From the holdings of the Department of Special Collections , Princeton University Library, Princeton, NJ

These documents are either in the public domain, or can only be used for educational and research purposes (Fair use) as per U.S. Copyright law (text below). By accessing this file, all users agree that their use falls in the public domain or within fair use as defined by the copyrightlaw. We do not charge any permission or usage fees for the publication of images of material inour collections, including those provided by Princeton University Library, via our website, catalog, or directly from staff.For more information regarding copyright, credit, and citation please review the information onthe Copyright, Credit and Citation Guidelines page of our website.
https://library.princeton.edu/special-collections/policies/copyright-credit-and-citation-guidelines. If you have any questions, including how to order higher-quality images from the collection, contact Special Collections Public Services via our Ask Us! Form. https://library.princeton.edu/special-collections/ask-us

---

Doctoral Dissertations, 2011-2026 Economics

2022

Essays in Microeconomic Theory: Framing and Complexity of Choice

Safonov, Evgenii

Princeton, NJ : Princeton University

---

http://arks.princeton.edu/ark:/88435/dsp01qf85nf46t

U.S. Copyright law: The copyright law of the United States (Title 17, §108, United States Code) governs the making of photocopies or other reproductions of copyrighted material. Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be used for any purpose other than private study, scholarship, or research. If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of fair use that user may be liable for copyright infringement. This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve a violation of copyright law. https://www.copyright.gov/title17/title17.pdf

ESSAYS IN MICROECONOMIC THEORY: FRAMING AND COMPLEXITY OF CHOICE

EVGENII SAFONOV

A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF ECONOMICS
ADVISER: WOLFGANG PESENDORFER

MAY 2022

© Copyright by Evgenii Safonov, 2022.

All rights reserved.

Abstract

This Dissertation comprises three chapters investigating the theory of framing effects and complexity of choice in the context of individual decision making.

In the first chapter, I consider a boundedly rational agent who employs a stochastic automaton decision rule to search for the best item drawn randomly with replacement from a menu of unknown composition. I study the minimal state and transition complexity of an automaton that allows the agent to choose the best alternative from any menu with probability arbitrarily close to one. Agents differ in their languages—collections of binary attributes used to describe alternatives. I provide the tight lower bounds and upper bounds on complexity among all languages. In the wide range of environments, the language is least complex if and only if it allows the agent to use an additive utility function with respect to the smallest possible set of attributes.

In the second chapter, I develop and analyze a model of framing under ambiguity. Frames are circumstances, unobservable to the analyst, that shape the perception of the relevant ambiguity by the agent. The analyst observes a choice correspondence that represents the set of possible choices under the various decision frames. My first result provides axioms that are equivalent to a multi-multiple prior model. I then show that the analyst can infer the unique minimum collection of frames from the choice behavior. Finally, I analyze the degree to which frames affect choice and the behavioral implications of combining several decision frames.

In the third chapter, I consider decision frames comprised of observable features of the environment and interpret each feature as an unobservable signal structure that provides the agent information regarding her own preferences. An analyst observes the resulting stochastic choice of a population of agents with heterogeneous state-dependent utility functions. I show that almost any stochastic choice that admits a random utility representation within each frame is consistent with such model. However, when the state space has limited cardinality, the resulting stochastic choice

should satisfy additional constraints on the sums of Block and Marschak (1960) polynomials constructed from choice frequencies under each decision frame.

Acknowledgements

I am deeply greatful to my advisor, Wolfgang Pesendorfer, for navigation through all my years during PhD studies, invaluable guidance, endless hours of discussions of good, not-so-good, and so-no-good research ideas, and for his overall support and encouragement in research and in life. I learned from him a lot about his approach to the Microeconomic Theory, and not only about that.

I am also indebted to Faruk Gul for his guidence and support. His genius always helped me to look at my research from a new angle and find more elegant presentations of questions and statements. I am also greatful to Faruk and Wolfgang for the opportunity to work on our joint project. Our work with a model of limited memory pushed me to think about an automaton model of choice, which ultimetely yielded the first chapter of this Dissertation.

I am also greateful to Pietro Ortoleva, who supported me a lot during my last years of PhD studies. He was always there to provide feedback on my research and to suggest numerous ways to develop my projects. I have been lucky to work with Pietro and Leeat Yariv on our joint project.

I am very thankful to Leeat: she has truly shown me an industrious approach to writing papers; I only hope to reach this perfection in the future, but at least I know the goal.

I would like to thank Stephen Morris, with whom I worked during the first years of my studies and whose wisdom contributed to my development as a researcher.

I am thankful to Laura Hedden for the fantastic support during all the years of my PhD studies; the Economics Department is lucky to have such a star administrator of the graduate program. I would like to also thank Ellen Graf and Laura Sciarrotta for their administrative support.

My special thanks is to Sarah-Jane Leslie, who was the Dean of the Graduate School during a hard period of my life, and who, with other stuff of the Graduate School, helped my family a lot.

Many thanks to my collegues in Economic Theory: Svetlana Kosterina, Lasse Mononen, Franz Ostrizek, Pellumb Reshidi, Kirill Rudov, Elia Sartori, Denis Shishkin, Ben Young, Mu Zhang, and others.

Also thanks to Vasyl Alba, Bruno Baranek, Stefano Baratuche, Francisco Cabezon, Grisha Chirkov, Anna Chorniy, Jonas Jin, Sergei and Lera Kiiashko, Alexander Kopytov, Alexey Lavrov, Nikita Melnikov, Paola Moscariello, Dasha Rudova, Olga Shishkina, Andrei Zeleneev, and many others for my joyful time in Princeton.

Finally, I cannot imagine the years of my PhD studies without my wife, Evgeniia Lambrinaki, and her love, support and enthusiasm. Our daughter Kira has illimunated the last years of our life here, in Princeton.

To my father Yurii and my newborn son Tim,
who didn't live to see it.

To my mother Natalia, my brother Kirill,
my wife Evgeniia, and my daughter Kira.

Contents

Abstract 3

Acknowledgements 5

List of Tables 12

List of Figures 13

1 Slow and Easy: a Theory of Browsing 14

1.1 Introduction 15

1.1.1 Related Literature 19

1.2 Model 21

1.2.1 Preference 21

1.2.2 Undirected search 21

1.2.3 Language 22

1.2.4 Automaton 23

1.2.5 Stochastic choice 23

1.2.6 Near optimal decision procedures 25

1.3 Complexity 27

1.4 Adaptive Languages 31

1.5 Sketches of the Proofs 38

1.5.1 Theorem 1.3.1—sketch of the proof 39

1.5.2 Theorem 1.3.4—sketch of the proof 40

1.5.3 Theorem 1.4.3—sketch of the proof 41

1.6 Extensions 44
1.6.1 Endogenous dismissal decision 44
1.6.2 Recall of the past investigations 47
1.7 Conclusion 49

2 Framing and Ambiguity **52**
2.1 Introduction 53
2.1.1 Related Literature 56
2.1.2 Structure of the Paper 58
2.2 Model 58
2.2.1 Choice Correspondence 59
2.2.2 Axioms 60
2.2.3 Representation 63
2.3 Redundant Frames 65
2.3.1 Identification of multi-multiple prior models in the literature . 68
2.4 Frame Sensitivity 69
2.5 Connection of Frames and Consistency of Choices 72
2.5.1 Decisiveness 75
2.5.2 Consistency 76
2.5.3 Identification of frames from binary choices 77
2.5.4 Application to the Social Choice Theory 79
2.5.5 Optimistic Learning 81

3 Random Choice with Framing Effects: a Bayesian Model **83**
3.1 Introduction 84
3.1.1 Related Literature 88
3.2 Model 91
3.2.1 Primitive 91

3.2.2 Bayesian Agents with Uncertain Preferences 92
3.2.3 Representation 93
3.3 Fixed State Space 98
3.3.1 Binary State Space 98
3.3.2 General State Space 102

A Appendices for Chapter 3 **108**
A.1 Lemmas for a more general setup 109
A.1.1 Preliminaries 109
A.1.2 Global dynamics 114
A.1.3 Local dynamics 121
A.1.4 A Graph Theory Approach 123
A.2 Proofs for the baseline model 140
A.2.1 Other Combinatorics statements 140
A.2.2 Universal decision rule 142
A.2.3 Additive languages and automata 149
A.2.4 Other Lemmas 164
A.2.5 Proofs of statements in Sections 2-4 164
A.2.6 Proofs of lemmas and claims from Section 5 168
A.2.7 Formalizing the dynamics 169
A.3 Proofs for the extensions 171
A.3.1 Proofs for the model with endogenous dismissal decision . . . 171
A.3.2 Proofs for the model with recall of the past investigations . . . 173

B Appendices for Chapter 2 **177**
B.1 Proofs of Main Results 177
B.1.1 Proof of Theorem 2.2.1 177
B.1.2 Proof of Theorem 2.2.3 194

B.1.3 Proof of Proposition 2.3.1 196
B.1.4 Proof of Proposition 2.3.2 and Corollary 2.3.3 199
B.1.5 Proof of Proposition 2.4.1 203
B.1.6 Proof of Proposition 2.5.1 205
B.1.7 Proof of Theorem 2.5.2 206
B.1.8 Proof of Proposition 2.5.3, Corollary 2.5.4 and Corollary 2.5.5 221
B.1.9 Proof of Proposition 2.5.6 222
B.2 Proofs of other statements 224
B.2.1 Proof of Proposition 2.2.2 224
B.2.2 Proof of Lemma B.1.10 228
B.2.3 Other Proofs 230

C Appendices for Chapter 1 **233**
C.1 Proofs 233
C.1.1 Proof of Proposition 3.2.1 233
C.1.2 Proof of Corollary 3.2.2 234
C.1.3 Proof of Theorem 3.2.3 234
C.1.4 Proof of Corollary 3.2.4 241
C.1.5 Proof of Proposition 3.3.1 242
C.1.6 Proof of Lemma 3.3.2 243
C.1.7 Proof of Theorem 3.3.3 244
C.1.8 Proof of Corollary 3.3.4 245
C.1.9 Proof of Theorem 3.3.5 246

References **248**

List of Tables

List of Figures

1.1 An automaton with one memory state. 24

1.2 A decision rule that solves the choice problem $(Q^{**}, \succeq)$. 28

1.3 The least-complex decision rule that solves the choice problem $(Q, \succeq)$. 34

1.4 A generic decision rule in the class Ψ_n^+. 36

2.1 Non-existence of the minimum representation of choice correspondence constrained to the set of binary menus. 78

3.1 Subsets of linear orders $\theta_1 = \{xyz, xzy, zxy, zyx\}$ (left plot) and $\theta_2 = \{xyz, yxz,\ yzx, zyx\}$ (right plot) induced by the maximization of expected utility with beliefs $\lambda \in [0, 1]$. 100

Chapter 1

Slow and Easy: a Theory of Browsing

1.1 Introduction

[1]Window shopping and its online cousin, browsing, are commonly observed behavioral patterns among consumers. Their defining features are a lack of urgency and a lack of direction. Consumers who browse are in no hurry to make a purchase and their attention jumps from item to item at random. In this paper, I interpret a browsing consumer as someone who sacrifices fast decision making to save on cognitive resources. Specifically, I show how consumers can achieve near optimality with very limited cognitive resources and no direction to their search as long as they take their time.

Consider, for example, the search for an appliance, for a toy or for an extracurricular activity for a child. Most consumers are not regular participants in these markets and, therefore, do not know what options are available. Furthermore, the choice objects are difficult to compare because they are differentiated along many attributes, and many of them are difficult to quantify. Finally, the decision is not urgent, that is, the decision can be delayed by days, weeks, or even longer. Because of this last feature, the decision maker can adopt slow procedures and is willing to tradeoff decision speed in favor of economizing on cognitive resources. My main results characterize the boundedly rational procedures that achieve near optimal decisions with very limited cognitive resources when time is not of the essence.

My model has two components, a *language* and an *automaton strategy*. The language is a list of yes-no questions such as *"Is this appliance rechargeable?"*. Thus, the language encapsulates how the agent describes the alternatives. The automaton strategy consists of memory states, which play the same role here as in other automaton models, an interrogation rule that specifies which questions to ask, and a

[1]This chapter is based on the paper "Slow and Easy: a Theory of Browsing". It has been presented in SET: Seminars in Economic Theory (NT2022TM).

stochastic transition rule that describes how to move between the memory states and when to choose.

The economic environment is one of undirected search. The decision maker's attention is randomly drawn to one of the available objects which she proceeds to examine. This examination takes the form of "asking questions" and transitioning between memory states upon hearing the answers. The examination stops when the decision maker selects the item for consumption or when a new alternative is brought to her attention. The search process assumes that the decision maker has no control over which alternatives captures her attention. More realistic search procedures will have aspects of undirected search but also feature some purposeful directed search. I choose the extreme case of undirected search to establish a benchmark. As my main results show, the agent can do remarkably well with very limited cognitive resources even in this extreme benchmark.

The decision maker's objective is to choose optimally while economizing on cognitive resources, that is, to find the least complex optimal strategy. As a measure of complexity, I consider *transitional complexity*. As the name suggests, transitional complexity counts the number of state-action pairs the decision maker employs. This notion of complexity measures the difficulty of executing the strategy. Banks and Sundaram (1990) discuss this notion of complexity in the context of repeated games played by finite automata. An example below illustrates the complexity and introduces my solution concept.

Suppose that the universe of items is the four element set $\{(1,1),(1,0),(0,1),(0,0)\}$, where each vector specifies the two attribute values of the item. The decision maker must search from an unknown (non-empty) subset of the four possible objects. Specifically, her task is to find the alternative that maximizes her preference $(1,1) \succ (0,1) \succ (1,0) \succ (0,0)$.

Consider a decision maker who has two memory states and employs the following strategy. In state 1, the strategy inquires about the first attribute: if the object has attribute 1, then the agent transitions to state 2—this counts as one transition—whereas if it does not have attribute 1, then the transition to state 2 occurs only with probability ϵ; with probability $1-\epsilon$ the agent remains in (transitions to) state 1—this counts for two more transitions. In state 2, the algorithm inquires about the second attribute: if the object has the second attribute, the agent chooses the object with probability 1—this counts for one transition—and if the object does not have the second attribute, the agent chooses it with probability ϵ^2 and remains in state 2 with probability $1-\epsilon^2$—this counts for two transitions. Thus, the complexity of the automaton strategy described above is $1+2+1+2=6$.

If the agent examines item $(1,1)$, then she will choose it first time she starts to investigate it, provided her attention is not drawn to a new alternative before that. When $\epsilon > 0$ is small, item $(0,1)$, on average, requires $\eta/((1-\eta)\epsilon)$ draws from the menu to be chosen, where η is the probability to draw a new alternative. For item $(1,0)$, this number is $\eta/((1-\eta)\epsilon^2)$ and for item $(0,0)$, it is $\eta^2/((1-\eta)\epsilon^3)$. If ϵ is small and the menu does not contain item $(1,1)$, then the decision maker is likely to cycle multiple times through all her available choices before settling on one of the items. Moreover, in that case, the decision maker will most likely settle on the best available choice. In the limit, as ϵ converges to zero, the probability of picking the best available item from any conceivable menu converges to one. I will refer to this property of a strategy as *solving* the decision problem.

In the example above, there are only 4 alternatives that can comprise a potential menu, and the strategy used to solve the choice problem employs 6 transitions. What happens when there are a lot of alternatives and they are differentiated along many attributes? My first main result, Theorem 1.3.1, shows that if the agent's language allows her to learn a convenient subset of attributes, she can solve the decision problem

using as few as $3\lceil \log_2(m) \rceil$ transitions, where m is the number of indifference classes of items that the agent distinguishes, and $\lceil x \rceil$ denotes the ceiling of number x—the smallest integer weakly greater than x. The example above illustrates one of these "simple" solutions, since $6 = 3 \cdot \log_2(4)$. Theorem 1.3.1 also shows that the total number of transitions required to solve a decision problem for an agent with *any* language is at least $3\lceil \log_2(m) \rceil$; thus, $3\lceil \log_2(m) \rceil$ is a tight lower bound on complexity of a strategy that can solve the choice problem.

My next main result, Theorem 1.3.4, shows that some languages provide much less convenient description of items for the decision maker: the minimum number of transitions required to solve the choice problem can be more than $2M - 2$, where M is the total number of possible items. Theorem 1.3.4 also provides an upper bound on complexity: any language that can correctly distinguish items from distinct indifference classes allows for a solution of the choice problem by using at most $3M-3$ transitions.

Theorems 1.3.1 and 1.3.4 show that language—that is, the way how the agent describes the alternatives—plays an important role in the demand for cognitive resources, which ranges from logarithmic to linear in the size of the choice problem. Thus, I study a particular class of languages that I call *adaptive* for the agent's preference relation. An adaptive language contains a small subset of questions—of the size of $\lceil \log_2(m) \rceil$—such that there exists a utility function additive with respect to the corresponding subset of attributes that correctly represents the strict part of the agent's preference relation. In the example above, the language that the agent uses to describe the items is adaptive because $m = 4$, and a two-attribute utility function $u(a) = a_1 + 2a_2$ represents the agent's preference relation.

My final main result, Theorem 1.4.3, shows that an agent with an adaptive language can always solve the choice problem by using the simplest possible strategy—a strategy that contains $3\lceil \log_2(m) \rceil$ transitions. Moreover, if the number of indiffer-

ence classes m satisfies $(3/4) \cdot 2^n < m \leq 2^n$ for some natural n, then agent who can solve the choice problem by using the simplest possible strategy must have an adaptive language. Thus, for this range of m, Theorem 1.4.3 fully characterizes all *simple* languages—the ones that allow for the simplest possible solution of the choice problem—as being adaptive.

1.1.1 Related Literature

This paper contributes to the literature on the search for a multi-attribute alternative, on the decision making with limited memory and information processing constraints, on stochastic choice rules, and, more broadly, on the complexity of the decision problems.

There is an extensive literature discussing the optimal search, including Weitzman (1979), Kohn and Shavell (1974), Morgan and Manning (1985). In Klabjan et al. (2014), the authors are concerned with the optimality of the search for a multi-attribute alternative. In their paper, the agent is fully rational, but the information acquisition is costly.

Sanjurjo (2014) builds a theoretical model of the memory-constrained search for a multi-attribute alternative. In his model, the agent memorises both the values of the discovered attributes and indexes of the searched alternative/attribute pairs. His paper is mainly focused on discussing a class of the exhaustive search procedures that deliver the least memory load, without emphasis on the optimality. Related papers Sanjurjo (2015), Sanjurjo (2017) study a multi-attribute search with values of items, additive with respect to the attributes, in the laboratory setting.

In Dow (1991), the author considers an agent with one bit of memory who can use it to solve two separate decision problems. His paper shows that it is optimal to remember information relevant to one of the problems, but not to both problems jointly.

In Cerreia-Vioglio et al. (2020), the authors also study a stochastic browsing procedure. In their continuous-time neuro-computational model, the agent compares a default item to a randomly drawn item from the menu, and the item that is default at the time of the deadline is chosen. The resulting random choice is characterized by a generalized version of the Multinomial Logit model. A discussion of the neuro-computational models of stochastic choice is given in Rustichini (2020).

In a series of papers Cover (1969), Cover and Hellman (1970), Hellman and Cover (1970), Hellman and Cover (1971), authors study hypothesis testing and learning with finite memory. They show that allowing for randomization improves the effectiveness of automata designed to deliver the best possible long-run payoff subject to the m-state memory constraint. They find upper bounds on optimality and construct "ϵ-optimal" sequences of automata that yield payoff, arbitrary close to the upper bound. My paper studies a different decision problem from theirs in many aspects, but also relies on near-optimal sequences of the stochastic finite-state automata.

In Economics literature, the automata are used to model bounded rationality in the game theory context in Abreu and Rubinstein (1988), Kalai and Stanford (1988), Banks and Sundaram (1990), and in the context of information processing in Börgers and Morales (2004), Wilson (2014), Kocer (2010). The benefits of using randomization in simple algorithms is also discussed in Kalai and Solan (2003) in the context of dynamic decision making.

In Oprea (2020), the author studies complexity of the implementation of an automaton decision procedure in the laboratory setting. His findings suggest that both the number of the automaton's states and the number of its transitions contribute to its perceived complexity. In the current paper, both measures are analyzed.

In Mandler et al. (2012)[2], the authors consider a decision procedure that eliminates, at each step, objects from the menu based on the value of one of the binary

[2]I'd like to thank Jawwad Noor for showing me this paper.

attributes. In their model, the agent does not know the model—as in my paper. A choice procedure that require the least amount of steps uses a collection of "yes-no" questions that is a special case of an "adaptive language", introduced in my paper. In general, only a very specific class of languages allows the agent to pick the best item by using a sequential elimination procedure, while the stochastic algorithm, considered in my paper, allows the agent to solve the choice problem wherever she can differentiate a pair of items if she is not indifferent between them.

1.2 Model

1.2.1 Preference

The set of alternatives, A, is finite with generic element $a \in A$. A complete and transitive binary relation $\succeq$ describes the agent's preferences over alternatives. The symbols $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succeq$. Let $C(a) = \{a' \in A | \, a' \sim a\}$ be an indifference class and let m be the number of distinct indifference classes, that is, m is the cardinality of the set $\{C(a) \,|\, a \in A\}$. I assume that the choice problem is not trivial: $m > 1$.

1.2.2 Undirected search

From the finite set of alternatives, A, nature chooses a non-empty menu, $B \subseteq A$, of available alternatives. The particular modality of nature's choice of a menu is unimportant for the subsequent analysis except for the fact that every non-empty subset B of A is a possible menu.

As long as the search continues, there is a fixed probability $\eta \in (0, 1)$ that, in any period, a new alternative draws the agent's attention. This alternative is chosen by the nature randomly with replacement from the set B according to the full-support

probability[3] $\rho^B(a) = \rho(a)/\sum_{b\in B}\rho(b)$, where $\rho(a)$ is a sampling likelihood of alternative a. If an alternative $a \in B$ has the agent's attention, she examines it by sequentially asking questions about it in the manner described below and may choose it. The search ends if the agent chooses an alternative. There is no cost of delay and, thus, the agent's objective is to find the best alternative in the menu.

1.2.3 Language

A *language* is a family of non-trivial binary partitions $Q = \{Q_{i0}, Q_{i1}\}_{i=1}^{n}$ of A. I interpret Q_i as the question "does a have property i?" It does if $a \in Q_{i1}$ and it does not if $a \in Q_{i0}$. Let $N = \{1, \ldots, n\}$ be the index set of questions.

For example, let $A = \{b, c, d\}$ be three cars where b is a red car from 2010 with manual transmission, c is a red car from 2011 with automatic transmission, and d is a green car from 2010 with automatic transmission. Then, $Q = \{Q_{i0}, Q_{i1}\}_{i=1}^{3}$ such that $Q_1 = \{\{b\}, \{c, d\}\}$, $Q_2 = \{\{c\}, \{b, d\}\}, Q_3 = \{\{d\}, \{b, c\}\}$ is a language that describes the three cars. In this example, question 1 asks whether the car has manual transmission, question 2 asks if it is from 2011, and question 3 asks if it is green.

Every language corresponds to a description of alternatives as n-dimensional vectors with coordinates taking values 1 (if the answer is "yes") and 0 (if the answer is "no"). In the example above, a description of items in language Q is $b = (1, 0, 0)$, $c = (0, 1, 0)$, $d = (0, 0, 1)$. I denote by $Q_i(a) = 1$ if $a_i = 1$ and $Q_i(a) = 0$ if $a_i = 0$ the value of the i-th attribute of alternative a according to the language Q.

In the example above, an alternative language might consist of question 3 alone; such a language is adequate for an agent who is indifferent between the two red cars b, c, but is not adequate for agents who are not indifferent between b and c. Formally, a language Q is *adequate* for $\succeq$ if for any $a, b \in A$ such that $a \succ b$ there exists $i \in N$ such that $a_i \neq b_i$. Thus, adequate languages facilitate distinction of any pair of non-

[3] My main results in Sections 3,4 hold for an arbitrary family of full-support distributions $\{\rho^B\}_{\varnothing\neq B\subseteq A}$.

indifferent alternatives. Henceforth, I assume that the language is adequate for $\succeq$. I refer to the pair $(Q, \succeq)$ as the choice problem.

1.2.4 Automaton

To solve the choice problem, the agent employs an automaton with state space $S = S^o \cup \{\diamond\}$, where $S^o = \{1, ..., k\}$ contains the flexible memory states, and $\diamond$ is the decision state. Upon reaching $\diamond$, the agent chooses the alternative and the search ends. Each time a new alternative draws the agent's attention, the state initializes at $s = 1$; we adopt a convention that the decision to choose an alternative is executed before the new alternative is drawn[4].

An interrogation rule $\iota : S^o \to N$ specifies which question the agent asks in a memory state s. A more general model would allow the agent to ask no question at a particular state. None of the subsequent results would change if we allowed this possibility; we omit it to simplify the notation. A stochastic transition rule $\tau : S^o \times \{0, 1\} \to \triangle(S)$ specifies the probability that the agent transitions from state $v \in S^o$ to state $s \in S$ as a function of the answer received in state v. Note that, if alternative a is under consideration, then this transition occurs with probability $\tau\Big(v, s, Q_{\iota(s)}(a)\Big)$.

Thus, an automaton $\sigma = (S, \iota, \tau)$ consists of the state space $S = S^o \cup \{\diamond\}$, the interrogation rule ι, and the stochastic transition rule τ. I denote by Σ the set of all automatons with a finite state space.

1.2.5 Stochastic choice

The realized menu $B \subseteq A$ and the sampling probability ρ^B from B determine the economic environment. The language Q and the automaton $\sigma = (S, \iota, \tau) \in$

[4]Alternatively, if the agent decides to choose the current item, i.e. to move to the state $\diamond$ in the next period, but her attention switches to a new item, she would investigate the new item instead of choosing the previous one. My results remain the same in this case.

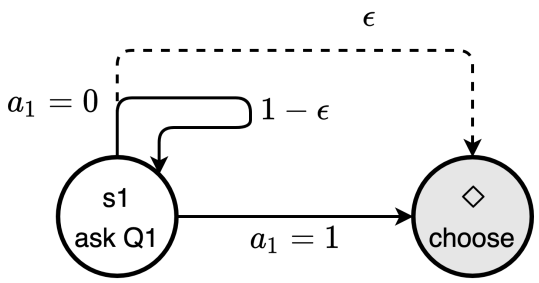
Σ determines the agent's decision procedure. Together, the economic environment and the decision procedure, induce a dynamic random choice in a straightforward way.[5] In this paper, I am concerned with the total probability that the agent chooses alternative a from menu B:

$$p^B(a) := Pr\Big(\text{alternative } a \text{ is chosen from menu } B \text{ in some period}) \tag{1.1}$$

When necessary, I will use the subscript σ to indicate the dependence of $p^B(a)$ on σ.

As an illustration, consider the car example, described above. The automaton $\sigma \in \Sigma$, described in Figure 1.1 below, has a single state which asks the question, "does

Figure 1.1: An automaton with one memory state.



the car have manual transmission?" If the answer is yes, then the agent chooses the car immediately; if the answer is no, then the agent chooses the car with probability $\epsilon \in (0, 1)$, and, otherwise, remains in state 1.

Suppose the menu $B = \{b, c\}$ consists of car b with manual transition and car c with automatic transmission. If car c is drawn from the menu in some period T, then with probability ϵ, it is chosen. With probability $(1-\epsilon)\cdot\eta$, it is not chosen and a new car catches the agent's attention in period $T+1$, which ends the investigation of c.[6] Finally, with probability $(1-\epsilon)\cdot(1-\eta)$, the agent continues to investigate car c and the process repeats. Summing the probabilities of choosing c in periods $T, T+1, ...,$

[5]The straightforward, but somewhat bulky formal description can be found in the Appendix A.2.7

[6]Note that the newly drawn car can be car c as well.

we get $q(c) = \epsilon/(1-(1-\eta)(1-\epsilon))$ for the probability of choosing c during a single uninterrupted investigation process. Similarly, $q(b) = 1$ is the probability of choosing car b during a single uninterrupted investigation process.

To calculate the total probability $p^B(c)$ that c is chosen, note that c is drawn from the menu with probability $\rho^B(c) = \rho(c)/(\rho(b)+\rho(c))$. Conditional on being drawn, c is chosen with probability $q(c)$. If c is not chosen then either b is drawn or c is drawn again. The former event occurs with probability $(1-q(c)) \cdot \rho^B(b)$ and the latter with probability $(1-q(c)) \cdot \rho^B(c)$. Summing the probabilities of choosing c after a single investigation, after two investigations, et cetera, we obtain $p^B(c) = q(c)\rho(c)/(q(c)\rho(c)+\rho(b))$.

A Luce rule (Luce (1959)) is a random choice rule in which each alternative a has a weight $V(a)$ and the probability of choosing a from menu B is $V(a)/\sum_B V(a')$. Calculating the probabilities $p^B(a)$ for all possible menus B and all possible alternatives reveals that the resulting random choice rule is a Luce choice rule with weight $\rho(a) \cdot q(a)$ assigned to car $a \in \{b, c, d\}$. Lemma 1.2.1, below, notes that this observation holds generally.

Lemma 1.2.1 *The family of choice probabilities $\rho^B(a)$ constitutes a Luce choice rule with the convention that if all weights of alternatives in menu B are zero, then $\rho^B(a) = 0$ for all $a \in B$.*

1.2.6 Near optimal decision procedures

The random choice rule described in Lemma 1.2.1 makes mistakes, that is, it may choose inferior choices from a menu. My notion of optimality allows such mistakes but requires that they can be made arbitrarily small. Specifically, I study sequences of automata for which the probability of choosing optimally converges to one. For $\sigma \in \Sigma$, let $\mathcal{T}_\sigma$ denote the *transitions of* σ, that is, the transitions that occur with

positive probability:

$$\mathcal{T}_\sigma := \left\{(v,s,j) \in S^o \times S \times \{0,1\} \;\middle|\; \tau(v,s,j) > 0\right\} \tag{1.2}$$

Let $\mathbb{N}$ be the set of natural numbers. A decision rule $\psi_{r\in\mathbb{N}}$ is a sequence of automata $\{\sigma_r\}$ such that all elements of the sequence share the same state space $S = S_\psi = S^o_\psi \cup \{\diamond\}$, the same interrogation rule $\iota = \iota_\psi$, and the same set of transitions $\mathcal{T} = \mathcal{T}_\psi$. I denote with Ψ the set of all decision rules; thus,

$$\Psi := \left\{\{\sigma_r\}_{r\in\mathbb{N}} \subset \Sigma^{\mathbb{N}} \;\middle|\; (S_{\sigma_r}, g_{\sigma_r}, \mathcal{T}_{\sigma_r}) \;=\; (S_{\sigma_t}, g_{\sigma_t}, \mathcal{T}_{\sigma_t})\ \forall r,t \in \mathbb{N}\right\} \tag{1.3}$$

When there is some parameter, $\epsilon = \epsilon_r \longrightarrow 0$, such that the transition probabilities τ_r depend on r only via ϵ_r, I denote the decision rule by $\psi = \{\sigma_\epsilon\}_{\epsilon\to 0}$ or simply $\psi = \sigma_\epsilon$ instead of $\psi = \{\sigma_r\}_{r=\mathbb{N}}$ to ease notation.

A decision rule $\psi = \{\sigma_r\}_{r\in\mathbb{N}} \in \Psi$ *solves* the choice problem $(Q, \succeq)$ if

$$\lim_r \sum_{a\in B:\ a\succeq b\ \forall b\in B} p^B_{\sigma_r}(a) \;=\; 1 \qquad\qquad \forall B:\ \varnothing \neq B \subseteq A \tag{1.4}$$

Thus, a decision rule solves the choice problem if, for every menu, the probability of choosing optimally converges to one.

To illustrate the solution concept, let us return to the car example discussed above. Suppose the agent prefers manual transmission but is otherwise indifferent between the features so that the agent's preference relation is $b \succ c \sim d$. Consider the decision rule $\psi^* = \sigma_\epsilon$, where $\epsilon \longrightarrow 0$, and for a fixed $\epsilon > 0$, $\sigma_\epsilon = \sigma$ is the automaton, shown in Figure 1, and discussed above. To see why the rule σ_ϵ solves the choice problem, consider menu $B = \{b, c\}$ and recall our calculations, performed above, to get

$$p^B(c) \;=\; \frac{q(c)\rho(c)}{q(c)\rho(c) + \rho(b)} \;=\; \frac{\epsilon\rho(c)}{\epsilon\rho(c) + (\eta + \epsilon - \eta\epsilon)\rho(b)} \;\longrightarrow\; 0$$

Thus, the agent chooses car b from menu $B = \{b, c\}$ with probability, converging to one in the limit. Similar calculations show that, in the limit $\epsilon \longrightarrow 0$, car b is always chosen with probability one from any menu that contains this car. If the choice set contains no car with automatic transmission then every choice is optimal and, thus, it suffices to note that the agent will eventually choose one of the cars.

Every choice problem with adequate language has a solution:

Lemma 1.2.2 *There exists a decision rule $\psi \in \Psi$ that solves the choice problem $(Q, \succeq)$.*

1.3 Complexity

A standard measure of complexity of an automaton is the cardinality of its state space[7]. We can interpret it as measuring the size of the "operational" memory that the decision maker requires to implement the choice procedure. However, two automata with the same state space may differ in terms of the complexity of their transition structures, as has been pointed out in Banks and Sundaram (1990). The importance of transition structures for perceived complexity has been shown experimentally in Oprea (2020). I define the complexity, κ, of the automaton $\sigma = (S, g, \tau) \in \Sigma$ as the number of distinct transitions that are a possible:

$$\kappa(\sigma) := \ |\mathcal{T}_\sigma| \ = \ \left|\left\{(v, s, j) \in S^o \times S \times \{0, 1\} \ \middle| \ \tau(v, s, j) > 0\right\}\right| \tag{1.5}$$

I interpret κ, in line with Banks and Sundaram (1990), to be the number of distinct events that the decision maker needs to distinguish to be able to use the automaton σ. Thus, it is a measure of the memory required to recall the decision procedure itself.

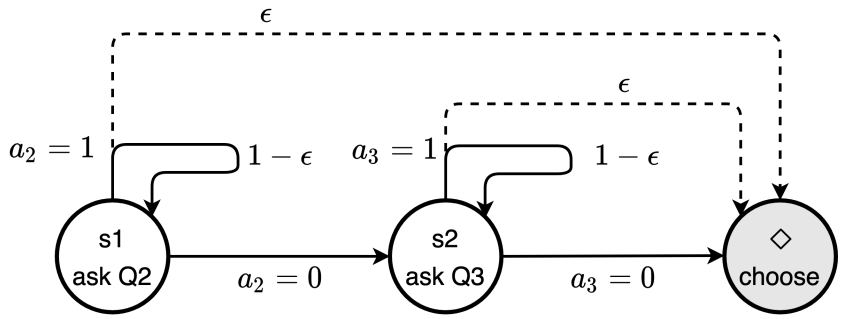
[7]See , for instance, Rubinstein (1986) in the context of repeated games

The complexity $\kappa(\psi)$ of the decision rule $\psi = \{\sigma_r\}_{r\in\mathbb{N}} \in \Psi$ is the complexity of the automaton σ_r for any r—recall that transitions $\mathcal{T}$ are the same along the sequence of automata that comprise a decision rule.

For example, the complexity of the decision rule ψ^*, depicted in Figure 1.1 is $\kappa(\psi^*) = |\mathcal{T}_{\psi^*}| = 3$, since $\mathcal{T}_{\psi^*} = \{(1, \diamond, 1), (1, \diamond, 0), (1, 1, 0)\}$; to calculate the complexity from the graphical representation of the automaton, we can simply count the number of arrows.

As the following example illustrates, some languages are more amenable to low complexity solutions than others. In the car example above, the agent might have the following language: $Q^{**} = \{Q_2, Q_3\}$, where $Q_2 = \{\{c\}, \{b, d\}\}$ and $Q_3 = \{\{d\}, \{b, c\}\}$. A decision rule $\psi^{**} = \sigma_\epsilon$ that solves the choice problem $(Q^{**}, \succeq)$, is shown in Figure 1.2.

Figure 1.2: A decision rule that solves the choice problem $(Q^{**}, \succeq)$.



The set of transitions of ψ^{**} is $\mathcal{T}_{\psi^{**}} = \{(1, 1, 1), (1, \diamond, 1), (1, 2, 0), (2, 2, 1), (2, \diamond, 1), (2, \diamond, 0)\}$ and hence, its complexity is $\kappa(\psi^{**}) = 6$. Note also that ψ^{**} uses $|S^o_{\psi^{**}}| = 2$ memory states. It can be shown that any decision rule that solves the choice problem $(Q^{**}, \succeq)$ in our example should have complexity at least 6 and use at least 2 memory states.

For a given preference relation $\succeq$, I define the complexity $\kappa(Q)$ of an adequate language Q to be the minimal complexity of the decision rules that solve the choice

problem $(Q, \succeq)$:

$$\kappa(Q) := \min_{\psi \in \Psi:\ \psi \text{ solves } (Q,\succeq)} \kappa(\psi)$$

My first main result provides a lower bound on complexity *for all languages* Q and, moreover, shows that there exists a language that achieves this lower bound. Recall that $\lceil x \rceil$ denotes the smallest integer weakly greater than x.

Theorem 1.3.1 *Let the preference relation* $\succeq$ *have* m *indifference classes. Then for any language* Q, $\kappa(Q) \geq 3\lceil \log_2(m) \rceil$, *and there exists a language* Q *such that* $\kappa(Q) = 3\lceil \log_2(m) \rceil$.

While Theorem 1 focuses on the transitional complexity, the size of the state space required to solve the choice problem is also an important measure of its complexity. For a given a preference relation $\succeq$, I define a memory load $\mathcal{M}(Q)$ of an adequate language Q to be the smallest size of the memory state space of a decision rule that solves the choice problem $(Q, \succeq)$:

$$\mathcal{M}(Q) := \min_{\psi \in \Psi:\ \psi \text{ solves } (Q,\succeq)} |S^o_\psi|$$

My next result gives the tight lower bound on the memory load of a language and says that decision rules that deliver the smallest complexity must also achieve the minimal memory load.

Proposition 1.3.2 *Let the preference relation* $\succeq$ *have* m *indifference classes. Then for any language* Q, $\mathcal{M}(Q) \geq \lceil \log_2(m) \rceil$, *and there exists a language* Q *such that* $\mathcal{M}(Q) = \lceil \log_2(m) \rceil$. *Moreover, if a decision rule* ψ *solves the choice problem* $(Q, \succeq)$, *and* $\kappa(\psi) = 3\lceil \log_2(m) \rceil$, *then* $|S^o_\psi| = \lceil \log_2(m) \rceil$.

Theorem 1 says that the minimal required complexity grows as a logarithm of the number of indifference classes of $\succeq$. To interpret this result, it is useful to understand how complex a language can be; that is, what is the upper bound on complexity?

Suppose that $C^1, C^2 \subset A$ are two indifference classes of the preference relation $\succeq$. If C^1 or C^2 contains many alternatives, then there are languages that require a lot of questions to figure out if a particular alternative belongs to C^1 or C^2. All these questions should be asked in a decision rule that solves the corresponding choice problem. Thus, in contrast to the lower bound provided by Theorem 1.3.1, the number of the indifference classes of the preference relation does not limit the maximum complexity of a language.

Proposition 1.3.3 *For any natural number k, there is a set of items A, a preference relation $\succeq$ on A with two indifference classes and an adequate language Q such that $\kappa(Q) > k$.*

My next main result shows that the maximum possible complexity of an adequate language grows linearly in the total number of alternatives in A.

Theorem 1.3.4 *Let $|A|$ be the total number of alternatives. Then for any preference relation $\succeq$ and language Q, $\kappa(Q) \leq 3|A| - 3$. Moreover, for any $m \in \{2, ..., |A|\}$, there is a preference relation $\succeq$ with m indifference classes and a language Q such that $\kappa(Q) \geq 2|A| + \lceil \log_2(m) \rceil - 2$.*

Theorem 1.3.4 says that for any preference relation $\succeq$ on A, for any language Q, adequate with respect to $\succeq$, there is a decision rule with no more than $3|A| - 3$ transitions that solves the choice problem $(Q, \succeq)$. Moreover, for any preference relation $\succeq$ there is an adequate language Q such that any decision rule that solves the choice problem $(Q, \succeq)$ must have at least $2|A| + \lceil \log_2(m) \rceil - 2$ transitions.

An analogous result holds for the memory load; moreover, the upper bound is tight.

Proposition 1.3.5 *Let $|A|$ be the total number of alternatives. Then for any preference relation $\succeq$ and adequate language Q, $\mathcal{M}(Q) \leq |A| - 1$. Moreover, for any*

$m \in \{2, ..., |A|\}$, there is a preference relation $\succeq$ with m indifference classes and a language Q such that $\mathcal{M}(Q) = |A| - 1$.

To compare the lower and upper bounds given by Theorems 1.3.1 and 1.3.4, suppose that set A contains 2^n alternatives, and there is no pair of indifferent alternatives; thus, $m = |A| = 2^n$. In this case, the tight lower bound on complexity of a language is $3 \cdot n$, while there are languages with complexity at least $2^{n+1} + n - 2$.

1.4 Adaptive Languages

What languages Q allow for the least complex solutions of the choice problem $(Q, \succeq)$? I call such languages simple, meaning that the description of the problem in a language maximally eases the agent's choice problem. Formally, given a preference relation $\succeq$ with m indifference classes, a language Q is *simple*, if

$$\kappa(Q) \;=\; 3\lceil \log_2(m) \rceil$$

Thus, a language is simple if there is a decision rule that achieves the lower bound on complexity, given by Theorem 1.3.1, and solves the choice problem for that language.

Intuitively, if a language describes a certain choice problem conveniently for the decision maker, it should contain as few attributes as possible and, moreover, the relationship between the attributes and the value of the alternatives should be "simple."

Let the preference relation $\succeq$ with m indifference classes be given. In line with the intuition discussed above, I say that a language Q is *adaptive* if it contains a subset $Q' = \{Q_i\}_{i \in N'} \subseteq Q$ of $|N'| = \lceil \log_2(m) \rceil$ questions such that there exists a vector $\lambda \in \mathbb{R}^{N'}$ with $\lambda_i \neq 0$ for all $i \in N'$ and for all $a, b \in A$,

$$a \succ b \quad \Longrightarrow \quad u(a) > u(b), \qquad \text{where} \quad u(a) := \sum_{i \in N'} \lambda_i a_i \tag{1.6}$$

Thus, a language is adaptive if it facilitates the usage of an additive utility function u with few attributes. Note that the utility function u does not necessary represent the preference relation $\succeq$: I allow for the cases when $a \sim b$, but $u(a) > u(b)$. In other words, the utility function u correctly differentiates the non-indifferent alternatives, but can break ties of the preference relation $\succeq$. Note also that an adequate language should contain at least $\lceil \log_2(m) \rceil$ questions to be able to differentiate any pair of non-indifferent alternatives, hence I use this bound in the definition of an adaptive language.

For example, let $A = \{b, c, d, e\}$, and $b \succ c \succ d \succ e$. Then $m = 4$, $\lceil \log_2(m) \rceil = 2$, and the following language is adaptive: $Q = \{Q_1, Q_2\}$ with

$$Q_1 \;=\; \{\{b, c\}, \{d, e\}\}, \qquad\qquad Q_2 \;=\; \{\{b, d\}, \{c, e\}\}.$$

Thus, $b = (1, 1)$, $c = (1, 0)$, $d = (0, 1)$, $e = (0, 0)$ is a description of the alternatives in the language Q. Take $\lambda_1 = 2$ and $\lambda_2 = 1$, then $u(a) = 2a_1 + a_2$:

$$\begin{aligned} u(b) &= 2 + 1 = 3, \qquad & u(c) &= 2 + 0 = 2, \\ u(d) &= 0 + 1 = 1, \qquad & u(e) &= 0 + 0 = 0, \end{aligned}$$

and $u(b) > u(c) > u(d) > u(e)$, showing that Q is indeed adaptive.

Examples of the non-adaptive, but adequate languages for the preference relation $\succeq$ are $Q^* = \{Q_1^*, Q_2^*\}$ and $Q^{**} = \{Q_1^{**}, Q_2^{**}, Q_3^{**}, Q_4^{**}\}$, where

$$\begin{aligned} Q_1^* &= \{\{b, c\}, \{d, e\}\}, \qquad & Q_1^{**} &= \{\{b\}, \{c, d, e\}\}, \\ Q_2^* &= \{\{b, e\}, \{c, d\}\}, \qquad & Q_2^{**} &= \{\{c\}, \{b, d, e\}\}, \\ & & Q_3^{**} &= \{\{d\}, \{b, c, e\}\}, \\ & & Q_4^{**} &= \{\{e\}, \{b, c, d\}\}. \end{aligned}$$

A description of items in language Q^* is $b = (1,1)$, $c = (1,0)$, $d = (0,0)$ and $e = (0,1)$. Would Q^* be adaptive, there should exists an additive utility function u^*, defined on the corresponding vectors of attributes, satisfying $u^*(1,1) > u^*(1,0) > u^*(0,0) > u^*(0,1)$. However, the function u^* is not monotone with respect to the second attribute: $u^*(0,1) < u^*(0,0)$, but $u^*(1,1) > u^*(1,0)$, and hence, u^* cannot be additive.

The language Q^{**} describes alternatives as $b = (1,0,0,0)$, $c = (0,1,0,0)$, $d = (0,0,1,0)$ and $e = (0,0,0,1)$ and hence, allows for an additive utility representation $u^{**}(a) = 3a_1 + 2a_2 + a_1$. However, this representation uses too many attributes: $3 > 2 = \lceil \log_2(m) \rceil$. Any pair of questions of the language Q^{**} do not allow the agent to differentiate some pair of alternatives from A and thus, is not sufficient for utility representation of an antisymmetric preference relation $\succeq$. It follows that the language Q^{**} is not adaptive.

Adaptive languages always exist:

Proposition 1.4.1 *For any preference relation, there is an adaptive language.*

Note that if we add new questions to an adaptive language, the language remains adaptive. Thus the following Corollary is straightforward.

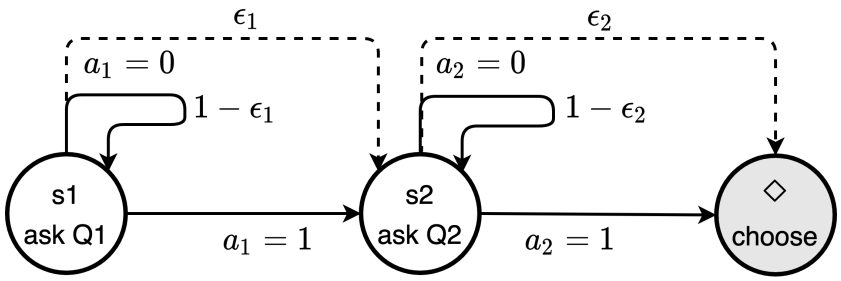
Corollary 1.4.2 *(i) The language that consists of all non-empty binary partitions of the set A is adaptive for every preference relation $\succeq$ on A; (ii) if language Q is adaptive for the preference relation $\succeq$, then language $Q' \supseteq Q$ is also adaptive for the preference relation $\succeq$.*

My main result in this section shows that all adaptive languages are simple; moreover, for a wide range of environments, adaptive languages are the only simple languages.

Theorem 1.4.3 *Let the preference relation $\succeq$ have m equivalence classes. If a language is adaptive, then it is simple. Moreover, if $(3/4)\cdot 2^n < m \leq 2^n$ for a natural n, then a language is simple if and only if it is adaptive.*

The first statement of Theorem 1.4.3 says that for any adaptive language, there is a decision rule ψ with $\kappa(\psi) = 3\lceil \log_2(m) \rceil$ that solves the choice problem $(Q, \succeq)$. As an illustration, recall the example from this section with $b \succ c \succ d \succ e$ and an adaptive language Q with $Q_1 = \{\{b, c\}, \{d, e\}\}$, $Q_2 = \{\{b, d\}, \{c, e\}\}$. A least-complex decision rule $\psi = \sigma_\epsilon$, $\epsilon \to 0$, that solves the choice problem $(Q, \succeq)$ is shown in Figure 1.3.

Figure 1.3: The least-complex decision rule that solves the choice problem $(Q, \succeq)$.



Recall that the utility function $u(a) = \lambda_1 a_1 + \lambda_2 a_2 = 2a_1 + a_2$ represents the preference relation $\succeq$ when alternatives are described in language Q. Consider $\epsilon_1 = \epsilon^{\lambda_1} = \epsilon^2$, and $\epsilon_2 = \epsilon^{\lambda_2} = \epsilon$ with $\epsilon \longrightarrow 0$. Let $q(a)$ be the probability that alternative $a \in \{b, c, d, e\}$ is chosen during a single investigation process.

To calculate $q(d)$, for instance, let the investigation of d starts at period T. Since $d_1 = 0$, with probability $\epsilon_1 = \epsilon^2$, the automaton transitions to state $s = 2$, and with the remaining probability $1 - \epsilon^2$, the state remains at $s = 1$, in which case, with total probability $(1-\epsilon^2)\cdot\eta$, the investigation ends, and with total probability $(1-\epsilon^2)\cdot(1-\eta)$, the agent continues the investigation process at state $s = 1$ in period $T + 1$, and the analysis repeats. Summing up the probabilities of transition to state $s = 2$ in periods T, $T + 1$,..., we obtain $\epsilon^2/(1 - (1 - \eta)(1 - \epsilon^2))$ for the total probability to transition to state $s = 2$ during a single investigation process of d. Once the automaton is in

state $s = 2$, with probability η, the investigation ends, and with probability $(1 - \eta)$, the state transitions to $\diamond$, since $d_2 = 1$. Therefore,

$$q(d) \;=\; \frac{(1-\eta)\epsilon^2}{1-(1-\eta)(1-\epsilon^2)}$$

If we express quantities $q(a)$ for $a \in \{b, c, d, e\}$ in terms of the power expansion with respect to a vanishing parameter ϵ, we get the following leading terms:

$$\begin{array}{lcllcl} q(b) & = & 1-\eta & \qquad q(c) & \approx & ((1-\eta)/\eta)\cdot\epsilon \\ q(d) & \approx & ((1-\eta)/\eta)\cdot\epsilon^2 & \qquad q(e) & \approx & ((1-\eta)/\eta^2)\cdot\epsilon^3 \end{array}$$

Note that for $a \in \{b, c, d, e\}$,

$$q(a) \;\approx\; constant(a)\cdot\epsilon^{u(b)-u(a)} \tag{1.7}$$

where the term $constant(a)$ does not depend on ϵ. Similarly to the car example discussed in Section 2.5, we can analyze the probability of choosing alternative $a \in \{b, c, d, e\}$ from menu $B \subseteq A$ after $1, 2, 3, ...$ investigation processes. The resulting random choice rule turns out to be a Luce rule with weights $V(a) = \rho(a)\cdot q(a)$. When $\epsilon \longrightarrow 0$, the ratio $q(a')/q(a)$ converges to zero for any pair of alternatives a, a' such that $a \succ a'$. Thus, in the limit, the agent chooses the best alternative with probability one from any menu.

In the general case, given an adaptive language Q, it is always possible to construct a decision rule that generalizes the rule ψ from the example above such that eq. (1.7) holds, where $u(b)$ is the utility of the best alternative in A. Let us describe a class of such decision rules.

For a given natural n and language Q with $|N| \geq n$ questions, consider a decision rule ψ^+ with the memory state space $S^o_\psi = \{1, ..., n\}$, a one-to-one interrogation function $\iota_\psi : S \to N$ and a set of transitions $\mathcal{T}_\psi$ defined as follows:

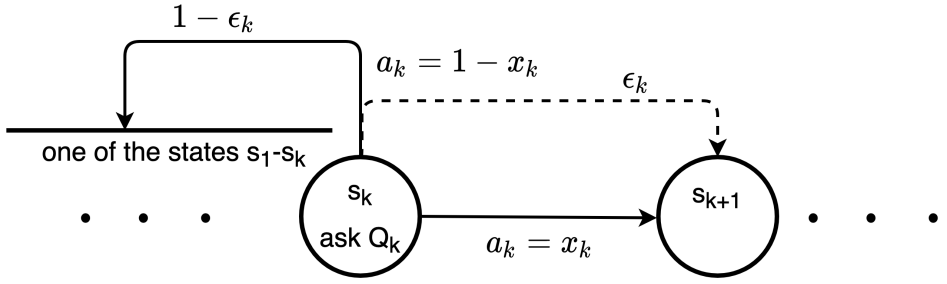
(a) $\tau(s, s+1, x_s) = 1$ with the convention that the state $|S|+1$ denotes $\diamond$;

(b) $\tau(s, s+1, 1-x_s) = \epsilon_s$ with the convention that the state $|S|+1$ denotes $\diamond$;

(c) There exists a unique $s' \in \{1, ..., s\}$ such that $\tau(s, s', 1-x_s) = 1-\epsilon_s$.

where $(x_1, ..., x_n) \in \{0,1\}^n$ and $(\epsilon_1, ..., \epsilon_n) \longrightarrow (0, ..., 0)$; and in particular, for all $i \in \{1, ..., n\}$, $\epsilon_i = (\epsilon_i)_r > 0$ for $r = 1, 2, ...$ is a sequence, converging to zero. A generic k-th state of the decision rule ψ^+ is represented in Figure 1.4, where the interrogation rule is assumed to be $\iota(k) = k$ for convenience.

Figure 1.4: A generic decision rule in the class Ψ_n^+.



I denote by Ψ_n^+ the set of decision rules ψ such that there exists a permutation $\pi : \{1, .., n\} \to \{1, ..., n\}$ with $\pi(1) = 1$ of the indexes of the states $s \in S^0$ such that $\pi(\psi) = \psi^+$ is represented by the described above way, where the action of π on ψ is assumed to be re-enumeration of the states of S^0 and, correspondingly, the states in the interrogation and transition rules. Note that

$$\kappa(\psi^+) \;=\; 3n \qquad \forall \psi \in \Psi_n^+$$

The following result characterizes the adaptive languages as those that allow for a solution of the choice problem via some decision rule from the set Ψ_n^+ and also characterizes the simplest decision rules as the rules from the set Ψ_n^+ in case when $(3/4) \cdot 2^n < m \leq 2^n$.

