Definition 3.1.2 is the image of some arithmetic point in $\mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight 1 and trivial character, which, by an abuse of notation, we still denote by $\mathrm{P}_\circ$.

Let $\breve{\mathscr{G}}$ be the $\mathbf{I}_{\mathscr{G}}$-adic cuspform passing through the ordinary p-stabilization of the test vector $\breve{\mathrm{g}}_\circ$ ([DR14] Section 2.6). We define a homomorphism of $\boldsymbol{\Lambda}_{L,\boldsymbol{\chi}}$-modules $d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]} : \mathbf{h}_L(M\mathcal{O}_L; O) \longrightarrow \mathbf{I}_{\mathscr{G}}$ by

$$d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\left(\langle z\rangle \mathbf{T}(y)\right) = \begin{cases} \breve{\mathscr{G}}\left(\langle z\rangle \mathbf{T}(y)\right) \phi_{\boldsymbol{\chi}}\big([y_{\mathfrak{p}}, 1]\big) y_{\mathfrak{p}}^{-1} & \text{if } y_p \in \mathcal{O}_{L,p}^\times; \\ 0 & \text{otherwise.} \end{cases}$$

Using diagonal restriction $\zeta : \mathbf{h}_{\mathbb{Q}}(M; O) \to \mathbf{h}_L(M\mathcal{O}_L; O)$, $\zeta([z, a]) = [\Delta(z), \Delta(a)]a^{-1}$, we define the ordinary $\mathbf{I}_{\mathscr{G}}$-adic cuspform

$$e_{\mathrm{ord}} \zeta^* \big(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger \in \overline{\mathbf{S}}_{\mathbb{Q}}^{\mathrm{ord}}\big(M, \psi_\circ^{-1}; \mathbf{I}_{\mathscr{G}}\big) \tag{3.3}$$

by setting

$$\zeta^* \big(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger \big(\langle z\rangle \mathbf{T}(y)\big) = \theta_{\mathbb{Q}}(\bar{y}) \cdot \phi_{\boldsymbol{\chi}}\big([\Delta(\xi_z)^{-1}\Delta(\xi_y)^{-1/2}, 1]\big) \cdot d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big(\zeta\left[\langle z\rangle \mathbf{T}(y)\right]\big),$$

where $\Delta : 1 + p\mathbb{Z}_p \hookrightarrow 1 + p\mathcal{O}_{L,p}$ is the diagonal embedding.

Proposition 3.1.3. Let $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ be an arithmetic point extending $\mathsf{w}_{\ell,\chi_\diamond}$. If we denote by $\chi_{\mathfrak{p}}\theta_{L,\mathfrak{p}}^{\ell-1} :\ \mathcal{O}_{L,\mathfrak{p}}^\times \longrightarrow O^\times$ the local character $x \mapsto \chi_\diamond \theta_L^{\ell-1}(x)$ ($\chi_\diamond$ is regarded as a character of $\mathcal{O}_{L,\mathfrak{p}}^\times$ via its natural identification with $\mathbb{Z}_p^\times$), then the specialization of the $\mathbf{I}_{\mathscr{G}}$-adic form in (3.3) at P is given by

$$e_{\mathrm{ord}} \zeta^* \big(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger(\mathrm{P}) = e_{\mathrm{ord}} \zeta^* \big(d_\mu^{1-\ell}(\breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]} \star \chi_{\mathfrak{p}}^{-1}\theta_{L,\mathfrak{p}}^{1-\ell})\big) \otimes \theta_{\mathbb{Q}}^{\ell-1}\chi_\diamond |-|_{\mathbb{A}_{\mathbb{Q}}}^{\ell-2},$$

which is a classical elliptic cuspform of weight $(2,1)$ and trivial character.

Proof. Let $\chi = \chi_\diamond \circ \mathrm{N}_{L/\mathbb{Q}}$. A direct computation

$$\begin{aligned}
&\mathrm{P} \circ e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger([z,a]) \\
=&\theta_{\mathbb{Q}}(a^{-1}) \cdot \mathrm{P} \circ \phi_{\boldsymbol{\chi}}([\Delta(\xi_z)^{-1}\Delta(\xi_a)^{1/2}, 1]) \cdot \mathrm{P} \circ d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big([\Delta(z), \Delta(a)]a^{-1}\big) \\
=&\theta_{\mathbb{Q}}^{-1}(a) \cdot \chi_\diamond^2(z)\eta_{\mathbb{Q}}^{2(2-\ell)}(z)\chi_\diamond^{-1}(a)\eta_{\mathbb{Q}}^{\ell-2}(a) \cdot \mathrm{P} \circ \phi_{\boldsymbol{\chi}}([\Delta(z)\Delta(a)_{\mathfrak{p}}^{-1}, \Delta(a)]) \\
=&\theta_{\mathbb{Q}}^{-1}(a) \cdot \chi_\diamond^2(z)\eta_{\mathbb{Q}}^{2(2-\ell)}(z)\chi_\diamond^{-1}(a)\eta_{\mathbb{Q}}^{\ell-2}(a) \cdot (\chi_\diamond \theta_L^{1-\ell}\chi^{-1})(\Delta(z)\Delta(a)_{\mathfrak{p}}^{-1})\varepsilon_L^{\ell-2}(\Delta(z)\Delta(a)_{\mathfrak{p}}^{-1}) \\
=&1
\end{aligned}$$

shows that $e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger(\mathrm{P})$ is a cuspform of weight $(2,1)$ and trivial character. Moreover,

$$\mathsf{a}_p(y, d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}(\mathrm{P})) = y_{\mathfrak{p}}^{1-\ell}\mathsf{a}_p(y, \breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]}) \cdot \chi^{-1}(y_{\mathfrak{p}})\theta_L^{1-\ell}(y_{\mathfrak{p}}),$$

which implies

$$\zeta^* d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}(\mathrm{P}) = \zeta^*\big(d_\mu^{1-\ell}\big(\breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]} \star \chi_{\mathfrak{p}}^{-1}\theta_{L,\mathfrak{p}}^{1-\ell}\big)\big).$$

Finally,

$$\begin{aligned}
\mathsf{a}_p\big(y, \zeta^*(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]})^\dagger(\mathrm{P})\big) &= \theta_{\mathbb{Q}}(y) \cdot \chi_\diamond(y)\eta_{\mathbb{Q}}^{2-\ell}(y) \cdot \mathsf{a}_p\big(y, \zeta^* d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}(\mathrm{P})\big) \\
&= \theta_{\mathbb{Q}}^{\ell-1}(y)\chi_\diamond(y)\varepsilon_{\mathbb{Q}}^{2-\ell}(y) \cdot \mathsf{a}_p\big(y, \zeta^* d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}(\mathrm{P})\big)
\end{aligned}$$

gives the desired explicit formula

$$e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger(\mathrm{P}) = e_{\mathrm{ord}}\zeta^*\big(d_\mu^{1-\ell}(\breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]} \star \chi_{\mathfrak{p}}^{-1}\theta_{L,\mathfrak{p}}^{1-\ell})\big) \otimes \theta_{\mathbb{Q}}^{\ell-1}\chi_\diamond|-|_{\mathbb{A}_{\mathbb{Q}}}^{\ell-2}.$$

□

Corollary 3.1.4. We have

$$e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet\breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger \in S_{2,1}^{\mathrm{ord}}\big(Mp;\mathbb{1};O\big)\otimes_O \mathbf{I}_{\mathscr{G}}.$$

Proof. By Proposition 3.1.3

$$e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet\breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger(\mathrm{P}) \in S_{2,1}^{\mathrm{ord}}\big(Mp;\mathbb{1};O\big)$$

for any arithmetic crystalline point $\mathrm{P}\in\mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight ℓ and trivial character, we have $e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet\breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger(\mathrm{P}) \in S_{2,1}^{\mathrm{ord}}(Mp;\mathbb{1};O)$. By the density of such crystalline points, the homomorphism $e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet\breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger$ factors through the reduction to weight $(2,1)$, level Mp and trivial character of $\mathbf{h}_{\mathbb{Q}}^{\mathrm{ord}}(M;O)$. □

3.2 The automorphic p-adic L-function

Now we are ready to define the p-adic L-function. The construction is an adaption of the one in Section 3.3 of [BCF], which in turn builds on [DR14]. Let $e_{\mathsf{f}_\circ^{(p)}}$ be the $\mathsf{f}_\circ^{(p)}$-isotypical projector, then we can write

$$e_{\mathsf{f}_\circ^{(p)}}e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet\breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger = \sum_{d|(M/N)}\boldsymbol{\lambda}_d\cdot\mathsf{f}_\circ^{(p)}(q^d)\qquad\text{for}\quad\{\boldsymbol{\lambda}_d\}_d\subset\mathbf{I}_{\mathscr{G}},$$

and define

$$\mathbf{L}_p(\breve{\mathscr{G}},\mathsf{f}_\circ) = \sum_{d|(M/N)}\boldsymbol{\lambda}_d\cdot\frac{\big\langle\mathsf{f}_\circ^{(p)}(q^d),\mathsf{f}_\circ^{(p)}(q)\big\rangle}{\big\langle\mathsf{f}_\circ^{(p)},\mathsf{f}_\circ^{(p)}\big\rangle}\in\mathbf{I}_{\mathscr{G}}.$$

Definition 3.2.1. The twisted triple product p-adic L-function attached to $\big(\breve{\mathscr{G}},\mathsf{f}_\circ\big)$

is the rigid-analytic function

$$\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ) : \boldsymbol{\mathcal{W}}(\mathbf{I}_{\mathscr{G}}) \longrightarrow \mathbb{C}_p$$

determined by $\mathbf{L}_p(\breve{\mathscr{G}}, \mathsf{f}_\circ) \in \mathbf{I}_{\mathscr{G}}$.

For any arithmetic point $\mathrm{P} \in \boldsymbol{\mathcal{A}}(\mathbf{I}_{\mathscr{G}})$ of weight ℓ and character $\chi_\diamond$, we have

$$\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)(\mathrm{P}) = \frac{\left\langle e_{\mathrm{ord}}\zeta^*\big(d_\mu^{1-\ell}(\breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]} \star \chi_{\mathfrak{p}}^{-1}\theta_{L,\mathfrak{p}}^{1-\ell})\big) \otimes \theta_{\mathbb{Q}}^{\ell-1}\chi_\diamond|-|_{\mathbb{A}_{\mathbb{Q}}}^{\ell-2},\ \mathsf{f}_\circ^{(p)}\right\rangle}{\langle \mathsf{f}_\circ^{(p)}, \mathsf{f}_\circ^{(p)}\rangle}. \tag{3.4}$$

Remark. The p-adic L-function $\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)$ changes sign with a different choice of μ.

3.3 Link to the complex L-value in weight one.

Recall $\mathrm{P}_\circ$ is the arithmetic point in $\boldsymbol{\mathcal{A}}(\mathbf{I}_{\mathscr{G}})$ of weight 1 and trivial character such that $\mathscr{G}(\mathrm{P}_\circ) = \mathsf{g}_\circ^{(p)}$. The value of the p-adic L-function at $\mathrm{P}_\circ$ is the complex L-values up to an explicit term consisting of Euler factors at p. Let $\alpha_{\mathsf{f}_\circ^*}$, $\beta_{\mathsf{f}_\circ^*}$ be the inverses of the roots of the $T(\varpi_p)$-Hecke polynomial $1 - \mathsf{a}_p(\varpi_p, \mathsf{f}_\circ^*)X + \psi_\circ^{-1}(p)pX^2$. If we set

$$\mathsf{E}(\mathsf{f}_\circ^*) = 1 - \beta_{\mathsf{f}_\circ^*}\alpha_{\mathsf{f}_\circ^*}^{-1},$$

then we can rewrite the values of the p-adic L-function at every arithmetic point $\mathrm{P} \in \boldsymbol{\mathcal{W}}_{\mathscr{G}}$ as

$$\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)(\mathrm{P}) = \frac{1}{\mathsf{E}(\mathsf{f}_\circ^*)}\frac{\left\langle e_{\mathrm{ord}}\zeta^*\big(d_\mu^{1-\ell}(\breve{\mathsf{g}}_{\mathrm{P}}^{[\mathcal{P}]} \star \chi_{\mathfrak{p}}^{-1}\theta_{L,\mathfrak{p}}^{1-\ell})\big) \otimes \theta_{\mathbb{Q}}^{\ell-1}\chi_\diamond|-|_{\mathbb{A}_{\mathbb{Q}}}^{\ell-2},\ \mathsf{f}_\circ^*\right\rangle}{\langle \mathsf{f}_\circ^*, \mathsf{f}_\circ^*\rangle}. \tag{3.5}$$

Lemma 3.3.1. We have

$$\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)(\mathrm{P}_\circ) = \frac{\mathcal{E}_p^{\mathrm{sp}}(\mathsf{g}_\circ, \mathsf{f}_\circ^*)}{\mathsf{E}(\mathsf{f}_\circ^*)\mathcal{E}_{1,p}(\mathsf{g}_\circ, \mathsf{f}_\circ^*)} \frac{\left\langle e_{\mathrm{ord}}\big(\zeta^*(\breve{\mathsf{g}}_\circ) \otimes |-|_{\mathbb{A}_\mathbb{Q}}^{-1}\big), \mathsf{f}_\circ^* \right\rangle}{\langle \mathsf{f}_\circ^*, \mathsf{f}_\circ^* \rangle}$$

where

$$\mathcal{E}_p^{\mathrm{sp}}(\mathsf{g}_\circ, \mathsf{f}_\circ^*) = \prod_{\bullet, \star \in \{\alpha, \beta\}} \left(1 - \bullet_1 \star_2 \alpha_{\mathsf{f}_\circ^*}^{-1}\right), \qquad \mathcal{E}_{1,p}(\mathsf{g}_\circ, \mathsf{f}_\circ^*) = 1 - \alpha_1\beta_1\alpha_2\beta_2\alpha_{\mathsf{f}_\circ^*}^{-2}.$$

where α_i, β_i are the inverses of the roots of the $T(\mathfrak{p}_i)$-Hecke polynomial for $\mathsf{g}_\circ$, $i = 1, 2$.

Proof. By Proposition 3.1.3, we have

$$e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger(\mathrm{P}_\circ) = e_{\mathrm{ord}}\big(\zeta^*\big(\breve{\mathsf{g}}_\circ^{[\mathcal{P}]}\big) \otimes |-|_{\mathbb{A}_\mathbb{Q}}^{-1}\big).$$

Then the desired claim follows directly from the explicit calculations in Lemmas 3.10 and 3.11 of [BCF]. □

Corollary 3.3.2. The p-adic L-value $\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}_{\mathrm{ord}}, \mathsf{f}_\circ)(\mathrm{P}_\circ) \neq 0$ if and only if the complex L-value $L\big(\frac{1}{2}, \pi_{\mathsf{g}_\circ, \mathsf{f}_\circ}\big) \neq 0$.

Proof. By Theorem 3.2 of [BCF] there exists a choice of the test vector $\breve{\mathsf{g}}_\circ$ such that

$$\left\langle e_{\mathrm{ord}}\big(\zeta^*(\breve{\mathsf{g}}_\circ) \otimes |-|_{\mathbb{A}_\mathbb{Q}}^{-1}\big), \mathsf{f}_\circ^* \right\rangle \neq 0$$

if and only if $L\big(\frac{1}{2}, \pi_{\mathsf{g}_\circ, \mathsf{f}_\circ}\big) \neq 0$. Then Lemma 3.3.1 gives the claim after noting that all the factors appearing in the formula are non-zero. Indeed, the p-adic valuation of $\mathsf{E}(\mathsf{f}_\circ^*)$ and $\mathcal{E}_{1,p}(\mathsf{g}_\circ, \mathsf{f}_\circ^*)$ is zero, and $\mathcal{E}_p^{\mathrm{sp}}(\mathsf{g}_\circ, \mathsf{f}_\circ^*)$ is non-zero because α_i, β_i for $i = 1, 2$ are roots of unity while α_{f^*} is not because of the Weil conjectures. □

Chapter 4

Hilbert modular varieties and Hirzebruch-Zagier classes

4.1 Basic notations

We keep the notation that L denotes a real quadratic field with ring of integers $\mathcal{O}_L$ and different $\mathfrak{d}$. Asumme the prime p splits in L as $p\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$, and assume that $\mathfrak{p}_1$ (and thus also $\mathfrak{p}_2$) is narrowly principal in $\mathcal{O}_L$, i.e. it is generated by a totally positive element. There are natural identifications $\mathcal{O}_{L,\mathfrak{p}_1} \simeq \mathbb{Z}_p$ and $\mathcal{O}_{L,\mathfrak{p}_2} \simeq \mathbb{Z}_p$, and we will often identify $\mathcal{O}_{L,p} = \mathcal{O}_{L,\mathfrak{p}_1} \times \mathcal{O}_{L,\mathfrak{p}_2}$ with $\mathbb{Z}_p \times \mathbb{Z}_p$ and represent its elements with pairs of the form (a_1, a_2) with $a_1, a_2 \in \mathbb{Z}_p$. We make a further technical assumption that there is no totally positive units in $\mathcal{O}_{L,+}^{\times}$ congruent to -1 modulo p.

Thanks to the excellent exposition in Section 2 of [LLZ16], we will partly follow the notations there. First recall the definitions

$$D = \mathrm{Res}^L_{\mathbb{Q}}\mathbb{G}_m, \qquad G = \mathrm{Res}^L_{\mathbb{Q}}\mathrm{GL}_2, \qquad G^* = G \times_D \mathbb{G}_m.$$

Note there are natural embeddings $\mathrm{GL}_2 \hookrightarrow G^* \hookrightarrow G$. Let $\mathfrak{H}$ be the Poincaré upper half plane. We will consider the following types of Shimura varieties.

Definition 4.1.1. Let $K \subseteq G(\mathbb{A}_{\mathbb{Q},f})$, $K^* \subseteq G^*(\mathbb{A}_{\mathbb{Q},f})$ and $K' \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ be open compact subgroups. We write

$$S(K)(\mathbb{C}) := G(\mathbb{Q})_+ \backslash \mathfrak{H}^2 \times G(\mathbb{A}_{\mathbb{Q},f})/K,$$

$$S^*(K^*)(\mathbb{C}) := G^*(\mathbb{Q})_+ \backslash \mathfrak{H}^2 \times G^*(\mathbb{A}_{\mathbb{Q},f})/K^*,$$

$$Y(K')(\mathbb{C}) := \mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathfrak{H} \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})/K'.$$

If K (respectively, K^*, K') is sufficiently small, then $S(K)(\mathbb{C})$ (respectively, $S^*(K^*)(\mathbb{C})$, $Y(K')(\mathbb{C})$) has a natural structure of smooth variety defined over $\mathbb{Q}$ with the set of complex points as given above. We will use $S(K)$, $S^*(K^*)$ and $Y(K')$ to denote the resulting Shimura varieties.

Remark. In the literature, both $S(K)$ and $S^*(K^*)$ are referred to as Hilbert modular surfaces. To distinguish between the two, we will always specify if it is for the algebraic group G or G^*, unless it is otherwise clear. In this paper we will mostly work with G as it admits a richer collection of Hecke actions and the corresponding automorphic representations satisfy the multiplicity one property. We will only use G^* when we later relate the computation of syntomic regulators to overconvergent modular forms, since the Shimura varieties $S^*(U^*)$ are of PEL type, and it is more natural to consider overconvergent forms there. Note that we will always use S to denote a surface, and Y to denote a curve.

For every $g \in G(\mathbb{A}_{\mathbb{Q},f})$, we have a map

$$\mathfrak{T}_g : S(K) \longrightarrow S(gKg^{-1}), \qquad [x,h] \mapsto [x, hg^{-1}],$$

which is a morphism of varieties defined over $\mathbb{Q}$. The same applies to $S^*(K^*)$ and $Y(K')$, and when there is no risk of confusion, we will use the same notation $\mathfrak{T}_g$ to denote the map.

Note that $\mathfrak{T}_g$ is only defined for $S^*(K^*)$ when $g \in G^*(\mathbb{A}_{\mathbb{Q},f})$, which is a rather small group compared to $G(\mathbb{A}_{\mathbb{Q},f}) = \mathrm{GL}_2(\mathbb{A}_{L,f})$. Indeed, the former only consists of the elements in the latter with determinant belonging to $\mathbb{A}_{\mathbb{Q},f}$. To partially remedy this, we enlarge the group of transformations on $S^*(K^*)$ following a construction of Shimura ([Shi78], page 643; see also [LLZ16], Definition 2.2.4).

Definition 4.1.2. Let $\mathcal{G}$ denote the subgroup $G(\mathbb{Q})_+ G^*(\mathbb{A}_{\mathbb{Q},f}) \subseteq G(\mathbb{A}_{\mathbb{Q},f})$.

Then for each K^* there is a bijection $Y^*(K^*) = G(\mathbb{Q})_+ \backslash \mathfrak{H}^2 \times \mathcal{G}/K^*$, and therefore the maps $\mathfrak{T}_g$ are defined for all $g \in \mathcal{G}$.

4.2 Hecke correspondences and operators

In this section we introduce Hecke correspondences and operators for G. All the discussions below apply to GL_2 as well. As for G^*, the general definitions of Hecke correspondences and double coset operators also work as long as $g \in \mathcal{G}$. We will revisit this topic for G^* when we introduce overconvergent Hilbert modular forms later. Let $g \in G(\mathbb{A}_{\mathbb{Q},f})$ and K be an open compact subgroup of $G(\mathbb{A}_{\mathbb{Q},f})$. The double coset KgK defines the following correspondence

$$\begin{array}{ccc} S(g^{-1}Kg \cap K) & \xrightarrow{\mathfrak{T}_g} & S(K \cap gKg^{-1}) \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}'} \\ S(K) & & S(K) \end{array} \tag{4.1}$$

and the double coset operator $[KgK]$ acts on the cohomology groups of $S(K)$ by $\mathrm{pr}'_* \circ \mathfrak{T}_{g,*} \circ \mathrm{pr}^*$. A straightforward double coset calculation shows that this geometrically

defined operator coincides with the algebraically defined double coset operator used in [Hid91]. Suppose K contains $U(\mathfrak{N})$ for some ideal $\mathfrak{N}$ of $\mathcal{O}_L$. For $a, b \in \mathcal{O}_{L,\mathfrak{N}}^{\times}$ and $z \in \mathbb{A}_{L,f}^{\times}$, we have the operators

$$T(a,b) = [K \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} K], \qquad \langle z \rangle = [K \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} K].$$

Furthermore, we let $\langle z, a \rangle \in \mathbb{G}_L(\mathfrak{N})$ act as $\langle z \rangle T(a^{-1}, 1)$ in accordance with definitions in [Hid91].

Remark. Note that since $z \cdot \mathbb{1}_2$ is central, the action of $\langle z \rangle$ is precisely $\mathfrak{T}_{z,*} = \mathfrak{T}^*_{z^{-1}}$. Moreover, if the matrix

$$D_{a,b} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{normalizes the compact open subgroup} \quad K,$$

then $T(a,b)$ acts as $\mathfrak{T}_{D_{a,b},*} = \mathfrak{T}^*_{D_{a,b}^{-1}}$.

The U_p-operator.

Let $\varpi_{\mathfrak{p}_1} \in \mathbb{A}_{L,f}^{\times}$ be the element whose $\mathfrak{p}_1$-component is p and every other component is 1, and similarly define $\varpi_{\mathfrak{p}_2}$. Set $\varpi_p = \varpi_{\mathfrak{p}_1}\varpi_{\mathfrak{p}_1}$.

Consider $g = \begin{pmatrix} \varpi_p & 0 \\ 0 & 1 \end{pmatrix}$, so that $g^{-1}K(p^\alpha)g \cap K(p^\alpha) = K(p^\alpha) \cap K_0(p^{\alpha+1})$. There are two natural projections

$$\pi_1 : S(K(p^\alpha) \cap K_0(p^{\alpha+1})) \longrightarrow S(K(p^\alpha)), \qquad \pi_2 : S(K(p^\alpha) \cap K_0(p^{\alpha+1})) \longrightarrow S(K(p^\alpha)),$$

where π_1 is the natural projection and π_2 is induced by $\mathfrak{T}_g$. In terms of the diagram in (4.1), π_1 is precisely pr and π_2 is just $\mathrm{pr}' \circ \mathfrak{T}_g$. Then the U_p operator is given by

$U_p = (\pi_2)_* \circ (\pi_1)^*$ and the adjoint U_p operator is $U_p^* = (\pi_1)_* \circ (\pi_2)^*$.

Analogously, we also have a $U_\mathfrak{p}$ operator for each prime above p. Let $\mathfrak{p}$ be one of $\mathfrak{p}_1$ and $\mathfrak{p}_2$, and $g_\mathfrak{p} = \begin{pmatrix} \varpi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}$. Then $g_\mathfrak{p}^{-1} K(p^\alpha) g_\mathfrak{p} = K(p^\alpha) \cap K_0(\mathfrak{p}^{\alpha+1})$, and similar to above we have the natural projection $\pi_{1,\mathfrak{p}}$ and the map $\pi_{2,\mathfrak{p}}$ induced by $\mathfrak{T}_{g_\mathfrak{p}}$, both maps from $S(K(p^\alpha) \cap K_0(\mathfrak{p}^{\alpha+1}))$ to $S(K(p^\alpha))$. The $U_\mathfrak{p}$ operator is defined as $U_\mathfrak{p} = (\pi_{2,\mathfrak{p}})_* \circ (\pi_{1,\mathfrak{p}})^*$ with its adjoint given by $U_\mathfrak{p}^* = (\pi_{1,\mathfrak{p}})_* \circ (\pi_{2,\mathfrak{p}})^*$. Note the above discussion of U_p and $U_\mathfrak{p}$ operators hold verbatim if we change $K(p^\alpha)$ to any other level subgroup defined in Definitions 4.3.1 and 4.3.2.

Remark. Hida has used two different conventions for the Hecke operator $T(x)$ for $x \in \mathbb{A}_{L,f}$. In [Hid91], $[UgU]$ is defined by decomposing $UgU = \coprod_\gamma \gamma U$ and setting $(f|[UgU])(h) = \sum_\gamma f(h\gamma)$, and $T(x)$ is defined as $[U \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} U]$. In [Hid88], [Hid89b] and [Hid04] (see in particular Section 4.2.6 of [Hid04], end of p.172 and top of p.174), however, the double coset operator $[UgU]'$ (we added a prime to indicate the difference with the previously defined double coset operator, and similarly for $T'(x)$ below) on the automorphic side is defined by decomposing $UgU = \coprod_\gamma U\gamma$ and setting $(f|[UgU]')(h) = \sum_\gamma f(h\gamma^{-1})$, at least in the weight $(2t_L, t_L)$ situation, and $T'(x)$ is defined as $[U \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} U]'$. The two notions differ by translation by the diagonal matrix $x \cdot \mathbb{1}_2$ so that $T(x) = \langle x, 1 \rangle T'(x)$, hence the Hida theory is the same. In this paper we have followed the convention in [Hid91].

4.3 Atkin-Lehner morphism

Our next goal is to define Atkin-Lehner morphisms at a prime above p. Let $\mathfrak{p}$ be one of $\mathfrak{p}_1$ and $\mathfrak{p}_2$, and $\mathfrak{p}'$ the other one.

The $\mathfrak{T}_{\tau_{\mathfrak{p}^\alpha}}$ morphism

Consider the matrix $\tau_{\mathfrak{p}^\alpha} \in \mathrm{GL}_2(\mathbb{A}_{L,f})$ defined by

$$(\tau_{\mathfrak{p}^\alpha})_{\mathfrak{p}} = \begin{pmatrix} 0 & -1 \\ \varpi_{\mathfrak{p}}^{\alpha} & 0 \end{pmatrix}, \qquad (\tau_{\mathfrak{p}^\alpha})_v = \mathbb{I}_2 \text{ for } v \neq \mathfrak{p}$$

which normalizes $K(p^\alpha)$. Then we have the morphism

$$\mathfrak{T}_{\tau_{\mathfrak{p}^\alpha}} : S(K(p^\alpha)) \longrightarrow S(K(p^\alpha)), \qquad [x,h] \mapsto [x, h\tau_{\mathfrak{p}^\alpha}^{-1}].$$

One can check how this operator commute with diamonds operators:

$$\mathfrak{T}_{\tau_{\mathfrak{p}^\alpha}} \circ \langle z, a \rangle = \langle z \cdot a_{\mathfrak{p}}, a \cdot a_{\mathfrak{p}}^{-2} \rangle \circ \mathfrak{T}_{\tau_{\mathfrak{p}^\alpha}} \tag{4.2}$$

if $(z,a) \in Z_{0,L}^{\alpha}(K) \times (\mathcal{O}_L/p^\alpha)^\times$. Furthermore,

$$\mathfrak{T}_{\tau_{\mathfrak{p}^\alpha}}^2 = \langle -\varpi_{\mathfrak{p}}^{\alpha}, 1 \rangle. \tag{4.3}$$

It will also be useful to look at how $\mathfrak{T}_{\tau_{\mathfrak{p}^\alpha}}$ interacts with the Hecke operators at p. We keep the notation of the previous section. We have the following commutative diagrams

$$\begin{array}{ccc} S(K(p^\alpha) \cap K_0(p^{\alpha+1})) & & \\ \downarrow{\scriptstyle \nu_{1,\mathfrak{p}}} & \searrow^{\pi_1} & \\ S(K(p^\alpha) \cap K_0(\mathfrak{p}^{\alpha+1})) & \xrightarrow[\pi_{1,\mathfrak{p}}]{} & S(K(p^\alpha)), \end{array} \qquad \begin{array}{ccc} S(K(p^\alpha) \cap K_0(p^{\alpha+1})) & & \\ \downarrow{\scriptstyle \nu_{2,\mathfrak{p}}} & \searrow^{\pi_2} & \\ S(K(p^\alpha) \cap K_0(\mathfrak{p}^{\alpha+1})) & \xrightarrow[\pi_{2,\mathfrak{p}}]{} & S(K(p^\alpha)), \end{array}$$

where $\pi_{1,\mathfrak{p}}$ and $\nu_{1,\mathfrak{p}}$ are canonical projections, while $\pi_{2,\mathfrak{p}}$ and $\nu_{2,\mathfrak{p}}$ are induced by $\mathfrak{T}_{g_{\mathfrak{p}}}$

and $\mathfrak{T}_{g_{\mathfrak{p}'}}$ respectively. A direct calculation shows we have the following relations

$$\pi_{1,\mathfrak{p}} \circ \mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha+1}}} = \mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha}}} \circ \pi_{2,\mathfrak{p}}, \qquad \pi_{2,\mathfrak{p}} \circ \mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha+1}}} = \langle \varpi_{\mathfrak{p}}^{-1}, 1 \rangle \circ \mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha}}} \circ \pi_{1,\mathfrak{p}} \tag{4.4}$$

and

$$\nu_{1,\mathfrak{p}'} \circ \mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha+1}}} = \mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha}}} \circ \nu_{2,\mathfrak{p}'}, \qquad \nu_{2,\mathfrak{p}'} \circ \mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha+1}}} = \langle \varpi_{\mathfrak{p}}^{-1}, 1 \rangle \circ \mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha}}} \circ \nu_{1,\mathfrak{p}'}. \tag{4.5}$$

It follows that

$$(\mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha}}})_* \circ U_{\mathfrak{p}} \circ (\mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha}}})^* = U_{\mathfrak{p}}^* \circ \langle \varpi_{\mathfrak{p}}, 1 \rangle. \tag{4.6}$$

On the other hand, it is clear that $U_{\mathfrak{p}'}$ commutes with $(\mathfrak{T}_{\tau_{\mathfrak{p}^{\alpha}}})^*$.

Special level subgroups

We will be interested in the following special types of open compact subgroups of $G(\mathbb{A}_{\mathbb{Q},f})$.

Definition 4.3.1. For any $\alpha \geq 1$ and any compact open $K \subseteq G(\mathbb{A}_{\mathbb{Q},f})$ hyperspecial at p we set

$$K_{\diamond}(p^{\alpha}) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(p^{\alpha}) : a_{\mathfrak{p}_1} \equiv a_{\mathfrak{p}_2},\ d_{\mathfrak{p}_1} \equiv d_{\mathfrak{p}_2} \pmod{p^{\alpha}} \},$$

$$K_{\diamond,t}(p^{\alpha}) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(p^{\alpha}) : a_{\mathfrak{p}_1} d_{\mathfrak{p}_1} \equiv a_{\mathfrak{p}_2} d_{\mathfrak{p}_2},\ d_{\mathfrak{p}_1} d_{\mathfrak{p}_2} \equiv 1 \pmod{p^{\alpha}} \}.$$

$$K_{\mathrm{x}}(p^{\alpha}) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(p^{\alpha}) : a_{\mathfrak{p}_1} d_{\mathfrak{p}_1} \equiv a_{\mathfrak{p}_2} d_{\mathfrak{p}_2},\ d_{\mathfrak{p}_1} a_{\mathfrak{p}_2} \equiv 1 \pmod{p^{\alpha}} \},$$

$$K_{\mathrm{X},1}(p^\alpha) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(p^\alpha) : a_{\mathfrak{p}_1} \equiv d_{\mathfrak{p}_2},\ d_{\mathfrak{p}_1} \equiv a_{\mathfrak{p}_2} \equiv 1 \pmod{p^\alpha} \},$$

and

$$K_{\diamond,1}(p^\alpha) := K_\diamond(p^\alpha) \cap V_1(p^\alpha).$$

Definition 4.3.2. For any $\alpha \geq 1$ and any compact open $K \subseteq G(\mathbb{A}_{\mathbb{Q},f})$ hyperspecial at p we set

Remark. Note that $K_{\mathrm{X}}(p^\alpha)$ is precisely the subgroup of $K_0(p^\alpha)$ generated by $K_{\mathrm{X},1}(p^\alpha)$ and matrices γ of the form

$$\gamma_v = \mathbb{1}_2 \quad \text{for } v \neq p, \qquad \gamma_p = \begin{pmatrix} d^{-1} & \\ & d \end{pmatrix} \quad \text{with} \quad d_{\mathfrak{p}_1} \equiv d_{\mathfrak{p}_2} \pmod{p^\alpha}.$$

These level subgroups are related by the following morphisms

$$\mathfrak{T}_{\tau_{\mathfrak{p}_2^\alpha}} : S(K_{\diamond,1}(p^\alpha)) \longrightarrow S(K_{\mathrm{X},1}(p^\alpha)), \qquad \mathfrak{T}_{\tau_{\mathfrak{p}_2^\alpha}} : S(K_{\mathrm{X}}(p^\alpha)) \longrightarrow S(K_{\diamond,t}(p^\alpha)).$$

The subgroups of the form $K_{\diamond,t}(p^\alpha)$ are related to arithmetic specializations of the ordinary Hida family $\mathscr{G}$.

The ν_α isomorphism

For $a \in \mathbb{Z}_p^\times$, let $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q})$ be the element corresponding to a by class field theory. More concretely, σ_a sends ζ_{p^α} to $\zeta_{p^\alpha}^{a^{-1}}$. Also recall our assumption made at the beginning of this chapter that there is no totally positive units in $\mathcal{O}_{L,+}^\times$ congruent to -1 modulo p.

Proposition 4.3.3. There is an isomorphism of $\mathbb{Q}(\zeta_{p^\alpha})$-varieties

$$\nu_\alpha : S(K_\diamond(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha}) \xrightarrow{\sim} S(K_{\mathrm{x}}(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha})$$

such that $\nu_\alpha \circ \sigma_a = \langle 1, (a,a)\rangle \sigma_a \circ \nu_\alpha$.

Proof. The component set of $S(K(p^\alpha))$, which is a 0-dimensional Shimura variety for $\mathrm{Res}_{L/\mathbb{Q}}\mathrm{GL}_1$, is given by

$$\pi_0(S(K(p^\alpha))) = L_+^\times \backslash \mathbb{A}_{L,f}^\times / \det(K(p^\alpha)).$$

Similarly we have

$$\pi_0(S(K_{\mathrm{x},1}(p^\alpha))) = L_+^\times \backslash \mathbb{A}_{L,f}^\times / \det(K_{\mathrm{x},1}(p^\alpha))$$

and

$$\pi_0(S(K_{\diamond,1}(p^\alpha))) = L_+^\times \backslash \mathbb{A}_{L,f}^\times / \det(K_{\diamond,1}(p^\alpha)).$$

Note that these are all defined over $\mathbb{Q}$, and by the reciprocity law, the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q})$ acts on the points via the Artin map and the diagonal embedding of $\mathbb{A}_{\mathbb{Q},f}^\times$ inside $\mathbb{A}_{L,f}^\times$. In particular, since

$$\det(K_{\mathrm{x},1}(p^\alpha)) = \det(K_{\diamond,1}(p^\alpha)) = (\mathbb{Z}_p + p^\alpha \mathcal{O}_{L,p})\widehat{\mathcal{O}}_L^{p,\times},$$

the image of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q})$ is trivial and hence all points of $\pi_0(S(K_{\mathrm{x},1}(p^\alpha)))$ and $\pi_0(S(K_{\diamond,1}(p^\alpha)))$ are defined over $\mathbb{Q}$. The natural projection $S(K(p^\alpha)) \to S(K_{\diamond,1}(p^\alpha))$ induces an isomorphism of $\mathbb{Q}$-varieties

$$S(K(p^\alpha)) \simeq S(K_{\diamond,1}(p^\alpha)) \times_{\pi_0(S(K_{\diamond,1}(p^\alpha)))} \pi_0(S(K(p^\alpha))). \tag{4.7}$$

The projection $\pi_0(S(K(p^\alpha))) \to \pi_0(S(K_{\diamond,1}(p^\alpha)))$ corresponds to the natural projection

$$L_+^\times \backslash \mathbb{A}_{L,f}^\times / (1 + p^\alpha \mathcal{O}_{L,p}) \widehat{\mathcal{O}}_L^{p,\times} \longrightarrow L_+^\times \backslash \mathbb{A}_{L,f}^\times / (\mathbb{Z}_p + p^\alpha \mathcal{O}_{L,p}) \widehat{\mathcal{O}}_L^{p,\times}$$

with fiber of the identity component identified with

$$(\mathbb{Z}_p + p^\alpha \mathcal{O}_{L,p}) \widehat{\mathcal{O}}_L^{p,\times} \big/ L_+^\times \cap (1 + p^\alpha \mathcal{O}_{L,p}) \widehat{\mathcal{O}}_L^{p,\times} \cong (\mathbb{Z}/p^\alpha \mathbb{Z})^\times,$$

since there is no totally positive units congruent to -1 modulo p. The Galois action of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q})$ is precisely given by the reciprocal of the Artin map, permuting the fiber. Since the complex points of the two connected components happen to carry the structure of abelian groups and the projection is a group homomorphism, we see this is true for all fibers of the projection. This identifies the 0-dimensional variety $\pi_0(S(K(p^\alpha)))$ with $\pi_0(S(K_{\diamond,1}(p^\alpha))) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha})$, and hence via (4.7), also identifies $S(K(p^\alpha))$ with $S(K_{\diamond,1}(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha})$.

Consider the action of $\begin{pmatrix} (a^{-1}, a^{-1}) & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{L,p})$ on $S(K(p^\alpha))$, where (a^{-1}, a^{-1}) is understood as an element of $\mathcal{O}_{L,p}^\times$. Through the isomorphism (4.7), it acts as $\langle 1, (a,a) \rangle \times \sigma_a$ on $S(K_{\mathrm{X},1}(p^\alpha)) \times_{\pi_0(S(K_{\mathrm{X},1}(p^\alpha)))} \pi_0(S(K_{11}(p^\alpha))) \simeq S(K_{\mathrm{X},1}(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha})$. On the other hand, for $S(K_{\diamond,1}(p^\alpha))$ we have completely analogous results. Identifying $S(K(p^\alpha))$ with $S(K_{\diamond,1}(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha})$, the action of $\begin{pmatrix} (a^{-1}, a^{-1}) & 0 \\ 0 & 1 \end{pmatrix}$ is just $1 \times \sigma_a$ by definition.

In summary, we have a canonical isomorphism

$$\nu_\alpha : S(K_{\diamond,1}(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha}) \xrightarrow{\sim} S(K_{\mathrm{X},1}(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha})$$

factoring through $S(K(p^\alpha))$, such that $\nu_\alpha \circ \sigma_a = \langle 1, (a,a) \rangle \sigma_a \circ \nu_\alpha$. If we further

quotient out the action of $\{\begin{pmatrix}(d^{-1},d^{-1}) & \\ & (d,d)\end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{L,p})\} \subset \mathrm{GL}_2(\mathbb{A}_{L,f})$ on $S(K(p^\alpha))$ (which acts trivially on $\pi_0(S(K(p^\alpha)))$ as the determinant is 1), we obtain $S(K_\diamond(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha})$ on the source of ν_α and $S(K_{\mathrm{x}}(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha})$ on the target. Slightly abusing the notation, we still denote by ν_α the resulting isomorphism

$$\nu_\alpha : S(K_\diamond(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha}) \xrightarrow{\sim} S(K_\square(p^\alpha)) \xrightarrow{\sim} S(K_{\mathrm{x}}(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha}), \tag{4.8}$$

where we define $K_\square(p^\alpha) := \{\begin{pmatrix}a & b \\ c & d\end{pmatrix} \in K_0(p^\alpha) : a_{\mathfrak{p}_1} \equiv a_{\mathfrak{p}_1} \equiv d_{\mathfrak{p}_1}^{-1} \equiv d_{\mathfrak{p}_2}^{-1} \pmod{p^\alpha}\}$. As the diamond operators are defined over $\mathbb{Q}$, this new quotient isomorphism ν_α satisfies the same twisted Galois equivariance property $\nu_\alpha \circ \sigma_a = \langle 1, (a,a)\rangle \sigma_a \circ \nu_\alpha$. $\square$

For future reference we record the following lemma.

Lemma 4.3.4. The identity

$$U_p \circ (\nu_\alpha)^* = (\nu_\alpha)^* \circ U_p$$

holds as maps on cohomology groups $\mathrm{H}^\bullet(S(K_{\mathrm{x}}(p^\alpha))_{\mathbb{Q}(\zeta_{p^\alpha})}) \longrightarrow \mathrm{H}^\bullet(S(K_\diamond(p^\alpha))_{\mathbb{Q}(\zeta_{p^\alpha})})$, where $\mathrm{H}^\bullet$ can be either étale cohomology with constant coefficient sheaf, or algebraic de Rham cohomology.

Proof. Recall the U_p operator on $S(K_?(p^\alpha))$, $? \in \{\diamond, \square, \mathrm{x}\}$ is given by $\pi_{2,p,*} \circ \pi_{1,p}^*$, where

$$\pi_{1,p}, \pi_{2,p} : S(K_?(p^\alpha) \cap K_0(p^{\alpha+1})) \longrightarrow S(K_?(p^\alpha))$$

are given by $\mathfrak{T}_{\mathbb{1}_2}$ and $\mathfrak{T}_{g_p}$, $g_p = \begin{pmatrix}\varpi_p & 0 \\ 0 & 1\end{pmatrix}$, respectively.

Repeating the argument in Proposition 4.3.3, we see there is also an isomorphism

$$\begin{aligned}\nu_\alpha : S(K_\diamond(p^\alpha) \cap K_0(p^{\alpha+1})) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha}) &\xrightarrow{\sim} S(K_\square(p^\alpha) \cap K_0(p^{\alpha+1})) \\ &\xrightarrow{\sim} S(K_{\mathrm{x}}(p^\alpha)) \cap K_0(p^{\alpha+1}) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha}),\end{aligned} \tag{4.9}$$

and both $\pi_0(S(K_\square(p^\alpha))$ and $\pi_0(S(K_\square(p^\alpha) \cap K_0(p^{\alpha+1}))$ are identified with

$$L_+^\times \backslash \mathbb{A}_{L,f}^\times / \{x \in \hat{\mathcal{O}}_L^\times | x_p \in 1 + p^\alpha \mathcal{O}_{L,p}\}.$$

Since the determinant of both I_2 and g_p lie in $L_+^\times \cdot \{x \in \hat{\mathcal{O}}_L^\times | x_p \in 1 + p^\alpha \mathcal{O}_{L,p}\}$, both

$$\pi_{1,p}, \pi_{2,p} : S(K_\square(p^\alpha) \cap K_0(p^{\alpha+1})) \longrightarrow S(K_\square(p^\alpha))$$

induce the identity map on the component group. Therefore via the isomorphisms in (4.9), they correspond to $\pi_{1,p} \times 1$ and $\pi_{2,p} \times 1$, respectively, on both $S(K_\diamond(p^\alpha) \cap K_0(p^{\alpha+1})) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha})$ and $S(K_{\mathrm{x}}(p^\alpha) \cap K_0(p^{\alpha+1})) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha})$. In other words, both $\pi_{1,p}$ and $\pi_{2,p}$ commute with ν_α. This implies

$$\begin{aligned}U_p \circ (\nu_\alpha)^* =& (\pi_{2,p})_* \circ (\pi_{1,p})^* \circ (\nu_\alpha)^* = (\pi_{2,p})_* \circ (\nu_\alpha)^* \circ (\pi_{1,p})^* \\ =& (\pi_{2,p})_* \circ (\nu_\alpha^{-1})_* \circ (\pi_{1,p})^* = (\nu_\alpha^{-1})_* \circ (\pi_{2,p})_* \circ (\pi_{1,p})^* \\ =& (\nu_\alpha)^* \circ U_p\end{aligned}$$

as desired.

□

Quotients of upper half plane

For any compact open subgroup $K \leq G(\hat{\mathbb{Z}})$ the determinant map $\det : G \to \mathrm{Res}_{\mathbb{Q}}^L(\mathbb{G}_m)$ induces a bijection between the set of geometric connected components of $S(K)$ and

$\mathrm{Cl}_L^+(K)$, the strict class group of K, $\mathrm{Cl}_L^+(K) = L_+^\times \backslash \mathbb{A}_{L,f}^\times / \det(K)$.

As in Section 2.3, we fix a set of representatives $\{t_1, ..., t_h\}$ in $\mathbb{A}_{L,f}^\times$ of $\mathrm{Cl}_L^+(K)$ with $t_1 = 1$ and let $g_i = \begin{pmatrix} t_i & 0 \\ 0 & 1 \end{pmatrix}$. Write

$$\Gamma(g_i, K) = g_i K g_i^{-1} \cap G(\mathbb{Q})_+.$$

By strong approximation there is a decomposition

$$S(K)(\mathbb{C}) = G(\mathbb{Q})_+ \backslash \mathfrak{H}^2 \times G(\mathbb{A}_{\mathbb{Q},f})/K = \coprod_{i=1}^{h} \Gamma(g_i, K) \backslash \mathfrak{H}^2, \tag{4.10}$$

where $z \mod \Gamma(g_i, K) \in \Gamma(g_i, K)\backslash\mathfrak{H}^2$ is identified with $[z, g_i] \in S(K)(\mathbb{C})$.

Remark. This is precisely the geometric counterpart of the decomposition of adelic Hilbert modular forms into tuples of classical modular forms: every classical form component is precisely the corresponding section on each connected component in (4.10).

We are interested in comparing the complex uniformizations for the subgroups $K_\diamond(p^\alpha)$ and $K_{\mathrm{x}}(p^\alpha)$. The first observation is that the determinant of these groups are equal to $(\mathbb{Z}_p^\times + p^\alpha \mathcal{O}_{L,p})\widehat{\mathcal{O}}_L^{p,\times}$, thus we can choose the same sets of matrices $\{g_i |\ i = 1, ..., h\}$ for both of them.

Lemma 4.3.5. If $\mathcal{O}_L^\times$ does not contain a totally positive unit congruent to -1 modulo p, then for every $\mathfrak{c}$ and every g_i we have

$$\Gamma(g_i, K_\diamond(p^\alpha)) = \Gamma(g_i, K_{\mathrm{x}}(p^\alpha)).$$

Proof. We only prove $\Gamma(g_i, K_\diamond(p^\alpha)) \subseteq \Gamma(g_i, K_{\mathrm{x}}(p^\alpha))$ because the other inclusion is analogous. Let γ be a matrix in $\Gamma(g_i, K_\diamond(p^\alpha))$, then its entries $a_\gamma, b_\gamma, c_\gamma, d_\gamma \in L$

satisfy $(a_\gamma)_p, (d_\gamma)_p \in \mathbb{Z}_p^\times + p^\alpha \mathcal{O}_{L,p}$, $(c_\gamma)_p \in p^\alpha \mathcal{O}_{L,p}$ and its determinant $\det(\gamma) \in \mathcal{O}_{L,+}^\times$ is a totally positive unit. In order to prove the claimed inclusion we need to show that

$$(a_\gamma d_\gamma)_{\mathfrak{p}_1} \equiv (a_\gamma d_\gamma)_{\mathfrak{p}_2} \pmod{p^\alpha} \qquad \text{and} \qquad (d_\gamma)_{\mathfrak{p}_1} \equiv (a_\gamma)_{\mathfrak{p}_2} \pmod{p^\alpha}.$$

Since $\mathrm{N}_{L/\mathbb{Q}}\big(\det(\gamma)\big) = 1$, one sees that either $a_\gamma d_\gamma - 1 \in p^\alpha \mathcal{O}_L$ or $a_\gamma d_\gamma + 1 \in p^\alpha \mathcal{O}_L$. The second possibility implies that $\det(\gamma) \equiv -1 \pmod{p}$, which is ruled out by the hypothesis of the Lemma. Hence we must have $a_\gamma d_\gamma - 1 \in p^\alpha \mathcal{O}_L$ which implies the claim. □

Corollary 4.3.6. The morphism

$$\nu_\alpha : S(K_\diamond(p^\alpha))(\mathbb{C}) \xrightarrow{\sim} S(K_{\mathrm{X}}(p^\alpha))(\mathbb{C})$$

is the identity with respect to the complex uniformizations (4.10). Hence, ν_α commutes with the natural projections $\nu_{1,\mathfrak{p}}$ and $\pi_{1,\mathfrak{p}}$ for $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2$, since they all are induced by the identity map on $\mathfrak{H}^2$. However, ν_α does not in general commute with $\nu_{2,\mathfrak{p}}$ nor $\pi_{2,\mathfrak{p}}$.

Atkin-Lehner morphism

We are finally ready to define the Atkin-Lehner morphism at a prime aove p.

Definition 4.3.7. The Atkin-Lehner morphism at $\mathfrak{p}_2$ is defined as

$$w_{\mathfrak{p}_2^\alpha} = \mathfrak{T}_{\tau_{\mathfrak{p}_2^\alpha}} \circ \nu_\alpha : S(K_\diamond(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha}) \xrightarrow{\sim} S(K_{\diamond,t}(p^\alpha)) \times_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^\alpha}).$$

Since $\mathfrak{T}_{\tau_{\mathfrak{p}_2^\alpha}}$ is defined over $\mathbb{Q}$ and $\nu_\alpha \circ \sigma_a = \langle 1, (a,a)\rangle \sigma_a \circ \nu_\alpha$, we see that

$$w_{\mathfrak{p}_2^\alpha} \circ \sigma_a = \mathfrak{T}_{\tau_{\mathfrak{p}_2^\alpha}} \circ \langle 1, (a,a)\rangle \sigma_a \circ \nu_\alpha = \langle (1, a^{-1}), (a, a^{-1})\rangle \sigma_a \circ w_{\mathfrak{p}_2^\alpha}. \tag{4.11}$$

Combining equation (4.6) with Lemma 4.3.4, we obtain the following

Corollary 4.3.8. The identity

$$U_p \circ (w_{\mathfrak{p}_2^\alpha})^* = (w_{\mathfrak{p}_2^\alpha})^* \circ U_{\mathfrak{p}_1} \circ U_{\mathfrak{p}_2}^* \circ \langle \varpi_{\mathfrak{p}_2}, 1\rangle$$

holds as maps on cohomology groups $\mathrm{H}^\bullet(S(K_{\diamond,t}(p^\alpha))_{\mathbb{Q}(\zeta_{p^\alpha})}) \longrightarrow \mathrm{H}^\bullet(S(K_\diamond(p^\alpha))_{\mathbb{Q}(\zeta_{p^\alpha})})$, where $\mathrm{H}^\bullet$ can be either étale cohomology with constant coefficient sheaf, or algebraic de Rham cohomology.

Lemma 4.3.9. There is a $G_\mathbb{Q}$-equivariant isomorphism

$$w_{\mathfrak{p}_2^\alpha,*} : \mathrm{H}^i_{\text{ét}}(S(K_\diamond(p^\alpha))_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \simeq \mathrm{H}^i_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)(\delta_\alpha),$$

where $\delta_\alpha : G_\mathbb{Q} \to O[\mathbb{G}_L^\alpha(K)]^\times$ is the character obtained by composing the projection $G_\mathbb{Q} \to \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q})$ with the homomorphism mapping σ_a to the group-like element $\langle (1, a^{-1}), (a, a^{-1})\rangle$, where $(a^{-1}, 1)$ and $(1, a^{-1})$ is understood as elements of $\mathcal{O}_{L,p}^\times$.

Proof. This follows immediately from equation (4.11). □

4.4 Hirzebruch-Zagier cycles

We keep the notations and assumptions made at the beginning of the previous chapter. Let $K' := \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \cap K$. Note that $K_0'(p^\alpha) = \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \cap K_\diamond(p^\alpha)$ and there is a

cartesian diagram of Shimura varieties over $\mathbb{Q}$

$$\begin{array}{ccc} Y(K_0'(p^{\alpha+1})) & \xrightarrow{\zeta(\mathfrak{p}_1)} & S(K_\diamond(p^\alpha) \cap V_0(\mathfrak{p}_1^{\alpha+1})) \\ \downarrow{\scriptstyle \pi_1} & \diamond & \downarrow{\scriptstyle \pi_{1,\mathfrak{p}_1}} \\ Y(K_0'(p^\alpha)) & \xrightarrow{\zeta} & S(K_\diamond(p^\alpha)) \end{array} \tag{4.12}$$

as the horizontal arrows are closed embeddings and the vertical ones are finite of degree p.

Definition 4.4.1. For $\alpha \geq 1$ we define cycle classes in the respective Chow groups

$$\Delta_\alpha^\flat := \zeta_* \left[Y(K_0'(p^\alpha)) \right] \in \mathrm{CH}^1\big(S(K_\diamond(p^\alpha)\big)(\mathbb{Q})$$

and

$$\Delta_\alpha^\flat(\mathfrak{p}_1) := \zeta(\mathfrak{p}_1)_* \left[Y(K_0'(p^{\alpha+1})) \right] \in \mathrm{CH}^1\big(S(K_\diamond(p^\alpha) \cap V_0(\mathfrak{p}_1^{\alpha+1}))\big)(\mathbb{Q}).$$

Lemma 4.4.2. We have

$$(\pi_{1,\mathfrak{p}_1})^* \Delta_\alpha^\flat = \Delta_\alpha^\flat(\mathfrak{p}_1) \tag{4.13}$$

in $\mathrm{CH}^1\big(S(K_\diamond(p^\alpha) \cap V_0(\mathfrak{p}_1^{\alpha+1}))\big)(\mathbb{Q})$.

Proof. Since the diagram (4.12) is cartesian, the push-pull formula

$$(\pi_{1,\mathfrak{p}_1})^* \circ \zeta_* = \zeta(\mathfrak{p}_1)_* \circ (\pi_1)^*$$

holds. The desired conclusion follows immediately. □

Twisting by Atkin-Lehner.

Definition 4.4.3. Let $\Delta_\alpha^\sharp$ be the following class of the codimension 1 cycle

$$\Delta_\alpha^\sharp = (w_{\mathfrak{p}_2^\alpha})_* \Delta_\alpha^\flat \in \mathrm{CH}^1\big(S(K_{\diamond,t}(p^\alpha))\big)(\mathbb{Q}(\zeta_{p^\alpha}))$$

defined over $\mathbb{Q}(\zeta_{p^\alpha})$.

Proposition 4.4.4. We have

$$(\varpi_2)_* \Delta_{\alpha+1}^\sharp = \langle \varpi_{\mathfrak{p}_2}^{-1}, 1\rangle_* \circ U_{\mathfrak{p}_1} \Delta_\alpha^\sharp.$$

Proof. Combining equation (4.5) with Corollary 4.3.6, we have the following commutative diagram

$$\begin{array}{ccccc}
Y(K_0'(p^{\alpha+1})) & \xhookrightarrow{\zeta} & S(K_\diamond(p^{\alpha+1})) & \xrightarrow{w_{\mathfrak{p}_2^{\alpha+1}}} & S(K_{\diamond,t}(p^{\alpha+1})) \\
\| & & \downarrow{\scriptstyle \nu_{1,\mathfrak{p}_1}\circ\mu} & & \downarrow{\scriptstyle \nu_{2,\mathfrak{p}_1}\circ\mu} \\
Y(K_0'(p^{\alpha+1})) & \xhookrightarrow{\zeta(\mathfrak{p}_1)} & S(K_\diamond(p^\alpha)\cap U_0(\mathfrak{p}_1^{\alpha+1})) & \xrightarrow{\langle \varpi_{\mathfrak{p}_2}^{-1},1\rangle \circ w_{\mathfrak{p}_2^\alpha}} & S(K_{\diamond,t}(p^\alpha)\cap U_0(\mathfrak{p}_1^{\alpha+1})) \\
 & & & & \downarrow{\scriptstyle \pi_{2,\mathfrak{p}_1}} \\
 & & & & S(K_{\diamond,t}(p^\alpha)).
\end{array} \tag{4.14}$$

By definition, $(\varpi_2)_* \Delta_{\alpha+1}^\sharp$ is precisely the pushforward of the cycle class $[Y(K_0'(p^{\alpha+1}))]$ along the top arrows and then the the right arrows, and therefore the commutativity

of the diagram shows that

$$\begin{aligned}(\varpi_2)_*\Delta^\sharp_{\alpha+1} &= (\pi_{2,\mathfrak{p}_1})_* \circ \langle \varpi_{\mathfrak{p}_2}^{-1}, 1\rangle_* \circ (w_{\mathfrak{p}_2^\alpha})_* \circ (\zeta(\mathfrak{p}_1))_* \left[Y(K_0'(p^{\alpha+1}))\right] \\ &= \langle \varpi_{\mathfrak{p}_2}^{-1}, 1\rangle_* \circ (\pi_{2,\mathfrak{p}_1})_* \circ (w_{\mathfrak{p}_2^\alpha})_* \Delta^\flat_\alpha(\mathfrak{p}_1) \\ &= \langle \varpi_{\mathfrak{p}_2}^{-1}, 1\rangle_* \circ (\pi_{2,\mathfrak{p}_1})_* \circ (w_{\mathfrak{p}_2^\alpha})_* \circ (\pi_{1,\mathfrak{p}_1})^* \Delta^\flat_\alpha \\ &= \langle \varpi_{\mathfrak{p}_2}^{-1}, 1\rangle_* \circ (\pi_{2,\mathfrak{p}_1})_* (\pi_{1,\mathfrak{p}_1})^* \circ (w_{\mathfrak{p}_2^\alpha})_* \Delta^\flat_\alpha \\ &= \langle \varpi_{\mathfrak{p}_2}^{-1}, 1\rangle_* \circ U_{\mathfrak{p}_1} \Delta^\sharp_\alpha,\end{aligned}$$

where the third equality is due to Lemma 4.4.2 and the second to last one from the fact that $w_{\mathfrak{p}_2^\alpha}$ is an isomorphism commuting with $\pi_{1,\mathfrak{p}_1}$ as noted in Corollary 4.3.6. This proves the claim. □

Next, we put the cycles in the products of modular curves and Hilbert modular surfaces.

Definition 4.4.5. We denote by

$$Z_\alpha(K) = S(K_{\diamond,t}(p^\alpha)) \times Y(K_0'(p)). \tag{4.15}$$

The Hirzebruch-Zagier cycles of level $\alpha \geq 1$ is defined by

$$\Delta_\alpha = (\langle \varpi_{\mathfrak{p}_2}^\alpha, 1\rangle \circ w_{\mathfrak{p}_2^\alpha} \circ \iota_\alpha,\ \pi_{1,\alpha})_*[Y(K_0'(p^\alpha))] \in \mathrm{CH}^2\big(Z_\alpha(K)\big)(\mathbb{Q}(\zeta_{p^\alpha})), \tag{4.16}$$

where $\pi_{1,\alpha} : Y(K_0'(p^\alpha)) \longrightarrow Y(K_0'(p))$ is the natural projection.

As in Proposition 4.4.4 we find the following relation for the new cycles.

Proposition 4.4.6. The following identity holds in $\mathrm{CH}^2(Z_\alpha(K))(\mathbb{Q}(\zeta_{p^\alpha}))$:

$$(\varpi_2, \mathrm{id})_*\Delta_{\alpha+1} = (U_{\mathfrak{p}_1}, \mathrm{id})\Delta_\alpha.$$

Galois twisting

Later on we will apply the p-adic Abel-Jacobi map to the cycles and put them in a family, so it is desirable to make them defined over $\mathbb{Q}$.

Following equation (4.11), the Galois group $G_{\mathbb{Q}}$ acts on Δ_α through the finite quotient $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q})$ by

$$\sigma_a^*\Delta_\alpha = (\langle (1,a),(a^{-1},a)\rangle, \mathrm{id})\Delta_\alpha.$$

Definition 4.4.7. Let $S^\dagger(K_{\diamond,t}(p^\alpha))$ to be the twist of the $\mathbb{Q}$-variety $S(K_{\diamond,t}(p^\alpha))$ by the cocycle

$$\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q}) \mapsto (\sigma_a \mapsto \langle (1,a^{-1}),(a,a^{-1})\rangle)$$

and define $Z_\alpha^\dagger(K) = S^\dagger(K_{\diamond,t}(p^\alpha)) \times Y(K_0'(p))$.

Then Δ_α can now be viewed as a codimension 2 cycle in $Z_\alpha^\dagger(K)$ defined over $\mathbb{Q}$. Furthermore,

$$\mathrm{H}^i_{\text{ét}}(S^\dagger(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \simeq \mathrm{H}^i_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)(\delta_\alpha), \tag{4.17}$$

where $\delta_\alpha : G_{\mathbb{Q}} \to O[\mathbb{G}_L^\alpha(K)]^\times$ is the character sending σ_a to the group-like element $\langle (1,a^{-1}),(a,a^{-1})\rangle$ as before.