Proposition 1.4.4 *Let the preference relation $\succeq$ have m equivalence classes. Then a language Q is adaptive if and only if there exists a decision rule $\psi \in \Psi^+_{\lceil \log_2(m) \rceil}$ that solves the choice problem $(Q, \succeq)$. Moreover, if $(3/4) \cdot 2^n < m \leq 2^n$ for a natural n, then a decision rule ψ such that $\kappa(\psi) = 3n$ solves the choice problem $(Q, \succeq)$ if and only if $\psi \in \Psi^+_n$.*

Note that for any $m \geq 2$, either $(1/2) \cdot 2^n < m \leq (3/4) \cdot 2^n$, or $(3/4) \cdot 2^n < m \leq 2^n$, where $n = \lceil \log_2(m) \rceil$—the two cases split the set of possible values of m such that $2^{n-1} < m \leq 2^n$ in half. Thus, the condition $(3/4) \cdot 2^n < m \leq 2^n$ serves as a richness assumption for the number of indifference classes in Theorem 1.4.3 and Proposition 1.4.4. Since my model operates with binary attributes, then powers of 2 play a special role in the analysis: note that the lower bound on complexity, given by Theorem 1.3.1, increases by 3 each time the number of indifference classes surpasses a power of 2.

Intuitively, a decision rule $\psi \in \Psi^+_n$ can be interpreted as a procedure that uses the probability to reach a certain memory state as a proxy for the alternative's utility based on the subset of attributes that have been already investigated. By encountering an alternative from the menu, the agent starts to learn its attributes sequentially; if the attribute's value is good, she jumps to the next memory state immediately and thus, the alternative endowed with all the good properties is quickly picked.

When the agent learns an undesirable property of the item, she becomes less enthusiastic and continues the investigation of other attributes only with a small probability. This penalty in the probability of transition to the next memory state applied upon learning the bad news reflects the relative importance of the corresponding attribute for the agent's preference which, by Proposition 1.4.4, has a representation via an additive utility function.[8]

Theorem 1.4.3 and Proposition 1.4.4 tell us that the agents who are able to describe alternatives using adaptive languages can save their cognitive resources by employing

[8]Precisely, the strict part of the agent's preference relation should be represented by such function.

an intuitively simple decision procedure, described above. If the agent has a language that admits an additive utility representation of the agent's preference, but contains $n > \lceil \log_2(m) \rceil$ attributes, she can use a decision rule $\psi \in \Psi_n^+$ to solve the choice problem[9]. Each additional attribute increases the procedure's complexity by 3, but so long as the number of attributes that require investigation remains small, the agent's procedure is not too complex.

1.5 Sketches of the Proofs

In this section, I provide the key ideas behind the proofs of my main results. I first provide two lemmas that allow us to transform the analysis of solvability of a choice problem $(Q, \succeq)$ by a decision rule ψ to the analysis of directed graphs with vertex sets S_ψ, and the sets of directed edges (links) $\mathcal{T}_\psi$.

Let $q_\sigma(a)$ be the probability to choose alternative $a \in A$ during a single uninterrupted investigation process using the automaton strategy σ—this quantity has been introduced in Section 2.5 when discussing the car example, and discussed later in Sections 2.6 and 4.

Lemma 1.5.1 *A decision rule* $\{\sigma_r\}_{r=1,2,...} \in \Psi$ *solves choice problem* $(Q, \succeq)$ *if and only if the following two conditions hold:*
(i) $a \succ b$ *implies* $q_{\sigma_r}(b)/q_{\sigma_r}(a) \longrightarrow 0$ *for all* $a, b \in A$*;*
(ii) $q_{\sigma_r}(a) > 0$ *for all* $a \in A$*.*

Thus, Lemma 1.5.1 allows us to focus on analyzing the limiting properties of $q_{\sigma_r}(a)$ and abstract away from consideration of random choice from various menus. Consider a sequence of states of automaton $s_1...s_k$ that goes from the starting state $s_1 = 1$ to the decision state $s_k = \diamond$, and assume that this *path* contains each state at most once. For every alternative $a \in A$ and every such path $s_1...s_k$, we can calculate the

[9]Lemma A.2.8 in the Appendix.

probability of realization of $s_1...s_k$ during a process of investigation of a. Let $\omega^*_\sigma(a)$ be the largest of these probabilities among all paths for a fixed alternative $a \in A$.

Lemma 1.5.2 *A decisions rule* $\{\sigma_r\}_{r=1,2,...} \in \Psi$ *solves choice problem* $(Q, \succeq)$ *if and only if the following two conditions hold:*
(i) $a \succ b$ *implies* $\omega^*_{\sigma_r}(b)/\omega^*_{\sigma_r}(a) \longrightarrow 0$ *for all* $a, b \in A$*;*
(ii) $\omega^*_{\sigma_r}(a) > 0$ *for all* $a \in A$*.*

Thus, Lemma 1.5.2 allows us to consider only one highest-probability path[10] from the starting state $s = 1$ to the decision state $s = \diamond$ when analyzing the limiting properties of $q_{\sigma_r}(a)$.

Note that $\omega^*(a)$ is the product of transitional probabilities between states s_l, s_{l+1} in the highest-probability path $s_1...s_k$ when alternative a is under investigation. Given a decision rule ψ, I say that a link $(s, s', j) \in \mathcal{T}_\psi$ is *weak* if the corresponding probability $\tau(s, s', j)$ converges to zero. Otherwise, the link is *strong*.

1.5.1 Theorem 1.3.1—sketch of the proof

Assume that ψ solves $(Q, \succeq)$. I show that ψ should contain at least $2 \cdot \lceil \log_2(m) \rceil$ strong links and at least $\lceil \log_2(m) \rceil$ weak links, proving statement (i) of the Theorem.

Indeed, if $Q' \subseteq Q$ is the set of questions that are used in some state according to the interrogation rule ι_ψ, then Q' should contain at least $\lceil \log_2(m) \rceil$ questions asked in different states of ψ to differentiate any pair of alternatives from distinct indifference classes of $\succeq$. There are at least two outgoing strong links from each of these states, providing the lower bound of $2 \cdot \lceil \log_2(m) \rceil$ on the number of strong links of ψ.

Next, if the agent is not indifferent between alternatives a and b, then Lemma 1.5.2 implies that the highest-probability paths for alternatives $a, b \in A$ discussed in

[10]The fact that it in my model, it is enough to analyze simple paths on the set of states to figure out the limiting properties of the dynamic system resembles the "Z-tree" technique from Kifer (1988) utilized in Kandori et al. (1993) in the context of evolutionary game theory.

the previous section should have different sets of weak links. If n_{weak} is the number of weak links of ψ, then it should be that $2^{n_{weak}} \geq m$, providing the lower bound of $\lceil \log_2(m) \rceil$ on the number of weak links of ψ.

To show that there exists a language with complexity $3\lceil \log_2(m) \rceil$, I first prove the existence of an adaptive language, that is, Proposition 1.4.1. For the proof of Proposition 1.4.1, it is without loss of generality to assume that $\succeq$ is antisymmetric and that $m = 2^n$ for some natural n. Let the first question of language Q be: "does alternative a belong to top 2^{n-1} alternatives in A?" The answer to this question divides A in two equal sets: Q_{11} consisting of top 2^{n-1} alternatives according to $\succeq$, and Q_{10}, consisting of bottom 2^{n-1} alternatives. Let the second question of language Q be: "does alternative a belong to top 2^{n-2} alternatives in the set Q_{11}, or to top 2^{n-2} alternatives in the set Q_{10}?" By continuing to design questions $1, 2, ..., n$ in a similar fashion, we design an adaptive language Q, where $\lambda_i = 2^{n-i}$ are weights of attributes $i = 1, ..., n$ in an additive utility representation of $\succeq$.

Finally, using an adaptive language Q designed above, we can construct a decision rule $\psi^+ \in \Psi^+$ discussed in Section 4 that solves that choice problem $(Q, \succeq)$. For that, it suffices to choose an appropriate vector of vanishing transition probabilities $\epsilon_i = \epsilon^{\lambda_i}$, $i = 1, .., n$, for some $\epsilon \longrightarrow 0$.

1.5.2 Theorem 1.3.4—sketch of the proof

To prove the first statement of Theorem 1.3.4, I use a universal procedure to design a decision rule ψ that solves the choice problem $(Q, \succeq)$ for an arbitrary adequate language Q. The set of flexible memory states of the decision rule ψ has a binary tree structure with root $s = 1$. Each node of this tree (memory state $s \in S^0_\psi$) is associated with a non-singleton set A^s comprised of alternatives $a \in A$ such that the investigation of a could lead to this node.

For an arbitrary node s of the tree, since Q is adequate, there always exists a question $i \in N$ such that both answers $j = 0$ and $j = 1$ are possible for alternatives in A^s. Set up the interrogation rule $\iota(s) = i$. If a subset of alternatives in A^s such that $a_i = j$ contains more than one alternative, create a successor node s' for this subset, and set up a transitional probability $\tau(s, s', j) = 1$. Otherwise, there is a single alternative $a \in A^s$ with $a_i = j$. In this case, create a loop link by setting up $\tau(s, s, 1) = 1 - \epsilon_s$, and a link that leads to the decision state by setting up $\tau(s, \diamond, 1) = \epsilon_s$.

Repeating the procedure described above for each node of the tree until there are no more successor nodes, we construct a decision rule ψ. By choosing appropriate sequences of probabilities ϵ_s, we can achieve a solution of $(Q, \succeq)$ by ψ. The number of links of ψ turns out to be $3|A| - 2$. In addition, it is possible to set up $\epsilon_s = 1$ for one of the states $s \in S^o$ to get $\tau(s, s, 1) = 0$ and decrease the number of links by one, proving the first statement of the Theorem.

To get the intuition behind the proof of statement (ii), suppose $m = |A|$; thus, there is no pair of indifferent alternatives for the agent. Consider language Q consisting of questions "is $a = b$?" for all alternatives $b \in A$ except of the best alternative. All of these questions should be asked in a decision rule ψ that solves the choice problem $(Q, \succeq)$. There are $2|A| - 2$ strong links associated. Lemma 1.5.2 implies that ψ should also contain at least $\lceil \log_2(m) \rceil$ weak links, proving the second statement of the Theorem.

1.5.3 Theorem 1.4.3—sketch of the proof

The intuition behind the first statement of Theorem 1.4.3 is discussed in Section 4. Let us, therefore, sketch the proof of the statement that all simple languages should be adaptive when $(3/4) \cdot 2^n < m \leq 2^n$ for a natural n. Note that $n = \lceil \log_2(m) \rceil$.

Let us assume that the first statement of Proposition 1.4.4 holds; the corresponding proof is relatively more straightforward, and some of its intuition has already been discussed. It suffices, therefore, to prove the only if part of the second statement of Proposition 1.4.4. Thus, we want to show the following assertion: for the considered range of m, if the decision rule ψ solves the choice problem $(Q, \succeq)$ and achieves the lowest possible complexity $\kappa(\psi) = 3\lceil \log_2(m) \rceil$, then $\psi \in \Psi_n^+$.

Let ψ be such a rule. Building on the proof of Theorem 1.3.1, we conclude that ψ should contain exactly $2n$ strong links—and, therefore, exactly n memory states—and also, exactly n weak links. Moreover, those weak links should not be loops—if a weak link is a loop, it cannot be used in the highest-probability path for any alternative.

Suppose that the allocation of the outgoing weak links among the memory states in ψ is $y = (y_1, ..., y_n)$, where y_s is the number of weak links, outgoing from state s. Then we can show that Lemma 1.5.2 implies the following Claim:

Claim 1.5.3 *If ψ solves the choice problem $(Q, \succeq)$, then $\prod_{s \in S} (y_s + 1) \geq m$.*

Note that if $y = (1, ..., 1)$, then the product in the Claim above is equal to 2^n. The next algebraic observation highlights the role of the richness assumption $(3/4) \cdot 2^n < m \leq 2^n$:

Claim 1.5.4 *Let $\sum_s y_s = n$. If the allocation of weak links y is such that $y \neq (1, ..., 1)$, then there is an allocation y' with $\sum_s y'_s = n$ such that $\prod_{s \in S} (y_s + 1) \leq (3/4) \cdot \prod_{s \in S} (y'_s + 1)$*

Since the number of weak links is n, any allocation of weak links except of $y = (1, ..., 1)$ fails to satisfy the necessary condition, given by Claim 1.5.3, whenever $(3/4) \cdot 2^n < m \leq 2^n$. Summarizing our observations, we obtain the following Claim:

Claim 1.5.5 *In the decision rule ψ described above, there are $|S_\psi^o| = n$ memory states, and at each memory state, there are exactly two outgoing strong links, and exactly one outgoing weak link.*

Thus, to prove our assertion, we need to analyze, for each state, where those two strong and one weak link go. Consider the state $s = 1$. Suppose that question $i = 1$ is asked in state 1, and there are two links, associated with the answer "$a_1 = 0$": a weak link $(1, s, 0)$ and a strong link $(1, s', 0)$. Thus, there is only one strong link $(1, s'', 1)$, associated with the answer "$a_1 = 1$". For this sketch, consider a situation when s and s' are memory states such that $s, s' \neq 1$. Let v be the weak link, outgoing from the state s'.

Claim 1.5.6 *The weak links* $(1, s, 0)$ *and* v *described above cannot appear together in the highest-probability path for any alternative* $a \in A$.

Using Lemma 1.5.2, we are able to conclude that in this situation, the number of indifference classes should be $m \leq (3/4) \cdot 2^n$, contradicting the richness assumption. Continuing this analysis, we obtain:

Claim 1.5.7 *The strong link* $(1, s', 0)$ *described above is a loop; that is,* $s' = 1$.

For the last part of the analysis, note that either the set of alternatives $a \in A$ such that $a_i = 1$, or the set of those with $a_i = 0$ should contain m' indifference classes that satisfy $(3/4) \cdot 2^{n-1} < m' \leq 2^{n-1}$. Suppose it is the set with $a_i = 1$; denote it by A'. Recall that a strong link $(1, s'', 1)$ goes to some state s''. Consider an auxiliary decision rule ψ' that uses the same state space except of the state $s = 1$; instead, the investigation process according to this rule starts at state s'' and uses the same transitions between states $v, v' \neq 1$ as the rule ψ. Next, we replace the transitions $(v, 1, j)$ of ψ going from states $v \neq 1$ to state 1 by loops (v, v, j) in ψ'. Finally, we delete the three transitions, outgoing from the state 1 in ψ. The next Claim is the key observation in the last part of our analysis.

Claim 1.5.8 *An auxiliary decision rule described above has exactly* $3n-3$ *transitions and it solves an auxiliary choice problem* $(Q, \succeq')$*, where* $\succeq'$ *is the restriction of* $\succeq$ *on the subset* $A' \subset A$.

We, therefore, can use an induction in n to argue that $\psi' \in \Psi^+_{n-1}$. This allows us to establish the configuration of links, outgoing from all states except of $s = 1$. Using this finding, Lemma 1.5.2, and the richness assumption $(3/4) \cdot 2^n < m \leq 2^n$, we can figure out that the strong link $(1, s'', 1)$ and the weak link $(1, s, 0)$ should go to the same state $s = s''$. It follows that $\psi \in \Psi^+_n$, proving our assertion.

1.6 Extensions

I call the model analyzed in Sections 2-5 *the baseline model,* and consider two natural extensions in this section.

1.6.1 Endogenous dismissal decision

In the baseline model, the agent needs to wait for an exogenous attention shock to bring a new item for her consideration. While this is a good benchmark, and my theorems show that the agent chooses correctly using a low amount of cognitive resources when she takes her time, there are economic environments for which it is naturally to assume that the agent may also dismiss the current item and switch her attention to some other alternative at will. In this section, I incorporate this extension to the model and show that all the results remain unchanged in this case.

Consider the same model as in Section 1.2 with only one change: the automaton's state space is $S = S^o \cup \{\diamond\} \cup \{new\}$, where $S^o = \{1, ..., k\}$ is the set of memory states, state $\diamond$ represents the decision to choose an item for consumption, and a new special state *new* represents the decision to stop an investigation of the current item and pick a new random item from the menu. In this case, the new item is drawn according to the same probability distribution $\rho^B(\cdot)$ as in the case when the attention of the agent is switched to a new item exogenously. The agent always starts the investigation of the new item from the memory state $s = 1$. With probability $\eta \in (0, 1)$, a new item

catches the agent's attention even if she wants to proceed with the examination of the current item.

The definition of the interrogation rule $\iota : S^0 \to N$ remains unchanged, and the stochastic transition rule, naturally, accounts for the possibility to dismiss an item with some probability. Thus, the stochastic transition rule is given by the same formula, $\tau : S^o \times \{0, 1\} \to \triangle(S)$, where S includes also the state *new*. The definition of the set of the positive-probability transitions remains unchanged as well, after taking into account that S includes the state *new*. The definitions of the decision rule, the language's complexity, and the memory load remain the same as in the baseline model. I call the model described above *a model with endogenous dismissal decision.*

The following proposition shows that the complexity and the memory load of the language does not depend on whether the agent can dismiss an item at will or not:

Proposition 1.6.1 *For any adequate language Q, its memory load and complexity in the baseline model coincide with its memory load and complexity in the model with endogenous dismissal decision.*

It follows that all results regarding complexity that have been shown in the baseline model continues to hold in a model with endogenous dismissal decision.

Corollary 1.6.2 *In the model with endogenous dismissal decision, Lemma 1.2.1, Lemma 1.2.2, Theorem 1.3.1, Theorem 1.3.4, Theorem 1.4.3, Proposition 1.3.2, Proposition 1.3.3, Proposition 1.3.5, Proposition 1.4.1, and Corollary 1.4.2 hold.*

The key idea behind Proposition 1.6.1 is that Lemma 1.5.2—which says that for our analysis, it is enough to consider only highest-probability paths from the starting state $s = 1$ to the consumption state $s = \diamond$—is valid for both variants of the model. Thus, for any decision rule that uses endogenous dismissal with positive probability, we can construct a decision rule where the agent returns to the same memory state

instead of a dismissal. This new rule will have the same size of the state space and at most the same number of transitions as the original, and it will not have endogenous dismissal decisions.

For the Proposition 1.4.4 to work in the model with endogenous dismissal decision, we need to expand the set of decision rules Ψ_n^+ defined in Section 1.4 as follows. Use the same definition of Ψ_n^+ as in the baseline model with the only one change: relax the original property

(c) There exists a unique $s' \in \{1, ..., s\}$ such that $\tau(s, s', 1 - x_s) = 1 - \epsilon_s$.

to a property

(c') There exists a unique $s' \in \{1, ..., s\} \cup \{new\}$ such that $\tau(s, s', 1 - x_s) = 1 - \epsilon_s$.

Call the set of such decision rules $\Psi_n^{+dismissal}$. Thus, instead of returning to the previously visited memory state with probability $1 - \epsilon_s$ upon learning that the value of the investigated attribute is bad, a decision rule $\psi \in \Psi_n^{+dismissal}$ may alternatively have a transition to the dismissal decision with the same probability under the same conditions.

Proposition 1.6.3 *Consider the model with endogenous dismissal decision. Let the preference relation $\succeq$ have m equivalence classes, and $n = \lceil \log_2(m) \rceil$. Then a language Q is adaptive if and only if there exists a decision rule $\psi \in \Psi_n^{+dismissal}$ that solves the choice problem $(Q, \succeq)$. Moreover, if $(3/4) \cdot 2^n < m \leq 2^n$, then a decision rule ψ such that $\kappa(\psi) = 3n$ solves the choice problem $(Q, \succeq)$ if and only if $\psi \in \Psi_n^{+dismissal}$.*

While for the purpose of our analysis with zero waiting costs, both the baseline model and the model with endogenous dismissal decision give the same results, when waiting costs are small but not negligible, the agent, intuitively, should be better off by dismissing an item instead of cycling via the same memory states multiple times.

1.6.2 Recall of the past investigations

An important assumption of the baseline model is that, each time the new alternative catches the agent's attention, the investigation process starts anew. Thus, the agent does not recall the results of the past investigations. One can think that the ability to do so is important: for instance, if we ignore the cognitive constraints, but assume non-zero costs of waiting (or information acquisition), the agent should take advantage of the accumulated knowledge and adapt her decision rule during the search if she is uncertain about the choice environment[11].

In this section, I relax an assumption that the agent does not recall results of the past investigations. Instead, I assume that the agent has the same constraints in recalling this information as in recalling the information regarding the current item. Namely, the agent's experience regarding the whole search process is summarized via one of the automaton's memory states.

To incorporate all extensions, let us also allow the agent to dismiss the current item at will, as in the previous section. Here, I will not use a special state *new* to represent the agent's dismissal decision because in this case, the information gathered by the agent before the dismissal decision would be lost. Instead, I consider the automaton with the same definition of the state space as in the baseline model, $S = S^o \cup \{\diamond\}$, and the same definition of the interrogation rule $\iota : S^o \times \{0,1\} \to N$, but with an appropriate extension of the stochastic transition rule.

For convenience, let us define a stochastic transition rule, τ, as two functions $\tau = (\tau^1, \tau^2)$. The first function is $\tau^1 : S^o \times \{0,1\} \longrightarrow \triangle\Big(\Big(S^o \times \{old, new\}\Big) \cup \{\diamond\}\Big)$ with the following interpretation. The variable $x \in \{old, new\}$ represents the decision to continue an investigation of the old item ($x = old$), or to draw a new item from the menu ($x = new$). Consider an agent who learns value $j \in \{0,1\}$ of attribute $\iota(s)$ in memory state $s \in S^o$. This agent chooses the current item with probability $\tau^1(s, \diamond, j)$.

[11] This point is made, for instance, in Kohn and Shavell (1974).

The probability to continue the investigation of the current item and transition to memory state $v \in S^o$ is given by $\tau^1(s, v, j, old)$. Finally, the probability to dismiss the current item at will, pick a new item from the menu, and start its investigation from the memory state v is given by $\tau^1(s, v, j, new)$.

The function $\tau^2 : S^o \longrightarrow \triangle(S^o)$ gives the probability $\tau^2(s, v)$ to transition from memory state s to memory state v in case of the exogenous attention shock. In this case, the investigation of the new alternative starts from the memory state v.

Let us define the set of positive-probability transitions[12] by including in it all tuples (s, v, j, new), (s, v, j, old), $(s, \diamond, j)$ for which τ^1 is positive, and all pairs of states (s, v) for which τ^2 is positive. The decision rule is then defined as in the baseline model as a sequence of automata that share the same state space, interrogation rule, and set of positive-probability transitions.

Let us analyze the size of the state space required for an optimal decision rule. Recall that the memory load of language Q is the minimum size of the state space of a decision rule that solves the choice problem $(Q, \succeq)$:

$$\mathcal{M}(Q) \;=\; \min_{\psi \in \Psi:\ \psi \text{ solves } (Q,\succeq)} |S^o_\psi|$$

Let us call the model described above *a model with recall of the past investigations.*

Theorem 1.6.4 *Consider the model with recall of the past investigations. Let the preference relation $\succeq$ have m indifference classes. Then for any language Q, $\mathcal{M}(Q) \geq \lceil \log_2(m) \rceil$, and there exists a language Q such that $\mathcal{M}(Q) = \lceil \log_2(m) \rceil$.*

Theorem 1.6.4 provides the same tight low bound on the languages' memory load for the model with recall of the past observations as Proposition 1.3.2 for the baseline model. Thus, for the languages with the lowest memory load in the baseline model, it

[12]Since in this section, I analyze the memory load and not the transitional complexity, the definition of the set of transitions is non-essential; in particular, one could alternatively define a decision rule via a sequence of automata without any constraints on the set of transitions.

is not possible to economize more on the memory by implementing optimal decision rules that retain some memory of the past observations. The reason behind this result is relatively clear, once we note that the lower bound given by Proposition 1.3.2 is tight. Indeed, an optimal decision procedure should have at least $\lceil \log_2(m) \rceil$ states to be able to inquire about $\lceil \log_2(m) \rceil$ different attributes to differentiate any pair of non-indifferent alternatives and choose optimally in the corresponding binary menus. A similar result holds for the upper bound:

Theorem 1.6.5 *Consider the model with recall of the past investigations. Let $|A|$ be the total number of alternatives. Then for any preference relation $\succeq$ and adequate language Q, $\mathcal{M}(Q) \leq |A|-1$. Moreover, for any $m \in \{2, ..., |A|\}$, there is a preference relation $\succeq$ with m indifference classes and a language Q such that $\mathcal{M}(Q) = |A| - 1$.*

Theorem 1.6.4 and Theorem 1.6.5 provide the same tight lower and upper bounds on the language's memory load for the model with recall of the past investigations, as those that are obtained for the baseline model. Therefore, for the corresponding languages that attain those bounds, one way how the agent can economize on the memory load is to focus on the investigation of the current item—as analyzed in the baseline model.

1.7 Conclusion

In this paper, I introduce a model of decision making that captures a highly inattentive consumer who nonetheless achieves near optimal behavior by virtue of making decisions slowly. This type of behavior is reminiscent of browsing or window shopping.

My model also exhibits a close connection to the Luce model from discrete choice theory (Luce (1959)). As in the discrete choice theory, agents in my model behave stochastically and with some (albeit small) error. The relative likelihood of choosing

one alternative over the other is independent of the menu of choices, as required in LuceâĂŹs choice axiom. In discrete choice theory, the source of randomness is typically interpreted as a utility shock or an error, while in my setting random choice is a tool that allows the agent to achieve near optimal outcomes with limited cognitive resources.

In Section 1.6.1, I consider a natural extension of the baseline model where I allow the agent to drop the currently investigated item and pick a new random item from the menu intentionally. I show that this does not change the language's complexity.

My baseline model assumes that the agent has no control over the search process, and also that when a new alternative arrives, the agent does not retain memory from the past examination of alternatives. In Section 1.6.2, I relax the latter assumption and show that the lower and upper bounds on the memory required for the solution of the choice problem remain the same as in the baseline model (and these bounds are tight as well). Whether the corresponding results hold for the transitional complexity[13] is a direction for future research[14].

Having a more directed search, intuitively, should not matter for the optimality criterion, considered in the paper. For instance, the ability to avoid multiple investigations of the obviously inferior alternatives could make the agent's choice faster, but would not matter for the infinitely-patient agent. Moreover, the ability to return to the previously visited alternatives at will may require additional cognitive resources, such as recalling the indexes of the visited alternatives.

[13]One should carefully generalize the transitional complexity; for instance, the instruction that says to always start an investigation of the new alternative from state $s = 1$—considered in the baseline model—is intuitively simpler than the instruction that says that a new investigation should start from state 1 if the previous state was 2 or 3, and it should start from state 2 if the previous state was 1 or 4. Just counting transitions in the setup presented in Section 1.6.2 may not reflect this intuition.

[14]However, since the memory load and transitional complexity are related measures, the paper's results regarding the memory load do put bounds on the transitional complexity in a more general setup as well. The challenge is to reproduce the sharp characterization of the lower bound and extend the results regarding the simplicity of the adaptive languages.

All of the extensions discussed above matter for the analysis of the resulting random choice away from the limiting case of the infinitely patient agent considered in this paper. This is an intriguing avenue for the future research.

To summarise, this paper shows that a very cognitively limited agent operating in an environment with an uncertain choice set and subjected to exogenous attention shocks is able to solve a complex choice problem nearly optimally. For that, the agent only need ample time and a convenient way—that is, an adaptive language—to think about the alternatives. Moreover, an intuitive procedure that consists of looking at each attribute sequentially and advancing in this investigation with higher probability upon learning good news turns out to be the simplest possible algorithm, and often—the only simplest algorithm. From a broader perspective, by characterizing simple languages as adaptive, this paper provides a foundation for the usage of the separable utility functions in models with patient agents.

Chapter 2

Framing and Ambiguity

2.1 Introduction

[1]In this paper, we present a model of decision making under uncertainty with framing effects. More specifically, we analyze decision-makers who confront Knightian uncertainty and whose reasoning about this uncertainty is influenced by the manner in which the choice problem is presented. Each frame suggests a different assessment of the underlying uncertainty and therefore, may yield a different choice from a given set of options. A modeler who cannot observe the frame cannot rule out any one of these choices. Once the frame is fixed, the decision maker is a maxmin expected utility maximizer. The frame determines the set of probabilities over which expected utility is minimized.

Our motivation is the observation that decision makers are constrained by limited cognitive resources; they suffer from limited attention, limited memory or computational ability or a coarse understanding of contingencies. Such decision makers cannot identify, assess and integrate all available payoff-relevant information into unified decision procedure. In the absence of such a coherent procedure, the decision maker may be prone to mistakes and biases and may end up making different choices in seemingly identical situations.

To illustrate our framework, we consider a hypothetical student, Alice, preparing for an exam of unknown difficulty. She needs to decide whether to spend the day studying for the exam or work on her history paper. She mentions the exam to one of her friends who asks whether Alice thinks the exam will be *hard* or *not hard*. Alice realizes that if the exam is hard, she is likely to fail no matter how she spends the day. However, if the exam is not hard, an extra day of studying is likely to make the difference between success and failure. Given a choice between failing for sure and failing if the exam is hard, Alice decides to devote her day to studying for the exam.

[1]This chapter is based on the paper "Framing and Ambiguity".

Consider the following alternative scenario in which Alice, instead of discussing the exam with a friend, mentions it to her brother who states that in his opinion, the key is to assess whether the exam will be *easy* or *not easy*. Alice guesses that if the exam is *not easy*, then there is a possibility of a failure whether she studies for the exam or not, and if the exam is *easy*, she will likely pass even if she spends the day on her history paper. Based on this reasoning, Alice concludes that she should devote her day to the history paper.

An outside observer studying Alice's behavior will know whether she spent day studying for the exam or working on her history paper and whether she passed the exam or not and might even learn whether the exam was easy, hard or neither (that is, in-between). However, the observer is unlikely to know whether Alice's framing of the problem was influenced by her friend or her sister. Hence, to the outside observer, Alice can take either action, study for the exam or work on her paper.

Notice that each frame yields an incomplete specification of the underlying uncertainty. We interpret this incompleteness as ambiguity. With the friend's framing, Alice contemplates the events *hard* and *not hard* and hence these events become unambiguous while the events *easy* and *neither easy nor hard* are suppressed and remain ambiguous. With the brother's framing, Alice contemplates the events *easy* and *not easy* and hence these events become unambiguous while the events *hard* and *neither easy nor hard* are suppressed and remain ambiguous.

The primitive of our *Framed Ambiguity* model is a choice correspondence which maps every non-empty compact set of Anscombe-Aumann acts to a subset of possible choices from that set. Our axioms on this correspondence yield the following representation:

$$\begin{aligned} c(A) &= \bigcup_i c_i(A) \\ c_i(A) &= \left\{ f \in A \;\middle|\; \min_{\mu \in P_i} E_\mu U(f) \;\geq\; \min_{\mu \in P_i} E_\mu U(g) \;\; \forall g \in A \right\} \end{aligned}$$

for all i. Hence, each frame i identifies a set of probability distributions P_i describing the uncertainty associated with that frame, while vNM expected utility function U remains the same for all frames. After adopting a particular frame, the agent becomes a maxmin expected utility maximizer (Gilboa and Schmeidler (1989)).

The three key axioms of our representation theorem (Theorem 2.2.1) are *Framed Uncertainty*, a weakening of WARP that retains Sen's α (sometimes called contraction consistency) but replaces Sen's β (expansion consistency) with a weaker alternative, a new axiom, *No-C-Hedging*, which extends Gilboa and Schmeidler's c-independence for decision makers who fail WARP, and *Indirect Ambiguity Aversion*, which is a version of decision maker's aversion to ambiguity, suitable for the environment with framing. Hence, *framed ambiguity* is a multi-multiple prior model that has maxmin expected utility as a special case; when there is a unique frame. This is the only case in which the frame ambiguity model satisfies *WARP*.

With multiple frames, in general, it is not possible to identify uniquely the parameters of the representation. In Theorem 2.2.3, we show that decision maker's expected utility function is unique up to a positive affine transformation and that there is a *minimum* and a *maximum* family of frames. The set of all representations has a lattice structure: a family of frames represents the choice function if and only if it includes the minimum family and is included in the maximum family. We provide a full characterization of all representations in Proposition 2.3.2. If all sets of beliefs in a representation are singletons; that is, if the decision maker is a subjective expected utility maximizer with frame-dependent beliefs, then the representation is unique (Corollary 2.3.3).

We consider two criteria for comparing framed-ambiguity averse agents: decision maker 1 is *more decisive than* decision maker 2, if from each choice set, decision maker 1 chooses a subset of the options that decision maker 2 chooses. Decision maker 1 is

more consistent than decision maker 2, if whenever decision maker 2 chooses a unique option from a choice set, so does decision maker 1.

In Proposition 2.5.1, we show that one decision maker is more decisive than another if and only if the two have the same expected utility functions and the former's maximal family of frames is a subset of the latter's maximal family of frames. Hence, a more decisive agent is one who has fewer frames. Theorem 2.5.2 provides a characterization of the more consistent than relationship for decision makers with finitely many frames. Unlike greater decisiveness, greater consistency does not require fewer frames; rather, it imposes restrictions on the type of additional frames that a more consistent agent can have.

2.1.1 Related Literature

Our paper contributes to the literature on framing, on ambiguity and on aggregation of beliefs. In this section, we briefly introduce the related papers. We provide more details in subsequent sections to discuss relation to our results.

Tversky and Kahneman (1981) introduced the notion of framing. In their formulation, framing introduces a benchmark that enables decision makers to identify some outcomes as *gains* and others as *losses*. In our formulation, a frame enables a decision maker to organize her reasoning about the uncertainty.

Framed ambiguity model extends Gilboa and Schmeidler (1989) maxmin model of ambiguity aversion. We enrich it by assuming multiple sets of beliefs and interpreting each set of beliefs as a frame in the spirit of Salant and Rubinstein (2008)[2].

The Anscombe-Aumann framework in Heller (2012) is a special case of our model under two restrictions: all sets of beliefs are singletons, and the family of frames (singleton beliefs) is convex. The same is true for Lehrer and Teper (2011) with the nuance that they consider preference relation as a primitive.

[2]We also assume that the analyst does not observe a frame, but observes only a resulting choice correspondence, as in section 3 of Salant and Rubinstein (2008).

Frick et al. (2019a)—and, independently, Chandrasekher (2019)[3]—also consider a decision maker whose perception of uncertainty is given by a family of sets of beliefs. They assume that decision maker considers all sets of beliefs in the family, and chooses the most optimistic set of beliefs and the most pessimistic belief in a set to evaluate a given act. Thus, their agent respects *WARP* in contrast to ours.

Kopylov (2019) also builds a model where agent may use maxmin models with different sets of beliefs to evaluate Anscombe-Aumann acts. In the language of our model, he assumes that frames are menu-dependent; moreover, each menu corresponds to exactly one frame. We, in contrast, assume that decision maker can use all frames from the family to make choices from each menu. Thus, the two models are not nested.

Lu (2014) studies a random maxmin model. His primitive is a stochastic choice from menus consisting of ex-ante lotteries over Anscombe-Aumann acts. In terms of his model, we assume much less choice information: only the support of the distribution of choices from menus consisting of degenerate lotteries over acts[4] is known.

Other papers that model framing of uncertainty include Bourgeois-Gironde and Giraud (2009), Ahn and Ergin (2010) and Caplin and Martin (2020). Outside of the uncertainty framework, preference identification in the environment with framing is studied, for instance, in Goldin and Reck (2020).

Other decision-theoretic papers that consider violations of rationality in decision making under uncertainty include Ok et al. (2012), Galaabaatar and Karni (2013) and Hara et al. (2019).

Our comparative statics result contributes to the literature on aggregation of preferences under uncertainty. In particular, in Corollary 2.5.5 we consider aggregation

[3]To the best of our knowledge, these two papers will be combined.

[4]To connect his (random choice) and our (choice correspondence) primitives, assume that an act is chosen from the menu if the corresponding degenerate lottery over acts is chosen from the menu of the corresponding degenerate lotteries with positive probability—i.e. we consider the *upper choice correspondence, induced by the stochastic choice* from Ok and Tserenjigmid (2019).

of opinions of ambiguity averse experts that agree on utilities, but disagree on sets of beliefs. We discuss relation of our results to Crès et al. (2011) and Hill (2011) in Section 5. Other related papers are Nascimento (2012) and Danan et al. (2016).

2.1.2 Structure of the Paper

The rest of the paper is organized as follows. Section 2 introduces our setup, axioms and framed ambiguity model and provides characterization and identification theorems. Section 3 elaborates on identification results, Section 4 studies behavioral implications of agent's sensitivity to framing. Section 5 provides our comparative statics results and application to the social choice theory. All proofs are given in the Appendix.

2.2 Model

We consider an Anscombe-Aumann setup with a finite set of prizes X and a finite set of states of the world S with $|S| \geq 2$. An act $f : S \to \triangle X$ is a mapping from the state space to the set of probabilities (objective lotteries) on X. Denote the set of acts by H with typical elements $f, g, h, ...$ and endow it with metric d consistent with a product topology induced by the topology of the point-wise convergence of probabilities on X. The set of acts H is a mixture space with mixture operation defined via $(\lambda f + (1-\lambda)g)(s) = \lambda f(s) + (1-\lambda)g(s)$. A constant act is an act that gives the same consequences in all states: $f(s) = p \ \forall s \in S$. We denote such acts by corresponding lotteries $p, q, r, ...$ whenever it does not cause confusion. The set of constant acts is $H_0 \subset H$. Decision maker (DM, she) chooses acts from menus $A, B, C, ...$. A menu is a non-empty compact subset of H. We denote the set of all menus by $\mathcal{K}$ and endow it with the Hausdorff metric. A mixture between a menu and

an act is defined via $\lambda A + (1-\lambda)g = \{\lambda f + (1-\lambda)g \mid f \in A\}$. The set of menus consisting of constant acts is $\mathcal{K}_0 \subset \mathcal{K}$.

2.2.1 Choice Correspondence

The analyst observes a *choice correspondence* $c : \mathcal{K} \to 2^H$ such that $\varnothing \neq c(B) \subseteq B$ for any $B \in \mathcal{K}$. We interpret $c(B)$ as a subset of acts that *could be chosen* from menu B by the DM. We make the following implicit assumptions:

(i) DM's perception of uncertainty could be influenced by the details of the decision problem unobserved by the analyst that do not provide DM with additional information about the state of the world. We call the collection of such details *a frame*.

(ii) An analyst is able to recover choice correspondence from the data by observing repeated choices from each menu. Inconsistency of choices emerges only either from varying framing of the problem or from indifferences between the alternatives.

(iii) The collection of frames that DM experiences is invariant across menus.

The first assumption is the premise of the project, as we are interested in the class of choice correspondences that emerge as a result of the *ambiguity framing*. Assumption (ii) in particular says that *no learning* occurs: notwithstanding the fact that DM faces multiple frames for a menu, she is unable to connect her past decisions to the current situation. Assumption (iii) says that we don't consider menu-dependent frames. Finally, we make an assumption about the DM's attitude toward uncertainty:

(iv) Decision maker is cautious toward the uncertainty that she could realize.

Assumption (iv) is common for models involving Knightian uncertainty. Moreover, if DM realizes that she is sensitive to framing, this gives her an additional novel motivation to use caution. DM might think that the uncertainty relevant for the

choice problem is framed in a way that erroneously inflates values of some acts—such as emphasizing possible priors that favor these acts. In this case the worst case scenario evaluation could be a good approach to counteract this type of exogenous framing.

2.2.2 Axioms

We introduce behavioral axioms on the choice correspondence in the spirit of the discussed above implicit assumptions (i)-(iv). Recall that *Weak Axiom of Revealed Preference (WARP)* is equivalent to conditions α and β taken together:

WARP: $\forall A, B \in \mathcal{K} \quad c(A) \cap B \neq \varnothing \implies c(B) \cap A \subseteq c(A)$

$\Updownarrow$

α: $\forall A, B \in \mathcal{K} \quad c(A \cup B) \cap A \subseteq c(A)$

β: $\forall A, B \in \mathcal{K} \quad c(A \cup B) \cap A \neq \varnothing \implies c(A) \subseteq c(A \cup B)$

Our first axiom relaxes *WARP* on the domain of menus that involve non-constant acts.

AXIOM 1 *(Framed Uncertainty):*

1.1 *(α)*: $\forall A, B \in \mathcal{K} \quad c(A \cup B) \cap A \subseteq c(A)$

1.2 *(C-β)*: $\forall A, B \in \mathcal{K}_0 \quad c(A \cup B) \cap A \neq \varnothing \implies c(A) \subseteq c(A \cup B)$

1.3 *(Aizerman's Property)*: $\forall A \in \mathcal{K} \ \forall f \in H \quad f \notin c(A \cup \{f\}) \implies c(A) \subseteq c(A \cup \{f\})$

Condition α is appealing, since if there is some framing that allows DM to choose an act from a menu, the same framing would allow DM to chose this act from the subset of this menu. Since we assume that framing influences only the perception

of uncertainty, we demand *WARP* to hold on the domain of menus, consisting of constant acts, hence *C-*β. Finally, *Aizerman's Property* relaxes β: one can see that if $f \notin c(A \cup \{f\})$, then $c(A \cup B) \cap A \neq \varnothing$ for $B = \{f\}$, and, hence, $c(A) \subseteq c(A \cup B) = c(A \cup \{f\})$ by β. *Aizerman's Property* says that if h is chosen from the menu, and adding f to this menu results in h not being chosen, then at least f should be good enough by itself to be chosen under some framing.

Our next axiom is a generalization of Gilboa and Schmeidler's c-independence axiom for preference relation to the choice setting:

AXIOM 2 *(C-Independence):* $\forall A \in \mathcal{K}\ \forall p \in H_0\ \forall \lambda \in (0,1)$

$$c(\lambda A + (1-\lambda)p) = \lambda c(A) + (1-\lambda)p$$

C-Independence says that uncertainty regarding $s \in S$ does not matter for evaluation of constant acts, and that DM is able to factorize Knightian uncertainty and objective uncertainty regarding the prize that lottery $f(s)$ gives in state s. Thus, even though DM may have different perceptions of uncertainty under different frames, her choices do not change once we scale payoffs by varying objective probabilities. Once β is relaxed, though, *C-Independence* does not fully capture this intuition, and we also need the following property:

AXIOM 3 *(No C-Hedging):* $\forall A \in \mathcal{K}\ \forall f \in H\ \forall p \in H_0\ \forall \lambda \in (0,1)$

$$f, p \in A \implies c(A) \subseteq c(A \cup \{\lambda f + (1-\lambda)p\})$$

To get the intuition behind *No C-Hedging*, suppose that under any circumstances act $\lambda f + (1-\lambda)p$ seems to be so attractive that DM cannot choose h from A in the presence of this act. Disregarding DM's perception of uncertainty either f or p should be at least as attractive as $\lambda f + (1-\lambda)p$, since constant act p cannot effectively hedge

against bad outcomes in f. Hence, the presence of both f and p in A would not allow DM to choose h from A under any circumstances as well.

Monotonicity is a natural assumption for the choice under uncertainty. Our version involves only choices from binary menus:

AXIOM 4 *(Strict Monotonicity):* $\forall f, g \in H$

$$g(s) \notin c(\{f(s), g(s)\}) \; \forall s \in S \implies g \notin c(\{f, g\})$$

Strict Monotonicity says that DM chooses consequentially. If she cannot find a reason to choose $g(s)$ versus $f(s)$ for any possible contingency, then she concludes that there is no reason to choose g versus f whatever uncertainty regarding $s \in S$ she thinks is possible.

Our last structural axiom formalizes assumption (iv):

AXIOM 5 *(Indirect Ambiguity Aversion):* $\forall A \in \mathcal{K} \;\; \forall h \in A \;\; \forall f, g \in H \;\; \forall \lambda \in (0,1)$

$$h \notin c(A \cup \{f\}) \text{ and } h \notin c(A \cup \{g\}) \implies h \notin c(A \cup \{\lambda f + (1-\lambda) g\})$$

If DM is averse to the ambiguity within the frame, she always considers $\lambda f + (1 - \lambda)g$ to be as good as at least one of acts f or g. If adding either f or g to the menu does not allow h to be chosen under any frame, so should do their mixture. Note that the *Indirect Ambiguity Aversion* axiom keeps track of an act that cannot be chosen, and not of the act that is chosen.

Finally, we introduce our technical axioms. We endow $\mathcal{K} \times H$ with a product topology.

AXIOM 6 *(Continuity):* $\{ \; (A, f) \in \mathcal{K} \times H \mid f \in c(A) \; \}$ is closed

AXIOM 7 *(C-Non-Degeneracy):* $\exists p, q \in H_0$ such that $\{p\} = c(\{p, q\})$

We use *C-Non-Degeneracy* axiom instead of a weaker statement $c(A) \neq A$ for some $A \in \mathcal{K}$ because if choices over constant acts are degenerate, *Strict Monotonicity* has no bite, and pathological models are consistent with axioms 1–6 and $c(A) \neq A$ for some A[5].

2.2.3 Representation

We denote by Π the set of all non-empty compact and convex subsets of beliefs $P \subseteq \triangle S$. We endow Π with a Hausdorff metric. Note that Π is compact. When we talk about a set of beliefs P, we assume that $P \in \Pi$.

DEFINITION 1. A *Framed Ambiguity* model is a pair $(U, \mathcal{A})$, where $U : \triangle X \to R$ is a non-degenerate vNM expected utility function, and $\mathcal{A} \subseteq \Pi$ is a non-empty closed family of non-empty compact and convex sets of beliefs $P \in \Pi$.

DEFINITION 2. A framed ambiguity model $(U, \mathcal{A})$ *represents* choice correspondence $c(\cdot)$ if

$$c(B) = \bigcup_{P \in \mathcal{A}} \arg\max_{f \in B} W_P(f) \qquad \forall B \in \mathcal{K} \tag{2.1}$$

where

$$W_P(f) = \min_{\mu \in P} \sum_{s \in S} \mu(s) U(f(s)) \tag{2.2}$$

Note that each framed ambiguity model represents a unique choice correspondence given by eq. (2.1), since $c(B) \subseteq B$, and taking some $P \in \mathcal{A} \neq \varnothing$ we can see that *arg max* is always non-empty. $W_P(\cdot)$ is a utility function of the maxmin model with

[5]Lemma B.2.1 in Appendix B.

vNM expected utility $U(\cdot)$ and set of beliefs P. An act belongs to $c(B)$ whenever it maximizes menu B with respect to at least one of utility functions $\{W_P\}_{P\in\mathcal{A}}$. Thus, a framed ambiguity model operationalizes frames via sets of beliefs $P \in \Pi$. This way $\mathcal{A}$ is a set of frames, and each of them could be used as a rational for choosing $f \in c(B)$. We maintain an assumption that a frame is an unobserved property of a decision problem and not a property of an individual act[6]. Thus, if an act is chosen, it should be the best under some framing, when all other acts in the menu are evaluated under the same framing.

Our first main result is a characterization theorem.

Theorem 2.2.1 *A choice correspondence has a framed ambiguity representation if and only if Axioms 1–7 hold.*

Theorem 2.2.1 says that framed ambiguity model is equivalent to axioms *Framed Uncertainty, C-Independence, No-C-Hedging, Strict Monotonicity*, *Indirect Ambiguity Aversion, Continuity* and *C-Non-Degeneracy*. Our model does not contain redundant axioms:

Proposition 2.2.2 *Axioms 1–7 are independent.*

Our second main result is identification of the parameters of the model that provides a minimum representation.

Theorem 2.2.3 *Let choice correspondence $c(\cdot)$ satisfy axioms 1–7. Then there is a unique minimum family of frames $\mathcal{A} \subseteq \Pi$, a unique maximum family of frames $\mathcal{B} \subseteq \Pi$ and vNM expected utility function U such that $(U', \mathcal{A}')$ represents $c(\cdot)$ if and only if U' is a positive affine transformation of U, and $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{B}$.*

[6]To connect the two notions, we can consider a set of frames with a product structure, where each coordinate corresponds to a framing of some act. However, if we want to maintain assumption (iii) about the invariance of the set of frames across menus, we then should consider, for instance, choice from menu $B = \{f, g\}$ with all possible framing of not only acts f and g, but all other acts as well.

Theorem 2.2.3 tells us that, although we cannot exclude some multiplicity of the representation, there is a well-defined unique minimum collection of frames that provides a framed ambiguity representation of choice correspondence $c(\cdot)$ that satisfies our axioms. The main idea behind this result is that unobserved framing of choice problems induces observed menu effects for the resulting choice correspondence.

To identify frames, for any given act f we find all menus A maximal in the sense that f is chosen from A, but not chosen from any larger menu. If DM satisfies *WARP*, there exists exactly one such menu for any act f—menu, consisting of all acts that are weakly worse than f. In our model, each frame from the minimum family gives rise to a maximum menu A for some act f; this menu consists of all acts that are weakly worse than f under the considered frame.

Theorem 2.2.3 also tells us that there exists a maximum representation that includes all possible frames *redundant* for the representation: once there is a minimum collection $\mathcal{A}$, adding or deleting frames in $\mathcal{B}\backslash\mathcal{A}$ does not change the induced choice correspondence. Proposition 2.3.2 in the next section characterizes frames in $\mathcal{B}$.

2.3 Redundant Frames

Consider the following simple example of multiplicity of representation in our model. Let $X = \{x, y\}$ and $S = \{1, 2\}$. Agents 1 and 2 have the same vNM expected utility function $U(p) = p(x)$ for $p \in \triangle X$. The first agent operates with two non-nested intersecting sets of beliefs: $\mathcal{A}_1 = \{P_1, P_2\}$. The second agent uses the same sets of beliefs and also their intersection: $\mathcal{A}_2 = \{P_1, P_2, P_1 \cap P_2\}$. Clearly, $W_{P_1 \cap P_2}(f) = \max\{W_{P_1}(f), W_{P_2}(f)\}$. Thus, if act f maximizes $W_{P_1 \cap P_2}(\cdot)$ in menu B, it also maximizes at least one of $W_{P_1}(\cdot)$ or $W_{P_2}(\cdot)$. It follows that both agents have the same observed choice correspondence. Thus, frame $P_1 \cap P_2$ is redundant for the

representation.

DEFINITION 3. Given a non-degenerate vNM expected utility function U and a collection of frames $\mathcal{A} \subseteq \Pi$, frame $P \in \Pi$ is *redundant* if model $(U, \mathcal{A} \cup \{P\})$ represents the same choice correspondence as $(U, \mathcal{A} \backslash \{P\})$.

As we can see from the example, an intersection of frames could be a redundant frame. To extend the idea for larger state spaces, we introduce a notion of *coherent intersection* of compact convex sets (sets of beliefs).

DEFINITION 4. Given a non-empty collection $\mathcal{C}$ of non-empty convex compact sets in $\triangle S \subset R^{|S|}$, a non-empty convex compact set P is a *coherent intersection* of $\mathcal{C}$ if the projection of P on any linear subspace of $R^{|S|}$ is equal to the intersection of projections of sets in the collection $\mathcal{C}$. We say that a coherent intersection is *non-trivial* if $P \notin \mathcal{C}$.

Clearly, if P is a coherent intersection of $\mathcal{C}$, then it is an intersection of frames in $\mathcal{C}$—to see this, take $R^{|S|}$ as a linear subspace of itself. When $\mathcal{C}$ has a coherent intersection P, we say that sets of beliefs in $\mathcal{C}$ intersect coherently. Notice also that any $P \in \Pi$ is a trivial coherent intersection of the collection consisting of P (i.e. $\mathcal{C} = \{P\}$). Recall that a support functional of a set of beliefs P is a function $I_P : R^{|S|} \to R$ given by $I_P(z) = \min_{\mu \in P} \sum_{s \in S} \mu(s) \cdot z_s$. The next proposition shows some properties of the coherent intersections.

Proposition 2.3.1 *(i) An intersection P of $\mathcal{C}$ is coherent if and only if $I_P(z) = \max_{P' \in \mathcal{C}} I_{P'}(z)$ for all $z \in R^{|S|}$. (ii) If $|S| = 2$, any non-empty intersection of a non-empty closed collection $\mathcal{C} \subseteq \Pi$ is a coherent intersection of $\mathcal{C}$. (iii) If $|S| > 2$, there*

is a non-empty closed collection $\mathcal{C} \subseteq \Pi$ *that has a non-empty intersection, but does not have a coherent intersection.*

Thus, Proposition 2.3.1 gives an alternative definition of the coherent intersection: P is a coherent intersection of $\mathcal{C}$ is when the support functional of P—which is a utility function in the corresponding maxmin model—is equal to the maximum of the support functionals of the sets of beliefs in $\mathcal{C}$. In the proofs, we use this alternative definition.

Let us introduce one more piece of notations.

DEFINITION 5. Let $\mathcal{A} \subseteq \Pi$ be a non-empty collection of sets of beliefs, then its *coherent closure* is given by

$$(\mathcal{A})_{coh} \quad = \quad \{\ P \in \Pi \mid P \text{ is a coherent intersection of some } \mathcal{C} \subseteq \mathcal{A}\ \} \tag{2.3}$$

Lemma B.2.2 in Appendix B shows that $(\mathcal{A})_{coh}$ is indeed a closure of $\mathcal{A}$ with respect to the operation of coherent intersection, hence the name. Coherent intersections give rise to the redundant frames. In fact, they are the only source of redundant frames.