Null-homologous cycles

Finally, we make the cycles null-homologous by applying suitable modular correspondence. We define a correspondence $\varepsilon_{\mathsf{f}_\circ}$ on $Y(K_0'(p))$ following [DR17] (after equation (47)). Since p is assumed to be non-Eisenstein for $\mathsf{f}_\circ$, i.e. $\mathsf{f}_\circ$ is not congruent to an Eisenstein series modulo p, there exists an auxiliary prime $\ell \nmid Np$ for which

$\ell + 1 - \mathsf{a}_p(\ell, \mathsf{f}_\circ)$ lies in $\mathbb{Z}_p^\times$. Then the correspondence

$$\varepsilon_{\mathsf{f}_\circ} = (\ell + 1 - T(\ell))/(\ell + 1 - \mathsf{a}_p(\ell, \mathsf{f}_\circ))$$

has coefficients in $\mathbb{Z}_p$, annihilates $\mathrm{H}^0(Y(K_0'(p)))$ and $\mathrm{H}^2(Y(K_0'(p)))$ and acts as the identity on the $\mathsf{f}_\circ$-isotypic subspace $\mathrm{H}^1(Y(K_0'(p)))[\mathsf{f}_\circ]$.

Definition 4.4.8. The modified Hirzebruch-Zagier cycle is given by

$$\Delta_\alpha^\circ := (\mathrm{id}, \varepsilon_{\mathsf{f}_\circ})_* \Delta_\alpha \in \mathrm{CH}^2(Z_\alpha^\dagger(K))(\mathbb{Q}) \otimes \mathbb{Z}_p.$$

Proposition 4.4.9. The cycle class Δ_α° is null-homologous.

Proof. The étale cycle class image $\mathrm{cl}_{\text{ét}}(\Delta_\alpha^\circ)$ belongs to $\mathrm{H}^4_{\text{et}}(Z_\alpha^\dagger(K)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2))$. As the integral cohomology of smooth projective curves is torsion free and Hilbert modular surfaces are simply connected, the integral étale cohomology of $Z_\alpha^\dagger(K)$ has a Künneth decomposition whose every term is annihilated by the correspondence $(\mathrm{id}, \varepsilon_{\mathsf{f}_\circ})$. □

4.5 Big cohomology classes

For any number field D, the p-adic étale Abel-Jacobi map

$$\begin{aligned}\mathrm{AJ}_p^{\text{ét}} : \mathrm{CH}^2(Z_\alpha^\dagger(K))_0(D) \longrightarrow & \mathrm{H}^1\big(D, \mathrm{H}^3_{\text{et}}\big(Z_\alpha^\dagger(K)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2)\big)\big) \\ = & \mathrm{H}^1\big(D, \mathrm{H}^3_{\text{et}}\big(Z_\alpha(K)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2+\delta_\alpha)\big)\big)\end{aligned}$$

sends null-homologous cycles to Galois cohomology classes. The modified Hirzebruch-Zagier cycles Δ_α° are null-homologous and give rise to cohomology classes in the

appropriate Künneth component

$$\kappa_\alpha^\circ := \mathrm{AJ}_p^{\text{ét}}(\Delta_\alpha^\circ) \in \mathrm{H}^1\big(\mathbb{Q}, \mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2+\eta_\alpha)) \otimes \mathrm{H}^1_{\text{ét}}(Y(K_0'(p))_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)\big).$$

Lemma 4.5.1. For $\alpha \geq 1$ we have

$$(\varpi_2, \mathrm{id})_* \kappa_{\alpha+1}^\circ = (U_{\mathfrak{p}_1}, \mathrm{id})\kappa_\alpha^\circ.$$

Proof. It follows immediately from Proposition 4.4.6 combined with the commutativity of cycle class map and correspondences. □

Let

$$\mathbb{V}_\alpha^{\text{n.o.}}(K) := e_{\text{n.o.}} \mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2+\eta_\alpha)) \otimes \mathrm{H}^1_{\text{ét}}(Y(K_0'(p))_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)$$

be the tensor product of the nearly ordinary part of the étale cohomology groups of $S(K_{\mathrm{X}}(p^\alpha))$ and the étale cohomology of $Y(K_0'(p))$. By a slight abuse of notation we also denote by

$$\kappa_\alpha^\circ \in \mathrm{H}^1(\mathbb{Q}, \mathbb{V}_\alpha^{\text{n.o.}}(K))$$

the nearly ordinary projection of κ_α°. Since $U_{\mathfrak{p}_1}$ acts invertibly on the nearly ordinary part, we may define

$$\kappa_\alpha^{\text{n.o.}} := \big(U_{\mathfrak{p}_1}^{-\alpha}, \mathrm{id}\big)\kappa_\alpha^\circ \in \mathrm{H}^1(\mathbb{Q}, \mathbb{V}_\alpha^{\text{n.o.}}(K)). \tag{4.18}$$

It follows immediately from Lemma 4.5.1 and the commutativity of $U_{\mathfrak{p}_1}$ with $\varpi_{2,*}$ that

$$(\varpi_2, \mathrm{id})_* \kappa_{\alpha+1}^{\text{n.o.}} = \kappa_\alpha^{\text{n.o.}}, \tag{4.19}$$

which justifies our following definition.

Definition 4.5.2. Let

$$\mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);\mathbb{Z}_p(2)) := \varprojlim_{\alpha} e_{\mathrm{n.o.}}\mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}},\mathbb{Z}_p(2)),$$

where the inverse limit is taken with respect to $(\varpi_2)_*$, and

$$\mathbf{V}^{\mathrm{n.o.}}_\infty(K) := \varprojlim_{\alpha} \mathbb{V}^{\mathrm{n.o.}}_\alpha(K) = \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);\mathbb{Z}_p(2))(\boldsymbol{\delta}) \otimes \mathrm{H}^1_{\text{ét}}(Y(K'(p))_{\bar{\mathbb{Q}}},\mathbb{Z}_p)$$

where the limit is with respect to $(\varpi_2,\mathrm{id})_*$, and $\boldsymbol{\delta} = \varprojlim_\alpha \delta_\alpha$.

Define

$$\boldsymbol{\kappa}^{\mathrm{n.o.}}_\infty = \lim_{\leftarrow,\alpha} \kappa^{\mathrm{n.o.}}_\alpha \in \mathrm{H}^1(\mathbb{Q},\mathbf{V}^{\mathrm{n.o.}}_\infty(K)).$$

Remark. $\mathbb{V}^{\mathrm{n.o.}}_\alpha(K)$ has a natural action of $\mathbb{G}_L(K)$ as diamond operators, and hence is a module over $\mathbb{Z}_p[\![\mathbb{G}_L(K)]\!]$. The character $\boldsymbol{\delta}: G_{\mathbb{Q}} \to \mathbb{G}_L(K)$ is the composition of the canonical projection $G_{\mathbb{Q}} \to \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ with the map $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \to \mathbb{G}_L(K)$ sending σ_a to $\langle(1,a^{-1}),(a,a^{-1})\rangle$.

4.6 A refinement of specialization formula

Our last goal of the section is to further analyze the weight $(2,1)$ specializations of the Λ-adic elliptic modular form $e_{\mathrm{ord}}\zeta^*\big(d^\bullet_\mu \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger$ defined before Proposition 3.1.3, so that we can obtain a refinement of the specialization formula (3.4). To do this we need to first consider the geometric morphisms corresponding to the various operations used in defining the Λ-adic form.

Let

$$K'_{0,\det}(p^\alpha) := K_{\mathrm{X}}(p^\alpha) \cap \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$$
$$= K' \cap \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbb{Z}}) : a_p d_p \equiv 1,\ c_p \equiv 0 \pmod{p^\alpha} \}.$$

Then we have the natural diagonal embedding $Y(K'_{0,\det}(p^\alpha)) \hookrightarrow S(K^t_\diamond(p^\alpha))$ of Shimura varieties induced by the natural diagonal inclusion of algebraic groups $\mathrm{GL}_2 \hookrightarrow G = \mathrm{Res}^L_{\mathbb{Q}} \mathrm{GL}_2$.

Now we identify the complex points of the modular curve $Y(K'_{0,\det}(p^\alpha))$ as a disjoint union of quotients of $\mathfrak{H}$. Note that $\det(K'_{0,\det}(p^\alpha)) = \{a \in \hat{\mathbb{Z}}^\times : a_p \in 1+p^\alpha \mathbb{Z}_p\}$. Fixing the set of representatives $\{t \in \mathbb{A}^\times_{\mathbb{Q},f} : t_p = 1, 2, ..., p^\alpha, p \nmid t_p,\ t_q = 1\, \forall q \neq p\}$ for the class group $\mathbb{Q}^\times_+ \backslash \mathbb{A}^\times_{\mathbb{Q},f} / \det(K'_{0,\det}(p^\alpha))$ and noting $\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} K'_{0,\det}(p^\alpha) \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma_1(M) \cap \Gamma_0(p^\alpha)$ independent of t, we identify

$$Y(K'_{0,\det}(p^\alpha))(\mathbb{C}) = \coprod_t \Gamma_1(M) \cap \Gamma_0(p^\alpha) \backslash \mathcal{H}.$$

On the other hand, clearly $\det(K'_0(p^\alpha)) = \hat{\mathbb{Z}}^\times$, so there is only one connected component for $Y(K'_0(p^\alpha))$. Since $K'_0(p^\alpha) \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma_1(M) \cap \Gamma_0(p^\alpha)$ we may canonically identify $Y(K'_0(p^\alpha))(\mathbb{C})$ with $\Gamma_1(M) \cap \Gamma_0(p^\alpha) \backslash \mathcal{H}$. Therefore we may define a natural embedding $i_{\det} : Y(K'_0(p^\alpha))(\mathbb{C}) \hookrightarrow Y(K'_{0,\det}(p^\alpha))(\mathbb{C})$ which is the identity map onto the identity component (the component indexed by $t = 1$) of the latter.

By construction we have the following commutative diagram

$$\begin{array}{ccc} Y(K'_0(p^\alpha))(\mathbb{C}) & \xhookrightarrow{i_{det}} & Y(K'_{0,\det}(p^\alpha))(\mathbb{C}) \\ \downarrow{\scriptstyle \iota_\alpha} & & \downarrow{\scriptstyle \iota_\alpha} \\ S(K_\diamond(p^\alpha))(\mathbb{C}) & \xrightarrow[\simeq]{\nu_\alpha} & S(K_{\mathrm{X}}(p^\alpha))(\mathbb{C}) \end{array} \tag{4.20}$$

Proposition 4.6.1. Let $\mathrm{P} \in \boldsymbol{\mathcal{A}}(\mathbf{I}_{\mathscr{G}})$ be an arithmetic point of weight 2 and character $\chi_\diamond$, a character on $1 + p\mathbb{Z}_p$ of conductor p^α. Then

$$e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger(\mathrm{P}) = \mathsf{a}_p(\varpi_\mathfrak{p}, \breve{\mathfrak{g}}_\mathrm{P})^\alpha \cdot G(\theta_{L,\mathfrak{p}}^{-1}\chi_\mathfrak{p}^{-1})^{-1} \cdot e_{\mathrm{ord}}\zeta^* d_\mu^{-1} w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathfrak{g}}_\mathrm{P}^{[\mathcal{P}]}).$$

Proof. In the proof we will work with the classical Fourier coefficients $\mathsf{a}(\xi, -)$ on each components rather than the adelic Fourier coefficients, both for classical Hilbert modular forms and p-adic Hilbert modular forms, since the former type of coefficients is more convenient here. Note that these two points of view are completely equivalent as one determine the other.

By Proposition 3.1.3, the specialization of $e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger$ at P is $e_{\mathrm{ord}}\zeta^*\big(d_\mu^{-1}(\breve{\mathfrak{g}}_\mathrm{P}^{[\mathcal{P}]} \star \theta_{L,\mathfrak{p}}^{-1}\chi_\mathfrak{p}^{-1})\big) \otimes \theta_\mathbb{Q}^{-1}\chi_\diamond$. Lemma 2.9.4 implies that

$$\breve{\mathfrak{g}}_\mathrm{P}^{[\mathcal{P}]} \star \theta_{L,\mathfrak{p}}^{-1}\chi_\mathfrak{p}^{-1} = \mathsf{a}_p(\varpi_\mathfrak{p}, \breve{\mathfrak{g}}_\mathrm{P})^\alpha \cdot G(\theta_{L,\mathfrak{p}}^{-1}\chi_\mathfrak{p}^{-1})^{-1} \cdot (\breve{\mathfrak{g}}_\mathrm{P}^{[\mathcal{P}]}|\tau_{\mathfrak{p}^\alpha}^{-1}).$$

To simplify notation, write $c_\mathfrak{g}$ for the explicit factor appearing in the right hand side of the equation above. Then we have

$$e_{\mathrm{ord}}\zeta^*\big(d_\mu^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)^\dagger(\mathrm{P}) = c_\mathfrak{g} \cdot \big(e_{\mathrm{ord}}\zeta^* d_\mu^{-1}(\breve{\mathfrak{g}}_\mathrm{P}^{[\mathcal{P}]}|\tau_{\mathfrak{p}^\alpha}^{-1})\big) \otimes \theta_\mathbb{Q}^{-1}\chi_\diamond.$$

Realizing the weight $(2t_L, t_L)$ Hilbert modular form $\breve{\mathfrak{g}}_\mathrm{P}^{[\mathcal{P}]}$ as a differential on the complex surface $S(K_{\diamond,t}(p^\alpha))(\mathbb{C})$, we may pull it back via $\mathfrak{T}_{\tau_{\mathfrak{p}^\alpha}}$ and obtain a weight $(2t_L, t_L)$ Hilbert modular form on $S(K_\mathrm{X}(p^\alpha))(\mathbb{C})$ which, by definition, is precisely $\breve{\mathfrak{g}}_\mathrm{P}^{[\mathcal{P}]}|\tau_{\mathfrak{p}^\alpha}^{-1}$. Further pulling back along ν_α we obtain a Hilbert modular form $\nu_\alpha^*(\breve{\mathfrak{g}}_\mathrm{P}^{[\mathcal{P}]}|\tau_{\mathfrak{p}^\alpha}^{-1})$ on $S(K_\diamond(p^\alpha))(\mathbb{C})$, which is by definition $w_{\mathfrak{p}_2^\alpha}^*\breve{\mathfrak{g}}_\mathrm{P}^{[\mathcal{P}]}$. Since ν_α is the identity map on $\mathfrak{H}^2$ on each connected component, ν_α^* preserves the classical Fourier expansions. Thus

for all $\xi \in L_+$ and every $i = 1, ..., h$ we have

$$\mathsf{a}(\xi, ((w_{\mathfrak{p}_2^\alpha})^* \breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]})_i) = \mathsf{a}(\xi, (\breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]} | \tau_{\mathfrak{p}^\alpha}^{-1})_i).$$

Now we regard $(w_{\mathfrak{p}_2^\alpha})^* \breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]}$ and $\breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]} | \tau_{\mathfrak{p}^\alpha}^{-1}$ as p-adic modular forms. In particular, $a(\xi, (\breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]} | \tau_{\mathfrak{p}^\alpha})_i)$ is now understood as elements in $\bar{\mathbb{Q}}_p$. It is clear from the definition of the differential operators that

$$\mathsf{a}\big(\xi, (d_\mu^{-1}(w_{\mathfrak{p}_2^\alpha})^* \breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]})_i\big) = \mathsf{a}\big(\xi, (d_\mu^{-1}(\breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]} | \tau_{\mathfrak{p}^\alpha}^{-1}))_i\big). \tag{4.21}$$

Applying diagonal restriction to $d_\mu^{-1}(\breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]} | \tau_{\mathfrak{p}^\alpha}^{-1})$ we obtain a p-adic elliptic modular form on $Y(K'_{0,\det}(p^\alpha))$ whose Fourier expansion on the identity component is

$$\sum_{n=1}^{\infty} q^n \sum_{\xi \in \mathfrak{d}_L^{-1}, N_{L/\mathbb{Q}}(\xi) = n} \mathsf{a}\big(\xi, (d_\mu^{-1}(\breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]} | \tau_{\mathfrak{p}^\alpha}^{-1}))_1\big),$$

which, by (4.21), is the same as the p-adic modular form $\zeta^* d_\mu^{-1}(w_{\mathfrak{p}_2^\alpha})^* \breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]}$ on $Y(K'_0(p^\alpha))$.

By Lemma 2.9.2, twisting by $\theta_{\mathbb{Q}}^{-1}\chi_\diamond$ does not change the classical Fourier expansion on the identity component. Thus $\big(\zeta^* d_\mu^{-1}(\breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]} | \tau_{\mathfrak{p}^\alpha})\big) \otimes \theta_{\mathbb{Q}}^{-1}\chi_\diamond$ and $\zeta^* d_\mu^{-1}(w_{\mathfrak{p}_2^\alpha})^* \breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]}$, which are now both modular forms of weight $(2,1)$ and character $(\psi_\circ^{-1}, \mathbb{I})$ (in particular they both live on $Y(K'_0(p^\alpha))$ with only one connected component), have the same Fourier expansions, and thus are the same modular forms. This completes the proof. □

Corollary 4.6.2. Let $\mathrm{P} \in \boldsymbol{\mathcal{A}}(\mathbf{I}_{\mathscr{G}})$ be as in Proposition 4.6.1 above. Then

$$\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)(\mathrm{P}) = \mathsf{a}_p(\varpi_{\mathfrak{p}}, \breve{\mathfrak{g}}_{\mathrm{P}})^\alpha \cdot G(\theta_{L,\mathfrak{p}}^{-1}\chi_{\mathfrak{p}}^{-1})^{-1} \cdot \frac{\Big\langle e_{\mathrm{ord}} \zeta^* d_\mu^{-1} w_{\mathfrak{p}_2^\alpha}^* (\breve{\mathfrak{g}}_{\mathrm{P}}^{[\mathcal{P}]}), \mathsf{f}_\circ^{(p)} \Big\rangle}{\Big\langle \mathsf{f}_\circ^{(p)}, \mathsf{f}_\circ^{(p)} \Big\rangle}.$$

Proof. This follows immediately after combining (3.5) and Proposition 4.6.1. □

Chapter 5

Galois representations

5.1 Galois representations at finite level

Let f be a p-nearly ordinary eigenform of weight (k, w) and level $\mathfrak{N}p^\alpha$ over a totally real field F. Then one has a 2-dimensional Galois representation associated to f.

Theorem 5.1.1. (Blasius-Rogawski-Taylor) Let f be as above, and E be a finite extension of $\mathbb{Q}_p$ containing its Hecke eigenvalues. Then there is an irreducible 2-dimensional Galois representation ϱ_f over E

$$\varrho_\mathsf{f} : G_F \to \mathrm{GL}_2(E),$$

such that for all primes $\mathfrak{q}$ not dividing $\mathfrak{N}p$, ϱ_f is unramified at $\mathfrak{q}$, and

$$\det(1 - X\varrho_\mathsf{f}(\mathrm{Frob}_\mathfrak{q})) = 1 - \lambda(T(\mathfrak{q}))X + \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})^{m+1}\lambda(\langle\mathfrak{q}\rangle)X^2.$$

Definition 5.1.2. The Asai Galois representation associated to g is

$$\mathrm{As}(\mathrm{V}_\mathsf{g}) = \otimes\text{-}\mathrm{Ind}_F^\mathbb{Q}(\mathrm{V}_\mathsf{g}) : G_\mathbb{Q} \to \mathrm{GL}_4(E).$$

Lemma 5.1.3. Suppose $\mathsf{g} \in S_{2t_L,t_L}(K;\bar{\mathbb{Q}})$ is a primitive eigenform. If g does not have complex multiplication nor does it arise from base change, then $\mathrm{As}(\mathrm{V}_{\mathsf{g}})$ is irreducible.

Proof. For a choice of element $\theta \in G_{\mathbb{Q}} \setminus G_L$,

$$\mathrm{As}(\mathrm{V}_{\mathsf{g}})_{|G_L} \cong \mathrm{V}_{\mathsf{g}} \otimes (\mathrm{V}_{\mathsf{g}})^{\theta},$$

where $(\mathrm{V}_{\mathsf{g}})^{\theta}$ is the representation given by $\varrho_{\mathsf{g}}^{\theta}(\gamma) = \varrho_{\mathsf{g}}(\theta\gamma\theta^{-1})$ for any $\gamma \in G_L$. Suppose by contradiction that $\mathrm{As}(\mathrm{V}_{\mathsf{g}})$ is reducible, then it must have a $G_{\mathbb{Q}}$-stable subspace of dimension one or two. On the one hand, when the Galois stable subspace is 1-dimensional, let α be the character of G_L describing the action. It follows that the representation

$$\mathrm{As}(\mathrm{V}_{\mathsf{g}})(\alpha^{-1})_{|G_L} \simeq \mathrm{Hom}\big(\mathrm{V}_{\mathsf{g}}^{\vee}, (\mathrm{V}_{\mathsf{g}})^{\theta}\big)(\alpha^{-1})$$

has a line with trivial G_L-action. From the irreducibility of V_{g} we deduce that $(\mathrm{V}_{\mathsf{g}})^{\theta}$ is isomorphic to a twist of V_{g} as G_L-representations applying Schur's lemma. From the irreducibility of V_{g} we deduce that $(\mathrm{V}_{\mathsf{g}})^{\theta}$ is isomorphic to a twist of V_{g} as G_L-representations applying Schur's lemma. Hence by Theorem 2 of [LR98], g arises from base-change, contradicting our assumption. On the other hand, when $\mathrm{As}(\mathrm{V}_{\mathsf{g}})$ has an irreducible sub-representation $\mathrm{W} \subset \mathrm{As}(\mathrm{V}_{\mathsf{g}})$ of dimension two, the representation $\bigwedge^2 \mathrm{As}(\mathrm{V}_{\mathsf{g}})$ has $\bigwedge^2 \mathrm{W}$ as a 1-dimensional $G_{\mathbb{Q}}$-stable subspace. We deduce that both $\mathrm{Sym}^2 \mathrm{V}_{\mathsf{g}}$ and $\mathrm{Sym}^2(\mathrm{V}_{\mathsf{g}})^{\theta}$ have a 1-dimensional G_L-stable subspace because

$$\bigwedge^2 \mathrm{As}(\mathrm{V}_{\mathsf{g}})_{|G_L} \simeq \Big(\mathrm{Sym}^2 \mathrm{V}_{\mathsf{g}} \otimes \bigwedge^2 (\mathrm{V}_{\mathsf{g}})^{\theta}\Big) \oplus \Big(\mathrm{Sym}^2(\mathrm{V}_{\mathsf{g}})^{\theta} \otimes \bigwedge^2 \mathrm{V}_{\mathsf{g}}\Big).$$

Moreover, it follows that the G_L-representation $\mathrm{Hom}\big(\mathrm{V}_{\mathsf{g}}^{\vee}, \mathrm{V}_{\mathsf{g}}\big)$ has two 1-dimensional

G_L-stable subspaces, one of which is $\bigwedge^2 \mathrm{V}_{\mathfrak{g}} = \det \mathrm{V}_{\mathfrak{g}}$ as

$$\mathrm{Hom}\big(\mathrm{V}_{\mathfrak{g}}^{\vee}, \mathrm{V}_{\mathfrak{g}}\big) \simeq \mathrm{V}_{\mathfrak{g}} \otimes \mathrm{V}_{\mathfrak{g}} \simeq \mathrm{Sym}^2 \mathrm{V}_{\mathfrak{g}} \oplus \bigwedge^2 \mathrm{V}_{\mathfrak{g}}.$$

Let β be the character of the stable subspace of $\mathrm{Sym}^2 \mathrm{V}_{\mathfrak{g}}$, then $\mathrm{Hom}_{G_L}\big(\mathrm{V}_{\mathfrak{g}}^{\vee}, \mathrm{V}_{\mathfrak{g}}(\beta^{-1})\big)$ is 1-dimensional since $\mathrm{V}_{\mathfrak{g}}$ is irreducible. Therefore $\beta \neq \det \mathrm{V}_{\mathfrak{g}}$ and $\mathrm{V}_{\mathfrak{g}}$ is isomorphic to a twist of itself by a nontrivial character. This means $\mathfrak{g}$ has complex multiplication ([LR98], Theorem 2), which contradicts our assumptions. □

Proposition 5.1.4. (Brylinski-Labesse, Nekovář, c.f. also Theorem 4.4.3 of [LLZ16]) The $\mathfrak{g}$-isotypic part $\mathrm{H}^2_{\text{ét}}(S(K)_{\bar{\mathbb{Q}}}, E(2))[\mathfrak{g}]$ is 4-dimensional and isomorphic to $\mathrm{As}(\mathrm{V}_{\mathfrak{g}})$ as a $G_{\mathbb{Q}}$-representation.

Hodge-Tate numerology.

If we normalize Hodge-Tate weights by stating that the character $\mathcal{N}_F$ has negative weights, then for every $\mathcal{O}_F$- prime ideal $\mathfrak{p} \mid p$, the restriction $\big(V_{\mathfrak{f}}\big)_{|D_{\mathfrak{p}}}$ has a one-dimensional subrepresentation with Hodge-Tate weights $\{-\frac{k_\tau}{2} - \frac{m}{2}\}_{\tau \in \mathrm{I}_{F,\mathfrak{p}}}$ and a one-dimensional quotient with Hodge-Tate weights $\{\frac{k_\tau - 2}{2} - \frac{m}{2}\}_{\tau \in \mathrm{I}_{F,\mathfrak{p}}}$ ([Hid91], Introduction). Therefore, if we twist by the $-m/2$-th power of the character η_F, the restriction at $D_{\mathfrak{p}}$ of the representation

$$\mathrm{V}_{\mathfrak{f}}^{\dagger} = \mathrm{V}_{\mathfrak{f}}\left(-\frac{m}{2}\right)$$

has Hodge-Tate weights $\{-\frac{k_\tau}{2}, \frac{k_\tau - 2}{2}\}_{\tau \in \mathrm{I}_{F,\mathfrak{p}}}$.

Let $\mathfrak{g}$ be a nearly ordinary Hilbert eigenform of weight (ℓ, x) over a real quadratic field L. The Galois representation $\mathrm{V}_{\mathfrak{g}}^{\dagger}$ is related to the cohomology of Hilbert modular varieties via tensor induction, which is why we are interested in the Asai representation.

When $p\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$ splits in the real quadratic field L, $\mathrm{As}(\mathrm{V}_{\mathfrak{g}}^{\dagger})_{|D_p} = (\mathrm{V}_{\mathfrak{g}}^{\dagger})_{|D_{\mathfrak{p}_1}} \otimes$

$(\mathrm{V}_{\mathsf{g}}^{\dagger})_{|D_{\mathfrak{p}_2}}$, so the restriction to the decomposition group at p has a 3-step filtration

$$\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger}) \supset \mathrm{Fil}^1\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger}) \supset \mathrm{Fil}^2\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger}) \supset \{0\}$$

in which the graded pieces have dimension $1, 2$ and 1. We can compute the Hodge-Tate weights as follows.

Graded piece	Hodge-Tate weights
$\mathrm{Gr}^0\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger})$	$\frac{\lvert\ell\rvert}{2} - 2$
$\mathrm{Gr}^1\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger})$	$(\frac{\ell_1-\ell_2}{2} - 1, \frac{\ell_2-\ell_1}{2} - 1)$
$\mathrm{Gr}^2\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger})$	$-\frac{\lvert\ell\rvert}{2}$

Let f be an elliptic nearly ordinary eigenform of weight (k, w) (+ other conditions). We are interested in the $G_{\mathbb{Q}}$-representation $V_{\mathsf{g},\mathsf{f}}^{\dagger} = \mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger})(-1) \otimes \mathrm{V}_{\mathsf{f}}^{\dagger}$ whose restriction at the decomposition group at p has a 4-step filtration

$$V_{\mathsf{g},\mathsf{f}}^{\dagger} \supset \mathrm{Fil}^1 V_{\mathsf{g},\mathsf{f}}^{\dagger} \supset \mathrm{Fil}^2 V_{\mathsf{g},\mathsf{f}}^{\dagger} \supset \mathrm{Fil}^3 V_{\mathsf{g},\mathsf{f}}^{\dagger} \supset \{0\}$$

with graded pieces of dimension $1, 3, 3$ and 1 given by

$$\mathrm{Gr}^0 V_{\mathsf{g},\mathsf{f}}^{\dagger} = \mathrm{Gr}^0(\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger})) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}}^{\dagger}(-1)$$

$$\mathrm{Gr}^1 V_{\mathsf{g},\mathsf{f}}^{\dagger} = [\mathrm{Gr}^0(\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger})) \otimes \mathrm{Gr}^1\mathrm{V}_{\mathsf{f}}^{\dagger}](-1) \oplus [\mathrm{Gr}^1(\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger})) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}}^{\dagger}](-1)$$

$$\mathrm{Gr}^2 V_{\mathsf{g},\mathsf{f}}^{\dagger} = [\mathrm{Gr}^1(\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger})) \otimes \mathrm{Gr}^1\mathrm{V}_{\mathsf{f}}^{\dagger}](-1) \oplus [\mathrm{Gr}^2(\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger})) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}}^{\dagger}](-1)$$

$$\mathrm{Gr}^3 V_{\mathsf{g},\mathsf{f}}^{\dagger} = \mathrm{Gr}^2(\mathrm{As}(\mathrm{V}_{\mathsf{g}}^{\dagger})) \otimes \mathrm{Gr}^1\mathrm{V}_{\mathsf{f}}^{\dagger}(-1).$$

The Hodge-Tate weights of the graded pieces are computed in the following table.

Graded piece	Hodge-Tate weights
$\mathrm{Gr}^0 V^\dagger_{\mathsf{g},\mathsf{f}}$	$\frac{\lvert\ell\rvert+k}{2}-2$
$\mathrm{Gr}^1 V^\dagger_{\mathsf{g},\mathsf{f}}$	$(\frac{\lvert\ell\rvert-k}{2}-1, \frac{\ell_1-\ell_2+k}{2}-1, \frac{\ell_2-\ell_1+k}{2}-1)$
$\mathrm{Gr}^2 V^\dagger_{\mathsf{g},\mathsf{f}}$	$(\frac{\ell_1-\ell_2-k}{2}, \frac{\ell_2-\ell_1-k}{2}, \frac{k-\lvert\ell\rvert}{2})$
$\mathrm{Gr}^3 V^\dagger_{\mathsf{g},\mathsf{f}}$	$1-\frac{\lvert\ell\rvert+k}{2}$

Corollary 5.1.5. The Hodge-Tate weights of $\mathrm{Fil}^2 V^\dagger_{\mathsf{g},\mathsf{f}}$ are all strictly negative if and only if the triple of weights (ℓ, k) is balanced.

5.2 Big Galois representations.

The ordinary family $\mathscr{G}$ passing through $\mathsf{g}_\circ^{(p)}$ has an associated big Galois representation $\boldsymbol{\varrho}_{\mathscr{G}} : G_L \to \mathrm{GL}_2(\mathbf{Q}_{\mathscr{G}})$ acting on a two dimensional $\mathbf{Q}_{\mathscr{G}}$-vector space $\mathbf{V}_{\mathscr{G}}$ ([Hid89a], Theorem 1), where $\mathbf{Q}_{\mathscr{G}}$ is the fraction field of $\mathbf{I}_{\mathscr{G}}$. The representation $\boldsymbol{\varrho}_{\mathscr{G}}$ is unramified outside $\mathfrak{Q}p$ with determinant

$$\det(\boldsymbol{\varrho}_{\mathscr{G}})(z) = \phi_{\boldsymbol{\chi}}([z,1])\varepsilon_L(z) = \chi_\circ(z)\eta_L(z)[\xi_z^{-t_L}], \qquad \forall\ z \in \mathbb{A}_{L,f}^\times. \tag{5.1}$$

The characteristic polynomial at any prime $\mathfrak{q} \nmid \mathfrak{Q}p$ is given by

$$\det(1-\boldsymbol{\varrho}_{\mathscr{G}}(\mathrm{Frob}_{\mathfrak{q}})X) = 1-\mathscr{G}(\mathbf{T}(\mathfrak{q}))X + \phi_{\boldsymbol{\chi}}([\varpi_{\mathfrak{q}},1])\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{q})X^2,$$

and for every prime $\mathfrak{p}$ above p, fixing a decomposition group $D_{\mathfrak{p}}$ in G_L, there is an unramified character $\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}} : D_{\mathfrak{p}} \to \mathbf{I}_{\mathscr{G}}^\times$, $\mathrm{Frob}_{\mathfrak{p}} \mapsto \mathscr{G}(\mathbf{T}(\varpi_{\mathfrak{p}}))$, such that ([Hid89a],

Proposition 2.3)

$$(\boldsymbol{\varrho}_{\mathscr{G}})_{|D_{\mathfrak{p}}} \sim \begin{pmatrix} \boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}}^{-1} \cdot \det(\boldsymbol{\varrho}_{\mathscr{G}})_{|D_{\mathfrak{p}}} & * \\ 0 & \boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}} \end{pmatrix}. \tag{5.2}$$

Definition 5.2.1. We let

$$\mathrm{As}(\mathbf{V}_{\mathscr{G}}) := \otimes\text{-}\mathrm{Ind}_L^{\mathbb{Q}}(\mathbf{V}_{\mathscr{G}})$$

denote the tensor induction of $\mathbf{V}_{\mathscr{G}}$.

Definition 5.2.2. We define

$$\boldsymbol{\eta}_{\mathbb{Q}} : G_{\mathbb{Q}} \longrightarrow O[\![\Gamma]\!]^{\times}, \qquad z \mapsto \left[\eta_{\mathbb{Q}}(z)\right] = \left[\xi_z^{-1}\right]$$

for any $z \in \mathbb{A}_{\mathbb{Q}}^{\times}$. Then $(\boldsymbol{\eta}_{\mathbb{Q}})_{|D_p}$ is the character associated by local class field theory to the homomorphism $\mathbb{Q}_p^{\times} \to \Gamma \to O[\![\Gamma]\!]^{\times}$, $x \mapsto \left[\langle x \rangle^{-1}\right]$.

Note that since $p\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$ splits in the real quadratic field L, the decomposition group $D_p \subset G_{\mathbb{Q}}$ at p is contained in G_L. Hence we can consider the characters $\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}}$ as characters of D_p and we can define $\boldsymbol{\Psi}_{\mathscr{G},p} = \boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_2}$. Furthermore, the fixed $\theta \in G_{\mathbb{Q}} \setminus G_L$ interchanges $\mathfrak{p}_1$ and $\mathfrak{p}_2$, inducing an isomorphism $L_{\mathfrak{p}_1} \simeq L_{\mathfrak{p}_2}$. Thanks to equation (5.2), we have a concrete description of the action of the decomposition group on $\mathrm{As}(\mathbf{V}_{\mathscr{G}})$.

First, by (5.1) we have

$$\mathrm{As}(\det(\boldsymbol{\varrho}_{\mathscr{G}})) = \eta_{\mathbb{Q}}^2\boldsymbol{\eta}_{\mathbb{Q}}^2.$$

Furthermore, we have the following filtration of $\mathrm{As}(\mathbf{V}_{\mathscr{G}})$ as a local Galois representation.

Proposition 5.2.3. The restriction of $\mathrm{As}(\mathbf{V}_{\mathscr{G}})$ to D_p admits a three step D_p-stable

filtration

$$\mathrm{As}(\mathbf{V}_{\mathscr{G}}) \supset \mathrm{Fil}^1\mathrm{As}(\mathbf{V}_{\mathscr{G}}) \supset \mathrm{Fil}^2\mathrm{As}(\mathbf{V}_{\mathscr{G}}) \supset 0$$

with graded pieces

$$\mathrm{Gr}^0\mathrm{As}(\mathbf{V}_{\mathscr{G}}) = \mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}_{\mathscr{G},p}\big), \qquad \mathrm{Gr}^2\mathrm{As}(\mathbf{V}_{\mathscr{G}}) = \mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}_{\mathscr{G},p}^{-1} \cdot (\eta_{\mathbb{Q}}^2\boldsymbol{\eta}_{\mathbb{Q}}^2)_{|D_p}\big)$$

and

$$\mathrm{Gr}^1\mathrm{As}(\mathbf{V}_{\mathscr{G}}) = \mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}^{-1}\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_2} \cdot (\chi_{\circ,\mathfrak{p}_1}\eta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}})_{|D_p}\big) \oplus \mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_2}^{-1} \cdot (\chi_{\circ,\mathfrak{p}_2}\eta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}})_{|D_p}\big)$$

where, for any prime $\mathfrak{p} \mid p$, we let $\chi_{\circ,\mathfrak{p}} : D_p \to O^\times$ be the unramified character determined by $\chi_{\circ,\mathfrak{p}}(\mathrm{Frob}_p) = \chi_\circ(\mathrm{Frob}_{\mathfrak{p}})$. In particular, $\chi_{\circ,\mathfrak{p}_1} \cdot \chi_{\circ,\mathfrak{p}_2} = 1$.

Proof. Let $\mathbf{V}_{\mathscr{G}}^+$ denote the $\mathbf{Q}_{\mathscr{G}}$-rank 1 submodule of $\mathbf{V}_{\mathscr{G}}$ coming from the upper left corner in equation (5.2), and $\mathbf{V}_{\mathscr{G}}^- = \mathbf{V}_{\mathscr{G}}/\mathbf{V}_{\mathscr{G}}^+$ the rank 1 quotient. If we define

$$\mathrm{Fil}^2\mathrm{As}(\mathbf{V}_{\mathscr{G}}) := \mathbf{V}_{\mathscr{G}}^+ \otimes (\mathbf{V}_{\mathscr{G}}^+)^\theta \quad \text{and} \quad \mathrm{Fil}^1\mathrm{As}(\mathbf{V}_{\mathscr{G}}) := \mathbf{V}_{\mathscr{G}}^+ \otimes (\mathbf{V}_{\mathscr{G}})^\theta + \mathbf{V}_{\mathscr{G}} \otimes (\mathbf{V}_{\mathscr{G}}^+)^\theta,$$

then $\mathrm{Fil}^2\mathrm{As}(\mathbf{V}_{\mathscr{G}})$ has rank 1 over $\mathbf{Q}_{\mathscr{G}}$ while $\mathrm{Fil}^1\mathrm{As}(\mathbf{V}_{\mathscr{G}})$ has rank 3. By the description in (5.2), D_p acts on $\mathrm{Fil}^2\mathrm{As}(\mathbf{V}_{\mathscr{G}})$ through the character $\boldsymbol{\Psi}_{\mathscr{G},p}^{-1} \cdot \big(\eta_{\mathbb{Q}}^2\boldsymbol{\eta}_{\mathbb{Q}}^2\big)_{|D_p}$, while it acts on the zero-th graded piece $\mathrm{Gr}^0\mathrm{As}(\mathbf{V}_{\mathscr{G}}) = \mathbf{V}_{\mathscr{G}}^- \otimes (\mathbf{V}_{\mathscr{G}}^-)^\theta$ through $\boldsymbol{\Psi}_{\mathscr{G},p}$. Finally, the first graded piece is

$$\mathrm{Gr}^1\mathrm{As}(\mathbf{V}_{\mathscr{G}}) = \mathbf{Q}_{\mathscr{G}}\left(\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}^{-1}\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}^{\theta} \cdot \det(\boldsymbol{\varrho}_{\mathscr{G}})_{|D_p}\right) \oplus \mathbf{Q}_{\mathscr{G}}\left(\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}(\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}^{\theta})^{-1} \cdot \det(\boldsymbol{\varrho}_{\mathscr{G}})_{|D_p}^{\theta}\right).$$

It is clear that $\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}^{\theta} = \boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_2}$ and that

$$\det(\boldsymbol{\varrho}_{\mathscr{G}})_{|D_p} = \big(\chi_{\circ,\mathfrak{p}_1}\eta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}}\big)_{|D_p}, \qquad \det(\boldsymbol{\varrho}_{\mathscr{G}})_{|D_p}^{\theta} = \big(\chi_{\circ,\mathfrak{p}_2}\eta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}}\big)_{|D_p}.$$

□

Definition 5.2.4. We define the following $G_{\mathbb{Q}}$-representation

$$\mathrm{As}(\mathbf{V}_{\mathscr{G}})^{\dagger} := \mathrm{As}(\mathbf{V}_{\mathscr{G}})(\theta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}}^{-1}).$$

Let $\mathrm{V}_{\mathsf{f}_\circ}$ denote the representation attached to the ordinary elliptic cuspform $\mathsf{f}_\circ$. Then

$$\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ} := \mathrm{As}(\mathbf{V}_{\mathscr{G}})^{\dagger}(-1) \otimes \mathrm{V}_{\mathsf{f}_\circ}.$$

interpolates Kummer self-dual Galois representations.

The explicit realization of that Galois representation in the cohomology of a tower of certain threefolds with increasing level at p plays an important role in the construction of big Hirzebruch-Zagier classes.

Lemma 5.2.5. The restriction to a decomposition group at p of the Galois representation $\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ}$ is endowed with a 4-steps filtration by $G_{\mathbb{Q}_p}$-stable submodules with graded pieces given by

$$\mathrm{Gr}^0\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ} = \mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}_{\mathscr{G},p}\delta_p(\mathsf{f}_\circ)\cdot(\eta_{\mathbb{Q}}^{-1}\boldsymbol{\eta}_{\mathbb{Q}}^{-1})_{|D_p}\big),$$

$$\begin{aligned}\mathrm{Gr}^1\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ} =&\mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}_{\mathscr{G},p}\delta_p(\mathsf{f}_\circ)^{-1}\cdot(\theta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}}^{-1})_{|D_p}\big)\oplus\mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}^{-1}_{\mathscr{G},\mathfrak{p}_1}\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_2}\delta_p(\mathsf{f}_\circ)\cdot(\chi_{\circ,\mathfrak{p}_1})_{|D_p}\big)\\ &\oplus\mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}\boldsymbol{\Psi}^{-1}_{\mathscr{G},\mathfrak{p}_2}\delta_p(\mathsf{f}_\circ)\cdot(\chi_{\circ,\mathfrak{p}_2})_{|D_p}\big),\end{aligned}$$

$$\begin{aligned}\mathrm{Gr}^2\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ} =&\mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}^{-1}_{\mathscr{G},\mathfrak{p}_1}\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_2}\delta_p(\mathsf{f}_\circ)^{-1}(\chi^{-1}_{\circ,\mathfrak{p}_2}\varepsilon_{\mathbb{Q}})_{|D_p}\big)\oplus\mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}\boldsymbol{\Psi}^{-1}_{\mathscr{G},\mathfrak{p}_2}\delta_p(\mathsf{f}_\circ)^{-1}\cdot(\chi^{-1}_{\circ,\mathfrak{p}_1}\varepsilon_{\mathbb{Q}})_{|D_p}\big)\\ &\oplus\mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}^{-1}_{\mathscr{G},p}\delta_p(\mathsf{f}_\circ)\cdot(\eta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}})_{|D_p}\big),\end{aligned}$$

$$\mathrm{Gr}^3\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ} = \mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}^{-1}_{\mathscr{G},p}\delta_p(\mathsf{f}_\circ)^{-1}\cdot(\theta_{\mathbb{Q}}\eta^2_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}})_{|D_p}\big)$$

Proof. As $\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ} = \mathrm{As}(\mathbf{V}_{\mathscr{G}})(\eta_{\mathbb{Q}}^{-1}\boldsymbol{\eta}_{\mathbb{Q}}^{-1}) \otimes \mathrm{V}_{\mathsf{f}_\circ}$, its graded pieces are given by

$$\mathrm{Gr}^0\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ} = \mathrm{Gr}^0\mathrm{As}(\mathbf{V}_{\mathscr{G}})(\eta_{\mathbb{Q}}^{-1}\boldsymbol{\eta}_{\mathbb{Q}}^{-1}) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}_\circ},$$

$$\mathrm{Gr}^1\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ} = [\mathrm{Gr}^0\mathrm{As}(\mathbf{V}_{\mathscr{G}})(\eta_{\mathbb{Q}}^{-1}\boldsymbol{\eta}_{\mathbb{Q}}^{-1}) \otimes \mathrm{Gr}^1\mathrm{V}_{\mathsf{f}_\circ}] \oplus [\mathrm{Gr}^1\mathrm{As}(\mathbf{V}_{\mathscr{G}})(\eta_{\mathbb{Q}}^{-1}\boldsymbol{\eta}_{\mathbb{Q}}^{-1}) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}_\circ}],$$

$$\mathrm{Gr}^2\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ} = [\mathrm{Gr}^1\mathrm{As}(\mathbf{V}_{\mathscr{G}})(\eta_{\mathbb{Q}}^{-1}\boldsymbol{\eta}_{\mathbb{Q}}^{-1}) \otimes \mathrm{Gr}^1\mathrm{V}_{\mathsf{f}_\circ}] \oplus [\mathrm{Gr}^2\mathrm{As}(\mathbf{V}_{\mathscr{G}})(\eta_{\mathbb{Q}}^{-1}\boldsymbol{\eta}_{\mathbb{Q}}^{-1}) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}_\circ}],$$

$$\mathrm{Gr}^3\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ} = \mathrm{Gr}^2\mathrm{As}(\mathbf{V}_{\mathscr{G}})(\eta_{\mathbb{Q}}^{-1}\boldsymbol{\eta}_{\mathbb{Q}}^{-1}) \otimes \mathrm{Gr}^1\mathrm{V}_{\mathsf{f}_\circ}.$$

Hence, the statement follows from Proposition 5.2.3 and a direct computation. □

Definition 5.2.6. We define the direct summand $\mathbf{V}^{\mathsf{f}_\circ}_{\mathscr{G}}$ of $\mathrm{Gr}^2\mathbf{V}^{\dagger}_{\mathscr{G},\mathsf{f}_\circ}$ by

$$\mathbf{V}^{\mathsf{f}_\circ}_{\mathscr{G}} := \mathbf{Q}_{\mathscr{G}}\big(\boldsymbol{\Psi}^{-1}_{\mathscr{G},p}\delta_p(\mathsf{f}_\circ)\cdot(\eta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}})_{|D_p}\big).$$

5.3 Étale cohomology of towers of Hilbert modular surfaces

We first show the ordinary Hida family introduced in Definition 3.1.2 is of $K_{\diamond,t}(p^\infty)$ level.

Let $\mathbf{h}_{2t_L,t_L}(K_{\diamond,t}(p^\alpha);\mathbb{Z}_p)$ be the Hecke algebra acting on $S_{2t_L,t_L}(K_{\diamond,t}(p^\alpha);\mathbb{Z}_p)$, which is a subspace of $S_{2t_L,t_L}(K(p^\alpha);\mathbb{Z}_p)$. Write

$$\mathbb{G}^{\alpha}_{L,0}(K) = \mathcal{O}_L^{\times}K_0(p^\alpha)/\mathcal{O}_L^{\times}K(p^\alpha), \qquad \mathbb{G}_{L,0}(K) = \varprojlim_{\alpha}\mathbb{G}^{\alpha}_{L,0}(K)$$

and

$$\mathbb{G}^{\alpha}_{L,0}(K_{\diamond,t}) = \mathcal{O}_L^{\times}K_0(p^\alpha)/\mathcal{O}_L^{\times}K_{\diamond,t}(p^\alpha), \qquad \mathbb{G}_{L,0}(K_{\diamond,t}) = \varprojlim_{\alpha}\mathbb{G}^{\alpha}_{L,0}(K_{\diamond,t}).$$

Recall the group $\mathbb{G}^{\alpha}_{L,0}(K)$ acts as diamond operators on $S_{2t_L,t_L}(K(p^\alpha);\mathbb{Z}_p)$, and $S_{2t_L,t_L}(K_{\diamond,t}(p^\alpha);\mathbb{Z}_p)$ is the subspace fixed by the subgroup $\mathcal{O}_L^\times K_{\diamond,t}(p^\alpha)/\mathcal{O}_L^\times K(p^\alpha)$. Therefore

$$\mathbf{h}_{2t_L,t_L}(K_{\diamond,t}(p^\alpha);\mathbb{Z}_p) = \mathbf{h}_{2t_L,t_L}(K(p^\alpha);\mathbb{Z}_p) \otimes_{\mathbb{Z}_p[\mathbb{G}^{\alpha}_{L,0}(K)]} \mathbb{Z}_p[\mathbb{G}^{\alpha}_{L,0}(K_{\diamond,t})].$$

Passing to inverse limit, we define as before

$$\mathbf{h}_L(K_{\diamond,t};\mathbb{Z}_p) := \varprojlim_{\alpha} \mathbf{h}_{2t_L,t_L}(K_{\diamond,t}(p^\alpha);\mathbb{Z}_p)$$

and

$$\mathbf{h}^{\mathrm{n.o.}}_L(K_{\diamond,t};\mathbb{Z}_p) := e_{\mathrm{n.o.}}\mathbf{h}_L(K_{\diamond,t};\mathbb{Z}_p).$$

Then we have

$$\mathbf{h}^{\mathrm{n.o.}}_L(K_{\diamond,t};\mathbb{Z}_p) = \mathbf{h}^{\mathrm{n.o.}}_L(K;\mathbb{Z}_p) \otimes_{\mathbb{Z}_p[\![\mathbb{G}_{L,0}(K)]\!]} \mathbb{Z}_p[\![\mathbb{G}_{L,0}(K_{\diamond,t})]\!]. \tag{5.3}$$

Lemma 5.3.1. The Hida family $\mathscr{G} : \mathbf{h}^{\mathrm{n.o.}}_L(K;\mathbb{Z}_p) \to \mathbf{I}_{\mathscr{G}}$ factors through $\mathbf{h}^{\mathrm{n.o.}}_L(K_{\diamond,t};\mathbb{Z}_p)$.

Proof. By the equation (5.3) above, it suffices to check that $\phi_{\boldsymbol{\chi}}$ factors through $\mathbb{Z}_p[\![\mathbb{G}_L(\mathfrak{Q})]\!] \otimes_{\mathbb{Z}_p[\![\mathbb{G}_{L,0}(K)]\!]} \mathbb{Z}_p[\![\mathbb{G}_{L,0}(K_{\diamond,t})]\!]$, which is straightforward from the definition of $K_{\diamond,t}(p^\alpha)$. □

Of course, the same also applies to our test vector $\breve{\mathscr{G}}$.

Definition 5.3.2. Let $K = V_1(M\mathcal{O}_L)$ be a compact open subgroup of $G(\mathbb{A}_{\mathbb{Q},f})$. We define the *anemic* Hecke algebra $\widetilde{\mathbf{h}}^{\mathrm{n.o.}}_L(K_{\diamond,t};\mathbb{Z}_p)$ to be the subalgebra of $\mathbf{h}^{\mathrm{n.o.}}_L(K_{\diamond,t};\mathbb{Z}_p)$ generated by the Hecke operators $\mathbf{T}(y)$ with $y_M = 1$. Moreover, the algebra homomorphism $\widetilde{\mathscr{G}} : \widetilde{\mathbf{h}}^{\mathrm{n.o.}}_L(K_{\diamond,t};\mathbb{Z}_p) \to \mathbf{I}_{\mathscr{G}}$ is the restriction of the Hida family $\mathscr{G}$ to the anemic Hecke algebra.

Recall the modules $\mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);\mathbb{Z}_p(2))$ and $\mathbf{V}^{\mathrm{n.o.}}_\infty(K)$ and the big cohomology class $\boldsymbol{\kappa}^{\mathrm{n.o.}}_\infty$ introduced in Definition 4.5.2. We would like to consider their projections to suitably defined $\mathscr{G}$-isotypic quotients. We note that $\mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);\mathbb{Z}_p(2))$ is naturally a module over $\widetilde{\mathbf{h}}^{\mathrm{n.o.}}_L(K_{\diamond,t};\mathbb{Z}_p)$ and the module structure commutes with Galois action.

Definition 5.3.3. Let

$$\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M) := \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty), O(2))(\boldsymbol{\delta}) \otimes_{\tilde{\mathscr{G}}} \mathbf{I}_{\mathscr{G}} \tag{5.4}$$

using the homomorphism $\tilde{\mathscr{G}} : \widetilde{\mathbf{h}}^{\mathrm{n.o.}}_L(K;O) \to \mathbf{I}_{\mathscr{G}}$, where $\boldsymbol{\delta}$ is as defined in Definition 4.5.2.

Note that

$$\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M) = \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty), O(2)) \otimes_{\tilde{\mathscr{G}}} \mathbf{I}_{\mathscr{G}}(\theta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}}^{-1})$$

because

$$\tilde{\mathscr{G}} \circ \boldsymbol{\delta}(\sigma_a) = \phi_{\boldsymbol{\chi}} \circ \boldsymbol{\delta}(\sigma_a) = \theta_{\mathbb{Q}}(a)\boldsymbol{\eta}_{\mathbb{Q}}^{-1}(a) \in \mathbf{I}^\times_{\mathscr{G}}.$$

The anemic Hecke algebra $\widetilde{\mathbf{h}}^{\mathrm{n.o.}}_L(K;O)$ is finite and torsion-free over the complete Noetherian local ring $\boldsymbol{\Lambda}$, hence it decomposes as the product of the localizations at its maximal ideals. If we denote by $\mathfrak{m}_{\tilde{\mathscr{G}}}$ the maximal ideal containing $\ker(\tilde{\mathscr{G}})$ then

$$\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M) = \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); O(2))_{\mathfrak{m}_{\tilde{\mathscr{G}}}} \otimes_{\tilde{\mathscr{G}}} \mathbf{I}_{\mathscr{G}}(\theta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}}^{-1}).$$

Specializations

Let $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ be an arithmetic point of weight 2 and character $\chi_\diamond$, a p-power order character of conductor p^α. Then by Theorem 2.4 of [Hid89b], the composition

$\mathscr{G}_{\mathrm{P}} = \mathrm{P} \circ \mathscr{G}$ factors through a $\mathbb{Z}_p$-linear map of finite level Hecke algebra

$$\mathscr{G}_{\mathrm{P}} : \mathbf{h}_L^{\mathrm{n.o.}}(K_{\diamond,t}; \mathbb{Z}_p) \longrightarrow O$$

and similarly for the map on anemic Hecke algebra

$$\tilde{\mathscr{G}}_{\mathrm{P}} : \tilde{\mathbf{h}}_L^{\mathrm{n.o.}}(K_{\diamond,t}; \mathbb{Z}_p) \longrightarrow O.$$

Definition 5.3.4. The specialization of $\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)$ at P is

$$\mathcal{V}_{\mathscr{G}_{\mathrm{P}}}(M) := \boldsymbol{\mathcal{V}}_{\mathscr{G}}(M) \otimes_{\mathbf{I}_{\mathscr{G}},\mathrm{P}} O.$$

Combining the above discussion with equation (5.3.3) we see that

$$\mathcal{V}_{\mathscr{G}_{\mathrm{P}}}(M) \simeq \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\alpha), O(2))(\delta_\alpha) \otimes_{\tilde{\mathscr{G}}_{\mathrm{P}}} O = \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\alpha), O(2)) \otimes_{\tilde{\mathscr{G}}_{\mathrm{P}}} O(\chi_\diamond \theta_{\mathbb{Q}})$$

as representations of $G_{\mathbb{Q}}$.

Lemma 5.3.5. Let $\mathbf{M}$ be a finitely generated $\mathbf{I}_{\mathscr{G}}$-module. If there is an infinite collection $\Sigma \subset \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of arithmetic points such that for all $\mathrm{P} \in \Sigma$ with image $O_{\mathrm{P}} \subset \bar{\mathbb{Q}}_p$, the specialization $\mathbf{M} \otimes_{\mathbf{I}_{\mathscr{G}},\mathrm{P}} O_{\mathrm{P}}$ is O_{P}-free of constant rank r, then $\mathbf{M}$ is also $\mathbf{I}_{\mathscr{G}}$-free of rank r.

Proof. Since $\mathbf{M}$ is finitely generated, it fits in a short exact sequence

$$0 \longrightarrow \mathbf{N} \longrightarrow \mathbf{I}_{\mathscr{G}}^{\oplus r} \longrightarrow \mathbf{M} \longrightarrow 0$$

of $\mathbf{I}_{\mathscr{G}}$-modules. For every $\mathrm{P} \in \Sigma$ we obtain an induced exact sequence

$$\mathbf{N} \otimes_{\mathrm{P}} O_{\mathrm{P}} \longrightarrow O_{\mathrm{P}}^{\oplus r} \xrightarrow{\sim} \mathbf{M} \otimes_{\mathrm{P}} O_{\mathrm{P}} \longrightarrow 0,$$

where the map $O_{\mathrm{P}}^{\oplus r} \twoheadrightarrow \mathbf{M} \otimes_{\mathrm{P}} O_{\mathrm{P}}$, a priori only a surjection, is actually an isomorphism being a surjective map between O_{P}-free modules of the same rank. Therefore,

$$\mathbf{N} \subseteq \bigcap_{\mathrm{P} \in \Sigma} \ker \mathrm{P} \cdot (\mathbf{I}_{\mathscr{G}})^{\oplus r} = \{0\}$$

because the intersection of the kernels of infinitely many arithmetic points of $\mathbf{I}_{\mathscr{G}}$ is trivial ([LV11], Proposition 9.4). □

Proposition 5.3.6. Let $\omega(M/\mathfrak{Q})$ be the number of $\mathcal{O}_L$-ideals dividing $M\mathcal{O}_L/\mathfrak{Q}$. The big Galois module $\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)$ is a finite free $\mathbf{I}_{\mathscr{G}}$-module of rank $4 \cdot \omega(M/\mathfrak{Q})$ and it satisfies exact control:

> for any arithmetic point $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight 2 there is a natural Galois equivariant surjection
>
> $$\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M) \twoheadrightarrow \mathrm{As}_O(\mathrm{V}_{\mathscr{G}_{\mathrm{P}}})(\chi_\diamond \theta_{\mathbb{Q}})^{\oplus \omega(M/\mathfrak{Q})},$$
>
> where $\mathrm{As}_O(\mathrm{V}_{\mathscr{G}_{\mathrm{P}}})$ is isomorphic to a $G_{\mathbb{Q}}$-stable O-lattice in the 4-dimensional Asai representation attached to $\mathscr{G}_{\mathrm{P}}$ in Definition 5.1.2.

Proof. For any $\alpha \geq 1$ there is a Hecke equivariant surjection

$$\mathrm{H}^2_{\text{ét}}(K_{\diamond,t}(p^\infty); O(2))_{\mathfrak{m}_{\mathscr{G}}} \twoheadrightarrow \mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}; O(2))_{\mathfrak{m}_{\mathscr{G}}} \tag{5.5}$$

onto a finite free O-module ([Dim13], Theorem 3.8(iii) & Corollary 3.9). Let $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ be an arithmetic point of weight 2 and level $p^\alpha > 1$, then by Proposition 5.1.4, the $\breve{\mathscr{G}}_{\mathrm{P}}$-isotypic submodule $\mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}, O(2))[\breve{\mathscr{G}}_{\mathrm{P}}]$ of $\mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}; O(2))_{\mathfrak{m}_{\mathscr{G}}}$

is free over O of rank $4\cdot\omega(M/\mathfrak{Q})$ and isomorphic to a $G_{\mathbb{Q}}$-stable lattice in the $G_{\mathbb{Q}}$-representation $\mathrm{As}(\mathrm{V}_{\mathscr{G}_{\mathrm{P}}})^{\oplus\omega(M/\mathfrak{Q})}$. Moreover, Dimitrov's result (5.5) together with the perfectness of the Hecke equivariant twisted Poincaré duality imply that

$$\mathrm{H}^2_{\text{ét}}(K_{\diamond,t}(p^\infty);O(2))_{\mathfrak{m}_{\mathscr{G}}}\otimes_{\mathscr{G}_{\mathrm{P}}}O\cong\mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}};O(2))_{\mathfrak{m}_{\mathscr{G}}}\otimes_{\mathscr{G}_{\mathrm{P}}}O$$

is a free O-module of rank $4\cdot\omega(M/\mathfrak{Q})$. Hence, if we set

$$\mathrm{As}_O(\mathrm{V}_{\mathscr{G}_{\mathrm{P}}}):=\mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}};O(2))_{\mathfrak{m}_{\mathscr{G}}}\otimes_{\mathscr{G}_{\mathrm{P}}}O,$$

then $\mathrm{As}_O(\mathrm{V}_{\mathscr{G}_{\mathrm{P}}})$ is a $G_{\mathbb{Q}}$-stable O-lattice in the 4-dimensional Asai representation attached to $\mathscr{G}_{\mathrm{P}}$ and

$$\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)\otimes_{\mathscr{G}_{\mathrm{P}}}O\cong\mathrm{As}_O(\mathrm{V}_{\mathscr{G}_{\mathrm{P}}})(\chi_\diamond\theta_{\mathbb{Q}})^{\oplus\omega(M/\mathfrak{Q})}.$$

Finally, we showed that for every arithmetic point $\mathrm{P}\in\mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight 2, the quotient $\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)\otimes_{\mathscr{G}_{\mathrm{P}}}O$ is O-free of rank $4\cdot\omega(M/\mathfrak{Q})$. By Lemma 5.3.5, the $\mathbf{I}_{\mathscr{G}}$-module $\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)$ is free of rank $4\cdot\omega(M/\mathfrak{Q})$. □

Proposition 5.3.7. There is an isomorphism of $G_{\mathbb{Q}}$-representations

$$\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)\otimes_{\mathbf{I}_{\mathscr{G}}}\mathbf{Q}_{\mathscr{G}}\cong\left(\mathrm{As}(\mathbf{V}_{\mathscr{G}})^\dagger\right)^{\oplus\omega(M/\mathfrak{Q})}. \tag{5.6}$$

Proof. For every prime $q\nmid Mp$ the traces of Frob_q acting the Galois modules $\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)$ and $\mathrm{As}(\mathbf{V}_{\mathscr{G}})^{\oplus\omega(M/\mathfrak{Q})}$ belong to $\mathbf{I}_{\mathscr{G}}$. Proposition 5.3.6 show that these traces become equal modulo every arithmetic point $\mathrm{P}\in\mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight 2, hence they must agree as elements of $\mathbf{I}_{\mathscr{G}}$. Since $\mathscr{G}$ does not have complex multiplication nor does it arise from base change, the two $G_{\mathbb{Q}}$-representations in (5.6) are semisimple by Lemma 5.1.3.

The theorem of Brauer-Nesbitt then implies they are isomorphic. □

Corollary 5.3.7 suggests that under some mild technical conditions we may transfer the filtration of the $G_{\mathbb{Q}_p}$-representation $\mathrm{As}(\mathbf{V}_{\mathscr{G}})^{\oplus\omega(M/\mathfrak{Q})}$ given by Proposition 5.2.3 to the free $\mathbf{I}_{\mathscr{G}}$-module $\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)$. The following lemma shows that $\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)$ inherits such a filtration when the Jordan-Holder factors of the residual representation $\mathrm{As}(\mathbf{V}_{\mathscr{G}})\otimes_{\mathbf{I}_{\mathscr{G}}}\overline{\mathbb{F}}_p$ are all distinct.

Lemma 5.3.8. Let $(R,\mathfrak{m})$ be a local ring, G a profinite group and M a finite free R-module with a continuous action of G. Suppose that $\widetilde{M} = M\otimes_R \mathrm{Frac}(R)$ has a G-stable filtration with graded pieces $\mathrm{Gr}^i\widetilde{M} = \mathrm{Frac}(R)(\chi_i)$, for characters $\chi_i : G \longrightarrow R^\times$, $i = 0,\dots,r-1$.

If the Jordan-Holder factors of the $R/\mathfrak{m}[G]$-module $M/\mathfrak{m}M$ are all distinct then there is a filtration

$$M \supset \mathrm{Fil}^1 M \supset \cdots \supset \mathrm{Fil}^r M = 0$$

of free R-submodules with graded pieces $\mathrm{Gr}^i M = R(\chi_i)$ of rank 1 over R.

Proof. Consider the natural projection $\widetilde{M} \to \mathrm{Gr}^0\widetilde{M}$ and define $\mathrm{Gr}^0 M = \mathrm{Im}(M \to \mathrm{Gr}^0\widetilde{M})$, $\mathrm{Fil}^1 M = \ker(M \to \mathrm{Gr}^0\widetilde{M})$, so that we have an exact sequence of $R[G]$-modules

$$0 \to \mathrm{Fil}^1 M \to M \to \mathrm{Gr}^0 M \to 0.$$

We claim that $\mathrm{Gr}^0 M$ is free of rank 1 over R. If that is true than $\mathrm{Fil}^1 M$ is a projective R-module, so free since R is local, and we conclude by descending induction.

Clearly $\mathrm{Gr}^0 M$ is torsion-free, so it suffices to show it is generated by one element. The reduction $\mathrm{Gr}^0 M\otimes_R R/\mathfrak{m}$ is non-zero because $\mathrm{Gr}^0 M\otimes_R \mathrm{Frac}R = \mathrm{Gr}^0\widetilde{M}$ and it is of dimension one over the residue field because the Jordan-Holder factors of $M/\mathfrak{m}M$ are all distinct. □

Remark. Recall the primitive eigenform $\mathtt{g}_\circ \in S_{t_L,t_L}(\mathfrak{Q};\chi_\circ;O)$ and denote by α_i,β_i the eigenvalues of $\varrho_{\mathtt{g}_\circ}(\mathrm{Frob}_{\mathfrak{p}_i})$ for $\mathfrak{p}_1,\mathfrak{p}_2$ the $\mathcal{O}_L$-prime ideals above p. Thanks to Proposition 5.2.3, the Jordan-Holder factors of the residual representation $\mathrm{As}(\mathbf{V}_{\mathscr{G}})\otimes_{\mathbf{I}_{\mathscr{G}}}\overline{\mathbb{F}}_p$ are all distinct if and only if the products $\alpha_1\alpha_2,\ \alpha_1\beta_2,\ \beta_1\alpha_2,\ \beta_1\beta_2$ are all distinct in $\overline{\mathbb{F}}_p$.

Proposition 5.3.9. Let $K/\mathbb{Q}$ be a non-totally real S_5-quintic extension whose Galois closure contain a real quadratic field L. Recall there is a parallel weight one Hilbert eigenform $\mathtt{g}_K$ over L such that $\mathrm{As}(\varrho_{\mathtt{g}_K})\cong\mathrm{Ind}_K^{\mathbb{Q}}\mathbb{1}-\mathbb{1}$ ([For18], Corollary 4.2). If $p\neq 5$ is a rational prime unramified in K whose Frobenius conjugacy class is that of 5-cycles in S_5, then p splits in L and the residual $G_{\mathbb{Q}_p}$-representation $\big(\mathrm{As}(\varrho_{\mathtt{g}_K})\otimes\overline{\mathbb{F}}_p\big)_{|D_p}$ has distinct Jordan-Holder factors.

Proof. The representation $\mathrm{As}(\varrho_{\mathtt{g}_K}):G_{\mathbb{Q}}\to\mathrm{GL}_2(O)$ factors through the Galois group of the Galois closure of K. As a representation of the symmetric group S_5 it is isomorphic to the irreducible 4-dimensional direct summand of the permutation representation of S_5 acting on 5 elements. If $p\neq 5$ is a rational prime unramified in K whose Frobenius conjugacy class is that of 5-cycles in S_5, then the decomposition group D_p is cyclic of order 5 and we can conclude by noting that $\big(\mathrm{As}(\varrho_{\mathtt{g}_K})\otimes\overline{\mathbb{F}}_p\big)(\mathrm{Frob}_p)$ has four distinct eigenvalues given by the non-trivial 5-th roots of unity. □

5.4 Local cohomology class

Let $\mathrm{V}_{\mathtt{f}_\circ}(M)$ the $\mathtt{f}_\circ$-isotypic quotient of $\mathrm{H}^1_{\text{ét}}(Y(K_0'(p))_{\bar{\mathbb{Q}}},O(1))$ which comes equipped with a natural projection $\mathrm{H}^1_{\text{ét}}(Y(K_0'(p))_{\bar{\mathbb{Q}}},O(1))\twoheadrightarrow\mathrm{V}_{\mathtt{f}_\circ}(M)$.

Definition 5.4.1. We define the Galois module

$$\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathtt{f}_\circ}(M):=\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)\otimes\mathrm{V}_{\mathtt{f}_\circ}(M)(-1).$$

There is a natural projection $\mathrm{pr}_{\mathscr{G},\mathsf{f}_\circ} : \mathbf{V}_\infty^{\mathrm{n.o.}}(M) \to \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)$.

Definition 5.4.2. The $\mathbf{I}_{\mathscr{G}}$-adic cohomology class attached to $\mathscr{G}$ and $\mathsf{f}_\circ$ is the class

$$\boldsymbol{\kappa}_{\mathscr{G},\mathsf{f}_\circ} = \mathrm{pr}_{\mathscr{G},\mathsf{f}_\circ}(\boldsymbol{\kappa}_\infty^{\mathrm{n.o.}}) \in \mathrm{H}^1(\mathbb{Q}, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)).$$

Let

$$\boldsymbol{\kappa}_p(\mathscr{G}, \mathsf{f}_\circ) := \mathrm{loc}_p(\boldsymbol{\kappa}_{\mathscr{G},\mathsf{f}_\circ}) \in \mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M))$$

denote the local cohomology class obtained by restricting to the subgroup $D_p \simeq \Gamma_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}}$.

By Corollary 5.3.7,

$$\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M) \otimes_{\mathbf{I}_{\mathscr{G}}} \mathrm{Frac}(\mathbf{I}_{\mathscr{G}}) \cong \left(\mathbf{V}^\dagger_{\mathscr{G},\mathsf{f}_\circ} \otimes_{\mathbf{I}_{\mathscr{G}}} \mathrm{Frac}(\mathbf{I}_{\mathscr{G}})\right)^{\oplus\omega(M/\mathfrak{Q})}.$$

Thanks to Lemma 5.3.8, we have a three step D_p-invariant filtration on $\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)$ coming from the filtration on $\mathbf{V}^\dagger_{\mathscr{G},\mathsf{f}_\circ}$ as in Lemma 5.2.5.

Lemma 5.4.3. The natural map

$$\mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^2\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M))$$

induced by the D_p-stable filtration is an injection.

Proof. Lemma 5.2.5 implies that

$$\mathrm{H}^0(\mathbb{Q}_p, \mathrm{Gr}^1\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)) = 0, \qquad \mathrm{H}^0(\mathbb{Q}_p, \mathrm{Gr}^0\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)) = 0.$$

Taking the long exact sequence in Galois cohomology associated with the short exact

sequence of D_p-modules

$$0 \longrightarrow \mathrm{Gr}^1 \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M) \longrightarrow \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)/\mathrm{Fil}^2 \longrightarrow \mathrm{Gr}^0 \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M) \longrightarrow 0,$$

we deduce that $\mathrm{H}^0(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)/\mathrm{Fil}^2) = 0$ and the lemma follows. □

Proposition 5.4.4. Consider the following element of the ring $\mathbf{I}_{\mathscr{G}}$

$$\boldsymbol{\xi}_{\mathscr{G},\mathfrak{f}_\circ} = \big(1 - \alpha_{\mathfrak{f}_\circ}\chi_\circ(\mathfrak{p}_1)\mathscr{G}\big(\mathbf{T}(\varpi_{\mathfrak{p}_1})^{-1}\mathbf{T}(\varpi_{\mathfrak{p}_2})\big)\big)\big(1 - \alpha_{\mathfrak{f}_\circ}\chi_\circ(\mathfrak{p}_2)\mathscr{G}\big(\mathbf{T}(\varpi_{\mathfrak{p}_1})\mathbf{T}(\varpi_{\mathfrak{p}_2})^{-1}\big)\big). \tag{5.7}$$

We have

$$\boldsymbol{\xi}_{\mathscr{G},\mathfrak{f}_\circ} \cdot \boldsymbol{\kappa}_p(\mathscr{G}, \mathfrak{f}_\circ) \in \mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^2 \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)).$$

Proof. We follow the argument in ([DR17], Proposition 2.2). By exact control, $\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M) \otimes_{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}_\alpha$ is realized as a quotient of

$$\mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}, O(1))(\delta_\alpha) \otimes \mathrm{H}^1_{\text{ét}}(Y(K'(p))_{\bar{\mathbb{Q}}}, O(1)),$$

a direct summand of $\mathrm{H}^3_{\text{ét}}(Z^\dagger_\alpha(K)_{\bar{\mathbb{Q}}}, O(2))$. Let

$$\mathrm{H}^1_f(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M) \otimes_{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}_\alpha) \subset \mathrm{H}^1_g(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M) \otimes_{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}_\alpha) \subset \mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M) \otimes_{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}_\alpha)$$

denote the finite part and geometric part of the Galois cohomology as in [BK90], Section 3. By the work of Saito ([Sai09]), later refined by Skinner ([Ski09]), the purity conjecture for the monodromy filtration holds for the middle cohomology of Hilbert modular varieties and hence for the middle cohomology of $Z^\dagger_\alpha(K)$. Therefore, by Theorem 3.1 of [Nek00], the natural image of $\mathrm{loc}_p(\boldsymbol{\kappa}_{\mathscr{G},\mathfrak{f}_\circ})$ in $\mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M) \otimes_{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}_\alpha)$ lies in $\mathrm{H}^1_g(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M) \otimes_{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}_\alpha)$, which coincides with $\mathrm{H}^1_f(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M) \otimes_{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}_\alpha)$ in this

case.

By Lemma 5.2.5, $\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)\otimes_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_\alpha$ is an ordinary $\Gamma_{\mathbb{Q}_p}$-representation in the sense of [Wes05], i.e. there is a decreasing exhaustive and separated $G_{\mathbb{Q}_p}$-stable filtration such that an open subgroup of the inertia group I_p acts on the j-th graded piece $\mathrm{Gr}^j\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)\otimes_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_\alpha$ via the j-the power of the cyclotomic character $\varepsilon_{\mathbb{Q}}^j$. According to Lemma 2 of [Fla90], which assumes ordinarity in the slightly more restrictive sense of [Gre89] but the same argument nonetheless applies in the setting of [Wes05],

$$\mathrm{H}^1_f(\mathbb{Q}_p,\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)\otimes_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_\alpha)=\ker\big(\mathrm{H}^1(\mathbb{Q}_p,\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)\otimes_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_\alpha)\\ \to\mathrm{H}^1(\mathrm{I}_p,(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)/\mathrm{Fil}^2)\otimes_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_\alpha)\big).$$

Hence, the natural image of $\mathrm{loc}_p(\boldsymbol{\kappa}_{\mathscr{G},\mathfrak{f}_\circ})$ in $\varprojlim_\alpha \mathrm{H}^1\big(\mathrm{I}_p,\big(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)/\mathrm{Fil}^2\big)\otimes_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_\alpha\big)$ is zero. Since

$$\mathrm{Fil}^2\big(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)\big)=\varprojlim_\alpha\big(\mathrm{Fil}^2\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)\big)\otimes_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_\alpha,$$

$$\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)=\varprojlim_\alpha\big(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)\big)\otimes_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_\alpha,$$

and the Mittag-Leffler condition is satisfied, we conclude

$$\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)/\mathrm{Fil}^2=\varprojlim_{\leftarrow,\alpha}\big(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)/\mathrm{Fil}^2\big)\otimes_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_\alpha,$$

and

$$\varprojlim_{\leftarrow,\alpha}\mathrm{H}^1\big(\mathrm{I}_p,\big(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)/\mathrm{Fil}^2\big)\otimes_{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_\alpha\big)\simeq\mathrm{H}^1\big(\mathrm{I}_p,\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)/\mathrm{Fil}^2\big).$$

We deduce that

$$\mathrm{loc}_p(\boldsymbol{\kappa}_{\mathscr{G},\mathfrak{f}_\circ})\in\ker\big(\mathrm{H}^1\big(\mathbb{Q}_p,\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)\big)\to\mathrm{H}^1\big(\mathrm{I}_p,\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathfrak{f}_\circ}(M)/\mathrm{Fil}^2\big)\big).$$

Therefore the image of $\mathrm{loc}_p(\boldsymbol{\kappa}_{\mathscr{G},\mathsf{f}_\circ})$ in $\mathrm{H}^1\big(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2\big)$ lies in

$$\mathrm{H}^1\big(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, \big(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2\big)^{\mathrm{I}_p}\big) \cong \ker\big(\mathrm{H}^1(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2) \to \mathrm{H}^1(\mathrm{I}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2)\big).$$

The choice of the arithmetic Frobenius Frob_p as a topological generator gives the identification $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \cong \widehat{\mathbb{Z}}$ and allows us to compute that

$$\mathrm{H}^1\big(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, \big(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2\big)^{\mathrm{I}_p}\big) \cong \big(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2\big)^{\mathrm{I}_p}/(\mathrm{Frob}_p - 1).$$

Since $\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2$ sits in the short exact sequence

$$0 \to \mathrm{Gr}^1\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M) \to \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2 \to \mathrm{Gr}^0\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M) \to 0$$

and Lemma 5.2.5 implies

$$\big(\mathrm{Gr}^0\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)\big)^{\mathrm{I}_p} = 0,$$

we see

$$\begin{aligned}\big(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2\big)^{\mathrm{I}_p} =& \big(\mathrm{Gr}^1\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)\big)^{\mathrm{I}_p} \\ =& \mathbf{I}_{\mathscr{G}}\big(\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}^{-1}\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_2}\delta_p(\mathsf{f}_\circ)\chi_{\circ,\mathfrak{p}_1}\big) \oplus \mathbf{I}_{\mathscr{G}}\big(\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_1}\boldsymbol{\Psi}_{\mathscr{G},\mathfrak{p}_2}^{-1}\delta_p(\mathsf{f}_\circ)\chi_{\circ,\mathfrak{p}_2}\big).\end{aligned}$$

Therefore if we define

$$\boldsymbol{\xi}_{\mathscr{G},\mathsf{f}_\circ} = \big(1 - \alpha_{\mathsf{f}_\circ}\chi_\circ(\mathfrak{p}_1)\mathscr{G}\big(\mathbf{T}(\varpi_{\mathfrak{p}_1})^{-1}\mathbf{T}(\varpi_{\mathfrak{p}_2})\big)\big)\big(1 - \alpha_{\mathsf{f}_\circ}\chi_\circ(\mathfrak{p}_2)\mathscr{G}\big(\mathbf{T}(\varpi_{\mathfrak{p}_1})\mathbf{T}(\varpi_{\mathfrak{p}_2})^{-1}\big)\big)$$

then

$$\boldsymbol{\xi}_{\mathscr{G},\mathsf{f}_\circ} \cdot \mathrm{H}^1\big(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, \big(\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2\big)^{\mathrm{I}_p}\big) = 0.$$

Hence the image of $\boldsymbol{\xi}_{\mathscr{G},\mathsf{f}_\circ} \cdot \boldsymbol{\kappa}_p(\mathscr{G},\mathsf{f}_\circ)$ in $\mathrm{H}^1\big(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)/\mathrm{Fil}^2\big)$ vanishes and the claim follows. □

In light of Proposition 5.4.4, from now to the end of this article, we replace the ring $\mathbf{I}_{\mathscr{G}}$ and the various modules over it with their respective localizations at the multiplicative set generated by $\boldsymbol{\xi}_{\mathscr{G},\mathsf{f}_\circ}$. Note that the arithmetic specializations of $\boldsymbol{\xi}_{\mathscr{G},\mathsf{f}_\circ}$ never vanish: for any $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$, the specialization of $\mathscr{G}\big(\mathbf{T}(\varpi_{\mathfrak{p}_1})^{-1}\mathbf{T}(\varpi_{\mathfrak{p}_2})\big)$ at P is an algebraic integer with complex absolute value 1, whereas $\alpha_{\mathsf{f}_\circ}$ has complex absolute value $p^{1/2}$.

Corollary 5.4.5. With the above convention in place,

$$\boldsymbol{\kappa}_p(\mathscr{G},\mathsf{f}_\circ) \in \mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^2\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)).$$

Definition 5.4.6. Let

$$\boldsymbol{\mathcal{V}}^{\mathsf{f}_\circ}_{\mathscr{G}}(M) := \mathrm{Fil}^2\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}_\circ}(M)(-1).$$

We denote by

$$\boldsymbol{\kappa}^{\mathsf{f}_\circ}_p(\mathscr{G}) \in \mathrm{H}^1\big(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}^{\mathsf{f}_\circ}_{\mathscr{G}}(M)\big)$$

the image of $\boldsymbol{\kappa}_p(\mathscr{G},\mathsf{f}_\circ)$ under the natural surjection $\mathrm{Fil}^2\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M) \twoheadrightarrow \boldsymbol{\mathcal{V}}^{\mathsf{f}_\circ}_{\mathscr{G}}(M)$.

Note that $\boldsymbol{\mathcal{V}}^{\mathsf{f}_\circ}_{\mathscr{G}}(M)$ is isomorphic a direct sum of copies of the $G_{\mathbb{Q}_p}$-representation

$$\boldsymbol{\mathcal{V}}^{\mathsf{f}_\circ}_{\mathscr{G}}(M) \cong \big(\mathbf{V}^{\mathsf{f}_\circ}_{\mathscr{G}}\big)^{\oplus\omega(M/N)}.$$

Specialization in parallel weight one.

The specialization at arithmetic point $\mathrm{P}_\circ \in \mathcal{A}_{\boldsymbol{\chi}}(\mathbf{I}_{\mathscr{G}})$ of the big Galois representation $\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)$ can be written as

$$\boldsymbol{\mathcal{V}}_{\mathscr{G}_{\mathrm{P}_\circ},\mathsf{f}_\circ}(M) = \mathrm{As}(\varrho_{\mathsf{g}_\circ})(M) \otimes \mathrm{V}_{\mathsf{f}_\circ}(Mp).$$

We denote by

$$\kappa_p(\mathsf{g}_\circ^{(p)}, \mathsf{f}_\circ) \in \mathrm{H}^1\big(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G}_{\mathrm{P}_\circ},\mathsf{f}_\circ}(M)\big) \tag{5.8}$$

the specialization of the big cohomology class $\boldsymbol{\kappa}_p(\mathscr{G}, \mathsf{f}_\circ)$. Consider the map

$$\partial_p : \mathrm{H}^1\big(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G}_{\mathrm{P}_\circ},\mathsf{f}_\circ}(M)\big) \longrightarrow \mathrm{H}^1\big(\mathbb{Q}_p, \mathrm{As}(\varrho_{\mathsf{g}_\circ})(M) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}_\circ}(Mp)\big)$$

induced by the projection $\boldsymbol{\mathcal{V}}_{\mathscr{G}_{\mathrm{P}_\circ},\mathsf{f}_\circ}(M) \to \mathrm{As}(\varrho_{\mathsf{g}_\circ}) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}_\circ}(Mp)$, then the local Selmer group at p arises as the kernel of ∂_p,

$$\mathrm{H}^1_f\big(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G}_{\mathrm{P}_\circ},\mathsf{f}_\circ}(M)\big) = \ker(\partial_p),$$

because of the Hodge-Tate weights.

Proposition 5.4.7. The class $\kappa_p(\mathsf{g}_\circ^{(p)}, \mathsf{f}_\circ)$ is not crystalline at p if and only if the specialization of $\boldsymbol{\kappa}_p^{\mathsf{f}_\circ}(\mathscr{G})$ at $\mathrm{P}_\circ$ is non-trivial:

$$\partial_p\big(\kappa_p(\mathsf{g}_\circ^{(p)}, \mathsf{f}_\circ)\big) \neq 0 \qquad \Longleftrightarrow \qquad \boldsymbol{\kappa}_p^{\mathsf{f}_\circ}(\mathscr{G})(\mathrm{P}_\circ) \neq 0.$$

Furthermore

$$\partial_p\big(\kappa_p(\mathsf{g}_\circ^{(p)}, \mathsf{f}_\circ)\big) \in \mathrm{H}^1\big(\mathbb{Q}_p, \mathrm{As}(\varrho_{\mathsf{g}_\circ})(M)^{\beta_p} \otimes \mathrm{V}_{\mathsf{f}_\circ}(Mp)\big)$$

where $\mathrm{As}(\varrho_{\mathsf{g}_\circ})^{\beta_p}$ is the subspace where Frob_p acts as multiplication by $\beta_p = \beta_1\beta_2$.

Proof. By Corollary 5.4.5,

$$\kappa_p(\mathsf{g}_\circ^{(p)}, \mathsf{f}_\circ) \in \mathrm{H}^1\big(\mathbb{Q}_p, \mathrm{Fil}^2\boldsymbol{\mathcal{V}}_{\mathscr{G}_{\mathrm{P}_\circ},\mathsf{f}_\circ}(M)\big).$$

As

$$\boldsymbol{\mathcal{V}}^{\mathsf{f}_\circ}_{\mathscr{G}_{\mathrm{P}_\circ}}(M) = \mathrm{Im}\big(\mathrm{Fil}^2\boldsymbol{\mathcal{V}}_{\mathscr{G}_{\mathrm{P}_\circ},\mathsf{f}_\circ}(M) \longrightarrow \mathrm{As}(\varrho_{\mathsf{g}_\circ})(M) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}_\circ}(Mp)\big)$$

the two classes $\partial_p\big(\kappa_p(\mathsf{g}_\circ^{(p)}, \mathsf{f}_\circ)\big), \boldsymbol{\kappa}_p^{\mathsf{f}_\circ}(\mathscr{G})(\mathrm{P}_\circ)$ coincide. For the last claim we invoke Proposition 5.2.3 which implies that

$$\mathrm{As}(\varrho_{\mathsf{g}_\circ})^{\beta_p} = \mathrm{Fil}^2\mathrm{As}(\varrho_{\mathsf{g}_\circ}).$$

□

Chapter 6

Big pairing

We now generalize some of the results in [Oht95], which will be essential in the construction of motivic p-adic L-functions later. Note that due to the complicated geometry of Hilbert modular surfaces, we are not able to generalize the so-called Λ-adic Eichler-Shimura isomorphism; instead, we can only obtain a very crude form of the map.

6.1 Algebra interlude

Recall $\Gamma = 1 + p\mathbb{Z}_p$, $\Gamma_\alpha = \Gamma/\Gamma^{p^\alpha}$ and $\mathbf{\Lambda} = \varprojlim_\alpha O[\Gamma_\alpha]$. The arithmetic points of the weight space $\boldsymbol{\mathcal{W}}$ are those corresponding to the continuous homomorphisms

$$\mathsf{w}_{\ell,\chi} : \mathbf{\Lambda} \longrightarrow \bar{\mathbb{Q}}_p, \qquad [u] \mapsto \chi(u)u^{\ell-2}$$

for $\ell \in \mathbb{Z}_{\geq 1}$ and $\chi : \Gamma \to \bar{\mathbb{Q}}_p^\times$ a finite order character.

Definition 6.1.1. We define

$$\mathbf{\Pi} := \varprojlim_\alpha E[\Gamma_\alpha].$$

For any $\mathbf{\Lambda}$-module $\mathbf{M}$, we write $\mathbf{M}_{\mathbf{\Pi}} = \mathbf{M} \otimes_{\mathbf{\Lambda}} \mathbf{\Pi}$.

Let $\mathrm{P} : \mathbf{\Lambda} \to \bar{\mathbb{Q}}_p$ be any weight point in $\boldsymbol{\mathcal{W}}$. Then we have a specialization map $\mathrm{P} : \mathbf{M} \to \mathbf{M} \otimes_{\mathbf{\Lambda},\mathrm{P}} \bar{\mathbb{Q}}_p$, which, by a slight abuse of notation, will again be denoted by P. Now assume $\mathbf{M} = \varprojlim_\alpha M_\alpha$ is given by a projective system $(M_\alpha)_{\alpha \geq 1}$ of *flat* $O[\Gamma_\alpha]$-modules, such that it satisfies $\mathbf{M} \otimes_{\mathbf{\Lambda}} O[\Gamma_\alpha] \simeq M_\alpha$ for all $\alpha \geq 1$. If $\mathrm{P} \in \boldsymbol{\mathcal{W}}$ is a weight 2 arithmetic point $\mathsf{w}_{2,\chi}$ induced by a finite order character $\chi : \Gamma \to \bar{\mathbb{Q}}_p^\times$ factoring through Γ_α, then the specialization map at $\mathsf{w}_{2,\chi}$ becomes the canonical projection $\mathsf{w}_{2,\chi} : \mathbf{M} \to M_\alpha \otimes_{O[\Gamma_\alpha],\chi} \bar{\mathbb{Q}}_p$. Note that here again we are slightly abusing the notation to use χ to denote the O-algebra homomorphism $O[\Gamma_\alpha] \to \bar{\mathbb{Q}}_p$ induced by the character with the same name.

Lemma 6.1.2. Let $\mathbf{M} = \varprojlim_\alpha M_\alpha$ be a $\mathbf{\Lambda}$-module satisfying $\mathbf{M} \otimes_{\mathbf{\Lambda}} O[\Gamma_\alpha] \simeq M_\alpha$ as above. Let $x = (x_\alpha)_\alpha \in \mathbf{M}$. Then $x = 0$ if and only if $\mathsf{w}_{2,\chi}(x) = 0$ for all finite order character $\chi : \Gamma \to \bar{\mathbb{Q}}_p^\times$.

Proof. The forward implication is trivial. We look at the other direction. The injectivity of the homomorphism $\oplus\chi : O[\Gamma_\alpha] \hookrightarrow \oplus_\chi \bar{\mathbb{Q}}_p$ where χ runs over all characters of Γ_α implies the injectivity of $\oplus \mathsf{w}_{2,\chi} : M_\alpha \hookrightarrow \oplus_\chi (M_\alpha \otimes_\chi \bar{\mathbb{Q}}_p)$. In particular, if $\mathsf{w}_{2,\chi}(x) = 0$ for all finite order characters, then $x_\alpha = 0$ for all $\alpha \geq 1$. □

Note that since $\mathbf{\Pi} \otimes_{\mathbf{\Lambda}} O[\Gamma_\alpha] \simeq E[\Gamma_\alpha]$, the above lemma in particular applies to $\mathbf{\Pi}$. Moreover, the next lemma shows it will also apply to any finitely generated $\mathbf{\Lambda}$-module.

Lemma 6.1.3. Suppose $\mathbf{M}$ is a finitely generated $\mathbf{\Lambda}$-module. Then we have a canonical isomorphism

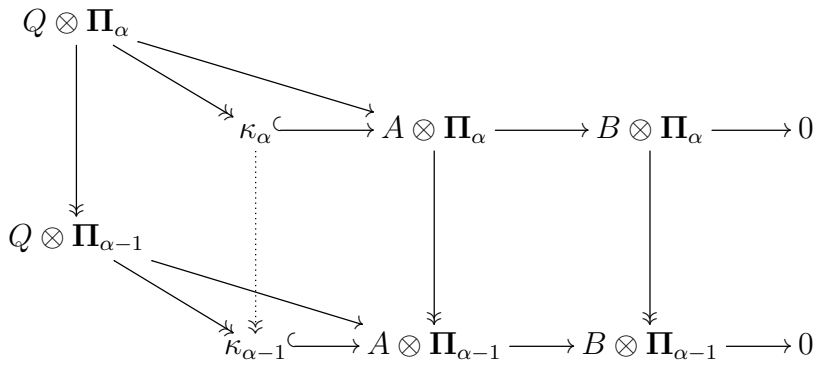
$$\mathbf{M} \simeq \varprojlim_\alpha \mathbf{M} \otimes_{\mathbf{\Lambda}} O[\Gamma_\alpha].$$

Proof. This is exactly Lemma 2.4.4 of the Iwasawa Theory notes by Sharifi. □

Lemma 6.1.4. If $A \twoheadrightarrow B$ is a surjective morphism of $\mathbf{\Lambda}$-modules then

$$\varprojlim_{\alpha} (A \otimes \mathbf{\Pi}_\alpha) \twoheadrightarrow \varprojlim_{\alpha} (B \otimes \mathbf{\Pi}_\alpha).$$

Proof. Let $Q = \ker(A \to B)$, and $\kappa_\alpha = \ker(A \otimes \mathbf{\Pi}_\alpha \to B \otimes \mathbf{\Pi}_\alpha)$ for every $\alpha \geq 1$, then there is a commutative diagram



Since tensor product is right exact, $Q \otimes \mathbf{\Pi}_\alpha \twoheadrightarrow \kappa_\alpha$ is surjective for every $\alpha \geq 1$. Therefore the transition map $\kappa_\alpha \to \kappa_{\alpha-1}$ is surjective for every $\alpha > 1$ and $\varprojlim^1_\alpha \kappa_\alpha = 0$ as required. □

Lemma 6.1.5. Suppose $\mathbf{M}$ is a finitely generated, flat $\mathbf{\Lambda}$-module and write $M_\alpha = \mathbf{M} \otimes_{\mathbf{\Lambda}} O[\Gamma_\alpha]$. Then

$$\mathbf{M}_{\mathbf{\Pi}} \xrightarrow{\sim} \varprojlim_{\alpha} (M_\alpha \otimes_O E).$$

Proof. Since $\mathbf{\Lambda}$ is Noetherian, $\mathbf{M}$ is finitely presented, and thus fits in an exact sequence of the form

$$(\mathbf{\Lambda})^{\oplus m} \to (\mathbf{\Lambda})^{\oplus n} \to \mathbf{M} \to 0. \tag{6.1}$$

Tensoring is right exact, hence we deduce presentations for $\mathbf{M}_{\mathbf{\Pi}}$ and $\mathbf{M} \otimes \mathbf{\Pi}_\alpha =$

$M_\alpha \otimes_O E$ (Lemma 6.1.3):

$$\mathbf{\Pi}^{\oplus m} \to \mathbf{\Pi}^{\oplus n} \to \mathbf{M}_{\mathbf{\Pi}} \to 0, \qquad \mathbf{\Pi}_\alpha^{\oplus m} \to \mathbf{\Pi}_\alpha^{\oplus n} \to \mathbf{M} \otimes \mathbf{\Pi}_\alpha \to 0.$$

Now it suffices to show that

$$\mathbf{\Pi}^{\oplus m} \to \mathbf{\Pi}^{\oplus n} \to \varprojlim_\alpha (\mathbf{M} \otimes \mathbf{\Pi}_\alpha) \to 0$$

is exact. The surjectivity of $\mathbf{\Pi}^{\oplus n} \twoheadrightarrow \varprojlim_\alpha (\mathbf{M} \otimes \mathbf{\Pi}_\alpha)$ follows from Lemma 6.1.4, so we are left to prove exactness at the middle of the sequence. Let $Q = \ker(\mathbf{\Lambda}^n \to \mathbf{M})$, since (6.1) is a presentation Q comes equipped with a surjection $\mathbf{\Lambda}^m \twoheadrightarrow Q$. Moreover,

$$Q \otimes \mathbf{\Pi}_\alpha = \ker(\mathbf{\Pi}_\alpha^{\oplus n} \to \mathbf{M} \otimes \mathbf{\Pi}_\alpha)$$

because $\mathrm{Tor}_1^{\mathbf{\Lambda}}(\mathbf{M}, \mathbf{\Pi}_\alpha) = 0$ as $\mathbf{M}$ is a flat $\mathbf{\Lambda}$-module. Therefore, we are left to show that the induced map $\mathbf{\Pi}^m \to \varprojlim_\alpha (Q \otimes \mathbf{\Pi}_\alpha)$ is surjective, which follows from another application of Lemma 6.1.4. □

In particular, since $\mathbf{I}_{\mathscr{G}}$ is a finite flat algebra over $\mathbf{\Lambda}$, Lemma 6.1.5 implies

$$\mathbf{I}_{\mathscr{G}} \otimes_{\mathbf{\Lambda}} \mathbf{\Pi} \simeq \varprojlim_\alpha \mathbf{I}_{\mathscr{G}} \otimes_{\mathbf{\Lambda}} E[\Gamma_\alpha] \simeq \varprojlim_\alpha (\mathbf{I}_{\mathscr{G}} \otimes_{\mathbf{\Lambda}} \mathbf{\Pi}) \otimes_{\mathbf{\Lambda}} O[\Gamma_\alpha]$$

as $\mathbf{\Lambda}$-modules. Thus the condition of Lemma 6.1.2 is also satisfied for $\mathbf{I}_{\mathscr{G}} \otimes_{\mathbf{\Lambda}} \mathbf{\Pi}$. This will be crucial later when we compare the automorphic and motivic p-adic L-functions so we record it here as a corollary.

Corollary 6.1.6. Let $x \in \mathbf{I}_{\mathscr{G}} \otimes_{\mathbf{\Lambda}} \mathbf{\Pi}$. Then $x = 0$ if and only if $\mathrm{P}(x) = 0$ for all arithmetic points $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$.

Proof. Since $\mathbf{I}_{\mathscr{G}} \otimes_{\mathbf{\Lambda}} \mathbf{\Pi} \simeq \varprojlim_\alpha \mathbf{I}_{\mathscr{G}} \otimes_{\mathbf{\Lambda}} \mathbf{\Pi}_\alpha$, an element $x \in \mathbf{I}_{\mathscr{G}} \otimes_{\mathbf{\Lambda}} \mathbf{\Pi}$ is equal to zero if

and only if its image $x_\alpha \in \mathbf{I}_\mathscr{G} \otimes_{\mathbf{\Lambda}} \mathbf{\Pi}_\alpha$ is equal to zero for every $\alpha \geq 1$. We have a natural injection $\mathbf{\Pi}_\alpha \hookrightarrow \oplus_\chi \bar{\mathbb{Q}}_p$, the sum over all the characters $\chi : \Gamma_\alpha \to \bar{\mathbb{Q}}_p^\times$. Hence

$$\mathbf{I}_\mathscr{G} \otimes_{\mathbf{\Lambda}} \mathbf{\Pi}_\alpha \hookrightarrow \oplus_\chi \left(\mathbf{I}_\mathscr{G} \otimes_{\mathbf{\Lambda}, \mathsf{w}_{2,\chi}} \bar{\mathbb{Q}}_p\right) \tag{6.2}$$

because $\mathbf{I}_\mathscr{G}$ is $\mathbf{\Lambda}$-flat. Since Hida families are finite étale over arithmetic points of weight at least 2, i.e. $\mathbf{I}_\mathscr{G} \otimes_{\mathbf{\Lambda}} O_\chi$ is finite étale over O_χ for every character $\chi : \Gamma_\alpha \to \bar{\mathbb{Q}}_p^\times$ (here O_χ is the image of $\mathsf{w}_{2,\chi}$), we see

$$(\mathbf{I}_\mathscr{G} \otimes_{\mathbf{\Lambda}} O_\chi) \otimes_{O_\chi} E_\chi \cong \mathbf{I}_\mathscr{G} \otimes_{\mathbf{\Lambda}, \chi} E_\chi$$

is a finite product of finite field extensions of $E_\chi = \mathrm{Frac}(O_\chi)$, the field of fractions of O_χ, indexed by the arithmetic points $\mathrm{P} \in \mathcal{A}(\mathbf{I}_\mathscr{G})$ above $\mathsf{w}_{2,\chi}$. In particular,

$$\mathbf{I}_\mathscr{G} \otimes_{\mathbf{\Lambda}, \mathsf{w}_{2,\chi}} \bar{\mathbb{Q}}_p \cong \oplus_{\mathrm{P}} \bar{\mathbb{Q}}_p \tag{6.3}$$

the sum over the arithmetic points $\mathrm{P} \in \mathcal{A}(\mathbf{I}_\mathscr{G})$ above $\mathsf{w}_{2,\chi}$. Combining (6.2) and (6.3) yields the desired claim immediately. □

6.2 A weak Λ-adic Eichler-Shimura map

Let $\mathscr{G} \in \overline{\mathbf{S}}_L^{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); \mathbf{\Lambda})$ be a $\mathbf{\Lambda}$-adic Hilbert cuspform, which corresponds to a $\mathbf{\Lambda}$-module homomorphism

$$\mathscr{G} : \mathbf{h}_L^{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); O) \longrightarrow \mathbf{\Lambda}.$$

The exact control for the nearly ordinary Hecke algebra implies that tensoring $\mathscr{G}$ with $O[\Gamma_\alpha]$ over $\mathbf{\Lambda}$ produces an $O[\Gamma_\alpha]$-module homomorphism

$$\underline{\mathscr{G}}_\alpha : \mathrm{h}^{\mathrm{n.o.}}_{2t_L,t_L}(K_{\diamond,t}(p^\alpha);O) \longrightarrow O[\Gamma_\alpha].$$

Since $O[\Gamma_\alpha] = \oplus_{\sigma\in\Gamma_\alpha} O\cdot[\sigma]$, we may write $\underline{\mathscr{G}}_\alpha = \oplus_{\sigma\in\Gamma_\alpha}\mathscr{G}_{\alpha,\sigma^{-1}}\cdot[\sigma]$, and the $O[\Gamma_\alpha]$-linearity implies $\mathscr{G}_{\alpha,\sigma^{-1}}(-) = \mathscr{G}_{\alpha,1}([\sigma]-)$. In order to lighten the notation we write $\mathscr{G}_\alpha$ for the Hilbert cuspform

$$\mathscr{G}_\alpha := \mathscr{G}_{\alpha,1} \in S^{\mathrm{n.o.}}_{2t_L,t_L}(K_{\diamond,t}(p^\alpha);O).$$

The compatibility

$$\begin{array}{ccc} \mathrm{h}^{\mathrm{n.o.}}_{2t_L,t_L}(K_{\diamond,t}(p^{\alpha+1});O) & \xrightarrow{\underline{\mathscr{G}}_{\alpha+1}} & O[\Gamma_{\alpha+1}] \\ \downarrow & & \downarrow \\ \mathrm{h}^{\mathrm{n.o.}}_{2t_L,t_L}(K_{\diamond,t}(p^{\alpha});O) & \xrightarrow{\underline{\mathscr{G}}_{\alpha}} & O[\Gamma_{\alpha}] \end{array}$$

translates into

$$\sum_{\sigma\in\ker(\Gamma_{\alpha+1}\to\Gamma_\alpha)} \mathscr{G}_{\alpha+1}([\sigma]-) = \mathscr{G}_\alpha(-),$$

or equivalently,

$$\mu_*\mathscr{G}_{\alpha+1} = \pi_1^*\mathscr{G}_\alpha. \tag{6.4}$$

Lemma 6.2.1. There is a Hecke-equivariant morphism

$$\mho_\infty : \overline{\mathbf{S}}^{\mathrm{n.o.}}_L(K_{\diamond,t}(p^\infty);\mathbf{I}_{\mathscr{G}}) \longrightarrow \mathbf{D}_{\mathrm{dR}}\big(\mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);E)\big)\otimes_{\mathbf{\Lambda}}\mathbf{I}_{\mathscr{G}},$$

where

$$\mathbf{D}_{\mathrm{dR}}(\mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);E)) := \varprojlim_{\alpha,\varpi_{2,*}} \mathrm{D}_{\mathrm{dR}}\big(e_{\mathrm{n.o.}}\mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}},E)\big).$$

Proof. Pushing forward equation (6.4) along π_2 gives $\varpi_{2,*}\mathscr{G}_{\alpha+1} = U_p\mathscr{G}_\alpha$. Hence, the collection $(U_p^{-\alpha}\mathscr{G}_\alpha)_\alpha$ is compatible under projection along ϖ_2. The map

$$\overline{\mathbf{S}}^{\mathrm{n.o.}}_L(K_{\diamond,t}(p^\infty);\mathbf{\Lambda}) \longrightarrow \varprojlim_{\alpha,\varpi_2} S^{\mathrm{n.o.}}_{2t_L,t_L}(K_{\diamond,t}(p^\alpha);O), \qquad \mathscr{G} \mapsto \big(U_p^{-\alpha}\mathscr{G}_\alpha\big)_\alpha$$

combined with the inclusions

$$S^{\mathrm{n.o.}}_{2t_L,t_L}(K_{\diamond,t}(p^\alpha);E) \simeq e_{\mathrm{n.o.}}\Omega^2(S^{\mathrm{tor}}(K_{\diamond,t}(p^\alpha))/E) \hookrightarrow \mathrm{D}_{\mathrm{dR}}(e_{\mathrm{n.o.}}\mathrm{H}^2_{\text{ét}}(S^{\mathrm{tor}}(K_{\diamond,t}(p^\alpha)),E))$$

gives

$$\mho_\infty : \overline{\mathbf{S}}^{\mathrm{n.o.}}_L(K_{\diamond,t}(p^\infty);\mathbf{\Lambda}) \longrightarrow \mathbf{D}_{\mathrm{dR}}\big(\mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);E)\big). \tag{6.5}$$

Since $\mathbf{\Lambda}$ is Noetherian and $\mathbf{h}^{\mathrm{n.o.}}_L(K_{\diamond,t}(p^\infty);O)$ is finite over $\mathbf{\Lambda}$, the Hecke algebra is also finitely presented as a $\mathbf{\Lambda}$-module. As $\mathbf{I}_{\mathscr{G}}$ is flat over $\mathbf{\Lambda}$ it follows that

$$\overline{\mathbf{S}}^{\mathrm{n.o.}}_L(K_{\diamond,t}(p^\infty);\mathbf{I}_{\mathscr{G}}) \simeq \overline{\mathbf{S}}^{\mathrm{n.o.}}_L(K_{\diamond,t}(p^\infty);\mathbf{\Lambda}) \otimes_{\mathbf{\Lambda}} \mathbf{I}_{\mathscr{G}}.$$

The claimed Hecke equivariant morphism is obtained from (6.5) by extension of scalars. □

Definition 6.2.2. For M a $\widetilde{\mathbf{h}}^{\mathrm{n.o.}}_L(K;O)$-module we denote by

$$M[\mathscr{G}] := \big\{m \in M : Tm = \mathscr{G}(T)m,\ \forall T \in \widetilde{\mathbf{h}}^{\mathrm{n.o.}}_L(K;O)\big\}$$

its $\mathscr{G}$-isotypic submodule.

Consider the Hecke module

$$\mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);B_{\mathrm{dR}}) := \varprojlim_{\alpha,\varpi_2} \mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}};E)\otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}.$$

As the Hecke action commute with the Galois action, there is an induced map

$$\mathbf{D}_{\mathrm{dR}}(\mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);O)[\mathscr{G}] \longrightarrow \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);B_{\mathrm{dR}})[\mathscr{G}]^{G_{\mathbb{Q}_p}}. \tag{6.6}$$

Definition 6.2.3. We let

$$\mho_{\mathscr{G}} : \overline{\mathbf{S}}^{\mathrm{n.o.}}_L(K_{\diamond,t}(p^\infty);\mathbf{I}_{\mathscr{G}})[\mathscr{G}] \longrightarrow \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty);B_{\mathrm{dR}})[\mathscr{G}]^{G_{\mathbb{Q}_p}}$$

be the composition of the Hecke equivariant map $\mho_\infty$ with (6.6).

6.3 Big pairing.

To simplify notation, write $\mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^\alpha);O\big) = \mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}};O)$. Recall the group $\mathbb{G}^\alpha_{\diamond,t,L}(K) = K_0(p^\alpha)\mathcal{O}_L^\times/K_{\diamond,t}(p^\alpha)\mathcal{O}_L^\times$ acts as diamond operators on $S(K_{\diamond,t}(p^\alpha))$. We have an inclusion

$$\Gamma_\alpha = (1+p\mathbb{Z}_p)/(1+p^\alpha\mathbb{Z}_p) \hookrightarrow \mathbb{G}^\alpha_{\diamond,t,L}(K), \qquad z \mapsto \begin{pmatrix} \Delta(z) & \\ & \Delta(z) \end{pmatrix},$$

where $\Delta : 1+p\mathbb{Z}_p \hookrightarrow 1+p\mathcal{O}_p$ is the diagonal inclusion. In this way, $z \in \Gamma_\alpha$ acts as the diamond operator $\langle \Delta(z), 1\rangle$. Let $m_\alpha \in \mathbb{A}^\times_{L,f}$ be the image of $Mp^\alpha \in L$ under the

diagonal embedding and consider the matrix

$$\tau_\alpha = \begin{pmatrix} 0 & -1 \\ m_\alpha & 0 \end{pmatrix} \in G(\mathbb{A}_{\mathbb{Q},f}).$$

If we denote by $(-)^* : G(\mathbb{A}_{\mathbb{Q},f}) \to G(\mathbb{A}_{\mathbb{Q},f})$ the involution $g^* = \det(g)^{-1}g$ then for every $\alpha \geq 1$ there is a morphism $\lambda_\alpha : S(K_{\diamond,t}(p^\alpha)) \to S(K_{\diamond,t}(p^\alpha))$ defined as the composition

$$\begin{array}{ccc} S(K_{\diamond,t}(p^\alpha)) & \xrightarrow{\quad\lambda_\alpha\quad} & S(K_{\diamond,t}(p^\alpha)) \\ & {\scriptstyle \mathfrak{T}_{\tau_\alpha}} \searrow \quad \nearrow {\scriptstyle (-)^*} & \\ & S(\tau_\alpha K_{\diamond,t}(p^\alpha)\tau_\alpha^{-1}) & \end{array} .$$

Lemma 6.3.1. The morphism λ_α is defined over $\mathbb{Q}(\zeta_{Mp^\alpha})$. Furthermore, let $\sigma \in G_{\mathbb{Q}}$ such that $a \in \mathbb{A}_{\mathbb{Q},f}^\times$ corresponds to σ via global Artin map. Then we have

$$\langle \Delta(a^{-1}), 1 \rangle \circ \lambda_\alpha \circ \sigma = \sigma \circ \lambda_\alpha$$

where $\Delta : \mathbb{A}_{\mathbb{Q},f} \hookrightarrow \mathbb{A}_{L,f}$ is the natural diagonal inclusion.

Proof. This is a standard computation using the reciprocity laws of Shimura varieties at CM points. We will sketch a proof below following the notations of [Mil05], in particular those in Chapters 12 and 13.

Let $x \in \mathcal{H}^2$ be an imaginary quadratic point defined over a CM field E. Suppose $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ and choose $s \in \mathbb{A}_E^\times$ so that s corresponds to $\sigma_{|\bar{\mathbb{Q}}}$ under the reciprocal of the global Artin map (this coincides with the convention [Mil05] uses). Then we

have the following commutative diagram

$$\begin{array}{ccc} [x,h] \in S(\tau_\alpha K_{\diamond,t}(p^\alpha)\tau_\alpha^{-1}) & \xrightarrow{\quad (-)^* \quad} & [x, \det(h)^{-1}h] \in S(K_{\diamond,t}(p^\alpha)) \\ \downarrow \sigma & & \downarrow \sigma \\ [x, r_x(s)h] \in S(\tau_\alpha K_{\diamond,t}(p^\alpha)\tau_\alpha^{-1}) & \xrightarrow{(-)^*\langle \det(r_x(s)),1\rangle} & [x, \det(h)^{-1}r_x(s)h] \in S(K_{\diamond,t}(p^\alpha)) \end{array}$$

where $r_x(s) = \mathrm{N}_{E/\mathbb{Q}}(\mu_x(s_f))$ as defined by equation (52) of Chapter 12 in [Mil05], and $\mu_x : \mathbb{G}_m \to G$ is the usual cocharacter of G associated to x characterized by $\mu_x(z) = h_{x\mathbb{C}}(z, 1)$ for $z \in \mathbb{C}$, which is defined over E. In the case of Hilbert modular varieties, we have $\det(\mu_x)$ is the natural embedding $\mathbb{G}_m \hookrightarrow \mathrm{Res}^L_{\mathbb{Q}}\mathbb{G}_m$ and thus $\det(r_x(s)) = \mathrm{N}_{E/\mathbb{Q}}(s_f)$, by which we mean the image of $s_f \in \mathbb{A}_{E,f}^\times$ in $\mathbb{A}_{L,f}^\times$ along the following commutative diagram

$$\begin{array}{ccc} \mathbb{A}_{E,f}^\times & \hookrightarrow & (\mathbb{A}_{E,f} \otimes_{\mathbb{Q}} L)^\times \\ \downarrow \mathrm{N}_{E/\mathbb{Q}} & & \downarrow \mathrm{N}_{E/\mathbb{Q}} \\ \mathbb{A}_{\mathbb{Q},f}^\times & \xrightarrow{i_{L/\mathbb{Q}}} & (\mathbb{A}_{\mathbb{Q},f} \otimes_{\mathbb{Q}} L)^\times = \mathbb{A}_{L,f}^\times \end{array}$$

where the horizontal maps are natural inclusions. Now by functoriality of class field theory, $\mathrm{N}_{E/\mathbb{Q}}(s_f) \in \mathbb{A}_{\mathbb{Q},f}^\times$ corresponds to under the reciprocal of the global Artin map for $\mathbb{Q}$ to $\sigma_{|\bar{\mathbb{Q}}}$ considered as an element in $G_{\mathbb{Q}}$

Since points of the form $[x, h]$ with fixed x are Zariski dense in the Shimura variety (see for example Lemma 13.5 of [Mil05]), taking into account that the map $\mathfrak{T}_{\tau_\alpha}$ is defined over $\mathbb{Q}$ and hence commutes with Galois action, we conclude that for any $\sigma \in G_{\mathbb{Q}}$ that fixes the reflex field of any CM point, if we choose $a \in \mathbb{A}_{\mathbb{Q},f}^\times$ so that a corresponds to σ via the global Artin map (note that here we switch back to our usual convention of translating between Galois and adelic characters, which is the inverse

of Milne's convention), we have

$$\langle \Delta(a^{-1}), 1\rangle \circ \lambda_\alpha \circ \sigma = \sigma \circ \lambda_\alpha.$$

Clearly such σ's generate $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, and thus the above relation holds for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Note that in particular if σ fixes $\mathbb{Q}(\zeta_{Mp^\alpha})$ then σ commutes with λ_α, implying λ_α is defined over $\mathbb{Q}(\zeta_{Mp^\alpha})$. □

A direct calculation shows that

$$\pi_{1,p} \circ \lambda_{\alpha+1} = \lambda_\alpha \circ \pi_{2,p}, \qquad \pi_{2,p} \circ \lambda_{\alpha+1} = \lambda_\alpha \circ \pi_{1,p}, \tag{6.7}$$

so that

$$U_p = \lambda_{\alpha,*} \circ U_p^* \circ \lambda_\alpha^*. \tag{6.8}$$

We define a twisted group-ring-valued pairing at finite level

$$\{\ ,\ \}_\alpha : \mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^\alpha); O(2)\big) \times \mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^\alpha); O\big) \longrightarrow O[\Gamma_\alpha]$$

by

$$\{x_\alpha, y_\alpha\}_\alpha = \sum_{z \in \Gamma_\alpha} \big\langle \langle z\rangle^* x_\alpha, \lambda_\alpha^* U_p^\alpha y_\alpha \big\rangle_\alpha [z^{-1}], \tag{6.9}$$

where the Poincaré pairing $\langle\ ,\ \rangle_\alpha$ is defined modulo torsion.

Proposition 6.3.2. The pairing $\{\ ,\ \}_\alpha$ is $O[\Gamma_\alpha]$-bilinear and all the Hecke operators are self-adjoint with respect to it. In particular, $\{\ ,\ \}_\alpha$ induces a pairing on nearly ordinary parts.

Proof. The Hecke operator U_p is self-adjoint with respect to $\{\ ,\ \}_\alpha$ because $U^*(\varpi_p)$ and U_p are adjoint with respect the Poincaré pairing and equation (6.8). A similar

argument works for all the other Hecke operators $T(\varpi_{\mathfrak{q}})$ (or $U(\varpi_{\mathfrak{q}})$ if $\mathfrak{q} \mid Mp$). To check $O[\Gamma_\alpha]$-bilinearity, let b be any element in Γ_α, then we have $\langle b\rangle\lambda_\alpha = \lambda_\alpha\langle b^{-1}\rangle$, which implies $\lambda_\alpha^*\langle b\rangle^* = \langle b\rangle_*\lambda_\alpha^*$. Therefore

$$\begin{aligned}\{x_\alpha, \langle b\rangle^* y_\alpha\}_\alpha &= \sum_{z\in\Gamma_\alpha} \big\langle \langle z\rangle^* x_\alpha, \lambda_\alpha^* U_p^\alpha \langle b\rangle^* y_\alpha \big\rangle_\alpha [z^{-1}] \\ &= \sum_{z\in\Gamma_\alpha} \big\langle \langle zb\rangle^* x_\alpha, \lambda_\alpha^* U_p^\alpha y_\alpha \big\rangle_\alpha [z^{-1}] \\ &= [b]\{x_\alpha, y_\alpha\}_\alpha.\end{aligned}$$

and similarly

$$\{\langle b\rangle^* x_\alpha, y_\alpha\}_\alpha = [b]\{x_\alpha, y_\alpha\}_\alpha.$$

□

Lemma 6.3.3. The finite Galois covering $\mu : S(K_{\diamond,t}(p^{\alpha+1})) \to S(K_{\diamond,t}(p^\alpha)\cap U_0(p^{\alpha+1}))$ induces an isomorphism

$$\mu^* : \mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^\alpha)\cap U_0(p^{\alpha+1}); E(2)\big) \longrightarrow \mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^{\alpha+1}); E(2)\big)^{\mathfrak{I}^{\alpha+1,\alpha}_{\diamond,t,L}(K)},$$

where $\mathfrak{I}^{\alpha+1,\alpha}_{\diamond,t,L}(K) = \ker(\mathbb{G}^{\alpha+1}_{\diamond,t,L}(K) \to \mathbb{G}^{\alpha}_{\diamond,t,L}(K))$ is the Galois group.

Proof. We claim that the part of the Hochschild-Serre spectral sequence,

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p\big(\mathfrak{I}^{\alpha+1,\alpha}_{\diamond,t,L}(K), \mathrm{H}^q_{\text{ét}}(K_{\diamond,t}(p^{\alpha+1}); E(2))\big) \implies \mathrm{H}^{p+q}_{\text{ét}}\big(K_{\diamond,t}(p^\alpha)\cap U_0(p^{\alpha+1}); E(2)\big),$$

computing $\mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^\alpha)\cap U_0(p^{\alpha+1}); E(2)\big)$ degenerates at the second page and the term

$$\mathrm{E}_2^{0,2} = \mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^{\alpha+1}); E(2)\big)^{\mathfrak{I}^{\alpha+1,\alpha}_{\diamond,t,L}(K)}$$

is the only non-zero subquotient. Indeed, $\mathrm{E}_2^{p,0} = 0$ for all $p > 0$ since $\mathfrak{I}^{\alpha+1,\alpha}_{\diamond,t,L}(K)$ is

a finite group and $\mathrm{H}^0_{\text{ét}}(K_{\diamond,t}(p^{\alpha+1}); E(2))$ is uniquely divisible, while $\mathrm{E}_2^{p,1} = 0$ for all $p \geq 0$ because Hilbert modular surfaces are simply connected. □

Proposition 6.3.4. Let $p_{\alpha+1} : O[\Gamma_{\alpha+1}] \to O[\Gamma_\alpha]$ be the homomorphism induced by the natural projection $\Gamma_{\alpha+1} \to \Gamma_\alpha$. Then the diagram

$$\begin{array}{ccc} \mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^{\alpha+1}); O(2)\big) \times \mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^{\alpha+1}); O\big) & \xrightarrow{\{,\}_{\alpha+1}} & O[\Gamma_{\alpha+1}] \\ \big\downarrow{\scriptstyle \varpi_{2,*}\times\varpi_{2,*}} & & \big\downarrow{\scriptstyle p_{\alpha+1}} \\ \mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^{\alpha}); O(2)\big) \times \mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^{\alpha}); O\big) & \xrightarrow{\{,\}_{\alpha}} & O[\Gamma_{\alpha}] \end{array}$$

commutes.

Proof. We prove the proposition through a direct computation. Since the pairing is defined modulo torsion, it suffices to prove the lemma after inverting p. We have

$$\begin{aligned} p_{\alpha+1}\big(\{x_{\alpha+1}, y_{\alpha+1}\}_{\alpha+1}\big) &= p_{\alpha+1}\Big(\sum_{z\in\Gamma_{\alpha+1}} \big\langle \langle z\rangle^* x_{\alpha+1}, \lambda_{\alpha+1}^* U_p^{\alpha+1} y_{\alpha+1}\big\rangle_{\alpha+1}[z^{-1}]\Big) \\ &= \sum_{b\in\Gamma_\alpha} \Big\langle \sum_{z\in\Gamma_{\alpha+1},\ z\mapsto b} \langle z\rangle^* x_{\alpha+1},\ \lambda_{\alpha+1}^* U_p^{\alpha+1} y_{\alpha+1}\Big\rangle_{\alpha+1}[b^{-1}]. \end{aligned}$$

Note that $\sum_{z\in\Gamma_{\alpha+1},\ z\mapsto b}\langle z\rangle^* x_{\alpha+1} = \sum_{z\in\Gamma_{\alpha+1},\ z\mapsto b}\langle z^{-1}\rangle_* x_{\alpha+1}$ is invariant under the action of $\mathfrak{I}^{\alpha+1,\alpha}_{\diamond,t,L}(K)$, and thus equals to $\mu^*\eta_b$ for some $\eta_b \in \mathrm{H}^2_{\text{ét}}\big(K_{\diamond,t}(p^\alpha)\cap U_0(p^{\alpha+1}); E(2)\big)$ by Lemma 6.3.3. We compute that

$$\begin{aligned} p_{\alpha+1}\big(\{x_{\alpha+1}, y_{\alpha+1}\}_{\alpha+1}\big) &= \sum_{b\in\Gamma_\alpha} \big\langle \mu^*\eta_b,\ \lambda_{\alpha+1}^* U_p^{\alpha+1} y_{\alpha+1}\big\rangle_{\alpha+1}[b^{-1}] \\ &= \sum_{b\in\Gamma_\alpha} \big\langle \eta_b,\ \mu_*\lambda_{\alpha+1}^* U_p^{\alpha+1} y_{\alpha+1}\big\rangle_{\alpha}[b^{-1}] \\ &= \sum_{b\in\Gamma_\alpha} \big\langle \eta_b,\ \lambda_{\alpha+1}^* U_p^{\alpha} \varpi_{2,*}\pi_1^* y_{\alpha+1}\big\rangle_{\alpha}[b^{-1}] \\ &= \sum_{b\in\Gamma_\alpha} \big\langle \pi_{2,*}\eta_b,\ \lambda_{\alpha+1}^* U_p^{\alpha} \varpi_{2,*} y_{\alpha+1}\big\rangle_{\alpha}[b^{-1}]. \end{aligned} \tag{6.10}$$

Observing that

$$\begin{aligned}\pi_{2,*}\eta_b &= \frac{1}{\deg(\mu)} \cdot \pi_{2,*}\mu_*\Big(\sum_{z\in\Gamma_{\alpha+1},\ z\mapsto b} \langle z^{-1}\rangle_* x_{\alpha+1}\Big)\\ &= \langle b^{-1}\rangle_* \varpi_{2,*} x_{\alpha+1}\\ &= \langle b\rangle^* \varpi_{2,*} x_{\alpha+1},\end{aligned}$$

we deduce

$$p_{\alpha+1}\big(\{x_{\alpha+1}, y_{\alpha+1}\}_{\alpha+1}\big) = \{\varpi_{2,*}x_{\alpha+1}, \varpi_{2,*}y_{\alpha+1}\}_\alpha.$$

□

Extending scalars in the pairing $\{,\}_\alpha$ in (6.9) and taking the projective limit yield

$$\{\ ,\}_\infty :\ \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); O(2))\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}} \times \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); B_{\mathrm{dR}}) \longrightarrow \varprojlim_\alpha B_{\mathrm{dR}}[\Gamma_\alpha]$$

with respect to which all Hecke operators in the anemic Hecke algebra are self-adjoint. By restricting to the $\mathscr{G}$-isotypic subspace in the second argument we obtain

$$\{\ ,\}_{\mathscr{G}} :\ \big(\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)(\theta_{\mathbb{Q}}^{-1}\boldsymbol{\eta}_{\mathbb{Q}})\big)\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}} \times \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); B_{\mathrm{dR}})[\mathscr{G}] \longrightarrow \big(\varprojlim_\alpha B_{\mathrm{dR}}[\Gamma_\alpha]\big) \otimes_{\boldsymbol{\Lambda}} \mathbf{I}_{\mathscr{G}}. \tag{6.11}$$

Proposition 6.3.5. Let $\boldsymbol{\Theta} = (\eta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}})_{|G_{\mathbb{Q}_p}}$, then the pairing

$$\{\ ,\}_{\mathscr{G}} :\ \boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)(\boldsymbol{\Theta}^{-1})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}} \times \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); B_{\mathrm{dR}})[\mathscr{G}] \longrightarrow \big(\varprojlim_\alpha B_{\mathrm{dR}}[\Gamma_\alpha]\big) \otimes_{\boldsymbol{\Lambda}} \mathbf{I}_{\mathscr{G}}(-1)$$

is $G_{\mathbb{Q}_p}$-equivariant.