Proposition 2.3.2 *A frame* $P \in \Pi$ *is redundant with respect to* $\mathcal{A} \subseteq \Pi$ *if and only if there is a subcollection* $\mathcal{C} \subseteq \mathcal{A}$ *such that* P *is a non-trivial coherent intersection of* $\mathcal{C}$*. Let axioms 1–7 hold for* $c(\cdot)$*. If* $(U', \mathcal{A}')$ *represents* $c(\cdot)$*, then the collection of frames* $\mathcal{B}$ *in the maximum representation* $(U, \mathcal{B})$ *of* $c(\cdot)$ *consists of all coherent intersections of subcollections* $\mathcal{C} \subseteq \mathcal{A}'$*.*

Corollary 2.3.3 *Let choice correspondence* $c(\cdot)$ *satisfying Axioms 1–7 be represented by* $(U, \mathcal{A})$*. If all frames* $P \in \mathcal{A}$ *are singletons, then the representation is unique up to a positive affine transformation of* U*.*

Proposition 2.3.2 characterizes all possible redundant frames. Moreover, it says that we can start with any representation $(U', \mathcal{A}')$ of a choice correspondence $c(\cdot)$ and

find out the maximum collection of frames $\mathcal{B} = (\mathcal{A}')_{coh}$ by coherently intersecting frames in $\mathcal{A}'$. Notice that we can start with $\mathcal{A}' = \mathcal{A}$ being the minimum collection of frames, that is, $\mathcal{B} = (\mathcal{A})_{coh}$. In the special case when DM is a subjective expected utility maximizer with frame-dependent belief, $\mathcal{A}$ is a unique collection of frames (singleton beliefs) in the representation of $c(\cdot)$.

2.3.1 Identification of multi-multiple prior models in the literature

In Frick et al. (2019a), the set of *relevant priors*—a convex hull of the priors that are used by DM to evaluate acts—is unique for the representation. This is true for our model as well: the set of relevant priors is a convex hull of the union of all frames in the minimum family[7]. Our model, therefore, suggests a stronger identification result, since our representation also exhibits a unique minimum family of sets of beliefs.

In Kopylov (2019), the analyst can identify the projections of sets of beliefs that account for the ambiguity relevant for given decision problems. Our model allows for the identification of each set of beliefs completely, but some redundant sets of beliefs may occur, although we fully characterize these redundancies.

Both Frick et al. (2019a) and Kopylov (2019) operate with a preference relation as a primitive[8]. Our model makes use of data on choices from larger menus as well, which is advantageous for the identification purposes—we clarify this point in Section 5.3.

Lu (2014) shows that the distribution of sets of beliefs is uniquely identified if the analyst knows frequencies of choices from doubleton menus consisting of an ex-

[7]This follows from Proposition 2.3.2. Moreover, the set of priors that are used by DM to evaluate acts—and not just the convex hull of this set—is also unique in our model, since redundant sets of beliefs consist of intersections of sets of beliefs from the minimum family.

[8]Kopylov (2019) considers also an extension of his model for choice functions.

ante lottery over Anscombe-Aumann acts and a constant act. His model, therefore, provides a superior identification result by extending the primitive.

Lu (2014)'s result comes at a cost of assuming linearity over the ex-ante mixtures of Anscombe-Aumnann acts, and implicitly assuming that the process of assignment of a menu is independent of the process governing the choice of the set of beliefs that DM uses. We, in contrast, can allow for a correlation between a menu and a set of beliefs as long as it is not perfect—we care only about the supports of the distributions.

Finally, Heller (2012) identifies a convex family of singleton beliefs uniquely from the choice correspondence, while our Corollary 2.3.3 shows that a non-convex family of singleton beliefs is also identified uniquely. Our model, even constrained to singleton beliefs, allows for a richer behavior. For instance, suppose the DM in our model has two frames: $P_1 = \{\mu_1\}$, that favors f to $0.5f + 0.5g$ to g, and $P_2 = \{\mu_2\}$, that favors g to $0.5f + 0.5g$ to f. Such DM chooses f and g, but not a compromise alternative $0.5f + 0.5g$ from menu $\{f, 0.5f + 0.5g, g\}$. In contrast, the DM in Heller (2012) should necessary choose a compromise alternative $0.5f + 0.5g$ if she chooses f and g, since this mixture is the best according to one of the intermediate beliefs $\lambda\mu_1 + (1 - \lambda)\mu_2$ for some $\lambda \in (0, 1)$.

2.4 Frame Sensitivity

In this section, we study the behavioral difference between agents who are *sensitive* to the framing of a choice problem—in the sense that they may change their decision depending on the frame that they have in mind—and agents who are not sensitive to the framing.

A Gilboa and Schmeidler (1989) maxmin model is a special case of a framed ambiguity model when the set of frames is singleton: $\mathcal{A} = \{P\}$. Intuitively, DM is

able to hold the same perception of uncertainty independently of varying irrelevant details of the choice problem—either because framing does not matter for her, or because she is able to recognize the whole set of frames and connect them into a single frame. Formally:

DEFINITION 6. The DM is *not sensitive to framing* if her choice correspondence $c(\cdot)$ admits a framed ambiguity representation with a single frame.

If DM is not sensitive to framing, the analyst can use choices from binary menus (preference relation) to identify the corresponding set of beliefs uniquely, while in this paper we use data on choices from larger menus to identify a unique minimum family of frames. Can we achieve the same identification result using just a preference relation?

Formally, consider the following property:

Normality: $\quad \forall A \in \mathcal{K} \quad c(A) = \{f \in A \mid f \in c(\{f, g\})\ \forall g \in A\}$

Normality says that once the analyst knows choices from binary menus, the analyst can reconstruct the full choice correspondence by a straightforward way; consideration of larger menus in this case is not advantageous. We, therefore, are interested in cases when *Normality* holds for our DM.

Next, we want to shed some light on our key axioms—*Framed Uncertainty*, *No-C-Hedging* and *Indirect Ambiguity Aversion*—by considering their alternative versions that bear the same meaning for DM with one frame, but, possibly, different meaning for DM with multiple frames.

If DM has only one frame, her choice correspondence should satisfy *WARP*, which is strengthening of the *Framed Uncertainty* axiom.

The ambiguity aversion axiom for the maxmin model translates from the language of preference relation to the language of choice correspondence as follows: $f \in c(\{f,g\})$ implies $\lambda f + (1-\lambda)g \in c(\{\lambda f + (1-\lambda)g, g\})$. It is easy to see that this property holds for agent who operates with multiple frames as well. However, this property does not clarify what happens when DM chooses from non-doubleton menus. For instance, $f \in c(\{f,g\} \cup A)$ does not imply that $\lambda f + (1-\lambda)g \in c(\{\lambda f + (1-\lambda)g\} \cup \{A\})$, since DM might dislike g. We, therefore, require that g is also good enough: $g \in c(A \cup \{g\})$.

Direct Ambiguity Aversion: $\forall A \in \mathcal{K} \;\; \forall f,g \in H \;\; \forall \lambda \in (0,1)$

$$f \in c(A \cup \{f\}) \text{ and } g \in c(A \cup \{g\}) \implies \lambda f + (1-\lambda)g \in c(A \cup \{\lambda f + (1-\lambda)g\})$$

A decision maker who satisfies *Direct Ambiguity Aversion* argues in the spirit of Axiom 5 (*Indirect Ambiguity Aversion*), but is able to connect her decisions under, possibly, different framing: if there is a reason to choose f versus acts in A and there is a reason to choose g versus acts in A, then there should be a reason to choose their mixture versus acts in A, since mixture hedges uncertainty in f and g. We are interested in cases when *Direct Ambiguity Aversion* holds for our agent.

Suppose agent has one frame, and she prefers act h to both act f and constant act p, i.e. $h \in c(\{h,f\})$ and $h \in c(\{p,h\})$. Then this agent respects a pairwise version of *No-C-Hedging*: it should be that $h \in c(\{\lambda f + (1-\lambda)p, h\})$. Can this property hold for DM with multiple frames as well? Formally,

Pairwise No C-Hedging: $\forall f,h \in H, \; \forall p \in H_0$

$$h \in c(\{h,f\}) \text{ and } h \in c(\{h,p\}) \implies h \in c(\{h, \lambda f + (1-\lambda)p\})$$

Intuitively, DM who satisfies *Pairwise No C-Hedging* argues in the spirit of Axiom 3 (*No C-Hedging*), but she can connect her decisions under different framing: if there is a reason to choose h versus f, and there is a reason to choose h versus p, then there should be a reason to choose h versus their mixture, since constant act p cannot effectively hedge against uncertainty in f.

The next proposition relates frame sensitivity to the violation of these properties:

Proposition 2.4.1 *Let choice correspondence $c(\cdot)$ satisfy Axioms 1–7. Then the following properties are equivalent for $c(\cdot)$: β, WARP, Normality, Direct Ambiguity Aversion, Pairwise No-C-Hedging, DM with choice correspondence $c(\cdot)$ being not sensitive to framing.*

Proposition 2.4.1 says that $c(\cdot)$ is inherently *not normal*: the only case when an analyst can reconstruct the whole choice correspondence using only information coming from the binary choices is when DM is not frame sensitive. Notice that an analyst can also verify frame sensitivity using only binary choices, since *Pairwise No-C-Hedging* operates with doubleton menus.

Intuitively, *Direct Ambiguity Aversion* and *Pairwise No-C-Hedging* are violated by a frame sensitive DM not because she doesn't respect the associated hedging reasoning, but because she fails to contemplate her choices under different frames, what is needed to build the premises of the corresponding arguments.

2.5 Connection of Frames and Consistency of Choices

Imagine two agents who agree on the ranking of prizes, say both prefer $x = \$100$ to $y = \$0$, but choose differently from menus consisting of acts that yield state-dependent lotteries over x and y. The ambiguity literature distinguishes two main reasons for

that: different perception of the relevant uncertainty—that is, ambiguity—and different sentiment regarding the perceived uncertainty—that is, ambiguity attitude[9].

Our model allows to distinguish a third reason: difference in the sensitivity to framing. All else equal, two agents may have different ability to connect the same information about the environment into a decision procedure. In our comparative statics analysis, we focus on the situations when one of the agents is more capable of connecting various decision frames than another. Naturally, the choice of such decision maker exhibits more consistency and is less frame-dependent.

One way to connect the two points of view is to extend the aversion of ambiguity to frames as well, and evaluate everything from the worst point of view—when frames are singleton beliefs, this aggregation idea corresponds to *Default to Certainty* in Gilboa et al. (2010), and for non-singleton beliefs this is considered in Crès et al. (2011) and Hill (2011).

DEFINITION 7. Given a non-empty closed collection of frames $\mathcal{C}$, its *convex union* is a set of beliefs P given by

$$P \quad = \quad conv\left(\bigcup_{P' \in \mathcal{C}} P'\right) \tag{2.4}$$

Intuitively, an agent who contemplates a subset of frames $\mathcal{C}$ at a moment, can think that any prior in any set of beliefs is possible and be cautious toward this opportunity. An aggregate frame P is then a convex hull[10] of the union of all sets of beliefs in $\mathcal{C}$.

While cautious method of aggregation suggests to consider all possible priors, an optimistic decision maker can do the opposite: consider only priors that belong

[9]See, for instance, Ghirardato et al. (2004).

[10]Non-convex sets of beliefs—as oppose to non-convex sets of frames—are not identifiable in our model; also, a decision maker who contemplates beliefs μ_1 and μ_2 within a given frame could contemplate as well an intermediate belief $\lambda\mu_1 + (1-\lambda)\mu_2$.

to every frame in the collection $\mathcal{C}$; that is, the intersection of frames. The choice behavior of such agent corresponds to contemplating a decision to choose an act versus a constant alternative within each frame as an argument in favor of this act—we call this type of reasoning, novel in the literature as far as we aware, an *optimistic learning* and analyze it in the end of the section. An obvious pitfall of this approach is that a collection $\mathcal{C}$ may have empty intersection. Some intersections of frames are coherent, and they play special role in our analysis, as one might guess from our identification results.

Finally, the agent can try to find a compromise between frames: interpret each frame as a point of view and assign a weight to it. For instance, if frames are singletons $P_1 = \{\mu_1\}$ and $P_2 = \{\mu_2\}$, then a singleton set $P = \lambda\{\mu_1\} + (1-\lambda)\{\mu_2\}$ corresponds to weighting[11] the first point of view—or expert's opinion—by λ, and second point of view by $1-\lambda$.

DEFINITION 8. Given a non-empty finite collection of frames $\mathcal{C} = \{P_1, ..., P_N\}$, its *convex combination* with respect to weights $\lambda \in \triangle(\{1, ..., N\})$ is a set of beliefs P given by

$$P = \sum_{i=1}^{N} \lambda_i P_i \equiv \left\{ \mu \in \triangle S \;\middle|\; \exists \mu_i \in P_i \text{ for } i = 1, ..., N : \; \mu = \sum_{i=1}^{N} \lambda_i \mu_i \right\} \quad (2.5)$$

A convex combination[12] P of a collection of frames $\mathcal{C}$ has a nice property that the value of an act calculated with respect P is a weighted average of its values with respect to each of the frames $\mathcal{C}$, and the weights are the same as used in the convex

[11] This way of aggregation different opinions is also considered in Crès et al. (2011) and Hill (2011).

[12] We consider finite collections for two reasons. First, a version of our results involving infinite collections lacks proofs at the moment. Second, infinite convex collection is a less intuitive object, since, for instance, we should require a selection of a prior in each frame to be measurable with respect to σ-algebra on $\mathcal{C}$.

combination of frames. Thus, the idea of a compromise between different points of view works simultaneously for values of acts, and for beliefs as well.

In the remaining part of this section, we study the choices of—potentially frame sensitive—agent 1 who connects some of the frames of agent 2. We say that a collection of frames $\mathcal{C}$ is connected in a non-trivial way in a frame P by one of the discussed three ways, if $P \notin \mathcal{C}$.

2.5.1 Decisiveness

We start by considering a natural decisiveness relation.

DEFINITION 9. DM 1 with choice correspondence $c_1(\cdot)$ is *more decisive* than DM 2 with choice correspondence $c_2(\cdot)$ if $c_1 \subseteq c_2$.

Thus, we say that DM 1 is more decisive than DM 2 if she chooses among a subset of alternatives that could be chosen by DM 2. Proposition 2.3.2 provides us with necessary tools to characterize this comparative decisiveness relation:

Proposition 2.5.1 *Let $c_1(\cdot)$ and $c_2(\cdot)$ are represented by $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$. Then DM 1 is more decisive than DM 2 if and only if U_1 is a positive affine transformation of U_2 and $\mathcal{A}_1 \subseteq (\mathcal{A}_2)_{coh}$.*

Recall that $(\mathcal{A}_2)_{coh}$ is a maximum collection of frames that represents $c_2(\cdot)$. Thus, Proposition 2.5.1 tells us that a more decisive agent is one who starts with the maximum collection of frames that the other agent may have and drops some of those. Therefore, the only non-trivial way how a more decisive agent may connect the frames of a less decision agent is by taking a coherent intersection of them.

2.5.2 Consistency

We proceed with a less demanding notion, which compares instances in which agents make the same choice consistently.

DEFINITION 10. DM 1 with choice correspondence $c_1(\cdot)$ is *more consistent* than DM 2 with choice correspondence $c_2(\cdot)$ if for all $A \in \mathcal{K}$ $|c_2(A)| = 1 \implies |c_1(A)| = 1$.

It is easy to see that if DM 1 is more decisive than DM 2, then DM 1 is more consistent then DM 2, since $c_2(A) = \{f\} \implies c_1(A) = \{f\}$. Therefore, we know that if DM 1 considers coherent intersections of frames of DM 2, then she is more consistent. Note also that the binary relation on the set of framed ambiguity agents of given comparative consistency is transitive—as well as the binary relation given by the comparative decisiveness.

DEFINITION 11. Let $\mathcal{A} \subseteq \Pi$ be a non-empty collection of frames. Then its closure with respect to the operations of convex union, convex combination and coherent intersection is a minimum family of frames $\mathcal{A}^*$ that satisfies (i) $\mathcal{A} \subseteq \mathcal{A}^*$, and (ii) if P is either a convex union, or a convex combination, or a coherent intersection of a collection of frames $\mathcal{C} \subseteq \mathcal{A}^*$, then $P \in \mathcal{A}^*$.

Lemma B.1.26 in Appendix A shows that the closure given by Definition 11 exists and is unique. As we have seen from Alice's example, connecting frames by taking convex unions, convex combinations or coherent intersections of frames (weakly) increases consistency of choices of DM. Our main result in this section is that, in fact, there operations of frame connection characterise the comparative consistency:

Theorem 2.5.2 *Let $c_1(\cdot)$ and $c_2(\cdot)$ be represented by $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$ respectively, and $|\mathcal{A}_2| < \infty$. Then the following statements are equivalent:*

(i) DM 1 is more consistent than DM 2.

(ii) U_1 *is a positive affine transformation of* U_2*, and* $\mathcal{A}_1$ *belongs to the closure of* $\mathcal{A}_2$ *with respect to operations of convex union, convex combination and coherent intersection.*

(iii) U_1 *is a positive affine transformation of* U_2*, and any* $P \in \mathcal{A}_1$ *is a coherent intersection of convex unions of convex combinations of frames in* $\mathcal{A}_2$*.*

Note that Theorem 2.5.2 assumes additionally that DM 2 has a finite number of frames[13]. If $\mathcal{A}_2$ is infinite, one has to generalize the definition of the convex combination of frames to work with closed collections of frames, making it less intuitive[14]. We conjecture that after this generalization Theorem 2.5.2 will remain true[15].

2.5.3 Identification of frames from binary choices

Despite the fact that—by Proposition 2.4.1—the data on binary choices is not sufficient for reconstruction of the whole choice correspondence, our next result shows that this data is sufficient to tell if one DM is more consistent than another according to Definition 11.

Proposition 2.5.3 *Let* $c_1(\cdot)$ *and* $c_2(\cdot)$ *satisfy axioms 1–7. Then DM 1 is more consistent than DM 2 if and only if* $\forall g, h \in H \; c_1(\{f, g\}) \subseteq c_2(\{f, g\})$*.*

While comparative consistency is defined via choices from arbitrary menus, there is a reason why this notion can be reduced to comparison of binary choices.

[13] We acknowledge that we don't introduce the corresponding behavioral property.

[14] Weights λ generalize to the probability distribution on the Borel sigma-algebra of the given collection of frames, and selection of priors $\mu_i \in P_i$ generalizes to the measurable selection from the given collection.

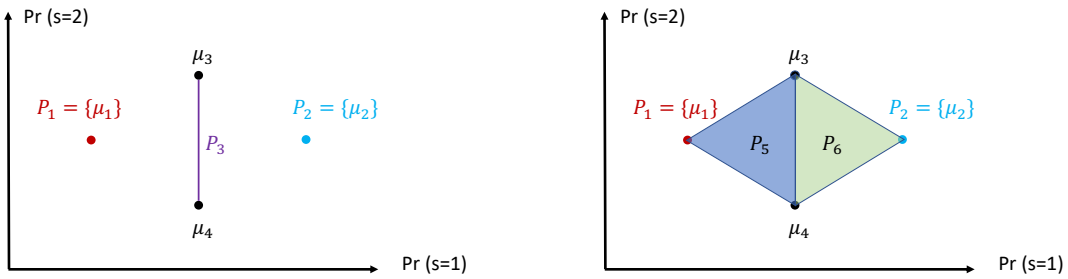
[15] To prove that it would be enough to show that eq. (3) in Frick et al. (2019a) works not only for functions $R^S \to R$, but also for functions $C(Y) \to R$ from the space of continuous functions on a compact set Y to R

This binary formulation of comparative consistency allows us to use Theorem 2.5.2 to get identification result for a variant of our model when the primitive is preference relation—that is, data on binary choices—as oppose to the full choice correspondence. The next result follows from the fact that $c_1(\cdot) = c_2(\cdot)$ for binary menus if and only if DM 1 is more consistent than DM 2 and the other way around.

Corollary 2.5.4 *Framed ambiguity models $(U, \mathcal{A})$ and $(U', \mathcal{A}')$ with $|\mathcal{A}|, |\mathcal{A}'| < \infty$ represent the same choice correspondence on the set of binary menus if and only if U and U' are positive affine transformations of each other, and closures of $\mathcal{A}$ and $\mathcal{A}'$ with respect to operations of convex union, convex combination and coherent intersection coincide.*

Corollary 2.5.4 tells us that observing choices only from binary menus leaves a big room for different representations. Moreover, in contrast to Theorem 2.2.3 for choice correspondence, an analyst may not be able to identify a unique minimum collection of frames that represents[16] a given preference relation[17], which is clear from the following example.

Figure 2.1: Non-existence of the minimum representation of choice correspondence constrained to the set of binary menus.



Let $|S| = 3$. Consider priors $\mu_1 = (0.1, 0.2, 0.7)$, $\mu_2 = (0.3, 0.2, 0.5)$, $\mu_3 = (0.2, 0.3, 0.5)$, $\mu_4 = (0.2, 0.1, 0.7)$ and sets of priors $P_1 = \{\mu_1\}$, $P_2 = \{\mu_2\}$,

[16]Nothing changes for the identification of the vNM expected utility function, since the restriction $c_0(\cdot)$ of choice correspondence on the set $\mathcal{K}_0$ of menus consisting of constant acts satisfies *WARP*.

[17]Where we define preference relation via choice from binary menus by $f \succeq g$ if $f \in c(\{f, g\})$.

$P_3 = conv(\mu_3, \mu_4)$, $P_5 = conv(\mu_1, \mu_3, \mu_4)$, $P_6 = conv(\mu_2, \mu_3, \mu_4)$. Let the first family of frames be $\mathcal{A}_1 = \{P_1, P_2, P_3\}$, and the second family of frames be $\mathcal{A}_2 = \{P_1, P_2, P_5, P_6\}$. Note that $P_5 = conv(P_1 \cup P_3)$, and $P_6 = conv(P_2 \cup P_3)$, and $P_3 = P_5 \cap P_6$, where the intersection of P_5 and P_6 is coherent. Hence, the closures of $\mathcal{A}_1$ and $\mathcal{A}_2$ with respect to the operations of convex union, convex combination and coherent intersection coincide.

It is straightforward to see that the corresponding choice over binary menus does not have a minimum representation, since in the model with the set of frames $\mathcal{A}_3 = \mathcal{A}_1 \cap \mathcal{A}_2 = \{P_1, P_2\}$, the DM's beliefs regarding $s = 2$ is fixed: $\mu(2) = \mu_1(2) = \mu_2(2) = 0.2$, while the original model allows for $\mu_3(2) = 0.3$ and $\mu_4(2) = 0.1$.

In the example above, if an analyst observes choices from non-binary menus as well, then $(U, \mathcal{A}_1)$ and $(U, \mathcal{A}_2)$ represent different choice correspondences $c_1(\cdot)$ and $c_2(\cdot)$. By Proposition 2.3.2, $\mathcal{A}_1$ is the unique family of frames representing $c_1(\cdot)$. Choice correspondence $c_2(\cdot)$ admits representations with two different families of frames: $\mathcal{A}_2$ is the minimum family, and $(\mathcal{A}_2)_{coh} = \mathcal{A}_2 \cup P_3$ is the maximum family.

2.5.4 Application to the Social Choice Theory

Our model has a population interpretation: we can think about a group of people that agree on the ranking of prizes (share the same utility), are averse to the ambiguity, but disagree on beliefs. Suppose they want to aggregate their judgements such that the resulting rule still respects ambiguity aversion, and a natural *Unanimity* criterion:

DEFINITION 12. Let $\{\succeq_i\}_{i=1,\dots,N}$ and $\succeq$ be preference relations on H. We say that $\succeq$ satisfies *Unanimity* with respect to $\{\succeq_i\}_{i=1,\dots,N}$ if for all $f, g \in H$

$[f \succeq_i g \ \forall i = 1, ..., N] \implies f \succeq g.$

Then Theorem 2.5.2 helps us to characterize the set of such rules:

Corollary 2.5.5 *Let $\{\succeq_i\}_{i=1,...,N}$ and $\succeq$ be preference relations on H that satisfy axioms of Gilboa and Schmeidler (1989) and have maxmin representations with the same vNM expected utility function, but different sets of beliefs $\{P_i\}_{i=1,...,N}$ and P. Then $\succeq$ satisfies Unanimity with respect to $\{\succeq_i\}_{i=1,...,N}$ if and only if P is a coherent closure of convex unions of convex combinations of $\{P_1, ..., P_N\}$.*

Notice also that taking arbitrary number and order of operations of coherent intersection, convex union and convex combination of $\{P_1, ..., P_N\}$ always produces a rule that satisfies Unanimity.

Aggregation of ambiguity averse opinions in the literature

In Crès et al. (2011), the authors study the aggregation of the opinions of a group of ambiguity-averse experts. They impose a stronger *Expert Uncertainty Aversion* axiom which is a bit less straightforward then the *Unanimity* axiom. In terms of our result, their aggregation procedure is equivalent to taking convex unions and convex combinations of sets of beliefs, but not coherent intersections. We show that usage of all three of these operations characterises the aggregation procedure that satisfies the unanimity criterion. To get this result, we use some of the instruments from proofs in Crès et al. (2011) and Frick et al. (2019a).

In Hill (2011), the author characterizes the same aggregation rule as Crès et al. (2011) in terms of the unanimity criterion plus the requirement that an aggregation rule should remain the same across all profiles of preferences. In terms of our model, his additional requirement rules out coherent intersections primarily because not all intersections of sets of beliefs are coherent, hence if a rule includes a coherent intersection, it cannot be applied for all profiles of preferences.

2.5.5 Optimistic Learning

Let us return to the idea that DM may consider frames as pieces of information about priors that cannot be correct. Can we characterise this type of learning via observed choices?

Suppose the agent considers act $h = 0.5f + 0.5g$; frame P_1 is favorable for act f, while frame P_2 is favorable for act g. DM may then optimistically evaluate act h as one half of the value of f in frame P_1 plus one half of the value of g on frame P_2. Moreover, if the agent is sophisticated enough, she may also consider consider, for instance, a decomposition[18] $\lambda h + (1-\lambda)q = \sigma f' + (1-\sigma)g'$ for some constant act q, convex wights λ, σ and acts f', g', and figure out the value of h by considering the most favorable frames for acts f', g' and then using the *C-Independence* principle. The next definition formalizes this idea.

DEFINITION 13. DM 1 optimistically learns from DM 2 if there exist $q \in H_0$ such that for all $f \in H$ there exist $\lambda \in (0,1]$, $f_1, ..., f_k \in H$ and $\sigma \in \triangle(\{1, ..., k\})$ such that:

$(i) \quad \lambda f + (1-\lambda)q = \sum_{i=1}^{k} \sigma_i f_i$

$(ii) \;\; \lambda f + (1-\lambda)q \in c_1(\{\lambda f + (1-\lambda)q, p\}) \iff f_i \in c_2(\{f_i, p\}) \; \forall i = 1, ..., N \quad \forall p \in H_0$

Intuitively, DM 2 interprets $f_i \in c_2(\{f_i, p\})$ as an argument in favor of f_i versus constant act p, and then optimistically claims that these arguments should work for the mixture of acts f_i as well. To present our last result, we need one of the two following technical assumptions.

[18] The reason why we need this extra step is that the set of acts H is compact. Thus, for instance, if act f is an extreme point of H in the space of utilities, it does not have a non-trivial decomposition.

CONDITION 1. A family of frames $\mathcal{A}$ is finite, its intersection is non-empty, and each $P \in \mathcal{A}$ is polyhedral.

Condition 1 says that each $P \in \mathcal{A}$ can be expressed via a finite number of linear inequalities[19]. An equivalent formulation is that each $P \in \mathcal{A}$ is a polytope—that is, there are finitely many priors $\mu \in P$ that DM uses to evaluate acts with respect to P. If we need to consider more general sets of priors, we can use the following condition, where we denote by $ri(P)$ the relative interior of P:

CONDITION 2. A family of frames $\mathcal{A}$ is finite, and $\bigcap_{P \in \mathcal{A}} ri(P)$ is non-empty.

Intuitively, Condition 2 says that the intersection of frames does not cause the ambiguity existing in each frame in some dimension to vanish. The following proposition make use of results from Convex Analysis[20]:

Proposition 2.5.6 *Let choice correspondences $c_1(\cdot)$ and $c_2(\cdot)$ be represented $(U_1, \{P_1\})$ and $(U_2, \mathcal{A}_2)$, and either Condition 1 or Condition 2 holds for $\mathcal{A}_2$. Then DM 1 optimistically learns from DM 2 if and only if U_1 is a positive affine transformation of U_2, and $P_1 = \bigcap_{P \in \mathcal{A}_2} P$.*

DM 1 in Proposition 2.5.6 has only one frame in mind, thus $c_1(\cdot)$ satisfies WARP and is represented by the maxmin model with vNM expected utility $U_1(\cdot)$ and set of beliefs P_1. Proposition 2.5.6 then says that if DM 1 becomes rational by optimistically connecting arguments $f_i \in c_2(\{f_i, p\})$ ("f_i is weakly better than p according to some frame"), then it is equivalent to learning about priors that cannot be correct—that is, intersecting sets of beliefs.

[19] And equations as well, since $a = b$ is equivalent to $a \geq b$ and $a \leq b$.
[20] Corollary 16.4.1 and Theorem 20.1 in Rockafellar (1970).

Chapter 3

Random Choice with Framing Effects: a Bayesian Model

3.1 Introduction

[1]This paper interprets decision frames as emerging from different unobservable information structures that signal about the values of the alternatives in the decision problem faced by the agent. The resulting distribution over posterior beliefs induces a stochastic choice that admits a random utility representation within each frame. A question that we address is: is there a connection between choice frequencies *across* various decision frames?

We motivate our analysis by two interpretation of this informational model of framing effects. The first interpretation is that the relevant state space encodes uncertainty regarding the intrinsic preference of the decision maker. Imagine a person who does not know precisely some aspect of her true preferences, such as the extent of her fear of height. She contemplates between spending a weekend at the seashore or going to a mountain trip. Her decision might then depend on a seemingly irrelevant detail: whether the day before, the featured Wikipedia article has been "Parachuting" or not. In the first case, being a Wikipedia fan, the agent reads about parachuting. Although the article does not provide any objective information regarding the benefits or costs of going to a particular seashore or mountain trip, reading about parachuting triggers the agent to think more about her own attitude toward great heights. As a result, the distribution of beliefs regarding the extend of fear or love of heights in the population of agents would be different depending on whether such learning of own preferences has been triggered or not, resulting in different frequencies of vacation choices.

The second interpretation is that the relevant state space encodes some objective unobservable properties of the choice problem that matter for the decision making. For instance, suppose a salesman can be either honest or dishonest; the true value of

[1]This chapter is based on the paper "Random Choice with Framing Effects: a Bayesian Model". An earlier draft of the paper has been awarded a Marimar & Cristina Torres Prize for the best 3rd year research paper.

the items offered by the salesman depends on his type. A buyer acquires information about the type of the salesman by observing fine details of his facial expression during a conversation. One decision frame in this example corresponds to buyer wearing her glasses, while the other—to buyer not wearing her glasses, and hence, having lower ability to differentiate between salesman's types.

The primitive of our model is an *extended random choice* that consists of choice frequencies from different menus of items under various decision frames. Each frame is characterized by a collection of observable features that we call *frame details*. Our model of Bayesian Agents with Uncertain Preferences (BAUP) introduces a state space with a commonly known prior and signal structures associated with each frame detail; the agents in the population have heterogeneous state-dependent utility functions. We say that a BAUP model represents an extended random choice if the probability that the agent from the population receives a signal that suggest to choose a certain item from the menu is equal to the corresponding observed frequency, and this holds for all decision frames.

Our results show that whether there is a connection between choice frequencies across different frames depends on the cardinality of the state space. We first analyse the case when the relevant state space could be arbitrary. Our first main result, Theorem 3.2.3, says that an extended random choice could be approximated by a BAUP model—meaning that there exists a model that induces the choice probabilities arbitrary close to the original extended random choice—if and only if the random choice within each frame admits a random utility representation. Thus, a BAUP model places almost no restrictions on choice frequencies observed under different decision frames. One corollary of this result is that the analyst cannot infer the relative informativeness of the decision frames from the observed extended random choice. Another application is to the dynamic random choice, where Theorem 3.2.3

replicates the "almost everything goes" result that could be inferred from the analysis made in Frick et al. (2019b).

We next analyse the case when the relevant state space is fixed. As an example, consider a stylized experiment where a population of agents chooses from menus of alternatives that provide different consequences depending on the random binary state $s \in \{1, 2\}$. There are three alternatives: x, y, and z. Item $x = (x_1, x_2)$ provides consequence x_1 in the first state of the world and x_2 in the second state, and similarly do items $y = (y_1, y_2)$ and $z = (z_1, z_2)$. There are 3 treatments in the experiment that differ in terms of information about the state revealed to the agents. An analyst observes choice frequencies of a population of agents from different subsets of items x, y, z. For each agent, first, Nature draws a state of the world $s \in \{1, 2\}$, and then reveals a random signal informative about the state according to the treatment's signal structure, after which the agent makes a choice.

Let $\rho_i^A(a)$ be the frequency with which item a is chosen from menu A under treatment i, and suppose $\rho_1^{\{x,y\}}(x) = 0.8$, $\rho_1^{\{x,y,z\}}(x) = 0.1$, $\rho_2^{\{y,z\}}(y) = 0.7$, and $\rho_2^{\{x,y,z\}}(y) = 0$. Therefore, under treatment $i = 1$, the resulting random choice admits a random utility representation that places probability $\rho_1^{\{x,y\}}(x) - \rho_1^{\{x,y,z\}}(x) = 0.7$ that the agent prefers z to x to y (we write it as $z \succ x \succ y$). Under treatment $i = 2$, a random utility representation should place probability $\rho_2^{\{y,z\}}(y) - \rho_2^{\{x,y,z\}}(y) = 0.7$ that the agent's preference is $x \succ y \succ z$. Thus, at least some agents in the population exhibit preference reversal—placing item z to the top or to the bottom of the preference order depending on the treatment.

The stochastic choice behavior in this example can be consistent with expected utility maximization and Bayesian updating. For instance, suppose that utility values of items in states 1 and 2 are $x = (-1, 9)$, $y = (0, 0)$, and $z = (5, -15)$. In this case, depending on the belief $p = Pr(s = 2)$, the preference is: $z \succ y \succ x$ if $p < 0.1$, $z \succ x \succ y$ if $p \in (0.1, 0.2)$, $x \succ z \succ y$ if $p \in (0.2, 0.25)$, and $x \succ y \succ z$ if $p > 0.25$. Let

the prior be $\pi = 0.233$, and treatments 1 and 2 induce distributions over posteriors $P_1 = 0.2\delta_0 + 0.7\delta_{0.19} + 0.1\delta_1$ and $P_2 = 0.1\delta_0 + 0.2\delta_{0.22} + 0.7\delta_{0.27}$. The resulting random utility model under treatment 1 places probability 0.2 on $z \succ y \succ x$, 0.7 on $z \succ x \succ y$, and 0.1 on $x \succ y \succ z$. Similarly, the resulting random utility model under treatment 2 places probability 0.1 on $z \succ y \succ x$, 0.2 on $x \succ z \succ y$, and 0.7 on $x \succ y \succ z$, consistent with the observed choice frequencies.

In fact, if we are concerned only with choice frequencies under treatments 1 and 2, almost any data could be explained by such informational model. However, our results show that, if the observed choice frequencies under the first two treatments are as in our example, and the agents in the population are expected utility maximizers, it must be the case that under treatment 3, $\rho_3^{\{x,z\}}(z) - \rho_3^{\{x,y,z\}}(z) \leq 0.6$. This restriction holds for a population of heterogeneous agents as well, and it does not depend even on the assumption of Bayesian updating. For this result, it is sufficient to know that the agents are expected utility maximizers, and the relevant state space is binary.

We provide two type of explicit constraints specific for the binary state space in Proposition 3.3.1. For instance, the constraint mentioned in the example above is given by

$$\rho_2^{\{y,z\}}(y) - \rho_2^{\{x,y,z\}}(y) + \rho_3^{\{x,z\}}(z) - \rho_3^{\{x,y,z\}}(z) + \rho_1^{\{x,y\}}(x) - \rho_1^{\{x,y,z\}}(x) \ \leq \ 2$$

Our next main results, Theorems 3.3.3 and 3.3.5, generalize Proposition 3.3.1 and formulate a way to calculate a set of constraints necessary for a BAUP representation with a given state space S. Theorem 3.3.3 does it indirectly, requiring the existence of a family of random utility representations coherent in a certain way across frames. Theorem 3.3.5, in contrast, formulates a set of explicit constraints. It requires to calculate, from the frequencies of choices within each frame, specific Block and Marschak (1960) polynomials, and restricts the values of certain sums of these polynomials. In

Corollary 3.3.4, we show that for any fixed state space, if the set of alternatives and the set of frames are rich enough, these additional constraints do have a bite: there is an extended random choice that admits random utility representations within each frame, but fails to be approximated by a BAUP model with the given state space.

The rest of the paper is organized as follows. In the remaining part of the Introduction, we discuss the relevant literature. In Section 2, we develop a BAUP model and provide results for an unrestricted state space. In Section 3, we analyze a BAUP model with a fixed state space. All proofs are given in the Appendix.

3.1.1 Related Literature

This paper contributes to the literature on random utility models, random choice with unobservable information, and to the literature on framing effects.

A random utility model has been considered in Block and Marschak (1960) where the authors have introduced necessary conditions in terms of linear inequalities on the choice frequencies that later has been called the *Block-Marschak polynomials*. In Falmagne (1978), it has been proven that these conditions are also sufficient; this result has been then discussed in Economics literature in Barberá and Pattanaik (1986). In our model, the random choice frequencies are consistent with the random utility model within each frame, and the Block-Marschak polynomials play an important role in our analysis when the state space has fixed cardinality.

Tversky and Kahneman (1981) introduced a notion of framing that serves as a benchmark that enables the agents to identify some outcomes as *gains* and others as *losses*. In our paper, the framing triggers a specific learning process and is associated with some unobservable signal structure. Framing effects has been studied in a great number of papers, for instance in Huber et al. (1982), Tversky and Simonson (1993), and Salant and Rubinstein (2008); our notion of the *extended random choice* could be viewed as a stochastic counterpart of the Salant and Rubinstein (2008)'s "extended

choice". The latter consists of deterministic choices from menus under various decision frames. Menu-dependent stochastic choice is considered in Manzini and Mariotti (2014).

Preference uncertainty has been discussed in Kreps (1979) and Dekel et al. (2001) in the context of choice over menus and preference for flexibility; in Karni and Safra (2016) in the setup of "mental acts". In Ahn and Sarver (2013), the authors connect the model of Dekel et al. (2001) and the random expected utility of Gul and Pesendorfer (2006) to identify probabilities over subjective state space and utilities. In Dillenberger et al. (2014), the authors show how the analyst can recover the unobservable private information from the preference relation on the set of menus of acts. In our model, the analyst has access to a much less rich choice data.

In Frick et al. (2019b), the authors consider a dynamic random choice in presence of unobservable private information, where the agents choose from lotteries of today's consumption and the continuation menu. The most close to our paper is their analysis of the model of "Bayesian Evolving Beliefs" with restriction to "Atemporal consumption problems" with "Consumption inertia and learning".[2] There are several differences between their and our setup: first, in their model, the analyst observes correlations between choices in various periods, while in our model, only marginal distributions within each frame are observable. However, the proof of our Theorem 3.2.3, in fact, allows us to extend the theorem to the setup where the analyst observes the joint distribution of choices under various decision frames as well.[3] Second, in Frick et al. (2019b), the lotteries over consumption (and future menus) are considered, while in our model, we focus on deterministic consumption.[4] Third, our setup allows us to consider a more general assumptions on how the agent's private information is

[2]Sections 4.2 and 6.2 of Frick et al. (2019b) correspondingly.

[3]We consider marginal probabilities in our model because it is a more natural primitive; in general, it is hard to justify the observability of choices of the same agent under different decision frames if the set of these frames within the model does not admit a linear Blackwell order.

[4]Thus, our model extends the random utility model of Falmagne (1978), while Frick et al. (2019b) is extends the random expected utility model of Gul and Pesendorfer (2006).

connected across the decision frames; an application to a dynamic choice is a special case of our model. Finally, we analyse the implications of a known size of the relevant state space, while in Frick et al. (2019b), the authors do not constraint the state space.

In Lu (2016), the author considers a stochastic choice from menus of Anscombe-Aumann acts defined on the objective state space. He assumes that the choice results from an unobservable private information and shows that the analyst can infer the agent's private information from her choices. In particular, the analyst can figure out if one agent is better informed than another.[5] In our model, the relevant state space is subjective, and the utilities of the alternatives depend on the state. An application of our Theorem 3.2.3 shows that in our setup, in contrast to insights from Lu (2016), the analyst almost always cannot make any inference about the informativeness of the signals associated with the particular decision frame from the observed extended random choice.

In Natenzon (2019), the author considers a decision maker who learns her preferences when she encounters a given choice problem. In his setup, signals are in the form of an additive Gaussian noise associated with each of the "real" alternatives that can be potentially chosen and each of the "phantom" alternatives in the menu that cannot be chosen. Equivalently, each "real menu" consisting of the "real alternatives" can be encountered under one of the decision frames that correspond to different sets of "phantom" alternatives (decoys). Our paper complements Natenzon (2019)'s analysis by considering a general non-parametric setup and a more general set of frames at the expense of having less sharp predictions.

Several papers study the role of unobserved information structures in explaining the realized choices/deterministic preferences as oppose to the stochastic choice primitive, considered in our model. In the setup considered in Shmaya and Yariv

[5]In the sense of the Blackwell information order.

(2016), the authors show that the Bayesian updating has a bite only when under all signal realizations, the agent chooses the same option; the counterpart of this result in the probabilistic world would say that the Bayesian updating put restriction only on the degenerate choice frequencies, which mirrors "almost everything goes" result in our Theorem 3.2.3. In Piermont (2017), the author studies a family of preference relations on the set of Anscombe-Aumann acts indexed by a "menu"; the "menu" serves as the set of potential consequences of the considered acts. He assumes that the agent has state-dependent utility function and that each menu is associated with an unobservable signal realization. Thus, in his model, the state of the world may have arbitrary correlation with the menu, which yields a version of "everything goes" result. In our model, in contrast, for any menu, the probability of the state of the world and the signal structures associated with each decision frame remain the same; thus, the decision frames are completely disentangled from the menus. Finally, a contemporary study Deb and Renou (2021) characterises, under various assumptions on the signal structures and state-dependent utilities, the set of possible realizations of the sequences of choices of a population of agents who receive the same unobservable signals regarding the state of the world.

3.2 Model

3.2.1 Primitive

There is a finite non-empty universal set of alternatives, X. A menu is a non-empty subset of X. Denote by $\mathcal{X}$ the set of all menus; thus, $\mathcal{X} = 2^X \backslash \{\varnothing\}$. A finite non-empty set D is a set of *frame details*. We interpret a frame detail as an observable by the analyst feature of the environment that may be present or absent in the particular choice problem—such as a certain article in Wikipedia being featured or not. The set D, hence, is the set of all such features. We call a *decision frame* a

subset of frame details $f \subseteq D$ and use letters $f, g, ...$ for a generic frame. We denote by F the set of frames that the agent may face; thus, $F \subseteq 2^D$, $F \neq \varnothing$.

A primitive of the model is a family of probability distributions $\{\rho_f^A\}_{A \in \mathcal{X}, f \in F}$ that, for each menu $A \in \mathcal{X}$ and each frame $f \in F$, gives the probability $\rho_f^A(a)$ that the agent chooses alternative $a \in A$ from menu A under the decision frame f. Thus, $\text{supp}(\rho_f^A) \equiv \{x \in X | \rho_f^A(x) > 0\} \subseteq A$. We call ρ an *extended random choice*.

3.2.2 Bayesian Agents with Uncertain Preferences

A model of Bayesian Agents with Uncertain Preferences (BAUP) consists of a tuple (S, π, σ, μ), where S is a finite state space, π is a prior probability distribution on S, $\sigma : D \to \Sigma(S, \mathbb{M})$ is a mapping from the set of frame details D to the set $\Sigma(S, \mathbb{M})$ of Blackwell experiments (signal structures) with state space S and some finite message space $\mathbb{M}$, and μ is a finite-support[6] probability distribution over state-dependent utility functions $u : S \times X \to \mathbb{R}$. For convenience, we call σ a *framing function*.

Thus, there is a population of agents who share a common prior regarding the state and interpret the details of the framing as signals regarding the state. Different types of agents in the population have different state-dependent utility functions[7].

Denote by M_d the random signal, associated with the Blackwell experiment $\sigma(d)$, and by m_d its generic realization. Denote by $\sigma(d)(s, m_d)$ the probability of signal realization m_d conditional on state $s \in S$; thus,

$$\sigma(d)(s, m) \;=\; Pr(M_d = m \mid s)$$

[6]Considering infinitely many population types does not change the analysis—this follows from the fact that the primitive is a finite data set.

[7]An alternative models would allow for a heterogeneity in the prior beliefs and/or in the signal structures as well. The corresponding analysis leads to similar results. We consider a less flexible model that still allows to represent a wide variety of primitives.

If the agent with state-dependent utility function u receives a collection of signals $m = \{M_d = m_d\}_{d \in D'}$ for some $D' \subseteq D$, she forms a posterior belief p using Bayes formula

$$p^m(s) \;=\; \frac{\pi(s) Pr\Big(\{M_d = m_d\}_{d \in D'} \mid s\Big)}{\sum_{t \in S} \pi(t) Pr\Big(\{M_d = m_d\}_{d \in D'} \mid t\Big)} \tag{3.1}$$

Denote by $u^m : X \to \mathbb{R}$ the corresponding expected utility function, that is:

$$u^m(x) \;\equiv\; \mathbb{E}[u(x) \mid m] \;=\; \sum_{s \in S} p^m(s) u_s(x) \tag{3.2}$$

We assume that signals associated with different frame details are independent.

CONDITION 1 (Conditional independence of signals):

$$Pr\Big(\{M_d = m_d\}_{d \in D} \mid s\Big) \;=\; \prod_{d \in D} Pr\Big(M_d = m_d \mid s\Big) \tag{3.3}$$

3.2.3 Representation

Consider a BAUP model (S, π, σ, μ). Given a subset of frame details $D' \subseteq D$, denote by $M^{D'} = \{M_d\}_{d \in D'}$ the collection of random signals associated with Blackwell experiments $\sigma(d)$ for $d \in D'$. We say that a BAUP model (S, π, σ, μ) *represents* an extended random choice ρ if for all $A \in \mathcal{X}$, $f \in F$, and $x \in A$,

$$\rho_f^A(x) \;=\; \sum_u \mu(u) \cdot Pr\Big(\Big\{u^{M(f)}(x) \geq u^{M(f)}(y)\ \forall y \in A\Big\}\Big) \tag{3.4}$$

where the summation is performed over all state-dependent utility functions $u : S \times X \to \mathbb{R}$ such that $\mu(u) > 0$, and $u^{M(f)}(\cdot)$ is a function given in eq. (3.2) where signal

$M(f)$ associated with frame f is random and distributed according to

$$\begin{aligned} Pr\Big(M(f) = \{m_d\}_{d\in f}\Big) &= \sum_{s\in S} \pi(s) \cdot Pr\Big(M(f) = \{m_d\}_{d\in f} \;\Big|\; s\Big) = \\ &= \sum_{s\in S} \pi(s) \cdot \prod_{d\in D} \sigma(d)(s, m_d) \end{aligned} \tag{3.5}$$

Note that the representation implies that the maximizer of the expected utility for each menu under each frame is unique with probability one; in other words, ties in the expected utility occur with probability zero.

We offer two interpretations of a BAUP model. In the first interpretation, each agent in the population does not know perfectly her own preferences regarding alternatives in X. We capture this uncertainty via a state $s \in S$; the agent knows her utility $u_s(x)$ of alternative x depending on s. Absent further information, she uses prior π regarding the state s. A frame detail d reveals some aspect of the agent's preferences in the form of a signal M_d with conditional distribution $Pr(M_d = m|s) = \sigma(d)(s, m)$; a frame consisting of several frame details reveals the corresponding aspects of the agent's preferences. Facing some menu under a particular frame, the agent chooses the best alternative from this menu, calculating expected utility according to her posterior belief about her preferences. The stochasticity of the observed choices comes from three sources. Two of them emerge from the population heterogeneity: a random state-dependent utility type u—known by the agent and distributed according to $\mu(\cdot)$—and a random sub-type $s \in S$—unknown to the agent and distributed according to $\pi(\cdot)$. The third source of stochasticity is a random signal $M(f)$ that induces a random posterior distribution $p^{M(f)}(\cdot)$ of agent's beliefs regarding her sub-type $s \in S$. Implicitly in the representation, we require coherency of beliefs regarding s, because prior π plays role both in the agent's updating rule (eq. 3.1) and in the calculation of the resulting extended random choice via formulas (3.4), (3.5).

In the second interpretation, the unknown state of the world $s \in S$ represents conditions that matter for the optimal choice; each choice problem, thus, belongs to one of $|S|$ categories, but agents do not know precisely what particular category they face. Each agent has a population type given by her state-dependent (category-dependent) utility function u. Frame details reveal information about the relevant category in the form of a random signal, and the agent evaluates an alternative by its expected utility across different categories of problems. Thus, the three sources of stochasticity in the representation are: first, a random category $s \in S$, second, a random agent type u, and third, a random signal $M(f)$ regarding the relevant category of the choice problem.

Following the literature, we say that a random choice that provides choice frequencies for various menus $A \in \mathcal{X}$ has a random utility representation if there is a distribution of linear orders[8] over X such that the probability that item $x \in A$ is the highest-rank alternative in menu A is equal to the probability with which x is chosen from A. It is straightforward to see that within each frame, a BAUP model reduces to a random utility model:

Proposition 3.2.1 *If an extended random choice ρ has a BAUP representation, then for each frame $f \in F$, the random choice ρ_f has a random utility representation.*

In particular, in a special case when the state space is singleton ($|S| = 1$), a BAUP model reduces a single random utility model:

Corollary 3.2.2 *If $|S| = 1$, then an extended random choice ρ has a BAUP representation if and only if the following two conditions hold:*

(1) $\rho_f = \rho_g$ for all $f, g \in F$;

(2) The random choice ρ_f has a random utility representation for all $f \in F$.

[8] A linear order over X is a complete transitive antisymmetric binary relation $\succeq \subseteq X \times X$. We use the symbol $\succ$ to denote the asymmetric part of $\succeq$.

Recall that a classical random utility model is characterized in terms of the system of linear inequalities constructed recursively from the observed random choice[9]. Given an extended random choice ρ_f^A, let us define the following *Block-Marschak* linear polynomials recursively:

$$q_f(x,\varnothing) = \rho_f^X(x) \;\; \forall x \in X, \quad q_f(x,A) = \begin{cases} \rho_f^{X\setminus A}(x) - \sum\limits_{B\subsetneq A} q_f(x,B) & \text{if } x \notin A \\ 0 & \text{if } x \in A \end{cases} \tag{3.6}$$

Note that the summation in the formula above is performed over all subsets B of the set X that are not subsets of the set A. If ρ_f has a random utility representation, the Block-Marschak polynomial $q_f(x,A)$ gives the probability that set A is a strict upper counter set of item x; thus, if a BAUP model represents ρ, it must be that $q_f(x,A) \geq 0$ for all x, A, f. Our first main result says that these conditions are *almost* sufficient for the representation in the sense, that a BAUP model can approximate choice frequencies arbitrary close when these conditions hold.

DEFINITION 1. An extended random choice ρ has a BAUP approximation if for any $\epsilon > 0$ there exists an extended random choice τ such that τ is represented by a BAUP model, and $\mid \rho_f^A(x) - \tau_f^A(x) \mid < \epsilon$ for all $A \in \mathcal{X}$, $x \in X$, $f \in F$.

Theorem 3.2.3 *An extended random choice ρ has a BAUP approximation if and only if $q_f(x,A) \geq 0$ for all $f \in F$, $A \in \mathcal{X}$, $x \in A$, where q are Block-Marschak polynomials defined in eq. (3.6). Moreover, if $q_f(x,A) > 0$ for all $f \in F$, $A \in \mathcal{X}$, $x \notin A$, then ρ has a BAUP representation.*

Corollary 3.2.4 *An extended random choice ρ has a BAUP approximation if and only if for each $f \in F$, the random choice ρ_f has a random utility representation.*

[9]Falmagne (1978), Barberá and Pattanaik (1986)

Theorem 3.2.3 is a negative result: a BAUP, as a general informational model of framing, (almost) does not put restrictions on choice frequencies across different frames. Moreover, it is clear from the proof of Theorem 3.2.3, that it continues to hold in case if the primitive of the model is not a family of marginal distributions, but a joint distribution of choices across all frames instead. The generality of our primitive—in particular, the flexibility of the set of frame details D and the set of observable frames F—allows us to use Theorem 3.2.3 to formulate this negative result in several contexts.

First, suppose F consists of singleton frames $f = \{d_f\}$. Then we may interpret Theorem 3.2.3's message as the statement that hidden belief updating that starts from a common prior *almost* does not bring constraints on the resulted frequencies of random choices—in the sense that any random choice that admits a random utility model within one frame can be approximated by BAUP models.

Second, considering a set of frames $F = \{f_i\}_{i=1}^{N}$ that is nested: $f_i = \{1, 2, ..., i\}$, we can interpret an extended random choice as a dynamic random choice where the agent gradually learns some payoff-relevant information. Thus, a similar "almost everything goes" result holds for a dynamic random choice with unobservable gradual learning—a result that one could infer from Proposition 6 of Frick et al. (2019b)[10] by restricting the menus to include only deterministic options and restricting the set of menus in each period to be determined in period 1 (period 0 in their model).

Third, assume the analyst knows that some frames are more informative than others. Thus, the analyst knows a transitive[11] binary relation $\succeq_{inf} \in F \times F$ on the set frames. It turns out, there is a structure of frame details that is coherent with $\succeq_{inf}$ in the sense that $f \succeq_{inf} g$ implies $g \subseteq f$. For instance, this holds when $D = \{d_f\}_{f \in F}$, and $f = \{d_f\} \cup \{d_g \in D | f \succeq_{inf} g\}$. In this context, Theorem 3.2.3 says that for a

[10]Their primitive is dynamic stochastic choice from menus of lotteries of items and the continuation menu.