Proof. Let $\sigma \in G_{\mathbb{Q}_p}$ such that $a \in \mathbb{Q}_p^\times$ corresponds to σ via the local Artin map.

Lemma 6.3.1 implies

$$(\lambda_\alpha)^* \circ \sigma = \langle \Delta(a^{-1}), 1\rangle^* \circ \sigma \circ (\lambda_\alpha)^*,$$

which, combined with (6.9) yields

$$\begin{aligned}\{\sigma x_\alpha, \sigma y_\alpha\}_\alpha &= \sum_{z\in\Gamma_\alpha} \big\langle \sigma\langle\Delta(z),1\rangle^* x_\alpha, \sigma\langle\Delta(a^{-1}),1\rangle_* \circ (\lambda_\alpha)^* \circ U_p^\alpha y_\alpha \big\rangle_\alpha [z^{-1}] \\ &= \sum_{z\in\Gamma_\alpha} \big\langle \langle\Delta(za^{-1}),1\rangle^* x_\alpha, (\lambda_\alpha)^* \circ U_p^\alpha y_\alpha \big\rangle_\alpha [z^{-1}] \\ &= \sum_{z\in\Gamma_\alpha} \big\langle \langle\Delta(a),1\rangle_* \circ \langle\Delta(z),1\rangle^* x_\alpha, (\lambda_\alpha)^* \circ U_p^\alpha y_\alpha \big\rangle_\alpha [z^{-1}]\end{aligned}$$

where in the second equality we used the Galois equivariance of the Poincaré pairing

$$\langle\ ,\rangle_\alpha : \mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}, O(2)) \times \mathrm{H}^2_{\text{ét}}(S(K_{\diamond,t}(p^\alpha))_{\bar{\mathbb{Q}}}, O) \to O.$$

As

$$\mathscr{G}([\Delta(a),1]) = \boldsymbol{\chi}(\Delta(a))\big[\xi_{\Delta(a)}^{-t_L}\big] = \theta_{\mathbb{Q}}^{-2}(a)\boldsymbol{\eta}_{\mathbb{Q}}^2(a),$$

we see that

$$\{\sigma x_\alpha, \sigma y_\alpha\}_\alpha = \theta_{\mathbb{Q}}^{-2}(a)\boldsymbol{\eta}_{\mathbb{Q}}^2(a)\{x_\alpha, y_\alpha\}_\alpha.$$

Therefore

$$\begin{aligned}\{\ ,\}_{\mathscr{G}} : \big(\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)(\theta_{\mathbb{Q}}^{-1}\boldsymbol{\eta}_{\mathbb{Q}})\big)\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}} \times \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); B_{\mathrm{dR}})[\mathscr{G}] \\ \longrightarrow \big(\varprojlim_\alpha B_{\mathrm{dR}}[\Gamma_\alpha]\big) \otimes_{\boldsymbol{\Lambda}} \mathbf{I}_{\mathscr{G}}(\theta_{\mathbb{Q}}^{-2}\boldsymbol{\eta}_{\mathbb{Q}}^2)\end{aligned}$$

is $G_{\mathbb{Q}_p}$-equivariant and the claim follows by twisting. □

6.4 On Dieudonné modules

Given $\mathsf{f}_\circ \in S_{2,1}^{\mathrm{ord}}(N;\mathbb{1};\bar{\mathbb{Q}})$ an ordinary elliptic cuspform, one can define a linear map

$$\varphi : S_{2,1}^{\mathrm{ord}}(Np;\mathbb{1};\bar{\mathbb{Q}}) \longrightarrow \bar{\mathbb{Q}}, \qquad \mathsf{h} \mapsto \frac{\left\langle \mathsf{h}, \mathsf{f}_\circ^{(p)} \right\rangle_{\mathrm{Pet}}}{\left\langle \mathsf{f}_\circ^{(p)}, \mathsf{f}_\circ^{(p)} \right\rangle_{\mathrm{Pet}}}$$

which satisfies $\varphi(T_\ell^*\mathsf{h}) = \mathsf{a}_p(\ell, \mathsf{f}_\circ) \cdot \varphi(\mathsf{h})$. The natural inclusion of cuspforms in de Rham cohomology induces an isomorphism

$$S_{2,1}^{\mathrm{ord}}(Np;\mathbb{1};E) \xrightarrow{\sim} \mathrm{D}_{\mathrm{dR}}\big(\mathrm{Gr}^0(\mathrm{V}_{\mathsf{f}_\circ}(Mp)(-1))\big), \qquad \mathsf{h} \mapsto \omega_{\mathsf{h}},$$

hence we can make the following definition as in ([DR17], Section 2.3 & Equation (118)).

Definition 6.4.1. Following (79) of [DR17], let $\eta_\circ \in \mathrm{H}^1_{\mathrm{dR}}(Y(K_0'(p)))^{\mathrm{ord,ur}}[\mathsf{f}_\circ]$ denote the unique class that satisfies

$$\Phi(\eta_\circ) = \alpha_{\mathsf{f}_\circ} \cdot \eta_\circ$$

and for any $\mathsf{h} \in S_{2,1}^{\mathrm{ord}}(Np;\mathbb{1};\bar{\mathbb{Q}})$

$$\left\langle \omega_{\mathsf{h}}, (\lambda_1)_* \eta_\circ \right\rangle_{\mathrm{dR}} = \varphi(\mathsf{h}).$$

Remark. For any $\phi \in S_{2,1}^{\mathrm{ord}}(Np;\mathbb{1};\bar{\mathbb{Q}})$ we have

$$\left\langle \eta_\circ, \omega_\phi \right\rangle_{\mathrm{dR}} = \left\langle (\lambda_1)_* \eta_\circ, (\lambda_1)_* \omega_\phi \right\rangle_{\mathrm{dR}} = \varphi\big((\lambda_1)_*\phi\big) = \frac{\left\langle (\lambda_1)_*\phi, \mathsf{f}_\circ^{(p)} \right\rangle_{\mathrm{Pet}}}{\left\langle \mathsf{f}_\circ^{(p)}, \mathsf{f}_\circ^{(p)} \right\rangle_{\mathrm{Pet}}}.$$

The Hecke equivariant twist of the Poincaré pairing

$$\{\ ,\}_{\mathbb{Q}} : \mathrm{H}^1_{\text{ét}}(Y(K_0'(p), O) \times \mathrm{H}^1_{\text{ét}}(Y(K_0'(p), O(1)) \longrightarrow O, \qquad \big\{x, y\big\}_{\mathbb{Q}} = \big\langle x, (\lambda_1)^* y\big\rangle_{\mathrm{dR}} \tag{6.12}$$

induces a $G_{\mathbb{Q}_p}$-equivariant perfect pairing on $\mathsf{f}_\circ$-isotypic components

$$\{\ ,\}_{\mathsf{f}_\circ} : \mathrm{V}_{\mathsf{f}_\circ}(Mp)(-1) \times \mathrm{V}_{\mathsf{f}_\circ}(Mp) \longrightarrow O.$$

Furthermore, by looking at the Galois action, one sees that $\mathrm{Fil}^1\mathrm{V}_{\mathsf{f}_\circ}(Mp)(-1)$ and $\mathrm{Fil}^1\mathrm{V}_{\mathsf{f}_\circ}(Mp)$ are orthogonal with respect to $\{\ ,\}_{\mathsf{f}_\circ}$. Therefore there is an induced perfect pairing

$$\{,\}_{\mathsf{f}_\circ} : \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}_\circ}(Mp)(-1) \times \mathrm{Fil}^1\mathrm{V}_{\mathsf{f}_\circ}(Mp) \longrightarrow O. \tag{6.13}$$

which we can use to make the identification

$$\mathrm{D}_{\mathrm{dR}}(\mathrm{Fil}^1\mathrm{V}_{\mathsf{f}_\circ}(Mp)) \xrightarrow{\sim} \mathrm{Hom}_E\big(\mathrm{D}_{\mathrm{dR}}\big(\mathrm{Gr}^0\mathrm{V}_{\mathsf{f}_\circ}(Mp)(-1)\big), E\big).$$

Definition 6.4.2. We denote by

$$\eta_\circ' \in \mathrm{D}_{\mathrm{dR}}(\mathrm{Fil}^1\mathrm{V}_{\mathsf{f}_\circ}(Mp))$$

the element corresponding to the homomorphism

$$\mathrm{D}_{\mathrm{dR}}\big(\mathrm{Gr}^0(\mathrm{V}_{\mathsf{f}_\circ}(Mp))(-1)\big) \longrightarrow E, \qquad \omega_\phi \mapsto \big\langle \omega_\phi, \eta_\circ \big\rangle_{\mathrm{dR}}.$$

It satisfies

$$\big\{\omega, \eta'_\circ\big\}_{\mathrm{f}_\circ} = \big\langle \omega, \eta_\circ \big\rangle_{\mathrm{dR}} \qquad \forall\ \omega \in \mathrm{D}_{\mathrm{dR}}\big(\mathrm{Gr}^0 \mathrm{V}_{\mathrm{f}_\circ}(Mp)(-1)\big).$$

Proposition 6.4.3. For any Λ-adic test vector $\breve{\mathscr{G}} \in \overline{\mathbf{S}}^{\mathrm{n.o.}}(K_{\diamond,t}; \mathbf{I}_{\mathscr{G}})[\mathscr{G}]$ there exists a homomorphism of $\mathbf{I}_{\mathscr{G}}$-modules

$$\langle\ , \omega_{\breve{\mathscr{G}}} \otimes \eta'_\circ \rangle : \mathbb{D}\big(\mathbf{U}^{\mathrm{f}_\circ}_{\mathscr{G}}(M)\big) \longrightarrow \mathbf{\Pi} \otimes_{\mathbf{\Lambda}} \mathbf{I}_{\mathscr{G}}$$

whose specialization at any arithmetic point $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight 2 is

$$\mathrm{P} \circ \langle -, \omega_{\breve{\mathscr{G}}} \otimes \eta'_\circ \rangle = \langle -, (\lambda_\alpha)^* \omega_{\breve{\mathscr{G}}_{\mathrm{P}}} \otimes \eta_\circ \rangle_{\mathrm{dR}} : \mathrm{D}_{\mathrm{dR}}\big(\mathrm{U}^{\mathrm{f}_\circ}_{\mathscr{G}_{\mathrm{P}}}(M)\big) \longrightarrow \mathbb{C}_p.$$

Proof. Recall that

$$\mathbf{U}^{\mathrm{f}_\circ}_{\mathscr{G}}(M) = \mathrm{Fil}^2 \boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)(\mathbf{\Theta}^{-1}) \otimes \mathrm{Gr}^0 \mathrm{V}_{\mathrm{f}_\circ}(M)(-1).$$

By tensoring the $G_{\mathbb{Q}_p}$-equivariant pairing in Proposition 6.3.5

$$\{\ , \}_{\mathscr{G}} :\ \boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)(\mathbf{\Theta}^{-1}) \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}^{\mathrm{ur}}_p \times \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); B_{\mathrm{dR}})[\mathscr{G}] \longrightarrow \big(\varprojlim_{\alpha} B_{\mathrm{dR}}[\Gamma_\alpha]\big) \otimes_{\mathbf{\Lambda}} \mathbf{I}_{\mathscr{G}}(-1)$$

with the $G_{\mathbb{Q}_p}$ equivariant pairing in (6.13), we obtain

$$\mathbf{U}^{\mathrm{f}_\circ}_{\mathscr{G}}(M) \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}^{\mathrm{ur}}_p \times \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); B_{\mathrm{dR}})[\mathscr{G}] \otimes_O \mathrm{Fil}^1 \mathrm{V}_{\mathrm{f}_\circ}(M) \longrightarrow (\varprojlim_{\alpha} B_{\mathrm{dR}}[\Gamma_\alpha]) \otimes_{\mathbf{\Lambda}} \mathbf{I}_{\mathscr{G}}(-1).$$

Restricting the pairing to $G_{\mathbb{Q}_p}$-invariants we obtain

$$\langle\ ,\rangle : \mathbb{D}\big(\mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}}(M)\big) \times \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); B_{\mathrm{dR}})[\mathscr{G}]^{G_{\mathbb{Q}_p}} \otimes_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{dR}}\big(\mathrm{Fil}^1\mathrm{V}_{\mathsf{f}_\circ}(M)\big) \longrightarrow \mathbf{\Pi} \otimes_{\mathbf{\Lambda}} \mathbf{I}_{\mathscr{G}}. \tag{6.14}$$

Let $\omega_{\breve{\mathscr{G}}} := \mathfrak{V}_{\mathscr{G}}(\breve{\mathscr{G}}) \in \mathrm{H}^2_{\mathrm{n.o.}}(K_{\diamond,t}(p^\infty); B_{\mathrm{dR}})[\mathscr{G}]^{G_{\mathbb{Q}_p}}$ be the class represented by the compatible collection $(U_p^{-\alpha}\breve{\mathscr{G}}_\alpha)_\alpha$ of cuspforms, and let $\eta'_\circ \in \mathrm{D}_{\mathrm{dR}}\big(\mathrm{Fil}^1\mathrm{V}_{\mathsf{f}_\circ}(M)\big)$ be the class of Definition 6.4.2. Then evaluating the pairing (6.14) at $\omega_{\breve{\mathscr{G}}} \otimes \eta'_\circ$ gives the homomorphism

$$\big\langle\ , \omega_{\breve{\mathscr{G}}} \otimes \eta'_\circ \big\rangle : \mathbb{D}\big(\mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}}(M)\big) \longrightarrow \mathbf{\Pi} \otimes_{\mathbf{\Lambda}} \mathbf{I}_{\mathscr{G}}. \tag{6.15}$$

Now we study the specialization of the pairing at arithmetic points $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight 2 and character $\chi_\diamond$ of conductor p^α. Let

$$\boldsymbol{z} = \sum_i x_i \otimes y_i \in \mathbb{D}\big(\mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}}(M)\big) = \big(\mathrm{Fil}^2\boldsymbol{\mathcal{V}}_{\mathscr{G}}(M)(\mathbf{\Theta}^{-1}) \otimes \mathrm{Gr}^0\mathrm{V}_{\mathsf{f}_\circ}(M)(-1)\big)^{G_{\mathbb{Q}_p}}$$

be any element, then by construction

$$\big\langle \boldsymbol{z},\ \omega_{\breve{\mathscr{G}}} \otimes \eta'_\circ \big\rangle = \sum_i \big\{x_i, \omega_{\breve{\mathscr{G}}}\big\}_{\mathscr{G}} \cdot \big\{y_i, \eta'_\circ\big\}_{\mathsf{f}_\circ}.$$

Firstly we note that

$$\big\{y_i, \eta'_\circ\big\}_{\mathsf{f}_\circ} = \big\langle y_i, \eta_\circ \big\rangle_{\mathrm{dR}},$$

then we observe that the projection of $\big\{x_i, \omega_{\breve{\mathscr{G}}}\big\}_{\mathscr{G}}$ to level α is

$$\begin{aligned}\big\{x_{i,\alpha}, U_p^{-\alpha}\breve{\mathscr{G}}_\alpha\big\}_\alpha &= \sum_{z\in\Gamma_\alpha} \big\langle \langle z\rangle^* x_{i,\alpha}, (\lambda_\alpha)^* U_p^\alpha U_p^{-\alpha}\breve{\mathscr{G}}_\alpha\big\rangle[z^{-1}] \\ &= \sum_{z\in\Gamma_\alpha} \big\langle x_{i,\alpha}, (\lambda_\alpha)^* \circ \langle z\rangle^*\breve{\mathscr{G}}_\alpha\big\rangle[z^{-1}].\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{P} \circ \left\{x_i, \omega_{\breve{\mathscr{G}}}\right\}_{\mathscr{G}} &= \left\langle x_{i,\alpha}, (\lambda_\alpha)^* \sum_z \chi_\diamond(z^{-1}) \langle z^{-1} \rangle_* \breve{\mathscr{G}}_\alpha \right\rangle \\ &= \left\langle x_{i,\alpha}, (\lambda_\alpha)^* \breve{\mathscr{G}}_{\mathrm{P}} \right\rangle = \left\langle x_{i,\mathrm{P}}, (\lambda_\alpha)^* \breve{\mathscr{G}}_{\mathrm{P}} \right\rangle\end{aligned}$$

where the last equality results from the $O[\Gamma_\alpha]$-equivariance of the twisted Poincaré pairing. It follows that

$$\mathrm{P} \circ \left\langle \boldsymbol{z},\ \omega_{\breve{\mathscr{G}}} \otimes \eta'_\circ \right\rangle = \left\langle \boldsymbol{z}_{\mathrm{P}},\ (\lambda_\alpha)^* \omega_{\breve{\mathscr{G}}_{\mathsf{P}}} \otimes \eta_\circ \right\rangle_{\mathrm{dR}}.$$

□

Chapter 7

Motivic p-adic L-functions

7.1 Perrin-Riou's regulator

Definition 7.1.1. We consider the characters

$$\mathbf{\Psi}_{\mathscr{G}}^{\mathsf{f}_\circ} := \mathbf{\Psi}_{\mathscr{G},p}^{-1}\delta_p(\mathsf{f}_\circ), \qquad \mathbf{\Theta} := (\eta_{\mathbb{Q}}\boldsymbol{\eta}_{\mathbb{Q}})_{|D_p},$$

of a decomposition group D_p at p of $G_{\mathbb{Q}}$. We denote by $\mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ}$ the representation corresponding to the unramified character $\mathbf{\Psi}_{\mathscr{G}}^{\mathsf{f}_\circ}$.

Note that the arithmetic Frobenius acts on $\mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ}$ as multiplication by $\mathbf{\Psi}_{\mathscr{G}}^{\mathsf{f}_\circ}(\mathrm{Frob}_p) = \alpha_{\mathsf{f}_\circ}\mathscr{G}(\mathbf{T}(\varpi_p))^{-1}$ and Lemma 5.2.5 implies $\mathbf{V}_{\mathscr{G}}^{\mathsf{f}_\circ} = \mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ}(\mathbf{\Theta})$.

Suppose $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ is an arithmetic point of weight ℓ and character $\chi_\diamond$, then

$$\mathbf{\Theta}(\mathrm{P}) = \chi_\diamond^{-1} \cdot (\eta_{\mathbb{Q}}^{\ell-1})_{|D_p}$$

has negative Hodge-Tate weight if $\ell \geq 2$. In this case the weights of the pair of modular forms $(\mathsf{g}_{\mathrm{P}}, \mathsf{f}_\circ)$ are balanced. For the arithmetic point $\mathrm{P}_\circ$ of weight 1 and trivial character,$\mathbf{\Theta}(\mathrm{P}_\circ) = 1$ has Hodge-Tare weight equal to zero and the weights

of the pair $(\mathfrak{g}_\circ^{(p)}, \mathsf{f}_\circ)$ are $\mathbb{Q}$-dominated. Therefore, if we let $\mathrm{V}_{\mathfrak{g}_\mathrm{P}}^{\mathsf{f}_\circ} := \big(\mathbf{V}_{\mathscr{G}}^{\mathsf{f}_\circ}\big)_\mathrm{P}$ be the specialization of $\mathbf{V}_{\mathscr{G}}^{\mathsf{f}_\circ}$ at P, we have isomorphisms

$$\begin{aligned} &\log_{\mathrm{BK}} : \mathrm{H}^1\big(\mathbb{Q}_p, \mathrm{V}_{\mathfrak{g}_\mathrm{P}}^{\mathsf{f}_\circ}\big) \xrightarrow{\sim} \mathrm{D}_{\mathrm{dR}}\big(\mathrm{V}_{\mathfrak{g}_\mathrm{P}}^{\mathsf{f}_\circ}\big), && \text{if P has weight } \ell \geq 2, \\ &\exp^*_{\mathrm{BK}} : \mathrm{H}^1\big(\mathbb{Q}_p, \mathrm{V}_{\mathfrak{g}_{\mathrm{P}_\circ}}^{\mathsf{f}_\circ}\big) \xrightarrow{\sim} \mathrm{D}_{\mathrm{dR}}\big(\mathrm{V}_{\mathfrak{g}_{\mathrm{P}_\circ}}^{\mathsf{f}_\circ}\big), && \text{if } \mathrm{P} = \mathrm{P}_\circ, \end{aligned} \tag{7.1}$$

since $\mathrm{V}_{\mathfrak{g}_\mathrm{P}}^{\mathsf{f}_\circ}$ is never isomorphic to $\mathbb{Q}(1)$ nor the trivial representation ($\boldsymbol{\Psi}_{\mathscr{G}}^{\mathsf{f}_\circ}(\mathrm{P})$ should always be of infinite order because of the Weil conjectures).

Lemma 7.1.2. Let $\beta : \mathbb{Z}_p^\times \to E_\beta^\times$ be a finite order character of conductor p^α, where E_β is a finite extension of $\mathbb{Q}_p$, and suppose β corresponds to a Galois character $\beta : G_{\mathbb{Q}_p} \to E_\beta^\times$ factoring through $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\alpha})/\mathbb{Q}_p)$. Consider the $G_{\mathbb{Q}_p}$-representation $E_\beta\big(\beta + j\big)$, then the E_β-vector space

$$\mathrm{D}_{\mathrm{dR}}\big(E_\beta(\beta + j)\big) = E_\beta \cdot b_{\beta,j}.$$

has a canonical basis $b_{\beta,j}$.

Proof. For any $j \in \mathbb{Z}$, the choice of a compatible sequence of p-power roots of unity $\boldsymbol{\zeta} := \{\zeta_{p^\alpha}\}_{\alpha \geq 0}$ determines a basis $\boldsymbol{\zeta}^j$ of the $G_{\mathbb{Q}_p}$-representation $\mathbb{Q}_p(j)$ and an element $t^{-j} \in \mathrm{B}_{\mathrm{dR}}$ such that the element $\boldsymbol{\zeta}^j \otimes t^{-j}$ gives a canonical basis of $\mathrm{D}_{\mathrm{dR}}\big(\mathbb{Q}_p(j)\big)$ independent of $\boldsymbol{\zeta}$. We consider models of the $G_{\mathbb{Q}_p}$-representations $E_\beta(\beta)$, $E_\beta(-\beta)$ appearing in the Galois modules $E_\beta \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^\alpha})$ where $G_{\mathbb{Q}_p}$ acts only on the second factor by Galois automorphisms. For a character $\alpha : \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\alpha})/\mathbb{Q}_p) \to E_\beta^\times$ the element

$$\theta_\alpha = \sum_{\tau \in \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\alpha})/\mathbb{Q}_p)} \alpha^{-1}(\tau) \otimes \zeta_{p^\alpha}^\tau \in E_\beta \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^\alpha})$$

satisfies $(\theta_\alpha)^\sigma = \alpha(\sigma)\theta_\varepsilon$ for all $\sigma \in G_{\mathbb{Q}_p}$. Then $E_\beta(\beta) \cong E_\beta \cdot \theta_\beta$ and $E_\beta(-\beta) \cong E_\beta \cdot \theta_{\beta^{-1}}$. We choose the model $E_\beta \cdot \theta_\beta \otimes \boldsymbol{\zeta}^j$ of the $G_{\mathbb{Q}_p}$-representation $E_\beta(\beta + j)$ and we note

that

$$b_{\beta,j} := (\theta_\beta \otimes \boldsymbol{\zeta}^j) \otimes_{E_\beta} (\theta_{\beta^{-1}} \otimes t^{-j}) \in E_\beta(\beta + j) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$$

is invariant under the $G_{\mathbb{Q}_p}$-action and it is independent of the choice of $\boldsymbol{\zeta}$. Therefore, we deduce that $\mathrm{D}_{\mathrm{dR}}\big(E_\beta(\beta + j)\big)$ has $b_{\beta,j}$ as canonical E_β-basis. □

Write $\boldsymbol{\Lambda}_\Gamma$ for $O[\![\mathbb{Z}_p^\times]\!]$, then by ([KLZ17], Theorem 8.2.3) and ([LZ14], Theorem 4.15 – Theorem B.5) there is a $\big(\mathbf{I}_{\mathscr{G}} \widehat{\otimes} \boldsymbol{\Lambda}_\Gamma\big)$-linear map $\boldsymbol{\mathcal{L}} : \mathrm{H}^1\big(\mathbb{Q}_p, \mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ} \widehat{\otimes} \boldsymbol{\Lambda}_\Gamma(-\mathbf{j})\big) \longrightarrow \mathbb{D}(\mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ}) \widehat{\otimes} \boldsymbol{\Lambda}_\Gamma$ such that for all points $\mathrm{P} \in \mathrm{Hom}(\mathbf{I}_{\mathscr{G}}, \bar{\mathbb{Q}}_p)$ and all characters of the cyclotomic Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ of the form $\eta \cdot \varepsilon_{\mathbb{Q}}^j$ where $j \in \mathbb{Z}$ and η has finite order, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1\big(\mathbb{Q}_p, \mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ} \widehat{\otimes} \boldsymbol{\Lambda}_\Gamma(-\mathbf{j})\big) & \xrightarrow{\boldsymbol{\mathcal{L}}} & \mathbb{D}(\mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ}) \widehat{\otimes} \boldsymbol{\Lambda}_\Gamma \\ \big\downarrow & & \big\downarrow \\ \mathrm{H}^1\big(\mathbb{Q}_p, \mathbf{U}_{\mathscr{G}_{\mathrm{P}}}^{\mathsf{f}_\circ}(-j-\eta)\big) & \longrightarrow & \mathrm{D}_{\mathrm{dR}}(\mathbf{U}_{\mathscr{G}_{\mathrm{P}}}^{\mathsf{f}_\circ}(-j-\eta)) \end{array}$$

where the rightmost vertical map is

$$\mathbb{D}(\mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ}) \widehat{\otimes} \boldsymbol{\Lambda}_\Gamma \longrightarrow \mathrm{D}_{\mathrm{dR}}(\mathbf{U}_{\mathscr{G}_{\mathrm{P}}}^{\mathsf{f}_\circ}(-j-\eta)), \quad \boldsymbol{x} \otimes [u] \mapsto \eta \varepsilon_{\mathbb{Q}}^j(u) \cdot \nu_{\mathrm{P}}(\boldsymbol{x}) \otimes b_{\eta^{-1},-j},$$

and the bottom horizontal map is given by (compare also with Theorem 8.2.8 of [KLZ17])

$$\begin{cases} \big(1 - \alpha_{1,\mathsf{g}_{\mathrm{P}}} \alpha_{2,\mathsf{g}_{\mathrm{P}}} \alpha_{\mathsf{f}_\circ}^{-1} \eta(\mathrm{Frob}_p) p^j\big) \big(1 - \alpha_{1,\mathsf{g}_{\mathrm{P}}}^{-1} \alpha_{2,\mathsf{g}_{\mathrm{P}}}^{-1} \alpha_{\mathsf{f}_\circ} \eta^{-1}(\mathrm{Frob}_p) p^{-j-1}\big)^{-1} & \mathrm{cond}(\eta) = 0 \\ \big(\alpha_{1,\mathsf{g}_{\mathrm{P}}} \alpha_{2,\mathsf{g}_{\mathrm{P}}} \alpha_{\mathsf{f}_\circ}^{-1} \eta(\mathrm{Frob}_p) p^{1+j}\big)^{\mathrm{cond}(\eta)} G(\eta^{-1})^{-1} & \mathrm{cond}(\eta) > 0 \end{cases} \tag{7.2}$$

$$\times \quad \begin{cases} \frac{(-1)^{-j-1}}{(-j-1)!} \cdot \log_{\mathrm{BK}} & j < 0 \\ \\ j! \cdot \exp^*_{\mathrm{BK}} & j \geq 0. \end{cases}$$

We pull back the map $\boldsymbol{\mathcal{L}}$ by the automorphism of $\mathbf{I}_{\mathscr{G}}\widehat{\otimes}\boldsymbol{\Lambda}_\Gamma$ given by

$$1 \otimes [z] \mapsto \boldsymbol{\Theta}(z)^{-1} \cdot 1 \otimes [z],$$

by functoriality of the construction of $\boldsymbol{\mathcal{L}}$ we obtain

$$\begin{array}{ccc} \mathrm{H}^1\big(\mathbb{Q}_p, \mathbf{V}^{\mathsf{f}_\circ}_{\mathscr{G}}\widehat{\otimes}\boldsymbol{\Lambda}_\Gamma(-\mathbf{j})\big) & \xrightarrow{\boldsymbol{\mathcal{L}}'} & \big(\mathbb{D}(\mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}})\widehat{\otimes}\boldsymbol{\Lambda}_\Gamma\big)\widehat{\otimes}_{\boldsymbol{\Theta}}\boldsymbol{\Lambda}_\Gamma \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\boldsymbol{\Theta}^{-1}} \\ \mathrm{H}^1\big(\mathbb{Q}_p, \mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}}\widehat{\otimes}\boldsymbol{\Lambda}_\Gamma(-\mathbf{j})\big) & \xrightarrow{\boldsymbol{\mathcal{L}}} & \mathbb{D}(\mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}})\widehat{\otimes}\boldsymbol{\Lambda}_\Gamma. \end{array} \tag{7.3}$$

Remark. The conventions of the local reciprocity map and Gauss sum are different in [LZ14] and [KLZ17], which causes the character in the Gauss sum to be inverted as compared to Theorem 8.2.8 of [KLZ17]. In fact, in [LZ14] they normalize the reciprocity map so that geometric Frobenius elements are sent to uniformizers, which means the identification of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ with $\mathbb{Z}_p^\times$ coincides with the cyclotomic character, which is the reciprocal of our convention (see Section 2.8 of loc.cit.). Furthermore, their Gauss sum is really a sum of Galois characters, whereas ours is a sum of Hecke characters, so implicitly there is a translation via class field theory involved. Our equation (7.2) has already taken this into account.

Proposition 7.1.3. There is an homomorphism

$$\boldsymbol{\mathcal{L}}^{\mathsf{f}_\circ}_{\mathscr{G}} : \mathrm{H}^1\big(\mathbb{Q}_p, \mathbf{V}^{\mathsf{f}_\circ}_{\mathscr{G}}\big) \longrightarrow \mathbb{D}(\mathbf{U}^{\mathsf{f}_\circ}_{\mathscr{G}})$$

satisfying the following properties:

(i) For all arithmetic points $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight $\ell \geq 2$, and character $\chi_\diamond$,

$$\nu_{\mathrm{P}} \circ \boldsymbol{\mathcal{L}}_{\mathscr{G}}^{\mathsf{f}_\circ} = \frac{(-1)^{\ell-2}}{(\ell-2)!} \cdot \Upsilon(\mathrm{P}) \cdot \big(\log_{\mathrm{BK}} \circ \nu_{\mathrm{P}} \big)$$

where $\Upsilon(\mathrm{P}) = \big(\alpha_{1,\mathsf{g}_\mathrm{P}} \alpha_{2,\mathsf{g}_\mathrm{P}} \alpha_{\mathsf{f}_\circ}^{-1} p^{2-\ell}\big)^{\alpha} \cdot G\big(\chi_\diamond \cdot \theta_{\mathbb{Q}|D_p}^{\ell-1}\big)^{-1}$.

(ii) For the arithmetic point $\mathrm{P}_\circ \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight one,

$$\nu_{\mathrm{P}_\circ} \circ \boldsymbol{\mathcal{L}}_{\mathscr{G}}^{\mathsf{f}_\circ} = \Upsilon(\mathrm{P}_\circ) \cdot \big(\exp^*_{\mathrm{BK}} \circ \nu_{\mathrm{P}_\circ} \big)$$

where $\Upsilon(\mathrm{P}_\circ) = \big(1 - \alpha_{1,\mathsf{g}_{\mathrm{P}_\circ}} \alpha_{2,\mathsf{g}_{\mathrm{P}_\circ}} \alpha_{\mathsf{f}_\circ}^{-1}\big)\big(1 - \alpha_{1,\mathsf{g}_{\mathrm{P}_\circ}}^{-1} \alpha_{2,\mathsf{g}_{\mathrm{P}_\circ}}^{-1} \alpha_{\mathsf{f}_\circ} \cdot p^{-1}\big)^{-1}$.

Proof. First, we note that the map $\boldsymbol{\Theta} \otimes \mathrm{id} : \boldsymbol{\Lambda}_\Gamma \widehat{\otimes}_{\boldsymbol{\Theta}} \boldsymbol{\Lambda}_\Gamma \xrightarrow{\sim} \boldsymbol{\Lambda}_\Gamma$ is an isomorphism, hence $\boldsymbol{\mathcal{L}}'$ can be seen as a homomorphism $\boldsymbol{\mathcal{L}}' : \mathrm{H}^1\big(\mathbb{Q}_p, \mathbf{V}_{\mathscr{G}}^{\mathsf{f}_\circ} \widehat{\otimes} \boldsymbol{\Lambda}_\Gamma(-\mathbf{j})\big) \longrightarrow \mathbb{D}(\mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ}) \widehat{\otimes} \boldsymbol{\Lambda}_\Gamma$. For any arithmetic point $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1\big(\mathbb{Q}_p, \mathbf{V}_{\mathscr{G}}^{\mathsf{f}_\circ} \widehat{\otimes} \boldsymbol{\Lambda}_\Gamma(-\mathbf{j})\big) & \xrightarrow{\boldsymbol{\mathcal{L}}'} & \mathbb{D}(\mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ}) \widehat{\otimes} \boldsymbol{\Lambda}_\Gamma \\ \big\downarrow & & \big\downarrow \\ \mathrm{H}^1\big(\mathbb{Q}_p, \mathrm{V}_{\mathsf{g}_\mathrm{P}}^{\mathsf{f}_\circ}\big) & \longrightarrow & \mathrm{D}_{\mathrm{dR}}\big(\mathrm{V}_{\mathsf{g}_\mathrm{P}}^{\mathsf{f}_\circ}\big) \end{array}$$

obtain by composing (7.3) with specialization at point P and character

$$\boldsymbol{\Theta}(\mathrm{P})^{-1} = \chi_\diamond \cdot \big(\varepsilon_{\mathbb{Q}}^{1-\ell} \theta_{\mathbb{Q}}^{\ell-1}\big)_{|D_p}.$$

Then, using (7.2), the bottom horizontal map can be computed to be

$$\frac{(-1)^{\ell-2}}{(\ell-2)!} \cdot \big(\alpha_{1,\mathsf{g}_\mathrm{P}} \alpha_{2,\mathsf{g}_\mathrm{P}} \alpha_{\mathsf{f}_\circ}^{-1} p^{2-\ell}\big)^{\alpha} G\big(\chi_\diamond \cdot \theta_{\mathbb{Q}|D_p}^{\ell-1}\big)^{-1} \cdot \log_{\mathrm{BK}}.$$

Similarly, when considering the non-arithmetic point $\mathrm{P}_\circ$, the relevant character is the trivial character $\boldsymbol{\Theta}(\mathrm{P}_\circ)^{-1} = \mathbb{1}$, and the bottom horizontal map can be seen to be

$$\big(1 - \alpha_{1,\mathrm{g}_{\mathrm{P}_\circ}}\alpha_{2,\mathrm{g}_{\mathrm{P}_\circ}}\alpha_{\mathsf{f}_\circ}^{-1}\big)\big(1 - \alpha_{1,\mathrm{g}_{\mathrm{P}_\circ}}^{-1}\alpha_{2,\mathrm{g}_{\mathrm{P}_\circ}}^{-1}\alpha_{\mathsf{f}_\circ}\cdot p^{-1}\big)^{-1}\cdot \exp^*_{\mathrm{BK}}\,.$$

In order to define the claimed homomorphism $\boldsymbol{\mathcal{L}}_{\mathscr{G}}^{\mathsf{f}_\circ}$, we note that if we consider

$$\beta : \mathbf{I}_{\mathscr{G}}\widehat{\otimes}\boldsymbol{\Lambda}_\Gamma \longrightarrow \mathbf{I}_{\mathscr{G}}, \qquad 1\otimes[u]\mapsto\langle u\rangle[u],$$

then for all $\mathrm{P}\in\mathcal{A}(\mathbf{I}_{\mathscr{G}})$ the following diagram commutes:

$$\begin{array}{ccc} \mathbf{I}_{\mathscr{G}}\widehat{\otimes}\boldsymbol{\Lambda}_\Gamma & \xrightarrow{\ \beta\ } & \mathbf{I}_{\mathscr{G}} \\ & {\scriptstyle \mathrm{P}\otimes\boldsymbol{\Theta}(\mathrm{P})^{-1}}\searrow & \downarrow{\scriptstyle \mathrm{P}} \\ & & O. \end{array}$$

Therefore, the composition

$$\begin{array}{ccc} \mathrm{H}^1\big(\mathbb{Q}_p,\mathbf{V}_{\mathscr{G}}^{\mathsf{f}_\circ}\big) & \overset{\boldsymbol{\mathcal{L}}_{\mathscr{G}}^{\mathsf{f}_\circ}}{\dashrightarrow} & \mathbb{D}(\mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ}) \\ \downarrow{\scriptstyle \mathrm{H}^1\left(\mathbb{Q}_p,\mathrm{id}\otimes 1\right)} & & \uparrow{\scriptstyle \beta} \\ \mathrm{H}^1\big(\mathbb{Q}_p,\mathbf{V}_{\mathscr{G}}^{\mathsf{f}_\circ}\widehat{\otimes}\boldsymbol{\Lambda}_\Gamma(-\mathbf{j})\big) & \xrightarrow{\ \boldsymbol{\mathcal{L}}'\ } & \mathbb{D}(\mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ})\widehat{\otimes}\boldsymbol{\Lambda}_\Gamma \end{array}$$

satisfies the claimed properties. □

7.2 The motivic p-adic L-function.

We are finally ready to define our motivic p-adic L-function. Because of the lack of a satisfactory generalization of the Λ-adic Eichler-Shimura isomorphism as in the modular curve case given by [Oht95], a priori it lacks certain integrality properties (in particular, it lies in a finite algebra over $\boldsymbol{\Pi}$ instead of $\boldsymbol{\Lambda}$). However, later on we

will still be able to compare it with our previously constructed automorphic p-adic L-function and show they are essentially the same.

Recall we have the class $\boldsymbol{\kappa}_p^{\mathsf{f}_\circ}(\mathscr{G}) \in \mathrm{H}^1\big(\mathbb{Q}_p, \boldsymbol{\mathcal{V}}_{\mathscr{G}}^{\mathsf{f}_\circ}(K)\big)$. Also recall the $\mathbf{I}_{\mathscr{G}}$-modules homomorphism

$$\langle\ , \omega_{\breve{\mathscr{G}}} \otimes \eta'_\circ\rangle : \mathbb{D}\big(\mathbf{U}_{\mathscr{G}}^{\mathsf{f}_\circ}(M)\big) \longrightarrow \boldsymbol{\Pi} \otimes_{\boldsymbol{\Lambda}} \mathbf{I}_{\mathscr{G}}$$

associated to any Λ-adic test vector $\breve{\mathscr{G}} \in \overline{\mathbf{S}}^{\mathrm{n.o.}}(K_{\diamond,t}; \mathbf{I}_{\mathscr{G}})[\mathscr{G}]$ given by Proposition 6.4.3.

Definition 7.2.1. Let $\breve{\mathscr{G}} \in \overline{\mathbf{S}}^{\mathrm{n.o.}}(K_{\diamond,t}; \mathbf{K}_{\mathscr{G}})[\mathscr{G}]$ be any Λ-adic test vector as above. The motivic p-adic L-function is defined as

$$\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}}, \mathsf{f}_\circ) := \Big\langle \boldsymbol{\mathcal{L}}_{\mathscr{G}}^{\mathsf{f}_\circ}\big(\boldsymbol{\kappa}_p^{\mathsf{f}_\circ}(\mathscr{G})\big),\ \omega_{\breve{\mathscr{G}}} \otimes \eta'_\circ \Big\rangle \in \boldsymbol{\Pi} \otimes_{\boldsymbol{\Lambda}} \mathbf{I}_{\mathscr{G}}.$$

Chapter 8

p-adic Gross-Zagier formulas

The main goal of this section is to give a formula for certain values of the syntomic Abel-Jacobi map in terms of p-adic modular forms. Given the definition of the motivic p-adic L-function in terms of the pairing introduced in Chapter 6, we will be interested in the values

$$\mathrm{AJ}_{\mathrm{syn}}\big(\Delta_\alpha^\circ\big)\big((\lambda_\alpha)^*\omega_{\tilde{\mathscr{E}}_{\mathrm{Fr}}}\otimes\eta_\circ\big)\in\mathbb{C}_p.$$

The basic tool is the so-called finite polynomial cohomology, as first introduced in [Bes00] and later generalized in [BLZ16]. Below we will first review the basic theory following [BLZ16], and then carry out the explicit calculations in our situation.

8.1 P-syntomic cohomology

Let $K/\mathbb{Q}_p$ be a finite extension, we denote by K_0 the maximal unramified subfield of K and by q be the cardinality of the residue field.

Definition 8.1.1. A filtered (φ, N, G_K)-module over K is a finite dimensional $\mathbb{Q}_p^{\mathrm{ur}}$-vector space D endowed with a $\mathbb{Q}_p^{\mathrm{ur}}$-semilinear bijective Frobenius endomorphism φ and a $\mathbb{Q}_p^{\mathrm{ur}}$-linear monodromy operator N satisfying $N\varphi = p\varphi N$. The absolute Galois

group G_K acts $\mathbb{Q}_p^{\mathrm{ur}}$-semilinearly on D and there is a decreasing, separated, exhaustive filtration of the K-vector space

$$D_K := \left(D \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} \bar{\mathbb{Q}}_p\right)^{G_K}$$

by K-vector subspaces $\mathrm{Fil}^i D_K$.

One writes D_{st} for the K_0-vector space D^{G_K} of G_K-invariant elements. A filtered (φ, N, G_K)-module D such that the G_K-action is unramified and $N = 0$ is said to be *crystalline*. In this case one can show that

$$D = D_{\mathrm{st}} \otimes_{K_0} \mathbb{Q}_p^{\mathrm{ur}} \qquad \text{and} \qquad D_K = D_{\mathrm{st}} \otimes_{K_0} K.$$

Definition 8.1.2. A crystalline filtered (φ, N, G_K)-module over K is said to be *convenient* for a choice of polynomial $P(T) \in 1 + TK[T]$ if $P(\Phi)$ and $P(q\Phi)$ are bijective endomorphisms of D_K, where Φ denotes the extension of scalars of the K_0-linear operator $\varphi^{[K_0:\mathbb{Q}_p]}$ on D_{st}.

For a variety $X_{/K}$, we let $\mathrm{H}^\bullet_{\mathrm{HK}}(X_h)$ and $\mathrm{H}^\bullet_{\mathrm{dR}}(X_h)$ be the extensions of Hyodo-Kato and de-Rham cohomology groups defined by Beilinson [Bei13] and by

$$\iota^{\mathrm{B}}_{\mathrm{dR}} : \mathrm{H}^\bullet_{\mathrm{HK}}(X_h) \otimes_{K_0} K \longrightarrow \mathrm{H}^\bullet_{\mathrm{dR}}(X_h)$$

the comparison morphism (which is an isomorphism if X has a semistable model over $\mathcal{O}_K$). For the filtered (φ, N, G_K)-modules $D^\bullet(X_h) = \mathbf{D}_{\mathrm{pst}}\left(\mathrm{H}^\bullet_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_p)\right)$ it was shown by Beilinson [Bei13] that

$$D^\bullet(X_h)_{\mathrm{st}} = \mathrm{H}^\bullet_{\mathrm{HK}}(X_h) \qquad \text{and} \qquad D^\bullet(X_h)_K = \mathrm{H}^\bullet_{\mathrm{dR}}(X_h).$$

Cohomology of filtered (φ, N, G_K)-modules.

For a polynomial $P(T) \in 1 + TK[T]$ one can define the complex

$$C^{\bullet}_{\mathrm{st},P}(D): \qquad D_{\mathrm{st},K} \oplus \mathrm{Fil}^0 D_K \longrightarrow D_{\mathrm{st},K} \oplus D_{\mathrm{st},K} \oplus D_K \longrightarrow D_{\mathrm{st},K}$$

where the first map is $(u, v) \mapsto (P(\Phi)u, Nu, u - v)$, and the second is $(w, x, y) \mapsto Nw - P(q\Phi)x$. The cohomology of this complex is denoted by

$$\mathrm{H}^{\bullet}_{\mathrm{st},P}(D) := \mathrm{H}^{\bullet}\big(C^{\bullet}_{\mathrm{st},P}(D)\big).$$

Theorem 8.1.3. ([BLZ16] Theorem 2.1.2) There is a P-syntomic descent spectral sequence

$$E_2^{i,j} = \mathrm{H}^i_{\mathrm{st},P}\big(D^j(X_h)(r)\big) \implies \mathrm{H}^{i+j}_{\mathrm{syn},P}(X_h, r)$$

compatible with cup products.

8.2 Syntomic Abel-Jacobi map

Let $X_{/K}$ be a proper and smooth d-dimensional variety. The commutativity of the following diagram

$$\begin{array}{ccc}
 & \mathrm{CH}^i(X) & \\
{\scriptstyle \mathrm{cl}_{\mathrm{syn}}}\swarrow & & \searrow{\scriptstyle \mathrm{cl}_{\mathrm{et}}} \\
\mathrm{H}^{2i}_{\mathrm{syn}}(X_{K,h}, i) & \xrightarrow{\ \rho_{\mathrm{syn}}\ } & \mathrm{H}^{2i}_{\mathrm{et}}(X_K, \mathbb{Q}_p(i)) \\
\downarrow & & \downarrow \\
\mathrm{Gr}^0_{\mathrm{syn}} & \longrightarrow & \mathrm{Gr}^0_{\mathrm{et}} \\
\downarrow & & \| \\
\mathrm{H}^0_{\mathrm{st},1-T}(D^{2i}(X_h)(i)) & \xrightarrow{\ \sim\ } & \mathrm{H}^{2i}_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_p(i))^{G_K}
\end{array} \tag{8.1}$$

where ρ_{syn} is Nekovář-Niziol period morphism, follows from the compatibility of the syntomic descent spectral sequence for syntomic cohomology and the Hochshield-Serre spectral sequence for étale cohomology ([BLZ16] Theorem 2.1.2).

Remark. The bottom horizontal map of diagram (8.1) is an isomorphism by ([BLZ16] Theorem 1.1.4), hence the middle horizontal map is injective.

Therefore, if we let

$$\mathrm{CH}^i(X)_0 := \ker\left(\mathrm{cl}_{\mathrm{et}} : \mathrm{CH}^i(X) \longrightarrow \mathrm{H}^{2i}_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_p(i))^{G_K}\right)$$

denote the subgroup of null-homologous cycles, then the syntomic and the p-adic étale Abel-Jacobi maps can be compared

$$\begin{array}{ccc} & \mathrm{CH}^i(X)_0 & \\ {\scriptstyle \mathrm{AJ}_{\mathrm{syn}}}\swarrow & & \searrow{\scriptstyle \mathrm{AJ}_p^{\mathrm{et}}} \\ \mathrm{H}^1_{\mathrm{st},1-T}(D^{2i-1}(X_h)(i)) & \xrightarrow{\exp_{\mathrm{st}}} & \mathrm{H}^1(K, \mathrm{H}^{2i-1}_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_p(i))) \end{array}$$

through the generalized Bloch-Kato exponential map.

Furthermore, if V is a quotient of $\mathrm{H}^{2i-1}_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_p(i))$ such that $D = \mathbf{D}_{\mathrm{pst}}(V)$ is a convenient quotient of $D^{2i-1}(X_h)(i)$ with respect to the polynomial $1-T$, then the natural inclusion $D_K \hookrightarrow C^1_{\mathrm{st},1-T}(D)$ induces an isomorphism

$$\frac{D_K}{\mathrm{Fil}^0 D_K} \cong \mathrm{H}^1_{\mathrm{st},1-T}(D),$$

and one can refine the comparison as follows

$$\begin{array}{ccc} & \mathrm{CH}^i(X)_0 & \\ {\scriptstyle \mathrm{AJ}_{\mathrm{syn},D}}\swarrow & & \searrow{\scriptstyle \mathrm{AJ}^{\mathrm{et}}_{p,V}} \\ D_K/\mathrm{Fil}^0 & \xrightarrow{\quad\exp_{\mathrm{BK}}\quad} & \mathrm{H}^1_e(K,V). \end{array}$$

Remark. According to Definition 3.10 of [BK90], the Bloch-Kato exponential surjects onto $\mathrm{H}^1_e(K,V)$ and its kernel is given by $\mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1}/\mathrm{H}^0(K,V)$. In particular, when $\mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1}=0$ we can write

$$\mathrm{AJ}_{\mathrm{syn},D} = \log_{\mathrm{BK}} \circ \mathrm{AJ}^{\mathrm{et}}_{p,V},$$

where $\log_{\mathrm{BK}} = \exp^{-1}_{\mathrm{BK}}$.

If we let $D^*(1)$ denote the Tate dual of D, which is a submodule of $D^{2(d-i)+1}(X_h)(d+1-i)$, then $D_K/\mathrm{Fil}^0 = \big(\mathrm{Fil}^0 D^*(1)_K\big)^\vee$ and we can write

$$\mathrm{AJ}_{\mathrm{syn},D} : \mathrm{CH}^i(X)_0 \longrightarrow \big(\mathrm{Fil}^0 D^*(1)_K\big)^\vee. \tag{8.2}$$

Evaluation using P-syntomic cohomology.

Let $\Delta \in \mathrm{CH}^i(X)_0$ be a null-homologous cycle. For any class

$$\eta \in \mathrm{Fil}^0 D^*(1)_K \subset \mathrm{Fil}^{d+1-i}\mathrm{H}^{2(d-i)+1}_{\mathrm{dR}}(X_h),$$

choose a polynomial $P(T) \in 1+TK[T]$ such that $P(1)\neq 0$, $P(q^{-1})\neq 0$ and $\eta \in \mathrm{H}^0_{\mathrm{st},P}(D^*(1))$ ([BLZ16] Proposition 1.4.3). Suppose that η is in the kernel of the "knight's move" map

$$\mathrm{H}^0_{\mathrm{st},P}\big(D^{2(d-i)+1}(X_h)(d+1-i)\big) \longrightarrow \mathrm{H}^2_{\mathrm{st},P}\big(D^{2(d-i)}(X_h)(d+1-i)\big),$$

so that η can be lifted to syntomic cohomology. Then for any choice of lift $\tilde{\eta} \in \mathrm{H}^{2(d-i)+1}_{\mathrm{syn},P}\big(X_h, d+1-i\big)$ we can write

$$\mathrm{AJ}_{\mathrm{syn},D}(\Delta)(\eta) = \mathrm{tr}_{X,\mathrm{syn},P}\big(\mathrm{cl}_{\mathrm{syn}}(\Delta) \cup \tilde{\eta}\big) \tag{8.3}$$

thanks to the compatibility of the P-syntomic descent spectral sequence with cup products

$$\begin{array}{ccccc}
\mathrm{Fil}^1\mathrm{H}^{2i}_{\mathrm{syn},1-T}\big(X_h, i\big) & \times & \mathrm{H}^{2(d-i)+1}_{\mathrm{syn},P}\big(X_h, d+1-i\big) & \longrightarrow & \mathrm{H}^{2d+1}_{\mathrm{syn},P}\big(X_h, d+1\big)/\mathrm{Fil}^2 \cong K \\
\downarrow & & \downarrow & & \downarrow \sim \\
\mathrm{H}^1_{\mathrm{st},1-T}\big(D^{2i-1}(X_h)(i)\big) & \times & \mathrm{H}^0_{\mathrm{st},P}\big(D^{2(d-i)+1}(X_h)(d+1-i)\big) & \longrightarrow & \mathrm{H}^1_{\mathrm{st},P}\big(\mathbf{D}_{\mathrm{pst}}(\mathbb{Q}(1))\big) \cong K \\
\downarrow & & \uparrow & & \| \\
\mathrm{H}^1_{\mathrm{st},1-T}(D) & \times & \mathrm{H}^0_{\mathrm{st},P}(D^*(1)) & \longrightarrow & \mathrm{H}^1_{\mathrm{st},P}\big(\mathbf{D}_{\mathrm{pst}}(\mathbb{Q}(1))\big) \cong K.
\end{array}$$

8.3 Abel-Jacobi map of Hirzebruch-Zagier cycles

For every arithmetic point $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ of weight 2 and level p^α, the Galois representation $\mathrm{V}_{\mathsf{g}_\mathrm{P}}$ is crystalline as a $G_{\mathbb{Q}_p(\zeta_{p^\alpha})}$-representation. Throughout this subsection, we will consider all our geometric structures, including the moduli schemes and the cycles, to be defined over $F_\alpha := \mathbb{Q}_p(\zeta_{p^\alpha})$. Similarly, we will regard all Galois representations, as well as Deudonneé functors, to be defined with respect to the absolute Galois group $G_{\mathbb{Q}_p(\zeta_{p^\alpha})}$.

Let $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ be an arithmetic point of weight 2 and character $\chi_\diamond$ of conductor p^α, and write $\chi = \chi_\diamond \circ \mathrm{N}_{L/\mathbb{Q}}$, then the specialization of $\boldsymbol{\mathcal{V}}_{\mathscr{G},\mathsf{f}_\circ}(M)$ at P is a quotient of $\mathrm{H}^3_{\text{ét}}(Z_\alpha(K)_{\bar{F}_\alpha}, \mathbb{Q}_p(2))$ such that $D_{\mathsf{g}_\mathrm{P},\mathsf{f}_\circ} := \mathbf{D}_{\mathrm{pst}}(\mathcal{V}_{\mathscr{G}_\mathrm{P},\mathsf{f}_\circ}(M))$ is a convenient quotient of $D^3(Z_\alpha(K))(2)$. Furthermore, $D_{\mathsf{g}_\mathrm{P},\mathsf{f}_\circ} \cong (D_{\mathsf{g}_\mathrm{P},\mathsf{f}_\circ})^*(1)$ since $\mathcal{V}_{\mathscr{G}_\mathrm{P},\mathsf{f}_\circ}(M)$ is Kummer self-dual.

Definition 8.3.1. Let $\omega_\mathrm{P} \in e_{\mathrm{n.o.}}\mathrm{Fil}^2\mathrm{H}^2_{\mathrm{dR}}(S(K_{\diamond,t}(p^\alpha))/F_\alpha)$ be the de Rham cohomol-

ogy class associated with the specialization $\breve{\mathfrak{g}}_{\mathrm{P}} \in S^{\mathrm{n.o.}}_{2t_L,t_L}\big(K_{\diamond,t}(p^\alpha);\chi_\circ\theta_L^{-1}\chi^{-1},\mathbb{1};O\big)$ of the $\mathbf{K}_{\mathscr{G}}$-adic cuspform $\breve{\mathscr{G}}$.

Recall the class $\eta_\circ$ of Definition 6.4.1. The tensor product $(\lambda_\alpha)^*\omega_{\mathrm{P}}\otimes\eta_\circ$ belongs to the convenient $(\varphi,N,\Gamma_{F_\alpha})$-module $(D_{\mathfrak{g}_{\mathrm{P}},\mathfrak{f}_\circ})^*(1)_{F_\alpha}\cong(D_{\mathfrak{g}_{\mathrm{P}},\mathfrak{f}_\circ})_{F_\alpha}$ and we will evaluate

$$\mathrm{AJ}_{\mathrm{syn}}\big(\Delta^\circ_\alpha\big)\big((\lambda_\alpha)^*\omega_{\mathrm{P}}\otimes\eta_\circ\big)=\mathrm{AJ}_{\mathrm{syn}}\big((\lambda_\alpha,\mathrm{id})_*\Delta^\circ_\alpha\big)\big(\omega_{\mathrm{P}}\otimes\eta_\circ\big).$$

Lemma 8.3.2. We have

$$\lambda_\alpha\circ w_{\mathfrak{p}_2^\alpha}=\langle\varpi_{\mathfrak{p}_2}^\alpha,1\rangle\circ w_{\mathfrak{p}_2^\alpha}\circ\lambda_\alpha$$

when restricted to the first connected component of $S(K_\diamond(p^\alpha))$.

Proof. Recall that by definition $w_{\mathfrak{p}_2^\alpha}=\mathfrak{T}_{\tau_{\mathfrak{p}_2}}\circ\nu_\alpha$ and $\lambda_\alpha=(-)^*\circ\mathfrak{T}_{\tau_\alpha}$. A direct calculation on the complex uniformization of Shimura varieties shows that

$$\lambda_\alpha\circ\mathfrak{T}_{\tau_{\mathfrak{p}_2}}=\langle\varpi_{\mathfrak{p}_2}^\alpha,1\rangle\circ\mathfrak{T}_{\tau_{\mathfrak{p}_2}}\circ\lambda_\alpha.$$

On the other hand, identifying both $S(K_\diamond(p^\alpha))(\mathbb{C})$ and $S(K_{\mathrm{x}}(p^\alpha))(\mathbb{C})$ with $\coprod_{i=1}^h\Gamma^i\backslash\mathcal{H}^2$, we see the first connected component $\Gamma^1\backslash\mathcal{H}^2$ is identified via

$$(z_1,z_2)\in\mathcal{H}^2\mapsto[(z_1,z_2),I_2]\in S(K_\diamond(p^\alpha))(\mathbb{C}),$$

so λ_α restricted to $\Gamma^1\backslash\mathcal{H}^2$ is precisely the map $(z_1,z_2)\mapsto(-1/(Mp^\alpha z_1),-1/(Mp^\alpha z_2)$, which clearly commutes with ν_α, the identity map on $\mathcal{H}^2$. □

The diagonal embedding $\zeta : Y(K_0'(p^\alpha)) \to S(K_\diamond(p^\alpha))$ naturally factors

$$\begin{array}{ccc} Y(K_0'(p^\alpha)) & \xrightarrow{\iota} & S^*(K_\diamond^*(p^\alpha)) \\ & \searrow^{\zeta} & \downarrow \xi \\ & & S(K_\diamond(p^\alpha)) \end{array}$$

through the natural diagonal embedding map to a Hilbert-Blumenthal variety

$$\iota : Y(K_0'(p^\alpha)) \to S^*(K_\diamond^*(p^\alpha)).$$

Lemma 8.3.3. Let $Y_\alpha := Y(K_0'(p^\alpha))$ and $Z_\diamond^*(p^\alpha) := S^*(K_\diamond^*(p^\alpha)) \times Y(K_0'(p))$. Consider the null-homologous cycle

$$\Xi_\alpha^\circ := (\mathrm{id}, \varepsilon_{\mathfrak{f}_\circ^*})_*(\iota \circ \lambda_\alpha, \pi_{1,\alpha})_*[Y_\alpha] \in \mathrm{CH}^2(Z_\diamond^*(p^\alpha))_0(F_\alpha) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Then

$$(\lambda_\alpha, \mathrm{id})_* \Delta_\alpha^\circ = (w_{\mathfrak{p}_2^\alpha} \circ \xi, \mathrm{id})_* \Xi_\alpha^\circ.$$

Proof. Since the image of the diagonal embedding only involves the first connected component of $S(K_\diamond(p^\alpha))$, Lemma 8.3.2 allows us to "push" $(\lambda_\alpha, \mathrm{id})_*$ all the way back:

$$\begin{aligned} (\lambda_\alpha, \mathrm{id})_* \Delta_\alpha^\circ &= (\lambda_\alpha, \mathrm{id})_* (\mathrm{id}, \varepsilon_{\mathfrak{f}_\circ})_* (\langle \varpi_{\mathfrak{p}_2}^\alpha, 1 \rangle \circ w_{\mathfrak{p}_2^\alpha} \circ \zeta,\ \pi_{1,\alpha})_* [Y_\alpha] \\ &= (\mathrm{id}, \varepsilon_{\mathfrak{f}_\circ^*})_* (\langle \varpi_{\mathfrak{p}_2}^{-\alpha}, 1 \rangle \circ \lambda_\alpha \circ w_{\mathfrak{p}_2^\alpha} \circ \zeta,\ \pi_{1,\alpha})_* [Y_\alpha] \\ &= (\mathrm{id}, \varepsilon_{\mathfrak{f}_\circ^*})_* (\langle \varpi_{\mathfrak{p}_2}^{-\alpha}, 1 \rangle \circ \langle \varpi_{\mathfrak{p}_2}^{\alpha}, 1 \rangle \circ w_{\mathfrak{p}_2^\alpha} \circ \lambda_\alpha \circ \zeta,\ \pi_{1,\alpha})_* [Y_\alpha] \\ &= (\mathrm{id}, \varepsilon_{\mathfrak{f}_\circ^*})_* (w_{\mathfrak{p}_2^\alpha} \circ \zeta \circ \lambda_\alpha,\ \pi_{1,\alpha})_* [Y_\alpha] \\ &= (w_{\mathfrak{p}_2^\alpha} \circ \xi,\ \mathrm{id})_* \Xi_\alpha^\circ. \end{aligned}$$

□

The functoriality of the construction of the syntomic Abel-Jacobi map gives

$$\mathrm{AJ}_{\mathrm{syn}}\big((\lambda_\alpha,\mathrm{id})_*\Delta_\alpha^\circ\big)\big(\omega_{\mathrm{P}}\otimes\eta_\circ\big)=\mathrm{AJ}_{\mathrm{syn}}\big(\Xi_\alpha^\circ\big)\big(\omega_{\mathrm{P}}^\diamond\otimes\eta_\circ\big) \tag{8.4}$$

where the de Rham cohomology class

$$\omega_{\mathrm{P}}^\diamond:=(w_{\mathfrak{p}_2^\alpha}\circ\xi)^*\omega_{\mathrm{P}}\in\mathrm{Fil}^2\mathrm{H}^2_{\mathrm{dR}}(S^*(K_\diamond^*(p^\alpha))/F_\alpha)$$

is associated with the cuspform

$$\breve{\mathfrak{g}}_{\mathrm{P}}^\diamond:=(\nu_\alpha\circ\xi)^*(\breve{\mathfrak{g}}_{\mathrm{P}}|\tau_{\mathfrak{p}_2^\alpha}^{-1})\in S_{2t_L,t_L}\big(K_\diamond^*(p^\alpha);O\big).$$

If $R(T),Q(T)\in 1+T\cdot F_\alpha[T]$ are polynomials such that $R(p^{-2}\Phi)\omega_{\mathrm{P}}^\diamond$ and $Q(\Phi)\eta_\circ$ are zero, then

$$\omega_{\mathrm{P}}^\diamond\in\mathrm{H}^0_{\mathrm{st},R}\big(D^2(S^*(K_\diamond^*(p^\alpha)))(2)\big),\qquad \eta_\circ\in\mathrm{H}^0_{\mathrm{st},Q}\big(D^1(Y_1)\big).$$

Lemma 8.3.4. Suppose that $Q(p)\neq 0$, then there exists lifts

$$\tilde{\omega}_{\mathrm{P}}^\diamond\in\mathrm{H}^2_{\mathrm{syn},R}(S^*(K_\diamond^*(p^\alpha)),2),\qquad \tilde{\eta}_\circ\in\mathrm{H}^1_{\mathrm{syn},Q}(Y_1,0)$$

of $\omega_{\mathrm{P}}^\diamond$ and $\eta_\circ$ to syntomic cohomology.

Proof. As the Hilbert-Blumenthal surface $S^*(K_\diamond^*(p^\alpha))$ is simply connected, we have $\mathrm{H}^2_{\mathrm{st},R}\big(D^1(S^*(K_\diamond^*(p^\alpha)))(2)\big)=0$, thus the descent spectral sequence induces a surjection $\mathrm{H}^2_{\mathrm{syn},R}(S^*(K_\diamond^*(p^\alpha)),2)\twoheadrightarrow\mathrm{H}^0_{\mathrm{st},R}\big(D^2(S^*(K_\diamond^*(p^\alpha)))(2)\big)$. In the modular curve case we can compute that

$$\mathrm{H}^2_{\mathrm{st},Q}\big(D^0(Y_1)\big)=\mathrm{H}^2_{\mathrm{st},Q}\big(\mathbf{D}_{\mathrm{pst}}(\mathbb{Q}_p)\big)=F_\alpha/Q(p)F_\alpha$$

is zero as long as $Q(p) \neq 0$, in which case we also obtain a surjection $\mathrm{H}^1_{\mathrm{syn},Q}(Y_1, 0) \twoheadrightarrow \mathrm{H}^0_{\mathrm{st},Q}\big(D^1(Y_1)\big)$. $\square$

It follows by equation (8.3) that if $(R \star Q)(1) \neq 0$ and $(R \star Q)(p^{-1}) \neq 0$ we can compute values of the syntomic Abel-Jacobi map as

$$\mathrm{AJ}_{\mathrm{syn}}\big((\lambda_\alpha, \mathrm{id})_* \Delta^\circ_\alpha\big)\big(\omega_{\mathrm{P}} \otimes \eta_\circ\big) = \mathrm{tr}_{Z^*_\diamond(p^\alpha), R\star Q}\big(\mathrm{cl}_{\mathrm{syn}}\big(\Xi^\circ_\alpha\big) \cup \big(\tilde{\omega}^\diamond_{\mathrm{P}} \otimes \tilde{\eta}_\circ\big)\big).$$

By the projection formula ([BLZ16], Theorem 2.5.3) and noting that $(\varepsilon_{\mathsf{f}_\circ})^* \eta_\circ = \eta_\circ$, we can compute the syntomic trace for $Z_\diamond(p^\alpha)$ as a syntomic trace for the curve $Y_\alpha = Y(K'_\diamond(p^\alpha))$:

$$\mathrm{AJ}_{\mathrm{syn}}\big((\lambda_\alpha, \mathrm{id})_* \Delta^\circ_\alpha\big)\big(\omega_{\mathrm{P}} \otimes \eta_\circ\big) = \big\langle (\iota \circ \lambda_\alpha)^* \tilde{\omega}^\diamond_{\mathrm{P}},\ (\pi_{1,\alpha})^* \tilde{\eta}_\circ \big\rangle_{Y_\alpha, R\star Q}. \tag{8.5}$$

Explicit cup product formulas.

In order to continue the computation of syntomic regulators it is convenient to make the choice of lifts more explicit. Any lift $\tilde{\omega}^\diamond_{\mathrm{P}} \in \mathrm{H}^2_{\mathrm{syn},R}(S^*(K^*_\diamond(p^\alpha)), 2)$ of $\omega^\diamond_{\mathrm{P}}$ can be explicitly described as in Section 2.4 of [BLZ16]:

$$\tilde{\omega}^\diamond_{\mathrm{P}} = \big[u, \omega^\diamond_{\mathrm{P}}; w\big] \qquad \text{where} \qquad u \in \mathbb{R}\Gamma^{B,2}_{\mathrm{HK}}(S^*(K^*_\diamond(p^\alpha))),\ w \in \mathbb{R}\Gamma^{B,1}_{\mathrm{HK}}(S^*(K^*_\diamond(p^\alpha)))$$

satisfy $du = 0$, $dw = R(p^{-2}\Phi)u$, and $\iota^B_{\mathrm{dR}}(u) = \omega^\diamond_{\mathrm{P}}$. Similarly, a lift $\tilde{\eta}_\circ \in \mathrm{H}^1_{\mathrm{syn},Q}(Y_1, 0)$ of $\eta_\circ$ can be described as

$$\tilde{\eta}_\circ = [u', \eta_\circ; w'] \qquad \text{where} \qquad u' \in \mathbb{R}\Gamma^{B,1}_{\mathrm{HK}}(Y_1),\ w' \in \mathbb{R}\Gamma^{B,0}_{\mathrm{HK}}(Y_1)$$

satisfy $du' = 0$, $dw' = Q(\Phi)u'$ and $\iota^B_{\mathrm{dR}}(u') = \eta_\circ$. Now we consider any two polynomials $a(T_1, T_2)$ and $b(T_1, T_2)$ satisfying $(R \star Q)(T_1T_2) = a(T_1, T_2)R(T_1) + b(T_1, T_2)Q(T_2)$.

Then by Proposition 2.4.1 of [BLZ16], the class

$$(\iota \circ \lambda_\alpha)^* \tilde{\omega}_{\mathrm{P}}^{\diamond} \cup (\pi_{1,\alpha})^* \tilde{\eta}_\circ \in \mathrm{H}^3_{\mathrm{syn},R\star Q}(Y_\alpha, 2)$$

can be described by $[u'', v''; w''] = [0, 0; w'']$ where,

$$u'' = (\iota \circ \lambda_\alpha)^* u \cup (\pi_{1,\alpha})^* u' = 0, \quad v'' = (\iota \circ \lambda_\alpha)^* \omega_{\mathrm{P}}^{\diamond} \cup (\pi_{1,\alpha})^* \eta_\circ = 0$$

for dimension reasons, and

$$\begin{aligned} w'' &= a(p^{-2}\Phi, \Phi)((\iota \circ \lambda_\alpha)^* w \cup (\pi_{1,\alpha})^* u') + b(p^{-2}\Phi, \Phi)((\iota \circ \lambda_\alpha)^* u \cup (\pi_{1,\alpha})^* w') \\ &= a(p^{-2}\Phi, \Phi)((\iota \circ \lambda_\alpha)^* w \cup (\pi_{1,\alpha})^* u'). \end{aligned}$$

Combining (8.5) with the definition of the syntomic trace map ([BLZ16], Definition 3.1.2 & Equation (2)) we obtain

$$\begin{aligned} \mathrm{AJ}_{\mathrm{syn}}\big((\lambda_\alpha, \mathrm{id})_* \Delta_\alpha^\circ\big)\big(\omega_{\mathrm{P}} \otimes \eta_\circ\big) &= \mathrm{tr}_{Y_\alpha, R\star Q}\big((\iota \circ \lambda_\alpha)^* \tilde{\omega}_{\mathrm{P}}^{\diamond} \cup (\pi_{1,\alpha})^* \tilde{\eta}_\circ\big) \\ &= -\iota_{\mathrm{dR}}^B\big((R \star Q)(\Phi)^{-1} w''\big) \\ &= \frac{-a(p^{-2}\Phi, \alpha_{\mathsf{f}_\circ})}{(R \star Q)(p^{-1})} \cdot \big[(\lambda_\alpha)^* \iota^* [\iota_{\mathrm{dR}}^B(w)] \cup_{\mathrm{dR}} (\pi_{1,\alpha})^* \eta_\circ\big] \end{aligned} \tag{8.6}$$

where the last equality follows from the facts that $\eta_\circ$ is a eigenvector for Φ of eigenvalue $\alpha_{\mathsf{f}_\circ}$ and the Frobenius endomorphism of $\mathbf{D}_{\mathrm{pst}}(\mathbb{Q}_p(1))_{\mathrm{st},F_\alpha}$ is multiplication by p^{-1}.

8.4 Overconvergent Hilbert Modular Forms

In the next section we will present the syntomic lift in terms of overconvergent Hilbert modular forms. In this section we briefly review the the theory of overconvergent

Hilbert modular forms following [KL05] (see also [TX16], which also has an excellent account of the material and we will also follow it closely), and slightly adapt their constructions to our situation.

Roughly speaking, Kisin and Lai [KL05] defined overconvergent Hilbert modular forms of level $K_1^*(p^\alpha)$ as overconvergent sections of automorphic vector bundles over a certain rigid analytic subspace of $S^*(K_1^*(p^\alpha))$. To more precisely introduce the overconvergent Hilbert modular forms, we need to first summarize the geometric definitions and constructions made in op.cit. Let $\mathfrak{c} \subset L$ be a fractional ideal of L and $\mathcal{M}(\mathfrak{c}, \mu_{\mathfrak{N}p^\alpha})$ be the Hilbert-Blumenthal moduli scheme classifying pairs $(\underline{A} = (A, \lambda), i)$ over $\mathbb{Z}_p$ with λ a $\mathfrak{c}$-polarization and $i : \mathfrak{d}^{-1} \otimes \mu_{\mathfrak{N}p^\alpha} \hookrightarrow A$ a $\mu_{\mathfrak{N}p^\alpha}$-level structure.

Following section 2.3 of [TX16], for any open compact subgroup $U_0(\mathfrak{N}) \subset K \subset \Gamma(\mathfrak{N})$, we may replace the tame $\mu_{\mathfrak{N}}$-level structure with the more general K-level structure. Let $\mathcal{M}(\mathfrak{c}, K, \mu_{p^\alpha})$ be the moduli scheme classifying triples $(\underline{A} = (A, \lambda), i_{\mathfrak{N}}, i_p)$ where $\underline{A}$ is above, $i_{\mathfrak{N}}$ is a K-level structure, and i_p is a μ_{p^α}-level structure.

The $\mathbb{Q}_p$-variety $\mathcal{M}(\mathfrak{c}, K, \mu_{p^\alpha})_{\mathbb{Q}_p}$ is canonically identified with a Shimura variety for G^*. Choose an adele $t \in \mathbb{A}_{L,f}$ generating the ideal $\mathfrak{c}\mathfrak{d}$. Let

$$K_{\mathfrak{c},?}(p^\alpha) = \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} K_?(p^\alpha) \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$K^*_{\mathfrak{c},?}(p^\alpha) = K_{\mathfrak{c},?}(p^\alpha) \cap G^*(\mathbb{A}_f)$$

for $? \in \{0, 1, \diamond\}$, so that the lower left corner lies in $\mathfrak{c}\mathfrak{d}\hat{\mathcal{O}}_L$ and the upper right in $(\mathfrak{c}\mathfrak{d})^{-1}\hat{\mathcal{O}}_L$. Note that clearly $K_{\mathfrak{c},1}(p^\alpha)$ does not depend on the choice of t. Then $\mathcal{M}(\mathfrak{c}, K, \mu_{p^\alpha})_{\mathbb{Q}_p}$ is the canonical $\mathbb{Q}_p$-model for the G^*-Shimura variety $S^*(K^*_{\mathfrak{c},1}(p^\alpha))$.

Denote by $\bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})$ the $\mathbb{Z}_p$-scheme obtained by gluing unramified cusps to

$\mathcal{M}(\mathfrak{c}, K, \mu_{p^\alpha})$ (c.f. 3.2.1 of op.cit.). Let $\bar{\mathfrak{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})$ be the formal completion of $\bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})$ along its special fiber and

$$\mathfrak{M}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha}) := \bar{\mathfrak{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha}) \cap \bar{\mathcal{M}}(\mathfrak{c}, K, \mu_{p^\alpha})$$

the open formal subscheme of $\bar{\mathfrak{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})$. Furthermore, let $\bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{ord}}$ be the rigid analytic generic fiber of $\bar{\mathfrak{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})$, and

$$\mathcal{M}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{ord}} := \bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{ord}} \cap \mathcal{M}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{rig}}_{\mathbb{Q}_p},$$

where $\mathcal{M}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{rig}}_{\mathbb{Q}_p}$ denotes the rigid analytic variety associated with the $\mathbb{Q}_p$-variety $\mathcal{M}(\mathfrak{c}, K, \mu_{p^\alpha})_{\mathbb{Q}_p}$. Kisin and Lai defined a system of admissible open strict neighborhoods $\bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{ord}}(r)$ given by pulling back the corresponding open neighborhoods on the tame level Hilbert modular varieties $\bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K)^{\mathrm{ord}}$ given by the Hasse invariant (3.2.2 of op.cit.), and set

$$\mathcal{M}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{ord}}(r) = \bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{ord}}(r) \cap \mathcal{M}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{rig}}_{\mathbb{Q}_p}.$$

Note that here the convention on the indexing parameter r is such that as $r \to 1^-$ the open neighborhood $\bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{ord}}(r)$ shrinks towards $\bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{ord}}$. Finally, let $\mathcal{A} \to \mathcal{M}(\mathfrak{c}, K, \mu_{p^\alpha})$ be the universal abelian scheme, which extends to a universal semi-abelian scheme $\bar{\mathcal{A}}$ over $\bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})$.

When $\mathfrak{c} = \mathfrak{d}^{-1}$, we will omit it from the notation and simply write $\mathcal{M}(K, \mu_{p^\alpha})$, etc. Then $\mathcal{M}(K, \mu_{p^\alpha})_{\mathbb{Q}_p}$ is the canonical $\mathbb{Q}_p$-model for $S^*(K_1^*(p^\alpha))$.

Now we are ready to define the overconvergent Hilbert modular forms on $\mathbb{Q}_p$-model for $S^*(K_1^*(p^\alpha))$. Here we use the automorphic bundles introduced in [TX16], which are slightly more general than the ones used in [KL05], as the former adopts the

more flexible double digit weights which makes it easier to later descend to Hilbert modular surfaces for G. Let $\underline{\omega}_{k,w}$ be the automorphic line bundle on $\bar{\mathcal{M}}(K, \mu_{p^\alpha})$ of weight (k, w) defined in 2.12 of [TX16]. Here we are using the weight convention consistent with the current paper, and the $\underline{k}$ in op.cit. is precisely our k, whereas their w, which is just an integer, is our $m + 2$. The line bundle $\underline{\omega}_{k,w}$ extends to $\bar{\omega}_{k,w}$ over $\bar{\mathcal{M}}^{\mathrm{ur}}(K, \mu_{p^\alpha})$. Slightly abusing the notations, we will also use the same symbols to denote their appropriate rigid-analytificatons.

Let $S^*(K_1^*(p^\alpha))^{\mathrm{rig}}$ be the rigid analytic variety associated to $S^*(K_1^*(p^\alpha))$, and let $j : \mathcal{M}^{\mathrm{ur}}(K, \mu_{p^\alpha})^{\mathrm{ord}} \hookrightarrow S^*(K_1^*(p^\alpha))^{\mathrm{rig}}$ be the natural inclusion of rigid analytic varieties.

Definition 8.4.1. (c.f. [KL05], Definition 4.2.1 and Section 3.3, [TX16]) The space of overconvergent Hilbert modular forms of weight (k, w) and level $K_1^*(p^\alpha)$ with coefficients in an extension E of $\mathbb{Q}_p$ is

$$M^\dagger_{k,w}(K_1^*(p^\alpha), E) := H^0(S^*(K_1^*(p^\alpha))^{\mathrm{rig}}_E, j^\dagger \underline{\omega}_{k,w}).$$

Note that each section of $\underline{\omega}_{k,w}$ on the (rigid-analytification of the) whole generic fiber $S^*(K_1^*(p^\alpha))$ clearly restricts to an overconvergent section on the strict neighborhoods of $\bar{\mathcal{M}}^{\mathrm{ur}}(\mathfrak{c}, K, \mu_{p^\alpha})^{\mathrm{ord}}$, and thus the space of classical Hilbert modular forms of weight (k, w) and level $K_1^*(p^\alpha)$ canonically embeds into the space of overconvergent forms $M^\dagger_{k,w}(K_1^*(p^\alpha))$.

Kisin and Lai showed that the usual Hecke correspondences defined on the Shimura variety $S^*(K_1^*(p^\alpha))$ restrict to correspondences on the strict neighborhoods (c.f. 1.9-1.11 and 4.1 of [KL05], in particular Lemma 4.1.10), and they induce well-defined Hecke-operators (4.2.3 of op.cit.). In particular, there is the $U(p)$ operator.

For completeness we now recall the definition of $U(p)$ on forms of weight $(2t_L, t_L)$ (i.e. working with trivial coefficient sheaves). Let $\mathcal{M}_p(K, \mu_{p^\alpha})$ denote the moduli

scheme classifying triples $(\underline{A}, i_{\mathfrak{Q}}, i_p, C)$ where $\underline{A}$ is a $\mathfrak{d}^{-1}$-polarized HBAV, $i_{\mathfrak{Q}}$ a K-level structure, $i_p : \mathfrak{d}^{-1} \otimes \mu_{p^\alpha} \hookrightarrow A[p^\alpha]$ a μ_{p^α}-level structure, and $C \subset A$ a closed subscheme stable under the action of $\mathcal{O}_L$ étale locally isomorphic to the constant group scheme $\underline{\mathcal{O}_L/p}$ and meets $i_p(\mu_{p^\alpha})$ only at the identity section. We have the natural projection

$$\mathrm{pr}_1 : \mathcal{M}_p(K, \mu_{p^\alpha}) \twoheadrightarrow \mathcal{M}(K, \mu_{p^\alpha}) : \ (\underline{A}, i_{\mathfrak{Q}}, i_p, C) \mapsto (\underline{A}, i_{\mathfrak{Q}}, i_p).$$

There is another projection

$$\mathrm{pr}_2 : \mathcal{M}_p(K, \mu_{p^\alpha}) \twoheadrightarrow \mathcal{M}(K, \mu_{p^\alpha})$$

sending $(\underline{A}, i_{\mathfrak{Q}}, i_p, C)$ to $(\underline{A}/C, i_{\mathfrak{Q}}/C, i_p/C)$, where $\underline{A}/C$ is the HBAV A/C together with the $\mathfrak{d}^{-1}$-polarization as specified in 1.9 of op.cit., and $i_{\mathfrak{Q}}/C$ (respectively, i_p/C) is the composition of $i_{\mathfrak{Q}}$ (respectively, i_p) with the canonical projection $A \twoheadrightarrow A/C$. The $U(p)$ operator on cohomology groups is induced by $\mathrm{pr}_{1,*}\mathrm{pr}_2^*$.

Remark.

1. Here we are slightly simplifying the construction in 1.9 of op.cit., as we are looking at $\mathfrak{a} = p\mathcal{O}_L$ in their notation, which clearly represents the trivial in the narrow class group. Thus implicitly we are omitting the data $\varepsilon : p\mathfrak{c} \simeq \mathfrak{c}$, which is understood to be multiplication by p^{-1}.

2. It can be shown the $U(p)$ correspondence and operator here is equivalent to the double coset operator associated to $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, where p is considered as an element in $\mathbb{A}_{L,f}$ via diagonal embedding, and thus the notation $U(p)$ is justified. Note that $\mathcal{M}_p(K, \mu_{p^\alpha})_{\mathbb{Q}}$ is precisely the G^*-Shimura variety $S^*(K_1^*(p^\alpha) \cap U^0(p)))$,

and pr_1 corresponds to the natural projection

$$S^*(K_1^*(p^\alpha) \cap U^0(p))) \twoheadrightarrow S^*(K_1^*(p^\alpha)),$$

whereas pr_2 is the composition

$$S^*(K_1^*(p^\alpha) \cap U^0(p))) \overset{\gamma}{\simeq} S^*(K_1^*(p^\alpha) \cap U_0(p^{\alpha+1}))) \to S^*(K_1^*(p^\alpha))$$

where $\gamma = \mathcal{T}_g$ with $g = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in G^*(\mathbb{A}_{\mathbb{Q},f})$, and the second map is the natural projection. On the other hand, the double coset operator $U(p)$ corresponds to $(\mathrm{pr}_1 \circ \gamma^{-1})_*(\mathrm{pr}_2 \circ \gamma^{-1})^*$, which equals to $\mathrm{pr}_{1,*}(\gamma^{-1})_*(\gamma^{-1})^*\mathrm{pr}_2^* = \mathrm{pr}_{1,*}\mathrm{pr}_2^*$.

Following 3.2.5 of op.cit, there is also a Frobenius morphism on the formal scheme $\mathrm{Frob}_p : \mathfrak{M}(K, \mu_{p^\alpha}) \to \mathfrak{M}(K, \mu_{p^\alpha})$ defined as follows. Let $(\underline{A}, i_{\mathfrak{Q}}, i_p) \in \mathfrak{M}(K, \mu_{p^\alpha})$. Denote by C_m the multiplicative part of $A[p^m]$ for each $m > 0$. Then i_p defines an isomorphism between $\mathfrak{d}^{-1} \otimes \mu_{p^m}$ and C_m for each $m \leq \alpha$. Let $\underline{A}'$ be the HBAV $\underline{A}/C_1$ with the induced $\mathfrak{d}^{-1}$-polarization, and $i'_p : \mathfrak{d}^{-1} \otimes \mu_{p^\alpha} \hookrightarrow K_{\alpha+1}$ any morphism satisfying $p \cdot i'_p = i_p$. Then i'_p is independent of the choice modulo C_1, and thus gives a well-defined embedding $i'_p : \mathfrak{d}^{-1} \otimes \mu_{p^\alpha} \hookrightarrow A'$. Furthermore, let $i'_{\mathfrak{Q}} = i_{\mathfrak{Q}}/C_1$ be the composition with the canonical projection as above. Then Frob_p is the map sending $(\underline{A}, i_p, i_{\mathfrak{Q}})$ to $(\underline{A}', i'_p, i'_{\mathfrak{Q}})$. Furthermore, thanks to Proposition 3.2.6 of op.cit., Frob_p induces morphisms on the ordinary locus as well as the strict neighborhoods. We then define the Frobenius operator Φ on the space of overconvergent modular forms as Frob_p^*.

Furthermore, we have the diamond operators at p. The group $K_0^*(p^\alpha)/K_1^*(p^\alpha)$ acts on $S^*(K_1^*(p^\alpha))$ via the morphism $\mathfrak{T}_g$ for $g \in K_0^*(p^\alpha)$. It is naturally identified with the Galois group of the covering $S^*(K_1^*(p^\alpha)) \twoheadrightarrow S^*(K_0^*(p^\alpha))$. Since the ordinary

locus $\mathcal{M}^{\mathrm{ur}}(K,\mu_{p^\alpha})^{\mathrm{ord}}$ and strict neighborhoods $\mathcal{M}^{\mathrm{ur}}(K,\mu_{p^\alpha})^{\mathrm{ord}}(r)$ are all pullbacks of the respective rigid spaces along the natural projection $S^*(K_1^*(p^\alpha)) \twoheadrightarrow S^*(K^*)$, which factors through $S^*(K_0^*(p^\alpha))$, we see $K_0^*(p^\alpha)/K_1^*(p^\alpha)$ acts on the ordinary locus and strict neighborhoods. In particular, $K_0^*(p^\alpha)/K_1^*(p^\alpha)$ acts on $M_{k,w}^\dagger(K_1^*(p^\alpha),E)$, extending the usual diamond operator action on the classical Hilbert modular forms.

Finally, we introduce the S_p operator defined in 3.10 of [TX16], which corresponds to the Hecke action of $\begin{pmatrix} \varpi_p^{-1} & 0 \\ 0 & \varpi_p^{-1} \end{pmatrix} \in G^*(\mathbb{A}_{\mathbb{Q},f})$, where we recall that $\varpi_p \in \mathbb{A}_{L,f}$ is the element which is p at places above p and 1 everywhere else. Let $\pi_{S_p} : \mathcal{M}(K,\mu_{p^\alpha}) \to \mathcal{M}(K,\mu_{p^\alpha})$ be the map sending $(\underline{A}, i_{\mathfrak{Q}}, i_p)$ to $(\underline{A}/A[p], i''_{\mathfrak{Q}}, i''_p)$, where i''_p is any embedding of $\mathfrak{d}^{-1}\otimes\mu_{p^\alpha}$ into A such that $p\cdot i''_p = i_p$, inducing a well-defined embedding $i'' : \mathfrak{d}^{-1}\otimes\mu_{p^\alpha} \hookrightarrow A/A[p]$, and $i''_{\mathfrak{Q}} = i_{\mathfrak{Q}}/A[p]$ as before. Then S_p operator is given by $\pi_{S_p}^*$.

Lemma 8.4.2. We have

$$\pi_{S_p}\circ \mathrm{pr}_1 = \mathrm{Frob}_p \circ \mathrm{pr}_2$$

as morphisms $\mathfrak{M}_p(K,\mu_{p^\alpha}) \to \mathfrak{M}(K,\mu_{p^\alpha})$.

Proof. Follow immediately from the definitions. □

Corollary 8.4.3. We have

$$U(p)\Phi = p^2 S_p$$

as operators acting on $M_{2t_l,t_L}^\dagger(K_1^*(p^\alpha))$.

Proof. This is essentially Lemma 3.20 of [TX16], and follows from a direct calculation:

$$U(p)\Phi = \mathrm{pr}_{1,*}\mathrm{pr}_2^*\mathrm{Frob}_p^* = \mathrm{pr}_{1,*}(\mathrm{Frob}_p\circ\mathrm{pr}_2)^* = \mathrm{pr}_{1,*}(\pi_{S_p}\circ\mathrm{pr}_1)^* = \mathrm{pr}_{1,*}\mathrm{pr}_1^*\pi_{S_p}^* = p^2 S_p.$$

□

Now we would like to descend to our Hilbert modular surface $S^*(K_\diamond^*(p^\alpha))$ in question. This is possible because the condition on determinant of G^* implies that the congruence condition on the upper left corner of $K_1^*(p^\alpha) = K_1(p^\alpha) \cap G^*(\mathbb{A}_f)$ requires its p-component to come from $\mathbb{Z}_p^\times$ modulo p^α, so that $K_1^*(p^\alpha) \subset K_\diamond^*(p^\alpha)$.

Recall we have $K_0^*(p^\alpha)/K_1^*(p^\alpha)$ acting as diamond operators. Consider the subgroup $K_\diamond^*(p^\alpha)/K_1^*(p^\alpha)$. Note that $\{\begin{pmatrix} 1 & 0 \\ 0 & \Delta(a) \end{pmatrix} : a \in \mathbb{Z}_p^\times/(1+p^\alpha\mathbb{Z}_p)\}$ is a set of representatives for $K_\diamond^*(p^\alpha)/K_1^*(p^\alpha)$, where $\Delta : \mathbb{Z}_p^\times/(1+p^\alpha\mathbb{Z}_p) \hookrightarrow \mathcal{O}_{L,p}^\times/1+p^\alpha\mathcal{O}_{L,p}$ is the diagonal embedding. We denote by D_α^* this group of representatives. Then D_α^* is naturally identified with the Galois group of the covering map $S^*(K_1^*(p^\alpha)) \to S^*(K_\diamond^*(p^\alpha))$.

We have seen that D_α^* act on the ordinary locus and the strict neighborhoods. In particular, by quotienting out the action of the Galois group, we obtain the ordinary locus $\mathcal{S}_\diamond^{\mathrm{ur,ord}}$ of $S^*(K_\diamond^*(p^\alpha))^{\mathrm{rig}}$, as well as a system of strict neighborhoods of $\mathcal{S}_\diamond^{\mathrm{ur,ord}}$. Therefore, still using $j : \mathcal{S}_\diamond^{\mathrm{ur,ord}} \hookrightarrow S^*(K_\diamond^*(p^\alpha))^{\mathrm{rig}}$ to denote the natural inclusion, we may define

$$M_{k,w}^\dagger(K_\diamond^*(p^\alpha), E) := H^0(S^*(K_\diamond^*(p^\alpha))_E^{\mathrm{rig}}, j^\dagger\underline{\omega}_{k,w}).$$

As noted in Remark 2.17 in [TX16], one has the Θ operators on the overconvergent modular forms. In our particular case of interest of forms of $((2,0),(1,0))$ and $((0,2),(0,1))$, the Θ operators become

$$d_1 : M_{(0,2),(0,1)}^\dagger(K_\diamond^*(p^\alpha), E) \to M_{2_L,t_L}^\dagger(K_\diamond^*(p^\alpha), E)$$

and

$$d_2 : M_{(2,0),(1,0)}^\dagger(K_\diamond^*(p^\alpha), E) \to M_{2_L,t_L}^\dagger(K_\diamond^*(p^\alpha), E),$$

where for an overconvergent form f with q-expansion $f = \sum_\lambda a_\lambda q^\lambda$ on one component one has $d_1 f = \sum_\lambda a_\lambda q^\lambda$ and $d_2 f = \sum_\lambda \bar{\lambda} a_\lambda q^\lambda$. Here implicitly we are using our fixed

embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ corresponding to a place of $\bar{\mathbb{Q}}$ above the place $\mathfrak{p}$ of L, and $\bar{\lambda} \in L$ is the algebraic conjugate. Translating to the p-adic Fourier coefficients, one sees that $a_p(y, d_1 f) = y_{\mathfrak{p}_1} a_p(y, d_1 f)$ and $a_p(y, d_2 f) = y_{\mathfrak{p}_2} a_p(y, d_1 f)$.

Replacing the real quadratic field L with $\mathbb{Q}$, one obtains the usual elliptic overconvergent modular forms of Coleman as in [Col97]. More precisely, writing $Y_\alpha := Y(K_0'(p^\alpha))$, one has the space of overconvergent modular forms of weight k and level $K_\diamond'(p^\alpha)$ given by

$$M_k^\dagger(K_\diamond'(p^\alpha), E) := \mathrm{H}^0(Y_\alpha^{\mathrm{rig}}, j^\dagger \underline{\omega}_k),$$

where $j : Y_\alpha^{\mathrm{ord}} \hookrightarrow Y_\alpha^{\mathrm{rig}}$ is again the natural inclusion of rigid spaces and $j^\dagger$ is with respect to a system of strict neighborhoods of Y_α^{ord}.

Lemma 8.4.4. The diagonal embedding $\iota : Y_\alpha \hookrightarrow S^*(K_\diamond^*(p^\alpha))$ maps ordinary locus to ordinary locus, restricting to a map $Y_\alpha^{\mathrm{ord}} \hookrightarrow S(K_\diamond(p^\alpha))^{\circ,\mathrm{ord}}$.

Proof. Note that ζ sits in the following commutative diagram

$$\begin{array}{ccc} Y('K_1(p^\alpha)) & \xhookrightarrow{\iota_1} & S^*(K_1^*(p^\alpha)) \\ \downarrow & & \downarrow \\ Y_\alpha & \xhookrightarrow{\iota} & S^*(K_\diamond^*(p^\alpha)). \end{array}$$

By the moduli description of the diagonal embedding ι_1 on the top, it extends to an embedding of the incomplete integral models, and thus ordinary loci. Moreover, the two vertical projections clearly map the respective ordinary locus of the modular variety on the top to the ordinary locus of the bottom, and since ordinary loci of both $Y(K_1'(p^\alpha))$ and Y_α are the pullback of the ordinary locus of $Y(K')$, we see the vertical map on the left induces a surjection on the ordinary loci. The claim of the lemma thus follows from the commutativity of the diagram. □

Thus a system of strict neighborhoods of $S^*(K_\diamond^*(p^\alpha))^{\mathrm{ord}}$ also restricts to a system

of strict neighborhoods of Y_α^{ord}. Furthermore, by the theory of dual BGG complexes (c.f. Section 2 of [TX16]), we have a quasi-isomorphic embedding of the dual BGG complex on $S(K_\diamond(p^\alpha))$

$$\underline{\omega}_{0,0} \to \underline{\omega}_{(0,2),(0,1)} \oplus \underline{\omega}_{(0,2),(0,1)} \to \underline{\omega}_{2t_L,t_L}$$

into the de Rham complex

$$\mathcal{O}(S^*(K_\diamond^*(p^\alpha))) \to \Omega^1(S^*(K_\diamond^*(p^\alpha))) \to \Omega^2(S^*(K_\diamond^*(p^\alpha))),$$

which in particular gives a restriction map

$$\iota^*(\underline{\omega}_{(0,2),(0,1)} \oplus \underline{\omega}_{(0,2),(0,1)}) \hookrightarrow \iota^*\Omega^1(S(K_\diamond(p^\alpha))) = \Omega^1(Y_\alpha) = \underline{\omega}_2.$$

Therefore we have a natural diagonal restriction map

$$\iota^* : M^\dagger_{(0,2),(0,1)}(K_\diamond(p^\alpha), E) \oplus M^\dagger_{(2,0),(1,0)}(K_\diamond(p^\alpha), E) \to M_2^\dagger(K'_\diamond(p^\alpha), E).$$

8.5 Relation to p-adic modular forms.

We are now in a situation very similar to that of [DR17]: both are trying to compute the pairing on a modular curve of a unit root lifting of an ordinary elliptic modular form with some de Rham cohomology class obtained from a syntomic cohomology class on a surface, only that in our case it comes from a Hilbert modular form on a Hilbert modular surface, instead of a product of elliptic modular forms on the product of two modular curves. We first briefly summarize the set-up of at the beginning of Chapter 4 of [DR17].

The compactified modular curve $X_0(p) := X(K_0'(p))$ admits a proper regular model $\mathscr{X}_0(p)_{/\mathbb{Z}_p}$ whose special fiber is the union of two curves, each isomorphic to the special fiber of the smooth integral model $\mathscr{X}$ of $X(K')$. One writes $\{\mathscr{W}_\infty, \mathscr{W}_0\}$ for the standard admissible covering of $\mathscr{X}_0(p)^{\mathrm{an}}$ by wide open neighborhoods obtained as the inverse image under the specialization map of the two distinguished curves in the special fiber. The two wide open neighborhoods are interchanged by the involution $\lambda_1 : \mathscr{X}_0(p)^{\mathrm{an}} \to \mathscr{X}_0(p)^{\mathrm{an}}$ defined over $\mathbb{Q}_p(\zeta_p)$. The de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(X_0(p)/\mathbb{Q}_p)$ is endowed with an action of a Frobenius map Φ commuting with the U_p Hecke operator. The ordinary unit root subspace

$$\mathrm{H}^1_{\mathrm{dR}}(X_0(p)/\mathbb{Q}_p)^{\mathrm{ord,ur}} \subseteq e_{\mathrm{ord}}\mathrm{H}^1_{\mathrm{dR}}(X_0(p)/\mathbb{Q}_p)$$

is spanned by the eigenvectors of Φ whose eigenvalue is a p-adic unit. The natural map

$$\mathrm{res}_{\mathscr{W}_\infty} : \mathrm{H}^1_{\mathrm{dR}}(X_0(p)/\mathbb{Q}_p)^{\mathrm{ord,ur}} \xrightarrow{0} \mathrm{H}^1_{\mathrm{rig}}(\mathscr{W}_\infty)$$

induced by restriction is the zero map, while

$$\mathrm{res}_{\mathscr{W}_0} : \mathrm{H}^1_{\mathrm{dR}}(X_0(p)/\mathbb{Q}_p)^{\mathrm{ord,ur}}[\phi] \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{rig}}(\mathscr{W}_0)[\phi]$$

is an isomorphism for any eigenform $\phi \in S_{2,1}(K';\bar{\mathbb{Q}})$ ([DR17], Lemma 4.2).

Lemma 8.5.1. For any $\omega \in \mathrm{H}^1_{\mathrm{dR}}(X(K_0'(p^\alpha))/\mathbb{Q}_p)$ the de Rham pairing

$$\left\langle (\lambda_\alpha)^*\omega, (\pi_{1,\alpha})^*\eta_\circ \right\rangle_{\mathrm{dR}} = \left\langle e_{\mathrm{ord}}\omega, (\pi_{2,\alpha})^*(\lambda_1)_*\eta_\circ \right\rangle_{\mathrm{dR}}$$

depends only on the p-adic cuspform associated to $e_{\mathrm{ord}}\omega$.

Proof. Since the class $\eta_\circ$ ordinary and $e_{\mathrm{ord}} \circ (\pi_{1,\alpha})^* = (\pi_{1,\alpha})^* \circ e_{\mathrm{ord}}$ we can compute

$$\begin{aligned}
\big\langle (\lambda_\alpha)^*\omega, (\pi_{1,\alpha})^*\eta_\circ \big\rangle_{\mathrm{dR}} &= \big\langle e_{\mathrm{ord}}^*(\lambda_\alpha)^*\omega, (\pi_{1,\alpha})^*\eta_\circ \big\rangle_{\mathrm{dR}} \\
&= \big\langle (\lambda_\alpha)^* e_{\mathrm{ord}}\omega, (\pi_{1,\alpha})^*\eta_\circ \big\rangle_{\mathrm{dR}} \\
&= \big\langle e_{\mathrm{ord}}\omega, (\lambda_\alpha)_*(\pi_{1,\alpha})^*\eta_\circ \big\rangle_{\mathrm{dR}} \\
&= \big\langle e_{\mathrm{ord}}\omega, (\lambda_\alpha^{-1})^*(\pi_{1,\alpha})^*\eta_\circ \big\rangle_{\mathrm{dR}} \\
&= \big\langle e_{\mathrm{ord}}\omega, (\pi_{2,\alpha})^*(\lambda_1^{-1})^*\eta_\circ \big\rangle_{\mathrm{dR}} \\
&= \big\langle e_{\mathrm{ord}}\omega, (\pi_{2,\alpha})^*(\lambda_1)_*\eta_\circ \big\rangle_{\mathrm{dR}},
\end{aligned}$$

where we have used the fact λ_α is an isomorphism and it intertwines π_1 and π_2 (see (6.7)). Finally, note the class $\eta_\circ$ is supported on the wide open $\mathscr{W}_0$. Hence, from the explicit description of the Poincaré pairing in Equation (109) of [DR17] and the fact that the involution $\lambda_1 : \mathscr{X}_0(p)^{\mathrm{an}} \to \mathscr{X}_0(p)^{\mathrm{an}}$ interchanges $\mathscr{W}_0$ with $\mathscr{W}_\infty$, we see that the pairing depends only on the restriction $\mathrm{res}_{\mathscr{W}_\infty(p^\alpha)}\big(e_{\mathrm{ord}}\omega\big)$. □

We now describe $e_{\mathrm{ord}}\zeta^*[\iota_{\mathrm{dR}}^B(w)]$ in terms of p-adic cuspforms. Recall $L/\mathbb{Q}$ is a real quadratic field and $p\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$ is a split prime, and that the cuspform $\breve{\mathbf{g}}_{\mathrm{P}} \in S_{2t_L,t_L}(K_{\diamond,t}(p^\alpha); O)$ is an eigenform for the Hecke operators $U_{\mathfrak{p}_1}, U_{\mathfrak{p}_2}^*$ with eigenvalues

$$U_{\mathfrak{p}_1}\breve{\mathbf{g}}_{\mathrm{P}} = \alpha_{1,\mathbf{g}_{\mathrm{P}}} \cdot \breve{\mathbf{g}}_{\mathrm{P}} \qquad \text{and} \qquad U_{\mathfrak{p}_2}^*\breve{\mathbf{g}}_{\mathrm{P}} = \overline{\alpha}_{2,\mathbf{g}_{\mathrm{P}}} \cdot \breve{\mathbf{g}}_{\mathrm{P}} = \alpha_{2,\mathbf{g}_{\mathrm{P}}}^{-1} p \cdot \breve{\mathbf{g}}_{\mathrm{P}}.$$

Thus, the cuspform $(w_{\mathfrak{p}_2^\alpha})^*\breve{\mathbf{g}}_{\mathrm{P}} \in S_{2t_L,t_L}(K_\diamond(p^\alpha); O)$ is an eigenform for the U_p-operator and Corollary 4.3.8 allows us to compute the Hecke eigenvalue by

$$\begin{aligned}
U_p \cdot (w_{\mathfrak{p}_2^\alpha})^*\breve{\mathbf{g}}_{\mathrm{P}} &= (w_{\mathfrak{p}_2^\alpha})^* U_{\mathfrak{p}_1} U_{\mathfrak{p}_2}^* \langle \varpi_{\mathfrak{p}_2}, 1\rangle \breve{\mathbf{g}}_{\mathrm{P}} \\
&= \big(\chi_\circ \theta_L^{-1}\chi^{-1}(\varpi_{\mathfrak{p}_2}) \cdot \alpha_{1,\mathbf{g}_{\mathrm{P}}}\alpha_{2,\mathbf{g}_{\mathrm{P}}}^{-1} p\big) \cdot (w_{\mathfrak{p}_2^\alpha})^*\breve{\mathbf{g}}_{\mathrm{P}} \qquad (8.7) \\
&= \chi_\circ \theta_L^{-1}\chi^{-1}(\varpi_{\mathfrak{p}_2}) \cdot \alpha_{1,\mathbf{g}_{\mathrm{P}}}\alpha_{2,\mathbf{g}_{\mathrm{P}}}^{-1} p \cdot (w_{\mathfrak{p}_2^\alpha})^*\breve{\mathbf{g}}_{\mathrm{P}}.
\end{aligned}$$

Working over Hilbert-Blumenthal varieties.

To further simplify notation, in the rest of this chapter we will write $S = S^*(K_\diamond^*(p^\alpha))$. We also fix a toroidal compactification of S over $\mathbb{Q}_p$, which we denote by S^{tor}.

We now need to slightly modify the operators $U_{\mathfrak{p}_1}$ and $U_{\mathfrak{p}_2}$ to overcome the minor technical issues that they do not commute with ν_α^* and they do not descent to the Hilbert-Blumenthal variety $S = S^*(K_\diamond^*(p^\alpha))$.

Suppose $\mathfrak{p}_1$ is narrowly principal in $\mathcal{O}_L$, $\mathfrak{p}_1 = p_1\mathcal{O}_L$ for a totally positive generator $p_1 \in \mathcal{O}_L$ and set $p_2 := p/p_1 \in \mathcal{O}_L$, where p is considered as an element in $\mathbb{A}_{L,f}^\times$ via diagonal embedding. Arguing as in Lemma 4.3.4, the Hecke operator $U(p_1)$, which is by definition the double coset operator associated with $\begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix}$, commutes with ν_α^*. Moreover, as $p_1\varpi_{\mathfrak{p}_1}^{-1} \in \hat{\mathcal{O}}_L^\times$, we see $U(p_1)$ and $U_{\mathfrak{p}_1}$ only differs by a diamond operator. More precisely,

$$U(p) = \langle 1, p^{-1}\varpi_p\rangle U_p, \qquad U(p_i) = \langle 1, p_i^{-1}\varpi_{\mathfrak{p}_i}\rangle U_{\mathfrak{p}_i} \qquad \text{for } i = 1, 2.$$

We also note that as $p_1, p_2 \in \mathcal{G} = G(\mathbb{Q})^+G^*(\mathbb{A}_{\mathbb{Q},f})$, the operators $U(p_1)$ and $U(p_2)$ are well-defined on S and commute with the natural map $\xi : S \to S(K_\diamond(p^\alpha))$.

Lemma 8.5.2. The cuspform $\breve{\mathbf{g}}_{\mathrm{P}}^\diamond := \xi^*(w_{\mathfrak{p}_2^\alpha})^*\breve{\mathbf{g}}_{\mathrm{P}}$ is an eigenform for the Hecke operators $U(p_1)$ and $U(p_2)$

$$U(p_1)\cdot\breve{\mathbf{g}}_{\mathrm{P}}^\diamond = \alpha_1^\diamond\cdot\breve{\mathbf{g}}_{\mathrm{P}}^\diamond, \qquad U(p_2)\cdot\breve{\mathbf{g}}_{\mathrm{P}}^\diamond = \alpha_2^\diamond\cdot\breve{\mathbf{g}}_{\mathrm{P}}^\diamond,$$

where

$$\alpha_1^\diamond := \chi_\circ\theta_L^{-1}\chi^{-1}((p_1)_{\mathfrak{p}_2}^{-1})\cdot\alpha_{1,\mathbf{g}_{\mathrm{P}}} \qquad \text{and} \qquad \alpha_2^\diamond := \chi_\circ\theta_L^{-1}\chi^{-1}((p_2)_{\mathfrak{p}_2}^{-1}\varpi_{\mathfrak{p}_2})\cdot\alpha_{2,\mathbf{g}_{\mathrm{P}}}^{-1}p.$$

Proof. We observe that

$$\begin{aligned} U(p_1)\cdot \breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond} &= \xi^* \circ U(p_1) \circ (w_{\mathfrak{p}_2^{\alpha}})^* \breve{\mathfrak{g}}_{\mathrm{P}} \\ &= \xi^* \circ (\nu_\alpha)^* \circ U(p_1) \circ (\mathfrak{T}_{\tau_{\mathfrak{p}_2}})^* \breve{\mathfrak{g}}_{\mathrm{P}}. \end{aligned}$$

As

$$\begin{aligned} U(p_1) \circ (\mathfrak{T}_{\tau_{\mathfrak{p}_2}})^* &= U_{\mathfrak{p}_1} \circ \langle 1, p_1^{-1}\varpi_{\mathfrak{p}_1}\rangle \circ (\mathfrak{T}_{\tau_{\mathfrak{p}_2}})^* \\ &= U_{\mathfrak{p}_1} \circ (\mathfrak{T}_{\tau_{\mathfrak{p}_2}})^* \circ \langle (p_1)_{\mathfrak{p}_2}^{-1}, p_1^{-1}\varpi_{\mathfrak{p}_1}\cdot (p_1)_{\mathfrak{p}_2}^{2}\rangle \\ &= (\mathfrak{T}_{\tau_{\mathfrak{p}_2}})^* \circ U_{\mathfrak{p}_1} \circ \langle (p_1)_{\mathfrak{p}_2}^{-1}, p_1^{-1}\varpi_{\mathfrak{p}_1}\cdot (p_1)_{\mathfrak{p}_2}^{2}\rangle, \end{aligned}$$

where in the second quality we used (4.2), we obtain

$$\begin{aligned} U(p_1)\cdot \breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond} &= \xi^* \circ (\nu_\alpha)^* \circ (\mathfrak{T}_{\tau_{\mathfrak{p}_2}})^* \circ U_{\mathfrak{p}_1} \circ \langle (p_1)_{\mathfrak{p}_2}^{-1}, \varpi_{\mathfrak{p}_1}\cdot (p_1)_{\mathfrak{p}_2}^{-2}\rangle \breve{\mathfrak{g}}_{\mathrm{P}} \\ &= \chi_\circ \theta_L^{-1}\chi^{-1}((p_1)_{\mathfrak{p}_2}^{-1})\cdot \alpha_{1,\mathfrak{g}_{\mathrm{P}}}\cdot \breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond}. \end{aligned}$$

Similarly,

$$U(p_2)\cdot \breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond} = \chi_\circ \theta_L^{-1}\chi^{-1}((p_2)_{\mathfrak{p}_2}^{-1}\varpi_{\mathfrak{p}_2})\cdot \alpha_{2,\mathfrak{g}_{\mathrm{P}}}^{-1} p\cdot \breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond}.$$

□

Remark. Note that

$$\alpha_1^{\diamond}\alpha_2^{\diamond} = \alpha_{1,\mathfrak{g}_{\mathrm{P}}}\alpha_{2,\mathfrak{g}_{\mathrm{P}}}^{-1} p,$$

which is the eigenvalue of $U(p)$ for $\breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond}$.

Similarly, if we define

$$V(p_1) := \langle 1, p_1\varpi_{\mathfrak{p}_1}^{-1}\rangle V_{\mathfrak{p}_1}, \qquad V(p_2) := \langle 1, p_2\varpi_{\mathfrak{p}_2}^{-1}\rangle V_{\mathfrak{p}_2} \quad \text{and} \quad V(p) := V(p_1)V(p_2),$$

then the action of $pV(p_i)$ is precisely right multiplication by $\begin{pmatrix} p_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{G}$ for $i = 1, 2$, which is again well-defined on S. Note that

$$U(p_i)V(p_i) = U_{\mathfrak{p}_i} V_{\mathfrak{p}_i} = 1,$$

whereas $V(p_i)U(p_i)$ is the operator of taking $\mathfrak{p}_i$-depletion, $i = 1, 2$.

Finally, we look at the action of the operator S_p on $\breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond}$.

Lemma 8.5.3. The operator S_p acts trivially on $\breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond}$.

Proof. By definition, S_p acts as $\langle \varpi_p, 1 \rangle$, which, by the same argument as in Lemma 4.3.4, commutes with ν_α. On the other hand, $\langle \varpi_p, 1 \rangle = \langle \varpi_p p^{-1}, 1 \rangle$ where p is regarded as an element of $\mathbb{A}_{L,f}$ via diagonal embedding, and as $\varpi_p p^{-1}$ has trivial components at p, it commutes with $\mathfrak{T}_{\tau_{\mathfrak{p}_2^\alpha}}$. Therefore S_p commutes with $w_{\mathfrak{p}_2^\alpha}$, and we have

$$S_p \cdot \breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond} = \xi^*(w_{\mathfrak{p}_2^\alpha})^* \langle \varpi_p p^{-1}, 1 \rangle \breve{\mathfrak{g}}_{\mathrm{P}} = \xi^*(w_{\mathfrak{p}_2^\alpha})^* \big(\chi_\circ \theta_L^{-1} \chi^{-1}(\varpi_p p^{-1}) \breve{\mathfrak{g}}_{\mathrm{P}} \big) = \breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond},$$

where in the last equality we used the facts that $\varpi_p p^{-1}$ has trivial p-component and $\chi_{\circ|\mathbb{Q}}$ is trivial. □

Choice of the polynomial.

Consider

$$\mathcal{R}_i(T_i) = (1 - \alpha_i^{\diamond} T_i) \quad \text{for} \quad i = 1, 2, \qquad \mathcal{R}(T) = 1 - \alpha_1^{\diamond} \alpha_2^{\diamond} T.$$

Setting $T = T_1 T_2$, one can write

$$\mathcal{R}(T) = \mathcal{R}_1(T_1)\mathcal{R}_2(T_2) + \alpha_2^{\diamond} T_2 \mathcal{R}_1(T_1) + \alpha_1^{\diamond} T_1 \mathcal{R}_2(T_2).$$

Thus by plugging in $T_1 = V(p_1)$, $T_2 = V(p_2)$, so that $T = V(p_1)V(p_2) = V(p)$, we see the cuspform $\mathcal{R}(V(p))\breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond}$ can be expressed as

$$\mathcal{R}(V(p))\breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond} = (\breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond})^{[\mathcal{P}]} + \alpha_2^{\diamond}V(p_2)(\breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond})^{[\mathfrak{p}_1]} + \alpha_1^{\diamond}V(p_1)(\breve{\mathfrak{g}}_{\mathrm{P}}^{\diamond})^{[\mathfrak{p}_2]}.$$

Definition 8.5.4. We choose $R(T) \in 1+T\cdot F_\alpha[T]$ to be the characteristic polynomial of the Frobenius endomorphism $p^{-2}\Phi$ acting on $\mathrm{H}^2_{\mathrm{dR}}(S)[\mathfrak{g}_{\mathrm{P}}^{\diamond}]$.

By construction, all roots of $R(T)$ have complex absolute value p^{-1}, and we have

$$R(p^{-2}\Phi)\omega_{\mathrm{P}}^{\diamond} = 0.$$

Moreover, $R(T)$ is divisible by $\mathcal{R}(T)$, i.e.,

$$R(T) = \mathscr{R}(T)\mathcal{R}(T)$$

for some polynomial $\mathscr{R}(T) \in 1 + T \cdot F_\alpha[T]$

Let $\iota_{\mathrm{rig}} : \mathbb{R}\Gamma^{\bullet}_{\mathrm{dR}}(S^{\mathrm{tor}}) \to \mathbb{R}\Gamma^{\bullet}_{\mathrm{rig}}(S^{\mathrm{ord}})$ be the natural map induced by restriction from S^{tor} to the ordinary locus S^{ord}, which is understood to be in the derived category. We may choose $\mathbb{R}\Gamma^{\bullet}_{\mathrm{dR}}(S^{\mathrm{tor}})$ to be an injective resolution of

$$0 \to S_{0,0}(K_\diamond^*(p^\alpha)) \overset{(d_1,d_2)}{\longrightarrow} S_{(2,0),(1,0)}(K_\diamond^*(p^\alpha)) \oplus S_{(0,2),(0,1)}(K_\diamond^*(p^\alpha)) \overset{d_2-d_1}{\longrightarrow} S_{2t_L,t_L}(K_\diamond^*(p^\alpha)) \to 0,$$

and $\mathbb{R}\Gamma^{\bullet}_{\mathrm{rig}}(S^{\mathrm{ord}})$ to be

$$0 \to S^\dagger_{0,0}(K_\diamond^*(p^\alpha)) \overset{(d_1,d_2)}{\longrightarrow} S^\dagger_{(2,0),(1,0)}(K_\diamond^*(p^\alpha)) \oplus S^\dagger_{(0,2),(0,1)}(K_\diamond^*(p^\alpha)) \overset{d_2-d_1}{\longrightarrow} S^\dagger_{2t_L,t_L}(K_\diamond^*(p^\alpha)) \to 0,$$

so that ι_{rig} respects the natural inclusion of classical modular forms into overconvergent forms.

Lemma 8.5.5. Let $\mathfrak{p}$ be an $\mathcal{O}_L$-prime ideal above p. For any overconvergent cuspform $\mathsf{g} \in S^{\dagger}_{2t_L,t_L}(K^*_{\diamond}(p^{\alpha}))$ there exist $\mathfrak{p}$-depleted overconvergent forms $\mathsf{g}_1 \in S^{\dagger}_{(2,0),(1,0)}(K^*_{\diamond}(p^{\alpha}))$ and $\mathsf{g}_2 \in S^{\dagger}_{(0,2),(0,1)}(K^*_{\diamond}(p^{\alpha}))$ such that

$$d_2\mathsf{g}_1 - d_1\mathsf{g}_2 = \mathsf{g}^{[\mathfrak{p}]}.$$

In particular, g_1 and g_2 both lie in the kernel of the $U(p)$-operator.

Proof. Since $U(p)\mathsf{g}^{[\mathfrak{p}]} = 0$ and $U(p)$ acts invertibly on $\mathrm{H}^2_{\mathrm{rig}}(S^{\mathrm{ord}})$, the cuspform $\mathsf{g}^{[\mathfrak{p}]}$ represents the trivial class in the rigid cohomology. Thus there exist cuspforms $\mathsf{g}_1 \in S^{\dagger}_{(2,0),(1,0)}(K^*_{\diamond}(p^{\alpha}))$ and $\mathsf{g}_2 \in S^{\dagger}_{(0,2),(0,1)}(K^*_{\diamond}(p^{\alpha}))$ satisfying $d_2\mathsf{g}_1 - d_1\mathsf{g}_2 = \mathsf{g}^{[\mathfrak{p}]}$. Since $\mathfrak{p}$-depletion is an idempotent operation commuting with the differential operators d_1 and d_2, then $\mathsf{g}_1^{[\mathfrak{p}]}$ and $\mathsf{g}_2^{[\mathfrak{p}]}$ satisfy all the conditions of the lemma. □

Hence, there are cuspforms

$$\mathsf{h}_1, \mathsf{b}_1, \mathsf{c}_1 \in S^{\dagger}_{(2,0),(1,0)}(K^*_{\diamond}(p^{\alpha})) \qquad \text{and} \qquad \mathsf{h}_2, \mathsf{b}_2, \mathsf{c}_2 \in S^{\dagger}_{(0,2),(0,1)}(K^*_{\diamond}(p^{\alpha}))$$

such that

- $\mathsf{h}_1, \mathsf{h}_2$ are $\mathcal{P}$-depleted, and $\mathscr{R}(V(p))(\breve{\mathsf{g}}^{\diamond}_{\mathrm{P}})^{[\mathcal{P}]} = d_2\mathsf{h}_1 - d_1\mathsf{h}_2$,
- $\mathsf{b}_1, \mathsf{b}_2$ are $\mathfrak{p}_1$-depleted, and $\mathscr{R}(V(p))V(p_2)(\breve{\mathsf{g}}^{\diamond}_{\mathrm{P}})^{[\mathfrak{p}_1]} = d_2\mathsf{b}_1 - d_1\mathsf{b}_2$,
- $\mathsf{c}_1, \mathsf{c}_2$ are $\mathfrak{p}_2$-depleted, and $\mathscr{R}(V(p))V(p_1)(\breve{\mathsf{g}}^{\diamond}_{\mathrm{P}})^{[\mathfrak{p}_2]} = d_2\mathsf{c}_1 - d_1\mathsf{c}_2$.

Lemma 8.5.6. Set

$$H_1 = \big(\mathsf{h}_1 + \alpha_2^{\diamond}\mathsf{b}_1 + \alpha_1^{\diamond}\mathsf{c}_1\big)$$

and

$$H_2 = \big(\mathsf{h}_2 + \alpha_2^{\diamond}\mathsf{b}_2 + \alpha_1^{\diamond}\mathsf{c}_2\big),$$

then there is an overconvergent function $f \in S^{\dagger}_{0,0}(K^*_{\diamond}(p^{\alpha}))$ such that

$$\iota^B_{\mathrm{dR}}(w) = \big(H_1 + d_1 f,\ H_2 + d_2 f\big).$$

Proof. By the construction of H_1 and H_2 we see

$$d_2 H_1 - d_1 H_2 = \mathscr{R}(V(p)) \cdot \big((\breve{\mathfrak{g}}^{\diamond}_{\mathrm{P}})^{[\mathcal{P}]} + \alpha_2^{\diamond} V(p_2)(\breve{\mathfrak{g}}^{\diamond}_{\mathrm{P}})^{[\mathfrak{p}_1]} + \alpha_1^{\diamond} V(p_1)(\breve{\mathfrak{g}}^{\diamond}_{\mathrm{P}})^{[\mathfrak{p}_2]}\big) = R(V(p))\breve{\mathfrak{g}}^{\diamond}_{\mathrm{P}}.$$

As $p^{-2}\Phi$ acts as $V(p)S_p^{-1} = V(p)$ on $\breve{\mathfrak{g}}^{\diamond}_{\mathrm{P}}$, we see $d(H_1, H_2) = d \cdot \iota^B_{\mathrm{dR}}(w)$, which means $\iota^B_{\mathrm{dR}}(w) - (H_1, H_2)$ defines a class in $\mathrm{H}^1_{\mathrm{rig}}(S^{\mathrm{tor}})$. Since (H_1, H_2) consists of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ depleted cuspforms, $U(p)(H_1, H_2) = 0$, and we can compute that

$$U(p)[\iota^B_{\mathrm{dR}}(w) - (H_1, H_2)] = [U(p)\iota^B_{\mathrm{dR}}(w)] = 0 \qquad \text{in} \qquad \mathrm{H}^1_{\mathrm{rig}}(S^{\mathrm{tor}})$$

since $[U(p)\iota^B_{\mathrm{dR}}(w)]$ is in the image of $\mathrm{H}^1_{\mathrm{dR}}(S^{\mathrm{tor}})$, which is trivial because the geometrically connected components of S^{tor} are simply connected. Therefore the class $[\iota^B_{\mathrm{dR}}(w) - (H_1, H_2)]$ is trivial because the $U(p)$-operator is invertible on $\mathrm{H}^1_{\mathrm{rig}}(S^{\mathrm{tor}})$, and the existence of such an f follows immediately. □

8.6 The formula

Now we are ready to carry out the explicit calculation of the syntomic regulator (8.6). To avoid overly cumbersome notations, we will use the same notation for an elliptic modular form of weight 2 and the corresponding differential on the modular curve, and it shall be clear from the context which one we are referring to.

We choose $Q(T) = 1 - \alpha_{\mathsf{f}_\circ}^{-1} T$, so that

$$\begin{aligned}(R \star Q)(T) &= R\big(\alpha_{\mathsf{f}_\circ}^{-1} T\big) \\ &= \mathscr{R}\big(\alpha_{\mathsf{f}_\circ}^{-1} T\big) \cdot (1 - \alpha_{1,\mathsf{g}_\mathrm{P}} \alpha_{2,\mathsf{g}_\mathrm{P}}^{-1} \alpha_{\mathsf{f}_\circ}^{-1} pT).\end{aligned}$$

Since $\alpha_{\mathsf{f}_\circ}$ has complex absolute value $\sqrt{p}$, while all roots of $R(T)$ has complex absolute value p^{-1}, we see $Q(p) \neq 0$ and $R \star Q(1) \neq 0$, $R \star Q(p^{-1}) \neq 0$. If we write

$$(R \star Q)(T_1 T_2) = a(T_1, T_2) R(T_1) + b(T_1, T_2) Q(T_2),$$

as in Section 8.3, then $a(T_1, \alpha_{\mathsf{f}_\circ}) = 1$ because plugging in $T_2 = \alpha_{\mathsf{f}_\circ}$ yields

$$R(T_1) = (R \star Q)(T_1 \cdot \alpha_{\mathsf{f}_\circ}) = a(T_1, \alpha_{\mathsf{f}_\circ}) R(T_1).$$

Combining with Lemma 8.5.1, we see that equation (8.6) simplifies to the following expression

$$\mathrm{AJ}_{\mathrm{syn}}\big((\lambda_\alpha, \mathrm{id})_* \Delta_\alpha^\circ\big)\big(\omega_\mathrm{P} \otimes \eta_\circ\big) = \frac{-1}{(R \star Q)(p^{-1})} \big\langle e_{\mathrm{ord}} \iota^* [\iota_{\mathrm{dR}}^B(w)], (\pi_{2,\alpha})^* (\lambda_1)_* \eta_\circ \big\rangle_{\mathrm{dR}, Y_\alpha}. \tag{8.8}$$

Combining the straightforward computation $e_{\mathrm{ord}} \iota^*(d_1 f, d_2 f) = e_{\mathrm{ord}} d\iota^* f = 0$ with Lemma 8.5.6, we have

$$e_{\mathrm{ord}} \iota^* [\iota_{\mathrm{dR}}^B(w)] = e_{\mathrm{ord}} \iota^*\big(H_1 + d_1 f, H_2 + d_2 f\big) = e_{\mathrm{ord}} \iota^*\big(H_1, H_2\big). \tag{8.9}$$

An important observation is that this last classical elliptic cuspform of level $K'_\diamond(p^\alpha)$ is independent of the choice of p-adic Hilbert modular forms $\mathsf{h}_1, \mathsf{h}_2, \mathsf{b}_1, \mathsf{b}_2, \mathsf{c}_1, \mathsf{c}_2$.