[11]Since the Blackwell order is transitive.

generic extended random choice consistent with random utility maximization within each frame, it is (almost) impossible to infer which frame is more informative than another—a conclusion, opposite to the results in the setup considered in Lu (2016). This follows from the fact that the set of constraints in Theorem 3.2.3 is invariant with respect to all permutations of the set of frames $F \to F$.

3.3 Fixed State Space

In this section, we consider the implications of the knowledge of the relevant state space. Most naturally, this corresponds to the interpretation of a BAUP model in the objective sense; that is, there are $|S|$ categories the represent various objective features of the choice situations that the agent may face. The second interpretation is subjective, in which case we are interested in the fixed set of the agent's states of the mind—for instance, whether the agent fears heights or not.

3.3.1 Binary State Space

Let us start our analysis with the binary state space, $S = \{1, 2\}$, and let $\{S, \pi, \sigma, \mu\}$ be a BAUP model. As it turns out to be, once we restrict the state space to be binary, there are more conditions necessary for a BAUP representation (approximation) than those stated in Theorem 3.2.3. In this section, we first provide calculations that illustrate reasons behind the emergence of these new constraints that are specific to the binary state space, and then formalize these constraints in Proposition 3.3.1. In the next section, we discuss how this intuition generalizes to non-binary state spaces.

Consider some state-dependent utility u that occur with positive probability; that is, $\mu(u) > 0$. Consider items $x, y, z \in X$. Since we assume away ties in the BAUP model[12], it is without loss to focus only on the linear orders induced by the expected

[12]According to the representation given by eq. (3.4).

utility function. Let $U^\lambda(a) = (1-\lambda)u_1(a) + \lambda u_2(a)$ be the expected utility of item a when the agent belief is $p(2) = \lambda$. If it is not the case that one of x, y is always preferred to the other, there is some belief λ_{xy} that divides the set of beliefs $[0,1]$ to the range of beliefs for which x is preferred to y and the range of beliefs for which y is preferred to x.

Let λ_{xz} and λ_{yz} be analogous beliefs that render items x and z, and items y and z correspondingly indifferent for the agent. The three points λ_{xy}, λ_{xz}, λ_{yz} divide the space of beliefs $[0,1]$ by at most 4 distinct regions; for each of these regions, the expected utility U^λ induces one linear order over x, y, z; the total number of such linear orders is, thus, no more than 4. Let $O = \{xyz, xzy, yxz, yzx, zxy, zyx\}$ be the set of all linear orders over alternatives x, y, z, where term xyz means a linear order $x \succ y \succ z$, etc. Given a state-dependent utility u, let

$$O(u) \;=\; \Big\{ \succeq \in O \;\Big|\; \exists \lambda \in [0,1] :\; a \succ b \iff U^\lambda(a) > U^\lambda(b)\ \forall a, b \in \{x, y, z\} \Big\}$$

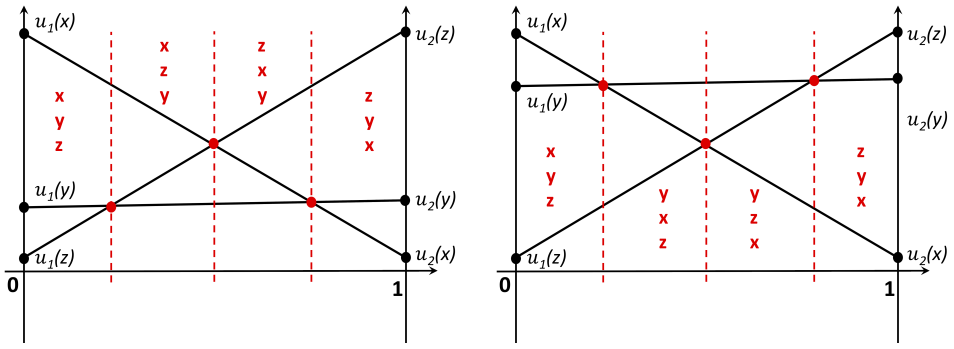
where the symbol $\succ$ denotes the asymmetric part of the linear order $\succeq$, and let

$$\Theta \;=\; \{\; O(u) \mid u \text{ is some state-dependent utility function } \}$$

One can figure out that Θ consists of the sets $\theta_1 = \{xyz, xzy, zxy, zyx\}$, $\theta_2 = \{xyz, yxz, yzx, zyx\}$, several subsets of θ_1, θ_2, and the sets that could be constructed from θ_1 and θ_2 and these subsets by permutations of x, y, z. The expected utilities that give rise to types θ_1 and θ_2 are illustrated in Figure 3.1. For instance, for θ_1, we have $0 < \lambda_{yz} < \lambda_{xz} < \lambda_{xy} < 1$, while for θ_2, we have $0 < \lambda_{xy} < \lambda_{xz} < \lambda_{yz} < 1$.

Thus, each type $\theta \in \Theta$ is a set of linear orders that could be exhibited by an agent with fixed state-dependent utility function under various frames. Let us now think about the extended random choice induced by a population of such agents. Take some set of linear orders $\psi \subseteq O$, and suppose that for all $\theta \in \Theta$, $\psi \not\subseteq \theta$. In other

Figure 3.1: Subsets of linear orders $\theta_1 = \{xyz, xzy, zxy, zyx\}$ (left plot) and $\theta_2 = \{xyz, yxz,\ yzx, zyx\}$ (right plot) induced by the maximization of expected utility with beliefs $\lambda \in [0, 1]$.



words, a subset of linear orders ψ is such that each possible type θ of agent misses at least one of the linear orders in ψ—that is, for all $\theta \in \Theta$, there is $\succeq \in \psi$ such that $\succeq \notin \theta$. One example of such set of linear orders is $\psi = \{xyz, yzx, zxy\}$. Indeed, one can see that $yzx \notin \theta_1$, $zxy \notin \theta_2$, and similarly for the other $\theta \in \Theta$.

Let us analyse further $\psi = \{xyz, yzx, zxy\}$. Consider an agent with state-dependent utility u such that $\mu(u) > 0$, and, for an arbitrary frame f, let p_f^u be the distribution of linear orders induced by the expected utility of this agent, that is, by a BAUP model $(S, \pi, \sigma, \delta_u)$ within this frame. Note that since $O(u)$ misses at least one of the linear orders xyz, yzx, zxy, it must be the case that at least one of the probabilities $p_f^u(xyz)$, $p_f^u(yzx)$, $p_f^u(zxy)$ is zero for all frames $f \in F$. It follows that for arbitrary frames $f, g, h \in F$,

$$p_f^u(xyz) + p_g^u(yzx) + p_h^u(zxy) \ \leq \ 2$$

Summing this inequality over all agents in the population with weights $\mu(u)$, we get

$$p_f(xyz) + p_g(yzx) + p_h(zxy) \ \leq \ 2$$

where p_f for $f \in F$ is a distribution of linear orders induced by a BAUP model (S, π, σ, μ). To formulate this constraint in terms of the observed extended random choice, note that $p_f(xyz) = \rho_f^{\{y,z\}}(y) - \rho_f^{\{x,y,z\}}(y)$, etc. For an arbitrary $f, g, h \in F$ and $x, y, z \in X$, denote by

$$W_{fgh}^{xyz} \;=\; \rho_f^{\{y,z\}}(y) - \rho_f^{\{x,y,z\}}(y) + \rho_g^{\{x,z\}}(z) - \rho_g^{\{x,y,z\}}(z) + \rho_h^{\{x,y\}}(x) - \rho_h^{\{x,y,z\}}(x) \quad (3.7)$$

then $W_{fgh}^{xyz} \leq 2$ is a necessary condition for a BAUP representation.

Continuing this analysis, one can proceed with $\psi = \{xzy, yxz, yzx, zxy\}$ and see that it also has the property that $\psi \not\subseteq \theta$ for all $\theta \in \Theta$. Repeating the same arguments, we consider, for arbitrary $f, g, h, j \in F$ and $x, y, z \in X$,

$$Q_{fghj}^{xyz} \;= \quad (3.8)$$

$$\rho_f^{\{y,z\}}(z) - \rho_f^{\{x,y,z\}}(z) + \rho_g^{\{x,z\}}(x) - \rho_g^{\{x,y,z\}}(x) + \rho_h^{\{x,z\}}(z) - \rho_h^{\{x,y,z\}}(z) + \rho_j^{\{x,y\}}(x) - \rho_j^{\{x,y,z\}}(x)$$

Again, since for any type u in the population, at least one of the probabilities $p_f^u(xzy)$, $p_g^u(yxz)$, $p_h^u(yzx)$, $p_j^u(zxy)$ must be zero, then $Q_{fghj}^{xyz} \leq 3$. We formalize our analysis in the next proposition. Naturally, we say that an extended random choice has a BAUP approximation with a binary state space if each model of the sequence that approximates the random choice according to Definition 1 has a binary state space.

Proposition 3.3.1 *Suppose an extended random choice ρ has a BAUP approximation (representation) with a binary state space. Then for any frames $f, g, h, j \in F$ and any items $x, y, z \in X$, $W_{fgh}^{xyz} \leq 2$ and $Q_{fghj}^{xyz} \leq 3$, where W and Q are linear polynomials of choice frequencies defined in eq. (3.7) and (3.8).*

Note that constraints given in Proposition 3.3.1 emerge solely because the set of potential linear orders induced by an expected utility is restricted when the state space is binary; this argument does not involve consideration of the beliefs' updat-

ing. Therefore, this result is robust to the assumptions regarding the actual updating procedure—for instance, if the agent uses Bayes formula or not—as well as assumptions regarding common prior and common framing function. Whenever the agents in the population are expected utility maximizers with binary state space, the observed choice frequencies should satisfy conditions given in Proposition 3.3.1.

A continuation of analysis of linear orders over three items performed above does not bring new constraints that are not implied by $W^{xyz}_{fgh} \leq 2$ and $Q^{xyz}_{fghj} \leq 3$. However, consideration of linear orders over more than 3 items, most likely, brings new constraints necessary for a BAUP representation with binary state space. The corresponding general analysis is performed in the next section for an arbitrary size of the state space.

Finally, note that constraint $W^{xyz}_{fgh} \leq 2$ has a bite only when all 3 frames are distinct; indeed, if, say, $f = g$, then $p_f(xyz) + p_f(yzx) \leq 1$, hence $W^{xyz}_{ffh} = p_f(xyz) + p_f(yzx) + p_h(zxy) \leq 2$ always holds. Similarly, a constraint $Q^{xyz}_{fghj} \leq 3$ has a bite only when all 4 frames f, g, h, j are distinct.

3.3.2 General State Space

Assume now that the relevant state space is known—meaning that the analyst knows $|S|$—but it may have cardinality larger than 2. Building on the intuition developed in the previous section, let O be the set of linear orders over items in X. For each state-dependent utility function u, define

$$O(u) \;=\; \Big\{ \succeq \in O \;\Big|\; \exists \lambda \in \triangle(S) :\; x \succ y \iff U^\lambda(x) > U^\lambda(y)\ \forall x, y \in X \Big\} \qquad (3.9)$$

where

$$U^\lambda(x) \;=\; \sum_{s \in S} \lambda_s u_s(x) \qquad \forall x \in X$$

is the corresponding expected utility function for belief $\lambda \in \triangle(S)$. An important observation is that if the set of alternative X is large enough, the set $O(u)$ cannot contain all linear orders over alternatives in X.

Lemma 3.3.2 *For any size of the state space $|S|$, there is n such that if $|X| \geq n$, then $O(u) \neq O$ for any state-dependent utility function u.*

The intuition behind Lemma 3.3.2 generalizes the observation made by us for $|S| = 2$ that the beliefs λ_{xy}, λ_{xz}, λ_{yz} divide the probability simplex by at most 4 regions where the induced linear orders remain constant, and uses a Combinatorics result[13] regarding the slicing of the space $\mathbb{R}^{|S|-1}$ by a set of hyperplanes of dimension $|S| - 2$.

We consider now a large enough set of alternatives. Define the collection of sets of linear orders Θ by

$$\Theta \;=\; \{\; O(u) \mid u \text{ is some state-dependent utility function } \} \tag{3.10}$$

and for $k = 1, 2, ..., |X|!$, define

$$\Psi^k \;=\; \left\{ \psi \subseteq O \;\middle|\; |\psi \backslash \theta| \geq k \;\; \forall \theta \in \Theta \right\} \tag{3.11}$$

where $|\psi \backslash \theta|$ is the number of elements of the set ψ that are not elements of the set θ. Thus, Ψ^k consists of the sets of linear orders ψ such that each type $\theta \in \Theta$ of the agent misses at least k linear orders in ψ.

Suppose an extended random choice ρ has a BAUP representation with state space S. As we know (Proposition 3.2.1), in this case, for each frame $f \in F$, there is a random utility representation given by $p_f \in \triangle(O)$.

[13]The result is given in the Introduction (page 1) of Orlik and Terao (2013), and, citing them, this result "was obtained by L. Schliifli on page 39 of his great posthumous work, Theorie der vielfachen Kontinuitiit (Denkschriften der Schweizerischen naturforschenden Gesellschaft, vol. 38, 1901)".

DEFINITION 2. Let O be the set of all linear orders on X and ρ be an extended random choice. A family of distributions over linear orders (random utility representations) $\{p_f\}_{f\in F}$, $p_f \in \triangle(O)$ is associated with ρ if for each frame $f \in F$,

$$p_f(\{\succeq \in O \mid x \succ y \ \forall y \in A \backslash x\}) = \rho_f^A(x).$$

Theorem 3.3.3 *Assume an extended random choice ρ has a BAUP approximation (representation) with state space S, and let Ψ^k for $k = 1, 2, ...$ be defined via eq. (3.9), (3.10), (3.11). Then there is a family of random utility representations $\{p_f\}_{f\in F}$ associated with ρ such that for any k, for any $\psi = (\succeq_1, ..., \succeq_n) \in \Psi^k$, for any $f_1, ..., f_n \in F$*

$$\sum_{i=1}^{n} p_{f_i}(\succeq_i) \ \leq \ n - k$$

Theorem 3.3.3 says that fixation of the state space of a BAUP model places a set of implicit restrictions on the observed choice frequencies across different frames that can be induced by such a model. Although the number of various restrictions placed on quantities $\{p_{f\in F}\}_{f\in F}$ might be large, the statement of the Theorem is relatively weak. Indeed, the result is formulated in terms of the random utility representations rather than in terms of the observed choice frequencies, and, most importantly, it does not deliver conditions necessary for all such representations, but rather says that there *exists* a family of representation that satisfies all considered conditions. The next Corollary, however, says that for rich enough choice data, these restrictions do have a bite.

Corollary 3.3.4 *For any finite state space S, there are numbers n, m such that if $|X| \geq n$ and $|F| \geq m$, there is an extended random choice ρ that admits a random utility representation within each frame, but does not have a BAUP approximation with state space S.*

Let us now show how to find some explicit constraints on the choice frequencies induced by a BAUP model with a fixed state space. The result given in Theorem 3.3.3 is relatively weak because the same random choice ρ_f within frame f may have multiple random utility representations p_f. In a special case when $|X| = 3$, the representation is always unique—the fact that we have used to derive Proposition 3.3.1. Note that the Block-Marschak polynomial $q_f(x, A)$ gives the probability that A is an upper counter set of x according to *any* random utility representation of ρ_f. Let Γ be the set of all pairs (x, A), $x \in A$, $A \subseteq X$ such that $x \notin A$:

$$\Gamma \;=\; \Big\{(x, A) \in X \times 2^X \;\Big|\; x \notin A\Big\} \tag{3.12}$$

Thus, a pair $(x, A) \in \Gamma$ indexes a Block-Marschak polynomial $q_f(A, x)$. We are not interested in $x \in A$, since in this case, $q_f(x, A) = 0$ by definition. For $k = 1, 2, ...,$ define

$$\Xi^k \;= \tag{3.13}$$

$$\Big\{\xi \subseteq \Gamma \;\Big|\; \forall \theta \in \Theta \; \exists \{(x_1, A_1)\}_{i=1}^k \in \xi : \; \theta \cap \Big\{ \succeq \in O \;\Big|\; y \succ x_i \;\Leftrightarrow y \in A_i\Big\} = \varnothing \; \forall i = 1, ..., k\Big\}$$

In words, an element $\xi \in \Xi^k$ is a collection of pairs (x, A) such that for any type $\theta = O(u)$ of the agent with some state-dependent utility function u, there are at least k distinct pairs $(x, A) \in \xi$ such that for each of these pairs, there is no linear order $\succeq \in \theta$[14] for which the set of items A is a strict upper counter set of the item x.

Theorem 3.3.5 *Assume extended random choice ρ has a BAUP approximation (representation) with state space S, and let Ξ^k for $k = 1, 2, ...$ be defined via eq. (3.9), (3.10), (3.12), (3.13). Then for any k, for any $\xi = ((x_1, A_1), ..., (x_n, A_n)) \in \Xi^k$, for any $f_1, ..., f_n \in F$*

$$\sum_{i=1}^{n} q_{f_i}(x_i, A_i) \;\leq\; n - k$$

[14]That is, for $\theta = O(u)$, an order that could be induced by an expectation of utility u_s with respect to some belief regarding the state $s \in S$.

where for each $f \in F$, q_f is a Block-Marschak polynomial constructed from the random choice frequencies ρ_f according to eq. (3.6).

Theorem 3.3.5 provides conditions necessary for a representation (or approximation) of a given extended random choice with a BAUP model with a fixed state space S. These constraints are given in terms of the familiar Block-Marschak polynomials that are used to characterize a classical random utility model. The advantage of this formulation is that the Block-Marschak polynomials are well-defined linear functions of the observed choice data. Theorem 3.3.5, thus, provides constraints on the possible joint values of the random choice frequencies—expressed via a Block-Marschak polynomial within each frame—across various frames in the form of a simple additive statistic.

As an illustration, let us show a way to derive Proposition 3.3.1 from Theorem 3.3.5. For that, let $X = \{x, y, z\}$, and consider

$$\xi \;=\; \{\; (y, \{x\}),\; (z, \{y\}),\; (x, \{z\})\; \}$$

Note that there is only one linear order such that $\{x\}$ is a strict upper counter set of item y, namely, $x \succ y \succ z$; and similarly, for pairs $(z, \{y\})$ and $(x, \{z\})$ the corresponding unique linear orders are $y \succ z \succ x$ and $z \succ x \succ y$. Thus, to prove that $\xi \in \Xi^1$, we only need to argue that each $\theta \in \Theta$ has empty intersection with one of the singleton sets of linear orders $\{xyz\}, \{yzx\}, \{zxy\} \subseteq O$; but we have already checked it in Section 3.3.1. Applying Theorem 3.3.5 for $\xi \in \Xi^1$, we get

$$q_f(y, \{x\}) + q_g(z, \{y\}) + q_h(x, \{z\}) \;\leq\; 3 - 1$$

Supstituting the expressions for the corresponding Block-Marschak polynomials, we get $W^{xyz}_{fgh} \leq 2$. We can get the inequality $Q^{xyz}_{fghj} \leq 3$ in a similar fashion by considering $\xi' = \{(x, \{z\}), (x, \{y\}), (y, \{z\}),\; (z, \{x\})\} \in \Xi^1$.

It is worth to mention that, analogous to Proposition 3.3.1, both Theorem 3.3.3 and Theorem 3.3.5 also hold when the agent's updating procedure is not Bayesian. In fact, both theorems hold whenever the agents in the population are expected utility maximizers with state space S. Finally, whether Theorem 3.3.3 describes a different set of constraints on the extended random choice than Theorem 3.3.5 is an open question.

Appendix A

Appendices for Chapter 3

Appendix

This Appendix has the following structure. In Appendix A.1, I analyze a more general setup that provides tools for the proofs of lemmas, propositions and theorems from Sections 2-4, and also helps to analyze the extensions discussed in Section 1.6.

In particular, Appendix A.1 introduces a more general setup and formalizes the dynamics of the random choice and investigation process performed by the agent in this setup. Section A.1.1 introduces and analyzes a class of Markov chains that facilitates the subsequent analysis, then Section A.1.2 introduces a general setup and analyzes the "global" dynamics that considers multiple investigation processes, performed by the agent. Next, Section A.1.3 considers a "local" dynamics that considers a single investigation process. Finally, Section A.1.4 shows how to transform a problem of solvability of a choice problem via a decision rule into a graph theory problem.

Next, Appendix A.2 returns to the setup, introduced in Section 2 of the paper, but uses tools, developed in Appendix A.1. Section A.2.1 considers useful Combinatorics statements that are built on the analysis, performed in Section A.1.4, including the proof of the lower bound on the language's complexity. Section A.2.2 introduces a

universal decision rule that solves the choice problem for any adequate language and analyzes its properties. Section A.2.3 discusses the automata that can be used for the solution of problems with languages that admit separable utility representation. Section A.2.3 also provides other technical results necessary for the proofs of statements in Section 1.4 of the paper. Section A.2.4 provides the proof of Proposition 1.4.1 in advance, since it is used in the proofs of other theorems.

Section A.2.5 uses tools developed in the sections before to prove all statements from Sections 2-4 of the paper. An interested reader may start from Section A.2.5 to see what lemmas in this Appendix are used in the proofs of a particular theorem or proposition from the main part of the paper. Section A.2.6 contains references for proofs of the statements given in Section 5 of the paper. Section A.2.7 formalises the dynamics of the system, although a more general version of this formalism is considered in Section A.1.2.

Section A.3 provides proofs for the extensions of the baseline model considered in Section 1.6.

A.1 Lemmas for a more general setup

A.1.1 Preliminaries

I first consider a more abstract setup that covers the baseline model together with the extensions.

Let $\mathbf{Z} = (Z_0, Z_1, ...)$ be a stationary Markov chain, where Z_t takes finitely many values $z_t \in \mathcal{Z}$. Let the sample space consists of all sequences of $(z_0, z_1, ...) \in \mathcal{Z}^{\mathbb{N}}$. Let Ω_0 be the algebra generated by events $\{Z_t = z_t \ \forall t \in \{0, ..., T\}\}$ for $T = 0, 1, 2, ...$ Consider probability μ on Ω_0, associated with the stochastic matrix P with elements $P^z_{z'} = Pr(Z_{t+1} = z \mid Z_t = z')$ and the distribution of the initial states $P^z_o = Pr(Z_0 =$

z) in a natural way:

$$\mu\Big(Z_t = z_t \ \forall t \in \{0, ..., T\}\Big) \ = \ P_o^{z_0} \cdot \prod_{t=1}^{T} P_{z_{t-1}}^{z_t}$$

The formula above defines μ for events $\{Z_t = z_t \ \forall t \in \{0, ..., T\}\}$; there is a unique extension of μ by additivity for all events in Ω_0. Let Ω be the smallest sigma-algebra that contains Ω_0. The Kolmogorov Extension Theorem allows for the unique extension of the probability μ to a probability measure on Ω. Similarly, for an event $E = \{Z_T = z_T\} \in \Omega$, I define a conditional probability measure $\mu(\cdot|E)$ as follows:

$$\mu\Big(Z_t = z_t \ \forall t \in \{T+1, ..., T'\} \ \Big| \ Z_T = z_T\Big) \ = \ \prod_{t=T+1}^{T'} P_{z_{t-1}}^{z_t}$$

and extend it to other events accordingly.

I analyze a specific class of stationary Markov chains $\mathcal{C}$ with finite state space $\mathcal{Z}$, which I call the set of *absorbing Markov chains with renewal property*, defined as follows. First, $\mathcal{Z}$ contains two non-empty disjoint subsets of states $\overline{\mathcal{Z}}$ and $\mathcal{Z}^o$. Set $\overline{\mathcal{Z}}$ admits a partition $\left(\overline{\mathcal{Z}}_a\right)_{a \in A}$, where A is a finite set. Elements of set $\mathcal{Z}^o$ are indexed by $d \in D$; that is, $\mathcal{Z}^o = \{z_d^o\}_{d \in D}$. I call elements of $\mathcal{Z}^o$ *starting states*. Next, for any $\mathbf{Z} \in \mathcal{C}$, there is a probability distribution f on D and number $\eta > 0$ such that the following conditions hold:

$$\begin{array}{lllcl}
(i) & \forall z \in \overline{\mathcal{Z}}: & \mu\Big(Z_{t+1} = z \mid Z_t = z\Big) & = & 1 \\
(ii) & & \mu\Big(Z_0 \in \mathcal{Z}^o\Big) & = & 1 \\
(iii) & \forall d \in D: & \mu\Big(Z_t = z_d^o \mid Z_t \in \mathcal{Z}^o\Big) & = & f(d) \\
(iv) & \forall z \in \mathcal{Z}: & \mu\Big(Z_{t+1} \in \mathcal{Z}^o \mid Z_t = z\Big) & \geq & \Big(1 - \mu\Big(Z_{t+1} \in \overline{\mathcal{Z}} \mid Z_t = z\Big)\Big) \cdot \eta
\end{array} \tag{A.1}$$

where I omit figure brackets in the notations of events such as $Z_{t+1} = z$ instead of $\{Z_{t+1} = z\}$, etc. for the ease of notations.

Condition (i) tells that set $\overline{\mathcal{Z}}$ is a set of absorbing states[1], and conditions (ii)-(iv) tell that each period, conditional on not hitting any absorbing state from set $\overline{\mathcal{Z}}$, with probability weakly greater than η, the Markov chain $\mathbf{Z}$ starts its evolution from the beginning, with initial distribution supported at $\mathcal{Z}^o = \{z_d^o\}_{d\in D}$ with likelihoods $f(d)$, $d \in D$. Denote by

$$\begin{aligned} q(d,a) &\equiv \mu\Big(\Big\{\exists r>0:\ Z_{t+r}\in\overline{\mathcal{Z}}_a \text{ and } Z_{t+r'}\notin\mathcal{Z}^o \text{ for all } 1\le r'<r \mid Z_t = z_d^o\Big\}\Big) \\ q(d,\varnothing) &\equiv \mu\Big(\Big\{\exists r>0:\ Z_{t+r}\in\mathcal{Z}^o \mid Z_t = z_d^o\Big\}\Big) \end{aligned} \tag{A.2}$$

Note that $q(d,b)$ and $q(d,\varnothing)$ do not depend on t, since Markov chain $\mathbf{Z}$ is stationary; hence, $q(d,b)$ and $q(d,\varnothing)$ are well-defined. Thus, $q(d,a)$ is the probability that $\mathbf{Z}$ hits an absorbing state $z \in \overline{\mathcal{Z}}_a$ conditional on starting at state z_d^o without going through starting states $z' \in \mathcal{Z}^o$; similarly, $q(d,\varnothing)$ is the probability that the state returns to some $z \in \mathcal{Z}^o$ without hitting any absorbing state from set $\overline{\mathcal{Z}}$.

Lemma A.1.1 *Let* $\mathbf{Z}$ *be an absorbing Markov chain with renewal property. Then*

$$\sum_{a\in A} q(d,a) \ + \ q(d,\varnothing) \ = \ 1 \qquad \forall d \in D$$

Proof. Let $E_d^t \equiv \Big\{Z_t = z_d^o\Big\}$. Consider events $E_{d\varnothing}^t \equiv \Big\{\exists r>0 : Z_{t+r}\in\mathcal{Z}^o\Big\}\cap E_d^t$, $E_{da}^t \equiv \Big\{\exists r>0 : Z_{t+r}\in\overline{\mathcal{Z}}_a \text{ and } Z_{t+r'}\notin\mathcal{Z}^o \text{ for all } 1\le r'<r\Big\}\cap E_d^t$, and $\widetilde{E}_d^t \equiv E_d^t \setminus \Big(\bigcup\limits_{a\in A\cup\{\varnothing\}} E_{da}^t\Big)$. Note that for any $r>0$, we have

$$\mu\Big(\widetilde{E}_d^t \mid E_d^t\Big) \ \le \ \mu\Big(\Big\{Z_{t+r'}\notin\overline{\mathcal{Z}}\cup\mathcal{Z}^0 \ \forall\ 0<r'\le r\Big\} \ \Big|\ E_d^t\Big) \ \le \ (1-\eta)^r \ \stackrel{r\to\infty}{\longrightarrow} \ 0$$

Since $q(d,a) = \mu(E_{da}^t|E_d^t)$ and $q(d,\varnothing) = \mu(E_{d\varnothing}^t|E_d^t)$, the lemma follows. ■

[1]Our definition allows for the case when the set of starting states $\mathcal{Z}^o = \{z^o\}$ is singleton, and z^o is an absorbing state. Thus we don't say that $\overline{\mathcal{Z}}$ is *the* set of all absorbing states.

Denote by $H_a \equiv \left\{\exists t : Z_t \in \overline{Z}_a\right\}$ and $H_\varnothing \equiv \Omega \setminus \bigcup_{a \in A} H_a$. Then $\mu(H_a \cap H_b) = 0$ for any $a, b \in A$ such that $a \neq b$, and $H_a \cap H_\varnothing = \varnothing$ for any $a \in A$.

Lemma A.1.2 *Let* $\mathbf{Z}$ *be an absorbing Markov chain with renewal property. If* $\exists d \in D$ *such that* $f(d) \cdot (1 - q(d, \varnothing)) > 0$*, then* $\mu(H_\varnothing) = 0$*, and*

$$\mu(H_a) \;=\; \frac{\sum_{d \in D} f(d) \cdot q(d, a)}{\sum_{d \in D} \sum_{b \in A} f(d) \cdot q(d, b)};$$

otherwise, $\mu(H_\varnothing) = 1$.

Proof. Suppose first $\exists d \in D$ such that $f(d) \cdot (1 - q(d, \varnothing)) > 0$. Let

$$T^k = \left\{(t_1, ..., t_k) \in (\{0\} \cup \mathbb{N})^k \;\middle|\; 0 = t_1 < ... < t_k\right\},$$

and for $(t_1, ..., t_k) \in T^k$, denote by

$$H^k(t_1, ..., t_k) = \left\{\forall t : 0 \leq t \leq t_k \;\; Z_t \in \mathcal{Z}^o \iff t \in \{t_1, ..., t_k\}\right\}$$

Thus, $H^k(t_1, ...t_k)$ is the event that the state returns to the set $\mathcal{Z}^o$ at periods $t_1, ..., t_k$, and does not return to $\mathcal{Z}^o$ at any other period before t_k. Next, for any $a \in A$, denote by

$$H^k_a = \left\{\exists (t_1, ..., t_k) \in T^k, t' > t_k \;\middle|\; [Z_t \in \mathcal{Z}^o \iff t \in \{t_1, ..., t_k\}] \text{ and } Z_{t'} \in \overline{\mathcal{Z}}_a\right\}$$

Thus, H^k_a is the event that the state has been at the set of starting states $\mathcal{Z}^o$ exactly k times at periods $t_1, ..., t_k$, and then went to the absorbing state $z \in \overline{\mathcal{Z}}_a$. By condition (iii) from eq. (A.1) for $\mathbf{Z} \in \mathcal{C}$,

$$\mu\Big(Z_{t_{k-1}} = z^0_d \;\Big|\; H^{k-1}(t_1, ..., t_{k-1})\Big) \;=\; f(d)$$

Then, using the definition of $q(d, \varnothing)$, we get

$$\sum_{t_k > t_{k-1}} \mu\Big(H^k(t_1, ..., t_k) \;\Big|\; H^{k-1}(t_1, ..., t_{k-1})\Big) \;=\; \sum_{d \in D} f(d) \cdot q(d, \varnothing)$$

Recursively,

$$\begin{aligned}
&\sum_{(t_1,...,t_k) \in T^k} \mu\Big(H^k(t_1, ..., t_k)\Big) \;= \\
&= \sum_{t_k > t_{k-1}} \sum_{(t_1,...,t_{k-1}) \in T^{k-1}} \mu\Big(H^k(t_1, ..., t_k) \;\Big|\; H^{k-1}(t_1, ..., t_{k-1})\Big) \cdot \mu\Big(H^{k-1}(t_1, ..., t_{k-1})\Big) \;= \\
&= \Big(\sum_{d \in D} f(d) \cdot q(d, \varnothing)\Big) \cdot \sum_{(t_1,...,t_{k-1}) \in T^{k-1}} \mu\Big(H^{k-1}(t_1, ..., t_{k-1})\Big) \;= \\
&= \;...\; = \\
&= \Big(\sum_{d \in D} f(d) \cdot q(d, \varnothing)\Big)^{k-1} \cdot \sum_{t_1 \in T^1} \mu\Big(H^1(t_1)\Big) \;=\; \Big(\sum_{d \in D} f(d) \cdot q(d, \varnothing)\Big)^{k-1}
\end{aligned}$$

where we used $T^1 = \{0\}$ by the definition of T^k and $\mu(H^1(0)) = 1$ by condition (ii) from eq. (A.1). Note that by our assumption,

$$\sum_{d \in D} f(d) \cdot q(d, \varnothing) \;=\; 1 - \sum_{d \in D} f(d)(1 - q(d, \varnothing)) \;<\; 1$$

Next, by the definition of $q(d, a)$ and condition (iii) from eq. (A.1),

$$\mu\Big(H_a^k \;\Big|\; H^k(t_1, ..., t_k)\Big) \;=\; \sum_{d \in D} f(d) q(d, a)$$

Then

$$\begin{aligned}
\mu\Big(H_a^k\Big) \quad &= \quad \sum_{(t_1,...,t_k) \in T^k} \mu\Big(H_a^k \;\Big|\; H^k(t_1, ..., t_k)\Big) \cdot \mu\Big(H^k(t_1, ..., t_k)\Big) \;= \\
&= \quad \Big(\sum_{d \in D} f(d) q(d, a)\Big) \cdot \Big(\sum_{d \in D} f(d) \cdot q(d, \varnothing)\Big)^{k-1}
\end{aligned}$$

Finally,

$$\begin{aligned}\mu\Big(H_a\Big) &= \sum_{k\in\mathbb{N}}\mu\Big(H_a^k\Big) = \sum_{k\in\mathbb{N}}\Big(\sum_{d\in D} f(d)q(d,a)\Big)\cdot\Big(\sum_{d\in D} f(d)\cdot q(d,\varnothing)\Big)^{k-1} = \\ &= \frac{\sum_{d\in D} f(d)q(d,a)}{1-\sum_{d\in D} f(d)q(d,\varnothing)} = \frac{\sum_{d\in D} f(d)q(d,a)}{\sum_{d\in D}\sum_{b\in A} f(d)q(d,b)}\end{aligned}$$

where I used Lemma A.1.1 and the fact that $\sum_{d\in D} f(d) = 1$ in the last equality. It follows that

$$\mu\Big(H_\varnothing\Big) = 1 - \sum_{a\in A}\mu\Big(H_a\Big) = 0$$

Suppose now that $f(d) > 0$ implies $q(d,\varnothing) = 1$; then by Lemma A.1.1, $\sum_{d\in D} f(d)q(d,a) = 0$ for any $a \in A$. Using the same analysis, we get

$$\mu\Big(H_a^k\Big) = \sum_{d\in D} f(d)q(d,a) = 0$$

It follows that $\mu(H_a) = \sum_{k\in\mathbb{N}}\mu(H_a^k) = 0$, hence $\mu\Big(H_\varnothing\Big) = 1$. ■

A.1.2 Global dynamics

I call "local dynamics" the behavior of state variables in a set of subsequent periods such that the decision maker investigates and, maybe, choose some alternative without drawing a new one. I call global dynamics the behavior of state variables in all periods. In this section, I apply Lemmas A.1.1 and A.1.2 to analyze the global dynamics.

To accommodate the extensions of the baseline model, I consider several generalizations. First, I allow no questions being asked at a state. Thus, the interrogation function becomes $\iota : S^o \to N \cup \{0\}$, where $\iota(s) = \varnothing$ represents no questions asked. Define a signal space $\mathcal{I}$ with generic element $\theta \in \mathcal{I}$ by $\mathcal{I} = \Big(N \times \{0,1\}\Big) \cup \{\varnothing\}$, where $\theta = (i,j)$ represents question $i \in N$ asked and answer $j \in \{0,1\}$ received, and $\theta = \varnothing$ represents no questions asked.

Second, I consider a stochastic interrogation rule given by $\iota : S^o \to \triangle(N \cup \{\varnothing\})$, where $\iota_s(i)$ is the probability to ask question $i \in N$ or do not ask any question ($i = \varnothing$) in state $s \in S$. A deterministic interrogation rule is a special case of a stochastic interrogation rule with ι being a delta-function: $\iota_s(i) = \delta^i_{\iota(s)}$ for $i \in N \cup \{\varnothing\}$, where $\iota(s)$ is the question asked in state s in the baseline model with the deterministic interrogation rule.

Third, I allow the agent to draw a new alternative intentionally. Thus, I consider an automaton with the state space $S = S^o \cup \{\diamond\} \cup \{new\}$, where state $s = new$ stands for agent's decision to draw a new alternative from the menu. The stochastic transition rule becomes $\tau : S^o \times \mathcal{I} \to S \cup \{\diamond\} \cup \{new\}$ in this case; $\tau(s', s, \theta)$ is the probability to go to state $s \in S^o \cup \{\diamond\} \cup \{new\}$ from state $s' \in S$ after signal $\theta \in \mathcal{I}$. The case when the agent cannot draw a new alternative intentionally is a special case when $\tau(s', new, \theta) = 0$ for all $s' \in S^o$, $\theta \in \mathcal{I}$. Recall that in the baseline model, the transitional probability is given by function $\tau : S^o \times \{0, 1\} \to S$. To connect the two versions, define $\tau(s', s, j) = \tau(s', s, \iota(s'), j)$, where the left hand side of the equation denotes the transitional probability in the baseline model, and the right hand side denotes the transitional probability in the general model with the restriction that the signal is $\theta = (\iota(s'), j)$, where $\iota(s')$ is the question, asked in state s'.

In the general model, the automaton strategy (or simply strategy) $\sigma = (S, \iota, \tau)$ consists of the state space S, the stochastic interrogation rule ι and the stochastic transition rule τ described above. As in the baseline model, I denote by Σ the set of all such strategies with finite state space.

The set of the positive-probability transitions in the general case becomes

$$\mathcal{T}_\sigma := \left\{(s', s, \iota) \;\middle|\; \iota_s(\iota) \cdot \tau(s', s, \iota) > 0\right\} \tag{A.3}$$

The set of transitions defined for the baseline model by eq. (1.2) is a special case of eq. (A.3), which one can see from

$$\mathcal{T}_\sigma \;=\; \Big\{(s',s,i,j)\;\Big|\;\iota_s(i)\cdot\tau(s',s,i,j)>0\Big\}\cup\Big\{(s',s,\varnothing)\;\Big|\;\iota_s(\varnothing)\cdot\tau(s',s,\varnothing)>0\Big\}$$

Define the global state space $\mathcal{Z} = (\mathcal{W},\mathcal{Y},\mathcal{I})$, where $w \in \mathcal{W} \equiv \{new, old, \diamond\}$ encodes whether some item has already been chosen ($w = \diamond$), or a new item is drawn ($w = new$), or the investigated item remains the same as in the previous period ($w = old$). Next, $y = (y, b) \in \mathcal{Y} \equiv S^o \times A$ describes the flexible state of the automaton and the investigated item (provided that the item has not been chosen previously—otherwise, it describes the last flexible state of the automaton from where it transitioned to $s = \diamond$ and the item that has been chosen). Finally, $\mathcal{I}$ is a signal space, defined above. The dynamics of the model is then described by a Markov chain $Z_0, Z_1, Z_2, ...$ I denote by $Z_t = (W_t, Y_t, \Theta_t)$ the corresponding t-th period random state of the chain, and by $z_t = (w_t, y_t, \theta_t)$ its realization.

The strategy $\sigma = (S, \iota, \tau)$ and menu B induce the stochastic $|\mathcal{Z}| \times |\mathcal{Z}|$ matrix $P^z_{z'} \equiv Pr(Z_{t+1} = z|Z_t = z')$ as follows:

$$P^z_{z'} \;=\; Pr\Big(Z_{t+1}=(w,s,a,\theta)\;\Big|\;Z_t=(w',s',a',\theta')\Big) \;=$$

$$\begin{cases} \delta^a_{a'}\delta^s_{s'}\Big(\delta^\diamond_{w'}+(1-\delta^\diamond_{w'})\tau(s',\diamond,\iota')\Big) & \text{if} \quad w=\diamond \\ (1-\delta^\diamond_{w'})\Big[(1-\tau(s',\diamond,\theta'))\eta+\tau(s',new,\theta')(1-\eta)\Big]\delta^1_s\rho^B(a)f^a_1(\theta) & \text{if} \quad w=new \\ (1-\delta^\diamond_{w'})\Big[1-\tau(s',\diamond,\theta')-\tau(s',new,\theta')\Big](1-\eta)\delta^{a'}_a\tau(s',s,\theta')f^a_s(\theta) & \text{if} \quad w=old \end{cases}$$

where

$$f^a_s(\theta) \;\equiv\; \begin{cases} \iota_s(i)\cdot\delta^{a_i}_j & \text{if} \quad i\in N \\ \iota_s(\varnothing) & \text{if} \quad i=\varnothing \end{cases}$$

The distribution of the initial state is given by a $|\mathcal{Z}|$-dimensional row vector P_o with components

$$P_o^z \;\equiv\; Pr\Big(Z_0 = (w, s, a, \varnothing)\Big) \;=\; \delta_w^{new} \cdot \delta_s^1 \cdot \rho^B(a) \cdot f_1^a(\theta)$$

Note that any result, proven for an arbitrary Markov chain induced by menu B and strategy $\sigma = (S, \iota, \tau)$ with generic functions $\iota : S \to \triangle(N \cup \{\varnothing\})$ and $\tau : S \times \mathcal{I} \to S \cup \{\diamond\} \cup \{new\}$ remains valid if we put restrictions on ι (such as ι being a delta-function, and $\iota_s(\varnothing) = 0$), or τ (such as $\tau(s', new, \iota) = 0$ for all $s' \in S$, $\theta \in \mathcal{I}$), or on both these functions. In particular, the results are applicable to the baseline model.

To apply Lemmas 1-2 to the Markov chain $\mathbf{Z}$ induced by menu B and the decision rule σ, I identify set A with the universal set of items A, and consider $D = A \times \mathcal{I}$. Thus, $\overline{\mathcal{Z}}_a = \Big\{(\diamond, s, a, \varnothing)\Big\}_{s \in S^o}$ and $z^o_{a,\theta} = (new, 1, a, \theta)$. Next, for a fixed menu B and strategy σ, consider $f(a, \varnothing) = \rho^B(a) \cdot \iota_1(\varnothing)$, $f(a, i, j) = \rho^B(a) \cdot \iota_1(i) \cdot \delta_j^{a_i}$ for $i \in N$, $j \in \{0, 1\}$.

Lemma A.1.3 *The Markov chain* $\mathbf{Z}$ *induced by menu* B *and strategy* $\sigma \in \Sigma$ *is an absorbing Markov chain with renewal property such that the set* $\overline{\mathcal{Z}} = \Big\{\overline{\mathcal{Z}}_a\Big\}_{a=A'}$, *where* $\overline{\mathcal{Z}}_a = \Big\{(\diamond, s, a, \varnothing)\Big\}_{s \in S^o}$, *is the set of absorbing states,* $D = A \times \mathcal{I}$, *the set* $\mathcal{Z}^o = \{z^o_d\}_{d \in D} = \Big\{(new, 1,\ a, \iota)\Big\}_{(a,\iota) \in A \times \mathcal{I}}$ *is the set of starting states, and* $f(a, \theta) = \rho^B(a) \cdot f_1^a(\theta)$.

Proof. First, note that sets $\overline{\mathcal{Z}}$ and $\mathcal{Z}^o$ are non-empty and disjoint. Second, let us verify conditions (i)-(iv) from eq. (A.1) for Markov chain $\mathbf{Z}$. Property (i) follows from $P^{\diamond,s,a}_{\diamond,s',a',\varnothing} = \delta^a_{a'} \cdot \delta^s_{s'}$. Property (ii) follows from the formula for P_o. For property (iii), recall that $z^o_{a,\theta} = (new, 1, a, \theta)$ and consider separately period $t = 0$ and periods $t > 0$. For $t = 0$, we have:

$$\mu\Big(Z_0 = z^o_d \;\Big|\; Z_0 \in \mathcal{Z}^o\Big) \;=\; \mu\Big(Z_0 = z^o_{a,\theta}\Big) \;=\; \rho^B(a) \cdot f_1^a(\theta) \;=\; f(a, \theta) \;=\; f(d)$$

For $t > 0$, we have:

$$\mu\Big(Z_t = z_d^o \;\Big|\; Z_{t-1} = z'\Big) \;=\; (1-\delta_{w'}^{\diamond}) \cdot \Big[\tau(s', new, \theta') \cdot (1-\eta) + (1-\tau(s', \diamond, \theta')) \cdot \eta\Big] \cdot f(d)$$

$$\mu\Big(Z_t \in \mathcal{Z}^o \;\Big|\; Z_{t-1} = z'\Big) \;=\; \sum_{e \in D} \mu\Big(Z_t = z_e^o \;\Big|\; Z_{t-1} = z'\Big) \;=\; \sum_{e \in D} x(z') \cdot f(e) = x(z')$$

where I denote by

$$x(z') \;\equiv\; (1-\delta_{w'}^{\diamond}) \cdot \Big[\tau(s', new, \theta') \cdot (1-\eta) + (1-\tau(s', \diamond, \theta')) \cdot \eta\Big]$$

Then

$$\mu\Big(Z_t \in \mathcal{Z}^o\Big) \;=\; \sum_{z' \in \mathcal{Z}} \mu\Big(Z_t \in \mathcal{Z}^o \;\Big|\; Z_{t-1} = z'\Big) \cdot \mu\Big(Z_{t-1} = z'\Big) \;=\; \sum_{z' \in \mathcal{Z}} x(z')$$

$$\mu\Big(Z_t = z_d^o\Big) \;=\; \sum_{z' \in \mathcal{Z}} \mu\Big(Z_t = z_d^o \;\Big|\; Z_{t-1} = z'\Big) \cdot \mu\Big(Z_{t-1} = z'\Big) \;=\; \sum_{z' \in \mathcal{Z}} x(z') \cdot f(d)$$

It follows

$$\mu\Big(Z_t = z_d^o \;\Big|\; Z_t \in \mathcal{Z}^o\Big) \;=\; \frac{\mu\Big(Z_t = z_d^o, Z_t \in \mathcal{Z}^o\Big)}{\mu\Big(Z_t \in \mathcal{Z}^o\Big)} \;=\; \frac{\mu\Big(Z_t = z_d^o\Big)}{\mu\Big(Z_t \in \mathcal{Z}^o\Big)} \;=$$

$$\frac{\sum_{z' \in \mathcal{Z}} x(z') \cdot f(d)}{\sum_{z' \in \mathcal{Z}} x(z')} \;=\; f(d)$$

To prove property (iv), we use the previous calculations:

$$\mu\Big(Z_t \in \mathcal{Z}^o \;\Big|\; Z_{t-1} = z'\Big) \;=\; x(z') \geq (1-\delta_{w'}^{\diamond}) \cdot \Big[(1-\tau(s', \diamond, \theta')) \cdot \eta\Big] \;=$$

$$\Big(1 - \mu\Big(Z_t \in \overline{\mathcal{Z}} \;\Big|\; Z_{t-1} = z'\Big)\Big) \cdot \eta$$

■

Lemma A.1.4 *Let* $\mathbf{Z}$ *be the Markov chain induced by menu* B *and strategy* σ*. Then* $q((b,\theta),a) = 0$ *for all* $a, b \in A$ *such that* $a \neq b$*.*

Proof. The lemma follows from

$$\mu\Big(\exists r > 0 : a_{t+r} \neq b, w_{t+r'} \neq new \ \forall 1 \leq r' \leq r \ \Big| \ w_t \neq \diamond, \ s_t = 1, \ a_t = b\Big) \ = \ 0$$

since $a' \neq a$ implies $Pr(Z_{t+1} = (old, s, a, \theta)|Z_t = (w', s', a', \theta')) = 0$ and $Pr(Z_{t+1} = (\diamond, s, a, \theta)|Z_t = (w', s', a', \theta')) = 0$. ■

Lemma A.1.5 *Let* $\mathbf{Z}$ *be the Markov chain induced by menu* B *and strategy* σ*. For* $a \in A$*, denote by*

$$q_\sigma(a) := \mu\Big(\exists r > 0 : w_{t+r} = \diamond, a_{t+r} = a, w_{t+r'} \neq new \ \forall 1 \leq r' \leq r \ \Big| \ w_t = new, s_t = 1, a_t = a\Big) \tag{A.4}$$

then: (i) $q_\sigma(\cdot)$ *does not depend on menu* B*, and (ii) the following holds:*

$$\sum_{d \in D} f(d) \cdot q(d,a) \ = \ \rho^B(a) \cdot q_\sigma(a) \qquad \forall a \in A$$

Proof. By Lemma A.1.4, we get

$$\sum_{d \in D} f(d) \cdot q(d,a) \ = \ \sum_{b \in A} \sum_{\theta \in I} \rho^B(b) \cdot q((b,\theta),a) \ = \ \sum_{\theta \in I} \rho^B(a) \cdot q((a,\theta),a) \ =$$

$$\sum_{\theta \in I} \rho^B(a) \mu\Big(\exists r > 0 : \ w_{t+r} = \diamond, a_{t+r} = a, w_{t+r'} \neq new \ \forall 1 \leq r' < r \ \Big| \ Z_t = (new, 1, a, \theta)\Big) \ =$$

$$\rho^B(a) \mu\Big(\exists r > 0 : \ w_{t+r} = \diamond, a_{t+r} = a, w_{t+r'} \neq new \ \forall 1 \leq r' \leq r \ \Big| \ w_t \neq \diamond, \ s_t = 1, \ a_t = a\Big) \ =$$

$$\rho^B(a) q_\sigma(a)$$

proving statement (ii). Statement (i) follows from the fact that $Pr(Z_{t+1} = (old, s, a, \theta)|Z_t = z')$ and $Pr(Z_{t+1} = (\diamond, s, a, \varnothing)|Z_t = z')$ do not depend on B for any $z' \in \mathcal{Z}$. ■

Lemma A.1.6 *Let* $\mathbf{Z}$ *be the Markov chain induced by menu* B *and strategy* σ*. If* $\exists a \in B$ *such that* $\rho^B(a) \cdot q_\sigma(a) > 0$*, then* $\mu(H_\varnothing) = 0$*, and*

$$\mu(H_a) \;=\; \frac{\rho^B(a) \cdot q_\sigma(a)}{\sum_{b \in B} \rho^B(b) \cdot q_\sigma(b)} \tag{A.5}$$

for any $a \in B$*; otherwise,* $\mu(H_\varnothing) = 1$*.*

Proof. The Lemma follows straightforwardly from Lemma A.1.2, Lemma A.1.3, Lemma A.1.5 and $\text{supp}(\rho^B) = B$. ■

Lemma A.1.7 *A decision rule* $\{\sigma_r\}_{r=1,2,\ldots} \in \Psi$ *solves choice problem* $(Q, \succeq)$ *if and only if the following two conditions hold: (i)* $a \succ b$ *implies* $q_{\sigma_r}(b)/q_{\sigma_r}(a) \longrightarrow 0$ *for all* $a, b \in A$*, and (ii)* $\exists \overline{r}$*:* $q_{\sigma_r}(a) > 0$ *for all* $r > \overline{r}$ *for all* $a \in A$*.*

Proof. Suppose first that σ_r solves $(Q, \succeq)$. Consider arbitrary $a \in A$. Towards a contradiction, assume there is a subsequence σ_{r^l} such that $q_{\sigma_{r^l}}(a) = 0$ and consider menu $B = \{a\}$. Then by Lemma A.1.3 and Lemma A.1.1, $q((a, \theta), \varnothing) = 0$ for all $\theta \in \mathcal{I}$. Since $B = \{a\}$, then $f((b, \theta)) = 0$ for $b \neq a$. Thus by Lemma A.1.6, $\mu(H_\varnothing) = 1$, contradicting $\mu(H_a) \overset{l \to \infty}{\longrightarrow} 1$. Thus, property (ii) holds; moreover, eq. (A.5) is well-defined.

Suppose $a \succ b$ and consider menu $B = \{a, b\}$. Since σ_r solves $(Q, \succeq)$, then

$$1 \;=\; \lim_{r \to \infty} \mu(H_a) \;=\; 1 - \frac{\rho^B(a)}{\rho^B(b)} \cdot \lim_{r \to \infty} \frac{q_\sigma(b)}{q_\sigma(a)}$$

and statement (ii) follows, since $\rho^B(a), \rho^B(b) > 0$.

Suppose now that conditions (i) and (ii) hold. For an arbitrary menu B, let $\overline{B} = \{a \in B | a \succeq b \; \forall b \in B\}$ be the set of maximizers of $\succeq$ over B, and $B_- = B \backslash \overline{B}$. Condition (ii) guarantees that $\rho^B(a) \cdot q_{\sigma_r}(a) > 0$ for any $a \in B$ for large enough r.

Then by Lemma A.1.6,

$$\lim_{r\to\infty}\mu(\overline{B}) \;=\; 1-\lim_{r\to\infty}\frac{\sum_{b\in B_-}\rho^B(b)\cdot q_{\sigma_r}(b)}{\sum_{b\in B_-}\rho^B(b)\cdot q_{\sigma_r}(b)+\sum_{a\in\overline{B}}\rho^B(a)\cdot q_{\sigma_r}(a)} \;=\; 1$$

where we used the fact that $\rho^B(a)>0$ for all $a\in B$ and $q_{\sigma_r}(b)/q_{\sigma_r}(a)\longrightarrow 0$ for any $b\in B_-$, $a\in\overline{B}$ by condition (i). Thus σ_r solves $(Q,\succeq)$. ■

A.1.3 Local dynamics

Lemma A.1.7 shows that for our analysis, it suffices to learn the limiting properties of function $q_{\sigma_r}(a)$ given by eq. (A.4). In this section, I assume that the current alternative $a\in A$ that the agent investigates fixed, and omit it if it does not cause confusion. By stationarity of the Markov chain $\mathbf{Z}$,

$$q_\sigma(a) \;=\; \mu\Big(\exists t>0: w_t=\diamond, a_t=a, w_{t'}\neq new\;\forall 1\leq t'\leq t \;\Big|\; w_0=new, s_0=1, a_0=a\Big)$$

It is useful to note that, when analysing $q_\sigma(a)$, we are interested only in the probability of realizations of Markov chain $\mathbf{Z}$ up to the point where either the current alternative $a\in A$ is chosen, or a new alternative is drawn. Thus, we may omit w_t and a_t in the description of the state $z\in\mathcal{Z}$, and instead focus on s_t and ι_t. To account for the cases when $w_r=\diamond$ or $w_r=new$—these cases correspond to the last period of our analysis relevant for the calculation of $q(a)$—I consider a local state space $\mathcal{X}=\Big(S^o\times\mathcal{I}\Big)\cup\{\diamond\}\cup new$.

To analyse $q_\sigma(a)$, I, therefore, consider Markov chain $\mathbf{X}=(X_0,X_1,...)$ with realizations $(x_0,x_1,...)\in\mathcal{X}^{\mathbb{N}}$ induced by strategy σ and alternative $a\in A$ via a mapping $\chi:\;\mathcal{Z}^{\mathbb{N}}\to\mathcal{X}^{\mathbb{N}}$ from the set of realizations of the Markov chain $\mathbf{Z}$ induced by menu $B=\{a\}$ to the set of realizations of the Markov chain $\mathbf{X}$ given by the following

formula:

$$\chi((z_0, z_1, ...)) = (x_0, x_1, ...) : \begin{cases} x_t = (s_t, \theta_t) & \text{if} \quad w_{t'} = old \ \forall 0 < t' \leq t \\ x_t = \diamond & \text{if} \quad \exists t'' \leq t : \ w_{t'} = old \ \forall 0 < t' < t'', \ w_{t''} = \diamond \\ x_t = new & \text{if} \quad \exists t'' \leq t : \ w_{t'} = old \ \forall 0 < t' < t'', \ w_{t''} = new \end{cases}$$

where $z_t = (w_t, s_t, a_t, \theta_t)$. Abusing notations denote by P be the $\mathcal{X} \times \mathcal{X}$ stochastic matrix, associated with the Markov chain $\mathbf{X}$, then:

$$P^x_{x'} \ = \ Pr\Big(X_{t+1} = (x, \theta) \ \Big| \ X_t = (x', \theta')\Big) \ =$$

$$= \begin{cases} \delta^{\diamond}_{x'} + (1 - \delta^{\diamond}_{x'} - \delta^{new}_{x'}) \cdot \tau(s', \diamond, \theta') & \text{if } x = \diamond \\ \delta^{new}_{x'} + (1 - \delta^{\diamond}_{x'} - \delta^{new}_{x'}) \cdot [\eta + (1 - \eta) \cdot \tau(s', new, \theta')] & \text{if } x = new \\ (1 - \delta^{\diamond}_{x'} - \delta^{new}_{x'}) \cdot (1 - \eta) \cdot \tau(s', s, \theta') \cdot f_s(\theta) & \text{otherwise} \end{cases}$$

where

$$f_s(\theta) \ \equiv \ \begin{cases} \iota_s(i) \cdot \delta^{a_i}_j & \text{if} \quad \theta = (i, j) \\ \iota_s(\varnothing) & \text{if} \quad \theta = \varnothing \end{cases}$$

The distribution of the initial state X_0 is given by

$$P^{s,\theta}_o \ \equiv \ Pr\Big(X_0 = (s, \theta)\Big) \ = \ \delta^1_s \cdot f_s(\theta)$$

The stochastic matrix P and initial distribution P_o induce a probability distribution μ on the sigma-algebra, generated by events $\{X_t = x_t \ \forall\{0, ..., T\}\}$ according to the formula below and its extension by countable additivity on other events:

$$\mu\Big(X_t = x_t \ \forall\{0, ..., T\}\Big) \ = \ P^{x_0}_o \cdot \prod_{t=1}^{T} P^{x_t}_{x_{t-1}} \tag{A.6}$$

This probability distribution is a pushforward measure given by $\mu(\chi(C)) = \mu(C)$ for event C of the original sigma-algebra of events Ω, associated with the global dynamics. In the new terms, quantity $q_\sigma(a)$ looks as

$$q_\sigma(a) \;=\; \tilde{\mu}\Big(\exists t > 0 : x_t = \diamond\Big)$$

Lemma A.1.8 *The probability that the state $X_t \in \mathcal{X}$ is not absorbing vanishes with each period at rate at least $(1-\eta)$; that is,*

$$\tilde{\mu}(X_t \neq \diamond, new) \;\leq\; (1-\eta)^t$$

Proof. The lemma is straightforward. ■

A.1.4 A Graph Theory Approach

In this section, I transform an analysis of the solution of the choice problem via a decision rule into the analysis of properties of specific directed graphs that represent transitions between the automaton's states, and in particular, into the analysis of simple paths going from the starting state $s = 1$ to the decision state $s = \diamond$.

Given a strategy σ, define a mapping $\omega :\ \mathcal{T} \to (0, 1]$ by

$$\omega(s', s, \theta) \;=\; \begin{cases} \iota_{s'}(i) \cdot \tau(s', s, i, j) & \text{if} \quad \theta = (i, j) \in N \times \{0, 1\} \\ \iota_{s'}(\varnothing) \cdot \tau(s', s, \varnothing) & \text{if} \quad \theta = \varnothing \end{cases} \tag{A.7}$$

I call $\omega(s', s, \iota)$ the *weight* of the link (s', s, ι). When needed, I use a subscript σ to show the dependence of ω on the strategy σ.

For an arbitrary alternative $a \in A$ and strategy $\sigma \in \Sigma$, define the set of links

$$\mathcal{T}_\sigma^a \;=\; \Big\{(s', s, \theta) \in \mathcal{T}_\sigma \;\Big|\; f_{s'}^a(\iota) \cdot \tau(s', s, \theta) > 0\Big\}$$

where

$$f^a_{s'}(\theta) \;=\; \begin{cases} \iota_{s'}(i)\cdot\delta^{a_i}_j & \text{if} \quad i\in N \\ \iota_{s'}(\varnothing) & \text{if} \quad i=\varnothing \end{cases}$$

Thus, $\mathcal{T}^a_\sigma$ is the set of links that can be potentially used for transitions while investigating alternative a using strategy σ. Note that

$$\mathcal{T}^a_\sigma \;=\; \Big\{(s',s,i,j)\in\mathcal{T}_\sigma \;\Big|\; \iota_s(i)\cdot\delta^{a_i}_j\cdot\tau(s',s,i,j)>0\Big\}\cup\Big\{(s',s,\varnothing)\in\mathcal{T}_\sigma \;\Big|\; \iota_s(\varnothing)\cdot\tau(s',s,\varnothing)\Big\}$$

Consider a directed weighted graph $\mathcal{G}=\mathcal{G}(a,\sigma)=(S_\sigma,\mathcal{T}^a_\sigma,\omega_\sigma)$, defined as follows. Set up $S=S\cup\{\diamond\}$ to be the set of vertexes, set up $\mathcal{T}^a=\mathcal{T}^a_\sigma$ to be the set of transitions, given by eq. (A.1.4), where we interpret $l=(s',s,\theta)\in\mathcal{T}^a_\sigma$ as a directed link from state $s'\in S^o$ to state $s\in S^o\cup\{\diamond\}$, labeled by a signal $\theta\in\mathcal{I}$. Finally, $\omega(s',s,\theta)$ is the weight of the link $(s',s,\theta)\in\mathcal{T}^a_\sigma$. Note that

$$(s',s,\theta)\in\mathcal{T}^a_\sigma \implies \omega(s',s,\theta)=\left\{\begin{array}{ll} \iota_{s'}(i)\delta^{a_i}_j\tau(s',s,i,j) & \text{if} \quad \iota=(i,j) \\ \iota_{s'}(\varnothing)\tau(s',s,\varnothing) & \text{if} \quad \iota=\varnothing \end{array}\right| = f^a_s(\iota)\cdot\tau(s',s,\theta)$$

Define a valid (directed) path l on the graph $\mathcal{G}$ to be a finite ordered set of labeled directed links $(l_1,...,l_T)\in\mathcal{T}^T$ such that the following properties hold:

(i) The first link, l_1, starts at state $s=1$;

(ii) If link $l_t=(s'_t,s_t,\iota_t)$, $t<T$, ends at state s_t, then link $l_{t+1}=(s'_{t+1},s_{t+1},\iota_{t+1})$ starts at state $s'_{t+1}=s_t$;

(iii) The last link, l_T, ends at $s=\diamond$;

Denote by $\mathcal{L}$ the set of all valid paths.

For an arbitrary valid path $l=(l_1,...,l_T)\in\mathcal{L}$, define its weight as a product of the weights of its links times $(1-\eta)^{T-1}$; that is, $\omega:\;\mathcal{L}\to\mathbb{R}$ given by

$$\omega(l) \;=\; \omega^a_\sigma(l)=(1-\eta)^{T-1}\cdot\prod_{t=1}^{T}\omega(l_t)$$

Lemma A.1.9 *Let $\mathcal{G} = (S, \mathcal{T}, \omega)$ be a directed graph, $\mathcal{L}$ be the set of valid paths, and ω be the weighting function defined above, then*

$$q_\sigma(a) \;=\; \sum_{l \in \mathcal{L}} \omega(l)$$

Proof. Consider event

$$(x_0, ..., x_T) \;=\; \{X_t = x_t \; \forall 0 \le t \le T\}$$

such that $x_T = \diamond$, and $x_t \neq \diamond, new$ for $t < T$. Let $\mathcal{D}$ be the set of such events, that is,

$$\mathcal{D}_T \;=\; \Big\{(x_0, ..., x_T) \;\Big|\; x_t \in \mathcal{X} \backslash \Big(\{\diamond\} \cup \{new\}\Big) \; \forall 0 \le t < T, \; x_T = \diamond\Big\}, \qquad \mathcal{D} \;=\; \bigcup_{t=0}^{\infty} \mathcal{D}_T$$

Then $\mathcal{D}$ is a partition of the event $\{\exists t > 0 : x_t = \diamond\}$; thus,

$$q_\sigma(a) \;=\; \sum_{D \in \mathcal{D}} \mu(D)$$

Let

$$\overline{\mathcal{D}} = \{D \in \mathcal{D} \mid \mu(D) > 0\}$$

Since the set $\mathcal{D}$ is countable, it follows that

$$q_\sigma(a) \;=\; \sum_{D \in \overline{\mathcal{D}}} \mu(D)$$

Define a mapping $\zeta : \overline{\mathcal{D}} \to \mathcal{L}$ as follows. For $D = \Big((s_0, \theta_0), (s_1, \theta_1), ..., (s_{T-1}, \theta_{T-1}), \diamond\Big) \in \overline{\mathcal{D}}$, $\zeta(D) = (l_1, ..., l_T) \in \mathcal{L}$ such that $l_t = (s_{t-1}, s_t, \theta_{t-1})$ for $t = 1, ..., T$. The mapping ζ is well-defined, since for $(x_0, ..., x_T) \in \overline{\mathcal{D}}$ we have $P^{x_t}_{x_{t-1}} > 0$, and hence, $(s_{t-1}, s_t, \theta_{t-1}) \in \mathcal{T}$ for all $t = 1, ..., T$; moreover, $\mu(D) > 0$ implies $\Big(P_o\Big)_{s_0, \theta_0} > 0$, hence

$s_0 = 1$. Next,

$$\mu(D) \;=\; \Big(P_o\Big)_{s_0,\theta_0} P^{s_1,\theta_1}_{s_0,\theta_0} ... P^{s_{T-1},\theta_{T-1}}_{s_{T-2},\theta_{T-2}} P^{\diamond}_{s_{T-1},\theta_{T-1}} \;=$$

$$f_s(\theta_0)(1-\eta)\tau(s_0,s_1,\theta_0) f_{s_1}(\theta_1) ... (1-\eta)\tau(s_{T-2},s_{T-1},\theta_{T-2}) f_{s_{T-1}}(\theta_{T-1})\tau(s_{T-1},s_T,\theta_{T-1}) =$$

$$(1-\eta)^{T-1} \prod_{t=1}^{T} f_{s_{t-1}}(\theta_{t-1})\tau(s_{t-1},s_t,\theta_{t-1}) \;=\; (1-\eta)^{T-1} \prod_{t=1}^{T} \omega(\zeta(D)_t) \;=\; \omega(\zeta(D))$$

The lemma follows. ■

I say that a valid path is simple if $t \neq r$ implies $s'_t \neq s'_r$, where $l_t = (s'_t, s_t, \theta_t)$, $l_r = (s'_r, s_r, \theta_r)$. Thus, a simple valid directed path goes via a particular state s at most one time. I denote by $\mathcal{L}_0 \subset \mathcal{L}$ the set of all simple valid directed paths.

For an arbitrary valid path $l \in \mathcal{L}$, denote by $\pi(l) = \Big(\pi(l)_1, ..., \pi(l)_{\overline{r}}\Big)$ the result of the application of the following recursive procedure π with steps $r = 1, 2, ..., \overline{r}$:

1. At the first step, $r = 1$, the procedure starts at state $s^1 = 1$;
2. At step r, assume the state $s^r \in S$ is given. For $l = (l_1, ..., l_T)$, define by t_r be maximum t such that the link l_t starts from the state s^r, that is,

$$t_r \;=\; \max\{t \in \{1, ..., T\} \mid s'_t = s^r \text{ for } l_t = (s'_t, s_t, \theta_t)\};$$

3. Define $\pi(l)_r = l_{t_r}$;
4. If the ending state of l_{t_r} is $s_{t_r} = \diamond$, then $r = \overline{r}$, and the procedure finishes;
5. If the ending state of l_{t_r} is $s_{t_r} \neq \diamond$, define $s^{r+1} = s_{t_r}$ and continue the procedure for step $r + 1$.

Lemma A.1.10 *Consider procedure π, defined above, then: (i) the procedure π always finishes at step $\overline{r} \leq T$, (ii) the result $\pi(l)$ is a simple valid path, i.e., $\pi(l) \in \mathcal{L}_0$, (iii) if $l \in \mathcal{L}_0$, then $\pi(l) = l$.*

Proof. Suppose r and $v = r + 1$ are two consecutive steps of the procedure. Since v is one of the steps, then $s^v = s^{r+1} \in S$. Moreover, $l_{t_r} = (s'_{t_r}, s_{t_r}, \theta_{t_r}) = (s'_{t_r}, s^{t_{r+1}}, \theta_{t_r})$, hence $l_{t_r+1} = (s^{t_{r+1}}, s_{t_r+1}, \theta_{t_r+1})$. It follows that $t_{r+1} \geq t_r + 1 > t_r$. Inductively, $v > r$ implies $t_v > t_r$. Since $t_r \in \{1, ..., T\}$, the procedure finishes after a finite number of steps $\overline{r} \leq T$, proving (i).

Next, since $s^1 = 1$, then $\pi(l)_1 = (1, s_{t_1}, \theta_{t_1})$, i.e., the first link of $\pi(l)$ begins at state $s = 1$. Since the procedure finishes at step $\overline{r}$ such that $s_{t_r} = \diamond$, then $\pi(l)_{\overline{r}} = (s'_{t_{\overline{r}}}, \diamond, \theta_{t_{\overline{r}}})$, i.e. the last link of $\pi(l)$ ends at $s = \diamond$. Consider $\pi(l)_r = (s'_{t_r}, s_{t_r}, \iota_{t_r})$. If $r < \overline{r}$, then $s^{r+1} = s_{t_r}$, hence $\pi(l)_{r+1} = (s_{t_r}, s_{t_{r+1}}, \theta_{t_{r+1}})$; thus, the next link of path $\pi(l)$ begins at the state at which the previous link ends. Thus, $\pi(l) \in \mathcal{L}$.

To show that $\pi(l) \in \mathcal{L}_0$, consider link $\pi(l)_r$ that begins at state s^r. Towards a contradiction, assume that $\pi(l)_{r'}$ for $r' > r$ begins at state $s^{r'} = s^r$ as well. Then $s'_{t_{r'}} = s^{r'} = s^r$, hence $t_{r'} \leq t_r$, in contradiction to the proven assertion $t_{r'} > t_r$. Statement (ii) is proven.

Suppose $l \in \mathcal{L}_0$. Then $l_t = (s'_t, s_t, \theta_t)$ implies $s'_v \neq s'_t$ for $v \neq t$. Then, inductively, $\pi(l)_r = l_r$ for all $r = 1, ..., T$, hence $\pi(l) = l$. ■

Lemma A.1.10 implies that π is a surjective mapping from $\mathcal{L}$ to $\mathcal{L}_0$. For $l \in \mathcal{L}_0$, denote by $\pi^{-1}(l)$ the pre-image of l with respect to π, that is,

$$\pi^{-1}(l) \;=\; \{l' \in \mathcal{L} \mid \pi(l') = l\}$$

Lemma A.1.11 *Let $k = |S|$ be the number of flexible memory states. Then for any $l \in \mathcal{L}_0$,*

$$\omega(l) \;\geq\; \eta^k \cdot \sum_{l' \in \pi^{-1}(l)} \omega(l')$$

Proof. We prove the lemma by induction in k.

First, suppose $k = 1$. The set of simple valid paths is then $\mathcal{L}_0 = \Big\{\{(1, \diamond, \theta)\} \;\Big|\; (1, \diamond, \theta) \in \mathcal{T}\Big\}$. Let $l = (l_1) \in \mathcal{L}_0$. For any path $l' = (l'_1, ..., l'_T) \in \pi^{-1}(l)$

we have $l'_T = l_1$ and $l_t = (1, 1, \theta_t)$ for all $t < T$. Then

$$\sum_{l' \in \pi^{-1}(l)} \omega(l') \;=$$

$$\begin{aligned}
&= \omega(l_1) \sum_{T=1}^{\infty} (1-\eta)^{T-1} \sum_{(\theta_1,...,\iota_{T-1})} \prod_{t=1}^{T-1} f_1(\theta_t)\tau(1,1,\theta_t) \\
&= \omega(l_1) \sum_{T=1}^{\infty} (1-\eta)^{T-1} \sum_{\theta_{T-1}} f_1(\theta_{T-1})\tau(1,1,\theta_{T-1}) \Big(\sum_{(\theta_1,...,\theta_{T-2})} \prod_{t=1}^{T-2} f_1(\theta_t)\tau(1,1,\theta_t) \Big) \;\leq \\
&\leq \omega(l_1) \sum_{T=1}^{\infty} (1-\eta)^{T-1} \Big(\sum_{(\theta_1,...,\theta_{T-2})} \prod_{t=1}^{T-2} f_1(\theta_t)\tau(1,1,\theta_t) \Big) \;\leq\; ... \;\leq \\
&\leq \omega(l_1) \sum_{T=1}^{\infty} (1-\eta)^{T-1} \;=\; \eta^{-1}\omega(l_1) \;=\; \eta^{-1}\omega(l)
\end{aligned}$$

where we used $\sum_{\theta \in \mathcal{I}} f_{s'}(\theta) = 1$ and $\tau(s', s, \theta) \leq 1$.

Second, suppose the lemma holds for $k = \overline{k}$, and consider $k = \overline{k} + 1$. Consider arbitrary $l \in \mathcal{L}_0$. Assume $|l| > 1$, then $l_1 = (1, v, \overline{\theta})$ for some $v \in S \backslash \{1\}$ and $\overline{\theta} \in \mathcal{I}$. Denote also $l = (l_1, ..., l_{\overline{T}})$. For $L \in \pi^{-1}(l)$, let

$$t_1(L) \;=\; \max\{t \in \{1, ..., T\} \mid s'_t = 1 \text{ for } L_t = (s'_t, s_t, \theta_t)\}$$

Thus, $t_1(L)$ is the last period when valid path $L \in \pi^{-1}(l)$ goes via state $s = 1$, and $t_1(L) = t_1$ when procedure π is applied to the path L.

For $L = (L_1, ..., L_T) \in \pi^{-1}(l)$, denote by $L_+ = (l_1, ..., L_{t_1(L)})$, and $L_- = (L_{t_1(L)+1}, ...L_T)$; thus, $L = (L_+, L_-)$. Let

$$\mathcal{L}_+(l) \;=\; \bigcup_{t=1}^{\infty} \Big\{ (\tilde{L}_1, ..., \tilde{L}_t) \;\Big|\; \exists L \in \pi^{-1}(l) : (\tilde{L}_1, ..., \tilde{L}_t) = L_+ \Big\}$$

and

$$\mathcal{L}_-(l) \;=\; \bigcup_{t=1}^{\infty} \Big\{ (\tilde{L}_1, ..., \tilde{L}_t) \;\Big|\; \exists L \in \pi^{-1}(l) : (\tilde{L}_1, ..., \tilde{L}_t) = L_- \Big\}$$

Claim A.1.12 *The following decomposition holds:*

$$\pi^{-1}(l) \;=\; \Big\{L \in \mathcal{L} \;\Big|\; \exists L^1 \in \mathcal{L}_+(l), L^2 \in \mathcal{L}_-(l) : L = (L^1, L^2)\Big\}$$

Proof of the Claim. For all paths $L \in \pi^{-1}(l)$, the last link of the sequence L_+ is $l_1 = (1, v, \theta)$, and the first link of sequence L_- starts from state v. It follows that if $L, \tilde{L} \in \pi^{-1}(l)$, then $(L_+, \tilde{L}_-) \in \pi^{-1}(l)$. Thus,

$$\pi^{-1}(l) \;\supseteq\; \Big\{L \in \mathcal{L} \;\Big|\; \exists L^1 \in \mathcal{L}_+(l), L^2 \in \mathcal{L}_-(l) : L = (L^1, L^2)\Big\}$$

To see that

$$\pi^{-1}(l) \;\subseteq\; \Big\{l \in \mathcal{L} \;\Big|\; \exists L^1 \in \mathcal{L}_+(l), L^2 \in \mathcal{L}_-(l) : L = (L^1, L^2)\Big\},$$

take $L^1 = L_+$ and $L^2 = L_-$ for $L \in \pi^{-1}(l)$. □

Claim A.1.13 *For $l \in \mathcal{L}_0$ and $\mathcal{L}_+(l)$ defined above,*

$$\sum_{L^1 \in \mathcal{L}_+(l)} (1-\eta)^{|L^1|-1} \cdot \prod_{t=1}^{|L^1|} \omega(L^1_t) \;\leq\; \eta^{-1} \cdot \omega(l_1)$$

Proof of the Claim. For $y = 1, 2, ...,$ denote by

$$\mathcal{L}^y_+(l) \;=\; \Big\{L^1 \in \mathcal{L}_+(l) \;\Big|\; |L^1| = y\Big\}$$

Then $\Big\{\mathcal{L}^y_+(l)\Big\}_{y=1,2,..}$ is a partition of $\mathcal{L}_+(l)$. Thus,

$$\sum_{L^1 \in \mathcal{L}_+(l)} (1-\eta)^{|L^1|} \cdot \prod_{t=1}^{|L^1|} \omega(L^1_t) \;=\; \sum_{y=1}^{\infty} (1-\eta)^y \cdot \sum_{L^1 \in \mathcal{L}^y_+(l)} \prod_{t=1}^{y} \omega(L^1_t)$$

Recall that $l_1 = (1, v, \theta) = L^1_y$ for all $L^1 \in \mathcal{L}^y_+(l)$. For any $L^1 = \Big((s^1_{t-1}, s^1_t, \theta^1_{t-1})\Big)_{t=1}^{y} \in \mathcal{L}^y_+(l)$, consider the event $\zeta^+(L^1)$ given by

$$\zeta^+(L^1) \;=\; \Big\{X_t = (s^1_t, \theta^1_t)\ \forall 0 \le t \le y-2,\ S_{y-1} = s^1_{y-1}\Big\}$$

Clearly, if $L^1, L^3 \in \mathcal{L}^y_+(l)$, $L^1 \neq L^3$, then $\zeta^+(L^1) \neq \zeta^+(L^3)$; moreover, $\zeta^+(L^1) \cap \zeta^+(L^3) = \varnothing$. Next, for $L^1 = \Big((s^1_{t-1}, s^1_t, \theta^1_{t-1})\Big)_{t=1}^{y} \in \mathcal{L}^y_+(l)$, if $y = 1$, then $\mu(\zeta^+(L^1)) = \mu(S_0 = 1) = 1$, and if $y > 1$, using the fact that $s^1_t \in S$ for all $t = 1, ..., y-1$ and the formula for the stochastic matrix P, we get

$$\mu(\zeta^+(L^1)) \;=\; f_1(\theta^1_0)\Big[\prod_{t=1}^{y-2}\Big((1-\eta)\tau(s^1_{t-1}, s^1_t, \theta^1_{t-1}) f_{s^1_t}(\theta^1_t)\Big)\Big](1-\eta)\tau(s^1_{y-1}, s^1_{y-1}, \theta^1_{y-2}) \;=$$

$$=\; (1-\eta)^{y-1}\prod_{t=1}^{y-1} f_{s^1_{t-1}}(\theta^1_{t-1})\tau(s^1_{t-1}, s^1_t, \theta^1_{t-1}) \;=$$

$$=\; (1-\eta)^{y-1}\prod_{t=1}^{y-1}\omega(s^1_{t-1}, s^1_t, \theta^1_{t-1}) = (1-\eta)^{y-1}\prod_{t=1}^{y-1}\omega(l^1)$$

Thus, for all $y = 1, 2, ...,$

$$\mu(\zeta^+(L^1)) \;=\; (1-\eta)^{y-1}\cdot\prod_{t=1}^{y-1}\omega(L^1)$$

Then, using $\zeta^+(L^1) \subset \{S_{y-1} = 1\}$ for all $L^1 \in \mathcal{L}^y_+(l)$, $\zeta^+(L^1) \cap \zeta^+(L^3) = \varnothing$ for $L^1 \neq L^3$ and Lemma A.1.8, we get

$$(1-\eta)^{y-1}\cdot\sum_{L^1\in\mathcal{L}^y_+(l)}\prod_{t=1}^{y-1}\omega(L^1_t) \;=\; \sum_{L^1\in\mathcal{L}^y_+(l)}\tilde{\mu}(\zeta^+(L^1)) \;\le\; \mu(S_{y-1}=1) \;\le\; (1-\eta)^{y-1}$$

It follows

$$\sum_{L^1\in\mathcal{L}_+(l)}(1-\eta)^{|L^1|-1}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t) \;\le\; \sum_{y=1}^{\infty}(1-\eta)^{y-1}\cdot\omega(L^1_y) \;=\; \eta^{-1}\cdot\omega(L_1)$$

proving the claim. □

Note that state v is the end of the link L_1, and it is the end of the link $L_{t_1(L)} = l_1$ for all $L \in \pi^{-1}(l)$. Let $\widehat{S} = \{1, ..., \overline{k}\} \cup \{\diamond\}$, and $\widehat{S}^o = \{1, ..., \overline{k}\}$. Define a mapping $\chi : \{2, ..., \overline{k}+1\} \cup \{\diamond\} \to \{1, ..., \overline{k}\} \cup \{\diamond\}\}$ as follows: $\chi(v) = 1$, $\chi(s) = s$ for all $s < v$, $\chi(s) = s - 1$ for all $s > v$, $\chi(\diamond) = \diamond$. Clearly, χ is a bijection. Let χ^{-1} be the inverse mapping, and

$$\widehat{\mathcal{T}}^a = \left\{(\widehat{r}, \widehat{s}, \theta) \in \widehat{S}^o \times \widehat{S} \times \mathcal{I} \;\middle|\; (\chi^{-1}(\widehat{r}), \chi^{-1}(\widehat{s}), \theta) \in \mathcal{T}^a\right\}$$

Let $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}_0$ be the set of valid paths and simple valid paths on $\widehat{S}$, $\widehat{\pi}$ be the corresponding mapping between $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}_0$, and $\widehat{\omega}$ be the corresponding weighting function, where

$$\widehat{\omega}(\widehat{r}, \widehat{s}, \theta) \;=\; f_{\chi^{-1}(\widehat{r})}(\theta) \cdot \tau(\chi^{-1}(\widehat{r}), \chi^{-1}(\widehat{s}), \theta) \;=\; \omega(\chi^{-1}(\widehat{r}), \chi^{-1}(\widehat{s}), \theta)$$

For a link $(s', s, \theta) \in \mathcal{T}^a$ such that $s', s \neq \{1\}$, define $\chi((s', s, \theta)) = (\chi(s'), \chi(s), \theta) \in \widehat{\mathcal{T}}^a$.

Claim A.1.14 *Mapping χ, applied to the set of links as defined above, is a bijection between the set $\{(s', s, \theta) \in \mathcal{T}^a \mid s', s \neq 1\}$ and the set $\widehat{\mathcal{T}}^a$.*

Proof of the Claim. The Claim follows from the definition of $\widehat{\mathcal{T}^a}$ and the fact that $\chi : \{2, ..., \overline{k}+1\} \to \{1, ..., \overline{k}\}$ is a bijection. □

For an ordered set of links $(L_{t'+1}, L_{t'+2}, ..., L_{T'})$ such that $(L_1, ..., L_{T'}) \in \mathcal{L}$, $L_{t'+1}$ begins at v, and L_t does not go via state $s = 1$ for $t > t'$, define $\chi((L_{t'+1}, L_{t'+2}, ..., L_{T'})) = (\chi(L_{t'+1}), \chi(L_{t'+2}), ..., \chi(L_{T'})) \in \widehat{\mathcal{L}}$.

Claim A.1.15 *The mapping χ, applied to the sequences of links as defined above, is a bijection between the set $\mathcal{L}_-(l)$ and the set $\widehat{\pi}^{-1}(\chi(L_2), ..., \chi(L_{\overline{T}}))$.*

Proof of the Claim. First, note that links $L_2, ..., L_{\overline{T}}$ do not go via $s = 1$, thus links $\chi(L_2), ..., \chi(L_{\overline{T}})$ are well-defined. The beginning of the link L_{t+1} is the end of the link L_t, thus the beginning of the link $\chi(L_{t+1})$ is the end of the link $\chi(L_t)$. Also, link L_2 starts at $s = v$, thus $\chi(L_2)$ starts at $\chi(v) = 1$, and $L_{\overline{T}}$ ends at $s = \diamond$, hence $\chi(L_{\overline{T}})$ ends at $\chi(\diamond) = \diamond$. Therefore, $(\chi(L_2), ..., \chi(L_{\overline{T}})) \in \widehat{\mathcal{L}}$. Next, since every $s \in S^o$ is a beginning of at most one of the links $L_2, ..., L_{\overline{T}}$, then every $\widehat{s} \in \widehat{S}^o$ is the beginning of at most one of the links $\chi(L_2), ..., \chi(L_{\overline{T}})$, thus $(\chi(L_2), ..., \chi(L_{\overline{T}})) \in \widehat{\mathcal{L}}_0$, and the set $\widehat{\pi}^{-1}(\chi(L_2), ..., \chi(L_{\overline{T}}))$ is well-defined.

Next, consider $L^2, L^3 \in \mathcal{L}_-(l)$ such that $\chi(L^2) = \chi(L^3)$. Then $|L^2| = |L^3| = K$ for some $K > 0$, and $\chi(L^2)_t = \chi(L^3)_t$ for all $t \in \{1, ..., K\}$, hence $\chi(L^2_t) = \chi(L^3_t)$ for all $t \in \{1, ..., K\}$, hence, by Claim A.1.14, $L^2_t = L^3_t$ for all $t \in \{1, ..., K\}$. Thus, $L^2 = L^3$, and χ is an injection from $\mathcal{L}_-(l)$ to $\widehat{\mathcal{L}}$.

Consider arbitrary $L^2 \in \mathcal{L}_-(l)$. For $r = 2, ..., \overline{T}$, let t_r be the maximum t such that L^2_t goes via state $(s')^r$ such that $L_r = ((s')^r, s^r, \theta^r)$. Then t_r is also the maximum t such that $\chi(L^2)_t$ goes via state $\chi((s')^r)$ such that $\chi(l)_r = (\chi((s')^r), \chi(s^r), \theta^r)$. It follows that $\chi(L^2) \in \widehat{\pi}^{-1}(\chi(l_2), ..., \chi(l_{\overline{T}}))$. Thus, χ is an injection from $\mathcal{L}_-(l)$ to $\widehat{\pi}^{-1}(\chi(l_2), ..., \chi(l_{\overline{T}}))$.

Finally, take arbitrary $L = (L_1, ..., L_T) \in \widehat{\pi}^{-1}(\chi(l_2), ..., \chi(l_{\overline{T}}))$ and consider $L^2 = (\chi^{-1}(L_1), ..., \chi^{-1}(L_T))$. For $r = 2, ..., \overline{T}$, let t_r be the maximum t such that $\chi(L^2)_t$ goes via state $\chi((s')^r)$ such that $\chi(l)_r = (\chi((s')^r), \chi(s^r), \theta^r)$, then t_r is also the maximum t such that L^2_t goes via state $(s')^r$ such that $l_r = ((s')^r, s^r, \theta^r)$. It follows that $L^2 \in \mathcal{L}_-(l)$. Since $\chi(L^2) = L$, then χ is also a surjection from $\mathcal{L}_-(l)$ to $\widehat{\pi}^{-1}(\chi(l_2), ..., \chi(l_{\overline{T}}))$, proving the claim. $\square$

Using Claims A.1.12, A.1.13, A.1.15 and the induction assumption, we get:

$$\sum_{L\in\pi^{-1}(l)}\omega(L) \;=\; \sum_{L^1\in\mathcal{L}_+(l)}\sum_{L^2\in\mathcal{L}_-(l)}\omega(L^1,L^2) \;=$$

$$= \sum_{L^1\in\mathcal{L}_+(l)}\sum_{L^2\in\mathcal{L}_-(l)}(1-\eta)^{|L^1|+|L^2|-1}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t)\cdot\prod_{t''=1}^{|L^2|}\omega(L^2_{t''}) =$$

$$= \sum_{L^1\in\mathcal{L}_+(l)}(1-\eta)^{|L^1|}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t)\cdot\Big[\sum_{L^2\in\mathcal{L}_-(l)}(1-\eta)^{|L^2|-1}\cdot\prod_{t''=1}^{|L^2|}\omega(L^2_{t''})\Big] \;=$$

$$= \sum_{L^1\in\mathcal{L}_+(l)}(1-\eta)^{|L^1|}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t)\cdot\Big[\sum_{\widehat{L}\in\widehat{\mathcal{L}}:\;\widehat{\pi}(\widehat{L})=(\chi(l_2),\ldots,\chi(l_{\overline{T}}))}(1-\eta)^{|\widehat{L}|-1}\cdot\prod_{t''=1}^{|\widehat{L}|}\widehat{\omega}(\widehat{L}_{t''})\Big] \;=$$

$$= \sum_{L^1\in\mathcal{L}_+(l)}(1-\eta)^{|L^1|}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t)\cdot\Big[\sum_{\widehat{L}\in\widehat{\pi}^{-1}(\chi(l_2),\ldots,\chi(l_{\overline{T}}))}\widehat{\omega}(\widehat{L})\Big] \;\leq$$

$$\leq \sum_{L^1\in\mathcal{L}_+(l)}(1-\eta)^{|L^1|}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t)\cdot\Big[\eta^{-\overline{k}}\cdot\omega'(\chi(l_2),\ldots,\chi(l_{\overline{T}}))\Big] \;\leq$$

$$\leq\; (1-\eta)\cdot\eta^{-1}\cdot\omega(l_1)\cdot\Big[\eta^{-\overline{k}}\cdot\widehat{\omega}(\chi(l_2),\ldots,\chi(l_{\overline{T}}))\Big] \;=$$

$$=\; (1-\eta)\cdot\eta^{-1}\cdot\omega(l_1)\cdot\Big[\eta^{-\overline{k}}\cdot(1-\eta)^{\overline{T}-2}\prod_{t=2}^{\overline{T}}\widehat{\omega}(\chi(l_t))\Big] \;=$$

$$=\; (1-\eta)\cdot\eta^{-1}\cdot\omega(l_1)\cdot\Big[\eta^{-\overline{k}}\cdot(1-\eta)^{\overline{T}-2}\prod_{t=2}^{\overline{T}}\omega(l_t)\Big] \;=$$

$$=\; \eta^{-\overline{k}-1}\cdot(1-\eta)^{\overline{T}-1}\prod_{t=1}^{\overline{T}}\omega(l_t) \;=\; \eta^{-\overline{k}-1}\cdot\omega(l)$$

which is the statement of the Lemma for $k=\overline{k}+1$.

When $l = \{l_1\}$, the same analysis applies, but this time $\mathcal{L}_-(l) = \varnothing$, and we can skip the usage of induction assumption and Claims A.1.14 and A.1.15. The Lemma is proven. ■

I now provide conditions on $\mathcal{L}_0$ and $\omega(\cdot)$ that are necessary and sufficient for a sequence of decision rules to solve the choice problem.

For an arbitrary alternative $a \in A$ and strategy $\sigma \in \Sigma$, define $l^*_\sigma(a) \in \mathcal{L}_0$ and $\omega^*_\sigma(a) \in [0,1]$ as follows:

$$\omega^*_\sigma(a) \;=\; \max_{l \in \mathcal{L}_0(a,\sigma)} \omega^a_\sigma(l), \qquad\qquad l^*_\sigma(a) = \arg\max_{l \in \mathcal{L}_0(a,\sigma)} \omega^a_\sigma(l)$$

where, if the set of maximizers of $\omega^a_\sigma(l)$ is not a singleton, then $l^*_\sigma(a)$ is chosen from this set according to a fixed total order on $\mathcal{L}_0$, that I don't introduce explicitly to ease the notations.

Lemma A.1.16 *A sequence decision rule $\{\sigma_r\}_{r=1,2,\ldots} \in \Psi$ solves choice problem $(Q, \succeq)$ if and only if the following conditions hold:*
(i) for all $a, b \in A$: $a \succ b$ implies $\omega^*_{\sigma_r}(b)/\omega^*_{\sigma_r}(a) \longrightarrow 0$;
(ii) for all $a \in A$, $\exists \overline{r}$: $\omega^*_{\sigma_r}(a) > 0 \;\forall r > \overline{r}$.

Proof. Since two links of a simple path cannot begin from the same state, the number of simple valid paths $|\mathcal{L}_0|$ is bounded from above by

$$K \;=\; \prod_{s \in S} \Big(\Big| \{(s,\theta) \in S \times \mathcal{I} \mid (s', s, \theta) \in \mathcal{T}\} \Big| + 1 \Big)$$

By Lemmas A.1.9, A.1.10, A.1.11 and the definition of $\omega^*_\sigma(a)$,

$$q_\sigma(a) \;=\; \sum_{l \in \mathcal{L}} \omega(l) \;=\; \sum_{l \in \mathcal{L}_0} \sum_{l' \in \pi^{-1}(l)} \omega(l') \;\leq\; \eta^{-k} \cdot \sum_{l \in \mathcal{L}_0} \omega(l) \;\leq\; \eta^{-k} \cdot K^{-1} \cdot \omega^*_\sigma(a)$$

From the other hand,

$$\omega^*_\sigma(a) \;=\; \omega(l^*_\sigma(a)) \;\leq\; \sum_{l\in\mathcal{L}}\omega(l) \;=\; q_\sigma(a)$$

Hence,

$$\omega^*_\sigma(a) \;\leq\; q_\sigma(a) \;\leq\; \eta^{-k}\cdot K^{-1}\cdot\omega^*_\sigma(a)$$

Since η^{-k} and K^{-1} are constant along the sequence of strategies σ_r that comprises the decision rule—recall that the set $\mathcal{T}$ remains constant for all σ_r—then $q_{\sigma_r}(b)/q_{\sigma_r}(a) \longrightarrow 0$ if and only if $\omega^*_{\sigma_r}(b)/\omega^*_{\sigma_r}(a) \longrightarrow 0$ and $\exists\overline{r}$: $\;q_{\sigma_r}(a) > 0\;\forall r > \overline{r}$ if and only if $\exists\overline{r}$: $\;\omega^*_{\sigma_r}(a) > 0\;\forall r > \overline{r}$. The statement of the Lemma then follows from Lemma A.1.7. ■

For a state $s \in S^o$, denote by $\mathcal{T}(s)$ the subset of links that begin in state s and end in some other state $v \neq s$; I omit index σ from $\mathcal{T}_\sigma$ and $\mathcal{T}_\sigma(s)$ when it does not cause confusion. Thus,

$$\mathcal{T}(s) \;=\; \Big\{(s',v,\theta)\in\mathcal{T} \mid s'=s, v\neq s\Big\} \tag{A.8}$$

and define similarly $\mathcal{T}^a(s)$:

$$\mathcal{T}^a(s) \;=\; \Big\{(s',v,\theta)\in\mathcal{T}^a \mid s'=s, v\neq s\Big\}$$

where I omit index σ for brevity as well. Note that

$$\mathcal{T}(s) \;=\; \bigcup_{a\in A}\mathcal{T}^a(s), \qquad \mathcal{T} \;=\; \bigcup_{a\in A}\mathcal{T}^a$$

Define also sets

$$\overline{\mathcal{T}}(s) \;=\; \mathcal{T}(s)\cup\{\varnothing\}, \qquad \mathbb{T} \;=\; \times_{s\in S}\,\overline{\mathcal{T}}(s) \tag{A.9}$$

and, similarly,

$$\overline{\mathcal{T}}^a(s) \;=\; \mathcal{T}^a(s) \cup \{\varnothing\}, \qquad\qquad \mathbb{T}^a \;=\; \times_{s\in S}\, \overline{\mathcal{T}}^a(s)$$

I interpret $\overline{\mathcal{T}}(s)$ as a set of links outgoing from state s that could potentially occur in a simple valid path $l \in \mathcal{L}(a,\sigma)$, where $\varnothing \in \overline{\mathcal{T}}(s)$ stands for path l not going via state s. For an element $e = (e_1, ..., e_{|S|}) \in \mathbb{T}$, define its weight by

$$\omega_\sigma(e) \;=\; \prod_{s\in S} \omega_\sigma(e_s), \qquad \text{where} \quad \begin{cases} \omega_\sigma(e_s) = \omega_\sigma(s', s, \theta) & \text{if} \quad e_s = (s', s, \theta) \\ \omega_\sigma(e_s) = 1 & \text{if} \quad e_s = \varnothing \end{cases}$$

I say that a decision rule $\{\sigma_r\}_{r=1,2,...} \in \Psi$ is *regular*, if the following two conditions hold:

(i) For all $a \in A$ for all r, r', $l^*_{\sigma_r}(a) = l^*_{\sigma_r}(a)$;

(ii) For any $e, e' \in \mathbb{T}$, there exists $\lim_{k\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e') \in [0,\infty) \cup \{\infty\}$

I denote by $\Psi_0 \subset \Psi$ the set of regular decision rules.

Lemma A.1.17 *Every decision rule $\{\sigma_r\}_{r=1,2,...} \in \Psi$ contains a regular decision rule as a subsequence.*

Proof. The Lemma follows from the finiteness of sets A, $\mathcal{L}_0(a,\sigma)$ and $\mathbb{T}$. ■

Given a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,...} \in \Psi_0$, define the following auxilliary binary relation $\hat{\succeq} \subseteq \mathbb{T} \times \mathbb{T}$ (where $\mathbb{T} = \mathbb{T}_\psi$ is given by eq. (A.9) with the state space $S = S_\psi$ and the set of transitions $\mathcal{T} = \mathcal{T}_\psi$):

$$e \;\hat{\succeq}_\psi\; e' \quad \Longleftrightarrow \quad \lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e') > 0 \tag{A.10}$$

where I use the convention $\{\infty\} > 0$.

Lemma A.1.18 *Let $\psi = \{\sigma_r\}_{r=1,2,...} \in \Psi_0$ be a regular decision rule, then the binary relation $\hat{\succeq}_\psi$ defined above, is complete and transitive.*

Proof. Since $\psi \in \Psi_0$, then for any $e, e' \in \mathbb{T}$, there exist the limits of the ratios $\omega_{\sigma_r}(e)/\omega_{\sigma_r}(e')$, $\omega_{\sigma_r}(e')/\omega_{\sigma_r}(e)$, and at least one of them is not zero. If $\lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e') > 0$ and $\lim_{r\to\infty} \omega_{\sigma_r}(e')/\omega_{\sigma_r}(e'') > 0$, then

$$\lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e'') \;=\; \Big(\lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e')\Big) \cdot \Big(\lim_{r\to\infty} \omega_{\sigma_r}(e')/\omega_{\sigma_r}(e'')\Big)$$

implying $\lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e'') > 0$, thus, $\hat{\succeq}_\psi$ is transitive. ■

Denote by symbols $\hat{\sim}_\psi$ and $\hat{\succ}_\psi$ the symmetric and asymmetric parts of $\hat{\succeq}_\psi$. Note that

$$e \;\hat{\sim}_\psi\; e' \quad\Longleftrightarrow\quad \lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e') \in (0,\infty),$$

$$e \;\hat{\succ}_\psi\; e' \quad\Longleftrightarrow\quad \lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e') = \infty \quad\Longleftrightarrow\quad \lim_{r\to\infty} \omega_{\sigma_r}(e')/\omega_{\sigma_r}(e) = 0$$

Define a mapping $\phi : A \to \mathbb{T}$ as follows:

$$\phi_s(a) \;=\; \begin{cases} (s', s, \theta) & \text{if} \quad (s', s, \theta) \text{ is a link of } l^*_\psi(a) \\ \varnothing & \text{if} \quad \nexists (s, \theta) : \; (s', s, \theta) \text{ is a link of } l^*_\psi(a) \end{cases}$$

Note that $\phi(\cdot)$ is well-defined, since there is at most one link (s', s, θ) outgoing from a state s' in a simple path $l^*_\psi(a)$.

Lemma A.1.19 *If a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,...} \in \Psi_0$ solves choice problem $(Q, \succeq)$, then $a \succ b$ implies $\phi(a) \;\hat{\succ}_\psi\; \phi(b)]$ for all $a, b \in A$.*

Proof. Consider arbitrary $a, b \in A$ such that $a \succeq b$. By Lemma A.1.16, $\omega^*_{\sigma_r}(a)/\omega^*_{\sigma_r}(b) \longrightarrow 0$. Thus,

$$\lim_{r\to\infty} \frac{\omega_{\sigma_r}(\phi(b))}{\omega_{\sigma_r}(\phi(a))} = \lim_{r\to\infty} \frac{\prod_{s\in S} \omega(\phi(b))}{\prod_{s\in S} \omega(\phi(a))} = \lim_{r\to\infty} \frac{\prod_{t=1}^{|l^*(b)|} \omega(l^*(b)_t)}{\prod_{t'=1}^{|l^*(a)|} \omega(l^*(a)_{t'})} = \lim_{r\to\infty} \cdot \frac{(1-\eta)^{|l^*(a)|} \cdot \omega(l^*(b))}{(1-\eta)^{|l^*(b)|} \cdot \omega(l^*(a))} = 0$$

It follows that $\phi(a) \hat{\succ}_{\psi} \phi(b)$. ■

Note that if a decision rule $\psi = \{\sigma_r\}_{r=1,2,...}$ is regular, then for any link (s', s, θ), there exists $\lim_{r\to\infty} \omega(s', s, \theta) \in [0, 1]$—to see this, notice that $\omega(\varnothing, ..., \varnothing) = 1$ for $(\varnothing, ..., \varnothing) \in \mathbb{T}$, and $e \in \mathbb{T}$, where $e_{s''} = \varnothing$ if $s'' \neq s$, and $e_s = (s', s, \theta)$.

Thus, for a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,...}$, I call a link (s', s, θ) *weak* if $\lim_{r\to\infty} \omega(s', s, \theta) = 0$; otherwise, I call a link *strong*. Thus, $\lim_{r\to\infty} \omega(s', s, \theta) \in (0, 1]$ for a strong link (s', s, θ). Define also $\varnothing \in \overline{\mathcal{T}}(s)$ for any s to be a strong link as well. Denote by

$$\mathcal{T}^{weak}(s) \;=\; \Big\{(s, v, \theta) \in \mathcal{T}(s) \;\Big|\; \lim_{r\to\infty} \omega(s, v, \theta) = 0\Big\}, \qquad \mathcal{T}^{weak} \;=\; \bigcup_{s\in S} \mathcal{T}^{weak}(s)$$

$$\mathcal{T}^{strong}(s) \;=\; \Big\{(s, v, \theta) \in \mathcal{T}(s) \;\Big|\; \lim_{r\to\infty} \omega(s, v, \theta) \in (0, 1]\Big\}, \qquad \mathcal{T}^{strong} \;=\; \bigcup_{s\in S} \mathcal{T}^{weak}(s)$$

Note that $\Big\{\mathcal{T}^{weak}(s), \mathcal{T}^{strong}(s)\Big\}$ is a partition of $\mathcal{T}(s)$.

For an arbitrary alternative $a \in A$, define

$$\mathcal{F}(a) \;=\; \bigcup_{s\in S:\ \phi_s(a)\in\mathcal{T}^{weak}} \phi_s(a)$$

Thus, the set $\mathcal{F}(a)$ consists of all weak links that are used in a simple path $l^*(a)$ that maximizes the probability to go from state 1 to state $\diamond$ among all simple paths for a. Denote also by

$$S^{weak}(a) \;=\; \{s \in S \mid (l^*(a))_s \in \mathcal{T}^{weak}\}$$

Thus, $S^{weak}(a)$ is the set of states $s \in S$ such that there is a link in the path $l^*(a)$, outgoing from state s, and this link is weak.

Lemma A.1.20 *Let a regular decision rule $\psi = \{\sigma_r\}_{r\in\mathbb{N}} \in \Psi_0$ be given. Suppose $e, e' \in \mathbb{T}_\psi$ be such that for all $s \in S$ if $e_s \in \mathcal{T}^{weak}(s)$ or $e'_s \in \mathcal{T}^{weak}(s)$, then $e'_s = e_s$. Then $e \hat{\sim}_\psi e'$.*

Proof. Note that $e_s \in \mathcal{T}^{weak}(s)$ if and only if $e'_s \in \mathcal{T}^{weak}(s)$. Let $\tilde{S} = \{s \in S | e_s \notin \mathcal{T}^{weak}(s)\}$. Then

$$\begin{aligned}\lim_{r\to\infty} \frac{\omega(e)}{\omega(e')} &= \lim_{r\to\infty} \prod_{s\in S\setminus\tilde{S}} \frac{\omega(e_s)}{\omega(e'_s)} \cdot \prod_{s\in\tilde{S}} \frac{\omega(e_s)}{\omega(e'_s)} = \lim_{r\to\infty} \prod_{s\in\tilde{S}} \frac{\omega(e_s)}{\omega(e'_s)} = \\ &= \prod_{s\in\tilde{S}} \frac{\lim_{r\to\infty}\omega(e_s)}{\lim_{r\to\infty}\omega(e'_s)} \in (0,\infty)\end{aligned}$$

since $\lim_{r\to\infty}\omega(e_s), \lim_{r\to\infty}\omega(e'_s) \in (0,1]$ for all $s \in \tilde{S}$. The statement of the Lemma follows from the definition of $\hat{\succeq}_\psi$. ■

Lemma A.1.21 *If a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,\ldots} \in \Psi_0$ solves the choice problem $(Q, \succeq)$, then for any $a, b \in A$, $a \succ b$ implies $\mathcal{F}(a) \neq \mathcal{F}(b)$.*

Proof. Towards a contradiction, assume that $a \succ b$, but $\mathcal{F}(a) = \mathcal{F}(b)$; hence, $S^{weak}(a) = S^{weak}(b) = \overline{S}$ for some $\overline{S} \subseteq S$. Then

$$\begin{aligned}\lim_{r\to\infty} \frac{\omega^*_{\sigma_r}(b)}{\omega^*_{\sigma_r}(a)} &= \lim_{r\to\infty} \prod_{s\in\overline{S}} \frac{\omega(\phi_s(b))}{\omega(\phi_s(a))} \cdot \prod_{s\in S\setminus\overline{S}} \frac{\omega(\phi_s(b))}{\omega(\phi_s(a))} = \lim_{r\to\infty} \prod_{s\in S\setminus S^o} \frac{\omega(\phi_s(b))}{\omega(\phi_s(a))} = \\ &= \prod_{s\in S\setminus\overline{S}} \frac{\lim_{r\to\infty}\omega(\phi_s(b))}{\lim_{r\to\infty}\omega(\phi_s(a))} > 0\end{aligned}$$

where we used $\phi_s(b) = \phi_s(a)$ for $s \in \overline{S}$, and $\lim_{r\to\infty}\omega(\phi_s(b)) > 0$, $\lim_{r\to\infty}\omega(\phi_s(a)) > 0$ for $s \in S\setminus\overline{S}$. Therefore, $\phi(b) \hat{\succeq}_\psi \phi(a)$, in contradiction to Lemma A.1.19. ■

Lemma A.1.22 *For a regular decision rule $\psi \in \Psi_0$, the number of indifference classes of the preference relation $\hat{\succeq}_\psi$ defined by eq. (A.10) is less than or equal to $\prod_{s\in S}\left(|\mathcal{T}^{weak}_\psi(s)| + 1\right)$.*

Proof. Consider a subset $\hat{\mathbb{T}} \subseteq \mathbb{T}$ given by $\hat{\mathbb{T}} = \times_{s \in S}\Big(\mathcal{T}^{weak}(s) \cup \{\varnothing\}\Big)$. By Lemma A.1.20, every element $e \in \mathbb{T}$ is indifferent to some element $e' \in \hat{\mathbb{T}}$. Since $|\hat{\mathbb{T}}| = \prod_{s \in S}\Big(|\mathcal{T}^{weak}_{\psi}(s)| + 1\Big)$, the Lemma follows. ■

A.2 Proofs for the baseline model

In this section, as well as in the rest of the Appendix, I consider a setup of the baseline model; that is, the interrogation rule is deterministic; and also $\iota : S \to N$. Thus, the agent always asks a question in a state. Next, the agent cannot draw a new alternative intentionally. Thus, the stochastic transition rule is $\tau : S^o \times \{0,1\} \to \triangle(S \cup \{\diamond\})$.