Lemma 8.6.1. Suppose $(\mathsf{g}_1, \mathsf{g}_2) \in M^{p\text{-adic}}_{(2,0),(1,0)}(K^*_\diamond(p^\alpha)) \oplus M^{p\text{-adic}}_{(0,2),(0,1)} K^*_\diamond(p^\alpha))$ be a pair

of p-adic Hilbert modular forms (not necessarily overconvergent) such that

$$d_2\mathfrak{g}_1 - d_1\mathfrak{g}_2 = 0,$$

and that either $\mathfrak{g}_1$ is $\mathfrak{p}_1$-depleted or $\mathfrak{g}_2$ is $\mathfrak{p}_2$-depleted. Then

$$e_{\mathrm{ord}}\iota^*(\mathfrak{g}_1, \mathfrak{g}_2) = 0.$$

Proof. Since this is a statement involving only sections on the identity component of the Hilbert-Blumenthal surface, we may carry out the computation using the usual Fourier expansion there.

First, suppose $\mathfrak{g}_1$ is $\mathfrak{p}_1$-depleted. For $\lambda \in L$, write $q^\lambda = \exp(2\pi i(\lambda z_1 + \bar{\lambda} z_2))$, where $\bar{\lambda} \in L$ is the algebraic conjugate. Suppose $\mathfrak{g}_1 = \sum_\lambda a_\lambda q^\lambda$ and $\mathfrak{g}_2 = \sum_\lambda b_\lambda q^\lambda$, where λ runs through the totally positive elements in some lattice in L. Then $d_2\mathfrak{g}_1 - d_1\mathfrak{g}_2 = 0$ implies

$$\lambda b_\lambda - \bar{\lambda} a_\lambda = 0, \qquad \text{or equivalently} \qquad b_\lambda = \frac{\bar{\lambda}}{\lambda} \cdot a_\lambda.$$

Here a_λ and b_λ are understood to be in $\bar{\mathbb{Q}}_p$, and so are λ and $\bar{\lambda}$ via our fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ which corresponds to a place of $\bar{\mathbb{Q}}_p$ above $\mathfrak{p}_1$.

On the other hand, as $\mathfrak{g}_1$ is $\mathfrak{p}_1$-depleted, $d_1^{-1}\mathfrak{g}_1 := \sum_\lambda \frac{a_\lambda}{\lambda} q^\lambda = \lim_{i\to\infty} d_1^{p^i-1}\mathfrak{g}_1$ is a well-defined p-adic modular form, and so is $\iota^*(d_1^{-1}\mathfrak{g}_1)$. Therefore

$$\iota^*(\mathfrak{g}_1, \mathfrak{g}_2)(z) = \sum_{n>0} q^n \sum_{\lambda+\bar{\lambda}=n} (a_\lambda + b_\lambda) = \sum_{n>0} nq^n \sum_{\lambda+\bar{\lambda}=n} \frac{a_\lambda}{\lambda} = d\iota^*\big(d_1^{-1}\mathfrak{g}_1\big)$$

is killed by e_{ord}. The case that $\mathfrak{g}_2$ is $\mathfrak{p}_2$-depleted is completely analogous. □

Corollary 8.6.2. We have the following identity

$$e_{\mathrm{ord}}\iota^*[\iota^B_{\mathrm{dR}}(w)] = \mathscr{R}(p^{-1}U_p^{-1}) \cdot e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(\nu_\alpha^*(\breve{\mathfrak{g}}_{\mathrm{P}}|\tau_{\mathfrak{p}_2^\alpha}^{-1}))^{[\mathcal{P}]}\big).$$

Proof. Lemma 8.6.1 gives us the freedom to choose any representatives h_1, h_2, b_1, b_2, c_1, c_2, even if they are not overconvergent. In particular, we may choose $\mathsf{h}_2 = \mathsf{b}_1 = \mathsf{c}_2 = 0$, and $\mathsf{h}_1 = d_2^{-1}\big(\mathscr{R}(V(p))(\breve{\mathfrak{g}}_{\mathrm{P}}^\diamond)^{[\mathcal{P}]}\big)$, $b_2 = -d_1^{-1}\big(\mathscr{R}(V(p))V(p_2)(\breve{\mathfrak{g}}_{\mathrm{P}}^\diamond)^{[\mathfrak{p}_1]}\big)$, $c_1 = d_2^{-1}\big(\mathscr{R}(V(p))V(p_1)(\breve{\mathfrak{g}}_{\mathrm{P}}^\diamond)^{[\mathfrak{p}_2]}\big)$.

Then equation (8.9) tells that

$$e_{\mathrm{ord}}\iota^*[\iota^B_{\mathrm{dR}}(w)] = e_{\mathrm{ord}}\iota^*\big(\mathsf{h}_1 + \alpha_1^\diamond\mathsf{c}_1, \alpha_2^\diamond\mathsf{b}_2\big).$$

Note that for $j \geq 0$, the y-th Fourier coefficient of $d_1^{-1}\big(V(p)^jV(p_2)(\breve{\mathfrak{g}}_{\mathrm{P}}^\diamond)^{[\mathfrak{p}_1]}\big)$ is zero unless $v_{\mathfrak{p}_1}(y) = j$ and $v_{\mathfrak{p}_2}(y) \geq j+1$, so that we may apply Lemma 5.3.6 of [LSZ16], which implies $e_{\mathrm{ord}}\iota^*\big(d_1^{-1}\big(V(p)^jV(p_2)(\breve{\mathfrak{g}}_{\mathrm{P}}^\diamond)^{[\mathfrak{p}_1]}\big)\big) = 0$, and hence $e_{\mathrm{ord}}\iota^*\mathsf{b}_2 = 0$. Similarly, $e_{\mathrm{ord}}\iota^*\mathsf{c}_1 = 0$. Therefore

$$e_{\mathrm{ord}}\iota^*[\iota^B_{\mathrm{dR}}(w)] = e_{\mathrm{ord}}\iota^*\mathsf{h}_1.$$

Now by Proposition 2.11 of [BCF], which tells $\iota^*V(p) = p^{-1}V(p)\iota^*$, we see

$$\iota^*\mathsf{h}_1 = \mathscr{R}(p^{-1}V(p)) \cdot \iota^*d_2^{-1}((\breve{\mathfrak{g}}_{\mathrm{P}}^\diamond)^{[\mathcal{P}]}) = \mathscr{R}(p^{-1}V(p)) \cdot \zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathfrak{g}}_{\mathrm{P}})^{[\mathcal{P}]})\big)$$

where we recall the definition $(\breve{\mathfrak{g}}_{\mathrm{P}}^\diamond)^{[\mathcal{P}]} = (\nu_\alpha \circ \xi)^*(\breve{\mathfrak{g}}_{\mathrm{P}}|\tau_{\mathfrak{p}_2^\alpha}^{-1}) = \xi^*(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathfrak{g}}_{\mathrm{P}})^{[\mathcal{P}]})$. The claimed identity now follows immediately from the fact that $e_{\mathrm{ord}}V(p) = U(p)^{-1}e_{\mathrm{ord}}$.

□

Finally we are ready to calculate the syntomic regulator in terms of p-adic cusp-forms.

Theorem 8.6.3. The following formula holds

$$\mathrm{AJ}_{\mathrm{syn}}\big((\lambda_\alpha,\mathrm{id})_*\Delta_\alpha^\circ\big)\big(\omega_{\mathrm{P}}\otimes\eta_\circ\big)=\frac{-\alpha_{\mathsf{f}_\circ}^{\alpha-1}}{1-\alpha_{1,\mathsf{g}_{\mathrm{P}}}\alpha_{2,\mathsf{g}_{\mathrm{P}}}^{-1}\alpha_{\mathsf{f}_\circ}^{-1}}\cdot\frac{\big\langle e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}),\mathsf{f}_\circ^{(p)}\big\rangle}{\big\langle\mathsf{f}_\circ^{(p)},\mathsf{f}_\circ^{(p)}\big\rangle}.$$

Proof. Combining (8.8) and Corollary 8.6.2 we have

$$\begin{aligned}
&\mathrm{AJ}_{\mathrm{syn}}\big((\lambda_\alpha,\mathrm{id})_*\Delta_\alpha^\circ\big)\big(\omega_{\mathrm{P}}\otimes\eta_\circ\big)\\
=&\frac{-1}{R(\alpha_{\mathsf{f}_\circ}^{-1}p^{-1})}\cdot\big\langle e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}),(\pi_{2,\alpha})^*(\lambda_1)^*\mathscr{R}(p^{-1}U_p^{-1})\eta_\circ\big\rangle_{\mathrm{dR},Y_\alpha}\\
=&\frac{-\mathscr{R}(\alpha_{\mathsf{f}_\circ}^{-1}p^{-1})}{R(\alpha_{\mathsf{f}_\circ}^{-1}p^{-1})}\cdot\big\langle e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}),(\pi_{2,\alpha})^*(\lambda_1)^*\eta_\circ\big\rangle_{\mathrm{dR},Y_\alpha}\\
=&\frac{-1}{1-\alpha_{1,\mathsf{g}_{\mathrm{P}}}\alpha_{2,\mathsf{g}_{\mathrm{P}}}^{-1}\alpha_{\mathsf{f}_\circ}^{-1}}\cdot\big\langle e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}),(\pi_{2,\alpha})^*(\lambda_1)^*\eta_\circ\big\rangle_{\mathrm{dR},Y_\alpha}.
\end{aligned}$$

Now Proposition 4.6.1 and Corollary 3.1.4 together imply that $e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}\big)$ is of level $K_0'(p)$, so we are really pairing with $(\pi_{1,\alpha})^*\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}\big)$ on Y_α. Thus

$$\begin{aligned}
&\big\langle e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}),(\pi_{2,\alpha})^*(\lambda_1)^*\eta_\circ\big\rangle_{\mathrm{dR},Y_\alpha}\\
=&\big\langle e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}),(\pi_{1,\alpha})_*(\pi_{2,\alpha})^*(\lambda_1)^*\eta_\circ\big\rangle_{\mathrm{dR},Y_1}\\
=&\big\langle e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}),(\lambda_1)^*U_p^{\alpha-1}\eta_\circ\big\rangle_{\mathrm{dR},Y_1}\\
=&\alpha_{\mathsf{f}_\circ}^{\alpha-1}\cdot\big\langle e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}),(\lambda_1)^*\eta_\circ\big\rangle_{\mathrm{dR},Y_1}\\
=&\alpha_{\mathsf{f}_\circ}^{\alpha-1}\frac{\big\langle e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathsf{g}}_{\mathrm{P}})^{[\mathcal{P}]}),\mathsf{f}_\circ^{(p)}\big\rangle}{\big\langle\mathsf{f}_\circ^{(p)},\mathsf{f}_\circ^{(p)}\big\rangle},
\end{aligned}$$

where in the second equality we used the fact that λ_1 intertwines U_p and U_p^*, and in the third one, $U_p^{\alpha-1}\eta_\circ=\alpha_{\mathsf{f}_\circ}^{\alpha-1}\cdot\eta_\circ$. This concludes the proof. □

8.7 An explicit reciprocity law

The main technical result of our work is the following theorem relating the automorphic and motivic p-adic L-functions constructed, which can be seen as an instance of the reciprocity laws:

Theorem 8.7.1. For any $\breve{\mathscr{G}} \in \overline{\mathbf{S}}^{\mathrm{n.o.}}(K_{\diamond,t}; \mathbf{K}_{\mathscr{G}})[\mathscr{G}]$, let

$$\boldsymbol{\zeta}_{\mathscr{G},\mathsf{f}_\circ} := -\alpha_{\mathsf{f}_\circ} \cdot \big(1 - \mathscr{G}(\mathbf{T}(\varpi_{\mathfrak{p}_1})\mathbf{T}(\varpi_{\mathfrak{p}_2}^{-1}))\alpha_{\mathsf{f}_\circ}^{-1}\big) \in \mathbf{I}_{\mathscr{G}}$$

then

$$\boldsymbol{\zeta}_{\mathscr{G},\mathsf{f}_\circ} \cdot \mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}}, \mathsf{f}_\circ) = \mathscr{L}_p^{\mathrm{an}}(\breve{\mathscr{G}}, \mathsf{f}_\circ) \qquad \text{in} \qquad \boldsymbol{\Pi} \otimes_{\boldsymbol{\Lambda}} \mathbf{I}_{\mathscr{G}}.$$

In particular, $\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)$ belongs to $\mathbf{I}_{\mathscr{G}}[\boldsymbol{\zeta}_{\mathscr{G},\mathsf{f}_\circ}^{-1}]$.

Proof. By Corollary 6.1.6 it suffices to show that both elements have the same specializations at all arithmetic points of $\mathbf{I}_{\mathscr{G}}$ of weight 2. Let $\mathrm{P} \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ be an arithmetic point of weight 2 and character $\chi_\diamond$ of conductor p^α. By Proposition 7.1.3,

$$\mathrm{P} \circ \boldsymbol{\mathcal{L}}_{\mathscr{G}}^{\mathsf{f}_\circ} = \Upsilon(\mathrm{P}) \cdot \big(\log_{\mathrm{BK}} \circ \mathrm{P}\big) \qquad \text{for} \qquad \Upsilon(\mathrm{P}) = \big(\alpha_{1,\mathsf{g}_\mathrm{P}}\alpha_{2,\mathsf{g}_\mathrm{P}}\alpha_{\mathsf{f}_\circ}^{-1}\big)^\alpha \cdot G\big(\chi_\diamond^{-1} \cdot \theta_{\mathbb{Q}|D_p}^{-1}\big)^{-1}.$$

By Propositions 6.4.3

$$\begin{aligned}\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)(\mathrm{P}) &= \big\langle \mathrm{P} \circ \boldsymbol{\mathcal{L}}_{\mathscr{G}}^{\mathsf{f}_\circ}\big(\boldsymbol{\kappa}_p^{\mathsf{f}_\circ}(\mathscr{G})\big), (\lambda_\alpha)^*\omega_{\breve{\mathscr{G}}_\mathrm{P}} \otimes \eta_\circ \big\rangle_{\mathrm{dR}} \\ &= \Upsilon(\mathrm{P}) \cdot \big\langle \log_{\mathrm{BK}}\big(\boldsymbol{\kappa}_p^{\mathsf{f}_\circ}(\mathscr{G})(\mathrm{P})\big),\ (\lambda_\alpha)^*\omega_{\breve{\mathscr{G}}_\mathrm{P}} \otimes \eta_\circ \big\rangle_{\mathrm{dR}}.\end{aligned}$$

From the definition of $\kappa_\alpha^{\mathrm{n.o.}\cdot}$ in (4.18) and Theorem 8.6.3 we continue with

$$\begin{aligned}\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}},\mathsf{f}_\circ)(\mathrm{P}) &= \alpha_{1,\mathbf{g}_\mathrm{P}}^{-\alpha}\cdot\Upsilon(\mathrm{P})\cdot\big\langle \mathrm{AJ}_{\mathrm{syn}}(\Delta_\alpha^\circ),(\lambda_\alpha)^*\omega_{\breve{\mathscr{G}}_\mathrm{P}}\otimes\eta_\circ\big\rangle_{\mathrm{dR}}\\ &= \alpha_{1,\mathbf{g}_\mathrm{P}}^{-\alpha}\cdot\Upsilon(\mathrm{P})\cdot\mathrm{AJ}_{\mathrm{syn}}\big((\lambda_\alpha,\mathrm{id})_*\Delta_\alpha^\circ\big)\big(\omega_{\breve{\mathscr{G}}_\mathrm{P}}\otimes\eta_\circ\big)\\ &= \frac{-\alpha_{2,\mathbf{g}_\mathrm{P}}^{-\alpha}\alpha_{\mathsf{f}_\circ}^{-1}\cdot G\big(\chi_\diamond^{-1}\cdot\theta_{\mathbb{Q}|D_p}^{-1}\big)^{-1}}{1-\alpha_{1,\mathbf{g}_\mathrm{P}}\alpha_{2,\mathbf{g}_\mathrm{P}}^{-1}\alpha_{\mathsf{f}_\circ}^{-1}}\cdot\frac{\big\langle e_{\mathrm{ord}}\zeta^*\big(d_2^{-1}(w_{\mathfrak{p}_2^\alpha}^*(\breve{\mathbf{g}}_\mathrm{P})^{[\mathcal{P}]}),\mathsf{f}_\circ^{(p)}\big\rangle}{\big\langle\mathsf{f}_\circ^{(p)},\mathsf{f}_\circ^{(p)}\big\rangle}.\end{aligned}\tag{8.10}$$

Comparing with Corollary 4.6.2, we obtain

$$\begin{aligned}\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}},\mathsf{f}_\circ)(\mathrm{P}) &= -\alpha_{\mathsf{f}_\circ}(1-\alpha_{1,\mathbf{g}_\mathrm{P}}\alpha_{2,\mathbf{g}_\mathrm{P}}^{-1}\alpha_{\mathsf{f}_\circ}^{-1})\cdot\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}},\mathsf{f}_\circ)(\mathrm{P})\\ &= \boldsymbol{\zeta}_{\mathscr{G},\mathsf{f}_\circ}(\mathrm{P})\cdot\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}},\mathsf{f}_\circ)(\mathrm{P}),\qquad\forall\ \mathrm{P}\in\mathcal{A}(\mathbf{I}_\mathscr{G}),\end{aligned}$$

as desired. □

Corollary 8.7.2. For any arithmetic point $\mathrm{P}\in\mathcal{A}(\mathbf{I}_\mathscr{G})$, the automorphic p-adic L-function is non-vanishing at P if and only if the motivic p-adic L-function is non-vanishing at P:

$$\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}},\mathsf{f}_\circ)(\mathrm{P})\neq 0\qquad\Longleftrightarrow\qquad\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}},\mathsf{f}_\circ)(\mathrm{P})\neq 0.$$

Proof. The claim follows from Theorem 8.7.1 and the fact that $\boldsymbol{\zeta}_{\mathscr{G},\mathsf{f}_\circ}(\mathrm{P})\neq 0$ for any arithmetic point $\mathrm{P}\in\mathcal{A}(\mathbf{I}_\mathscr{G})$. □

Chapter 9

On the equivariant BSD-conjecture

9.1 Artin representations and Selmer groups

Let (W, ϱ) be a d-dimensional self-dual Artin representation with coefficients in a number field D. Suppose ϱ factors through the Galois group $\mathrm{Gal}(H/\mathbb{Q})$ of a number field H

$$\begin{array}{ccc} G_{\mathbb{Q}} & \xrightarrow{\quad\varrho\quad} & \mathrm{GL}_d(D) \\ & \searrow \quad \nearrow & \\ & \mathrm{Gal}(H/\mathbb{Q}) & \end{array} .$$

Let $E_{/\mathbb{Q}}$ be a rational elliptic curve, then its algebraic rank with respect to the Artin representation ϱ is defined as

$$r_{\mathrm{alg}}(E, \varrho) = \dim_D E(H)^{\varrho}_D,$$

the dimension of the ϱ-isotypic component $E(H)^{\varrho}_D = \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(\varrho, E(H) \otimes D)$ of the Mordell-Weil group. For p a rational prime and $\wp \mid p$ an O_D-prime ideal, we can

consider the p-adic Galois representations

$$V_\wp(E) = \mathrm{H}^1_{\text{ét}}(E_{\bar{\mathbb{Q}}}, D_\wp(1)), \qquad \mathrm{W}_\wp = \mathrm{W} \otimes_D D_\wp.$$

As the Artin representation ϱ is self-dual, the Kummer map allows us to identify the group $E(H)^\varrho_D$ with a subgroup of the Bloch-Kato Selmer group $\mathrm{H}^1_f(\mathbb{Q}, \mathrm{W}_\wp \otimes V_\wp(E))$. Then, local Tate duality together with the global Poitou-Tate exact sequence can be used to show that global cohomology classes not crystalline at p bound the size of the ϱ-isotypic component of the Mordell-Weil group of $E_{/\mathbb{Q}}$ ([DR17], Section 6.1).

Lemma 9.1.1. Let $\kappa_1, \ldots, \kappa_d \in \mathrm{H}^1(\mathbb{Q}, \mathrm{W}_\wp \otimes V_\wp(E))$ be global cohomology classes with linearly independent images in the singular quotient $\mathrm{H}^1_{\text{sing}}(\mathbb{Q}_p, \mathrm{W}_\wp \otimes V_\wp(E))$ at p. Then the ϱ-isotypic part of $E(H)$ is trivial:

$$r_{\text{alg}}(E, \varrho) = 0.$$

Proof. By Lemma 6.1 of [DR17], the local cohomology group $\mathrm{H}^1(\mathbb{Q}_p, \mathrm{W}_\wp \otimes V_\wp(E))$ is a $2d$-dimensional $D_\wp$-vector space and the local Tate pairing induces a perfect duality of d-dimensional spaces

$$\langle\ ,\rangle : \mathrm{H}^1_f(\mathbb{Q}_p, \mathrm{W}_\wp \otimes V_\wp(E)) \times \mathrm{H}^1_{\text{sing}}(\mathbb{Q}_p, \mathrm{W}_\wp \otimes V_\wp(E)) \longrightarrow D_\wp.$$

The global Poitou-Tate exact sequence implies that the image of the localization at p

$$\mathrm{loc}_p : \mathrm{H}^1(\mathbb{Q}, \mathrm{W}_\wp \otimes V_\wp(E)) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathrm{W}_\wp \otimes V_\wp(E))$$

is d-dimensional ([DR17], Lemma 6.2). Therefore, the existence of global cohomology classes $\kappa_1, \ldots, \kappa_d$ whose localizations generate the singular quotient at p implies that

the restriction of loc_p to the Bloch-Kato Selmer group $\mathrm{H}^1_f(\mathbb{Q}, V_\wp(E)\otimes \mathrm{W}_\wp)$ is the zero map. The commutativity of the diagram

$$\begin{array}{ccc} E(H)^\varrho_D & \longrightarrow & \oplus_{\mathfrak{p}|p}\mathrm{Hom}_{G(H_\mathfrak{p}/\mathbb{Q}_p)}(\mathrm{W}_\wp, E(H_\mathfrak{p})\otimes D_\wp) \\ \downarrow & & \downarrow \\ \mathrm{H}^1_f(\mathbb{Q}, \mathrm{W}_\wp\otimes V_\wp(E)) & \xrightarrow{0} & \mathrm{H}^1(\mathbb{Q}_p, \mathrm{W}_\wp\otimes V_\wp(E)) \end{array}$$

and the injectivity of the vertical Kummer maps imply the triviality of the top horizontal morphism. Since $E(H)\otimes D \hookrightarrow E(H_\mathfrak{p})\otimes D_\wp$ is injective for all O_H-prime ideal $\mathfrak{p} \mid p$, we deduce that $\dim_D E(H)^\varrho_D = 0$. □

9.2 Existence of prime p

The goal of this last section is to apply the idea of p-adic deformation to the setting where the self-dual Artin representation is 4-dimensional and arises as the tensor induction

$$\mathrm{As}(\varrho) = \otimes\text{-}\mathrm{Ind}_L^{\mathbb{Q}}(\varrho)$$

of a totally odd, irreducible two-dimensional Artin representation $\varrho : G_L \to \mathrm{GL}_2(D)$ of the absolute Galois group of a real quadratic field L. We suppose that ϱ has conductor $\mathfrak{Q}$ and that the tensor induction of the determinant $\det(\varrho)$ is the trivial character. Let $E_{/\mathbb{Q}}$ be a rational elliptic curve of conductor N, and for any rational prime p, we consider the Kummer self-dual p-adic Galois representation of $G_\mathbb{Q}$

$$\mathrm{V}_{\varrho,E} = \mathrm{As}(\varrho)\otimes \mathrm{V}_p(E).$$

By modularity ([Wil95],[TW95], [PS16]) there is a primitive Hilbert cuspform $\mathtt{g}_\varrho \in S_{t_L,t_L}(\mathfrak{Q}; D)$ of parallel weight one, and a primitive elliptic cuspform $\mathtt{f}_E \in S_{2,1}(N;\mathbb{Q})$

of weight 2 associated to ϱ and E respectively. The twisted L-function $L\big(E, \mathrm{As}(\varrho), s\big)$ has meromorphic continuation to $\mathbb{C}$ and a functional equation centered at $s = 1$, at which the L-function is holomorphic.

Before we delve into the arithmetic results, let us first show that we can pick a prime p satisfying all assumptions. Recall that up till now we have the following assumptions on p in proving the explicit reciprocity law:

Assumption 9.2.1. Suppose p is a rational prime satisfying the following:

- $\mathsf{f}_\circ$ is ordinary and non-Eisenstein at p;
- $p\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$ splits into the product of two narrowly principal ideals, and
- there are no totally positive units of L congruent to -1 modulo p.

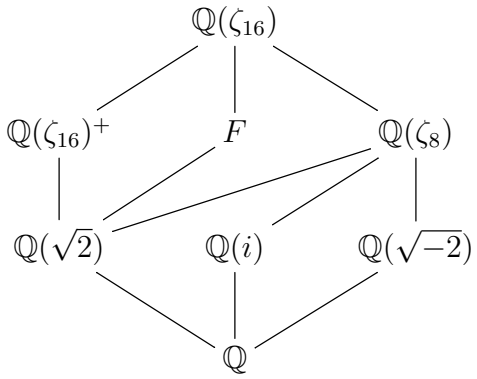
We first show that (for any given real quadratic field L) the set of prime p satisfying the last two conditions has positive density.

First of all, we record the following lemma regarding narrow class fields of real quadratic fields, whose proof follows immediately from some elementary and explicit calculations and is thus omitted:

Lemma 9.2.2. Let $L = \mathbb{Q}(\sqrt{d})$ be a real quadratic field, and let H_L^+ denote the narrow class field of L. Then

- if $d \equiv 1 \pmod 4$ then $H_L^+ \cap \mathbb{Q}(\zeta_8) = \mathbb{Q}$;
- if $d \equiv 3 \pmod 4$ then $\mathbb{Q}(i) \subseteq H_L^+$ and $H_L^+ \cap \mathbb{Q}(\sqrt{\pm 2}) = \mathbb{Q}$;
- if $d \equiv 6 \pmod 8$ then $H_L^+ \cap \mathbb{Q}(i) = \mathbb{Q}$, $H_L^+ \cap \mathbb{Q}(\sqrt{2}) = \mathbb{Q}$ and $\mathbb{Q}(\sqrt{-2}) \subseteq H_L^+$;
- if $d \equiv 2 \pmod 8$ then $H_L^+ \cap \mathbb{Q}(i) = \mathbb{Q}$, $H_L^+ \cap \mathbb{Q}(\sqrt{-2}) = \mathbb{Q}$ and $\mathbb{Q}(\sqrt{2}) \subseteq H_L^+$.

Recall we have the lattice of subfields of $\mathbb{Q}(\zeta_{16})$



(9.1)

where $F/\mathbb{Q}$ is the splitting field of the polynomial $X^4 + 4X^2 + 2$.

Proposition 9.2.3. Let $L = \mathbb{Q}(\sqrt{d})$ be a real quadratic field, then the primes $p \equiv 9 \pmod{16}$ that also split in L into narrowly principal factors have positive density.

Proof. It follows directly from Lemma 9.2.2 that:

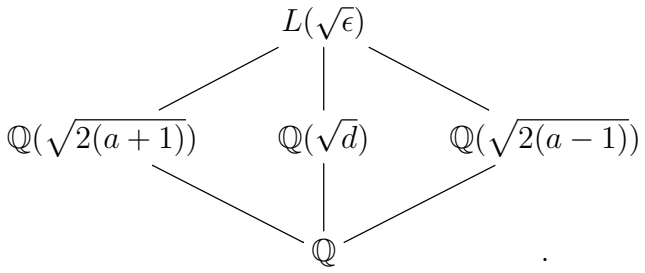
- if $d \equiv 1 \pmod 4$ then $\mathbb{Q}(\zeta_{16}) \cap H_L^+ = \mathbb{Q}$;
- if $d \equiv 3 \pmod 4$ then $\mathbb{Q}(\zeta_{16}) \cap H_L^+ = \mathbb{Q}(i)$;
- if $d \equiv 6 \pmod 8$ then $\mathbb{Q}(\zeta_{16}) \cap H_L^+ = \mathbb{Q}(\sqrt{-2})$.

When $d \equiv 2 \pmod 8$ we claim that $\mathbb{Q}(\zeta_{16}) \cap H_L^+ = \mathbb{Q}(\sqrt{2})$. Indeed, in this case the intersection could be either $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\zeta_{16})^+$ or F and we show that the latter two options cannot occur. Let A denote either $\mathbb{Q}(\zeta_{16})^+$ or F. When $d = 2$ then $\mathbb{Q}(\zeta_{16}) \cap H_L^+ = \mathbb{Q}(\sqrt{2})$ because $A/\mathbb{Q}(\sqrt{2})$ is ramified at 2. When $d \neq 2$, then $L(\sqrt{2})/\mathbb{Q}(\sqrt{2})$ is a proper extension unramified at 2. It follows that $L(\sqrt{2}) \cdot A/L(\sqrt{2})$ is ramified at 2 and cannot be contained in H_L^+.

Since the rational primes $p \equiv 9 \pmod{16}$ are precisely those totally split in $\mathbb{Q}(\zeta_8)$ and inert in the extension $\mathbb{Q}(\zeta_8) \subseteq \mathbb{Q}(\zeta_{16})$, the analysis above of the intersection $\mathbb{Q}(\zeta_{16}) \cap H_L^+$ together with Chebotarev's density theorem finishes the proof. □

Let $\epsilon \in \mathcal{O}_{L,+}^{\times}$ be a generator of the group of totally positive units and p a rational prime split in L. Then the requirement that there is no totally positive unit congruent to -1 modulo p is satisfied when the subgroup $\langle \bar{\epsilon} \rangle$ of $(\mathcal{O}_L/p)^{\times}$, generated by the reduction of ϵ, has odd order.

Lemma 9.2.4. Let ϵ and L be as above, then the totally real number field $L(\sqrt{\epsilon})$ is either equal to L or it is biquadratic over $\mathbb{Q}$. In the latter case, write $\epsilon = a + b\sqrt{d}$ for $a, b \in \mathbb{Z}^+$, then the subfields of $L(\sqrt{\epsilon})$ are given by



Proof. If the fundamental unit of L is not totally positive, then ϵ is a square in L and $L(\sqrt{\epsilon}) = L$. If ϵ is the fundamental unit, then the number field $L(\sqrt{\epsilon})$ is the splitting field of the polynomial

$$X^4 - \mathrm{Tr}_{L/\mathbb{Q}}(\epsilon)X^2 + 1 = (X^2 - \epsilon)(X^2 - 1/\epsilon),$$

hence it is biquadratic over $\mathbb{Q}$ and totally real. Using the relation $\mathrm{N}_{L/\mathbb{Q}}(\epsilon) = 1$ one sees that

$$\left(\sqrt{\frac{a+1}{2}} + \sqrt{\frac{a-1}{2}}\right)^2 = \epsilon$$

and the claim follows. □

Remark. The number field $L(\sqrt[8]{\epsilon})$ is not Galois over $\mathbb{Q}$. Its Galois closure is obtained by adding an 8-th root of unity. Indeed $J = L(\sqrt[8]{\epsilon}, \zeta_8)$ is the splitting field of the

polynomial

$$X^{16} - \mathrm{Tr}_{L/\mathbb{Q}}(\epsilon)X^8 + 1 = (X^8 - \epsilon)(X^8 - 1/\epsilon).$$

It is clear from this description that $J/\mathbb{Q}$ is a solvable extension.

Lemma 9.2.5. Let ϵ, L and J be as above. Then for all but finitely many primes $p \equiv 9 \pmod{16}$ that are also totally split in J, there is no totally positive unit in L congruent to -1 modulo p.

Proof. Suppose that $p \equiv 9 \pmod{16}$ and totally split in J. If $\mathfrak{p}$ is an $\mathcal{O}_L$-prime ideal above p then $(\mathcal{O}_L/\mathfrak{p})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times$ and for all but finitely many such primes the reduction $\bar{\epsilon}$ of ϵ modulo $\mathfrak{p}$ is an 8-th power. It follows that $\bar{\epsilon}$ generates a subgroup of order dividing $(p-1)/8$. Since $p \equiv 9 \pmod{16}$ this order is odd and the subgroup cannot contain -1. □

Corollary 9.2.6. Let L be a real quadratic field, then the set of primes p split in L with narrowly principal factors and such that there is no totally positive unit congruent to -1 mod p has positive density.

Proof. By Proposition 9.2.3 and Lemma 9.2.5, all the primes which are totally split in J, H_L^+, $\mathbb{Q}(\zeta_8)$ and inert in the extension $\mathbb{Q}(\zeta_8) \subseteq \mathbb{Q}(\zeta_{16})$ satisfy the requirements. Clearly the splitting conditions for J and H_L^+ are compatible, and from Proposition 9.2.3 we know that also the splitting conditions for H_L^+ and $\mathbb{Q}(\zeta_{16})$ are compatible too. We are left to understand $J \cap \mathbb{Q}(\zeta_{16})$. Clearly $\mathbb{Q}(\zeta_8)$ is contained in the intersection because $J = L(\sqrt[8]{\epsilon}, \zeta_8)$. One can check that

$$[J : \mathbb{Q}] = \begin{cases} 16 \cdot 2 & \text{if } \mathbb{Q}(\sqrt{2}) \subseteq L(\sqrt{\epsilon}) \\ 16 \cdot 4 & \text{if } \mathbb{Q}(\sqrt{2}) \not\subseteq L(\sqrt{\epsilon}), \end{cases}$$

$[L(\sqrt[4]{\epsilon}, i) : \mathbb{Q}] = 16$, and

$$L(\sqrt[4]{\epsilon}, i) \cap \mathbb{Q}(\zeta_{16}) = \begin{cases} \mathbb{Q}(\zeta_8) & \text{if } \mathbb{Q}(\sqrt{2}) \subseteq L(\sqrt{\epsilon}) \\ \mathbb{Q}(i) & \text{if } \mathbb{Q}(\sqrt{2}) \not\subseteq L(\sqrt{\epsilon}). \end{cases}$$

Suppose by contradiction that $\mathbb{Q}(\zeta_{16}) \subseteq J$. Then $J = \mathbb{Q}(\zeta_{16}) \cdot L(\sqrt[4]{\epsilon}, i)$ because $\mathbb{Q}(\zeta_{16}) \cdot L(\sqrt[4]{\epsilon}, i)$ is a subfield of the same degree as J. Therefore the natural injection

$$\mathrm{Gal}(J/\mathbb{Q}) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{16})/\mathbb{Q}) \times \mathrm{Gal}(L(\sqrt[4]{\epsilon}, i)/\mathbb{Q})$$

produces a contradiction because $\mathrm{Gal}(J/\mathbb{Q})$ contains an element of order 8 while the other two Galois groups have exponent 4. In summary, we showed that $J \cap \mathbb{Q}(\zeta_{16}) = \mathbb{Q}(\zeta_8)$ so that all the required splitting conditions are compatible. Chebotarev's density theorem finishes the proof. □

Proposition 9.2.7. Let $K/\mathbb{Q}$ be an S_5-quintic extension whose Galois closure $\widetilde{K}/\mathbb{Q}$ contains a real quadratic field L. Suppose $E_{/\mathbb{Q}}$ is a rational elliptic curve, then there are infinitely many ordinary primes p for $E_{/\mathbb{Q}}$ such that

- p splits in L into narrowly principal factors;
- there is no totally positive unit in L congruent to -1 modulo p;
- the conjugacy class of Fr_p in $G(\widetilde{K}/\mathbb{Q}) \cong S_5$ is that of 5-cycles.

Proof. Since $\widetilde{K}/L$ is a non-solvable extension, we deduce that $\widetilde{K} \cap J = L$ and $\widetilde{K} \cap H_L^+ = L$. Moreover, $\widetilde{K} \cap \mathbb{Q}(\zeta_{16})$ is either $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}$ according to whether $L = \mathbb{Q}(\sqrt{2})$ or not. Given that 5-cycles are in the kernel of the surjection $\mathrm{Gal}(\widetilde{K}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(L/\mathbb{Q})$, one can prove the existence of a set of positive density consisting of rational primes satisfying the listed conditions as in Corollary 9.2.6. It then remains to show

that infinitely many of such primes are of good ordinary reduction for the given elliptic curve. When $E_{/\mathbb{Q}}$ does not have complex multiplication, the ordinary primes have density one so there are infinitely many ordinary primes that satisfy the listed conditions. When the elliptic curves $E_{/\mathbb{Q}}$ has complex multiplication by a quadratic imaginary field B, a prime is ordinary for $E_{/\mathbb{Q}}$ if it splits in B. As this new splitting requirement is compatible with those coming from the conditions above, Chebotarev's density theorem gives the claim. □

9.3 Proof of main theorems

Theorem 9.3.1. Let p be a rational prime satisfying Assumption 9.2.1 such that the eigenvalues of Frob_p on $\mathrm{As}(\varrho)$ are all distinct modulo p. If $\mathsf{g}_\circ^{(p)}$ is any ordinary p-stabilization of $\mathsf{g}_\circ$, then the special L-value $L\big(E, \mathrm{As}(\varrho), 1\big)$ does not vanish if and only if the global cohomology class $\kappa(\mathsf{g}_\circ^{(p)}, \mathsf{f}_\circ)$ is not crystalline at p.

Proof. Corollary 3.3.2 shows that the non-vanishing of $L\big(E, \mathrm{As}(\varrho), 1\big)$ is equivalent to the non-vanishing of the automorphic p-adic L-function $\mathscr{L}_p^{\mathrm{aut}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)$ at the weight 1 arithmetic point $\mathrm{P}_\circ \in \mathcal{A}(\mathbf{I}_{\mathscr{G}})$ corresponding to $\mathsf{g}_\circ^{(p)}$, which in turn is equivalent to $\mathscr{L}_p^{\mathrm{mot}}(\breve{\mathscr{G}}, \mathsf{f}_\circ)(\mathrm{P}_\circ) \neq 0$ by Corollary 8.7.2. Unwinding the definitions in Proposition 7.1.3, this last statement holds if and only if $\boldsymbol{\kappa}_p^{\mathsf{f}_\circ}(\mathscr{G})(\mathrm{P}_\circ) \neq 0$, which, in light of Proposition 5.4.7, is the same as $\partial_p\big(\kappa_p(\mathsf{g}_\circ^{(p)}, \mathsf{f}_\circ)\big) \neq 0$. □

Corollary 9.3.2. Let p be as above. If $\mathsf{g}_\varrho^{(p)}$ is any ordinary p-stabilization of g_ϱ, then

$$L\big(E, \mathrm{As}(\varrho), 1\big) \neq 0 \qquad \Longleftrightarrow \qquad \partial_p\big(\kappa(\mathsf{g}_\varrho^{(p)}, \mathsf{f}_E)\big) \neq 0 \ \in \ \mathrm{H}^1_{\mathrm{sing}}\big(\mathbb{Q}_p, \mathrm{V}_{\varrho,E}\big)$$

Proof. This is a special case of Theorem 9.3.1. □

Theorem 9.3.3. Suppose that N is coprime to $\mathfrak{Q}$ and split in L, and p be as above. Then

$$r_{\mathrm{an}}\big(E, \mathrm{As}(\varrho)\big) = 0 \quad \Longrightarrow \quad r_{\mathrm{alg}}\big(E, \mathrm{As}(\varrho)\big) = 0.$$

Proof. As the eigenvalues of Frob_p on $\mathrm{As}(\varrho)$ are all distinct modulo p, the cuspform g_ϱ has 4 distinct p-stabilizations

$$\mathsf{g}_\varrho^{(\alpha_1\alpha_2)}, \quad \mathsf{g}_\varrho^{(\alpha_1\beta_2)}, \quad \mathsf{g}_\varrho^{(\beta_1\alpha_2)} \quad \text{and} \quad \mathsf{g}_\varrho^{(\beta_1\beta_2)}.$$

Therefore if $r_{\mathrm{an}}\big(E, \mathrm{As}(\varrho)\big) = 0$, by applying Corollary 9.3.2 to each of these four p-stabilization, we obtain four global cohomology classes

$$\kappa\big(\mathsf{g}_\varrho^{(\alpha_1\alpha_2)}, \mathsf{f}_E\big), \quad \kappa\big(\mathsf{g}_\varrho^{(\alpha_1\beta_2)}, \mathsf{f}_E\big), \quad \kappa\big(\mathsf{g}_\varrho^{(\beta_1\alpha_2)}, \mathsf{f}_E\big), \quad \kappa\big(\mathsf{g}_\varrho^{(\beta_1\beta_2)}, \mathsf{f}_E\big) \quad \in \mathrm{H}^1\big(\mathbb{Q}, \mathrm{V}_{\varrho,E}\big)$$

which are not crystalline at p. Given the assumption on the Frobenius eigenvalues of $\mathrm{As}(\varrho)$, Proposition 5.4.7 implies that the images of the global classes are linearly independent in the singular quotient $\mathrm{H}^1_{\mathrm{sing}}\big(\mathbb{Q}, \mathrm{V}_{\varrho,E}\big)$. The result follows by invoking Lemma 9.1.1. □

Corollary 9.3.4. Let $K/\mathbb{Q}$ be a non-totally real S_5-quintic extension of positive discriminant, such that it contains a real quadratic field L whose fundamental unit has norm -1. Suppose N is odd, unramified in $K/\mathbb{Q}$ and split in L, then

$$r_{\mathrm{an}}(E/K) = r_{\mathrm{an}}(E/\mathbb{Q}) \quad \Longrightarrow \quad r_{\mathrm{alg}}(E/K) = r_{\mathrm{alg}}(E/\mathbb{Q}).$$

Proof. By Corollary 4.2 of [For18] there exists a parallel weight one Hilbert eigenform

$\mathfrak{g}_K$ over L of level $\mathfrak{Q}$ prime to N such that $\mathrm{As}(\varrho_{\mathfrak{g}_K}) \cong \mathrm{Ind}_K^{\mathbb{Q}} \mathbb{1} - \mathbb{1}$. Then

$$r_{\mathrm{an}}\big(E, \mathrm{As}(\varrho_{\mathfrak{g}_K})\big) = r_{\mathrm{an}}(E/K) - r_{\mathrm{an}}(E/\mathbb{Q})$$

$$r_{\mathrm{alg}}\big(E, \mathrm{As}(\varrho_{\mathfrak{g}_K})\big) = r_{\mathrm{alg}}(E/K) - r_{\mathrm{alg}}(E/\mathbb{Q}).$$

Let $\widetilde{K}/\mathbb{Q}$ denote the Galois closure of K satisfying $G(\widetilde{K}/\mathbb{Q}) \cong S_5$, and write L for the discriminant field of $\widetilde{K}$. Thanks to Proposition 9.2.7 we can choose a rational prime $p \nmid 5N\mathfrak{Q}$ split in L such that its Frobenius conjugacy class in $G(\widetilde{K}/\mathbb{Q})$ is that of 5-cycles and satisfies Assumption 9.2.1. Then the eigenvalues of Frob_p on $\mathrm{As}(\varrho_{\mathfrak{g}_K})$ are all distinct modulo p by Proposition 5.3.9 and the corollary follows by applying Theorem 9.3.3.

□

Part II

Local computations for a $\mathrm{GSp}_4 \times \mathrm{GL}_2$ Euler system

Chapter 10

Introduction

10.1 Recent progress in Euler systems

The theory of Euler systems is one of the most powerful tools available for studying the arithmetic of Galois representations, and especially for controlling Selmer groups. It was first introduced by Kolyvagin [Kol90], inspired by works of Thaine [Tha88] and his own [Kol88], and later formalized by Rubin [Rub00]. However, the construction of Euler systems is a difficult problem, and there are relatively few known examples of Euler systems.

A common strategy for constructing Euler systems is pushing forward cohomology classes on certain special cycles of Shimura varieties of varying levels. The prototypical example is Kolyvagin's Euler system of Heegner points [Kol88], which uses codimension one cycles (Heegner points) in modular curves, although one should note that this is not exactly an Euler system in the sense of [Rub00], but rather a so-called "anticyclotomic" Euler system. Another related construction is Kato's Euler system [Kat04], which utilizes the cup product of Siegel units on modular curves.

Recently a series of examples have been constructed using this technique, includ-

ing the work of [LLZ14] (embedding of Shimura varieties corresponding to $\mathrm{GL}_2 \hookrightarrow \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$), [LLZ18] ($\mathrm{GL}_2 \hookrightarrow \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ with F a real quadratic field), and [LSZ17] ($\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \hookrightarrow \mathrm{GSp}_4$). The common starting point for all these recent constructions is Siegel units, realized in the appropriate motivic cohomology of (products of) modular curves. Pushing forward these cohomology classes via the corresponding closed embedding of Shimura varieties one gets motivic cohomology classes in the target Shimura variety. The embeddings are suitably "twisted" in order to get the desired levels in the Shimura variety, which can be then realized as one base Shimura variety base-changed to various cyclotomic extensions of $\mathbb{Q}$. One then considers the étale realization of such motivic cohomology classes to find elements in étale cohomology. Finally, an application of the Hochschild-Serre spectral sequence yields Galois cohomology classes satisfying the desired norm relations.

In [LLZ14] and [LLZ18], the proof of the norm relations involves explicit yet laborious double coset computations. In [LSZ17], a novel approach using a multiplicity-one argument in local representation theory is adopted, bypassing the complicated calculations on GSp_4 that would have been practically intractable. More specifically, Loeffler, Skinner and Zerbes use methods of smooth representation theory to reduce the norm-compatibility statement to a far easier, purely local statement involving Bessel models of unramified representations of $\mathrm{GSp}_4(\mathbb{Q}_l)$. This reduction is possible thanks to a case of the local Gan-Gross-Prasad conjecture due to Kato, Murase and Sugano [KMS03], showing that the space of $\mathrm{SO}_4(\mathbb{Q}_l)$-invariant linear functionals on an irreducible spherical representation of $\mathrm{SO}_4(\mathbb{Q}_l) \times \mathrm{SO}_5(\mathbb{Q}_l)$ has dimension as most one. This technique promises to be applicable in many other settings where local multiplicity one results of this type are known. In [Gro18], the norm relations for the Euler system of Hilbert modular surfaces as first constructed in [LLZ18] is improved and reproved using this type of multiplicity one arguments.

Exactly the same multiplicity one result can also be applied in the setting of the natural embedding of $\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ into $\mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, as suggested in [LSZ17] and [LZ18]. In this part of the thesis, we set up the analogous local representation tools and establish the key technical results that will lead to norm compatibility relations for an Euler system for $\mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, which is a joint work in progress with Chi-Yun Hsu and Ryotaro Sakamoto.

10.2 Outline of the proof and main results

We now give a brief overview of this part of the thesis. The main goal is to establish results in the setting of $\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \hookrightarrow \mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ that are analogous to those in Section 3 of [LSZ17]. In particular, we would like analogues of Proposition 3.10.4 of *op.cit.*, which will lead to norm relations in the vertical (i.e., p-) direction, and of Corollary 3.10.5, which will lead to tame norm relations. The latter one is the most crucial, as the vertical norm relations can still be proved relatively easily by explicit geometric arguments, but a similar argument for tame norm relations would be extremely tedious.

Construction of Euler system classes

To provide a little bit more of context, we first give a very brief description of the construction of the proposed Euler system for $\mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, which follows very closely the construction in [LSZ17], and therefore our account will also closely follow the introduction there.

Let $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ and $G = \mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$. We will construct a $G(\mathbb{A}_{\mathbb{Q},f})$-

equivariant map, the Lemma-Eisenstein map,

$$\mathcal{LE} : I \otimes_{\mathcal{H}(H(\mathbb{A}_{\mathbb{Q},f}))} \mathcal{H}(G(\mathbb{A}_{\mathbb{Q},f})) \longrightarrow \varinjlim_{U} \mathrm{H}^5_{\mathrm{mot}}(Y_G(U), \mathscr{D})$$

where I is a certain explicit representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ regarded as an $H(\mathbb{A}_{\mathbb{Q},f})$-representation via the projection to its first factor $H \twoheadrightarrow \mathrm{GL}_2$, $\mathcal{H}(-)$ denotes the Hecke algebra, U runs over all open compact subgroups of $G(\mathbb{A}_{\mathbb{Q},f})$, $Y_G(U)$ is the Shimura variety for G of level U, $\mathscr{D}$ is some relative Chow motive over Y_G arising from algebraic representations of G. Let K be a level subgroup of $G(\mathbb{A}_{\mathbb{Q},f})$, unramified outside p and a finite set S of primes and with a certain special form at p. For any integer n coprime to S, the base-extension variety $Y_G(K) \times_{\mathrm{Spec}\, Q} \mathrm{Spec}\, \mathbb{Q}(\mu_n)$ is again a Shimura variety for G of some level $K_n \subset K$. Combining this identification with some explicit choices of test data as input to $\mathcal{LE}$, we can define the motivic Euler system classes

$$z_{M,m} \in \mathrm{H}^5_{\mathrm{mot}}(Y_G(K) \times_{\mathrm{Spec}\, Q} \mathrm{Spec}\, \mathbb{Q}(\mu_{Mp^m}), \mathscr{D}),$$

for $m \geq 0$ and M coprime to pS.

To get an Euler system associated to a Galois representation in the usual sense, we choose $\Pi = \sigma \otimes \tau$ an automorphic representation of G satisfying certain conditions, where σ (respectively, τ) is an automorphic representation of GSp_4 (respectively, GL_2). Let W_Π be the p-adic Galois representation associated to Π, and W_Π^* be its dual representation. Then $\Pi_f^* \otimes W_\Pi^*$ appears with multiplicity 1 as a direct summand of $\varinjlim_U \mathrm{H}^4_{\text{ét}}(Y_G(U)_{\bar{\mathbb{Q}}}, \mathscr{D})$. Choosing a vector $\varphi \in \Pi_f$ thus results in a homomorphism of Galois representations $\Pi_f^* \otimes W_\Pi^* \to W_\Pi^*$, which factors through $(\Pi_f^*)^K \otimes W_\Pi^*$ if φ is K-invariant. This, combined with the Hochschild-Serre spectral sequence gives a map

$$\mathrm{H}^5_{\text{ét}}(Y_G(K) \times_{\mathrm{Spec}\, Q} \mathrm{Spec}\, \mathbb{Q}(\mu_{Mp^m}), \mathscr{D}) \longrightarrow \mathrm{H}^1(\mathbb{Q}(\mu_{Mp^m}), W_\Pi^*).$$

The images of the étale realizations of $z_{M,m}$ give a collection of cohomology classes $z^{\Pi}_{M,m} \in \mathrm{H}^1(\mathbb{Q}(\mu_{Mp^m}), W^*_{\Pi})$, and we will prove that they satisfy the norm-compatibility relations of an Euler system.

Norm relations

We will focus on the tame norm relations. The vertical norm relations can be dealt with in a similar fashion and are easier. Proving the norm relation amounts to comparing $z^{\Pi}_{M,m}$ and $\mathrm{norm}(z^{\Pi}_{lM,m})$, where l is a prime away from Mp. We have constructed a $G(\mathbb{A}_{\mathbb{Q},f})$-equivariant bilinear pairing

$$\left(I \otimes_{\mathcal{H}(H(\mathbb{A}_{\mathbb{Q},f}))} \mathcal{H}(G(\mathbb{A}_{\mathbb{Q},f}))\right) \otimes \Pi_f \longrightarrow \mathrm{H}^1(\mathbb{Q}(\mu_{Mp^m}), W^*_{\Pi}),$$

which, via Frobenius reciprocity (for more details see Section 13.2 below), corresponds to an $H(\mathbb{A}_{\mathbb{Q},f})$-equivariant pairing

$$\mathfrak{z} : I \otimes \Pi_f \longrightarrow \mathrm{H}^1(\mathbb{Q}(\mu_{Mp^m}), W^*_{\Pi}).$$

Now the classes $z^{\Pi}_{M,m}$ and $\mathrm{norm}(z^{\Pi}_{lM,m})$ are given by the values of $\mathfrak{Z}$ with different choices of test data $v_M, v_{lM} \in I \otimes \Pi_f$. In our case the representation I will be a direct sum of principal series τ for GL_2, and our choice of the test data is such that v_{lM} and v_M coincide at all places except for l. Moreover, the target of $\mathfrak{z}$ has trivial $H(\mathbb{A}_{\mathbb{Q},f})$-action, it suffices to replace it with the 1 dimensional trivial representation $\mathbb{C}$. Therefore we are in the situation of comparing the values of different local test data under the $H(\mathbb{Q}_l)$-equivariant pairing

$$\mathfrak{z}_l : \tau_l \otimes \Pi_l \longrightarrow \mathbb{C}. \tag{10.1}$$

By a known case of the Gan-Gross-Prasad conjecture due to [KMS03], the space of such $H(\mathbb{Q}_l)$-equivariant maps is 1-dimensional (the "multiplicity one" input), and we may construct an explicit basis element $\mathfrak{z}_{\chi,\psi}$ using zeta integrals (Proposition 13.1.3). Thus the desired norm relations are reduced to local statements involving our explicit pairing.

Discussion of main results and outline

In the main results of this part of the thesis (Corollary 13.3.3 and Theorem 13.4.1), we find the appropriate local test data that lead to norm-compatibility and establish the corresponding local relations. These will constitute the foundation and technical core of the anticipated Euler system: not only does the proof of the norm compatibility relations rely heavily on these local results, the local data we constructed so that we have the correct form of relations also essentially dictates how the Euler system should be constructed.

The strategy for constructing the local input data to the pairing (10.1) is as follows. The test vector in τ_l will be some form of Siegel sections, which is fairly standard. For the test vector in Π_l, we will look at a linear combination of the images of the normalized spherical vector φ_0 in Π_l under various Hecke operators. The intuition behind looking at such linear combinations is that the normalized zeta integral (12.2) of φ_0 at s is, up to a constant factor of $L(s, \sigma\otimes\tau)^{-1}$, a double sum $\sum_{i\geq 0}\sum_{j\geq 0}$ of the Whittaker function of φ_0 at certain values (for a precise statement see (12.3)), and (in an oversimplified situation) hitting φ_0 with the Klingen, Siegel, or both, U_l-operator will result in double sums $\sum_{i\geq 0}\sum_{j\geq 1}$, $\sum_{i\geq 1}\sum_{j\geq 1}$, or $\sum_{i\geq 1}\sum_{j\geq 1}$, of the same summands as before. Thus a linear combination $\sum_{i\geq 0}\sum_{j\geq 0} - \sum_{i\geq 0}\sum_{j\geq 1} - \sum_{i\geq 1}\sum_{j\geq 0} + \sum_{i\geq 1}\sum_{j\geq 1}$ will leave us with precisely the single term corresponding to $i = j = 0$, which in our case is 1, which, after taking the normalizing factor $L(s, \sigma \otimes \tau)^{-1}$ into consideration,

gives us exactly what we want. In reality, because of the way our explicit bilinear form $\mathfrak{z}_{\chi,\psi}$ is constructed, we can only use Hecke operators in the Klingen bi-invariant Hecke algebra to simplify the calculation (this is one of the main technical differences compared to the situation of [LSZ17]; see the end of the proof of Theorem 13.1.5), and therefore there will be extra terms showing up, forcing us to use more Hecke operators to compensate for them.

Finally we give a brief overview of all the chapters. Chapter 2 is preliminary in nature, where we collect the standard tools in the theory of local representations of GL_2 and GSp_4. Chapter 3 is a preparation for constructing the desired local data and proving the main results. We give explicit presentations of the Hecke operators we need, and compute their actions on the local zeta integral (Proposition 12.3.2), thus coming up with the desired linear combination (Corollary 12.3.3), which is the technical heart of this part of the thesis. Finally in Chapter 4, we put everything together to deduce the main results.

10.3 Notations and conventions

Throughout this part of the thesis, we will use the following notations.

- Let J be the skew-symmetric 4×4 matrix over $\mathbb{Z}$ given by $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$.

  We let GSp_4 be the group scheme over $\mathbb{Z}$ defined by

  $$\mathrm{GSp}_4(R) := \{(g, \mu) \in \mathrm{GL}_4(R) \times \mathrm{GL}_1(R) \mid g^{\mathrm{T}} \cdot J \cdot g = \mu J\}$$

  for any commutative ring R. We write $\mu \colon \mathrm{GSp}_4 \to \mathrm{GL}_1$ for the symplectic

multiplier map.

- Put $G := \mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ and $H := \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, with $\mathrm{GSp}_4 \to \mathrm{GL}_1$ the symplectic multiplier map μ and $\mathrm{GL}_2 \to \mathrm{GL}_1$ the determinant map. We also denote by μ the obvious induced maps $G \to \mathrm{GL}_1$ and $H \to \mathrm{GL}_1$.

- Let ι denote the embedding $H \hookrightarrow G$ given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) \mapsto \left(\begin{pmatrix} a & & & b \\ & a' & b' & \\ & c' & d' & \\ c & & & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right).$$

- For any prime number l, the Klingen congruence subgroup of level l^n is the subgroup of $\mathrm{GSp}_4(\mathbb{Z}_l)$ defined by

$$\mathrm{Kl}(l^n) = \{g \in \mathrm{GSp}_4(\mathbb{Z}_l) : g \equiv \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix} \mod l^n\},$$

and the Borel congruence subgroup of level l^n is the subgroup $B(l^n)$ of $\mathrm{GSp}_4(\mathbb{Z}_l)$ consisting of matrices which are upper-triangular modulo l^n.

- For integers $m, n \geq 0$, we define the following compact open subgroups of $G(\mathbb{Q}_l)$:

$$K_{m,n} := \{g \in G(\mathbb{Q}_\ell) : g \equiv (\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix}, \begin{pmatrix} * & * \\ * & * \end{pmatrix}) \bmod \ell^n, \mu(g) \equiv 1 \bmod \ell^m\}$$

$$B_{m,n} := \{g \in G(\mathbb{Q}_\ell) : g \equiv (\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & 1 \end{pmatrix}, \begin{pmatrix} * & * \\ & * \end{pmatrix}) \bmod \ell^n, \mu(g) \equiv 1 \bmod l^m\}.$$

Chapter 11

Local representation theory

We first review the basic theory and tools of local representations of GL_2. The basic reference is Chapter 4 of [Bum97]. Chapter 3 of [LSZ17] and Chapter 1 of [Gro18] also have excellent summaries of the materials, and we follow their accounts closely.

11.1 Principal series of $\mathrm{GL}_2(\mathbb{Q}_l)$

Let $|.|$ denote the standard l-adic norm on $\mathbb{Q}_l$ normalized such that $|l| = 1/l$. For χ a smooth character of $\mathbb{Q}_l^\times$, we write $L(\chi, s)$ for the local L-factor defined as

$$L(\chi, s) = L(\chi|.|, 0) = \begin{cases} (1 - \chi(l)l^{-s})^{-1} & \text{if } \chi|_{\mathbb{Z}_l^\times} = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 11.1.1. Given two smooth characters χ and ψ of $\mathbb{Q}_l^\times$, let $I(\chi, \psi)$ be the space of smooth functions $f : \mathrm{GL}_2(\mathbb{Q}_l) \to \mathbb{C}$ such that

$$f(\begin{pmatrix} a & b \\ & d \end{pmatrix} g) = \chi(a)\psi(d)|a/d|^{1/2} f(g),$$

equipped with a $\mathrm{GL}_2(\mathbb{Q}_l)$-action of right translation.

This is the normalized induction from the standard Borel group. It is well-known that if $\chi/\psi \neq |.|^{\pm 1}$, then $I(\chi, \psi)$ is an irreducible representation. If $\chi/\psi = |.|$, then $I(\chi, \psi)$ has an irreducible codimension one invariant subspace. If $\chi/\psi = |.|^{-1}$, then $I(\chi, \psi)$ has a one-dimensional invariant subspace and the quotient representation is irreducible.

Write dx (respectively, $d^\times x$, dg) for the Haar measure on $\mathbb{Q}_l$ (respectively, $\mathbb{Q}_l^\times$, $\mathrm{GL}_2(\mathbb{Q}_l)$) normalized so that $\mathbb{Z}_l$ (respectively, $\mathbb{Z}_l^\times$, $\mathrm{GL}_2(\mathbb{Z}_l)$) has unit volume. There is a natural pairing $I(\chi, \psi) \times I(\chi^{-1}, \psi^{-1}) \to \mathbb{C}$ defined by

$$\langle f_1, f_2 \rangle = \int_{\mathrm{GL}_2(\mathbb{Z}_l)} f_1(g) f_2(g) dg,$$

under which $I(\chi^{-1}, \psi^{-1})$ is identified with the dual of $I(\chi, \psi)$.

Definition 11.1.2. Let χ and ψ be smooth characters of $\mathbb{Q}_l$. A flat section of the family of representations $I(\chi|.|^{s_1}, \psi|.|^{s_2})$ indexed by $s_1, s_2 \in \mathbb{C}$ is a function $\mathrm{GL}_2(\mathbb{Q}_l) \times \mathbb{C} \to \mathbb{C}$, $(g, s) \mapsto f_{s_1,s_2}(g)$ such that for all fixed s_1, s_2, the function $g \mapsto f_{s_1,s_2}(g)$ is in $I(\chi|.|^{s_1}, \psi|.|^{s_2})$), and the restriction of f_{s_1,s_2} to $\mathrm{GL}_2(\mathbb{Z}_l)$ is independent of s_1 and s_2.

Remark. From the Iwasawa decomposition (Proposition 4.5.2 of [Bum97]), one sees that every $f \in I(\chi, \psi)$ extends to a unique flat section.

Next we recall the definition of the intertwining operator. Fix characters χ, ψ of $\mathbb{Q}_l^\times$ and write them as

$$\chi = \xi_1 |.|^{s_1}, \quad \psi = \xi_2 |.|^{s_2},$$

where ξ_1, ξ_2 are unitary characters of $s_1, s_2 \in \mathbb{C}$. Let $f \in I(\chi, \psi)$. For $g \in \mathrm{GL}_2(\mathbb{Q}_l)$,

consider

$$Mf(g) = \int_{\mathbb{Q}_l} f(w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g)dx,$$

where $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. This integral is well-defined if $\mathrm{Re}(s_1 - s_2) > 0$:

Proposition 11.1.3. ([Bum97], Proposition 4.5.6) If $\mathrm{Re}(s_1 - s_2) > 0$, then the integral defining $Mf(g)$ is absolutely convergent and defines a nonzero intertwining map

$$M : I(\chi, \psi) \to I(\psi, \chi), \qquad f \mapsto Mf.$$

The next proposition allows us to extend the intertwining operator to all $s_1, s_2 \in \mathbb{C}$ such that $\chi \neq \psi$.

Proposition 11.1.4. ([Bum97], Proposition 4.5.7) Fix $f \in I(\chi, \psi)$, and let f_{s_1,s_2} denote the flat section passing through f. Then for fixed $g \in \mathrm{GL}_2(\mathbb{Q}_l)$, the integral $Mf_{s_1,s_2}(g)$, originally defined for $\mathrm{Re}(s_1 - s_2) > 0$, has analytic continuation to all $s_1, s_2 \in \mathbb{C}$ where $\chi \neq \psi$, and defines a nonzero intertwining operator

$$M : I(\chi, \psi) \to I(\psi, \chi).$$

In particular, by Schur's lemma, when $I(\chi, \psi)$ is irreducible (and hence so is $I(\psi, \chi)$), the intertwining operator M is an isomorphism. On the other hand, when $\chi/\psi = |.|$ (respectively, $\chi/\psi = |.|^{-1}$), the kernel of M is precisely the unique nontrivial irreducible sub-representation of $I(\chi, \psi)$, and M maps the quotient isomorphically onto the irreducible sub-representation of $I(\psi, \chi)$.

11.2 Siegel sections

We follow the account of [LSZ17], Section 3.2. See also Section 1.5 of [Gro18].

Definition 11.2.1. Let $\mathcal{S}(\mathbb{Q}_l^2, \mathbb{C})$ be the space of Schwartz functions on $\mathbb{Q}_l^2$. For $\phi \in \mathcal{S}(\mathbb{Q}_l^2, \mathbb{C})$, we use $\hat{\phi}$ to denote its Fourier transform

$$\hat{\phi}(x, y) = \int \int e_l(xv - yu)\phi(u, v)dudv,$$

where $e_l(x)$ is the standard additive character of $\mathbb{Q}_l$ taking $1/l^n$ to $\exp(2\pi i/l^n)$.

Proposition 11.2.2. ([LSZ17], Proposition 3.2.2) Let $\phi \in \mathcal{S}(\mathbb{Q}_l^2, \mathbb{C})$ and χ, ψ be characters of $\mathbb{Q}_l^\times$. There is a well-defined element $f_{\phi,\chi,\psi} \in I(\chi, \psi)$ defined by integrals satisfying

$$f_{g\cdot\phi,\chi,\psi}(h) = \chi(\det g)^{-1}|\det g|^{-1/2} f_{\phi,\chi,\psi}(hg),$$
$$f_{\widehat{g\cdot\phi},\chi,\psi}(h) = \psi(\det g)^{-1}|\det g|^{-1/2} f_{\hat{\phi},\chi,\psi}(hg).$$

In particular, if $\psi = |.|^{-1/2}$, the map

$$\mathcal{S}(\mathbb{Q}_l^2, \mathbb{C}) \to I(\chi, \psi), \qquad \phi \mapsto F_{\phi,\chi,\psi} := f_{\hat{\phi},\chi,\psi}$$

is $\mathrm{GL}_2(\mathbb{Q}_l)$-equivariant.

Proposition 11.2.3. ([LSZ17], Proposition 3.2.3) We have

$$M(f_{\phi,\chi,\psi}) = \frac{\varepsilon(\psi/\chi)}{L(\chi/\psi, 1)} f_{\hat{\phi},\chi,\psi},$$

where $\varepsilon(\psi/\chi)$ is the local ε-factor (a non-zero scalar, equal to 1 if χ/ψ is unramified).

Definition 11.2.4. For integers $t \geq 0$, we define functions $\phi_t \in \mathcal{S}(\mathbb{Q}_l^2, \mathbb{C})$ as follows:

- for $t = 0$, $\phi_0 := \mathrm{ch}(\mathbb{Z}_l)\mathrm{ch}(\mathbb{Z}_l)$;
- for $t > 0$, $\phi_t := \mathrm{ch}(l^t\mathbb{Z}_l)\mathrm{ch}(\mathbb{Z}_l^\times)$.

Note that ϕ_t is preserved by the action of the group $K_0(l^t) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_l) : c \equiv 0 \mod l^t\}$.

Lemma 11.2.5. Let χ, ψ be unramified characters. Then the function $f_{\phi_t,\chi,\psi}$ is supported on $B(\mathbb{Q}_l)K_0(l^t)$, where $B \subset \mathrm{GL}_2$ is the standard Borel subgroup, and

$$f_{\phi_t,\chi,\psi}(1) = \begin{cases} 1 & \text{if } t = 0; \\ L(\chi/\psi, 1)^{-1} & \text{if } t \geq 1. \end{cases}$$

11.3 Principal series of $\mathrm{GSp}_4(\mathbb{Q}_l)$

Next we collect some basic results on principal series representations of $\mathrm{GSp}_4(\mathbb{Q}_l)$. We follow closely the account of [LSZ17], Section 3.5. The standard reference is [RS07].

Definition 11.3.1. Let χ_1, χ_2, ρ be smooth characters of $\mathbb{Q}_l^\times$ such that

$$|.|^{\pm 1} \notin \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1/\chi_2\}. \tag{11.1}$$

We define $\chi_1 \times \chi_2 \rtimes \rho$ to be the representation of $\mathrm{GSp}_4(\mathbb{Q}_l)$ given by the space of smooth functions $f : \mathrm{GSp}_4(\mathbb{Q}_l) \to \mathbb{C}$ satisfying

$$f(\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} g) = \frac{a^2 b|}{|c|^{3/2}} \chi_1(a)\chi_2(b)\rho(c)f(g),$$

with $\mathrm{GSp}_4(\mathbb{Q}_l)$ acting by right translation. The representation $\chi_1 \times \chi_2 \rtimes \rho$ is called an irreducible principal series.

Remark. This representation has central character $\chi_1\chi_2\rho^2$. The condition on χ_1, χ_2 implies it is irreducible and generic. In fact, this is the only type among the 11 groups of irreducible, admissible, non-supercuspidal representations of $\mathrm{GSp}_4(\mathbb{Q}_l)$ which is both generic and spherical (see table A.1 of [RS07] and table 3 of [Sch05]).

If η is a smooth character of $\mathbb{Q}_l^\times$, we may regard it as a character of $\mathrm{GSp}_4(\mathbb{Q}_l)$ via the multiplier map. Then twisting $\chi_1 \times \chi_2 \rtimes \rho$ by η results in the representation $\chi_1 \times \chi_2 \rtimes \rho\eta$.

Lemma 11.3.2. Let $\sigma = \chi_1 \times \chi_2 \rtimes \rho$ be an irreducible principal series as above, and η a smooth character of $\mathbb{Q}_l^\times$. Then the twist $\sigma \otimes \eta$ of σ by η is equivalent to η if and only if at least one of the following conditions is satisfied:

- $\eta = 1$;
- $\eta = \chi_1$ and $\chi_1^2 = 1$;
- $\eta = \chi_2$ and $\chi_2^2 = 1$;
- $\eta = \chi_1\chi_2$ and $\chi_1^2 = \chi_2^2 = 1$.

Proof. By Theorem 4.2 of [ST93], $\sigma \otimes \eta = \chi_1 \times \chi_2 \rtimes \rho\eta$ is isomorphic to σ if and only if (χ_1, χ_2, ρ) and $(\chi_1, \chi_2, \rho\eta)$ are in the same orbit of the Weyl group acting on characters of the diagonal torus. Using the explicit representatives of the 8-element Weyl group as given in Section 2.1 of [RS07], we see $(\chi_1, \chi_2, \rho\eta)$ must be one of the following: (χ_1, χ_2, ρ), (χ_2, χ_1, ρ), $(\chi_1^{-1}, \chi_2, \rho\chi_1)$, $(\chi_2, \chi_1^{-1}, \rho\chi_1)$, $(\chi_1, \chi_2^{-1}, \rho\chi_2)$, $(\chi_2^{-1}, \chi_1, \rho\chi_2)$, $(\chi_1^{-1}, \chi_2^{-1}, \rho\chi_1\chi_2)$, $(\chi_2^{-1}, \chi_1^{-1}, \rho\chi_1\chi_2)$. The desired claim then follows immediately. $\square$

Definition 11.3.3. Let $\sigma = \chi_1 \times \chi_2 \rtimes \rho$ be an irreducible principal series as defined above. The local spin L-factor of σ is defined as

$$L(\sigma, s) = L(\sigma \otimes |.|^s, 0) = L(\rho, s)L(\rho\chi_1, s)L(\rho\chi_2, s)L(\rho\chi_1\chi_2, s).$$

We record the following characterization of irreducible generic unramified representations.

Proposition 11.3.4. ([LSZ17], Proposition 3.5.3; see also [RS07], Section 2.2) Let $\sigma = \chi_1 \otimes \chi_2 \rtimes \rho$ be an irreducible principal series. Then σ is unramified if and only if all three characters $\chi_1.\chi_2, \rho$ are all unramified. Moreover, every irreducible, generic, unramified representation of $\mathrm{GSp}_4(\mathbb{Q}_l)$ is isomorphic to $\chi_1 \otimes \chi_2 \rtimes \rho$ for a unique Weyl-group orbit of unramified characters (χ_1, χ_2, ρ) satisfying (11.1).

Proposition 11.3.5. Let $\sigma = \chi_1 \times \chi_2 \rtimes \rho$ and $\tau = I(\chi, \psi)$ be irreducible principal series representations, of $\mathrm{GSp}_4(\mathbb{Q}_l)$ and $\mathrm{GL}_2(\mathbb{Q}_l)$, respectively. Suppose neither of the following cases happens:

- $\chi_i = \chi/\psi$ is quadratic for either $i = 1$ or $i = 2$;
- $\chi_1\chi_2 = \chi/\psi$ and both χ_1, χ_2 are quadratic.

Then $\sigma \otimes \tau$ remains irreducible as a representation of $G(\mathbb{Q}_l) \subset \mathrm{GSp}_4(\mathbb{Q}_l) \times \mathrm{GL}_2(\mathbb{Q}_l)$.

Proof. Let $Z = \{(aI_4, bI_2) \in \mathrm{GSp}_4(\mathbb{Q}_l) \times \mathrm{GL}_2(\mathbb{Q}_l) : a, b \in \mathbb{Q}_l^\times\}$ be the center of $\mathrm{GSp}_4(\mathbb{Q}_l) \times \mathrm{GL}_2(\mathbb{Q}_l)$. As $\sigma \otimes \tau$ is irreducible as a $\mathrm{GSp}_4(\mathbb{Q}_l) \times \mathrm{GL}_2(\mathbb{Q}_l)$-representation, Z acts as scalars. Thus $\sigma \otimes \tau$ is irreducible as a $G(\mathbb{Q}_l)$-representation if and only if it is also irreducible as an $Z \cdot G(\mathbb{Q}_l)$-representation.

Now we are in the situation of [GK82], Lemma 2.1. By part (d) of the cited lemma, it suffices to prove there is no character $\nu = \nu_1\nu_2$ of $\mathrm{GSp}_4(\mathbb{Q}_l) \times \mathrm{GL}_2(\mathbb{Q}_l)$

trivial on $Z \cdot G(\mathbb{Q}_l)$, such that $\sigma \otimes \tau \simeq (\sigma \otimes \nu_1) \otimes (\tau \otimes \nu_2)$. Here ν_1 (respectively, ν_2) is a character of $\mathbb{Q}_l^\times$, regarded as a character of $\mathrm{GSp}_4(\mathbb{Q}_l)$ via the multiplier map (respectively, a character of $\mathrm{GL}_2(\mathbb{Q}_l)$ via determinant).

The condition that ν is trivial on $G(\mathbb{Q}_l)$ implies that $\nu_1\nu_2 = 1$. Thus neither of ν_1 and nu_2 can be trivial. Moreover, $\tau \simeq \tau \otimes \nu_2$ implies $\nu_2 = \chi/\psi$ and is quadratic. On the other hand, Lemma 11.3.2 implies either ν_1 is trivial, or equals to one of χ_1 and χ_2 and is quadratic, or equals to $\chi_1\chi_2$ and both χ_1 and χ_2 is quadratic. It is clear none of the cases is possible given our assumption. □

From now on, we will assume $\sigma \otimes \tau$ satisfies the assumption of Proposition 11.3.5.

Chapter 12

Hecke operators and zeta integrals

12.1 Local zeta integrals

Let σ (respectively, τ) be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{Q}_l)$ (respectively, $\mathrm{GL}_2(\mathbb{Q}_l)$) with central character ω_σ (respectively, ω_τ), and suppose σ is generic. We fix a character Ψ of $\mathbb{Q}_l$, a Whittaker models W of σ with respect to Ψ, and a Whittaker model $\tilde{W}$ of τ with respect to Ψ^{-1}.

For $\varphi_1 \in \sigma$ and $\varphi_2 \in \tau$, let W_{φ_1} and $\tilde{W}_{\varphi_2}$ be their respective Whittaker functions.

For any unramified character η of $\mathbb{Q}_l^\times$, consider the following simplified zeta integral

$$\ell(\varphi_1 \otimes \varphi_2, \eta, s) := \int_{\mathbb{Q}_l^\times \times \mathbb{Q}_l^\times} W_{\varphi_1}\left(\begin{pmatrix} xy & & & \\ & x & & \\ & & y & \\ & & & 1 \end{pmatrix}\right) \tilde{W}_{\varphi_2}\left(\begin{pmatrix} x & \\ & y \end{pmatrix}\right) \eta(xy)|x|^{s-2}|y|^s d^\times x d^\times y. \tag{12.1}$$

More generally, for any $\varphi \in \sigma \otimes \tau$, we may write it as a finite sum of simple tensors and define $\ell(\varphi, \eta, s)$ by linearity.

Definition 12.1.1. Let σ, τ and η be as above. For every $\varphi \in \sigma \otimes \tau$, define

$$Z(\varphi, \eta, s) = L(s, \sigma \otimes \tau \otimes \eta)^{-1} \ell(\varphi, \eta, s). \tag{12.2}$$

Suppose φ_1 and φ_2 are the normalized spherical vectors in the respective automorphic representations. To simplify notations, for $i, j \in \mathbb{Z}$ put

$$W_1(i,j) = W_{\varphi_1}\left(\begin{pmatrix} l^{i+2j} & & & \\ & l^{i+j} & & \\ & & l^j & \\ & & & 1 \end{pmatrix}\right), \quad W_2(i,j) = \tilde{W}_{\varphi_2}\left(\begin{pmatrix} l^{i+j} & \\ & l^j \end{pmatrix}\right),$$

and $X = \eta(l) l^{-s}$. Note that $W_1(i,j) \neq 0$ if and only if $i, j \geq 0$, while $W_2(i,j) \neq 0$ if and only if $i \geq 0$. Then we have

$$\ell(\varphi_1 \otimes \varphi_2, \eta, s) = \sum_{i \geq 0} \sum_{j \geq 0} W_1(i,j) W_2(i,j) l^{2(i+j)} X^{i+2j}. \tag{12.3}$$

Proposition 12.1.2. The integral $\ell(\varphi, \eta, s)$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$, and it has analytic continuation to all $s \in \mathbb{C}$ which lies in $\mathbb{C}[l^s, l^{-s}]$. Furthermore, if $\varphi = \varphi_1 \otimes \varphi_2$ is the normalized spherical vector, we have

$$Z(\varphi, \eta, s) = L(2s, \eta^2 \omega)^{-1},$$

where $\omega = \omega_\sigma \omega_\tau$ is the central character of $\sigma \otimes \tau$.

Proof. Suppose $\tau = I(\chi_2, \psi_2)$, and write $\alpha = \chi_2(l)$, $\beta = \psi_2(l)$. Then $\omega_\tau(l) = \alpha\beta$. We have

$$l^{i/2} W_2(i,j) = \alpha^j \beta^j \frac{\alpha^{i+1} - \beta^{i+1}}{\alpha - \beta} = \mathrm{tr}(\mathrm{Sym}^i \otimes \det^j)(t_\tau),$$

the trace of the irreducible algebraic representation of GL_2 with highest weight (i,j), where $t_\tau = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ is the semisimple conjugacy class of the L-group assciated to τ.

Similarly, for the $\mathrm{GSp}_4(\mathbb{Q}_l)$ representation σ, let $t_\sigma = \begin{pmatrix} \xi_1\xi_2 & & & \\ & \xi_1\xi_3 & & \\ & & \xi_2\xi_4 & \\ & & & \xi_3\xi_4 \end{pmatrix}$ be the semisimple conjugacy class in the L-group. By equation (3.2.4) in [Bum89],

$$l^{3i/2+2j}W_1(i,j) = (\xi_1\xi_2\xi_3\xi_4)^j T_{i,j}(\xi_1,\xi_2,\xi_3,\xi_4) = \mathrm{tr}(\rho_{i,j}\otimes\mu^j)(t_\sigma),$$

where $T_{i,j}$ is the polynomial as defined in equation (3.2.2) of [Bum89], $\rho_{i,j}$ is the finite dimensional irreducible algebraic representation of GSp_4 as defined on p.91 of [Bum89], and μ is the usual similitude character of GSp_4. Here we remark that $\rho_{i,j}\otimes\mu^j$ is precisely $V^{j,i}$ in the notation of [LSZ17]. Therefore

$$\ell(\varphi_1\otimes\varphi_2,\eta,s) = \sum_{i\geq 0}\sum_{j\geq 0}\mathrm{tr}(\mathrm{Sym}^i\otimes\det^j)(t_\tau)\mathrm{tr}(\rho_{i,j}\otimes\mu^j)(t_\sigma)X^{i+2j}.$$

This last sum is precisely $\zeta(s,W,f)$ as defined in the equation above (A.1.3) in the appendix of [GPSR87]. By Proposition A.1 there, this last quantity equals to

$$L(2s,\wedge^2(\tau\otimes\eta)\otimes\omega_\sigma)^{-1}L(s,\sigma\otimes\tau\otimes\eta) = L(2s,\eta^2\omega)^{-1}L(s,\sigma\otimes\tau\otimes\eta)$$

as desired. □

12.2 Double coset computations

In this section we calculate decomposition of various double cosets into right cosets, in preparation of the computation of the actions of associated Hecke operators. We will be mostly interested in double cosets of $K_{m,n}$ defined in the introduction. However, calculations involving the Klingen congruence subgroups $\mathrm{Kl}(l^n)$ are generally easier. We first prove a technical lemma allowing us to reduce the decomposition of $K_{m,n}$-double cosets into that of Kl_n-double cosets.

Lemma 12.2.1. Let G be a group, $H \subset K \subset G$ two subgroups, and $g \in G$ any element. Then the natural map $HgH/H \to KgK/K$ is injective if and only if $gHg^{-1} \cap H = gKg^{-1} \cap H$. It is surjective if and only if $HgK = KgK$.

Proof. We have a natural bijection of right coset spaces

$$K/(gKg^{-1} \cap K) \cong KgK/K, \qquad \gamma \mapsto \gamma g,$$

and similarly a bijection $\gamma \mapsto \gamma g$ for K replaced by H, which implies we may look at the natural map $H/(gHg^{-1} \cap H) \to K/(gKg^{-1} \cap K)$ instead. This last map is injective if and only if $gHg^{-1} \cap H = gKg^{-1} \cap H$. The claim on surjectivity is straightforward. □

Taking $G = \mathrm{GSp}_4(\mathbb{Q}_l) \times \mathrm{GL}_2(\mathbb{Q}_l)$, $K = \mathrm{Kl}(l^n) \times \mathrm{GL}_2(\mathbb{Z}_l)$ and $H = K_{m,n}$, which is precisely the subgroup of K consisting of elements of the form (k_1, k_2) with $\mu(k_1) = \det(k_2) \in 1 + l^m\mathbb{Z}_l$. Then the condition $gHg^{-1} \cap H = gKg^{-1} \cap H$ is satisfied for all g as conjugation does not change the value of μ on GSp_4 or the determinant on GL_2.

On the other hand, if g is diagonal, then $HgK = KgK$, as we can always write any element of K as the product of an element in H and a diagonal element, and the latter

commutes with g. Thus by the above lemma the natural map $HgH/H \to KgK/K$ is a bijection. In particular, we have proven the following:

Corollary 12.2.2. Suppose $g = (g_1, g_2) \in \mathrm{GSp}_4(\mathbb{Q}_l) \times \mathrm{GL}_2(\mathbb{Z}_l)$ and we have the following decompositions of double cosets:

$$\mathrm{Kl}(l^n) g_1 \mathrm{Kl}(l^n) = \coprod_i \gamma_i \mathrm{Kl}(l^n), \qquad \mathrm{GL}_2(\mathbb{Z}_l) g_2 \mathrm{GL}_2(\mathbb{Z}_l) = \coprod_j \gamma_j' \mathrm{GL}_2(\mathbb{Z}_l).$$

If $(\gamma_i, \gamma_j') \in K_{m,n} g K_{m,n}$ for all pairs (i, j), then

$$K_{m,n} g K_{m,n} = \coprod_{i,j} (\gamma_i, \gamma_j') K_{m,n}.$$

Therefore under mild hypotheses we can build up decomposition of $K_{m,n}$-double cosets from the respective coset decomposition for $\mathrm{Kl}(l^n)$ and $\mathrm{GL}_2(\mathbb{Z}_l)$.

First we record deal with the case of $\mathrm{GL}_2(\mathbb{Z}_l)$.

Lemma 12.2.3. We have the following decomposition of $\mathrm{GL}_2(\mathbb{Z}_l)$ double cosets:

$$\mathrm{GL}_2(\mathbb{Z}_l) \begin{pmatrix} l & \\ & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l) = \coprod_{0 \le u < l} \begin{pmatrix} l & u \\ & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l) \sqcup \begin{pmatrix} 1 & \\ & l \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l),$$

$$\mathrm{GL}_2(\mathbb{Z}_l) \begin{pmatrix} l^2 & \\ & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l) = \coprod_{0 \le u < l^2} \begin{pmatrix} l^2 & u \\ & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l) \sqcup \coprod_{0 < t < l} \begin{pmatrix} l & t \\ & l \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l) \sqcup \begin{pmatrix} 1 & \\ & l^2 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l).$$

Proposition 12.2.4. Assume $n \ge 4$. We have the following decompositions of double

cosets of the Klingen parabolic $\mathrm{Kl}(l^n)$ into disjoint union of left cosets:

$$\mathrm{Kl}(l^n)\begin{pmatrix} l & & & \\ & l & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n) = \coprod_{0\le u,v,w<l}\begin{pmatrix} l & & u & w \\ & l & v & u \\ & & 1 & \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n)$$
$$\sqcup \coprod_{0\le\lambda,\kappa<l}\begin{pmatrix} l & \lambda & & \kappa \\ & 1 & & \\ & & l & -\lambda \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n),$$

$$\mathrm{Kl}(l^n)\begin{pmatrix} l^2 & & & \\ & l & & \\ & & l & \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n) = \coprod_{0\le\lambda,\mu<l,0\le\kappa<l^2}\begin{pmatrix} l^2 & l\lambda & l\mu & \kappa \\ & l & & \mu \\ & & l & -\lambda \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n),$$

$$\mathrm{Kl}(l^n)\begin{pmatrix} l^3 & & & \\ & l^2 & & \\ & & l & \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n) = \coprod_{0\le t,\lambda<l,0\le\mu<l^2,0\le\kappa<l^3}\begin{pmatrix} l^3 & l^2\lambda & l\mu & \kappa \\ & l^2 & lt & \mu-t\lambda \\ & & l & -\lambda \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n)$$
$$\sqcup \coprod_{0\le\mu<l,0\le\lambda<l^2,0\le\kappa<l^3}\begin{pmatrix} l^3 & l\lambda & l^2\mu & \kappa \\ & l & & \mu \\ & & l^2 & -\lambda \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n),$$

$$\mathrm{Kl}(l^n)\begin{pmatrix} l^4 & & & \\ & l^2 & & \\ & & l^2 & \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n) = \coprod_{0\le\lambda,\mu<l^2,0\le\kappa<l^4} \begin{pmatrix} l^4 & l^2\lambda & l^2\mu & \kappa \\ & l^2 & & \mu \\ & & l^2 & -\lambda \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n),$$

$$\mathrm{Kl}(l^n)\begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n) = \coprod_{0\le u,v,w<l^2} \begin{pmatrix} l^2 & & u & w \\ & l^2 & v & u \\ & & 1 & \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n)$$

$$\sqcup \coprod_{0\le\lambda,\kappa<l^2} \begin{pmatrix} l^2 & \lambda & & \kappa \\ & 1 & & \\ & & l^2 & -\lambda \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n)$$

$$\sqcup \coprod_{0<t<l,0\le\lambda,\mu<l,0\le\kappa<l^2} \begin{pmatrix} l^2 & l\lambda & l\mu+t\lambda & \kappa \\ & l & t & \mu \\ & & l & -\lambda \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n).$$

Proof. The decompositions follow the same line of argument as in the proof of part (i) of Lemma 6.1.1 of [RS07], except that instead of the paramodular subgroup $K(l^n)$ the stricter condition on the upper right corner of the Klingen congruence subgroup means we cannot get rid of the upper right corner in the decomposition.

We sketch the proof of the last decomposition. By Iwahori factorization (c.f.

equation (2.7) of [RS07] we have

$$\mathrm{Kl}(l^n)\begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n)$$

$$= \begin{pmatrix} 1 & \mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l \\ & 1 & & \mathbb{Z}_l \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix} \begin{pmatrix} \mathbb{Z}_l^\times & & & \\ & \mathbb{Z}_l & \mathbb{Z}_l & \\ & \mathbb{Z}_l & \mathbb{Z}_l & \\ & & & \mathbb{Z}_l^\times \end{pmatrix} \begin{pmatrix} 1 & & & \\ l^n\mathbb{Z}_l & 1 & & \\ l^n\mathbb{Z}_l & & 1 & \\ l^n\mathbb{Z}_l & l^n\mathbb{Z}_l & l^n\mathbb{Z}_l & 1 \end{pmatrix} \begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \mathrm{Kl}(l^n)$$

$$= \begin{pmatrix} 1 & \mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l \\ & 1 & & \mathbb{Z}_l \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix} \begin{pmatrix} \mathbb{Z}_l^\times & & & \\ & \mathbb{Z}_l & \mathbb{Z}_l & \\ & \mathbb{Z}_l & \mathbb{Z}_l & \\ & & & \mathbb{Z}_l^\times \end{pmatrix} \begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \mathrm{Kl}(l^n)$$

$$= \begin{pmatrix} 1 & \mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l \\ & 1 & & \mathbb{Z}_l \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & \mathbb{Z}_l & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \mathrm{Kl}(l^n)$$

$$\cup \begin{pmatrix} 1 & \mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l \\ & 1 & & \mathbb{Z}_l \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^2 & & & \\ & 1 & & \\ & & l^2 & \\ & & & 1 \end{pmatrix} \mathrm{Kl}(l^n)$$

$$\cup \begin{pmatrix} 1 & \mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l \\ & 1 & & \mathbb{Z}_l \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & l^{-1}\mathbb{Z}_l^\times & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^2 & & & \\ & l & & \\ & & l & \\ & & & 1 \end{pmatrix} \mathrm{Kl}(l^n).$$

Here at each step we try to push $\begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ to the left. The second equality follows from the fact that

$$\begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & & & \\ l^n\mathbb{Z}_l & 1 & & \\ l^n\mathbb{Z}_l & & 1 & \\ l^n\mathbb{Z}_l & l^n\mathbb{Z}_l & l^n\mathbb{Z}_l & 1 \end{pmatrix} \begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ l^n\mathbb{Z}_l & 1 & & \\ l^{n+2}\mathbb{Z}_l & & 1 & \\ l^{n+2}\mathbb{Z}_l & l^{n+2}\mathbb{Z}_l & l^n\mathbb{Z}_l & 1 \end{pmatrix}$$

is a subgroup of $\mathrm{Kl}(l^n)$, while the last one follows from the decomposition

$$\begin{aligned} \mathrm{GL}_2(\mathbb{Z}_l) \begin{pmatrix} l^2 & \\ & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l) = &\begin{pmatrix} 1 & \mathbb{Z}_l \\ & 1 \end{pmatrix} \begin{pmatrix} l^2 & \\ & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l) \\ &\sqcup \begin{pmatrix} 1 & l^{-1}\mathbb{Z}_l^\times \\ & 1 \end{pmatrix} \begin{pmatrix} l & \\ & l \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l) \sqcup \begin{pmatrix} 1 & \\ & l^2 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_l). \end{aligned}$$

We now analyze each of the three parts in the union. For the first one, since

$$\begin{pmatrix} 1 & \mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l \\ & 1 & \mathbb{Z}_l & \mathbb{Z}_l \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l \\ & 1 & & \mathbb{Z}_l \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & \mathbb{Z}_l & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \mathbb{Z}_l & \mathbb{Z}_l \\ & 1 & \mathbb{Z}_l & \mathbb{Z}_l \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathbb{Z}_l & & \\ & 1 & & \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix}$$

and $\begin{pmatrix} 1 & \mathbb{Z}_l & & \\ & 1 & & \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix}$ commutes with $\begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, we see the first part reduces to

$$\begin{pmatrix} 1 & & \mathbb{Z}_l & \mathbb{Z}_l \\ & 1 & \mathbb{Z}_l & \mathbb{Z}_l \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \mathrm{Kl}(l^n).$$

For the second one, note

$$\begin{pmatrix} 1 & \mathbb{Z}_l & \mathbb{Z}_l & \mathbb{Z}_l \\ & 1 & & \mathbb{Z}_l \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbb{Z}_l & & \mathbb{Z}_l \\ & 1 & & \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \mathbb{Z}_l & \\ & 1 & & \mathbb{Z}_l \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and $\begin{pmatrix} 1 & & \mathbb{Z}_l & \\ & 1 & & \mathbb{Z}_l \\ & & 1 & \\ & & & 1 \end{pmatrix}$ commutes with $\begin{pmatrix} l^2 & & & \\ & 1 & & \\ & & l^2 & \\ & & & 1 \end{pmatrix}$, so we are left with

$$\begin{pmatrix} 1 & \mathbb{Z}_l & & \mathbb{Z}_l \\ & 1 & & \\ & & 1 & \mathbb{Z}_l \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^2 & & & \\ & 1 & & \\ & & l^2 & \\ & & & 1 \end{pmatrix} \mathrm{Kl}(l^n).$$

For the last one, it is a union of left cosets of the form

$$\begin{pmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & t/l & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^2 & & & \\ & l & & \\ & & l & \\ & & & 1 \end{pmatrix} \mathrm{Kl}(l^n)$$

with $\lambda, \mu, \kappa \in \mathbb{Z}_l$ and $t \in \mathbb{Z}_l^\times$. By multiplying the coset representatives by elements in $\mathrm{Kl}(l^n)$ it suffices to use coset representatives with $0 \leq \lambda < l$ and $0 < t < l$, and furthermore we may restrict to those with $0 \leq \mu < l$ and $0 \leq \kappa < l^2$.

Putting them together, we see

$$\mathrm{Kl}(l^n)\begin{pmatrix} l^2 & & \\ & l^2 & \\ & & 1 \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n) = \bigcup_{0\le u,v,w<l^2}\begin{pmatrix} l^2 & & u & w \\ & l^2 & v & u \\ & & 1 & \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n)$$

$$\cup \bigcup_{0\le \lambda,\kappa<l^2}\begin{pmatrix} l^2 & \lambda & & \kappa \\ & 1 & & \\ & & l^2 & -\lambda \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n)$$

$$\cup \bigcup_{0<t<l,0\le\lambda,\mu<l,0\le\kappa<l^2}\begin{pmatrix} l^2 & l\lambda & l\mu+t\lambda & \kappa \\ & l & t & \mu \\ & & l & -\lambda \\ & & & 1 \end{pmatrix}\mathrm{Kl}(l^n).$$

It is clear that all of these left cosets are mutually disjoint. □

Remark.

1. Let $\mathrm{Kl}(l^n)g_1\mathrm{Kl}(l^n)$ be any of the five double cosets considered in Proposition 12.2.4 and $\gamma\mathrm{Kl}(l^n)$ be any of the left cosets in the coset decomposition, then γ can be written as $\gamma = k_1g_1k_2$ with $k_1, k_2 \in \mathrm{Kl}(l^n)$ and $\mu(k_1) = \mu(k_2) = 1$.

2. Similarly, let $\mathrm{GL}_2(\mathbb{Z}_l)g_2\mathrm{GL}_2(\mathbb{Z}_l)$ be either of the two double cosets considered in Lemma 12.2.3 and $\gamma'\mathrm{GL}_2(\mathbb{Z}_l)$ any of the left cosets in the coset decomposition, then γ' can be written as $\gamma' = k_1'g_2k_2'$ with $k_1', k_2' \in \mathrm{GL}_2(\mathbb{Z}_l)$ and $\det(k_1') = \det(k_2') = 1$.

In particular, as elements of $\mathrm{Kl}(l^n)\times\mathrm{GL}_2(\mathbb{Z}_l)$, (k_1, k_1') and (k_2, k_2') both belong to the subgroup $K_{m,n}$, so $(\gamma,\gamma') \in K_{m,n}(g_1, g_2)K_{m,n}$. Therefore the assumption of Corollary 12.2.2 is satisfied and we may compute decomposition of $K_{m,n}(g_1, g_2)K_{m,n}$ into

left $K_{m,n}$-cosets by combining the respective decompositions for $\mathrm{Kl}(l^n)$ and $\mathrm{GL}_2(\mathbb{Z}_l)$.

12.3 Hecke operators and local zeta integrals

Definition 12.3.1. We define Hecke operators at l given by the following double coset opertors:

- $T_1^{\mathrm{Kl}}(l) = \frac{1}{\mathrm{Vol}(K_{m,n})}\mathrm{ch}(K_{m,n}(\begin{pmatrix} l & & & \\ & l & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l & \\ & 1 \end{pmatrix})K_{m,n})$;
- $T_2^{\mathrm{Kl}}(l) = \frac{1}{\mathrm{Vol}(K_{m,n})}\mathrm{ch}(K_{m,n}(\begin{pmatrix} l^2 & & & \\ & l & & \\ & & l & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l & \\ & l \end{pmatrix})K_{m,n})$;
- $T_3^{\mathrm{Kl}}(l) = \frac{1}{\mathrm{Vol}(K_{m,n})}\mathrm{ch}(K_{m,n}(\begin{pmatrix} l^2 & & & \\ & l & & \\ & & l & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l^2 & \\ & 1 \end{pmatrix})K_{m,n})$;
- $T_4^{\mathrm{Kl}}(l) = \frac{1}{\mathrm{Vol}(K_{m,n})}\mathrm{ch}(K_{m,n}(\begin{pmatrix} l^2 & & & \\ & l^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l & \\ & l \end{pmatrix})K_{m,n})$;

- $T_5^{\mathrm{Kl}}(l) = \frac{1}{\mathrm{Vol}(K_{m,n})}\mathrm{ch}(K_{m,n}(\begin{pmatrix} l^3 & & & \\ & l^2 & & \\ & & l & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l^2 & \\ & l \end{pmatrix})K_{m,n})$;

- $T_6^{\mathrm{Kl}}(l) = \frac{1}{\mathrm{Vol}(K_{m,n})}\mathrm{ch}(K_{m,n}(\begin{pmatrix} l^4 & & & \\ & l^2 & & \\ & & l^2 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l^2 & \\ & l^2 \end{pmatrix})K_{m,n})$.

- $U_1(l) = \frac{1}{\mathrm{Vol}(B_{m,n})}\mathrm{ch}(B_{m,n}(\begin{pmatrix} l & & & \\ & l & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l & \\ & 1 \end{pmatrix})B_{m,n})$;

- $U_2(l) = \frac{1}{\mathrm{Vol}(B_{m,n})}\mathrm{ch}(B_{m,n}(\begin{pmatrix} l^2 & & & \\ & l & & \\ & & l & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l & \\ & l \end{pmatrix})B_{m,n})$.

We will compute $\ell(T_i^{\mathrm{Kl}}(l)(\varphi_1 \otimes \varphi_2), \eta, s)$ for $i = 1, .., 6$ using the double coset decompositions given in last section.

Proposition 12.3.2. We have the following identities:

$$\begin{aligned}
\ell(T_1^{\mathrm{Kl}}(l)(\varphi_1\otimes\varphi_2),\eta,s) &= l^2X^{-1}\Big(\sum_{i\geq 0}\sum_{j\geq 1}+\sum_{i\geq 1}\sum_{j\geq 0}\Big)W_1(i,j)W_2(i,j)l^{2(i+j)}X^{i+2j}\\
&\quad + lX^{-1}\sum_{i\geq 2}\sum_{j\geq 0}W_1(i,j)W_2(i-2,j+1)l^{2(i+j)}X^{i+2j}\\
&\quad + l^3X^{-1}\sum_{i\geq 0}\sum_{j\geq 1}W_1(i,j)W_2(i+2,j-1)l^{2(i+j)}X^{i+2j},\\
\ell(T_2^{\mathrm{Kl}}(l)(\varphi_1\otimes\varphi_2),\eta,s) &= l^2X^{-2}\sum_{i\geq 0}\sum_{j\geq 1}W_1(i,j)W_2(i,j)l^{2(i+j)}X^{i+2j},\\
\ell(T_3^{\mathrm{Kl}}(l)(\varphi_1\otimes\varphi_2),\eta,s) &= l^2X^{-2}\Big((l-1)\sum_{i\geq 1}\sum_{j\geq 1}-\sum_{i=0,j\geq 1}\Big)W_1(i,j)W_2(i,j)l^{2(i+j)}X^{i+2j}\\
&\quad + l^2X^{-2}\sum_{i\geq 2}\sum_{j\geq 1}W_1(i,j)W_2(i-2,j+1)l^{2(i+j)}X^{i+2j}\\
&\quad + l^4X^{-2}\sum_{i\geq 0}\sum_{j\geq 1}W_1(i,j)W_2(i+2,j-1)l^{2(i+j)}X^{i+2j},\\
\ell(T_4^{\mathrm{Kl}}(l)(\varphi_1\otimes\varphi_2),\eta,s) &= l^2X^{-2}\Big((l-1)\sum_{i\geq 1}\sum_{j\geq 1}-\sum_{i=0,j\geq 1}\Big)W_1(i,j)W_2(i,j)l^{2(i+j)}X^{i+2j}\\
&\quad + l^2X^{-2}\sum_{i\geq 2}\sum_{j\geq 0}W_1(i,j)W_2(i-2,j+1)l^{2(i+j)}X^{i+2j}\\
&\quad + l^4X^{-2}\sum_{i\geq 0}\sum_{j\geq 2}W_1(i,j)W_2(i+2,j-1)l^{2(i+j)}X^{i+2j},\\
\ell(T_5^{\mathrm{Kl}}(l)(\varphi_1\otimes\varphi_2),\eta,s) &= l^4X^{-3}\Big(\sum_{i\geq 1}\sum_{j\geq 1}+\sum_{i\geq 0}\sum_{j\geq 2}\Big)W_1(i,j)W_2(i,j)l^{2(i+j)}X^{i+2j}\\
&\quad + l^3X^{-3}\sum_{i\geq 2}\sum_{j\geq 1}W_1(i,j)W_2(i-2,j+1)l^{2(i+j)}X^{i+2j}\\
&\quad + l^5X^{-3}\sum_{i\geq 0}\sum_{j\geq 2}W_1(i,j)W_2(i+2,j-1)l^{2(i+j)}X^{i+2j},\\
\ell(T_6^{\mathrm{Kl}}(l)(\varphi_1\otimes\varphi_2),\eta,s) &= l^4X^{-4}\sum_{i\geq 0}\sum_{j\geq 2}W_1(i,j)W_2(i,j)l^{2(i+j)}X^{i+2j}.
\end{aligned}$$

Proof. We will prove the identity for $T_4^{\mathrm{Kl}}(l)$. The others are completely analogous.

By Proposition 12.2.4 together with the remark following Lemma 12.2.3, we see

$$T_4^{\mathrm{Kl}}(l)(\varphi_1 \otimes \varphi_2) = (\sum_{\gamma \in C} \gamma \cdot \varphi_1) \otimes (\begin{pmatrix} l & \\ & l \end{pmatrix} \cdot \varphi_2),$$

where γ runs through the disjoint union $C = C_1 \sqcup C_2 \sqcup C_3$ with

$$C_1 = \{ \begin{pmatrix} l^2 & & u & w \\ & l^2 & v & u \\ & & 1 & \\ & & & 1 \end{pmatrix} \mid 0 \leq u, v, w < l^2 \},$$

$$C_2 = \{ \begin{pmatrix} l^2 & \lambda & & \kappa \\ & 1 & & \\ & & l^2 & -\lambda \\ & & & 1 \end{pmatrix} \mid 0 \leq \lambda, \kappa < l^2 \},$$

$$C_3 = \{ \begin{pmatrix} l^2 & l\lambda & l\mu + t\lambda & \kappa \\ & l & t & \mu \\ & & l & -\lambda \\ & & & 1 \end{pmatrix} \mid 0 < t < l, 0 \leq \lambda, \mu < l, 0 \leq \kappa < l^2 \},$$

and "$\cdot$" denotes the action of right multiplication. Thus the zeta integral $\ell(T_4^{\mathrm{Kl}}(l)(\varphi_1 \otimes \varphi_2), \eta, s)$ becomes the following double sum

$$\sum_{\gamma \in C} \sum_{i,j \in \mathbb{Z}} W_{\varphi_1}(\begin{pmatrix} l^{i+2j} & & & \\ & l^{i+j} & & \\ & & l^j & \\ & & & 1 \end{pmatrix} \gamma) W_{\varphi_2}(\begin{pmatrix} l^{i+j+1} & \\ & l^{j+1} \end{pmatrix}) l^{2(i+j)} X^{i+2j}. \tag{12.4}$$

We will break up the above sum into the contributions of γ in each of C_1, C_2 and C_3 separately.

- For $\gamma \in C_1$, we have

$$\begin{pmatrix} l^{i+2j} & & & \\ & l^{i+j} & & \\ & & l^j & \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^2 & & u & w \\ & l^2 & v & u \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & l^{i+j}u & l^{i+2j}w \\ & 1 & l^i v & l^{i+j}u \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^{i+2+2j} & & & \\ & l^{i+2+j} & & \\ & & l^j & \\ & & & 1 \end{pmatrix}$$

so by definition of Whittaker functions

$$W_{\varphi_1}(\begin{pmatrix} l^{i+2j} & & & \\ & l^{i+j} & & \\ & & l^j & \\ & & & 1 \end{pmatrix} \gamma) = \Psi(l^i v) W_1(i+2, j).$$

Note that

$$\sum_{0 \le v < l^2} \Psi(l^i v) = \begin{cases} l^2 & \text{if } i \ge 0; \\ 0 & \text{if } i = -1 \text{ or } -2. \end{cases}$$

Thus the contribution to equation (12.4) is

$$\begin{aligned} & l^6 \sum_{i \ge 0} \sum_{j \ge 0} W_1(i+2, j) W_2(i, j+1) l^{2(i+j)} X^{i+2j} \\ =& l^2 X^{-2} \sum_{i \ge 2} \sum_{j \ge 0} W_1(i, j) W_2(i-2, j+1) l^{2(i+j)} X^{i+2j}. \end{aligned} \tag{12.5}$$

Here note that the terms with $i < -2$ in the sum on the left vanishes because $W_1(i+2, j) = 0$, and similarly for terms with $j < 0$.

- For $\gamma \in C_2$, we have

$$\begin{pmatrix} l^{i+2j} & & & \\ & l^{i+j} & & \\ & & l^{j} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^2 & \lambda & & \kappa \\ & 1 & & \\ & & l^2 & -\lambda \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & l^j\lambda & & l^{i+2j}\kappa \\ & 1 & & \\ & & 1 & -l^j\lambda \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^{i+2j+2} & & & \\ & l^{i+j} & & \\ & & l^{j+2} & \\ & & & 1 \end{pmatrix}$$

so we are summing

$$\Psi(l^j\lambda)W_1(i-2,j+2)W_2(i,j+1)l^{2(i+j)}X^{i+2j}.$$

Since

$$\sum_{0\le\lambda<l^2}\Psi(l^j\lambda) = \begin{cases} l^2 & \text{if } j \ge 0; \\ 0 & \text{if } j = -1 \text{ or } -2; \end{cases}$$

the contribution to equation (12.4) is

$$\begin{aligned} & l^4\sum_{i\ge 2}\sum_{j\ge 0} W_1(i-2,j+2)W_2(i,j+1)l^{2(i+j)}X^{i+2j} \\ =& l^4X^{-2}\sum_{i\ge 0}\sum_{j\ge 2} W_1(i,j)W_2(i+2,j-1)l^{2(i+j)}X^{i+2j}. \end{aligned} \tag{12.6}$$

- For $\gamma \in C_3$, we have

$$\begin{pmatrix} l^{i+2j} & & & \\ & l^{i+j} & & \\ & & l^{j} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^2 & l\lambda & l\mu+t\lambda & \kappa \\ & l & t & \mu \\ & & l & -\lambda \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & l^j\lambda & l^{i+j}(\mu+\frac{t\lambda}{l}) & l^{i+2j}\kappa \\ & 1 & l^{i-1}t & l^{i+j}\mu \\ & & 1 & -l^j\lambda \\ & & & 1 \end{pmatrix} \begin{pmatrix} l^{i+2j+2} & & & \\ & l^{i+j+1} & & \\ & & l^{j+1} & \\ & & & 1 \end{pmatrix}$$

so we are summing

$$\Psi(l^{i-1}t+l^j\lambda)W_1(i,j+1)W_2(i,j+1)l^{2(i+j)}X^{i+2j}.$$

We have

$$\sum_{0\le\lambda<l,0<t<l} \Psi(l^{i-1}t+l^j\lambda) = \sum_{0<t<l}\Psi(l^{i-1}t)\sum_{0\le\lambda<l}\Psi(l^j\lambda)$$
$$= \begin{cases} l^2-l & \text{if } i\ge 1 \text{ and } j\ge 0; \\ -l & \text{if } i=0 \text{ and } j\ge 0; \\ 0 & \text{if } j=-1. \end{cases}$$

Thus the contribution to equation (12.4) is

$$\begin{aligned} & l^4(l-1)\sum_{i\ge1}\sum_{j\ge0} W_1(i,j+1)W_2(i,j+1)l^{2(i+j)}X^{i+2j} \\ & - l^4\sum_{j\ge0} W_1(0,j+1)W_2(0,j+1)l^{2j}X^{2j} \\ =& l^2(l-1)X^{-2}\sum_{i\ge1}\sum_{j\ge1} W_1(i,j)W_2(i,j)l^{2(i+j)}X^{i+2j} \\ & - l^2X^{-2}\sum_{j\ge1} W_1(0,j)W_2(0,j)l^{2j}X^{2j}. \end{aligned} \tag{12.7}$$

Combining equations (12.5), (12.6) and (12.7) in the above three cases, we obtain the identity as desired. □

Corollary 12.3.3. Let T_s be the operator

$$\begin{aligned} T_s =& 1 - l^{-s-2}\eta(l)T_1^{\mathrm{Kl}}(l) + 2l^{-2s-3}\eta(l)^2T_2^{\mathrm{Kl}}(l) + l^{-2s-3}\eta(l)^2T_3^{\mathrm{Kl}}(l) \\ & + l^{-2s-3}\eta(l)^2T_4^{\mathrm{Kl}}(l) - l^{-3s-4}\eta(l)^3T_5^{\mathrm{Kl}}(l) + l^{-4s-4}\eta(l)^4T_6^{\mathrm{Kl}}(l). \end{aligned}$$

Then

$$\ell(T_s(\varphi_1\otimes\varphi_2),\eta,s) = 1.$$

Proof. A direct calculation using Proposition 12.3.2 shows

$$\begin{aligned}&\ell(\varphi_1 \otimes \varphi_2, \eta, s) - l^{-2}X\ell(T_1^{\mathrm{Kl}}(l)(\varphi_1 \otimes \varphi_2), \eta, s) + 2l^{-3}X^2\ell(T_2^{\mathrm{Kl}}(l)(\varphi_1 \otimes \varphi_2), \eta, s)\\ &+ l^{-3}X^2\ell(T_3^{\mathrm{Kl}}(l)(\varphi_1 \otimes \varphi_2), \eta, s) + l^{-3}X^2\ell(T_4^{\mathrm{Kl}}(l)(\varphi_1 \otimes \varphi_2), \eta, s)\\ &- l^{-4}X^3\ell(T_5^{\mathrm{Kl}}(l)(\varphi_1 \otimes \varphi_2), \eta, s) + l^{-4}X^4\ell(T_6^{\mathrm{Kl}}(l)(\varphi_1 \otimes \varphi_2), \eta, s) = 1,\end{aligned}$$

which gives the desired result. □

Later on we will be interested in the case where η is the trivial character, so we will make the following definition

Definition 12.3.4. Let $\mathbf{T} = T_0$ denote the Hecke operator

$$1 - l^{-2}T_1^{\mathrm{Kl}}(l) + 2l^{-3}T_2^{\mathrm{Kl}}(l) + l^{-3}T_3^{\mathrm{Kl}}(l) + l^{-3}T_4^{\mathrm{Kl}}(l) - l^{-4}T_5^{\mathrm{Kl}}(l) + l^{-4}T_6^{\mathrm{Kl}}(l).$$

Chapter 13

Towards norm relations

13.1 A local bilinear form

We keep the notations of the last section. Let χ, ψ be two characters of $\mathbb{Q}_l^\times$ such that $\chi/\psi \neq |.|^{\pm 1}$ and $\chi\psi = \omega^{-1}$. Take $\eta = \psi|.|^{1/2}$.

Lemma 13.1.1. For $\varphi \in \tau \otimes \sigma$, let $z_{s,\varphi} : H(\mathbb{Q}_l) \to \mathbb{C}$ be the function defined by

$$z_{s,\varphi}(h) = Z(h \cdot \varphi, \eta, s). \tag{13.1}$$

Then $z_{s,\varphi} \in I(|.|^{-s}\psi^{-1}, |.|^s\chi^{-1}) \otimes I(|.|^{\frac{1}{2}}, |.|^{-\frac{1}{2}})$.

Proof. This follows from a straightforward calculation. □

In particular, we have an $H(\mathbb{Q}_l)$-equivariant map

$$z_s : \tau \otimes \sigma \to I(|.|^{-s}\psi^{-1}, |.|^s\chi^{-1}) \otimes I(|.|^{\frac{1}{2}}, |.|^{-\frac{1}{2}}), \qquad \varphi \mapsto z_{s,\varphi}.$$

Proposition 13.1.2. Let $\varphi_0 \in \tau \otimes \sigma$ denote the normalized spherical vector. Then

$$z_{s,\varphi_0}(1) = L(\psi/\chi, 2s+1)^{-1}$$
$$z_{s,T_s\cdot\varphi_0}(1) = L(s, \sigma \otimes \tau \otimes \eta)^{-1}$$

Proof. The first part follows directly from Proposition 12.1.2 and the fact $\eta^2\omega = |.|\psi/\chi$. The second part is a direct consequence of Corollary 12.3.3. □

Proposition 13.1.3. Let

$$\langle , \rangle : I(\psi|.|, \chi|.|) \otimes I(|.|^{-1/2|}, |.|^{1/2}) \times I(\psi^{-1}|.|, \chi^{-1}|.|) \otimes I(|.|^{1/2|}, |.|^{-1/2}) \longrightarrow \mathbb{C}$$

be the canonical duality pairing. Then the bilinear form $\mathfrak{z}_{\chi,\psi} \in \mathrm{Hom}_H(I(\chi,\psi) \otimes \tau \otimes \sigma, \mathbb{C})$ defined by

$$\mathfrak{z}_{\chi,\psi}(f \otimes \phi) = \langle M(f) \otimes 1, z_{0,\varphi} \rangle,$$

is nonzero.

Proof. As $\chi/\psi \neq |.|^{\pm 1}$, according to equation (13.2) below, $\mathfrak{z}_{\chi,\psi}(F_{\phi_0}, \varphi_0) \neq 0$, which in particular implies $\mathfrak{z}_{\chi,\psi}$ is nonzero. □

Remark. The assumption $\chi/\psi \neq |.|$ can be lessened just as in [LSZ17]. In fact, as we are only using the constant function $1 \in I(|.|^{-1/2|}, |.|^{1/2})$ in the pairing defining $\mathfrak{z}_{\chi,\psi}$, it factors through $I(\psi|.|, \chi|.|) \otimes \mathbb{C} \times I(\psi^{-1}|.|, \chi^{-1}|.|) \otimes \mathbb{C} \longrightarrow \mathbb{C}$, where the first $\mathbb{C}$ is trivial sub-representation of $I(|.|^{-1/2|}, |.|^{1/2})$, and the second $\mathbb{C}$ is the trivial quotient representation of $I(|.|^{1/2|}, |.|^{-1/2})$. Now write $\tau = I(\chi', \psi')$, and assume the functions $L(\sigma \otimes \psi\psi', s)$ and $L(\psi/\chi, s+1/2)L(\psi'/\chi', s+1/2)$ do not both have a pole at $s = 1/2$. Then the $H(\mathbb{Q}_l)$-equivariant map

$$\sigma \otimes \tau \xrightarrow{z_0} I(\psi^{-1}, \chi^{-1}) \otimes I(|.|^{\frac{1}{2}}, |.|^{-\frac{1}{2}}) \twoheadrightarrow I(\psi^{-1}, \chi^{-1}) \otimes \mathbb{C}$$

factors through the unique irreducible quotient of $\sigma \otimes \tau$, and has its image landing in the unique irreducible sub-representation of $I(\psi^{-1}, \chi^{-1})$. This follows directly from Lemma 3.7.2 of [LSZ17], which in turn builds on an explicit computation using Mackey theory. Then we may define $\mathfrak{z}_{\chi,\psi}(f \otimes \phi) = \lim_{s\to 0} L(\psi/\chi, 2s+1)\langle M(f_s) \otimes 1, z_{s,\varphi}\rangle$. However, for ease of exposition, we will keep assuming $\chi/\psi \neq |.|$ and τ is irreducible.

Exactly the same as in [LSZ17], we have the following multiplicity one result:

Theorem 13.1.4. (Kato-Murase-Sugano) Write $\tau = I(\chi', \psi')$. Suppose neither χ/ψ nor χ'/ψ' is quadratic or $|.|^{\pm 1}$. Then

$$\dim \mathrm{Hom}_{H(\mathbb{Q}_l)}(I(\chi,\sigma) \otimes \tau \otimes \sigma, \mathbb{C}) \leq 1.$$

Proof. This is exactly Theorem 3.7.5 of [LSZ17]. □

To further simplify notation, fixing the characters χ, ψ, for every $\phi \in \mathcal{S}(\mathbb{Q}_l^2, \mathbb{C})$, we write

$$F_\phi := F_{\phi,\chi,\psi} = f_{\hat{\phi},\chi,\psi} \in I(\chi,\psi).$$

We choose $\psi = |.|^{-1/2}$ and $\chi = |.|^{1/2+k}\tau$, where $k \geq 1$ is an integer and τ is a finite-order unramified character. Then $\eta = 1$, and by Proposition 11.2.2, the map $\phi \mapsto F_{\phi,\chi,\psi}$ is $\mathrm{GL}_2(\mathbb{Q}_l)$-equivariant. Moreover, the assumptions

Theorem 13.1.5. Let $\mathfrak{z} \in \mathrm{Hom}_{H(\mathbb{Q}_l)}(I(\chi,\sigma) \otimes \tau \otimes \sigma, \mathbb{C})$. Then for any $t \geq 1$,

$$\mathfrak{z}(F_{\phi_t}, \varphi_0) = \frac{1}{l^{t-1}(l+1)}(1 - \frac{l^k}{\tau(l)})\mathfrak{z}(F_{\phi_0}, \varphi_0),$$
$$\mathfrak{z}(F_{\phi_t}, U \cdot \varphi_0) = \frac{1}{l^{t-1}(l+1)} L(\tau \otimes \sigma, 0)^{-1}\mathfrak{z}(F_{\phi_0}, \varphi_0).$$

Proof. We know $\mathrm{Hom}_{H(\mathbb{Q}_l)}(I(\chi,\psi) \otimes \tau \otimes \sigma, \mathbb{C})$ is 1-dimensional and $\mathfrak{z}_{\chi,\psi}$ spans it, so it suffices to prove the claims for $\mathfrak{z} = \mathfrak{z}_{\chi,\psi}$

By definition, F_{ϕ_t} is the value at $s=0$ of the Siegel section $f_{\hat{\phi}_t,\chi|.|^{-s},\psi|.|^s}$, and by Proposition 11.2.3, we have

$$M(f_{\hat{\phi}_t,\chi|.|^{-s},\psi|.|^s}) = L(\chi/\psi, 1-2s)^{-1} f_{\phi_t,\psi|.|^s,\chi|.|^{-s}}.$$

On the other hand, by Lemma 11.2.5, $f_{\phi_t,\psi|.|^s,\chi|.|^{-s}}$ restricted to $\mathrm{GL}_2(\mathbb{Z}_l)$ is supported on $K_0(l^t)$. Since ϕ_t is invariant under $K_0(l^t)$, it follows that $f_{\phi_t,\psi|.|^s,\chi|.|^{-s}} \otimes 1$ restricted to $H(\mathbb{Z}_l)$ is a scalar multiple of $\mathrm{ch}(K_{0,H}(l^t))$, where $K_{0.H}(l^t) = \{g \in H(\mathbb{Z}_l) : g \equiv (\begin{pmatrix} * & * \\ & * \end{pmatrix}, \begin{pmatrix} * & * \\ * & * \end{pmatrix}) \mod l^t\}$. Moreover, since φ_0 is the spherical vector, it is fixed by $H(\mathbb{Z}_l)$, which means z_{s,φ_0} is constant on $H(\mathbb{Z}_l)$. Therefore

$$\begin{aligned}
\langle M(f_{\hat{\phi}_t,\chi|.|^{-s},\psi|.|^s}), z_{s,\varphi_0}\rangle &= L(\chi/\psi, 1-2s)^{-1} \int_{K_{0,H}(l^t)} f_{\phi_t,\psi|.|^s,\chi|.|^{-s}}(h) z_{s,\varphi_0}(h) dh \\
&= L(\chi/\psi, 1-2s)^{-1} f_{\phi_t,\psi|.|^s,\chi|.|^{-s}}(1) z_{s,\varphi_0}(1) \mathrm{Vol}(K_{0,H}(l^t)) \\
&= \begin{cases} L(\chi/\psi, 1-2s)^{-1} L(\psi/\chi, 1+2s)^{-1} \mathrm{Vol}(H(\mathbb{Z}_l)) & \text{if } t=0; \\ L(\chi/\psi, 1-2s)^{-1} L(\psi/\chi, 1+2s)^{-2} \mathrm{Vol}(K_{0,H}(l^t)) & \text{if } t>0. \end{cases}
\end{aligned}$$

Here in the last equality we have used Lemma 11.2.5 and Proposition 13.1.2. It follows that

$$\begin{aligned}
\mathfrak{z}_{\chi,\psi}(F_{\phi_t}, \varphi_0) &= \lim_{s\to 0} L(\psi/\chi, 2s+1) \langle M(f_{\hat{\phi}_t,\chi|.|^{-s},\psi|.|^s}), z_{s,\varphi_0}\rangle \\
&= \begin{cases} L(\chi/\psi, 1)^{-1} \mathrm{Vol}(H(\mathbb{Z}_l)) & \text{if } t=0; \\ L(\chi/\psi, 1)^{-1} L(\psi/\chi, 1)^{-1} \mathrm{Vol}(K_{0,H}(l^t)) & \text{if } t>0. \end{cases} \qquad (13.2)
\end{aligned}$$

As $\frac{\mathrm{Vol}(K_{0,H}(l^t))}{\mathrm{Vol}(H(\mathbb{Z}_l))} = \frac{1}{[H(\mathbb{Z}_l):K_{0,H}(l^t)]} = \frac{1}{l^{t-1}(l+1)}$ and $L(\psi/\chi,1)^{-1} = 1 - l^{-1}\psi(l)/\chi(l) = 1 - \frac{l^k}{\tau(l)}$, this proves the first part of the claim.

For the second part, note that each $T_i^{\mathrm{Kl}}(l)$ is given by double coset operators which are invariant under $K_{0,H}(l^t)$, which means $z_{s,T_0\cdot\varphi_0}$ is constant on $K_{0,H}(l^t)$. Then a similar argument as above shows

$$\begin{aligned}\mathfrak{z}_{\chi,\psi}(F_{\phi_t}, T_0\cdot\varphi_0) &= \lim_{s\to 0} L(\chi/\psi, 1-2s)^{-1} z_{s,T_0\cdot\varphi_0}(1)\mathrm{Vol}(K_{0,H}(l^t)) \\ &= L(\chi/\psi,1)^{-1}L(0,\sigma\otimes\tau)^{-1}\mathrm{Vol}(K_{0,H}(l^t)),\end{aligned}$$

and the desired conclusion follows immediately. □

13.2 An application of Frobenius reciprocity

Let $\mathcal{H}(G(\mathbb{Q}_l))$ denote the Hecke algebra of locally constant compactly supported $\mathbb{C}$-valued functions on $G(\mathbb{Q}_l)$, with the algebra defined by convolution:

$$(\xi_1 * \xi_2)(g) = \int_{G(\mathbb{Q}_l)} \xi_1(gh^{-1})\xi_2(h)dh$$

for $\xi_1, \xi_2 \in \mathcal{H}(\mathbb{Q}_l)$. We regard the $G(\mathbb{Q}_l)$-representation $\sigma\otimes\tau$ as a left $\mathcal{H}(\mathbb{Q}_l)$-module via

$$\xi\cdot\varphi = \int_{G(\mathbb{Q}_l)} \xi(g)(g\cdot\varphi)dg$$

for any $\xi\in\mathcal{H}(\mathbb{Q}_l)$ and $\varphi\in\sigma\otimes\tau$, so that $g_1\cdot(\xi\cdot(g_2\cdot\varphi)) = \xi(g_1^{-1}(-)g_2^{-1})\cdot\varphi$.