A.2.1 Other Combinatorics statements

Lemma A.2.1 *For $k \in \mathbb{N}$, let functions $f^k, g^k : (\mathbb{N} \cup \{0\})^k \to \mathbb{N}$ be given by*

$$f^k(x_1, ..., x_k) \;=\; \prod_{i=1}^{k}(x_i + 1), \qquad g^k(x_1, ..., x_k) \;=\; \sum_{i=1}^{k} x_i$$

If $x_i = 0$, $x_j > 1$ for some $i, j \in \{1, ..., k\}$ then $f^k(x') \geq (4/3) \cdot f^k(x)$ and $g^k(x') = g^k(x')$, where $x'_l = x_l$ for $l \neq i, j$, $x'_i = 1$, $x'_j = x_j - 1$.

Proof. Let x, x' are as above, then $f^k(x') = f^k(x) \cdot 2 \cdot (x_j/(x_j + 1)) \geq (4/3) \cdot f^k(x)$, since $(x_j/(x_j + 1)) \geq 2/3$ for $x_j \geq 2$. ■

Lemma A.2.2 *Let the preference relation $\succeq$ have m indifference classes, and a decision rule $\psi = \{\sigma_r\}_{r=1,2,...} \in \Psi$ solves choice problem $(Q, \succeq)$. Let $Q' \subseteq Q$ be a collection of questions that are asked with positive probability in some state $s \in S_\psi$. Then $|Q'| \geq \lceil \log_2(m) \rceil$.*

Proof. Pick one representative a^k from each indifference class of the preference relation $\succeq$. Towards a contradiction, assume that $|Q'| < n \equiv \lceil \log_2(m) \rceil$. Then the maximum number of different vectors of binary answers is bounded above by 2^{n-1}. Since $m > 2^{n-1}$, then by the pigeonhole principle, there is a couple of alternatives a^k, $a^{k'}$ such that $a^k_i = a^{k'}_i$ for all i in the index set of Q'. It follows that all probabilities of transitions between the states of the automaton ψ are equal for a and b. Therefore, $q_\sigma(a) = q_\sigma(b)$ and hence, by Lemma A.1.7, ψ does not solve $(Q, \succeq)$, contradiction. ■

Lemma A.2.3 *Let $\succeq$ have m indifference classes and $n = \lceil \log_2(m) \rceil$. Suppose that the decision rule $\psi = \{\sigma_r\}_{r=1,2,\ldots} \in \Psi$ solves the choice problem $(Q, \succeq)$, then $\kappa(\psi) \geq 3n$.*

Proof. Assume first that a decision rule $\psi = \{\sigma_r\}_{r=1,2,\ldots} \in \Psi$ solves the choice problem $(Q, \succeq)$. By Lemma A.1.17, there is a regular subsequence $\psi_0 \subseteq \psi$ that also solves $(Q, \succeq)$. Without loss of generality, assume $\psi_0 = \psi$. Pick one representative a^k from each indifference class of the preference relation $\succeq$. By Lemma A.1.21, the sets of weak links corresponding to the alternatives from different indifference classes should be distinct, that is, $\mathcal{F}(a^k) \neq \mathcal{F}(a^{k'})$ for $k \neq k'$. Since $\mathcal{F}(a) \subseteq \mathcal{T}^{weak}$ for all $a \in A$, then by the pigeonhole principle, $2^{|\mathcal{T}^{weak}|} \geq m$, hence $|\mathcal{T}^{weak}| \geq \log_2(m)$, implying $|\mathcal{T}^{weak}| \geq \lceil \log_2(m) \rceil$, since $|\mathcal{T}^{weak}|$ is an integer.

Next, by Lemma A.2.2, there should be at least $n = \lceil \log_2(m) \rceil$ questions being asked in different states of S_ψ. Each question can have one of the two answers: 1 or 0, and, conditional on receiving one of the answers, there should be at least one transition with positive in the limit probability to some other (or the same) state—otherwise transitional probabilities from the corresponding state conditional on answer received would not sum up to one. Recall that, given a regular sequence of decision rules $\psi\{\sigma_r\}_{r=1,2,\ldots} \in \Psi_0$, a link $(s, s', \iota) \in \mathcal{T}$ is strong if the corresponding

transitional probability converges to non-zero number. It follows that the number of strong links is bounded above by $|\mathcal{T}^{strong}| \geq 2 \cdot n = 2 \cdot \lceil \log_2(m) \rceil$. Then

$$\kappa(\psi) \;=\; |\mathcal{T}_\psi| \;=\; |\mathcal{T}_\psi^{weak}| + |\mathcal{T}_\psi^{strong}| \;\geq\; \lceil \log_2(m) \rceil + 2 \cdot \lceil \log_2(m) \rceil \;=\; 3 \cdot \lceil \log_2(m) \rceil$$

proving that the lower bound on complexity of an arbitrary language Q is $3 \cdot \lceil \log_2(m) \rceil$.

■

A.2.2 Universal decision rule

In this section, I introduce a universal decision rule that serves for the proof of Lemma 1.2.2 and provides an upper bound for Theorem 1.3.4.

For an arbitrary preference relation $\succeq$ on A and language Q, adequate for $\succeq$, consider the automaton $\overline{\psi} \in \Psi$, defined via the recursive procedure $\mathcal{R}$ described below; I write $\overline{\psi} = \mathcal{R}(Q, \succeq)$ for the resulting automaton.

Recall that $C(a) = \{a' \in A | a' \sim a\}$ is the indifference class of alternative a with respect to $\succeq$. Define $m(a)$ to be the number of indifference classes C of $\succeq$ such that $a' \succeq a$ for any $a' \in C$; thus, $m(a) = 1$ for any first-best alternative, $m(b) = 2$ for any second-best alternative, etc. Take some parameter $\epsilon \in (0, 1)$—we will consider $\epsilon \longrightarrow 0$ later on.

Procedure $\mathcal{R}$: define recursively by steps $r = 1, 2, ...$ the following sets, interrogation and stochastic transition rules, and auxiliary functions

1. For $r = 1$, define $s(r) = 1$, $\mathcal{K}^1 = \{1\}$, $S^1 = \{1\}$, $\mathcal{J}^1 = \varnothing$, $V^1 = \varnothing$, $A^1 = A$.
2. For $r \geq 1$, if $\mathcal{K}^r = \varnothing$, finish the procedure. Otherwise, define $s(r) = \min \mathcal{K}^r$, denote by $s = s(r)$ and assume that sets $\mathcal{J}^s$, V^s and $A^s \subseteq A$, such that $|A^s| > 1$ are given.

Case $\alpha: A^s = \{a, b\}$ for some $a, b \in A$.

Let $i \in \{1, ..., N\}$ be the smallest index such that $a_i \neq b_i$. Define

$$\begin{aligned}
type(s) &= \alpha \\
\iota(s) &= i \\
\tau(s, \diamond, a_i) &= \epsilon^{m(a)-1} \\
\tau(s, s, a_i) &= 1 - \epsilon^{m(a)-1} \\
\tau(s, \diamond, b_i) &= \epsilon^{m(b)-1} \\
\tau(s, s, b_i) &= 1 - \epsilon^{m(b)-1} \\
\mathcal{K}^{r+1} &= \mathcal{K}^r \backslash \{s\} \\
S^{r+1} &= S^r
\end{aligned}$$

and repeat the procedure for step $r + 1$.

Case $\beta: |A^s| > 2$, and $\exists i \in N$, $a \in A^s$ such that $a_i \neq b_i$ for all $b \in A^s \backslash \{a\}$.

Let $s' = \max S^r$, and let $i \in N$ be the smallest index such that there exists $a \in A^s$ such that $a_i \neq b_i$ for all $b \in A^s \backslash \{a\}$. Define

$$\begin{aligned}
type(s) &= \beta \\
\iota(s) &= i \\
\tau(s, \diamond, a_i) &= \epsilon^{m(a)-1} \\
\tau(s, s, a_i) &= 1 - \epsilon^{m(a)-1} \\
\tau(s, s' + 1, 1 - a_i) &= 1 \\
\mathcal{K}^{r+1} &= \Big(\mathcal{K}^r \backslash \{s\}\Big) \cup \{s' + 1\} \\
S^{r+1} &= S^r \cup \{s' + 1\} \\
\mathcal{J}^{s'+1} &= \mathcal{J}^s \cup \{(i, 1 - a_i)\} \\
V^{s'+1} &= V^s \cup \{s\} \\
A^{s'+1} &= A^s \backslash \{a\}
\end{aligned}$$

and repeat the procedure for step $r + 1$.

Case $\gamma|A^s| > 2$, and $\nexists i \in N$, $a \in A^s$ such that $a_i \neq b_i$ for all $b \in A^s \backslash \{a\}$.

Let $s' = \max S^r$, and let $i \in N$ be the smallest index such that there exists $a, b \in A^s$ such that $a_i \neq b_i$. Define

$$\begin{aligned}
type(s) &= \gamma \\
\iota(s) &= i \\
\tau(s, s'+1, 1) &= 1 \\
\tau(s, s'+2, 0) &= 1 \\
\mathcal{K}^{r+1} &= \left(\mathcal{K}^r \backslash \{s\}\right) \cup \{s'+1\} \cup \{s'+2\} \\
S^{r+1} &= S^r \cup \{s'+1\} \cup \{s'+2\} \\
\mathcal{J}^{s'+1} &= \mathcal{J}^s \cup \{(i,1)\} \\
\mathcal{J}^{s'+2} &= \mathcal{J}^s \cup \{(i,0)\} \\
V^{s'+1} &= V^s \cup \{s\} \\
V^{s'+2} &= V^s \cup \{s\} \\
A^{s'+1} &= \{a \in A^s | a_i = 1\} \\
A^{s'+2} &= \{a \in A^s | a_i = 0\}
\end{aligned}$$

and repeat the procedure for step $r+1$.

3. If the procedure finishes at step $\overline{r}$, define $S = S^{\overline{r}}$.

Thus, the procedure $\mathcal{R}$ defines an automaton with a binary tree structure, where the vertexes are the states $s \in S$; for each s, the set V^s encodes all predecessors of s up to the root $s = 1$, the set A^s is the set of alternatives for which the automaton can reach state s, the set $\mathcal{J}$ is the set of signals that partitions A into A^s and $A \backslash A^s$. The sets $\mathcal{K}^r$ and S^r with $r < \overline{r}$ are auxiliary for the procedure: the set S^r is a set of states that have been introduced by the procedure, and the set $\mathcal{K}^r \subseteq S^r$ is the set of states such that the procedure hasn't applied one of the cases α, β or γ yet and, in particular, hasn't defined the interrogation rule g and transition rule τ. Finally,

type $\in \{\alpha, \beta, \gamma\}$ stands for the type of a vertex that is defined by the corresponding condition for cases α, β, γ.

Lemma A.2.4 *The decision rule* $\psi = \{(S, g, \tau)_\epsilon\}_{\epsilon \to 0}$*, where* $(S, g, \tau)_\epsilon = \mathcal{R}(Q, \succeq)$ *is well-defined:*

(i) If $s = s(r)$ *for some step* r*, then* $\mathcal{J}^s$*,* V^s*,* $A^s \subseteq A$ *have been already defined, and* $|A^s| > 1$*;*

(ii) The cases α*,* β*,* γ *are mutually exclusive and exhaust all possibilities;*

(iii) For each of the cases α*,* β*,* γ*, the index* i *is well-defined;*

(iv) The procedure finishes after a finite number of steps $\overline{r}$*;*

(v) The interrogation rule ι *and the stochastic transition rule* τ *are defined for all states* $s \in S$*;*

(vi) The state space S*, the interrogation rule* ι *and the set of positive-probability transitions* $\mathcal{T}$ *do not depend on* $\epsilon \in (0, 1)$*.*

Proof. The proofs of statements (i),(ii),(v),(vi) are straightforward, but bulky, and omitted. The statement (iii) is true since the language Q is adequate for the preference relation $\succeq$. To prove statement (iv), consider

$$h(r) \;=\; \sum_{s \in \mathcal{K}^r} 10^{|A^s|}$$

Let us show that $h(r+1) < h(r)$ if the procedure was not finished at steps r and $r+1$. Suppose Case α realized at step r, then $\mathcal{K}^{r+1} = \mathcal{K}^r \backslash \{s\}$, thus $h(r) - h(r+1) = 10^{|A^s|} > 0$. Suppose Case α realized at step r, then $\mathcal{K}^{r+1} = \Big(\mathcal{K}^r \backslash \{s\}\Big) \cup \{s'+1\}$ and $A^{s'+1} = A^s \backslash \{a\}$, hence $h(r) - h(r+1) = 10^{|A^s|} - 10^{|A^s \backslash \{a\}|} > 0$. Suppose Case γ realized at step r, then $\mathcal{K}^{r+1} = \Big(\mathcal{K}^r \backslash \{s\}\Big) \cup \{s'+1\} \cup \{s'+2\}$, $A^{s'+1} = \{a \in A^s | a_i = 1\}$, $A^{s'+2} = \{a \in A^s | a_i = 0\}$. Since function h' given by $h'(x) = 10^x$ is strictly convex, then $h(r) - h(r+1) = 10^{|A^{s'+1}| + |A^{s'+2}|} - 10^{|A^{s'+1}|} - 10^{|A^{s'+2}|} > 0$.

Therefore, h is a decreasing integer-valued function of the step r. It follows that $r \leq h(1) = 10^{|A|}$ for any step r of procedure $\mathcal{R}$, proving statement (iv). ■

Lemma A.2.5 *Let $\psi = \{\sigma_\epsilon\}_{\epsilon\to 0} = \mathcal{R}(Q, \succeq)$ be the decision rule, constructed via procedure $\mathcal{R}$. Then:*

(i) ψ solves the choice problem $(Q, \succeq)$.

(ii) $\kappa(\psi) = 3|A| - 2 - x$, where $x = |\{a \in A | a \succeq a' \ \forall a' \in A\}|$;

(iii) $|S_\psi| = |A| - 1$.

Proof. First, let us prove statement (i). Note that for an arbitrary alternative $b \in A$, there is a unique valid simple path $l = l^*(b)$; indeed, for each memory state of the automaton that is reached with positive probability while investigating $a \in A$, there is only one link to the other state. Next, this path goes via states for which the procedure $\mathcal{R}$ applies either case γ or case α for some $a \neq b$; in this cases, the probability of a transition to the next state in the path is 1. Finally, finally $l^*(b)$ reaches a state s' such that either case β applies, or case α applies with $a = b$. In both cases, the last link in the path $l^*(b)$ has probability $\tau_{s'}(\diamond, g(s'), b_{g(s')}) = \epsilon^{m(b)-1}$. Thus, for all $b \in A$,

$$\omega^*(b) \ = \ \epsilon^{m(b)-1}$$

By Lemma A.1.16, the decision rule $\psi = \mathcal{R}(Q, \succeq)$ solves the choice problem $(Q, \succeq)$, proving statement (i).

Second, let us prove statement (ii). Note that $type(s) = \alpha$ if and only if there are no $s' \in S$ such that $s \in V^{s'}$; thus, states where case α applies are leafs of the binary tree of transitions between the flexible memory states, and only such states are leafs. For a state $s \in S$, denote by $\xi(s)$ the maximum length of the path to the state $\diamond$; in other words, define $\xi(s)$ recursively as follows:

(1) $\xi(s) = 1$ for all s such that $type(s) = \alpha$;

(2) if $type(s) = \beta$ or $type(s) = \gamma$, then $\xi(s) = \max_{v:\ s \in V^v} \xi(s) + 1$.

Next, for $s \in S_\psi$, define

$$D(s) \ = \ \bigcup_{s':\ s \in V(s)} |\mathcal{T}_\psi(s)|, \qquad \overline{k}(B) \ = \ \{a \in B \mid m(a) = 1\}$$

Thus, $D(s)$ is the number of transitions outgoing from the states of the branch of the tree that starts at state s, and $\overline{k}(A^s)$ is the number of first-best alternatives that reach this branch of the tree.

Claim A.2.6 *If $D(s)$ and $\overline{k}(A^s)$ are as defined above, then:*

$$D(s) = 3|A^s| - 2 - \overline{k}(A^s)$$

Proof of the Claim. First, note that the Claim holds for all s such that $\xi(s) = 1$. Indeed, if $\theta(s) = \alpha$, then $D(s) = 4$ if $m(a), m(b) > 1$, $D(s) = 3$ if $m(a) = 1$, $m(b) > 1$ or if $m(a) > 1$, $m(b) = 1$, and $D(s) = 2$ is $m(a) = m(b) = 1$. Next, assume that the Claim holds for all s such that $\xi(s) \leq \overline{\xi}$, and consider arbitrary state with $\xi(s) = \overline{\xi}+1$.

Consider the case when $type(s) = \beta$, then $|\mathcal{T}(s)| = 3$ if $m(a) > 1$ and $|\mathcal{T}(s)| = 2$ if $m(a) = 1$. Therefore,

$$D(s) \ = \ 3 - \overline{k}(\{a\}) + D(v) \ = \ 3 - \overline{k}(\{a\}) + 3|A^v| - 2 - \overline{k}(A^v) \ = \ 3|A^s| - 2 - \overline{k}(A^s)$$

where $v = s' + 1$, and $s' = \max S^r$ is the state defined according to the case β by procedure $\mathcal{R}$; i.e. $s' + 1$ is the successor of s such that $\tau_s(s' + 1, g(s), 1 - a_i) = 1$. In the above equation, we used $A^v = A^{s'+1} = A^s \backslash \{a\}$ and the fact that $\xi(s) = \xi(v) + 1$, hence $\xi(v) = \overline{\xi}$ and the induction assumption applies.

Consider the case when $type(s) = \gamma$, then $|\mathcal{T}(s)| = 2$, and

$$D(s) \ = \ 2 + D(s' + 1) + D(s' + 2) \ =$$

$$= \ 2 + 3|A^{s'+1}| - 2 + \overline{k}(S^{s'+1}) + 3|A^{s'+2}| - 2 + \overline{k}(S^{s'+2}) \ = \ 3|A^s| - 2 - \overline{k}(A^s)$$

where $s' = \max S^r$ is the state defined according to the case γ by procedure R, and states $s'+1$ and $s'+2$ are the successors of the state s. In the above equation, we used the fact that $\{S^{s'+1}, S^{s'+2}\}$ is a partition of S^s, and the fact that $\xi(s) = \max\{\xi(s'+1), \xi(s'+2)\}+1$, hence $\xi(s'+1), \xi(s'+2) \leq \overline{\xi}$ and the induction assumption applies. □

Statement (ii) of the Lemma follows from Claim A.2.6 applied to the root of the tree: $\kappa(\psi) = D(1) = 3|A| - 2 - \overline{k}(A)$.

Finally, let us prove statement (iii). Define

$$E(s) \ = \ |\{s' : \ s \in V^{s'}\}| + 1$$

Thus, $E(s)$ is the number of successors of the state s in a tree plus 1.

Claim A.2.7 *If $E(s)$ is as defined above, then $E(s) = |A^s| - 1$*

Assume $\xi(s) = 1$, then $E(s) = 1$. Suppose now that the Claim is true for all s such that $\xi(s) \leq \overline{\xi}$. If $type(s) = \beta$, then

$$E(s) \ = \ 1 + E(s'+1) \ = \ 1 + |A^{s'+1}| - 1 \ = \ |A^s| - 1$$

where state $s'+1$ is a unique successor of the state s defined by case β of the procedure $\mathcal{R}$, and we used the induction assumption for $s'+1$. If $type(s) = \gamma$, then

$$E(s) \ = \ 1 + E(s'+1) + E(s'+2) \ = \ 1 + |S^{s'+1}| - 1 + |S^{s'+2}| - 1 \ = \ |A^s| - 1$$

where states $s'+1$ and $s'+2$ are successors of the state s defined by case γ of the procedure $\mathcal{R}$, and we used the induction assumption for $E(s'+1)$ and $E(s'+2)$. □

Statement (iii) of the Lemma follows from Claim A.2.6 applied to the root of the tree: $|S_\psi| = E(1) = |A| - 1$. ■

A.2.3 Additive languages and automata

I say that language Q with the index set of questions N is *additive* with respect to preference relation $\succeq$ if there exists a vector $\lambda \in R^N$ such that $\lambda_i \neq 0$ for all $i \in N$, and

$$a \succeq b \iff v(a) \geq v(b), \qquad \text{where } v(a) = \sum_{i \in N} \lambda_i a_i \text{ for } a \in A$$

For an additive language Q, let $\psi^+(Q, \succeq) = \{\sigma_\epsilon\}_{\epsilon \to 0} \in \Psi$ be the decision rule, defined as follows: $\sigma_\epsilon = (S, \iota, \tau_\epsilon)$, where $|S| = |N|$, $\iota(i) = i$, and the stochastic transition rule τ is

$$\begin{cases} \tau(s, s+1, 1) = 1, & \tau(s, s+1, 0) = \epsilon^{\lambda_s}, & \tau(s, s, 0) = 1 - \epsilon^{\lambda_s} & \text{if} \quad \lambda_s > 0 \\ \tau(s, s+1, 0) = 1, & \tau(s, s+1, 1) = \epsilon^{-\lambda_s}, & \tau(s, s, 1) = 1 - \epsilon^{-\lambda_s} & \text{if} \quad \lambda_s < 0 \\ \text{where} & \tau(|S|, |S|+1, \cdot) \equiv (|S|, \diamond, \cdot) & & \end{cases}$$

The set of positive-probability transitions of $\psi^+(Q, \succeq)$ is

$$\mathcal{T} = \Big\{(s, s+1, 1)\Big\} \cup \Big\{(s, s+1, 0)\Big\} \cup \Big\{(s, s, 1) | \lambda_s < 0\Big\} \cup \Big\{(s, s, 0) | \lambda_s > 0\Big\}$$

where $(|S|, |S|+1, \cdot) \equiv (|S|, \diamond, \cdot)$. Clearly, $\mathcal{T}$ does not depend on ϵ, so the decision rule $\psi^+(Q, \succeq) \in \Psi$ is well-defined.

Lemma A.2.8 *Let Q be an additive language for preference relation $\succeq$ and $\psi^+(Q, \succeq)$ be the decision rule, described above, then:*

(i) $\psi^+(Q, \succeq)$ solves the choice problem $(Q, \succeq)$;

(ii) The complexity of the decision rule $\psi^+(Q,\succeq)$ is $3|N|$, i.e.

$$\kappa(\psi^+(Q,\succeq)) \;=\; 3|N|$$

Proof. For the decision rule ψ^+, a simple path l for any alternative $a \in A$ should go via states $1, 2, ..., |S|, \diamond$ consequently, and also, for every state s and any alternative $a \in A$, there is only one link outgoing from state s that can be a part of the simple path, namely, link $(s, s+1, a_{\iota(s)})$. It follows that for any alternative $a \in A$, the set of simple paths is a singleton: $\mathcal{L}_0(a, \sigma_\epsilon) = \{l^*(a)\}$, where $l^*(a) = (l_1^*(a), ..., l_{|S|}^*(a))$, $l_s^*(a) = (s, s+1, a_{\iota(s)})$ with the convention that $(|S|, |S|+1, a_{\iota(s)}) = (|S|, \diamond, a_{\iota(s)})$. Therefore,

$$\begin{aligned}
(1-\eta)^{1-|S|}\omega^*(a) \;&=\; \prod_{s\in S:}\omega((s,s+1,a_{\iota(s)})) \;= \\
&=\; \Big(\prod_{s\in S:\ \lambda_{g(s)}>0,\ a_i=0}\epsilon^{\lambda_{\iota(s)}}\Big)\cdot\Big(\prod_{s\in S:\ \lambda_{g(s)}<0,\ a_i=1}\epsilon^{-\lambda_{\iota(s)}}\Big) \;= \\
&=\; \Big(\prod_{s\in S}\epsilon^{\lambda_{\iota(s)}\cdot(1-a_{\iota(s)})}\Big)\cdot\Big(\prod_{s\in S:\ \lambda_{\iota(s)}<0}\epsilon^{-\lambda_{\iota(s)}}\Big)
\end{aligned}$$

Thus, $\omega^*(a) > 0$ for all $a \in A$. Moreover, if $a \succ b$, then $v(a) > v(b)$, and

$$\omega^*(b)/\omega^*(a) \;=\; \prod_{s\in S}\epsilon^{\lambda_{\iota(s)}\cdot(a_{\iota(s)}-b_{\iota(s)})} \;=\; \epsilon^{v(a)-v(b)} \;\longrightarrow\; 0$$

By Lemma A.1.16, the decision rule $\psi^+(Q,\succeq)$ solves the choice problem $(Q,\succeq)$, proving statement (i).

Statement (ii) follows from

$$\mathcal{T}_{\psi^+(Q,\succeq)} \;=\; \Big\{(s,s+1,1)\Big\}\cup\Big\{(s,s+1,0)\Big\}\cup\Big\{(s,s,1)\Big|\lambda_{\iota(s)}<0\Big\}\cup\Big\{(s,s,0)\Big|\lambda_{\iota(s)}>0\Big\}$$

where $(|S|, |S|+1, \cdot) \equiv (|S|, \diamond, \cdot)$. ■

For a given preference relation $\succeq$ and language Q with $|N| \geq n = \lceil \log_2(m) \rceil$ questions, I denote by Ψ_n^{++} the set of decision rules $\psi = \sigma_\epsilon$, where $\epsilon = (\epsilon_1, ..., \epsilon_n) \in \mathbb{R}^n_{++}$ with $\epsilon_i \longrightarrow 0$ for all $i \in \{1, ..., n\}$, that have the following properties:

(i) The state space is $S = \{1, ..., n\}$;

(ii) The interrogation rule $\iota : S \to N$ is an injection;

(iii) There is $x = (x_1, ..., x_n) \in \{0, 1\}^n$ and permutation $h : \{1, ..., n\} \to \{1, ..., n\}$ such that $h(1) = 1$, and the transitional probabilities satisfy the following properties for all $s \in S$:

(a) $\tau(s, h^{-1}(h(s) + 1), x_s) = 1$, where $h^{-1}(n + 1)$ denotes $\diamond$;

(b) $\tau(s, h^{-1}(h(s) + 1), 1 - x_s) = \epsilon_s$, where $h^{-1}(n + 1)$ denotes $\diamond$;

(c) There exists a unique s' such that $h(s') \leq h(s)$ such that $\tau(s, s', 1-x_s) = 1-\epsilon_s$.

Note that ψ is well-defined, since transitional probabilities conditional on any event (s, ι) that may occur sum up to one. Note also that there are no other positive-probability transitions of ψ except those described in (a),(b),(c).

Note also that $\Psi_n^{++} = \Psi_n^+$, where Ψ_n^+ the set of decision rules, defined in Section 4 of the main part of the paper.

Lemma A.2.9 *Let the preference relation $\succeq$ have m indifference classes, and $n = \lceil \log_2(m) \rceil$. Then for any $\psi \in \Psi_n^{++}$, $\kappa(\psi) = 3n$*

Proof. In each of the n states, there are exactly 3 transitions outgoing: $(s, h^{-1}(h(s)+1), x_s)$, $(s, h^{-1}(h(s) + 1), 1 - x_s)$ and $(s, s', 1 - x_s)$. Hence, $\kappa(\psi) = 3n$. ■

Lemma A.2.10 *Let $\psi \in \Psi_n^{++}$. Then:*

(i) For an arbitrary alternative $a \in A$ for arbitrary valid simple path $l \in \mathcal{L}_0(a)$, l goes consequently via states $h(1) = 1$, $h(2)$,...,$h(n)$, $\diamond$;

(ii) For $a \in A$, any simple path that consists of positive-probability links starting from a state s such that $s = h(k)$ and ending in state $\diamond$, consequently goes via states $h(k), h(k+1), ..., h(n), \diamond$; moreover, the total number of different simple paths with the described properties is 2^{n-k+1}.

Proof. A valid simple path should start at $s = 1$. The only two positive-probability transitions outgoing from state 1 and ending not in s, end at state $s' = h(2)$, hence one of these to links should be l_1. Recursively, there are exactly 2 positive-probability transitions outgoing from state $s = h(k)$ that end at state s' such that $h^{-1}(s') > k$, and these transitions end at state $s' = h(k+1)$, where $h(n+1)$ denotes $\diamond$. The third transition outgoing from state $s = h(s)$ goes via one of the already visited states $\{s \in S | h^{-1}(s) \leq k\}$ and, hence, cannot be a part of a simple path. Therefore, one of these two transitions should be a link l_k of $l \in \mathcal{L}_0$.

The same argument works for a simple path that starts from an arbitrary state $s \in S$. Note that in state s, there are exactly two possible links that can be used in such simple path, and a path goes consequently via $h(k), h(k+1), ..., h(n), \diamond$, hence the total number of such paths is 2^{n-k+1}. ■

The next lemma facilitates the proof of Theorem 1.4.3. Let us first introduce the following notation:

$$\tilde{\mathbb{N}} \;=\; \Big\{ m \in \mathbb{N} \backslash \{1\} \;\Big|\; \exists n \in \mathbb{N} :\; (3/4) \cdot 2^n < m \leq 2^n \Big\}$$

Thus, $\tilde{\mathbb{N}} = \{2, 4, 7, 8, 13, 14, 15, 16, ...\}$ is the set of natural numbers that represents the condition on m given in statement (ii) of Theorem 1.4.3.

Lemma A.2.11 *Let the number of indifference classes of preference relation $\succeq$ be $m \in \tilde{\mathbb{N}}$, and $n = \lceil \log_2(m) \rceil$. If a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,...} \in \Psi_0$ solves the choice problem $(Q, \succeq)$, and $\kappa(\psi) = 3n$, then $\psi \in \Psi^{++}$.*

Proof. First, let's prove the following claims.

Claim A.2.12 *If a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,...} \in \Psi_0$ solves the choice problem $(Q, \succeq)$ above, and $\kappa(\psi) = 3n$, then:*

(1) The state space of ψ is $S_\psi = \{1, ..., n\}$;

(2) The interrogation rule $\iota : S \to N$ is an injection;

(3) There are exactly n weak links, and from each state there is exactly one outgoing weak link.

Proof of the Claim. First, note that by Lemma A.2.2, language Q should contain at least n questions. Each question is associated with a state at which it is asked and with at least 2 strong links. Hence, the number of strong links is at least $2n$. By lemma A.1.21 and the pigeonhole principle, the number of weak links should satisfy $2^{|\mathcal{T}^{weak}|} \geq m$, hence $|\mathcal{T}^{weak}| \geq n$. Since $\kappa(\psi) = 3n$, the number of weak links should be exactly n, and the number of strong links should be exactly $2n$. Since n state are associated with $3n$ links, and $\kappa(\psi) = 3n$, then $S_\psi = \{1, ..., n\}$; since there are n difference questions asked, then $g : S \to N$ is an injection. This proves statements (1) and (2).

Next, Let x_s be the number of weak links outgoing from state s. By Lemmas A.1.20 and A.1.19, for ψ to solve $(Q, \succeq)$, it should be that $f^n(x) \geq m$, where $f^n : (\mathbb{N} \cup \{0\})^n \to \mathbb{N}$ is given by $f^n(x) = \prod_{s=1}^n (x_s + 1)$.

Towards a contradiction, assume that $x_s \neq 1$ for some s. Since $\sum_{s=1}^n x_i = n$, then there are s', s'' such that $x_{s'} = 0$ and $x_{s''} > 1$. Consider $z \in (\mathbb{N} \cup \{0\})^n$ given by $z_l = x_l$ for $l \neq s', s''$, $z_{s'} = 1$, and $z_{s''} = x_{s''} - 1$. By Lemma A.2.1, $f^n(z) \geq (3/4) \cdot f^n(x)$. Note that the number of states such that $z_s = 0$ is the number of states such that $x_s = 0$ minus one. Keep replacing z by $\hat{z} \in (\mathbb{N} \cup \{0\})^n$ in the manner described above, until $z_s > 0$ for all s—this process stops in no more than n steps. Note that the result of this procedure is $z = (1, ..., 1)$. Let k be the number of steps of the described above procedure that is needed to transform x into $(1, ..., 1)$. Then

$f^n(1,...,1) \geq (4/3)^k \cdot f^n(x) \geq (4/3) \cdot f^n(x)$, since $k > 0$ for $x \neq (1,...,1)$. Then,

$$f^n(x) \;\leq\; (3/4) \cdot f^n(1,..,1) \;=\; (3/4) \cdot 2^n \;<\; m$$

in contradiction to $f^n(x) \geq m$, proving statement (3) of the Claim is proven. □

Claim A.2.13 *Suppose $\psi = \{\sigma_r\}_{r\in\mathbb{N}} \in \Psi_0$ solves the choice problem $(Q, \succeq)$. Define the decision rule $\psi' = \{\sigma'_r\}_{r\in\mathbb{N}}$ as follows:*

1. *$S_{\psi'} = S_\psi$ and $\iota_{\psi'} = \iota_\psi$;*
2. *If $s = 1$ or $s' \neq 1, s$, then $\tau_{\psi'}(s, s', j) = \tau_\psi(s, s', j)$;*
3. *$\tau_{\psi'}(s, 1, j) = 0$ for $s \neq 1$;*
4. *If $s \neq 1$, then $\tau_{\psi'}(s, s, j) = \tau_\psi(s, 1, j) + \tau_\psi(s, s, j)$ for all $s \neq 1$.*

Then the decision rule ψ' solves the choice problem $(Q, \succeq)$, and $\kappa(\psi') \leq \kappa(\psi)$.

Proof of the Claim. Since links (s, s, j) and $(s, 1, j)$ cannot appear in the simple path $l^*(a)$ for any alternative a, then $\omega^*_{\psi'}(a) = \omega^*_\psi(a)$ for all $a \in A$. Since ψ solves $(Q, \succeq)$, then by Lemma A.1.16, ψ' solves $(Q, \succeq)$ as well. Next, if $(s, s, j) \in \mathcal{T}_{\psi'} \backslash \mathcal{T}_\psi$, then $(s, 1, j) \in \mathcal{T}_\psi \backslash \mathcal{T}_\psi$. It follows that $|\mathcal{T}_{\psi'}| \leq |\mathcal{T}_\psi|$, proving the Claim. □

Consider $m = 2$, then $n = \lceil \log_2(m) \rceil = 1$. Suppose ψ solves $(Q, \succeq)$, and $\kappa(\psi) = 3n$. By Lemma A.1.21, ψ should have at least one weak link, and by lemma A.2.2, at least one question i should be asked, thus there are at least two strong links, associated with this question at a state s. It follows that ψ should have one state $s = 1$ with $g(1) = i$ for some $i \in N$, one transition—strong link—corresponding to some answer $Q_i(a) = x_1$, i.e. link $(1, \diamond, x_1)$, and two transitions—strong and weak links—corresponding to the opposite answer $Q_i(a) = 1 - x_1$. If link $(1, \diamond, 1 - x_1)$ is strong, then $q(a) > 0$ in the limit for all alternatives, and by Lemma A.1.7, ψ does not solve $(Q, \succeq)$, in contradiction. Hence, link $(1, \diamond, 1 - x_1)$ is strong, and $(1, 1, 1 - x_1)$ is weak. Denote by $\epsilon_1 = \tau(1, \diamond, 1 - x_1)$, then $\psi \in \Psi^{++}$.

Suppose that the statement of the Lemma holds for all $m_0 \in \mathbb{N}$ such that $\lceil \log_2(m_0) \rceil \leq n_0$, where $n_0 \geq 1$ and consider arbitrary $m \in \mathbb{N}$ such that $n_0 < \lceil \log_2(m) \rceil \leq n_0 + 1$. Consider state $s = 1$. By Claim A.2.12, there is exactly one weak link, outgoing from this state. Without loss of generality, this link is $v_1 = (1, \hat{s}, 1 - x_1)$ for some $\hat{s} \in S \backslash \{1\}$ and $x_1 \in \{0, 1\}$—recall that a weak link cannot end at the same state as it begins. Since there are two strong links associated with each state $s \in S$, then the other two outgoing links from $s = 1$ are $v_2 = (1, s', 1 - x_1)$ and $v_3 = (1, s'', x_1)$ for some $s', s'' \in S$.

If $s'' = 1$, then an alternative with $a_{g(1)} = x_1$ is not chosen from a singleton menu, consisting of this alternative—note that by the definition of a language, there should be at least one such alternative, since the partition $\{Q_{g(1)x_1}, Q_{g(1)1-x_1}\}$ of A is non-trivial. Hence, $s'' \neq 1$.

Towards a contradiction, assume that $s'' = \diamond$, then for all alternatives $a \in A$ such that $a_{g(1)} = x_1$ we have $q(a) = 1$ and $\omega(\phi(a)) = 1$. Consider $\hat{\mathbb{T}} \subseteq \mathbb{T}$ given by $\hat{\mathbb{T}} = \Big\{(1, \hat{s}, 1 - x_1)\Big\} \times \Big[\times_{s>1} \Big(\mathcal{T}^{weak} \cup \{\varnothing\}\Big)\Big]$. Since $a_{\iota(1)} = 1 - x_1$ implies $(\phi(a))_1 = (1, \hat{s}, 1 - x_1)$, then by Lemma A.1.20, for all $a \in A$ such that $a_{g(1)} = 1 - x_1$ there exists $e \in \hat{\mathbb{T}}$ such that $\phi(a) \hat{\sim}_\psi e$. Thus, $\phi(A) = \Big\{e \in \mathbb{T} \Big| \exists a \in A : \ e = \phi(a)\Big\}$ contains at most $1 + \prod_{s>1} \Big(|\mathcal{T}^{weak}| + 1\Big) = 1 + 2^{n-1}$ indifference classes, then by Lemma A.1.19, $\succeq$ cannot contain more than $1 + 2^{n-1}$ indifference classes. Thus, $m \leq 1 + 2^{n-1} \leq (3/4) \cdot 2^n < m$ in contradiction, where we used $1 + 2^{n-1} \leq (3/4) \cdot 2^n$ for $n = n_0 + 1 > 1$. Hence, $s'' \neq \varnothing$.

Claim A.2.14 *If $(1, s', 1 - x_1)$ is a strong link described above, then $s' = 1$.*

Proof of the Claim. Towards a contradiction, assume that $s' \neq 1$. If $s' = \diamond$, then for all $a \in A$ such that $a_{\iota(1)} = 1 - x_1$, we have $\omega(\phi(a)) = 1$. Then, we can repeat the argument above with $\hat{\mathbb{T}} = \Big\{(1, s'', x_1)\Big\} \times \Big[\times_{s>1} \Big(\mathcal{T}^{weak} \cup \{\varnothing\}\Big)\Big]$ and get a contradiction $m \leq 1 + 2^{n-1} \leq (3/4) \cdot 2^n < m$. Thus, $s' \in \overline{S} \backslash \{1, \diamond\}$.

Let v_4 be a weak link, outgoing from state s'; that is, $\mathcal{T}^{weak} = \{v_4\}$. Towards a contradiction, assume that there is $a \in A$ such that the path $l^*(a)$ contains both a weak link $v_1 = (1, \hat{s}, 1 - x_1)$ and a weak link v_4. Thus, $l^*(a) = v_1 l^1 v_4 l^2$ for some (possibly, empty) paths l^1, l^2. Note that an alternative path $\tilde{l} \in \mathcal{L}(a)$ is $v_2 v_4 l^2$. Then

$$\frac{\omega(l^*(a))}{\omega(\tilde{l})} \;\leq\; \frac{\omega(v_1)}{\omega(v_2)} \;\longrightarrow\; 0$$

since v_2 is a strong link, and v_1 is a weak link. Hence, $\omega(\tilde{l}) > \omega(l^*(a))$, contradicting the definition of $l^*(a)$. Thus, for all alternatives $a \in A$, both weak links v_1 and v_4 are never in use in a path $l^*(a)$. Let $\hat{\mathbb{T}} \subset \mathbb{T}$ be the following set:

$$\hat{\mathbb{T}} \;=\; \Big\{ e \in \mathbb{T}' \;\Big|\; (e_1, e_{s'}) \neq (v_1, v_4) \Big\}, \quad \text{where} \;\; \mathbb{T}' = \times_{s \in S} \Big(\mathcal{T}^{weak}(s) \cup \{\varnothing\} \Big)$$

By Lemma A.1.20 and the proven fact that v_1, v_4 are never used together in $l^*(a)$, for any $a \in A$, there is $e \in \hat{\mathbb{T}}$ such that $\phi(a) \;\hat{\sim}_\psi\; e$. Note that $|\hat{\mathbb{T}}| = 3 \cdot 2^{n-2}$, since only 3 out of four combinations $\{v_1, \varnothing\} \times \{v_4, \varnothing\}$ are possible, and $|\mathcal{T}^{weak}(s)| = 1$ for all s. By Lemma A.1.19, $(3/4) \cdot 2^n = 3 \cdot 2^{n-2} \geq m$, contradicting $(3/4) \cdot 2^n < m$. The Claim is proven. □

Thus, the 3 links outgoing from state 1 are: a weak link $v_1 = (1, \hat{s}, 1 - x_1)$ with transitional probability $\tau(s, \hat{s}, 1 - x_1) \equiv \epsilon_1 \longrightarrow 0$, a strong link $v_2 = (1, 1, 1 - x_1)$ with transitional probability $\tau(1, 1, 1 - x_1) = 1 - \epsilon_1 \longrightarrow 1$, and a strong link $v_3 = (1, s'', x_1)$ with transitional probability $\tau(1, s'', x_1) = 1$.

Let $A_1 = \{a \in A | a_{\iota(1)} = x_1\}$, and $A_2 = \{a \in A | a_{\iota(1)} = 1 - x_1\}$, then (A_1, A_2) is a partition of A. Let m_1 be the number of indifference classes of $\succeq$ on A_1, and m_2 be the number of indifference classes of $\succeq$ on A_2, then $m_1 + m_2 \geq m$. It follows that there is $i \in \{1, 2\}$ such that $m_i > (3/4) \cdot 2^{n-1}$.

Suppose $m_2 > (3/4) \cdot 2^{n-1}$. Let ψ' be the decision rule constructed according to Claim A.2.13, then it has the same set of links, outgoing from state 1 such that ψ', it solves $(Q, \succeq)$, and $\kappa(\psi') \leq \kappa(\psi) = 3n$. By Lemma A.2.3, $\kappa(\psi') \geq 3n$, hence $\kappa(\psi') = 3n$. Moreover, by Lemma A.1.17, there is a regular decision rule ψ'' that consists of a subsequence of automatons of ψ' and solves the choice problem $(Q, \succeq)$. Without loss of generality, $\psi'' = \psi'$.

Consider the following auxiliary choice problem: the set of alternatives is A_2, the preference relation is $\succeq$, restricted on A_2, and the language is Q. Construct the decision rule $\hat{\psi}$ as follows. Let $\chi : \{2, ..., n\} \to \{1, ..., n-1\}$ be the following bijection (recall that state $\hat{s}$ is the state at which the weak link beginning from state 1 in the decision rules ψ and ψ' ends): $\chi(\hat{s}) = 1$, $\chi(s) = s$ for all $s < \hat{s}$, and $\chi(s) = s - 1$ for all $s > \hat{s}$. Then, the flexible state space of $\hat{\psi}$ is $\hat{S} = \{1, ..., n-1\}$, the interrogation function is $\hat{\iota}(s) = \iota(h^{-1}(s))$ for $s \in \{1, ..., n-1\}$, and the transitional probabilities are $\hat{\tau}(s, s', j) = \tau'(\chi^{-1}(s), \chi^{-1}(s'), j)$, where τ' are the transitional probabilities of the decision rule ψ', described above.

Claim A.2.15 *The decision rule $\hat{\psi}$ solves the auxiliary choice problem with the set of alternatives A_2, preference relation $\succeq$ and language Q.*

Proof of the Claim. For an arbitrary alternative $a \in A_2$, let $\hat{\mathcal{L}}_0(a)$ be the set of simple paths defined with respect to the decision rule $\hat{\psi}$, and $\mathcal{L}_0(a)$ be the set of simple paths defined with respect to the decision rule ψ'. Note that for any $a \in A_2$ for any $l \in \mathcal{L}_0$, $l_1 = v_1 = (1, \hat{s}, 1 - x_1)$. Then $\chi : \{2, ..., n\} \to \{1, ..., n-1\}$ defined above induces a natural a bijection between $\mathcal{L}_0(a)$ and $\hat{\mathcal{L}}_{0(a)}$: for $l = v_1 v_{j_1} ... v_{j_k}$, $\chi(l) = \chi(v_{j_1}) ... \chi(v_{j_k})$, where $\chi(s, s', j) = (\chi(s), \chi(s'), j)$. Note that

$$(1 - \eta) \cdot \tau(1, \hat{s}, 1 - x_1) \cdot \omega(\chi(l)) \;=\; \omega(l)$$

Denote by $\hat{\omega}^*(a) = \max_{\hat{l} \in \hat{\mathcal{L}}_0}(\omega(\hat{l}))$. Then for all $a \in A_2$ we have

$$\hat{\omega}^*(a) \;=\; (\omega^*(a))/((1-\eta)\cdot\tau(1,\hat{s},1-x_1))$$

Since ψ' solves the original choice problem, then by Lemma A.1.16, $\omega^*(a) > 0$ for all $a \in A_2$, and for all $a, b \in A_2$ such that $a \succ b$, $(\omega^*(b))/(\omega^*(a)) \to 0$. It follows that $\hat{\omega}^*$ has the same properties, thus by Lemma A.1.16, $\hat{\psi}$ solves the auxiliary choice problem. □

Note that $\kappa(\hat{\psi}) = 3n-3$, since it has the same (after re-numeration of states via mapping χ) set of links as ψ', except of the 3 links, outgoing from the state $s=1$ in ψ'. Since $\hat{\psi}$ solves the auxiliary choice problem, by Lemma A.2.3, it should be that $m_2 \leq 2^{n-1}$. Thus, $(3/4)\cdot 2^{n-1} < m_2 \leq 2^{n-1}$, and we can apply the induction assumption to show that $\hat{\psi} \in \Psi^{++}_{n-1}$. In particular, there is $\hat{x} = (\hat{x}_1, ..., \hat{x}_{n-1}) \in \{0,1\}$, $(\hat{\epsilon}_1, ...\hat{\epsilon}_{n-1}) \in (0,1)^{n-1}$ with $\hat{\epsilon}_i \to 0$, and a permutation $\hat{h} : \{1,...,n-1\} \to \{1,...,n-1\}$ such that $\hat{h}(1) = 1$, and

(a) For all $s \in \{1,...,n-1\}$, $\hat{\tau}(s, \hat{h}^{-1}(\hat{h}(s)+1), \hat{x}_s) = 1$, where $\hat{h}^{-1}(n+1)$ denotes $\diamond$;

(b) For all $s \in \{1,...,n-1\}$, $\hat{\tau}(s, \hat{h}^{-1}(\hat{h}(s)+1), 1-\hat{x}_s) = \hat{\epsilon}_s$, where $\hat{h}^{-1}(n+1)$ denotes $\diamond$;

(c) For all $s \in \{1,...,n-1\}$, there exists a unique $s''' \in \{1,...,n-1\}$ such that $\hat{h}(s''') \leq \hat{h}(s)$ such that $\tau(s, s''', 1-\hat{x}_s) = 1-\hat{\epsilon}_s$.

where $\hat{\tau}$ are the transitional probabilities of $\hat{\psi}$.

Claim A.2.16 *Let $v_1 = (1, s', 1-x_1)$, $v_3 = (1, s'', x_1)$ be links of ψ defined above. Then $s' = s''$.*

Proof of the Claim. Let $\hat{x}$, $\hat{\epsilon}$, $\hat{h}$ are defined above. Towards a contradiction, assume that $s'' \neq s'$, then $k = \hat{h}(\chi(s'')) \neq \hat{h}(\chi(s')) = 1$, hence $k > 1$. Lemma A.2.10 applied

for the decision rule $\hat{\psi} \in \Psi_{n-1}^{++}$ implies that, when we consider a decision rule ψ', a valid simple path $l \in \mathcal{L}_0(a)$ for arbitrary alternative $a \in A_1$ should go consequently via states 1, $\chi(\hat{h}(k))$, $\chi(\hat{h}(k+1)), ..., \chi(\hat{h}(n)), \diamond$; moreover, the total number of different simple paths that can be used by all alternatives $a \in A_1$ is $2^{n-1+k+1}$. It follows that

$$\Big|\Big\{z \in [0,1] \;\Big|\; \exists a \in A_1 : \omega^*(a) = z\Big\}\Big| \;\leq\; 2^{n-1+k+1} \leq 2^{n-2}$$

where we used $k > 1$, because $s'' \neq s'$. Since ψ' solves the original choice problem, By Lemma A.1.16, A_1 should contain no more than 2^{n-2} indifference classes, that is, $m_1 \leq 2^{n-2}$. Since $m_2 \leq 2^{n-1}$, it follows that $m \leq m_1 + m_2 \leq 2^{n-2} + 2^n \leq (3/4) \cdot 2^n$, in contradiction. □

Thus, $s' = s''$. Define a permutation h as follows: $h(1) = 1$, $h(s) = \hat{h}(\chi(s)) + 1$ for $s \in \{2, ..., n-1\}$. Note that $h^{-1}(1) = 1$, and $h^{-1}(k) = \chi^{-1}(\hat{h}(k-1))$ for $k \in \{2, ..., n\}$. Next, x_1 and ϵ_1 has already been defined. Define $x_2, ..., x_n$ by $x_s = \hat{x}_{\chi(s)}$, and $\epsilon_2, ..., \epsilon_n$ by $\epsilon_s = \hat{\epsilon}_{\chi(s)}$ for all $s \in \{2, ..., n\}$.

Claim A.2.17 *The decision rule ψ' with h, x, ϵ defined as above, satisfy conditions (i),(ii),(iii)(a,b,c) in the definition of $\psi \in \Psi_n^{++}$.*

Proof of the Claim. We already know that $S_{\psi'} = \{1, ..., n\}$ and $g_{\psi'}$ is an injection. Next, let us check (iii). For $s = 1$, we get $h^{-1}(h(1) + 1) = h^{-1}(2) = \chi^{-1}(\hat{h}^{-1}(1)) = \chi^{-1}(1) = s' = s''$. The set of links, outgoing from $s = 1$ is $v_1 = (1, 2, 1 - x_1)$ with probability $\tau(1, 2, 1-x_1) = \epsilon_1$, $v_2 = (1, 1, 1-x_1)$ with probability $\tau(1, 1, 1-x_1) = 1-\epsilon_1$ and $v_3 = (1, 2, x_1)$ with probability $\tau(1, 2, x_1) = 1$. Hence, properties (iii)(a,b,c) hold for state $s = 1$.

Third, let $s > 1$, then, by the definition of the decision rule $\hat{\psi}$, the transitional probabilities of the decision rule ψ', outgoing from state s, are:

(a) $1 = \tau\Big(s, \chi^{-1}[\hat{h}^{-1}(\hat{h}(\chi(s)) + 1)], \hat{x}_{\chi(s)}\Big) = \tau\Big(s, h^{-1}(h(s) + 1), x_s\Big)$;

(b) $\epsilon_s = \hat{\epsilon}_{\chi(s)} = \tau\Big(s, \chi^{-1}[\hat{h}^{-1}(\hat{h}(\chi(s))+1)], 1-\hat{x}_{\chi(s)}\Big) = \tau\Big(s, h^{-1}(h(s)+1), 1-x_s\Big)$;

(c) $1-\epsilon_s = 1-\hat{\epsilon}_{\chi(s)} = \tau\Big(s, \chi^{-1}[\hat{h}^{-1}(\hat{h}(\chi(s)))], 1-\hat{x}_{\chi(s)}\Big) = \tau\Big(s, s, 1-x_s\Big)$.

Thus, property (iii) in the definition of Ψ_n^{++} holds for ψ' as well. □

Claim A.2.18 *If $\psi' \in \Psi_n^{++}$, where ψ' is constructed via $\psi \in \Psi_0$ according to Claim A.2.13, and ψ satisfies the properties stated in Claim A.2.12, then $\psi \in \Psi_n^{++}$.*

Proof of the Claim. Clearly, properties (i) and (ii) in the definition of Ψ_n^{++} hold for ψ. We show that if h, x and ϵ are such that property (iii) of the definition of Ψ_n^{++} holds for ψ' than this property also holds for ψ with the same h, x and ϵ.

Since $\kappa(\psi') = 3n = \kappa(\psi)$, then there is a bijection between the set of links $\mathcal{T}^1 \equiv \mathcal{T}_\psi \backslash \mathcal{T}_{\psi'}$ and $\mathcal{T}^2 \equiv \mathcal{T}_{\psi'} \backslash \mathcal{T}_\psi$. The set $\mathcal{T}^1$ is comprised from the links that go from states $s \in S'$ to state 1 for some $S' \subset S\backslash\{1\}$, and the set of links $\mathcal{T}^2$ is comprised of links that goes from states $s \in S'$ to themselves. Thus, for any $s \in S'$, the link in $\mathcal{T}^2$ outgoing from s is a strong that correspond to case (c) in the definition of Ψ^{++}, but since $h(1) = 1$, the link in $\mathcal{T}^1$, outgoing from s corresponds to case (c) in the definition of Ψ^{++} as well. Since the other links of ψ and ψ' coincide, we conclude that $\psi \in \Psi_n^{++}$ □

The Claim A.2.18 finishes the proof for the case when $m_2 > (3/4) \cdot 2^{n-1}$. The case when $m_1 > (3/4) \cdot 2^{n-1}$ is analyzed by the analogous way. ■

Lemma A.2.19 *Let the preference relation $\succeq$ have m equivalence classes, and $n = \lceil \log_2(m) \rceil$. Then language Q is adaptive if and only if there exists a decision rule $\psi \in \Psi_n^{++}$ that solves the choice problem $(Q, \succeq)$.*

Proof. First, suppose that language Q is adaptive for a preference relation $\succeq$. Then there is a subset $Q' = \{Q_i\}_{i \in N'}$ of questions such that $|N'| = n \equiv \lceil \log_2(m) \rceil$ and

$\lambda \in \mathbb{R}^{N'}$ such that

$$a \succ b \quad \Longrightarrow \quad v(a) > v(b), \qquad \text{where} \quad v(a) \;=\; \sum_{i \in N'} \lambda_i a_i$$

Claim A.2.20 *Let preference relations $\succeq'$ and $\succeq$ on A be such that $a \succ b$ implies $a \succ' b$ for all $a, b \in A$, and let Q be an arbitrary language, adequate for $\succeq'$. If a decision rule $\psi = \{\sigma_r\}_{r=1,2,...} \in \Psi$ solves the choice problem $(Q, \succeq')$, then ψ solves the choice problem $(Q, \succeq)$.*

Proof of the claim. Consider arbitrary $a, b \in A$ such that $a \succ b$, then $a \succ' b$. Since ψ solves the choice problem $(Q, \succeq')$, by Lemma A.1.7, $q_{\sigma_r}(b)/q_{\sigma_r}(b) \longrightarrow 0$. Therefore, by Lemma A.1.7, ψ solves the choice problem $(Q, \succeq)$. □

Let $\succeq'$ be the preference relation on A, induced by the utility function $v(\cdot)$; that is,

$$v(a) \geq v(b) \quad \Longleftrightarrow \quad a \succeq' b \qquad \forall a, b \in A$$

It follows that the language Q' is additive for a preference relation $\succeq'$. Then, by Lemma A.2.8, the decision rule $\psi^+(Q', \succeq')$ solves the choice problem $(Q', \succeq')$, and the complexity of this decision rule is $\kappa(\psi^+(Q', \succeq')) = 3n$. By Claim A.2.20, $\psi^+(Q', \succeq')$ solves the choice problem $(Q', \succeq)$ as well, and hence it solves the choice problem $(Q, \succeq)$, since $Q' \subseteq Q$.

Claim A.2.21 *Let $\psi = \psi^+(Q', \succeq')$ be the decision rule, defined for an additive (with respect to preference relation $\succeq'$) language Q' as described above, where language Q' contains exactly $|N'| = n = \lceil \log_2(m) \rceil$ questions. Then $\psi^+(Q', \succeq') \in \Psi_n^{++}$.*

Proof of the Claim. First, note that $S_\psi = \{1, ..., n\}$, and the interrogation rule $g(i) = i$ is an injection; thus, properties (i) and (ii) in the definition of Ψ_n^{++} hold. Second, consider a unit permutation h given by $h(i) = i$, let $x_i = \mathbb{1}\{\lambda_i > 1\}$, and

$\epsilon_i = \epsilon^{|\lambda_i|}$ for all $i \in \{1, ..., n\}$. Then it is straightforward to verify that ψ satisfies conditions (iii)(a,b,c) in the definition of Ψ_n^{++}. $\square$

Thus, $\psi^+(Q', \succeq') \in \Psi_n^{++}$ solves the choice problem $(Q, \succeq)$, proving the only if direction.

Suppose now that $\psi \in \Psi_n^{++}$ solves the choice problem $(Q, \succeq)$ with m indifference classes, and $n = \lceil \log_2(m) \rceil$. Let x, h, ϵ be as in the definition of Ψ_n^{++}. Note that for any alternative $a \in A$, its set of simple paths is a singleton: $\mathcal{L}_0 = \{l^*(a)\}$, where $l_k = (h(k), h(k+1), Q_{\iota(h(k))(a)})$ for $k \in \{1, ..., n\}$ with the convention that $h(k+1) = \diamond$. Hence,

$$\begin{aligned}
\omega^*(a) &= (1-\eta)^{n-1} \cdot \prod_{k=1}^{n} (h(k), h(k+1), Q_{\iota(h(k))}(a)) \; = \\
&= (1-\eta)^{n-1} \cdot \prod_{k=1}^{n} \left[\mathbb{1}\{Q_{\iota(h(k))}(a) = x_{h(k)}\} + \mathbb{1}\{Q_{\iota(h(k))}(a) = 1 - x_{h(k)}\} \cdot \epsilon_{h(k)} \right] \\
&= (1-\eta)^{n-1} \cdot \prod_{k:\; Q_{\iota(h(k))}(a) = 1 - x_{h(k)}} \epsilon_{h(k)} \;=\; (1-\eta)^{n-1} \cdot \prod_{s:\; Q_{\iota(s)}(a) = 1 - x_s} \epsilon_s
\end{aligned}$$

It follows that

$$\log(\omega^*(a)) \;=\; (n-1) \cdot \log(1-\eta) + \sum_{s:\; Q_{\iota(s)}(a) = 1 - x_s} \log(\epsilon_s)$$

Since ψ solves $(Q, \succeq)$, then by Lemma A.1.16, for all $a, b \in A$, $a \succ b$ implies

$$\omega^*(b)/\omega^*(a) \;\longrightarrow\; 0$$

Thus,

$$\log(\omega^*(b))/\log(\omega^*(a)) \;=\; \frac{\sum_{s:\; Q_{g(s)}(b) = 1 - x_s} \log(\epsilon_s)}{\sum_{s:\; Q_{g(s)}(a) = 1 - x_s} \log(\epsilon_s)} \;\longrightarrow\; 0$$

Recall that $\epsilon = (\epsilon_1, ..., \epsilon_n)$ is, in fact, a sequence converging to $(0, ..., 0)$. Since the number of pairs (a, b) with $a, b \in A$ is finite, there is an element of this sequence such

that

$$\sum_{s:\ Q_{\iota(s)}(b)=1-x_s} \log(\epsilon_s) < \sum_{s:\ Q_{\iota(s)}(a)=1-x_s} \log(\epsilon_s) \quad \forall a, b \in A:\ a \succ b \tag{A.11}$$

For $i \in N' = \iota(S)$, for this element of the sequence ϵ, define

$$\lambda_i \ = \ \lambda_{\iota(s)} \ = \ \begin{cases} -\log(\epsilon_s) & \text{if} \quad x_s = 1 \\ \log(\epsilon_s) & \text{if} \quad x_s = 0 \end{cases}$$

Note that λ_i are well-defined, since $\iota : S \to N$ is an injection. Then for any $a \in A$

$$\sum_{s:\ Q_{\iota(s)}(a)=1-x_s} \log(\epsilon_s) \ = \sum_{s:\ x_s=1,\ Q_{\iota(s)}(a)=0} \log(\epsilon_s) + \sum_{s:\ x_s=0,\ Q_{\iota(s)}(a)=1} \log(\epsilon_s) \ =$$

$$= \sum_{s:\ x_s=1} \log(\epsilon_s) \sum_{s:\ x_s=1,\ Q_{\iota(s)}(a)=1} \Big(-\log(\epsilon_s) \Big) + \sum_{s:\ x_s=0,\ Q_{\iota(s)}(a)=1} \log(\epsilon_s) \ =$$

$$= \ -\sum_{s:\ x_s=1} \lambda_s + \sum_{s \in S} \lambda_s a_{\iota(s)}$$

Thus, $a \succ b$ implies

$$\sum_{s \in S} \lambda_s a_{\iota(s)} \ > \ \sum_{s \in S} \lambda_s b_{\iota(s)} \quad \Longrightarrow \quad \sum_{i \in N'} \lambda_s a_i \ > \ \sum_{i \in N'} \lambda_s b_i$$

Since $|S| = n = \lceil \log_2(m) \rceil$, $g(S) = N'$ and ι is an injection, then $|N'| = \lceil \log_2(m) \rceil$. If $\lambda_i = 0$ for some $i \in N'$, then

$$\Big| \Big\{ \sum_{s \in S''} \log \epsilon_s \ \Big| \ \varnothing \subseteq S'' \subseteq S \Big\} \Big| \ \leq \ 2^{|S|-1} = 2^{|N'|-1} = 2^{n-1} < m$$

in contradiction to eq. (A.11). Thus, Q is an adaptive language, proving the if direction of the Lemma. ■

A.2.4 Other Lemmas

The next lemma proves a generalized version of Proposition 1.4.1.

Lemma A.2.22 *For any preference relation $\succeq$ with $m \geq 2$ indifference classes, there is an additive language Q with $n = \lceil \log_2(m) \rceil$ questions.*

Proof. Enumerate indifference classes of preference relation $\succeq$ by $C^0, C^1, ..., C^m$ such that $a \in C^j$, $b \in C^k$ implies $a \succ b$ iff $j > k$. Denote by $k(a)$ the index of the indifference class of a, i.e. $k(a) = k'$ iff $a \in C^{k'}$. Denote by $n = \lceil \log_2(m) \rceil$. Let $x_1 x_2 ... x_n$ be the expression of $k \in \{0, 1, ..., n-1\}$ via a base-2 number—it is well defined because of our choice of n—and denote by $k_i = x_i$ the corresponding digit of the base-2 expression of k. Consider the language Q with $Q_{ij} = \{a \in A \mid k_i(a) = j\}$ for $j \in \{1, 2\}$, and let $\lambda_i = 2^{n-i}$. Consider an additive utility function $v : A \to \mathbb{R}$ with weights λ, then

$$v(a) \;=\; \sum_{i=1}^{n} \lambda_i a_i \;=\; \sum_{i=1}^{n} 2^{n-i} k_i(a) \;=\; k(a)$$

It follows that $v(\cdot)$ represents $\succeq$. ■

A.2.5 Proofs of statements in Sections 2-4

In this section, I use previous calculations to prove statements from the main part of the paper.

Proof of Lemma 1.2.1

The Lemma follows from Lemma A.1.6 by substituting $\rho^B(a) = \rho(a) / \sum_{b \in B} \rho(b)$.

■

Proof of Lemma 1.2.2

By Lemma A.2.5, for any preference relation $\succeq$ and adequate language Q, the decision rule $\mathcal{R}(Q, \succeq)$ solves the choice problem $(Q, \succeq)$. ■

Proof of Theorem 1.3.1

The lower bound is proven in Lemma A.2.3.

Consider an arbitrary preference relation $\succeq$ on A with $m \geq 2$ indifference classes. By Lemma A.2.22, there is an additive with respect to preference relation $\succeq$ language Q with $n = \lceil \log_2(m) \rceil$ questions. By Lemma A.2.8, the sequence of decision rules $\psi^+(Q, \succeq)$ solves $(Q, \succeq)$; moreover, $\kappa(\psi^+(Q, \succeq)) = 3|N| = 3 \cdot \lceil \log_2(m) \rceil$ proving the tightness of the lower bound. ■

Proof of Proposition 1.3.2

First, by Lemma A.2.2, if $\psi \in \Psi$ solves $(Q, \succeq)$, then at least $n = \lceil \log_2(m) \rceil$ should be asked in various states, thus $|S| \geq \lceil \log_2(m) \rceil$.

Second, assume that $\psi \in \Psi$ solves $(Q, \succeq)$ solves $(Q, \succeq)$, and $\kappa(\psi) = 3n = 3 \cdot \lceil \log_2(m) \rceil$. Towards a contradiction, assume $|S| > n = \lceil \log_2(m) \rceil$. Similar to the proof of Theorem 1.3.1 given above, by Lemma A.1.21 and pigeonhole principle, $2^{|\mathcal{T}^{weak}|} \geq m$, thus $|\mathcal{T}^{weak}| \geq \lceil \log_2(m) \rceil = n$. Next, by Lemma A.2.2, there should be at least n states with question asked in each state, and there are at least two strong links outgoing from each such state. Since from every state there is an outgoing strong link, then $|\mathcal{T}^{strong}| > 2n$, and hence,

$$\kappa(\psi) \;=\; |\mathcal{T}^{weak}| + |\mathcal{T}^{strong}| > n + 2n = 3n$$

in contradiction to our assumption that $\kappa(\psi) = 3n = 3 \cdot \lceil \log_2(m) \rceil$ ■

Proof of Proposition 1.3.3

Consider set $A = \{a\} \cup \{b^1, ..., b^k\}$ with $k+1$ alternatives, and preference relation $\succeq$ on A such that $a \succ b^i$ and $b^i \sim b^{i'}$ for all i, i'. Then $\succeq$ has $m = 2$ indifference classes. Consider language $Q = \{Q_i\}_{i=1}^k$ with $Q_{i1} = \{b^i\}$ and $Q_{i0} = A \backslash \{b^i\}$ for all $i = 1, ..., k$. Note that the language Q is adequate, since for any $i \in \{1, ..., k\}$, $b_i^i \neq c_i'$ for any $c' \neq b^i$.

Suppose $\psi \in \Psi$ solves $(Q, \succeq)$. Towards a contradiction, assume that question Q_i is not asked in ψ, i.e. $\nexists s \in S : \ \iota(s) = i$. Then the set of links that can be used for transitions between the states for alternatives a and b^i coincide, that is, $\mathcal{T}^a(s) = \mathcal{T}^{b^i}(s)$ for all $s \in S$. It follows that $q(a) = q(b^i)$, which is a contradiction by Lemma A.1.7, since $a \succ b^i$. Thus, questions $Q_1, ..., Q_k$ should be asked in ψ. Note that each question i asked in state s is associated with at least two links $(s, s', 0), (s, s'', 1) \in \mathcal{T}$, and different questions are associated with different links, thus

$$\kappa(\psi) \ = \ |\mathcal{T}| \geq 2k \quad \forall \psi \in \Psi : \ \psi \text{ solves } (Q, \succeq)$$

Thus, $\kappa(Q) > k$, proving the Proposition. ■

Proof of Theorem 1.3.4

Statement (i) of the Theorem follows from the Lemma A.2.5, since the universal decision rule $\overline{\psi} = \mathcal{R}(Q, \succeq)$ solves the choice problem $(Q, \succeq)$ and uses $\kappa(\overline{\psi}) = 3|A| - 3$ transitions.

Let $m \in \{2, ..., |A|\}$. Let $\succeq$ be a preference relation such that $\tilde{a} \succ b$ for all $b \in A$; that is, $\tilde{a}$ is a unique $\succeq$-best element in A, and let $\succeq$ have m indifference classes; clearly, such preference relation exists. Enumerate alternatives in $A \backslash \{\tilde{a}\}$ by $a^1, ..., a^{|A|-1}$, and consider language $Q = \{Q_i\}_{i=1}^{|A|-1}$ with $Q_{i1} = \{a^i\}$, $Q_{i0} = A \backslash \{a^i\}$.

Assume a decision rule ψ solves the choice problem $(Q, \succeq)$; by Lemma A.1.17, it is without loss of generality to assume that ψ is regular. Note that all questions of the language Q should be asked in ψ; otherwise, if Q_i is not asked, then $l^*(\tilde{a}) \in \mathcal{L}(a^i)$, $\omega^*(\tilde{a}) \leq \omega^*(a^i)$, and by Lemma A.1.16, ψ does not solve $(Q, \succeq)$, in contradiction. Thus, $\mathcal{T}_\psi$ contains at least $2 \cdot (|A| - 1)$ strong links. Next, since $\succeq$ has m indifference classes, then by Lemma A.1.21, the sets of weak links used by alternatives from distinct indifference classes should be different. It follows that $2^{|\mathcal{T}_\psi^{weak}|} \geq m$, hence $|\mathcal{T}_\psi^{weak}| \geq \lceil \log_2(m) \rceil$. Hence,

$$\kappa(\psi) \;=\; |\mathcal{T}_\psi| \;=\; |\mathcal{T}_\psi^{strong}| + |\mathcal{T}_\psi^{weak}| \;\geq\; 2|A| - 2 + \lceil \log_2(m) \rceil$$

Since the above inequality holds for arbitrary decision rule ψ that solves the choice problem $(Q, \succeq)$, then $\kappa(Q) \geq 2|A| - 2 + \lceil \log_2(m) \rceil$, proving statement (ii) of the Theorem. ■

Proof of Proposition 1.3.5

The proof of the Proposition mirrors the proof of Theorem 1.3.4 above. Statement (i) of the Proposition follows from the Lemma A.2.5, since the universal decision rule $\overline{\psi} = \mathcal{R}(Q, \succeq)$ solves the choice problem $(Q, \succeq)$ and uses $|S_{\overline{\psi}}| = |A| - 1$ memory states.

Let $m \in \{2, ..., |A|\}$. Let $\succeq$ be a preference relation such that $\tilde{a} \succ b$ for all $b \in A$; that is, $\tilde{a}$ is a unique $\succeq$-best element in A, and let $\succeq$ have m indifference classes; clearly, such preference relation exists. Enumerate alternatives in $A \backslash \{\tilde{a}\}$ by $a^1, ..., a^{|A|-1}$, and consider language $Q = \{Q_i\}_{i=1}^{|A|-1}$ with $Q_{i1} = \{a^i\}$, $Q_{i0} = A \backslash \{a^i\}$.

Assume a decision rule ψ solves the choice problem $(Q, \succeq)$; by Lemma A.1.17, it is without loss of generality to assume that ψ is regular. Note that all questions of the language Q should be asked in ψ; otherwise, if Q_i is not asked, then $l^*(\tilde{a}) \in \mathcal{L}(a^i)$, $\omega^*(\tilde{a}) \leq \omega^*(a^i)$, and by Lemma A.1.16, ψ does not solve $(Q, \succeq)$, in contradiction.

Since all questions of the language Q must be asked in Ψ, then $|S_\psi| \geq |Q| = |A| - 1$. It follows that $\mathcal{M}(Q) \geq |A| - 1$. By the proven statement (i), $\mathcal{M}(Q) \leq |A| - 1$, hence $\mathcal{M}(Q) = |A| - 1$, proving statement (ii) of the Proposition. ■

Proof of Proposition 1.4.1

The Proposition follows from Lemma A.2.22. ■

Proof of Theorem 1.4.3

Suppose that language Q is adaptive for a preference relation $\succeq$ with m indifference classes, and $n = \lceil \log_2(m) \rceil$. By Lemma A.2.19, there is a decision rule $\psi \in \Psi_n^{++}$ that solves the choice problem $(Q, \succeq)$. By Lemma A.2.9, $\kappa(\psi) = 3n$, thus the language Q is simple.

Suppose now that $(3/4) \cdot 2^n < m \leq 2^n$ for some natural n, $\psi \in \Psi$ solves the choice problem $(Q, \succeq)$, and $\kappa(\psi) = 3n$. By Lemma A.2.11, $\psi \in \Psi_n^{++}$, and thus by Lemma A.2.19, the language Q is adaptive. ■

Proof of Proposition 1.4.4

Statement (i) of the Proposition is proved in Lemma A.2.19. Statement (ii) of the Proposition proved in Lemma A.2.11 ■

A.2.6 Proofs of lemmas and claims from Section 5

Lemma 1.5.1 is proven in Lemma A.1.7 in this Appendix, Lemma 1.5.2 is proven in Lemma A.1.16 in this Appendix, Claim 1.5.3 follows from Lemma A.1.19 and Lemma A.1.22 in the Appendix, Claim 1.5.4 is proven in Lemma A.2.1 in the Appendix, Claim 1.5.5 follows from Claim A.2.12 in the Appendix, Claim 1.5.6 is proven in Claim A.2.14 in the Appendix, Claim 1.5.8 follows from Claim A.2.15 in the Appendix.

A.2.7 Formalizing the dynamics

The realized menu $B \subseteq A$ and the sampling probability ρ^B from B determine the economic environment. The language Q and the automaton $\sigma = (S, \iota, \tau) \in \Sigma$ determines the agent's decision procedure. Together, the economic environment and the decision procedure, induce a dynamic random choice in a straightforward way.

Let $\mathcal{Y}^B = B \times S$ be the global state space; denote by $y = (a, s)$ its generic element. Thus, $a \in B$ represents the currently drawn alternative, and $s \in S$ represents the state of the automaton. In the following, I will omit a superscript B in the notations whenever the menu B is fixed. Let $\mathbf{Y} = (Y_1, Y_2, ...)$ be a Markov chain with realizations $\mathbf{y} = (y_1, y_2, ...) \in \mathcal{Y}^{\mathbb{N}}$ that describes the dynamics of the model in periods $t = 1, 2, ...$ as follows.

In the first period, a random alternative is drawn and the automaton's state is $s = 1$:

$$Pr\Big(Y_1 = (a, s)\Big) \quad = \quad \rho^B(a) \cdot \delta_1^s$$

where I denote by $\delta_i^k = 1$ if $i = k$ and $\delta_i^k = 0$ if $i \neq k$. The dynamics of the Markov chain $\mathbf{Y}$ in the subsequent periods ($t > 1$) is described by the following stochastic $\mathcal{Y} \times \mathcal{Y}$ matrix:

$$Pr\Big(Y_t = (a, s) \;\Big|\; Y_{t-1} = (b, v)\Big) \;=$$

$$\begin{cases} \Big(1 - \tau\Big(v, \diamond, Q_{\iota(v)}(b)\Big)\Big)\eta\rho^B(a)\delta_1^s + (1-\eta)\delta_b^a\tau\Big(v, s, Q_{\iota(v)}(b)\Big) & \text{if} \quad v, s \neq \diamond \\ \tau\Big(v, \diamond, Q_{\iota(v)}(b)\Big)\delta_b^a & \text{if} \quad v \neq \diamond, s = \diamond \\ \delta_a^b\delta_\diamond^s & \text{if} \quad v = \diamond \end{cases} \tag{A.12}$$

Thus, the first case in eq. (A.12) tells that a transition between the two memory states occurs in one of the two ways. First, if the alternative has not been chosen in the previous period, then with probability η a new alternative is drawn according

to the probability distribution $\rho^B(a)$, and the state moves to $s = 1$, hence the term δ_1^s. Second, it can be that the decision procedure prescribes to go to state s with probability $\tau\Big(v, s, Q_{\iota(v)}(b)\Big)$ as a result of the information "$Q_{\iota(v)}(b) = j$" with $j \in \{0, 1\}$ acquired in the previous period. If this transition occurs, it must be that a new alternative is not drawn, hence the multiple $(1 - \eta)$, and the current alternative remains the same, hence the multiple δ_b^a. The other two cases in eq. (A.12) are interpreted similarly.

For each period $t \in \{1, 2, ...\}$, the probability that alternative $a \in B$ is chosen in some period $t' < t$ is given by

$$Pr\Big(a \text{ is chosen from } B \text{ in period } t' < t\Big) \;=\; Pr\Big(Y_t^B = (a, \diamond)\Big)$$

In this paper, I am concerned with the total probability of an alternative to be chosen in some period from the menu. Thus, I analyze the stochastic choice rule given by

$$p^B(a) := \lim_{t \to \infty} Pr\Big(Y_t^B = (a, \diamond)\Big) \tag{A.13}$$

Note that $p^B(a)$ depends on the language Q and the automaton σ via the stochastic matrix given by eq. (A.12) that governs the evolution of the Markov chain $\mathbf{Y}$. I will use a subscript σ to show this dependence when needed.

A.3 Proofs for the extensions

A.3.1 Proofs for the model with endogenous dismissal decision

Proof of Proposition 1.6.1

First, note that analysis, performed in the Appendix A.1, applies for the model with endogenous dismissal decision. Let Ψ be the set of decision rules in the baseline model, and Ψ' be the set of decision rules in the model with endogenous dismissal decision. Then $\Psi \subset \Psi'$, and, hence, for any language, its memory load and complexity in a model with endogenous dismissal decision should be weakly less than in a baseline model. To prove the reverse statement, let us consider a mapping $\zeta : \Psi' \to \Psi$ defined as follows.

Let $\psi' = \zeta(\psi)$. Then $S_{\psi'} = S_\psi$ and $\iota_{\psi'} = \iota_\psi$. Next, for all $s \in S^o$, $j \in \{0, 1\}$,

$$\tau_{\psi'}(s, v, j) = \begin{cases} \tau_\psi(s, v, j) & \text{if} \quad v \neq s \\ \tau_\psi(s, s, j) + \tau_\psi(s, new, j) & \text{if} \quad v = s \end{cases}$$

Thus, the decision rule $\psi' = \zeta(\psi)$ has the same state space and the same interrogation rule as ψ. Moreover, it has the same probability of transitions between two distinct memory states. The only difference is that, instead of transitioning to the dismissal decision with probability $\tau_\psi(s, new, j)$, the agent remains in the same memory state. Note that

$$\mathcal{M}(\psi') = \mathcal{M}(\psi), \qquad \kappa(\psi') \leq \kappa(\psi)$$

Note that the highest-probability paths of the decision rules ψ' and ψ are identical, since non of them can contain transitions to the dismissal state or a loop transition to the same memory state. Thus, by Lemma 1.5.2, if ψ' solves $(Q, \succeq)$, then ψ solves

$(Q, \succeq)$ as well. Hence, for any language, its memory load and complexity in a baseline model is weakly less than in a model with endogenous dismissal decision, completing the proof of the proposition. ■

Proof of Corollary 1.6.2

The corollary follows immediately from Proposition 1.6.1. ■

Proof of Proposition 1.6.3

For convenience, denote by Ψ' the set of decision rules in the model with endogenous dismissal decision, and by Ψ the set of decision rules in the baseline model. Then $\Psi \subset \Psi'$. Let us prove an analogue of Lemma A.2.19:

Lemma A.3.1 *Consider the model with endogenous dismissal decision. Let the preference relation $\succeq$ have m equivalence classes, and $n = \lceil \log_2(m) \rceil$. Then language Q is adaptive if and only if there exists a decision rule $\psi \in \Psi_n^{+dismissal}$ that solves the choice problem $(Q, \succeq)$.*

Proof of Lemma A.3.1. Suppose first that language Q is adaptive for preference relation $\succeq$, then Q is additive for $\succeq$ as defined in Section A.2.3. In this case, $\psi^+(Q, \succeq) \in \Psi \subset \Psi'$, where $\psi^+(Q, \succeq)$ is defined in Section A.2.3. Next, it is straightforward to see that Claim A.2.20 continues to hold in the model with endogenous dismissal decision as well. It follows that the decision rule $\psi^+(Q, \succeq)$ solves the choice problem $(Q, \succeq)$. Moreover, $\kappa(\psi^+(Q, \succeq)) = 3n$, hence $\psi^+(Q, \succeq) \in \Psi_n^{+dismissal}$, proving the if direction of the lemma.

The proof of the only if direction of the lemma is identical to the corresponding part of the proof of Lemma A.2.19. □

Lemma A.3.1 proves the first statement of the proposition. By Corollary 1.6.2, Theorem 1.4.3 continue to hold for the model with endogenous dismissal decision,

which in conjunction with the proven first statement of this Proposition implies its second statement. ■

A.3.2 Proofs for the model with recall of the past investigations

In the model with recall of the past investigations, analysis, performed in Section A.1 is no longer valid, since the Markov chain that describes the evolution of the system is no longer a Markov chain with renewal property according to the corresponding definition. Let us define formally the Markov chain that we are interested in.

As in the baseline model, let $\mathcal{Y}^B = B \times S \times \{new, old\}$ be the global state space, where $a \in B$ denotes the currently drawn alternative, $s \in S = S^o \cup \{\diamond\}$ denotes the automaton's state, and $x \in \{new, old\}$ denotes if the current alternative has been just drawn from the menu ($x = new$), or not ($x = old$). Denote by $y = (a, s, x)$ a generic element of the global state space. In the first period, a random alternative is drawn and the automaton's state is $s = 1$:

$$Pr\Big(Y_1^B = (a, s, x)\Big) \quad = \quad \rho^B(a) \cdot \delta_1^s \cdot \delta_{new}^x$$

The dynamics of the Markov chain $\mathbf{Y}$ in the subsequent periods ($t > 1$) in the model with recall of past investigations is described by the following stochastic $\mathcal{Y}\times\mathcal{Y}$ matrix:

$$Pr\Big(Y_t^B = (a,s,x) \;\Big|\; Y_{t-1}^B = (b,v,z)\Big) \;=$$

$$\begin{cases} \Big[\Big(1-\tau\Big(v,\diamond,Q_{\iota(v)}(b)\Big)\Big)\eta\tau^2(v,s) + \tau^1\Big(v,s,Q_{\iota(v)}(b),new\Big)\Big]\rho^B(a) & \text{if} \quad v,s\neq\diamond,\ x=new \\ (1-\eta)\delta_b^a\tau^1\Big(v,s,Q_{\iota(v)}(b),old\Big) & \text{if} \quad v,s\neq\diamond,\ x=old \\ \tau\Big(v,\diamond,Q_{\iota(v)}(b)\Big)\delta_b^a\delta_{old}^x & \text{if} \quad v\neq\diamond, s=\diamond \\ \delta_a^b\delta_\diamond^s\delta_{old}^x & \text{if} \quad v=\diamond \end{cases}$$

As usual, I analyze the stochastic choice rule given by

$$p^B(a) := \lim_{t\to\infty} Pr\Big(Y_t^B = (a,\diamond,old) + Pr\Big(Y_t^B = (a,\diamond,new)\Big) \;=\; \lim_{t\to\infty} Pr\Big(Y_t^B = (a,\diamond,old)\Big)$$

Lemma A.3.2 *Consider the model with recall of the past investigations. If the decision rule ψ solves the choice problem $(Q,\succeq)$, and N' is the image of the interrogation rule τ_ψ, then for any $c,d\in A$ such that $c\succ d$ there exists $i\in N'$ such that $c_i\neq d_i$.*

Proof of Lemma A.3.2. Towards a contradiction, assume that ψ solves $(Q,\succeq)$, and there are $c,d\in A$ such that $c\succ d$ and $c_i = d_i$ for all N'. Consider the menu $B=\{c,d\}$.

Claim A.3.3 *Consider the model with recall of the past investigations. Let N' be the image of the interrogation rule τ_ψ of a decision rule ψ. If $c_i = d_i$ for all $i\in N'$, then $p^{\{c,d\}}(c)/\rho^{\{c,d\}}(c) = p^{\{c,d\}}(d)/\rho^{\{c,d\}}(d)$.*

Proof of Claim A.3.3. Denote by $B = \{c, d\}$. Note that since $Q_{\iota(v)}(c) = Q_{\iota(v)}(d)$ for all $v \in S^o_\psi$, then

$$Pr\Big(Y^B_t = (a, s, old) \;\Big|\; Y^B_{t-1} = (a, v, x)\Big) \;=\; Pr\Big(Y^B_t = (b, s, old) \;\Big|\; Y^B_{t-1} = (b, v, x')\Big)$$

for all $x, x' \in \{new, old\}$ for all $v \in S^o$, $s \in S$. Let $q(a, s)$ be the probability to choose alternative $a \in \{c, d\}$ during a single investigation process when starting from memory state $s \in S^o$. Applying the above equation recursively, we get

$$q(c, s) \;=\; q(d, s) \qquad \forall s \in S^o$$

Thus,

$$\begin{aligned}
\frac{p^B(c)}{\rho^B(c)} &= \frac{1}{\rho^B(c)} \sum_{t=1}^{\infty} \sum_{y^t \in \mathcal{Y}^t} \sum_{v \in S^o} Pr\Big(Y^t = y^t\Big) \cdot \mathbb{1}\{x_t = new,\ a_t = c,\ s_t = v\} \cdot q(c, v) = \\
&= \sum_{t=1}^{\infty} \sum_{y^t \in \mathcal{Y}^t} \sum_{v \in S^o} Pr\Big(Y^t = y^t\Big) \cdot \mathbb{1}\{x_t = new,\ s_t = v\} \cdot q(c, v) = \\
&= \frac{1}{\rho^B(d)} \sum_{t=1}^{\infty} \sum_{y^t \in \mathcal{Y}^t} \sum_{v \in S^o} Pr\Big(Y^t = y^t\Big) \cdot \mathbb{1}\{x_t = new,\ a_t = d,\ s_t = v\} \cdot q(d, v) = \\
&= \frac{p^B(d)}{\rho^B(d)}
\end{aligned}$$

where $Y^t = (Y_1, ..., Y_t)$, and $y^t = (y_1, ..., y_t) \in \mathcal{Y}^t$. □

By Claim A.3.3, $p^{\{c,d\}}(c)/\rho^{\{c,d\}}(c) = p^{\{c,d\}}(d)/\rho^{\{c,d\}}(d)$, hence $p^{\{c,d\}}(c) \leq \rho^{\{c,d\}}(c) < 1$, hence $p^{\{c,d\}}(c)$ does not converge to one in the limit, contradicting the assumption that ψ solves $(Q, \succeq)$, and proving the lemma. ■

Proof of Theorem 1.6.4

Let us prove the first statement of the theorem, that is, if ψ solves $(Q, \succeq)$, then $\mathcal{M}(\psi) \geq n$, where $n = \lceil \log_2(m) \rceil$, and m is the number of the indifference classes of

$\succeq$. Towards a contradiction, assume that ψ solves $(Q, \succeq)$ and $\mathcal{M}(\psi) < n$. Let N be a set of attributes about which the agent inquires when using the decision rule ψ, then $|N| < n$. It follows that there is a pair of items $a, b \in A$ such that $a \succ b$, and $a_i = b_i$ for all $i \in N$, in contradiction to Lemma A.3.2.

Note that any decision rule that the agent can use in the baseline model is feasible for the agent in the model with the recall of past investigations. Hence, the second statement of the theorem follows from Proposition 1.3.2.

Proof of Theorem 1.6.5

Since any decision rule that the agent can use in the baseline model is feasible for the agent in the model with the recall of past investigations, the first statement of the theorem follows from Proposition 1.3.5.

Let us prove the second statement of the theorem.Let $m \in \{2, ..., |A|\}$. Let $\succeq$ be a preference relation such that $\tilde{a} \succ b$ for all $b \in A$; that is, $\tilde{a}$ is a unique $\succeq$-best element in A, and let $\succeq$ have m indifference classes; clearly, such preference relation exists. Enumerate alternatives in $A \backslash \{\tilde{a}\}$ by $a^1, ..., a^{|A|-1}$, and consider language $Q = \{Q_i\}_{i=1}^{|A|-1}$ with $Q_{i1} = \{a^i\}$, $Q_{i0} = A \backslash \{a^i\}$.

Towards a contradiction, assume that ψ solves $(Q, \succeq)$, and $S_\psi < |A| - 1$. Then there is $i \in N = \{1, ..., |A| - 1\}$ such that the corresponding attribute is not investigated in any memory state of ψ. Thus, $\tilde{a} \succ a^i$ and $\tilde{a}_j = a^i_j$ for all $j \in N$, contradicting Lemma A.3.2. Hence, $\mathcal{M}(Q) \geq |A| - 1$, which in conjunction with the proven first statement of the theorem proves the second statement of the theorem. ■

Appendix B

Appendices for Chapter 2

B.1 Proofs of Main Results

In this section, we provide proofs of all main statements—theorems, propositions and corollaries—except of Proposition 2.2.2, which is proven, among some other statements, in section B.2.

B.1.1 Proof of Theorem 2.2.1

The proof that axioms follow from the representation is straightforward. We omit it except of the proof for the *Continuity* axiom.

Lemma B.1.1 *Function* $W : \Pi \times H \to \mathbb{R}$ *given by eq. (2.2) is continuous.*

Proof. Follows from the Berge theorem (Theorem of Maximum). □

Assume choice correspondence $c(\cdot)$ has representation $(U, \mathcal{A})$. Consider $(A^k, f^k) \longrightarrow (A, f)$ such that $f^k \in c(A^k)$ for all $k = 1, 2, \ldots$. Then $f^k \in A^k$, and $\rho(\{f\}, A) \leq d(f, f^k) + \rho(\{f^k\}, A) \leq d(f, f^k) + \rho(A^k, A) \longrightarrow 0$, hence $f \in cl(A) = A$. Since $f^k \in c(A^k)$ for $k = 1, 2, \ldots$, then there exists $P^k \in \mathcal{A}$ such that

$W_{P^k}(f^k) \geq W_P(g)$ for all $g \in A^k$. Since Π is compact, there is a convergent subsequence $P^{k_m} \longrightarrow P \in \Pi$. Since $\mathcal{A}$ is closed, $P \in \mathcal{A}$. Since $A^{k_m} \longrightarrow A$, for arbitrary $g \in A$ there exists $g^{k_m} \in A^{k_m}$ such that $g^{k_m} \longrightarrow g$. Then $W_{P^{k_m}}(f^{k_m}) \geq W_{P^{k_m}}(g^{k_m})$ for all $m = 1, 2, \ldots$. Since function $(P, f, g) \to W_P(f) - W_P(g)$ is continuous by Lemma B.1.1, we conclude that $W_P(f) \geq W_P(g)$. Since this holds for all $g \in A$, then $f \in c(A)$. Thus, *Continuity* holds for $c(\cdot)$.

Assume now that choice correspondence $c(\cdot)$ satisfies Axioms 1-7. We show that $c(\cdot)$ admits a framed ambiguity representation.

Lemma B.1.2 *Continuity (Axiom 6) implies axioms defined below:*
Act Continuity: $\forall A \in \mathcal{K} \;\; \{f \in H \mid f \in c(A \cup \{f\})\}$ *is closed*
Menu Continuity: $\forall f \in H \;\; \{A \in \mathcal{K} \mid f \in c(A)\}$ *is closed*

Proof. *Act Continuity* follows from $f^n \to f \implies (A \cup \{f^n\}, f^n) \longrightarrow (A \cup \{f\}, f)$.
Menu Continuity follows from $A^n \longrightarrow A \implies (A^n, f) \to (A, f)$. □

We say that the restriction of choice correspondence c_0 on the set of compact menus of constant acts $\mathcal{K}_0$ has an expected utility representation, if there is a vNM expected utility $U : \triangle X \to R$ such that $\forall A \in \mathcal{K}_0 \quad [(p, ..., p) \in c(A) \iff [U(p) \geq U(q) \;\; \forall q \in A]]$. Note that if $c(\cdot)$ satisfies Axioms 1-7, so does $c_0(\cdot)$.

Lemma B.1.3 *Given Axioms 1-7, c_0 has a non-degenerate expected utility representation that is unique up to a positive affine transformation.*

Proof. Consider binary relation $\succeq_{c_0} \subseteq H_0 \times H_0$ given by $p \succeq_{c_0} q$ if $p \in c_0(\{p, q\})$. Axioms α and *C-*β imply *WARP* for c_0, thus $\succeq_{c_0}$ is complete and transitive. The *C-Independence* for $c_0(\cdot)$ implies Independence for $\succeq_{c_0}$. *Act Continuity* implies $\{q \in H \mid p \succeq_{c_0} q\}$ is closed, and *Menu Continuity* implies $\{q \in H \mid p \preceq_{c_0} q\}$ is closed

$\forall p \in H_0$. *C-Non-Degeneracy* implies $p \not\succeq q$ for some p, q. The Lemma follows from the Mixture Space Theorem. $\square$

Lemma B.1.3 allows us to identify $U(\cdot)$ from $c(\cdot)$. Since X and $\triangle X$ are compact, there is the best prize $x \in X$ and the worst prize $y \in Y$. Abusing notations, we denote $\delta_x \in \triangle X$ by x and δ_y by y. We normalize $U(\cdot)$ by $1 = U(x) \geq U(p) \geq U(y) = 0$ for all $p \in \triangle X$.

Lemma B.1.4 *If $c(\cdot)$ satisfies Axioms 1-7, then it satisfies the following axiom:*
Monotonicity: $\forall f, g \in H \qquad f(s) \in c(\{f(s), g(s)\})\ \forall s \in S$ *implies*
$\forall A \in \mathcal{K}$ *(i)* $[g \in c(A) \implies f \in c(A \cup \{f\})]$ *and (ii)* $[f \in A \implies c(A) \subseteq c(A \cup \{g\})]$

Proof. For all menus $A \in \mathcal{K}$ and acts $f \in H$ denote

$$A_{0.5} \equiv 0.5A + 0.5(0.5x + 0.5y), \qquad f_{0.5} \equiv 0.5f + 0.5(0.5x + 0.5y)$$

Assume $f(s) \in c(\{f(s), g(s)\})\ \ \forall s \in S$. Then by Lemma B.1.3, $U(f(s)) \geq U(g(s))$ $\forall s \in S$.

Proof of (i) Let $g \in c(A)$, then by *C-Independence* $g_{0.5} \in c(A_{0.5})$. Consider $f^n_{0.5} \equiv (1 - 1/n)\, f_{0.5} + (1/n)x$ for $n = 1, 2, ...$ Then $\forall s \in S$ we have:

$$U(f^n_{0.5}(s)) = (1 - 1/n)\,(0.5U(f(s)) + 0.25) + 1/n \ >$$
$$0.5U(f(s)) + 0.25 = U(f_{0.5}(s)) \geq U(g_{0.5}(s))$$

By Lemma B.1.3, $g_{0.5}(s) \notin c(\{g_{0.5}(s), f^n_{0.5}(s)\})\ \forall s \in S$. *Strict Monotonicity* implies $g_{0.5} \notin c(\{g_{0.5}, f^n_{0.5}\})$, then by α we have $g_{0.5} \notin c(A_{0.5} \cup \{f^n_{0.5}\})$. By *Aizerman's Property* $f^n_{0.5} \in c(\{A_{0.5} \cup f^n_{0.5}\})$. Since $f^n_{0.5} \longrightarrow f_{0.5}$, by *Act Continuity* $f_{0.5} \in c(A_{0.5} \cup \{f_{0.5}\})$. By *C-Independence* $f \in c(A \cup \{f\})$.

Proof of (ii) Assume $f \in A$ and consider $h \in c(A)$. By *C-Independence* $h_{0.5} \in$

$c(A_{0.5})$. Consider $g^n_{0.5} \equiv (1-1/n)\, f_{0.5} + (1/n)y$ for $n = 1, 2, ...$ Then $\forall s \in S$ we have:

$$U(g^n_{0.5}(s)) = (1-1/n)\,(0.5U(f(s)) + 0.25) \;<$$
$$0.5U(g(s)) + 0.25 = U(g_{0.5}(s)) \leq U(f_{0.5}(s))$$

By Lemma B.1.3, $g^n_{0.5}(s) \notin c(\{g^n_{0.5}(s), f_{0.5}(s)\})$ for all $s \in S$, by *Strict Monotonicity* $g^n_{0.5} \notin c(\{g^n_{0.5}, f_{0.5}\})$, by α we have $g^n_{0.5} \notin c(A_{0.5} \cup \{g^n_{0.5}\})$. By *Aizerman's Property* $h_{0.5} \in c(A_{0.5} \cup \{g^n_{0.5}\})$. Next, $A_{0.5} \cup \{g^n_{0.5}\} \longrightarrow A_{0.5} \cup \{g_{0.5}\}$, then by *Menu Continuity* $h_{0.5} \in c(A_{0.5} \cup \{g_{0.5}\})$. Finally, by *C-Independence* $h \in c(A \cup \{g\})$. □

Define binary relations $\succeq^*, \sim \;\subseteq H \times H$ by

$$f \succeq^* g \;\text{ if }\; f(s) \in c(\{f(s), g(s)\})\; \forall s \in S, \qquad f \sim g \;\text{ if }\; [f \succeq^* g \text{ and } g \succeq^* f]$$

Using Lemma B.1.3, we get $f \succeq^* g \iff U(f(s)) \geq U(g(s)) \forall s \in S$ and $f \sim g \iff U(f(s)) = U(g(s))\; \forall s \in S$. Thus, $\sim$ is an equivalence relation on the set of acts. Consider a subset of acts that yield a convex combination of x and y at each state of the world:

$$\widehat{H} \;\equiv\; \left\{\; h \in H \mid \exists \mathbf{z} \in [0,1]^{|S|} \;:\; f(s) = z_s x + (1-z_s)y \;\forall s \in S \;\right\}$$

For each act $f \in H$ construct a canonical act $\widehat{f} \in \widehat{H}$ as follows:

$$\widehat{f}(s) \;\equiv\; U(f(s))x + (1 - U(f(s)))y$$

Act $\widehat{f}$ is well-defined, since $0 \leq U(f(s)) \leq 1$. We can see that if $f \sim g$, then $\widehat{f} = \widehat{g}$ and also $\widehat{f} = \widehat{\widehat{f}}$, since $U(x) = 1$, $U(y) = 0$. Thus, a canonical act $\widehat{f}$ represents an equivalence class of relation $\sim$. Given menu $A \in \mathcal{K}$, define canonical menu $\widehat{A}$ to be the set consisting of canonical acts corresponding to all equivalence classes of relation

$\sim$ in A:

$$\widehat{A} \equiv \{ h \in \widehat{H} \mid \exists f \in A : f \sim h \} \tag{B.1}$$

Lemma B.1.5 *Assume Axioms 1-7 hold. Then for any menu $A \in \mathcal{K}$ the corresponding canonical menu is a menu, i.e. $\widehat{A} \in \mathcal{K}$. Moreover,*

$$f \in c(A) \iff \widehat{f} \in c(\widehat{A})$$

Proof. We first show that $\widehat{A}$ is a menu. Consider an arbitrary sequence $\{\widehat{f}^n\} \subseteq \widehat{A}$, where $\{f^n\} \subseteq A$. Since A is compact, there is a convergent subsequence $f^{n_k} \longrightarrow f \in A$. Therefore, by continuity of $U(\cdot)$, $U(f^{n_k}(s)) \longrightarrow U(f(s))$ for all s, hence $\widehat{f}^{n_k} \longrightarrow \widehat{f} \in \widehat{A}$. Since an arbitrary sequence in $\widehat{A}$ contains a convergent subsequence, $\widehat{A}$ is compact, and since $\widehat{A}$ non-empty, $\widehat{A} \in \mathcal{K}$.

Claim B.1.6 *Let $\{h_1, ..., h_k\} \subset H$. Assume $\forall i \; \exists g_i \in A$: $g_i \sim h_i$. Then:*

$$c(A \cup \{h_1, ..., h_k\}) = c(A) \cup \{h \in \{h_1, ..., h_k\} \mid \exists g \in c(A) : h \sim g\}$$

Proof of Claim B.1.6. Let $h \sim g \in B$. Case (a): $g \notin c(B)$. If $h \in c(B \cup \{h\})$, then by *Monotonicity*, $g \in c(B \cup \{h\})$, and by α, $g \in c(B)$, contradiction. Therefore, $h \notin c(B \cup \{h\})$, and by *Aizerman's Property* and α, $c(B \cup \{h\}) = c(B)$.

Case (b): $g \in c(B)$. If $h \notin c(B \cup \{h\})$, then by *Aizerman's Property*, $g \in c(B \cup \{h\})$, and by *Monotonicity*, $h \in c(B \cup \{h\})$. Next, since $g \succeq^* h_{i+1}$, by *Monotonicity*, $c(B) \subseteq c(B \cup \{h\})$. By α and the above calculations, $c(B \cup \{h\}) = c(B) \cup \{h\}$.

Note that by *Monotonicity* and α, if $h_{k+1} \sim h_j$ for $j \leq k$ and $h_j \in c(A \cup \{h_1, ..., h_k\})$, then $g_j \in c(A \cup \{h_1, ..., h_k\})$ for $g_j \in A$. A straightforward induction argument that uses cases (a) and (b) completes the proof of the Claim. $\square$

For any menu A, since it is compact, there exist a finite set of acts F_n such that $F_n \subseteq A \subseteq \bigcup_{f \in F_n} \{g \in H |\ d(g,f) < 1/n\}$. Denote by $F^n \equiv \bigcup_{m=1}^{n} F_m$, $\widehat{F}^n = \{h \in \widehat{H} \mid \exists f \in F^n : h = \widehat{f}\}$.

Claim B.1.7 *For any menu A: (i) $F^n \cup \widehat{A} \longrightarrow A \cup \widehat{A}$, (ii) $A \cup \widehat{F}^n \longrightarrow A \cup \widehat{A}$.*

Proof of Claim B.1.7. Statement (i) is obvious, we prove statement (ii). We know that $\widehat{F}^n \subseteq \widehat{A}$. Consider $h \in \widehat{A}$. Then $\exists f \in A$ such that $f \sim h$. Since each of F_n induces a cover of A by balls of radius $1/n$, for any n there is $f^n \in F_n$, $d(f, f^n) < 1/n$, hence $f^n \longrightarrow f$; note also that $\{\widehat{f}^m\}_{m=1}^n \subseteq \widehat{F}^n$. By continuity of $U(\cdot)$, $U(f^n(s)) \longrightarrow U(f(s))$, hence $\widehat{f}^n \longrightarrow \widehat{f} = h$. Since $U(\cdot)$ is uniformly continuous on a compact set $\triangle X$, and $A \cup \hat{F}^n \subseteq A \cup \widehat{A}$, then

$$\rho(A \cup \widehat{F}^n, A \cup \widehat{A}) \;=\; \max_{h \in A \cup \widehat{A}} \min_{g \in A \cup \widehat{F}^n} d(g,h) \;\leq\; \max_{h \in \widehat{A}} \min_{g \in \widehat{F}^n} d(g,h) \;\longrightarrow\; 0 \qquad \square$$

Assume $f \in c(A)$. Without loss of generality (WLOG), $f \in F_1 \subseteq F^n$. By Claim B.1.6, $f \in c(A \cup F^n)$ for all n. By *Menu Continuity* and Claim B.1.7, $f \in c(A \cup \widehat{A})$. By *Monotonicity*, $\widehat{f} \in c(A \cup \widehat{A})$. By α, $\widehat{f} \in c(\widehat{A})$.

Assume $\widehat{f} \in c(\widehat{A})$. WLOG, $f \in F_1 \subseteq F^n$. By Claim B.1.6, $\widehat{f} \in c(\widehat{A} \cup F^n)$. By *Menu Continuity* and Claim B.1.7, $\widehat{f} \in c(\widehat{A} \cup A)$. By *Monotonicity*, $f \in c(\widehat{A} \cup A)$. By α, $f \in c(A)$. $\square$

Lemma B.1.5 allows us to focus on finding a representation for the choice correspondence restricted to the subset of canonical menus. Once we find a framed ambiguity representation of the restriction, it will automatically represent the whole choice correspondence as well, since $U(f(s)) = U(\widehat{f}(s)))$ for all $s \in S$ and $f \in H$. Thus, we assume for now that $H = \widehat{H}$. The set of canonical acts is isomorphic to the $|S|$- dimensional hypercube $[0,1]^{|S|}$ via identification of $\mathbf{z} \in [0,1]^{|S|}$ with $f \in \widehat{H}$ by $z_s = f(s)(x)$. We will use both $\mathbf{z} \in [0,1]^{|S|}$ and $f \in H$ as generic acts in the sub-

sequent calculations. Denote $\iota = (1, ..., 1) \in R^{|S|}$, then act $\mathbf{z} \in [0,1]^{|S|}$ is constant if and only if $\mathbf{z} = \overline{z}\iota$ for some $\overline{z} \in [0,1]$. Next, the premise of the *Strict Monotonicity* Axiom for acts $\mathbf{z}, \mathbf{z}'$ looks as $\mathbf{z} >> \mathbf{z}'$, and the premise of the *Monotonicity* Axiom looks as $\mathbf{z} \geq \mathbf{z}'$.