Definition 13.2.1. For smooth representations τ_1, τ_2 of $\mathrm{GL}_2(\mathbb{Q}_l)$ and σ of $\mathrm{GSp}_4(\mathbb{Q}_l)$, we define

$$\mathfrak{X}(\tau_1, (\sigma\otimes\tau_2)^\vee) := \mathrm{Hom}_{\mathcal{H}(G(\mathbb{Q}_l))}(\tau_1\otimes_{\mathcal{H}(H(\mathbb{Q}_l))}\mathcal{H}(G(\mathbb{Q}_l)), (\sigma\otimes\tau_2)^\vee),$$

where τ_1 is regarded as an $H(\mathbb{Q}_l)$-representation via projection of H to its first GL_2-factor.

Proposition 13.2.2. With notations as above, there is a canonical bijection

$$\mathfrak{X}(\tau_1, (\sigma \otimes \tau_2)^\vee) \cong \mathrm{Hom}_{H(\mathbb{Q}_l)}(\sigma \otimes \tau_1 \otimes \tau_2, \mathbb{C}).$$

Proof. This is analogous to Proposition 3.9.1 of [LSZ17]. Note that $\tau_1 \otimes_{\mathcal{H}(H(\mathbb{Q}_l))} \mathcal{H}(G(\mathbb{Q}_l))$ is the Hecke module corresponding to the smooth $G(\mathbb{Q}_l)$-representation $c\text{-Ind}_H^G \tau_1$ ([Ren10],III.2.6), where c-Ind denotes compact induction. Thus

$$\begin{aligned}\mathfrak{X}(\tau_1, (\sigma \otimes \tau_2)^\vee) &= \mathrm{Hom}_{G(\mathbb{Q}_l)}(c\text{-Ind}_H^G \tau_1, (\sigma \otimes \tau_2)^\vee) \\ &\cong \mathrm{Hom}_{G(\mathbb{Q}_l)}(\sigma \otimes \tau_2, (c\text{-Ind}_H^G \tau_1)^\vee).\end{aligned}$$

According to [Ren10], III.2.7, there is a canonical isomorphism

$$(c\text{-Ind}_H^G \tau_1)^\vee \cong \mathrm{Ind}_H^G(\tau_1^\vee).$$

Finally, by Frobenius reciprocity for non-compact induction ([Ren10], III.2.5), we see

$$\begin{aligned}\mathrm{Hom}_{G(\mathbb{Q}_l)}(\sigma \otimes \tau_2, \mathrm{Ind}_H^G(\tau_1^\vee)) &\cong \mathrm{Hom}_{H(\mathbb{Q}_l)}((\sigma \otimes \tau_2)|_H, \tau_1^\vee) \\ &\cong \mathrm{Hom}_{H(\mathbb{Q}_l)}(\sigma \otimes \tau_1 \otimes \tau_2, \mathbb{C})\end{aligned}$$

as desired. □

Unwinding the definitions of all the isomorphisms used above, the bijection in Proposition 13.2.2 is given explicitly as follows: given $\mathfrak{Z} \in \mathfrak{X}(\tau_1, (\sigma \otimes \tau_2)^\vee)$, the corresponding $\mathfrak{z} \in \mathrm{Hom}_{H(\mathbb{Q}_l)}(\sigma \otimes \tau_1 \otimes \tau_2, \mathbb{C})$ is characterized by

$$\mathfrak{Z}(f \otimes \xi)(\varphi \otimes g) = \mathfrak{z}(f \otimes \xi \cdot (\varphi \otimes g)). \tag{13.3}$$

Lemma 13.2.3. Let $\sigma \otimes \tau$ be a smooth representation of $G(\mathbb{Q}_l)$, where σ (respectively,

τ) is a smooth representation of $\mathrm{GSp}_4(\mathbb{Q}_l)$ (respectively, $\mathrm{GL}_2(\mathbb{Q}_l)$). For every $R \in \mathcal{H}(G(\mathbb{Q}_l))$, let $R' \in \mathcal{H}(G(\mathbb{Q}_l))$ be the function defined by $R'(g) = R(g^{-1})$. Then for any $\Phi \in (\sigma \otimes \tau)^\vee$ and $\varphi \in \sigma \otimes \tau$, we have

$$\Phi(R \cdot \varphi) = R' \cdot \Phi(\varphi).$$

Proof. By linearity of Φ we have

$$\Phi(R \cdot \varphi) = \Phi(\int_G (\mathbb{Q}_l) R(g) g \cdot \varphi dg) = \int_{G(\mathbb{Q}_l)} R(g) \Phi(g \cdot \varphi) dg.$$

On the other hand,

$$R' \cdot \varphi = \int_{G(\mathbb{Q}_l)} R(g^{-1}) g \cdot \Phi(\varphi) dg = \int_{G(\mathbb{Q}_l)} R(g^{-1}) \Phi(g^{-1} \cdot \varphi) dg$$

which equals to $\Phi(R \cdot \varphi)$ as required. □

In light of the above lemma, we let $U_i'(l)$ denote the Hecke operator associated to $U_i(l)$ as in the lemma, $i = 1, 2$. In particular, if we write $U_i(l)$ as

$$\frac{1}{\mathrm{Vol}(B_{m,n})} \mathrm{ch}(B_{m,n} u_i B_{m,n})$$

where $u_i \in G(\mathbb{Q}_l)$ is given as in Definition 12.3.1, then $U_i'(l)$ is given by

$$\frac{1}{\mathrm{Vol}(B_{m,n})} \mathrm{ch}(B_{m,n} u_i^{-1} B_{m,n}).$$

We slightly generalize the notation in Definition 11.2.4:

Definition 13.2.4. For integers $s, t \geq 0$, let $\phi_{1,s,t} \in \mathcal{S}(\mathbb{Q}_l^2, \mathbb{C})$ denote the function

$\mathrm{ch}(l^s\mathbb{Z}_l \times (1 + l^t\mathbb{Z}_l))$, and let

$$K_1(l^s, l^t) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_l) : c \equiv 0 \mod l^s, d \equiv 1 \mod l^t \}.$$

Remark. Note that when $s \geq t$, $K_1(l^s, l^t)$ is a subgroup of $\mathrm{GL}_2(\mathbb{Z}_l)$ preserving $\phi_{1,s,t}$.

Now we take τ_1 to be $\mathcal{S}(\mathbb{Q}_l^2, \mathbb{C})$, where $\mathrm{GL}_2(\mathbb{Q}_l)$ acts via $(g \cdot \phi)(x, y) = \phi((x, y)g)$. Let W be an arbitrary smooth complex representation of $G(\mathbb{Q}_l)$. We denote by $\mathfrak{X}(W)$ the space of homomorphisms $\mathcal{S}(\mathbb{Q}_l^2, \mathbb{C}) \otimes_{\mathcal{H}(H(\mathbb{Q}_l))} \mathcal{H}(G(\mathbb{Q}_l)) \to W$ which are $\mathrm{G}(\mathbb{Q}_l)$-equivariant. In other words, in the definition of $\mathfrak{X}(\tau_1, (\sigma \otimes \tau_2)^\vee)$ we take $\tau_1 = \mathcal{S}(\mathbb{Q}_l^2, \mathbb{C})$ and replace $(\sigma \otimes \tau_2)^\vee$ with W.

Lemma 13.2.5. Let $\xi \in \mathcal{H}(G(\mathbb{Q}_l))$ be invariant under left-translation by the principal congruence subgroup of level l^T in $H(\mathbb{Z}_l)$ for some integer $T \geq 0$. Then for any $\mathfrak{Z} \in \mathfrak{X}(W)$ and integers $s \geq t \geq T$, the expression

$$\frac{1}{\mathrm{Vol}(K_1(l^s, l^t))} \mathfrak{Z}(\phi_{1,s,t} \otimes \xi)$$

is independent of s and t.

Proof. This is analogous to Lemma 3.10.2 of [LSZ17]. Note that $l^s\mathbb{Z}_l \times (1 + l^t\mathbb{Z}_l)$ is precisely the orbit of $(0, 1) \in \mathbb{Q}_l^2$ under the right action of $K_1(l^s, l^t)$, which implies that

$$\phi_{1,T,T} = \sum_{\gamma \in J} \gamma \cdot \phi_{1,s,t},$$

where J is a set of representatives for the quotient $K_1(l^T)/K_1(l^s, l^t)$. We may take J to be a subset of the principal congruence subgroup of $\mathrm{GL}_2(\mathbb{Z}_l)$ modulo l^T, so by assumption for all $\gamma \in J$, the element $(\gamma, \gamma) \in H(\mathbb{Z}_l)$ will act trivially on ξ. By

$H(\mathbb{Q}_l)$-equivariance of $\mathfrak{Z}$, we see

$$\begin{aligned}\mathfrak{Z}(\phi_{1,T,T}\otimes\xi) &= \sum_{\gamma\in J}\mathfrak{Z}(\gamma\cdot\phi_{1,s,t}\otimes\xi) = \sum_{\gamma\in J}\mathfrak{Z}(\phi_{1,s,t}\otimes(\gamma,\gamma)^{-1}\cdot\xi) = (\#J)\cdot\mathfrak{Z}(\phi_{1,s,t}\otimes\xi)\\ &= \frac{\mathrm{Vol}(K_1(l^T))}{\mathrm{Vol}(K_1(l^s,l^t))}\mathfrak{Z}(\phi_{1,s,t}\otimes\xi)\end{aligned}$$

which concludes the proof. □

In light of Lemma 13.2.5, we will write $\mathfrak{Z}(\phi_{1,\infty}\otimes\xi)$ for this limiting value.

13.3 Main results towards vertical norm relations

The goal of this section is to prove Corollary 13.3.3, which is an analogue of Proposition 3.10.4 of [LSZ17] and serves as the building block for the vertical norm relation of our anticipated Euler system. Results obtained in this part do not reply on the multiplicity one argument and are proved directly.

Before stating the results, it is a good time to review all the assumptions we have made on the representations. Except explicitly stated otherwise, the same assumptions will also be effective in the next section.

- $\sigma = \chi_1 \times \chi_2 \rtimes \rho$ is an irreducible principal series of $\mathrm{GSp}_4(\mathbb{Q}_l)$, where χ_1, χ_2, ρ are smooth characters of $\mathbb{Q}_l^\times$ such that $|.|^{\pm 1} \notin \{\chi_1, \chi_2, \chi_1\chi, \chi_1/\chi_2\}$;
- $\tau = I(\chi', \psi')$ is an irreducible principal series of $\mathrm{GL}_2(\mathbb{Q}_l)$, where χ', ψ' are smooth characters of $\mathbb{Q}_l^\times$ satisfying χ'/ψ' is neither $|.|^{\pm 1}$ nor quadratic, so that Proposition 11.3.5 applies.
- $\chi = |.|^{1/2+k}\tau$ and $\psi = |.|^{-1/2}$ are smooth characters of $\mathbb{Q}_l^\times$ where $k \geq 1$ is an integer, and τ is an unramified character of finite order. Moreover, $\chi\psi\chi'\psi'\chi_1\chi_2\rho^2 = 1$.

Again we remark here that as discussed before, some of the above assumptions can be lessened, especially the requirements $\chi'/\psi' \neq |.|$ and $k \geq 1$.

For $a, b, c \in \mathbb{Z}_l$ and integer $m \geq 1$, let

$$\eta_m^{a,b,c} = \left(\begin{pmatrix} 1 & al^{-m} & bl^{-m} & \\ & 1 & & bl^{-m} \\ & & 1 & -al^{-m} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & cl^{-m} \\ & 1 \end{pmatrix} \right) \in G(\mathbb{Q}_l).$$

Proposition 13.3.1. Let $k \geq \max\{m, n\}$ be an integer, and $c \in \mathbb{Z}_l$ such that $c^2 \in l^k\mathbb{Z}_l$. For any $\mathfrak{Z} \in \mathfrak{X}(W)$, we have

$$\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,lc} B_{m,n})) = \begin{cases} \frac{U_2'(l)}{l^2} \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_k^{0,1,c} B_{m,n})) & \text{if } k \geq 1; \\ \frac{U_2'(l)-1}{l^2-1} \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_k^{0,1,c} B_{m,n})) & \text{if } k = 0. \end{cases}$$

Proof. A direct computation shows that when $k \geq \max\{m, n\}$, the conjugate of the principal congruent subgroup of level l^{2k} in $H(\mathbb{Z}_l) \subset G(\mathbb{Z}_l)$ by $\eta_k^{0,1,c}$ is a subgroup of $B_{m,n}$. This implies both $\mathrm{ch}(\eta_{k+1}^{0,1,c} B_{m,n})$ and $\mathrm{ch}(\eta_k^{0,1,c} B_{m,n})$ are invariant under the left action of the principal congruent subgroup of level l^{2k+2} in $H(\mathbb{Z}_l)$. By Lemma 13.2.5, we may choose any $s \geq t \geq T := 2k + 2$ and replace $\phi_{1,\infty}$ with $\phi_{1,s,t}$ in the computations. We will make an additional assumption $s \geq t + \max\{k, 1\}$ to further simplify the calculations later on.

The double coset decomposition in Proposition 12.2.4 together with Lemma 13.2.3

implies that

$$U_2'(l)\cdot\mathfrak{Z}(\phi_{1,s,t}\otimes \mathrm{ch}(\eta_k^{0,1,c}B_{m,n}))$$
$$=\sum_{0\le\lambda,\mu<l,0\le\kappa<l^2}\mathfrak{Z}(\phi_{1,s,t}\otimes \mathrm{ch}(\eta_k^{0,1,c}(\begin{pmatrix} l^2 & l\lambda & l\mu & \kappa\\ & l & & \mu\\ & & l & -\lambda\\ & & & 1\end{pmatrix},\begin{pmatrix} l & \\ & l\end{pmatrix})B_{m,n})).$$

Since

$$\eta_k^{0,1,c}(\begin{pmatrix} l^2 & l\lambda & l\mu & \kappa\\ & l & & \mu\\ & & l & -\lambda\\ & & & 1\end{pmatrix},\begin{pmatrix} l & \\ & l\end{pmatrix})=(\begin{pmatrix} l^2 & \kappa-\lambda l^{-k}\\ & 1\end{pmatrix},\begin{pmatrix} l & \\ & l\end{pmatrix})\eta_{k+1}^{l^k\lambda,1+l^k\mu,lc},$$

where on the right hand side we are implicitly using the natural inclusion $H\hookrightarrow G$, by $H(\mathbb{Q}_l)$-equivariance we see that each term in the last summation equals

$$\mathfrak{Z}((\begin{pmatrix} l^2 & \kappa-\lambda l^{-k}\\ & 1\end{pmatrix},\begin{pmatrix} l & \\ & l\end{pmatrix})^{-1}\cdot\phi_{1,s,t}\otimes \mathrm{ch}(\eta_k^{0,1,c}B_{m,n})).$$

Since $s\ge k+t$, we have $\begin{pmatrix} l^2 & \kappa-\lambda l^{-k}\\ & 1\end{pmatrix}^{-1}\cdot\phi_{1,s,t}=\phi_{1,s+2,t}$. Thus

$$U_2'(l)\cdot\mathfrak{Z}(\phi_{1,s,t}\otimes \mathrm{ch}(\eta_k^{0,1,c}B_{m,n}))=l^2\sum_{0\le\lambda,\mu<l}\mathfrak{Z}(\phi_{1,s+2,t}\otimes \mathrm{ch}(\eta_{k+1}^{l^k\lambda,1+l^k\mu,lc}B_{m,n})).\quad(13.4)$$

First we deal with the case $k \geq 1$. Write

$$A = \left(\begin{pmatrix} 1+l^k\mu & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1+l^k\mu & \\ -l^k\lambda & 1 \end{pmatrix} \right) \in H(\mathbb{Z}_l),$$

which fixes $\phi_{1,s+2,t}$, and

$$B = \left(\begin{pmatrix} 1+l^k\mu & & \\ & 1+l^k\mu & \\ & -l^k\lambda & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1+l^k\mu+c\lambda & c\mu+vc^2l^{-k} \\ -l^k\lambda & 1-c\lambda \end{pmatrix} \right) \in B_{m,n}$$

which lies in $B_{m,n}$ as $k \geq \max\{m,n\}$ and $c^2 \in l^k\mathbb{Z}_l$. We have

$$A^{-1}\eta_{k+1}^{l^k\lambda,1+l^k\mu,lc}B = \eta_{k+1}^{0,1,lc}.$$

Therefore (13.4) becomes

$$U_2'(l) \cdot \mathfrak{Z}(\phi_{1,s,t} \otimes \mathrm{ch}(\eta_k^{0,1,c}B_{m,n})) = l^4\mathfrak{Z}(\phi_{1,s+2,t} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,lc}B_{m,n})),$$

which implies

$$\begin{aligned} U_2'(l) \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_k^{0,1,c}B_{m,n})) &= l^4 \frac{\mathrm{Vol}(K_1(l^T))}{\mathrm{Vol}(K_1(l^s,l^t))}\mathfrak{Z}(\phi_{1,s+2,t} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,lc}B_{m,n})) \\ &= l^2 \frac{\mathrm{Vol}(K_1(l^T))}{\mathrm{Vol}(K_1(l^{s+2},l^t))}\mathfrak{Z}(\phi_{1,s,t} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,lc}B_{m,n})) \\ &= l^2\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,lc}B_{m,n})) \end{aligned}$$

as desired.

Next assume $k = 0$, which forces $m = n = 0$, i.e. $B_{m,n} = G(\mathbb{Z}_l)$. The same

argument works as in the previous case except when $\mu = l-1$, in which case A no longer lies in $H(\mathbb{Z}_l)$. Since

$$\eta_1^{\lambda,l,lc} = (\begin{pmatrix} 1 & -\lambda/l \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix})\eta_1^{\lambda,0,0}\eta_1^{0,l,lc},$$

where $\eta_1^{0,l,lc} \in B_{0,0} = G(\mathbb{Z}_l)$ and $(\begin{pmatrix} 1 & -\lambda/l \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix})$ fixes $\phi_{1,s+2,t}$ as $s+2 > t+1$, we see that each summand with $\mu = l-1$ in the right hand side of (13.4) equals to $\mathfrak{Z}(\phi_{1,s+2,t} \otimes \mathrm{ch}(\eta_1^{\lambda,0,0}B_{0,0}))$.

For $\lambda \neq 0$, writting $B = (\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ -\lambda & \end{pmatrix})$ we have

$$\eta_1^{\lambda,0,0} = B\eta_1^{0,1,0}B^{-1}$$

and $B \in H(\mathbb{Z}_l) \subset B_{0,0}$ fixes $\phi_{1,s+2,t}$, so

$$\mathfrak{Z}(\phi_{1,s+2,t} \otimes \mathrm{ch}(\eta_1^{\lambda,0,0}B_{0,0})) = \mathfrak{Z}(\phi_{1,s+2,t} \otimes \mathrm{ch}(\eta_1^{0,1,0}B_{0,0}))$$

just as in the cases where $\mu \neq l-1$. Finally for $\mu = 0$, we clearly have $\eta_1^{0,0,0}B_{0,0} = \eta_0^{0,1,c}B_{0,0} = B_{0,0}$. Combining above we see (13.4) becomes

$$\begin{aligned} &U_2'(l) \cdot \mathfrak{Z}(\phi_{1,s,t} \otimes \eta_0^{0,1,c}\mathrm{ch}(B_{0,0})) \\ =&l^2(l^2-1)\mathfrak{Z}(\phi_{1,s+2,t} \otimes \mathrm{ch}(\eta_1^{0,1,lc}B_{0,0})) + l^2\mathfrak{Z}(\phi_{1,s+2,t} \otimes \mathrm{ch}(B_{0,0,0})) \end{aligned}$$

which immediately implies the desired claim by passing to $\phi_{1,\infty}$ as in the previous case.

□

Proposition 13.3.2. Let $k \geq \max\{m, 1\}$ be an integers, and $0 \neq c \in \mathbb{Z}_l$ such that $l^k c^{-1} \in l\mathbb{Z}_l$. For any $\mathfrak{Z} \in \mathfrak{X}(W)$, we have

$$\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,c} B_{m,n})) = \frac{U_1'(l)}{l^3} \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_k^{0,1,c} B_{m,n})).$$

Moreover, in the case $c = l^k$, we have

$$\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,l^k} B_{m,n})) = \frac{U_1'(l) \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_k^{0,1,0} B_{m,n})) - l^2 \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,0} B_{m,n}))}{l^2(l-1)}.$$

Proof. The proof is very similar to that of Proposition 13.3.1. We sketch it below.

Similar to above we can choose $s \geq t \geq T := 2k + 2 + n$ and replace $\phi_{1,\infty}$ with $\phi_{1,s,t}$ in the computation.

By the double coset decomposition for $U_1(l)$ we see

$$\begin{aligned} & U_1'(l) \cdot \mathfrak{Z}(\phi_{1,s,t} \otimes \mathrm{ch}(\eta_k^{0,1,c} B_{m,n})) \\ &= \sum_{0 \leq u,v,w,x < l} \mathfrak{Z}(\phi_{1,s,t} \otimes \mathrm{ch}(\eta_k^{0,1,c}(\begin{pmatrix} l & & u & v \\ & l & w & u \\ & & l & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l & x \\ & 1 \end{pmatrix}) B_{m,n})). \end{aligned}$$

As

$$\eta_k^{0,1,c}(\begin{pmatrix} l & & u & v \\ & l & w & u \\ & & l & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l & x \\ & 1 \end{pmatrix}) = (\begin{pmatrix} l & v \\ & 1 \end{pmatrix}, \begin{pmatrix} l & w \\ & 1 \end{pmatrix}) \eta_{k+1}^{0,1+l^k u, c+l^k(x-w)},$$

the sum become

$$l \sum_{0 \le u,w,x < l} \mathfrak{Z}(\phi_{1,s+1,t} \otimes \mathrm{ch}(\eta_{k+1}^{0,1+l^k u, c+l^k(x-w)} B_{m,n})). \tag{13.5}$$

Since $k \ge 1$, $\eta_{k+1}^{0,1+l^k u, c+l^k(x-w)}$ is conjugate to $\eta_{k+1}^{0,1,c}$ via the element

$$\left(\begin{pmatrix} (1+l^k u)^2 & \\ & 1+l^k c^{-1}(x-w) \end{pmatrix}, \begin{pmatrix} (1+l^k u)(1+l^k c^{-1}(x-w)) & \\ & 1+l^k u \end{pmatrix} \right)$$

in $H(\mathbb{Q}_l) \cap B_{m,n}$, which also fixes $\phi_{1,s+1,t}$. Thus we obtain

$$U_1'(l) \cdot \mathfrak{Z}(\phi_{1,s,t} \otimes \mathrm{ch}(\eta_k^{0,1,c} B_{m,n})) = l^4 \mathfrak{Z}(\phi_{1,s+1,t} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,c} B_{m,n}))$$

which gives the desired result as in the previous proof.

Finally, we quickly dispose of the case when $c = l^k$. The above argument still holds except when l divides $1 + x - w$, so that the conjugating matrix is no longer invertible over $\mathbb{Z}_l$. In this case, $\eta_{k+1}^{0,1+l^k u, c+l^k(x-w)}$ and $\eta_{k+1}^{0,1+l^k u,0}$ only differ at the GL_2-component by an element in $\mathrm{GL}_2(\mathbb{Q}_l)$, so the two define the same left coset of $B_{m,n}$. As $\eta_{k+1}^{0,1+l^k u,0}$ is conjugate to $\eta_{k+1}^{0,1,0}$ via $\left(\begin{pmatrix} 1+l^k u & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1+l^k u & \\ & 1 \end{pmatrix} \right) \in H(\mathbb{Q}_l) \cap B_{m,n}$, we see the contribution of the terms with l dividing $1 + x - w$ to the sum (13.5) is $l^3 \mathfrak{Z}(\phi_{1,s+1,t} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,0} B_{m,n}))$, and in total we have

$$\begin{aligned} U_1'(l) \cdot \mathfrak{Z}(\phi_{1,s,t} \otimes \mathrm{ch}(\eta_k^{0,1,l^k} B_{m,n})) =& l^3(l-1)\mathfrak{Z}(\phi_{1,s+1,t} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,l^k} B_{m,n})) \\ &+ l^3 \mathfrak{Z}(\phi_{1,s+1,t} \otimes \mathrm{ch}(\eta_{k+1}^{0,1,0} B_{m,n})), \end{aligned}$$

which completes the proof. □

Now choose $k = 2m$, $c = l^m$ in Proposition 13.3.1 and $k = 2m+1$, $c = l^{m+1}$ in Proposition 13.3.2. To further simplify notation let $U(l) = U_1(l)U_2(l)$ and $\eta_m := \eta_{2m}^{0,1,l^m}$. We immediately obtain the following

Corollary 13.3.3. Assume $n \leq 2m$. Then for any $\mathfrak{Z} \in \mathfrak{X}(W)$, we have

$$\begin{aligned}
&\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_{m+1} B_{m,n})) \\
&= \begin{cases} \frac{U'(l)}{l^5} \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_m B_{m,n})) & \text{if } m \geq 1; \\ \frac{(U'(l)-1)\cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(B_{m,n})) - l^2(l^2-1)\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_2^{0,1,0} B_{m,n})) - l^2(l^2-1)}{l^2(l-1)(l^2-1)} & \text{if } m = 0. \end{cases}
\end{aligned}$$

Remark. This is the analogue of Proposition 3.10.4 of [LSZ17], and will be the main technical tool to prove the norm relations in the vertical (i.e. p-) direction. The reason we are interested in the $U(l)$ operator instead of only $U_1(l)$ or $U_2(l)$ as in the previous two propositions is as follows. Consider the case where we take a cuspidal Siegel modular eigenform of genus 2 and weight 3 which corresponds to our σ, and an elliptic eigenform of weight 2 which corresponds to our τ. The degree 8 Galois representation $V = V_\sigma^* \otimes V_\tau^*$ we will be interested in, which is the dual of the Galois representation associated to $\sigma \otimes \tau$, has Hodge-Tate weights $\{0, 1, 1, 2, 2, 3, 3, 4\}$ (recall our convention is the cyclotomic character has Hodge-Tate weight 1). Conjecture 7.4 of [LZ17] predicts that to get a rank-1 Euler system we would need to look at $V(-1)$, which would also need to satisfy the so-called rank-1 Panchishkin condition (see Definitions 7.1 and 7.2 of [LZ17]). In particular, we would need $V(-1)$ to have a $G_{\mathbb{Q}_p}$-stable subspace V^+ with all Hodge-Tate weights ≥ 1, and $V(-1)/V^+$ has all Hodge-Tate weights ≤ 0. As $V(-1)$ has Hodge-Tate weights $\{-1, 0, 0, 1, 1, 2, 2, 3\}$, this would require V^+ to be the 5-dimensional subspace with Hodge-Tate weight $\{1, 1, 2, 2, 3\}$. In light of [Urb05], Corollary 1, which says that V_σ has a $G_{\mathbb{Q}_p}$-stable line if σ is Siegel-ordinary, and a $G_{\mathbb{Q}_p}$-stable plane if σ is Klingen-ordinary, the existence

of such a V^+ is guaranteed by $\sigma \otimes \tau$ being both $U_1(l)$-ordinary and $U_2(l)$-ordinary.

13.4 Main result towards tame norm relations

We are now ready to state and prove the following analogue of Corollary 3.10.5 of [LSZ17], from which the tame norm relations of the anticipated Euler system would follow directly.

We now go back to the situation where $W = (\sigma \otimes \tau)^\vee$. We keep all the assumptions made at the beginning of the last section. Recall the Hecke operator defined in Definition 12.3.4:

$$\mathbf{T} = 1 - l^{-2}T_1^{\mathrm{Kl}}(l) + 2l^{-3}T_2^{\mathrm{Kl}}(l) + l^{-3}T_3^{\mathrm{Kl}}(l) + l^{-3}T_4^{\mathrm{Kl}}(l) - l^{-4}T_5^{\mathrm{Kl}}(l) + l^{-4}T_6^{\mathrm{Kl}}(l).$$

We will also refer to $\mathbf{T}$ as the function on $G(\mathbb{Q}_l)$ underlying the Hecke operator.

Theorem 13.4.1. Let $K = G(\mathbb{Z}_l)$. There exists an element $\xi \in \mathcal{H}(G(\mathbb{Q}_l), \mathbb{Q})$ of the form $\xi = \sum_{\eta \in J} m_\eta \mathrm{ch}(\eta K)$ with J s finite subset of $G(\mathbb{Q}_l)$ and each $m_\eta \in \mathbb{Z}$, so that

$$\mathfrak{Z}(\phi_{1,\infty} \otimes \xi) = l^4 L(\tau \otimes \sigma, 0)^{-1} \mathfrak{Z}(\phi_0 \otimes \mathrm{ch}(K)).$$

Proof. Translating the second identity of Theorem 13.1.5 in terms of $\mathfrak{Z}$ together with the fact that the representation $\sigma \otimes \tau$ is irreducible and thus generated by φ_0 gives

$$\mathfrak{Z}(\phi_t \otimes \mathbf{T}) = \frac{1}{l^{t-1}(l+1)\mathrm{Vol}(K_{m,n})} L(\tau \otimes \sigma, 0)^{-1} \mathfrak{Z}(\phi_0 \otimes \mathrm{ch}(K_{m,n}))$$

as elements of $\mathfrak{X}((\sigma \otimes \tau)^\vee)$.

Recall $\phi_t = \mathrm{ch}(l^t\mathbb{Z}_l) \times \mathbb{Z}_l^\times$ and $\phi_{1,t,t} = \mathrm{ch}(l^t\mathbb{Z}_l) \times (1 + l^t\mathbb{Z}_l)$ are related via

$$\phi_t = \sum_{a \in (\mathbb{Z}_l/l^t)^\times} \begin{pmatrix} 1 & \\ & a \end{pmatrix} \cdot \phi_{1,t,t},$$

hence by $H(\mathbb{Q}_l)$-equivariance of $\mathfrak{Z}$,

$$\mathfrak{Z}(\phi_t \otimes U)(\varphi_0) = \sum_{a \in (\mathbb{Z}_l/l^t)^\times} \mathfrak{Z}(\phi_{1,t,t}) \otimes \mathbf{T}\left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & a & \\ & & & a \end{pmatrix} \cdot (-)\right) = l^{t-1}(l-1)\mathfrak{Z}(\phi_{1,t,t} \otimes \mathbf{T}),$$

where in the last identity we used the fact that $\mathbf{T}$ is a linear combination of double coset operators of $K_{m,n}$, and diagonal matrices are in $K_{m,n}$. Therefore

$$\mathrm{Vol}(K_{m,n})\mathfrak{Z}(\phi_{1,\infty} \otimes \mathbf{T}) = L(\tau \otimes \sigma, 0)^{-1}\mathfrak{Z}(\phi_0 \otimes \mathrm{ch}(K_{m,n})), \tag{13.6}$$

where we have used the fact that $[\mathrm{GL}_2(\mathbb{Z}_l) : K_1(l^t, l^t)] = l^{2t-2}(l^2 - 1)$.

We would like to make the elements in the Hecke algebra appearing as the second argument of $\mathfrak{Z}$ right invariant under $K = G(\mathbb{Z}_l)$. To achieve it we sum over the action of representatives of $K/K_{m,n}$. On the right hand side of (13.6) we obtain (omitting the factor $L(\tau \otimes \sigma, 0)^{-1}$ for brevity)

$$\sum_{\gamma \in K/K_{m,n}} \gamma \cdot \mathfrak{Z}(\phi_0 \otimes \mathrm{ch}(K_{m,n})) = \mathfrak{Z}(\phi_0 \otimes \sum_{\gamma \in K/K_{m,n}} \mathrm{ch}(K_{m,n})((-) \cdot \gamma)) = \mathfrak{Z}(\phi_0 \otimes \mathrm{ch}(K)).$$

On the left hand side, recall that $l^4\mathbf{T}$ is a $\mathbb{Z}$-linear combination of 1 (the identity operator) and $T_i^{\mathrm{Kl}}(l) = \frac{1}{\mathrm{Vol}(K_{m,n})}\mathrm{ch}(K_{m,n}t_iK_{m,n})$ for $i = 1, ..., 6$ (where $t_i \in G(\mathbb{Q}_l)$ is as given in Definition 12.3.1), each of which is a sum of the form $\sum_{\eta \in J_i} \frac{1}{\mathrm{Vol}(K_{m,n})}\mathrm{ch}(\eta K_{m,n})$,

where each J_i is the set of left $K_{m,n}$-coset representatives of $K_{m,n}t_iK_{m,n}$ as given by Lemma 12.2.3 and Proposition 12.2.4 (see also the remark thereafter; 1 can be interpreted as $\frac{1}{\mathrm{Vol}(K_{m,n})}\mathrm{ch}(K_{m,n})$, so we set $J_0 = (I_4, I_2)$ the set with only the identity element of $G(\mathbb{Q}_l)$). Therefore we may write

$$\mathbf{T} = \frac{1}{l^4\mathrm{Vol}(K_{m,n})}\sum_{i=0}^{6} m_i \sum_{\eta\in J_i}\mathrm{ch}(\eta K_{m,n})$$

with $m_i \in \mathbb{Z}$ the corresponding coefficient as given in Corollary 12.3.3, and we have

$$\begin{aligned}\sum_{\gamma\in K/K_{m,n}} \gamma\cdot\mathrm{Vol}(K_{m,n})\mathfrak{Z}(\phi_{1,\infty}\otimes\mathbf{T}) &= \frac{1}{l^4}\sum_{i=0}^{6} m_i\sum_{\eta\in J_i}\mathfrak{Z}(\phi_{1,\infty}\otimes\sum_{\gamma\in K/K_{m,n}}\gamma\cdot\mathrm{ch}(\eta K_{m,n}))\\ &= \frac{1}{l^4}\sum_{i=0}^{6} m_i\sum_{\eta\in J_i}\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta K)).\end{aligned}$$

Comparing the sum of the two sides of (13.6) gives an identity of exactly the form as required. This completes the proof. □

13.5 Divisibility of m_η

In this last section, we record a divisibility result of the multiplicities m_η that will be useful in the future construction of the Euler system.

In fact, once we move on to the actual construction of the Euler system, it will be clear that in order to obtain p-adically integral cohomology classes, the Hecke element ξ needs to satisfy the following:

Assumption 13.5.1. Let $V = \{h = (h_1, h_2) \in H(\mathbb{Z}_l) : h_1 = \begin{pmatrix} * & * \\ 0 & 1\end{pmatrix}\}$ be the stabilizer of $\phi_{1,\infty}$ in $H(\mathbb{Z}_l)$, and put $V_\eta = V \cap \eta K_{0,1}\eta^{-1}$, where recall $K_{0,1} = \{h \in H(\mathbb{Z}_l) : \mu(h) = 1 \pmod{l}\}$. Then one needs the index $[V : V_\eta]$ to divide the multiplicity m_η,

as elements of $\mathbb{Z}_p$.

One can easily see that the prime-to-l part of V has order $(l+1)(l-1)^2$. For the purpose of arithmetic applications (i.e. bounding Selmer groups), we will be only interested at primes l congruent to 1 modulo p, so $l+1$ will not pose a problem. On the other hand, in general it could (and indeed often) happen that m_η needs to be divisible by $(l-1)^2$.

We will prove that ξ can be chosen such that the $l-1$ part of the assumption holds for every η appearing in ξ. More precisely:

Proposition 13.5.2. In the notation of Theorem 13.4.1, the element ξ can be chosen such that each m_η is divisible by $l-1$. Furthermore, if the index $[V : V_\eta]$ is divisible by $(l-1)^2$, then so is m_η.

The rest of this section is devoted to the proof of this proposition. The overall idea is simple and has been used in the computations in the last section: many elements in J_i, the set of coset representatives associated to the double coset operator $T_i^{\mathrm{Kl}}(l)$, are conjugate to each other via elements in V, and thus have the same action on $\mathfrak{Z}$. We will group the representatives into conjugacy classes and show the combined multiplicities have the desired divisibility. In fact, we will show that most of those combined multiplicities are divisible by $(l-1)^2$, and then examine the few rest to show the corresponding index $[V : V_\eta]$, after divided by $l-1$, is coprime to $l-1$ (up to a power of 2).

Recall the notation

$$\eta_m^{a,b,c} = \left(\begin{pmatrix} 1 & al^{-m} & bl^{-m} & \\ & 1 & & bl^{-m} \\ & & 1 & -al^{-m} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & cl^{-m} \\ & 1 \end{pmatrix} \right).$$

In addition, we will also write

$$\theta_m^{a,b,c} = \left(\begin{pmatrix} 1 & al^{-m} & bl^{-m} & \\ & 1 & & bl^{-m} \\ & & 1 & -al^{-m} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l & cl^{1-m} \\ & l^{-1} \end{pmatrix} \right)$$

and

$$\iota_m^{a,b,c} = \left(\begin{pmatrix} 1 & al^{-m} & bl^{-m} & \\ & 1 & & bl^{-m} \\ & & 1 & -al^{-m} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} l^{-1} & cl^{1-m} \\ & l \end{pmatrix} \right).$$

Lemma 13.5.3. For $i = 0, 1, ..., 6$, let $t_i \in G(\mathbb{Q}_l)$ be the matrix defining the double coset associated to $T_i^{\mathrm{Kl}}(l)$ as in Definition 12.3.1 (as before t_0 is understood to be the identity element (I_4, I_2)), and J_i the set of left $K_{m,n}$-coset representatives of $K_{m,n} t_i K_{m,n}$ as given by Lemma 12.2.3, Proposition 12.2.4 and the remark thereafter. Then for each $i = 0, 1, ..., 6$, the sum

$$S_i := \sum_{\eta \in J_i} \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta K)) \tag{13.7}$$

can be put into the form of

$$S_i = \sum_{\eta' \in J_i'} n_{\eta',i} \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta' K)), \tag{13.8}$$

where J_i' is a finite subset of $G(\mathbb{Q}_l)$, and $n_{\eta',i} \in (l-1)\mathbb{Z}$, except when η is one of $\eta_1^{0,0,0}, \theta_1^{0,0,0}, \iota_1^{0,0,0}$, in which case we only have $n_{\eta',i} \in \mathbb{Z}$.

Proof. This lemma follows from computations similar to those in Proposition 13.3.1.

We will only prove it for $T_5^{\mathrm{Kl}}(l)$, and the rest follows analogously.

Note that J_5 is the disjoint union of the following four subsets:

$$J_5^1 = \{(A_1^{\lambda,\mu,t,\kappa}, B_1^x) : 0 \le t, \lambda, x < l, 0 \le \mu < l^2, 0 \le \kappa < l^3\},$$
$$J_5^2 = \{(A_1^{\lambda,\mu,t,\kappa}, B_2) : 0 \le t, \lambda < l, 0 \le \mu < l^2, 0 \le \kappa < l^3\},$$
$$J_5^3 = \{(A_2^{\lambda,\mu,\kappa}, B_1^x) : 0 \le \lambda, x < l, 0 \le \mu < l^2, 0 \le \kappa < l^3\},$$
$$J_5^4 = \{(A_2^{\lambda,\mu,\kappa}, B_2) : 0 \le \lambda < l, 0 \le \mu < l^2, 0 \le \kappa < l^3\},$$

where to simplify notation we write

$$A_1^{\lambda,\mu,t,\kappa} = \begin{pmatrix} l^3 & l^2\lambda & l\mu & \kappa \\ & l^2 & lt & \mu - t\lambda \\ & & l & -\lambda \\ & & & 1 \end{pmatrix}, \qquad A_2^{\lambda,\mu,\kappa} = \begin{pmatrix} l^3 & l\mu & l^2\lambda & \kappa \\ & l & & \lambda \\ & & l^2 & -\mu \\ & & & 1 \end{pmatrix},$$

and

$$B_1^x = \begin{pmatrix} l^2 & lx \\ & l \end{pmatrix}, \qquad B_2 = \begin{pmatrix} l & \\ & l^2 \end{pmatrix}.$$

We also let $\eta_m^{a,b,c}$ denote $\left(\begin{pmatrix} 1 & al^{-m} & bl^{-m} & \\ & 1 & & bl^{-m} \\ & & 1 & -al^{-m} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & cl^{-m} \\ & 1 \end{pmatrix}\right)$.

For representatives in J_5^1, we have

$$(A_1^{\lambda,\mu,t,\kappa}, B_1^x) = \left(\begin{pmatrix} l^3 & \kappa \\ & 1 \end{pmatrix}, \begin{pmatrix} l^2 & lt \\ & l \end{pmatrix}\right) \cdot \eta_2^{l\lambda,\mu,l(t-x)}$$

and $\begin{pmatrix} l^3 & \\ & 1 \end{pmatrix}^{\kappa} \cdot \phi_{1,s,t} = \phi_{1,s+3,t}$, so like in the proof of Proposition 13.3.1 the contribution of J_5^1 to (13.7) is

$$l \cdot \sum_{0 \le \lambda, y < l, 0 \le \mu < l^2} \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_2^{l\lambda,\mu,ly} K)).$$

We split the above sum into four pieces based on whether λ and y are zero or not.

When $\lambda y \neq 0$, $\eta_2^{l\lambda,\mu,ly}$ is conjugate to $\eta_2^{l,\mu y^{-1}\lambda^{-1},l}$ via $(\begin{pmatrix} \lambda^2 y & \\ & 1 \end{pmatrix}, \begin{pmatrix} \lambda y & \\ & \lambda \end{pmatrix})$, which fixes $\phi_{1,s,t}$ and belongs to K. Thus the contribution is

$$\begin{aligned} & l \sum_{0<\lambda<l} \sum_{0<y<l} \sum_{0 \le \mu < l^2} \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_2^{l,\mu y^{-1}\lambda^{-1},l} K)) \\ =& l(l-1)^2 \sum_{0 \le \mu < l^2} \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_2^{l,\mu,l} K)). \end{aligned}$$

When $\lambda \neq 0$ but $y = 0$, $\eta_2^{l\lambda,\mu,ly}$ is conjugate to $\eta_2^{l,1,0}$ via $(\begin{pmatrix} \lambda\mu & \\ & 1 \end{pmatrix}, \begin{pmatrix} \mu & \\ & \lambda \end{pmatrix})$ if $l \nmid \mu$, and conjugate to $\eta_2^{l,l,0} = \eta_1^{1,1,0}$ via $(\begin{pmatrix} \lambda\mu/l & \\ & 1 \end{pmatrix}, \begin{pmatrix} \mu/l & \\ & \lambda \end{pmatrix})$ if $l | \mu$ but $l^2 \nmid \mu$, and conjugate to $\eta_2^{l,0,0} = \eta_1^{1,0,0}$ via $(\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & \lambda \end{pmatrix})$ if $\mu = 0$, so by the same argument as in the previous case the contribution is

$$\begin{aligned} & l \sum_{0<\lambda<l} \sum_{0 \le \mu < l^2} \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_2^{l,\mu,0} K)) \\ =& l^2(l-1)^2 \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_2^{l,1,0} K)) + l(l-1)^2 \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1^{1,1,0} K)) \\ & + l(l-1) \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1^{1,0,0} K)). \end{aligned}$$

When $\lambda = 0$ and $y \neq 0$, $\eta_2^{l\lambda,\mu,ly}$ is conjugate to $\eta_2^{0,1,l}$ via $\left(\begin{pmatrix} \mu & \\ & y\mu \end{pmatrix}, \begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)$ if $l \nmid \mu$, and conjugate to $\eta_2^{0,l,l} = \eta_1^{0,1,1}$ via $\left(\begin{pmatrix} \mu/l & \\ & y\mu/l \end{pmatrix}, \begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)$ if $l|\mu$ but $l^2 \nmid \mu$, and conjugate to $\eta_2^{0,0,l} = \eta_1^{0,0,1}$ via $\left(\begin{pmatrix} 1 & \\ & y \end{pmatrix}, \begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)$ if $\mu = 0$. Thus like above the contribution in this case is

$$\begin{aligned} & l \sum_{0<y<l} \sum_{0 \leq \mu < l^2} \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_2^{0,\mu,ly} K)) \\ =& l^2(l-1)^2 \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_2^{0,1,l} K)) + l(l-1)^2 \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1^{0,1,1} K)) \\ & + l(l-1)\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1^{0,0,1} K)). \end{aligned}$$

Next assume $\lambda = y = 0$. When $l \nmid \mu$ a completely analogous argument shows we have

$$l(l^2 - l)\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_2^{0,1,0} K)),$$

and when $\mu = l\mu'$ with $0 < \mu' < l$, we have $\eta_2^{0,\mu,0} = \eta_1^{0,\mu',0}$ and analogously we obtain

$$l(l-1)\mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta_1^{0,1,0} K)).$$

Finally we are left with $\eta_2^{0,0,0}$, which is the identity matrix.

Combining the above four cases, we see the contribution of J_5^1 to the sum in (13.7)

is given by

$$l(l-1)^2\Big(\sum_{0\leq\mu<l^2}\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_2^{l,\mu,l}K))+l\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_2^{l,1,0}K))+\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_1^{1,1,0}K))$$
$$+l\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_2^{0,1,l}K))+\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_1^{0,1,1}K))\Big)$$
$$+l(l-1)\Big(\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_1^{1,0,0}K))+\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_1^{0,0,1}K))+l\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_2^{0,1,0}K))$$
$$+\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(\eta_1^{0,1,0}K))\Big)+l\mathfrak{Z}(\phi_{1,\infty}\otimes\mathrm{ch}(K)) \tag{13.9}$$

In summary, the can be put into the desired form. A similar computation shows that this is true for J_5^2, J_5^3 and J_5^4 as well. This completes the proof. □

We will next examine, for each of $i=0,1,...,6$, the elements $\eta'\in J_i'$ appearing in the sum S_i such that $n_{\eta',i}$ is not divisible by $(l-1)^2$. From the calculations in the proof of Lemma 13.5.3 (i.e. equation (13.9) and the analogues for other representatives and other $T_i^{\mathrm{Kl}}(l)$), the only such η' are the following:

- For $i=0$: clearly there is only $\eta_1^{0,0,0}=(I_4,I_2)$ with multiplicity 1.
- For $i=1$: $\eta'=\eta_1^{1,0,0},\eta_1^{0,1,0},\eta_1^{0,0,1},\theta_1^{1,0,0},\theta_1^{0,0,1},\iota_1^{0,1,0},\iota_1^{0,0,1}$, with multiplicity $n_{1,\eta'}$ equal to $l-1$ up to a power of l, and $\eta'=\eta_1^{0,0,0},\theta_1^{0,0,0},\iota_1^{0,0,0}$ with $n_{1,\eta'}$ equal to power of l.
- For $i=2$: $\eta'=\eta_1^{1,0,0},\eta_1^{0,1,0}$ with $n_{2,\eta'}$ equal to $l-1$ up to a power of l, and $\eta'=\eta_1^{0,0,0}$ with $n_{2,\eta'}$ equal to a power of l.
- For $i=3$: $\eta'=\eta_1^{0,0,1},\theta_1^{1,0,0},\theta_1^{0,1,0},\theta_1^{0,0,1},\theta_2^{0,0,1},\iota_1^{1,0,0},\iota_1^{0,1,0}$ with $n_{3,\eta'}$ equal to $l-1$ up to a power of l, and $\eta'=\eta_1^{0,0,0},\theta_1^{0,0,0},\iota_1^{0,0,0}$ with $n_{3,\eta'}$ equal to a power of l.
- For $i=4$: $\eta'=\eta_1^{0,0,1},\theta_1^{1,0,0},\theta_2^{1,0,0},\iota_1^{0,1,0},\iota_1^{0,0,1},\iota_2^{0,1,0},\iota_2^{0,0,1}$ with $n_{4,\eta'}$ equal to $l-1$ up to a power of l, and $\eta'=\theta_1^{0,0,0},\iota_1^{0,0,0}$ with with $n_{4,\eta'}$ equal to a power of l.

- For $i = 5$: $\eta' = \eta_1^{1,0,0}, \eta_1^{0,1,0}$ with $n_{5,\eta'}$ equal to $2(l-1)$ up to a power of l, $\eta' = \eta_1^{0,0,1}, \eta_2^{1,0,0}, \eta_2^{0,1,0}, \theta_1^{1,0,0}$, $\theta_1^{0,1,0}, \theta_1^{0,0,1}, \theta_2^{1,0,0}, \iota_1^{1,0,0}, \iota_1^{0,1,0}, \iota_1^{0,0,1}, \iota_2^{0,1,0}$ with $n_{5,\eta'}$ equal to $l-1$ up to a power of l, $\eta' = \theta_1^{0,0,0}, \iota_1^{0,0,0}$ with $n_{5,\eta'}$ equal to a power of l, and $\eta' = \eta_1^{0,0,0}$ with with $n_{5,\eta'}$ equal to 2 up to a power of l.

- For $i = 6$: $\eta' = \eta_1^{1,0,0}, \eta_1^{0,1,0}, \eta_2^{1,0,0}, \eta_2^{0,1,0}$ with $n_{6,\eta'}$ equal to $l-1$ up to a power of l, and $\eta' = \eta_1^{0,0,0}$ with $n_{6,\eta'}$ equal to a power of l.

From the proof of Theorem 13.4.1, we see that $\mathfrak{Z}(\phi_{1,\infty} \otimes \xi)$ is given by, up to a l-power factor,

$$\sum_{i=0}^{6} m_i S_i = \sum_{\eta'} (\sum_{i=0}^{6} m_i n_{\eta',i}) \cdot \mathfrak{Z}(\phi_{1,\infty} \otimes \mathrm{ch}(\eta' K)),$$

where as before, $m_i \in \mathbb{Z}$ is the coefficient of $T_i^{\mathrm{Kl}}(l)$ in the linear combination $\mathbf{T}$ as given in Corollary 12.3.3. In particular, modulo $l-1$ we have $m_0 = m_3 = m_4 = m_6 = 1$, $m_1 = m_5 = -1$ and $m_2 = 2$.

Lemma 13.5.4. In the above sum:

1. if η' is one of $\theta_1^{0,0,1}, \theta_2^{0,0,1}, \iota_1^{0,0,1}, \iota_2^{0,0,1}, \eta_1^{0,0,0}, \theta_1^{0,0,0}, \iota_1^{0,0,0}$, then the combined multiplicity $\sum_{i=0}^{6} m_i n_{i,\eta'}$ is divisible by $l-1$, and

2. if η' is any other element appearing on the list above, then $\sum_{i=0}^{6} m_i n_{i,\eta'}$ is divisible by $(l-1)^2$.

Proof. Both parts follow from straightforward computations according to the list above. □

This last lemma implies that in the notation of Theorem 13.4.1, the only η appearing in $\xi = \sum_{\eta \in J} m_\eta \mathrm{ch}(\eta K)$ (after combining conjugates) with $(l-1)^2 \nmid m_\eta$ are $\theta_1^{0,0,1}, \theta_2^{0,0,1}, \iota_1^{0,0,1}, \iota_2^{0,0,1}, \eta_1^{0,0,0}, \theta_1^{0,0,0}, \iota_1^{0,0,0}$. It is then straightforward to check that if η is

any of the above seven, then $l-1$ part of the index $[V : V_\eta]$ is precisely $l-1$: we will only demonstrate with $\theta_2^{0,0,1}$ and the rest should be clear.

Let $\eta = \theta_2^{0,0,1}$, and $h = (\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix})$ be any element in V (so that in particular $a = a'd' - b'c'$). Then

$$\eta^{-1}h\eta = (\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a'-c' & (a'+b'-c'-d')l^{-2} \\ l^2c' & c'+d' \end{pmatrix}),$$

which lies in $K_{0,1}$ if and only if $b' = c' + d' - a' \pmod{l^2}$. In particular,

$$\{(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & b' \\ & 1 \end{pmatrix}) : b' = 0, 1, ..., l^2 - 1\}$$

is a complete set of representatives for $(V \cap K_{0,1})/(V \cap \eta K_{0,1}\eta^{-1})$. On the other had, it is clear$[V : V \cap K_{0,1}] = l - 1$. Thus $[V : V_\eta] = (l-1)l^2$.

This concludes the proof of Proposition 13.5.2.

Bibliography

[BCF] Iván Blanco-Chacón and Michele Fornea. Twisted triple product p -adic l-functions and hirzebruch–zagier cycles. *Journal of the Institute of Mathematics of Jussieu*, page 1–46.

[BDP13] Massimo Bertolini, Henri Darmon, and Kartik Prasanna. Generalized Heegner cycles and p-adic Rankin L-series. *Duke Math. J.*, 162(6):1033–1148, 2013. With an appendix by Brian Conrad.

[Bei13] A. Beilinson. On the crystalline period map. *Camb. J. Math.*, 1(1):1–51, 2013.

[Bes00] Amnon Besser. A generalization of Coleman's p-adic integration theory. *Invent. Math.*, 142(2):397–434, 2000.

[BK90] Spencer Bloch and Kazuya Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[BLZ16] Amnon Besser, David Loeffler, and Sarah Livia Zerbes. Finite polynomial cohomology for general varieties. *Ann. Math. Qué.*, 40(1):203–220, 2016.

[Bum89] Daniel Bump. The Rankin-Selberg method: a survey. In *Number theory, trace formulas and discrete groups (Oslo, 1987)*, pages 49–109. Academic Press, Boston, MA, 1989.

[Bum97] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[CH18] Francesc Castella and Ming-Lun Hsieh. Heegner cycles and p-adic L-functions. *Math. Ann.*, 370(1-2):567–628, 2018.

[Col97] Robert F. Coleman. Classical and overconvergent modular forms of higher level. *J. Théor. Nombres Bordeaux*, 9(2):395–403, 1997.

[CW77] J. Coates and A. Wiles. On the conjecture of Birch and Swinnerton-Dyer. *Invent. Math.*, 39(3):223–251, 1977.

[Dim13] Mladen Dimitrov. Automorphic symbols, p-adic L-functions and ordinary cohomology of Hilbert modular varieties. *Amer. J. Math.*, 135(4):1117–1155, 2013.

[DR14] Henri Darmon and Victor Rotger. Diagonal cycles and Euler systems I: A p-adic Gross-Zagier formula. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(4):779–832, 2014.

[DR17] Henri Darmon and Victor Rotger. Diagonal cycles and Euler systems II: The Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-functions. *J. Amer. Math. Soc.*, 30(3):601–672, 2017.

[Fla90] Matthias Flach. A generalisation of the Cassels-Tate pairing. *J. Reine Angew. Math.*, 412:113–127, 1990.

[For18] M. Fornea. Growth of the analytic rank of rational elliptic curves over quintic fields. *To appear in Math. Res. Letters*, February 2018.

[GK82] S. S. Gelbart and A. W. Knapp. L-indistinguishability and R groups for the special linear group. *Adv. in Math.*, 43(2):101–121, 1982.

[GPSR87] Stephen Gelbart, Ilya Piatetski-Shapiro, and Stephen Rallis. *Explicit constructions of automorphic L-functions*, volume 1254 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[Gre89] Ralph Greenberg. Iwasawa theory for p-adic representations. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 97–137. Academic Press, Boston, MA, 1989.

[Gro18] Giada Grossi. On norm relations for asai-flach classes. *ArXiv e-prints*, 2018.

[Hid88] Haruzo Hida. On p-adic Hecke algebras for GL_2 over totally real fields. *Ann. of Math. (2)*, 128(2):295–384, 1988.

[Hid89a] Haruzo Hida. Nearly ordinary Hecke algebras and Galois representations of several variables. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 115–134. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Hid89b] Haruzo Hida. On nearly ordinary Hecke algebras for $GL(2)$ over totally real fields. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 139–169. Academic Press, Boston, MA, 1989.

[Hid91] Haruzo Hida. On p-adic L-functions of $GL(2) \times GL(2)$ over totally real fields. *Ann. Inst. Fourier (Grenoble)*, 41(2):311–391, 1991.

[Hid04] Haruzo Hida. *p-adic automorphic forms on Shimura varieties*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2004.

[HT01] Michael Harris and Jacques Tilouine. p-adic measures and square roots of special values of triple product L-functions. *Math. Ann.*, 320(1):127–147, 2001.

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. Number 295, pages ix, 117–290. 2004. Cohomologies p-adiques et applications arithmétiques. III.

[KL05] Mark Kisin and King Fai Lai. Overconvergent Hilbert modular forms. *Amer. J. Math.*, 127(4):735–783, 2005.

[KLZ17] Guido Kings, David Loeffler, and Sarah Livia Zerbes. Rankin-Eisenstein classes and explicit reciprocity laws. *Camb. J. Math.*, 5(1):1–122, 2017.

[KMS03] Shin-ichi Kato, Atsushi Murase, and Takashi Sugano. Whittaker-Shintani functions for orthogonal groups. *Tohoku Math. J. (2)*, 55(1):1–64, 2003.

[Kol88] V. A. Kolyvagin. The Mordell-Weil and Shafarevich-Tate groups for Weil elliptic curves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 52(6):1154–1180, 1327, 1988.

[Kol90] V. A. Kolyvagin. Euler systems. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 435–483. Birkhäuser Boston, Boston, MA, 1990.

[LLZ14] Antonio Lei, David Loeffler, and Sarah Livia Zerbes. Euler systems for Rankin-Selberg convolutions of modular forms. *Ann. of Math. (2)*, 180(2):653–771, 2014.

[LLZ16] A. Lei, D. Loeffler, and S. Livia Zerbes. Euler systems for Hilbert modular surfaces. *ArXiv e-prints*, July 2016.

[LLZ18] Antonio Lei, David Loeffler, and Sarah Livia Zerbes. Euler systems for Hilbert modular surfaces. *Forum Math. Sigma*, 6:e23, 67, 2018.

[LR98] Erez Lapid and Jonathan Rogawski. On twists of cuspidal representations of GL(2). *Forum Math.*, 10(2):175–197, 1998.

[LSZ16] D. Loeffler, C. Skinner, and S. L. Zerbes. Syntomic regulators of Asai–Flach classes. *ArXiv e-prints*, aug 2016.

[LSZ17] David Loeffler, Chris Skinner, and Sarah Zerbes. Euler systems for GSp(4). *ArXiv e-prints*, 2017.

[LV11] Matteo Longo and Stefano Vigni. Quaternion algebras, Heegner points and the arithmetic of Hida families. *Manuscripta Math.*, 135(3-4):273–328, 2011.

[LZ14] David Loeffler and Sarah Livia Zerbes. Iwasawa theory and p-adic L-functions over $\mathbb{Z}_p^2$-extensions. *Int. J. Number Theory*, 10(8):2045–2095, 2014.

[LZ17] David Loeffler and Sarah Zerbes. Euler systems with local conditions. *ArXiv e-prints*, 2017.

[LZ18] David Loeffler and Sarah Zerbes. Euler systems (arizona winter school 2018 notes), 2018.

[Mil05] J. S. Milne. Introduction to Shimura varieties. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 265–378. Amer. Math. Soc., Providence, RI, 2005.

[Nek00] Jan Nekovář. p-adic Abel-Jacobi maps and p-adic heights. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 24 of *CRM Proc. Lecture Notes*, pages 367–379. Amer. Math. Soc., Providence, RI, 2000.

[Oht95] Masami Ohta. On the p-adic Eichler-Shimura isomorphism for Λ-adic cusp forms. *J. Reine Angew. Math.*, 463:49–98, 1995.

[PS16] V. Pilloni and B. Stroh. Surconvergence, ramification et modularité. *Astér.*, 382:195–266, 2016.

[Ren10] David Renard. *Représentations des groupes réductifs p-adiques*, volume 17 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2010.

[RS07] Brooks Roberts and Ralf Schmidt. *Local newforms for GSp(4)*, volume 1918 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[Rub00] Karl Rubin. *Euler systems*, volume 147 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study.

[Sai09] Takeshi Saito. Hilbert modular forms and p-adic Hodge theory. *Compos. Math.*, 145(5):1081–1113, 2009.

[Sch05] Ralf Schmidt. Iwahori-spherical representations of GSp(4) and Siegel modular forms of degree 2 with square-free level. *J. Math. Soc. Japan*, 57(1):259–293, 2005.

[Shi78] Goro Shimura. The special values of the zeta functions associated with Hilbert modular forms. *Duke Math. J.*, 45(3):637–679, 1978.

[Ski09] Christopher Skinner. A note on the p-adic Galois representations attached to Hilbert modular forms. *Doc. Math.*, 14:241–258, 2009.

[Ski20] Christopher Skinner. A converse to a theorem of gross, zagier, and kolyvagin. *Annals of Mathematics*, 191(2):329–354, 2020.

[ST93] Paul J. Sally, Jr. and Marko Tadić. Induced representations and classifications for $\mathrm{GSp}(2, F)$ and $\mathrm{Sp}(2, F)$. *Mém. Soc. Math. France (N.S.)*, (52):75–133, 1993.

[Tha88] Francisco Thaine. On the ideal class groups of real abelian number fields. *Ann. of Math. (2)*, 128(1):1–18, 1988.

[TW95] R. Taylor and A. Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995.

[TX16] Yichao Tian and Liang Xiao. p-adic cohomology and classicality of overconvergent Hilbert modular forms. *Astérisque*, (382):73–162, 2016.

[Urb05] Eric Urban. Sur les représentations p-adiques associées aux représentations cuspidales de $\mathrm{G}Sp_{4/\mathbb{Q}}$. Number 302, pages 151–176. 2005. Formes automorphes. II. Le cas du groupe $\mathrm{G}Sp(4)$.

[Wes05] Tom Weston. Iwasawa invariants of Galois deformations. *Manuscripta Math.*, 118(2):161–180, 2005.

[Wil88] A. Wiles. On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988.

[Wil95] A. Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.