Lemma B.1.8 *Let Menu Continuity hold. Then $\forall f \in H, \forall A \in \mathcal{K}$ if $f \in c(A)$, then there is a menu $D \in \mathcal{K}$ such that $A \subseteq D$, $f \in c(D)$ and D is maximal subject to this; that is, for any menu $D' \neq D$ such that $D \subseteq D'$, $f \notin c(D')$.*

Proof. Consider the set of menus that contain A such that f is chosen:

$$\mathcal{Q}(A, f) \equiv \{B \in \mathcal{K} \mid f \in c(B) \text{ and } A \subseteq B\},$$

partially ordered by set inclusion. Notice that $A \in \mathcal{Q}(A, f) \neq \varnothing$. Let $\mathcal{T}$ be an arbitrary non-empty chain (totally ordered subset) in $\mathcal{Q}(f, A)$. We argue that $\mathcal{T}$ has an upper bound in $\mathcal{Q}(f, A)$ equal to

$$C = cl\left(\bigcup_{B \in \mathcal{T}} B\right),$$

where closure is taken with respect to the topology on the space of acts H, and that $C \in \mathcal{Q}(A, f)$. Notice that $H \simeq [0,1]^{|S|}$ is compact. C is a closed subset of a compact set, hence it is compact, and $C \in \mathcal{K}$. Next, $B \subseteq C$ for all $B \in \mathcal{T}$ by construction. It remains to show that $f \in c(C)$. Let us show that for any $\epsilon > 0$ there is $B \in \mathcal{T}$ such that for any $B' \in \mathcal{T}$ such that $B \subseteq B'$ it is true that $\rho(B', C) < \epsilon$. Consider an open cover of C by a family of open balls of radius $\epsilon/3$ centered at all points of C:

$$\mathcal{G} = \{\{g \in H \mid \rho(g, f) < \epsilon/3\}_f\}_{f \in C}$$

Since C is a compact set, there is a finite subcover

$$\{\{g \in H \mid \rho(g, f_i) < \epsilon/3\}\}_{i=1}^{N} \subseteq \mathcal{G}, \qquad C \subseteq \bigcup_{i \in \{1,...,N\}} \{g \in H \mid \rho(g, f_i) < \epsilon/3\}$$

Recall that $B \subseteq C$ for any $B \in \mathcal{T}$, hence

$$\rho(B, C) = \sup_{g \in C} \left(\inf_{h \in B} d(g, h) \right) \leq \epsilon/3 + \max_{i \in \{1,...,N\}} \left(\inf_{h \in B} d(f_i, h) \right)$$

Since $f_i \in C$, for any $i = 1, ..., N$ there is $B_i \in \mathcal{T}$ such that there is $g_i \in B_i$ such that $d(f_i, g_i) < \epsilon/3$. Since $\mathcal{T}$ is totally ordered by set inclusion, there is $\widetilde{B} \in \{B_1, ..., B_N\}$ such that $B_i \subseteq \widetilde{B}$ for all $i = 1, ..., N$ (every totally ordered finite set has a maximum element). Therefore, $\{g_1, ..., g_n\} \subseteq \widetilde{B}$, hence $\rho(\widetilde{B}, C) < 2\epsilon/3$. Since we can find such $\widetilde{B} \in \mathcal{T}$ for arbitrary $\epsilon > 0$, it follows that $C \in cl(\mathcal{B})$. Therefore, by *Menu Continuity* $f \in c(C)$. We proved that every non-empty chain in $\mathcal{Q}(A, f)$ has an upper bound in $\mathcal{Q}(A, f)$, hence by Zorn's Lemma $\mathcal{Q}(A, f)$ has at least one maximal element D. ■

For a given act $\mathbf{z} \in H$ denote

$$\mathcal{L}(\mathbf{z}) \equiv \{ A \in \mathcal{K} \mid \mathbf{z} \in c(A) \text{ and } [A \subseteq A', \ \mathbf{z} \in c(A') \implies A' = A] \} \tag{B.2}$$

In other words, $\mathcal{L}(\mathbf{z})$ is a set of menus that are maximal with respect to $\mathbf{z}$ being chosen from a menu. Lemma B.1.8 implies that $\mathcal{L}(\mathbf{z}) \neq \varnothing$ for any $\mathbf{z} \in H$, since $\mathbf{z} \in c(\{\mathbf{z}\})$. For $\lambda \in (0, 1]$ denote

$$Z_\lambda = \lambda[0, 1]^{|S|} + (1 - \lambda)\{0.5\iota\} \tag{B.3}$$

Thus, Z_λ is a hypercube homothetic to $Z_1 = H = [0, 1]^{|S|}$ with the same center point 0.5ι. Similarly, for arbitrary menu $B \in \mathcal{K}$ denote $B_\lambda = \lambda B + (1-\lambda)\{0.5\iota\}$. Notice that $B_\lambda \subseteq Z_\lambda$. Next, by *C-Independence* $c(B_\lambda) = \lambda c(B) + (1-\lambda)\{0.5\iota\}$. Therefore, we are able to reconstruct the full choice correspondence knowing choice from menus

$B_\lambda \subseteq Z_\lambda$. For arbitrary $\mathbf{z}$ and maximal menu $A \in \mathcal{L}(\mathbf{z})$ define menu's "certainty equivalent"

$$a(A) \;=\; \max\{b \in [0,1] \mid b \cdot \iota \in A\} \tag{B.4}$$

Function $a(.)$ is well-defined, since by *Monotonicity* any maximal menu contains the worst act $(0, ..., 0)$, and compactness of A insures the existence of maximum. Consider the set of menus that are maximal for some act $\mathbf{z} \in Z_{0.2}$:

$$\mathcal{D}_{0.2} = \{A \in \mathcal{K} \mid \exists \mathbf{z} \in Z_{0.2} :\; A \in \mathcal{L}(\mathbf{z})\} \tag{B.5}$$

Since $0.4\iota \leq \mathbf{z} \leq 0.6\iota$ for all $\mathbf{z} \in Z_{0.2}$, by *Monotonicity* $a(A) \geq 0.4$, and by *Strict Monotonicity* $a(A) \leq 0.6$ for any $A \in \mathcal{D}_{0.2}$. For a vector $\mathbf{v} \in R^{|S|}$ denote $|\mathbf{v}| = \max_{s \in S} |v_s|$. We do not use *Continuity* in calculations involving $|\cdot|$, thus we don't care about the compatibility of $|\cdot|$ with metric d on H. Let

$$O \;=\; \left\{ \mathbf{v} \in R^{|S|} \;\middle|\; \sum_{s \in S} v_s = 0 \text{ and } |\mathbf{v}| = 1 \right\} \tag{B.6}$$

Define function $J :\; \mathcal{D}_{0.2} \times O \to R$ by the following formula:

$$J_A(\mathbf{v}) \;\equiv\; -\max\{b \in [-1,1] \mid a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot b \cdot \iota \in A\} \tag{B.7}$$

Notice that $b \in [-1,1]$ and $\mathbf{v} \in O$, thus for $\mathbf{z} = a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot b \cdot \iota$ we have $0.2\iota \leq \mathbf{z} \leq 0.8\iota$, thus $\mathbf{z} \in Z_{0.6} \subset Z_1 = H$, and $\mathbf{z}$ is an act; and the maximum exists, since A is compact. Next note that $a(A)\iota + 0.1 \cdot \mathbf{v} - 0.1\iota \leq a(A)\iota$, hence by *Monotonicity* $a(A)\iota + 0.1 \cdot \mathbf{v} - 0.1\iota \in A$. Similarly, for any $\epsilon > 0$ we have $a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot (1+\epsilon)\iota >> a(A)\iota$, hence by *Strict Monotonicity* $a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot (1+\epsilon)\iota \notin A$. Thus, there is no loss of generality to consider $b \in [-1,1]$ versus $b \in R$ in the definition of $J.(\cdot)$. Therefore, for any $\mathbf{v} \in O$, $A \in \mathcal{D}_{0.2}$ and any b such that $a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot b \cdot \iota \in Z_1$

is an act, using *Monotonicity* and maximality of $A \in \mathcal{D}_{0.2}$, we may conclude:

$$b \leq -J_A(\mathbf{v}) \iff a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot b \cdot \iota \in A \tag{B.8}$$

For $\mathbf{z} \in R^{|S|}$ consider its unique decomposition

$$\mathbf{z} = \overline{z} \cdot \iota + \mathbf{z}_\perp \quad , \quad \text{where} \quad \overline{z} = |S|^{-1} \cdot \sum_{s \in S} (\mathbf{z})_s \quad , \quad \text{and} \quad \mathbf{z}_\perp = \mathbf{z} - \overline{z} \cdot \iota \tag{B.9}$$

Then $\sum_{s \in S} (\mathbf{z}_\perp)_s = 0$. Define function $I : \mathcal{D}_{0.2} \times R^{|S|} \to R$ as follows:

$$I_A(z) \equiv \begin{cases} \overline{z} + |\mathbf{z}_\perp| \cdot J_A\left(\dfrac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}\right) & \text{if} \quad \mathbf{z}_\perp \neq \mathbf{0} \\ \overline{z} & \text{if} \quad \mathbf{z}_\perp = \mathbf{0} \end{cases} \tag{B.10}$$

The following lemma is an analogue of Lemma 3.3. in Gilboa and Schmeidler (1989).

Lemma B.1.9 *Let Axioms 1-7 hold. Then for all $A \in \mathcal{D}_{0.2}$ and all $\mathbf{z} \in Z_{0.4}$ we have $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq a(A)$. Moreover, function $I_A(\cdot)$ is:*

(i) Monotone: $\mathbf{z} \geq \mathbf{z}' \implies I_A(\mathbf{z}) \geq I_A(\mathbf{z}')$.

(ii) Positively homogeneous: $I_A(\alpha \mathbf{z}) = \alpha I_A(\mathbf{z})$ for all $\alpha \geq 0$.

(iii) Concave: $I_A(\lambda \mathbf{z} + (1-\lambda)\mathbf{z}') \geq \lambda I_A(\mathbf{z}) + (1-\lambda) I_A(\mathbf{z}')$ for all $\lambda \in (0,1)$.

(iv) C-additive: $I_A(\mathbf{z} + \beta \cdot \iota) = I_A(\mathbf{z}) + I_A(\beta \cdot \iota)$ for all $\beta \in R$.

(v) Normalized: $I_A(\iota) = 1$.

Proof. Statements (ii), (iv) and (v) hold by construction. Let $f \in Z_{0.2}$ be an act such that $A \in \mathcal{L}(f)$. It exists by the definition of $\mathcal{D}_{0.2}$. Let' prove the main statement $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq a(A)$, so assume $\mathbf{z} \in Z_{0.4}$.

Case 1. $|\mathbf{z}_\perp| = 0.1$, then $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq a(A)$ by eq. (B.8).

Case 2. $|\mathbf{z}_\perp| = 0$, then $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq a(A)$ by the definition of $a(A)$.

Case 3. $|\mathbf{z}_\perp| \neq 0, 0.1$ and $|\overline{z} - a(A)| \leq |\mathbf{z}_\perp|$. Denote

$$\mathbf{y} \;\equiv\; \left[a(A) + 0.1\frac{\overline{z}}{|\mathbf{z}_\perp|} - 0.1\frac{a(A)}{|\mathbf{z}_\perp|}\right]\iota \;+\; 0.1\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}$$

Since $a(A) \in [0.4, 0.6]$, we have $0.2\iota = (0.4-0.1-0.1)\iota \leq \mathbf{y} \leq (0.6+0.1+0.1)\iota = 0.8\iota$, thus $\mathbf{y} \in Z_{0.6} \subseteq Z_1$. Since $\dfrac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|} \in O$, by eq. (B.8) we have:

$$\mathbf{y} \in A \iff \frac{\overline{z}}{|\mathbf{z}_\perp|} - \frac{a(A)}{|\mathbf{z}_\perp|} \;\leq\; -J_A\left(\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}\right) \iff I_A\left(\frac{\mathbf{z}}{|\mathbf{z}_\perp|}\right) \leq \frac{a(A)}{|\mathbf{z}_\perp|} \iff I_A(\mathbf{z}) \leq a(A)$$

Case 3.1. $|\mathbf{z}_\perp| < 0.1$. Denote $\lambda = 10|\mathbf{z}_\perp| \in (0, 1)$. Then we have

$$\mathbf{z} \;=\; \lambda\mathbf{y} + (1-\lambda)a(A)\cdot\iota$$

Let $I_A(\mathbf{z}) \leq a(A)$, then $\{y, a(A)\cdot\iota\} \subset A$, and, since $f \in A$, by *No C-Hedging* $f \in c(A \cup \{\mathbf{z}\})$, thus $\mathbf{z} \in A$ by maximality of $A \in \mathcal{L}(f)$. Now assume that $I_A(\mathbf{z}) > a(A)$. Take $\epsilon = 0.01\min\left\{1, \dfrac{1-\lambda}{\lambda}, \dfrac{I_A(\mathbf{z}) - a(A)}{\lambda}\right\} > 0$. Then $\mathbf{y} - \epsilon\iota \in Z_1$, and by eq. (B.8) and our choice of ϵ also $\mathbf{y} - \epsilon\iota \notin A$. Similarly, $\left(a(A) + \dfrac{\lambda}{1-\lambda}\epsilon\right)\iota \in Z_1$, and $\left(a(A) + \dfrac{\lambda}{1-\lambda}\epsilon\right)\iota \notin A$ by definition of $a(A)$. Finally,

$$\mathbf{z} \;=\; \lambda(\mathbf{y} - \epsilon\iota) + (1-\lambda)\left(a(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota$$

By maximality of $A \in \mathcal{L}(f)$ we get $f \notin c(A \cup \{\mathbf{y} - \epsilon\iota\})$ and $f \notin c\left(A \cup \left\{\left(a(A) + \dfrac{\lambda}{1-\lambda}\epsilon\right)\iota\right\}\right)$. Therefore, we can apply *Indirect Ambiguity Aversion* to get $f \notin c(A \cup \{\mathbf{z}\})$, implying $\mathbf{z} \notin A$.

Case 3.2: $|\mathbf{z}_\perp| > 0.1$. Denote $\lambda = (10|\mathbf{z}_\perp|)^{-1} \in (0, 1)$. Then

$$\mathbf{y} \;=\; \lambda\mathbf{z} + (1-\lambda)a(A)\cdot\iota$$

Consider $I_A(\mathbf{z}) > a(A)$. Then $\mathbf{y} \notin A$. Assume for a contradiction that $\mathbf{z} \in A$, then by *No C-Hedging* $f \in c(A \cup \{\mathbf{y}\})$, hence by maximality of A we get $\mathbf{y} \in A$, contradiction. Thus, $\mathbf{z} \notin A$. Consider now $I_A(\mathbf{z}) \leq A$. Then $\mathbf{y} \in A$. Assume for a contradiction that $\mathbf{z} \notin A$. Since $A \cup \{\mathbf{z} - (1/n)\iota\} \to A \cup \{z\}$, then if for arbitrary large n we have $f \in c(A \cup \{\mathbf{z} - (1/n)\iota\})$, the by *Menu Continuity* $f \in c(A \cup \{\mathbf{z}\})$, contradicting maximality of $A \in \mathcal{L}(f)$. Therefore, there exists N such that $\mathbf{z} - (1/n)\iota \notin A$ for all $n \geq N$. Take $\epsilon = 0.01\text{min } \{1/N, (1-\lambda)/\lambda\} > 0$. Then $\mathbf{z} - \epsilon \cdot \iota \in Z_1 \backslash A$ and $\left(a(A) + \dfrac{\lambda}{1-\lambda}\epsilon\right)\iota \in Z_1 \backslash A$. Next,

$$\mathbf{y} \;=\; \lambda(\mathbf{z} - \epsilon\iota) + (1-\lambda)\left(a(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota$$

Since $f \notin c(A \cup \{\mathbf{z} - \epsilon\iota\})$ and $f \notin c\left(A \cup \left\{\left(a(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota\right\}\right)$, the *Indirect Ambiguity Aversion* implies $f \notin c(A \cup \{\mathbf{y}\})$, contradicting $\mathbf{y} \in A$.

Case 4. $|\mathbf{z}_\perp| \neq 0, 0.1$ and $\overline{z} - a(A) > |\mathbf{z}_\perp|$. Then $\mathbf{z} = \overline{z}\iota + \mathbf{z}_\perp >> (a(A) + 0.5(\overline{z} - a(A) - |\mathbf{z}_\perp|))\iota >> a(A)\iota$. Thus $f \notin c(A \cup \{(a(A) + 0.5(\overline{z} - a(A) - |\mathbf{z}_\perp|))\iota\})$ by the definition of $a(A)$, and by *Monotonicity* $f \notin c(A \cup \{\mathbf{z}\})$, implying $\mathbf{z} \notin A$. In this case $I_A(\mathbf{z}) \geq \overline{z} - |\mathbf{z}_\perp| > a(A)$, so the main statement of the Lemma holds.

Case 5. $|\mathbf{z}_\perp| \neq 0, 0.1$ and $\overline{z} - a(A) < -|\mathbf{z}_\perp|$. Then $\mathbf{z} = \overline{z}\iota + \mathbf{z}_\perp \leq a(A)\iota \in A$. Hence, $f \in c(A \cup \{\mathbf{z}\})$ by *Monotonicity*, and $\mathbf{z} \in A$ by maximality of A. From the other hand, in this case $I_A(\mathbf{z}) \leq \overline{z} + |\mathbf{z}_\perp| \leq a(A)$, so the main statement of the Lemma holds.

Case 1-5 cover all possibilities, thus $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq a(A)$ for all $\mathbf{z} \in Z_{0.4}$, $A \in \mathcal{D}_{0.2}$.

Proof of (iii). It is easy to see from the definition of $I_A(\cdot)$ that $|I_A(\mathbf{z})| \leq 2|\mathbf{z}|$ for any $\mathbf{z} \in R^{|S|}$. Consider arbitrary $\mathbf{z}, \mathbf{z}' \in R^{|S|}$. If $\mathbf{z} = \mathbf{z}' = \mathbf{0}$, then

$I_A(\mathbf{z}+\mathbf{z}') = I_A(0) = 0 = I_A(0)+I_A(0) = I_A(\mathbf{z})+I_A(\mathbf{z}')$. Otherwise, denote $t = 0.01\cdot(\max\{|z|,|z'|\})^{-1} > 0$. Then $-0.02 \leq I_A(t\mathbf{z}), I_A(t\mathbf{z}') \leq 0.02$. Consider

$$\tilde{\mathbf{z}} = t\mathbf{z} + (a(A) - I_A(t\mathbf{z}))\,\iota, \qquad \tilde{\mathbf{z}'} = t\mathbf{z}' + (a(A) - I_A(t\mathbf{z}'))\,\iota$$

Take arbitrary $\epsilon \in (0, 0.01)$. Since $a(A) \in [0.4, 0.6]$, then $0.3\iota \leq \tilde{\mathbf{z}} + \epsilon\iota, \tilde{\mathbf{z}'} + \epsilon\iota \leq 0.7\iota$, hence $\tilde{\mathbf{z}} + \epsilon\iota, \tilde{\mathbf{z}'} + \epsilon\iota \in Z_{0.4}$, thus the main statement of the Lemma applies to these acts. Next, $I_A(\tilde{\mathbf{z}}+\epsilon\iota) = I_A(\tilde{\mathbf{z}'}+\epsilon\iota) = a(A)+\epsilon > a(A)$, hence $\tilde{\mathbf{z}}+\epsilon\iota, \tilde{\mathbf{z}'}+\epsilon\iota \notin A$. Thus, $f \notin c(A\cup\{\tilde{\mathbf{z}}+\epsilon\iota\})$ and $f \notin c(A\cup\{\tilde{\mathbf{z}}'+\epsilon\iota\})$, hence by *Indirect Ambiguity Aversion*, maximality of A and the fact that $Z_{0.4}$ is convex, for all $\lambda \in (0,1)$ we have:

$$\lambda(\tilde{\mathbf{z}}+\epsilon\iota)+(1-\lambda)(\tilde{\mathbf{z}'}+\epsilon\iota) \in Z_{0.4}\backslash A \implies I_A(\lambda(\tilde{\mathbf{z}}+\epsilon\iota)+(1-\lambda)(\tilde{\mathbf{z}'}+\epsilon\iota)) > a(A) \implies$$

$$\implies I_A(\lambda\tilde{\mathbf{z}} + (1-\lambda)\tilde{\mathbf{z}'}) + \epsilon > a(A) \implies I_A(\lambda\tilde{\mathbf{z}} + (1-\lambda)\tilde{\mathbf{z}'}) \geq a(A) \implies$$

$$\implies I_A\left(\lambda\left[t\mathbf{z} + (a(A) - I_A(t\mathbf{z}))\,\iota\right] + (1-\lambda)\left[t\mathbf{z}' + (a(A) - I_A(t\mathbf{z}'))\,\iota\right]\right) \geq a(A) \implies$$

$$\implies I_A(\lambda t\mathbf{z} + (1-\lambda)t\mathbf{z}') \geq \lambda I_A(t\mathbf{z}) + (1-\lambda)I_A(t\mathbf{z}') \implies$$

$$\implies I_A(\lambda\mathbf{z} + (1-\lambda)\mathbf{z}') \geq \lambda I_A(\mathbf{z}) + (1-\lambda)I_A(\mathbf{z}')$$

Thus, $I_A(\cdot)$ is a concave function.

Proof of (i). Consider arbitrary $\mathbf{z}, \mathbf{z}' \in R^{|S|}$ and let $\mathbf{z} \geq \mathbf{z}'$. If $\mathbf{z} = \mathbf{z}' = \mathbf{0}$, then the statement holds. Otherwise denote $t = 0.01\cdot(\max\{|z|,|z'|\})^{-1} > 0$ and

$$\widetilde{\mathbf{z}} = t\mathbf{z} + (a(A) - I_A(t\mathbf{z}))\,\iota, \qquad \widetilde{\mathbf{z}'} = t\mathbf{z}' + (a(A) - I_A(t\mathbf{z})\,\iota$$

Notice that we use $I_A(t\mathbf{z})$ and not $I_A(t\mathbf{z}')$ in the definition of $\widetilde{\mathbf{z}'}$, like we did to prove concavity of $I_A(\cdot)$. This way $I_A(\widetilde{\mathbf{z}}) = a(A)$, $\widetilde{\mathbf{z}'} \leq \widetilde{\mathbf{z}}$, and $\widetilde{\mathbf{z}}, \widetilde{\mathbf{z}'} \in Z_{0.4}$. Therefore, $\widetilde{\mathbf{z}} \in A$,

and by *Monotonicity* $f \in c(A \cup \{\widetilde{\mathbf{z}}'\})$, thus by maximality of A we get $\widetilde{\mathbf{z}}' \in A$, hence $I_A(\widetilde{\mathbf{z}}') \leq a(A) = I_A(\widetilde{\mathbf{z}})$. Using statements (ii),(iv) and (v), we get $I_A(\mathbf{z}') \leq I_A(\mathbf{z})$. ■

Lemma B.1.10 *Let $\mathcal{I}$ be the set of monotone positively homogeneous concave constant additive and normalized functions $I:\ R^{|S|} \to R$, and Π be the set of non-empty convex and closed sets of probabilities $P \subseteq \triangle S$. Then there is a bijection $\tau:\ \Pi \to \mathcal{I}$ given by*

$$\tau(P)(\mathbf{z}) \quad \equiv \quad \min_{\mu \in P} \sum_{s \in S} \mu(s) z_s \qquad \forall \mathbf{z} \in R^{|S|} \tag{B.11}$$

$$\tau^{-1}(I) \quad = \quad \sigma(I) \quad \equiv \quad \left\{ \mu \in \triangle S \;\middle|\; \sum_{s \in S} \mu(s) z_s \ \geq\ I(\mathbf{z})\ \forall \mathbf{z} \in R^{|S|} \right\} \tag{B.12}$$

Proof. This is a well-known statement. We provide a full proof in Section B.2. ■

Recall the definition of the set of menus $\mathcal{D}_{0.2}$ given by eq. (B.5) and function I_A given by eq. (B.10). Define a collection of sets of beliefs $\mathcal{A}$ by

$$\mathcal{A} \ =\ \{P \in \Pi \mid \exists A \in \mathcal{D}_{0.2}:\ I_A = \tau(P)\} \tag{B.13}$$

Notice that $\mathcal{A} \neq \varnothing$, since $0.5\iota \in Z_{0.2}$, hence there exists $A \in \mathcal{L}(0.5\iota) \subseteq \mathcal{D}_{0.2}$, and $I_A \in \mathcal{I}$, thus $\tau(P) = I_A$ for $P = \sigma(I_A)$. Next, by lemma B.1.3 $U(\cdot)$ is non-degenerate. Therefore, $(U, cl(\mathcal{A}))$ is a well-defined framed ambiguity model, where $cl(\mathcal{A})$ is a closure of $\mathcal{A}$ with respect to the Hausdorff metric on Π. Let us abuse notation and say that model $(U, \mathcal{A})$, where U is a vNM expected utility function, and $\varnothing \neq \mathcal{A} \subseteq \Pi$ represents a choice correspondence $c(\cdot)$ if $c(A)$ is given by eq. (2.1) but $\mathcal{A}$ is not necessary closed.

Lemma B.1.11 *Let U be a vNM expected utility function that represents c_0 and $\mathcal{A}$ be defined via eq. (B.13). Then model $(U, \mathcal{A})$ represents $c(\cdot)$ on the set of menus consisting of canonical acts $\widehat{f} \in H = Z_1 = [0,1]^{|S|}$.*

Proof. Let $c_{(U,\mathcal{A})}$ be choice correspondence induced by the model $(U, \mathcal{A})$, given by eq. (2.1). By definition $\mathbf{z} \in c_{(U,\mathcal{A})}(B)$ if and only if there exists $P \in \mathcal{A}$ such that $W_P(\mathbf{z}) \geq W_P(\mathbf{z}')$ for all $\mathbf{z}' \in B$, where we implicitly use $U(\cdot)$ in the definition of $W_P(\cdot)$. Lemma B.1.10 implies $\forall B \in \mathcal{K}$

$$\mathbf{z} \in c_{(U,\mathcal{A})}(B) \quad \Longleftrightarrow \quad \exists A \in \mathcal{D}_{0.2} : \; I_A(\mathbf{z}) \geq I_A(\mathbf{z}') \; \forall \mathbf{z}' \in B$$

Step 1. We'll firstly show that the statement is true for all menus $B \subseteq Z_{0.2}$.

Step 1.1. We want to show that $\mathbf{z} \in Z_{0.2}$ and $A \in \mathcal{L}(\mathbf{z})$ imply $I_A(\mathbf{z}) = a(A)$. By lemma B.1.9 $I_A(\mathbf{z}) \leq a(A)$. Assume for a contradiction that $I_A(\mathbf{z}) < a(A)$. Consider $\mathbf{z}' = \mathbf{z} + 0.5 \cdot \min\{0.01, a(A) - I_A(\mathbf{z})\}\iota$. Then $\mathbf{z}' \in Z_{0.4}$ and also $I_A(\mathbf{z}') \leq 0.5 I_A(\mathbf{z}) + 0.5 a(A) < a(A)$, hence $\mathbf{z}' \in A$. Since $\mathbf{z} \in c(A)$, α implies $\mathbf{z} \in c(\{\mathbf{z}, \mathbf{z}'\})$, but $\mathbf{z}' >> \mathbf{z}$, contradicting *Strict Monotonicity.*

Step 1.2. We want to show that $c(B) \subseteq c_{(U,\mathcal{A})}(B)$. Let $\mathbf{z} \in c(B)$. Then there is $A \in \mathcal{L}(\mathbf{z})$ such that $B \subseteq A$ and $\mathbf{z} \in c(A)$. Using Lemma B.1.9, Lemma B.1.10 and Step 1.1, we get $I_A(\mathbf{z}') \leq a(A) = I_A(\mathbf{z})$ for all $z' \in B \subseteq A$. Therefore, $W_{\sigma(I_A)}(\mathbf{z}) \geq W_{\sigma(I_A)}(\mathbf{z}')$ for all $\mathbf{z}' \in B$, hence $\mathbf{z} \in c_{(U,\mathcal{A})}(B)$.

Step 1.3. We want to show that $c_{(U,\mathcal{A})}(B) \subseteq c(B)$. Let $\mathbf{z} \in c_{(U,\mathcal{A})}(B)$. Then there is $f \in Z_{0.2}$ and $A \in \mathcal{L}(f)$ such that $I_A(\mathbf{z}) \geq I_A(\mathbf{z}')$ for all $\mathbf{z}' \in B$.

Case 1: $I_A(\mathbf{z}) = a(A)$. Then for all $\mathbf{z}' \in B$ we have $I_A(\mathbf{z}') \leq a(A)$, hence by Lemma A.6 $\mathbf{z}' \in A$, thus $B \subseteq A$. Consider $\mathbf{z}^n = \mathbf{z} + (1/100n)\iota \in Z_{0.4}$ for $n = 1, 2, ...$ Notice that $\mathbf{z}^n \longrightarrow \mathbf{z}$. Next, $I_A(\mathbf{z}^n) = I_A(\mathbf{z}) + (1/100n) = a(A) + (1/100n)$, therefore by Lemma A.6 $\mathbf{z}^n \notin A$. By *Aizerman's Property* $\mathbf{z}^n \in c(A \cup \{\mathbf{z}^n\})$. By *Act Continuity*

$\mathbf{z} \in c(A \cup \{\mathbf{z}\}) = c(A)$, and α implies $\mathbf{z} \in c(B)$.

Case 2: $I_A(\mathbf{z}) > a(A)$. Denote $\lambda = \dfrac{0.7 - a(A)}{0.7 - I_A(\mathbf{z})}$. Observe that $\lambda \in (0,1)$, since $a(A) \in [0.4, 0.6]$. Then for all $\mathbf{z}' \in B$ we have

$$I_A(\lambda \mathbf{z}' + (1-\lambda)(0.7\iota)) \;=\; \lambda I_A(\mathbf{z}') + (1-\lambda)\cdot 0.7 \;\leq$$

$$\leq\; \lambda I_A(\mathbf{z}) + (1-\lambda)\cdot 0.7 \;=\; a(A) = I_A(\lambda \mathbf{z} + (1-\lambda)(0.7\iota))$$

We can apply Case 1 to conclude that $\lambda \mathbf{z} + (1-\lambda)(0.7\iota) \in c(\lambda B + (1-\lambda)\{0.7\iota\})$. Then by *C-Independence* $\mathbf{z} \in c(B)$.

Case 3: $I_A(\mathbf{z}) > a(A)$. Take $\lambda = \dfrac{a(A) - 0.3}{I_A(\mathbf{z}) - 0.3} \in (0,1)$, and then repeat the proof used for Case 2 with menu $\lambda B + (1-\lambda)\{0.3\iota\}$ to get $\mathbf{z} \in c(B)$.

Step 2. We finally show that the statement is true for all $B \in \mathcal{K}$

$$\mathbf{z} \in c_{(U,\mathcal{A})}(B) \iff \exists f \in Z_{0.2} :\; \exists A \in \mathcal{L}(f) :\; I_A(\mathbf{z}) \geq I_A(\mathbf{z}')\; \forall \mathbf{z}' \in B \iff$$

$$\iff \exists f \in Z_{0.2} :\; \exists A \in \mathcal{L}(f) :\; I_A(0.2\mathbf{z}+0.8\cdot(0.5\iota)) \geq I_A(0.2\mathbf{z}'+0.8\cdot(0.5\iota))\; \forall \mathbf{z}' \in B \iff$$

$$\iff 0.2\mathbf{z} + 0.8\cdot(0.5\iota) \in c_{(U,\mathcal{A})}(0.2B + 0.8\cdot(0.5\iota)) \iff$$

$$\iff 0.2\mathbf{z} + 0.8\cdot(0.5\iota) \in c(0.2B + 0.8\cdot(0.5\iota)) \iff \mathbf{z} \in c(B)$$

where we used properties of $I_A(\cdot)$ given by Lemma B.1.9, the fact that $0.2B + 0.8\cdot(0.5\iota) \subseteq Z_{0.2}$, Step 1 and *C-Independence*. ■

Lemma B.1.12 *Model $(U, \mathcal{A})$ represents a choice correspondence $c(\cdot)$ if and only if it represents choice correspondence $c(\cdot)$ restricted on the set of menus, consisting of canonical acts.*

Proof. We have already established that $f \sim g$ is equivalent to $U(f(s)) = U(g(s))$ for all $s \in S$, which implies $W_P(f) = W_P(\widehat{f})$ for all $f \in H$, $P \in \mathcal{A}$. Therefore, $\widehat{f} \in c_{(U,\mathcal{A})}(\widehat{B}) \iff f \in c_{(U,\mathcal{A})}(B)$ for a canonical act $\widehat{f} \sim f$ and canonical menu $\widehat{B}$. From the other hand, lemma B.1.5 says that $f \in c(B) \iff \widehat{f} \in c(\widehat{B})$. ■

Lemmas B.1.11 and B.1.12 show that $c(\cdot)$ admits a representation given by eq. (2.1) with non-degenerate $U(\cdot)$ (by lemma B.1.3), but we don't know if $\mathcal{A}$ is closed.

Lemma B.1.13 *If axioms 1-7 hold for $c(\cdot)$, and model $(U, \mathcal{A})$ represents $c(\cdot)$, then model $(U, cl(\mathcal{A}))$ represent $c(\cdot)$ as well.*

Proof. Let $P^k \in \mathcal{A}$ for all $k = 1, 2, ...$, and $P^k \longrightarrow P \in \Pi$. Notice that $(U, \{P\})$ is a framed ambiguity model, and let $c_{\{P\}}(\cdot)$ be the choice correspondence represented by this model. By the proven first part of the Theorem, $c_{\{P\}}(\cdot)$ satisfies Axioms 1-7. Consider an arbitrary menu $A \in \mathcal{K}$, and let $f \in c_{\{P\}}(A)$. Then by *C-Independence* we get $f_{0.5} \in c_{\{P\}}(A_{0.5})$, where $f_{0.5} = 0.5f + 0.5(0.5x + 0.5y)$ and $A_{0.5} = 0.5A + 0.5(0.5x + 0.5y)$. Therefore, $W_P(f_{0.5}) \geq W_P(g_{0.5})$ for all $g \in A$. Take $f^n_{0.5} = (1 - 1/n)f_{0.5} + (1/n)x$. Then $W_P(f^n_{0.5}) - W_P(g_{0.5}) \geq 1/(4n)$ for all $g \in A$. By lemma B.1.1, $W_P(h)$ is continuous w.r.t. (P, h), thus for large enough $N(k)$ we have $W_{P^k}(f^n_{0.5}) \geq W_{P^k}(g)$ for all $n \geq N(k)$ and all $g \in A$. Therefore, $f^n_{0.5} \in c(A_{0.5} \cup \{f^n_{0.5}\})$ for all $n \geq N(k)$. Take $n(k) = \max\{N(k), k\}$, then $f^{n(k)}_{0.5} \longrightarrow f_{0.5}$, hence by *Menu Continuity* $f_{0.5} \in c(A_{0.5} \cup \{f_{0.5}\}) = c(A_{0.5})$. *C-Independence* implies $f \in c(A)$. Thus $c_{\{P\}}(A) \subseteq c(A)$ for all $A \in \mathcal{K}$, and we conclude that $c'(A) \subseteq c(A)$, where $c'(\cdot)$ is a choice correspondence, represented by the framed ambiguity model $(U, \mathcal{A})$. From the other hand, since model $(U, \mathcal{A})$ represents $c(\cdot)$ and $\mathcal{A} \subseteq cl(\mathcal{A})$, then $c(\cdot) \subseteq c'(\cdot)$. We

conclude that $c'(\cdot) = c(\cdot)$, thus framed ambiguity model $(U, cl(\mathcal{A}))$ represents $c(\cdot)$. ■

Part 2 of Theorem 2.2.1 follows from lemmas B.1.11, B.1.12 and B.1.13. ■

B.1.2 Proof of Theorem 2.2.3

Step 1. Let $(U, cl(\mathcal{A}))$ be a framed ambiguity representation of $c(\cdot)$ constructed in the proof of Theorem 2.2.1, and $(U', \mathcal{A}')$ be an arbitrary framed ambiguity representation. Since both U and U' represent the same restriction of choice correspondence on menus consisting of constant acts, then U' is a positive affine transformation of U by Lemma B.1.3. Since for a maxmin utility function $W_P(\cdot)$ for any $P \in \Pi$ and any acts $f, g \in H$ statement $W_P(f) \geq W_P(g)$ is invariant with respect to the positive affine transformations of the associated vNM expected utility function, we can normalize $U' = U$.

Step 2. By lemma B.1.12 we may assume $H = [0,1]^{|S|}$. Recall the definitions of $\mathcal{L}(\cdot)$, $a(\cdot)$, $I_A(\cdot)$, $\tau(\cdot)$, $\sigma(\cdot)$, $\mathcal{A}$ given by eqs. (B.2), (B.4), (B.10), (B.11), (B.12), (B.13), and properties of function $I_A(\cdot)$ given by Lemma B.1.9. Let $P \in \mathcal{A}$. Then $\exists f \in Z_{0.2}$ and $A \in \mathcal{L}(f)$ such that $I_A = \tau(P)$ and $A = \{\widetilde{\mathbf{z}} \mid I_A(\widetilde{\mathbf{z}}) \leq a(A)\}$. Moreover, $I_A(f) = a(A)$ by Step 1.1. of Lemma B.1.11, and $a(A) \in [0.4, 0.6]$. Let

$$\mathcal{C}' = \{ P' \in \mathcal{A}' \mid A \subseteq B(P') \} \quad , \quad \text{where } B(P') = \{ \widetilde{\mathbf{z}} \in H \mid \tau(P')(f) \geq \tau(P')(\widetilde{\mathbf{z}}) \}$$

Since $f \in c(A)$, and $(U, \mathcal{A}')$ represents $c(\cdot)$, then there is some $P' \in \mathcal{A}'$ such that $W_{P'}(f) \geq W_{P'}(\widetilde{\mathbf{z}})$ for all $\widetilde{\mathbf{z}} \in A$, where we use function $U(\cdot)$ in the definition of $W_{P'}(\cdot)$ given by eq. (2.2). Thus, $W_{P'}(\widetilde{\mathbf{z}}) = \tau(P')(\widetilde{\mathbf{z}})$ for all $\widetilde{\mathbf{z}} \in H$. We conclude that $\mathcal{C}' \neq \varnothing$.

Step 3. Assume for a contradiction that $B(P') \not\subseteq A$ for some $P' \in \mathcal{C}'$. Then by Lemma B.1.8 there exists $D \in \mathcal{L}(f)$ such that $A \subset B(P') \subseteq D$ and $A \neq D$, contradicting the maximality of $A \in \mathcal{L}(f)$. We conclude that $A = B(P')$ for all $P' \in \mathcal{C}'$. In particular, $a(A) = I_A(f) = \tau(P')(f)$, since $b\iota \in A \iff b\iota \in B(P')$, and $I_A(b\iota) = b = \tau(P')(b\iota)$ for all $b \in [0,1]$.

Step 4. Let us show that $I_A(\widetilde{\mathbf{z}}) = \tau(P')(\widetilde{\mathbf{z}})$ for all $\widetilde{\mathbf{z}} \in R^{|S|}$ for all $P' \in \mathcal{C}'$. If $\widetilde{\mathbf{z}} = \mathbf{0}$, then $I_A(\mathbf{0}) = 0 = \tau(P')(\mathbf{0})$. Otherwise consider $\mathbf{w} = 0.1\dfrac{\widetilde{\mathbf{z}}}{|\widetilde{\mathbf{z}}|} + 0.5\iota \in Z_{0.2}$. Then $\mathbf{w} + 0.4b\iota \in Z_1 = H$ for all $b \in [-1,1]$. Notice also that $\mathbf{w} - 0.4\iota << 0.4\iota \leq a(A)\iota \leq 0.6\iota << \mathbf{w} + 0.4\iota$. Therefore, since I_A and $\tau(P')$ are monotone, $I_A(\mathbf{w} - 0.4\iota), \tau(P')(\mathbf{w} - 0.4\iota) < a(A) < I_A(\mathbf{w} - 0.4\iota), \tau(P')(\mathbf{w} - 0.4\iota)$. Next, by C-additivity $I_A(\mathbf{w} + 0.4b\iota) = I_A(\mathbf{w}) + 0.4b$ and $\tau(P')(\mathbf{z} + 0.4b\iota) = \tau(P')(\mathbf{w}) + 0.4b$. Finally, by Step 3 $I_A(\mathbf{w} + 0.4b\iota) \geq a(A) \iff \tau(P')(\mathbf{w} + 0.4b\iota) \geq a(A)$ for all $b \in [-1,1]$. Therefore, there is $b \in [-1,1]$ such that $I_A(\mathbf{w} + 0.4b\iota) = a(A) = \tau(P')(\mathbf{w} + 0.4\iota)$. By C-additivity and positive homogeneity of I_A and $\tau(P')$ we get $I_A(\mathbf{w}) = \tau(P')(\mathbf{w})$ and $I_A(\widetilde{\mathbf{z}}) = \tau(P')(\widetilde{\mathbf{z}})$. Since this holds for arbitrary $\widetilde{\mathbf{z}}$, then $I_A = \tau(P')$. Hence, $P = \sigma(\tau(P)) = \sigma(I_A) = \sigma(\tau(P')) = P'$. We conclude that $P \in \{P\} = \mathcal{C}' \subseteq \mathcal{A}'$. Since it holds for all $P \in \mathcal{A}$, we get $\mathcal{A} \subseteq \mathcal{A}'$. Therefore, $cl(\mathcal{A}) \subseteq cl(\mathcal{A}') = \mathcal{A}'$. We conclude that $(U, cl(\mathcal{A}))$ is a minimum framed ambiguity representation of $c(\cdot)$.

Step 5. Consider

$$\mathcal{B} = \bigcup_{\mathcal{A}' \subseteq \Pi:\ (U,\mathcal{A}') \text{ is a framed ambiguity model representing } c(\cdot)} \mathcal{A}' \tag{B.14}$$

We show that model $(U, \mathcal{B})$ represents $c(\cdot)$. Let act f maximize $W_P(\cdot)$ over menu B for some $P \in \mathcal{B}$, where we use $U(\cdot)$ in the definition of $W_P(\cdot)$ according to eq. (2.1). Then there is a framed ambiguity representation $(U, \mathcal{A}')$ of $c(\cdot)$ such that $P \in \mathcal{A}'$,

hence $f \in c(B)$. For the converse, let $f \in c(B)$. Take arbitrary representation $(U, \mathcal{A}')$, then there is $P \in \mathcal{A}' \subseteq \mathcal{B}$ such that f maximizes W_P over menu B. Thus indeed $(U, \mathcal{B})$ represents $c(\cdot)$. Finally, for any framed ambiguity representation $(U, \mathcal{A}')$ we get $\mathcal{A}' \subseteq \mathcal{B}$. By lemma B.1.13 a framed ambiguity model $(U, cl(\mathcal{B}))$ represents $c(\cdot)$, thus $\mathcal{B} = cl(\mathcal{B})$, and we conclude that $(U, \mathcal{B})$ is a maximum framed ambiguity representation of $c(\cdot)$.

Step 6. Finally, let $cl(\mathcal{A}) \subseteq \mathcal{A}' \subseteq \mathcal{B}$, and $\mathcal{A}'$ be closed. Then $c'(\cdot) \subseteq c''(\cdot) \subseteq c'''(\cdot)$, where $c'(\cdot)$, $c''(\cdot)$ and $c'''(\cdot)$ are choice correspondences represented by framed ambiguity models $(U, cl(\mathcal{A}))$, $(U, \mathcal{A}')$ and $(U, \mathcal{B})$. Since by previous steps $c'(\cdot) = c'''(\cdot) = c(\cdot)$, then $c''(\cdot) = c(\cdot)$, hence $(U, \mathcal{A}')$ represents $c(\cdot)$. Theorem 2 is proven. ■

B.1.3 Proof of Proposition 2.3.1

Consider the following alternative definition of the coherent intersection:

DEFINITION 4*: Given a non-empty collection of sets of beliefs $\mathcal{C} \subseteq \Pi$, a set of beliefs $P \in \Pi$ is a *coherent-* intersection* of $\mathcal{C}$ if its support functional is equal to the maximum of the support functionals of sets of beliefs in $\mathcal{C}$, and this maximum exists:

$$I_P(z) \;=\; \max_{P' \in \mathcal{C}} I_{P'}(z) \qquad \forall z \in R^{|S|} \tag{B.15}$$

$$\text{where } \; I: \; R^{|S|} \to R \quad \text{ is given by } \; I_P(z) \;=\; \min_{\mu \in P} \sum_{s \in S} \mu(s) \cdot z_s$$

Consider a natural embedding of $\triangle S$ in R^n. Lemma B.1.14 proves statement (i) of the Proposition.

Lemma B.1.14 *There exists a coherent-* intersection P^* of a non-empty collection of sets of beliefs $\mathcal{C} \subseteq \Pi$ if and only if there exists a coherent intersection P of $\mathcal{C}$, where*

sets of beliefs are considered as subsets of R^n with the requirement that $\sum_{i=1}^n \mu(i) = 1$; moreover, $P^* = P$.

Proof. Suppose that P is a coherent intersection of $\mathcal{C}$. For an arbitrary $z \in R^{|S|}$, let z' be its component, orthogonal to $(1, ..., 1)$. The projections of P and $\mathcal{C}$ on the linear subspace $R_{z'} = \{x \in R^n \mid \exists \lambda \in R : \ x = \lambda z'\}$ are an interval $[a(P), b(P)]$ and a collection of intervals $\left\{[a(P'), b(P')]\right\}_{P' \in \mathcal{C}}$. Then, without loss of generality, $I_{P'}(z) = a(P')$ for all $P' \in \mathcal{C} \cup P$. Since P is a coherent intersection, then

$$[a(P), b(P)] \ = \ \bigcap_{P' \in \mathcal{C}} [a(P'), b(P')]$$

It follows that

$$I_P(z) - \sum_{i=1}^{S} z_s \ = \ I_P(z') \ = \ a(P) \ = \ \max_{P' \in \mathcal{C}} a(P') \ = \ \max_{P' \in \mathcal{C}} I(P')(z') \ = \ \max_{P' \in \mathcal{C}} I_{P'}(z') - \sum_{i=1}^{S} z_s$$

Thus, P is a coherent-* intersection of $\mathcal{C}$.

Claim B.1.15 *Let $P \subseteq R^n$ be the intersection of sets in a collection $\mathcal{C}$, P_o and $\mathcal{C}_o$ are projections of P and $\mathcal{C}$ on an arbitrary linear subspace R_o and let $\tilde{P}$ be an intersection of sets in $\mathcal{C}_o$, then $P_o \subseteq \tilde{P}$.*

Proof of the claim. Consider arbitrary $\hat{x} \in P_o$, then there is $x \in P$ such that $x'' = x_o$, where x_o is a projection of x on R_o. It follows that $x \in P'$ for all $P' \in \mathcal{C}$, hence $x'' = x_o \in P'_o$ for all $P' \in \mathcal{C}$, hence $x'' \in \tilde{P}$. The Claim is proven. ■

Suppose now that P^* is a coherent-* intersection of $\mathcal{C}$. Let P^*_o and $\mathcal{C}_o$ be projections of P and $\mathcal{C}$ on an arbitrary linear subspace R_o of R^n that can be shifted to a subspace of $\triangle S$; that is, $\sum_{i=1}^n x_i = const$ for all $x \in R_o$. Towards a contradiction, assume that P^*_o is not the intersection of sets in $\mathcal{C}_o$. By Claim B.1.15,

$$\bigcap_{P'_o \in \mathcal{C}_o} P'_o \ \supsetneq \ P^*_o$$

Since all sets are convex, by the Separating Hyperplane Theorem, the above equation implies that there exists $z \in R^s$ such that

$$I_{\bigcap\limits_{P'_o \in \mathcal{C}_o} P'_o}(z) \;<\; I_{P^*_o}(z)$$

It follows that

$$\max_{P'_o \in \mathcal{C}_o} I_{P'_o}(z) \;\leq\; I_{\bigcap\limits_{P'_o \in \mathcal{C}_o} P'_o}(z) \;<\; I_{P^*_o}(z)$$

Let $z_\perp$ be a component of z that is orthogonal to R_o, and $z' = z - z_\perp$. Since $I_{P'_o}(z_\perp) = I_{P'_o}(z_\perp)$ for any $P'_o, P''_o \in \mathcal{C}_o \cup \{P_o\}$, then

$$\max_{P'_o \in \mathcal{C}_o} \; I_{P'_o}(z') \;<\; I_{P^*_o}(z')$$

and

$$\max_{P' \in \mathcal{C}} I_{P'}(z') \;<\; I_{P^*}(z')$$

in contradiction to P^* being a coherent-* intersection of $\mathcal{C}$. For a linear subspace R_o that cannot be shifted to a subspace of $\triangle S$, consider the additional projection to $\{x \in R_o | \sum_{i=1}^n x_i = 1\}$, and repeat the argument above. ■

Consider a binary state space $|S| = 2$ and a non-empty closed collection $\mathcal{C} \subseteq \Pi$. Then $\triangle S$ is isomorphic to $[0,1] \subset R$, where $\mu \in [0,1]$ corresponds to $\mu(1)$, and $\mu(2) = 1 - \mu(1)$. Thus we can consider each set of beliefs as an interval $P' = [a', b'] \subseteq [0,1]$ with $a' \leq b'$. Assume that there is a non-empty intersection $\overline{P}$ of sets of beliefs in $\mathcal{C}$. Then

$$\overline{P} \;=\; \bigcap_{P' \in \mathcal{C}} P' \;=\; \bigcap_{[a',b'] \in \mathcal{C}} [a', b'] \;= \tag{B.16}$$

$$\left[\sup_{[a',b'] \in \mathcal{C}} a' \;,\; \inf_{[a',b'] \in \mathcal{C}} b' \right] \;=\; \left[\max_{[a',b'] \in \mathcal{C}} a' \;,\; \min_{[a',b'] \in \mathcal{C}} b' \right] \;\neq\; \varnothing$$

where we used the closedness of $\mathcal{C}$ to get maximum and minimum instead of suprenum and infinum. Consider $\mathbf{z} = (z_1, z_2) \in R^2$ with $z_1 \geq z_2$. Then

$$I_{\overline{P}}(z_1, z_2) = \left(\max_{[a',b'] \in \mathcal{C}} a' \right) z_1 + \left(1 - \left(\max_{[a',b'] \in \mathcal{C}} a' \right) \right) z_2 = \max_{[a',b'] \in \mathcal{C}} I_{[a',b']}(z_1, z_2)$$

A symmetric argument works when $z_1 < z_2$. Thus $\overline{P}$ is a coherent intersection of $\mathcal{C}$, proving (ii).

Consider now $|S| > 2$. Let $\mu_1 = (0.1, 0.1, 0.8, ...)$, $\mu_2 = (0.3, 0.1, 0.6, ...)$, $\mu_3 = (0.2, 0.2, 0.6, ...)$, where ... denotes zeros for all s except of the first 3 states. Let $\mathcal{C} = \{P_1, P_2\}$ with $P_1 = conv(\{\mu_1, \mu_3\})$ and $P_2 = conv(\{\mu_2, \mu_3\})$. Then $P_1 \cap P_2 = \{\mu_3\}$. From the other hand, consider $\mathbf{z} \in R^{|S|}$ with $z_2 = 1$ and $z_i = 0$ for all $i \neq 2$. Then $I_{P_1}(\mathbf{z}) = I_{P_2}(\mathbf{z}) = 0.1 < 0.2 = I_{\{\mu_3\}}$. Thus, $\mathcal{C} \subset \Pi$ is a closed collection of sets of beliefs with non-empty intersection, but it fails to have a coherent intersection, otherwise it should have been $\mu_3 = P_1 \cap P_2$ by Step 2, but this is not the case. Proposition 2.3.1 is proven. ■

B.1.4 Proof of Proposition 2.3.2 and Corollary 2.3.3

Step 1. For a fixed vNM expected utility function $U(\cdot)$ and some non-empty collection of sets of beliefs $\mathcal{C} \subseteq \Pi$ denote $c_{\mathcal{C}}(\cdot)$ to be a choice correspondence that is represented by $(U, \mathcal{C})$. Such choice correspondences exist as we have argued in Section 2.3. Notice that by eq. (2.1) $\quad \forall B \in \mathcal{K}$

$$c_{\mathcal{C} \cup \mathcal{D}}(B) \quad = \quad c_{\mathcal{C}}(B) \cup c_{\mathcal{D}}(B), \qquad \mathcal{C} \subseteq \mathcal{A} \quad \Longrightarrow \quad c_{\mathcal{C}}(B) \subseteq c_{\mathcal{A}}(B),$$

$$c_{\mathcal{C}}(B) \quad = \quad \bigcup_{P \in \mathcal{C}} c_{\{P\}}(B)$$

Let P be a non-trivial coherent intersection of a non-empty subcollection of sets of beliefs $\mathcal{C} \subseteq \mathcal{A}$. Then for any $f \in H$ and $B \in \mathcal{K}$ we have

$$f \in c_{\{P\}}(B) \implies W_P(f) \geq W_P(g)\, \forall g \in B \implies \max_{P' \in \mathcal{C}} W_{P'}(f) \geq \max_{P' \in \mathcal{C}} W_{P'}(g)\, \forall g \in B \implies$$

$$\exists P'' \in \mathcal{C} :\ W_{P''}(f) \geq \max_{P' \in \mathcal{C}} W_{P'}(g) \geq W_{P''}(g)\ \forall g \in B \implies f \in c_{\mathcal{C}}(B)$$

which implies $c_{\{P\}}(B) \subseteq c_{\mathcal{C}}(B)$. Since P is a non-trivial coherent intersection of $\mathcal{C}$, then $\{P\} \notin \mathcal{C}$. Therefore, $\mathcal{C} \subseteq \mathcal{A} \backslash \{P\}$, and for any $B \in \mathcal{K}$ we get

$$c_{\mathcal{A} \cup \{P\}}(B) \quad = \quad c_{\mathcal{A} \backslash \{P\}}(B) \cup c_{\{P\}}(B) \quad \subseteq \quad c_{\mathcal{A} \backslash \{P\}}(B) \cup c_{\mathcal{C}}(B) \quad = \quad c_{\mathcal{A} \backslash \{P\}}(B)$$

Therefore, $c_{\mathcal{A} \cup \{P\}}(B) = c_{\mathcal{A} \backslash \{P\}}(B)$ for all $B \in \mathcal{K}$, hence P is a redundant frame given U and $\mathcal{A}$.

Step 2. Assume that, given a non-degenerate vNM expected utility $U(\cdot)$, frame P is redundant for a collection $\mathcal{A}$. Since $c_{\mathcal{A} \cup \{P\}}(B) = c_{\mathcal{A} \backslash \{P\}}(B) \cup c_{\{P\}}(B) = c_{\mathcal{A} \backslash \{P\}}(B)$ for any $B \in \mathcal{K}$, then for any $B \in \mathcal{K}$ and $f \in B$ we have

$$W_P(f) \geq W_P(g)\ \forall g \in B \quad \iff \quad f \in c_{\{P\}}(B) \quad \implies$$

$$\implies \quad f \in c_{\mathcal{A} \backslash \{P\}}(B) \quad \implies \quad \exists P' \in \mathcal{A} \backslash \{P\} :\quad W_{P'}(f) \geq W_{P'}(g)\ \forall g \in B$$

Since $U(\cdot)$ is non-degenerate, WLOG $1 = U(x) > U(y) = 0$ for some $x, y \in \triangle X$. The above equation is true in particular for menus consisting only of canonical acts $f :\ S \to \triangle(\{x, y\})$. Thus we assume for now that $H = [0,1]^{|S|}$. Recall the definition of sets $Z_\lambda, O \subset R^{|S|}$ given by eqs. (B.3), (B.6) and support functional $I_P(\cdot)$ given by

eq. (B.10). Denote

$$B = \{\mathbf{z} \in H \mid I_P(\mathbf{z}) \leq 0.5\},\ \mathbf{z}^{\mathbf{v}} = (0.5 - 0.01 I_P(\mathbf{v}))\iota + 0.01\mathbf{v} \in Z_{0.2} \subset Z_1 = H \text{ for } \mathbf{v} \in O$$

where we used $|I_P(\mathbf{v})| \leq |\mathbf{v}| = 1$ implied by the properties of $I_P(\cdot) \in \mathcal{I}$ proven in lemma A.6. Recall that with the normalization $U(x) = 1, U(y) = 0$ we have $W_P(\cdot) = I_P(\cdot)$ for acts in $H = [0,1]^{|S|}$. Then for all $\mathbf{v} \in O$ we have $I_P(\mathbf{z}^{\mathbf{v}}) = 0.5$, thus $\mathbf{z}^{\mathbf{v}} \in c_{\{P\}}(B) \subseteq c_{\mathcal{A}\backslash\{P\}}(B)$. Therefore, for every $\mathbf{v} \in O$ there is $P^{\mathbf{v}} \in \mathcal{A}\backslash\{P\}$ such that $\mathbf{z}^{\mathbf{v}} \in c_{\{P^{\mathbf{v}}\}}(B)$.

Step 2.1. Assume for a contradiction that $I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}}) < 0.5$. Then $I_{P^{\mathbf{v}}}(0.5 \cdot \mathbf{z}^{\mathbf{v}} + 0.5 \cdot 0.5\iota) > I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}})$, but $I_P(0.5 \cdot \mathbf{z}^{\mathbf{v}} + 0.5 \cdot 0.5\iota) = 0.5$ and $0.5 \cdot \mathbf{z}^{\mathbf{v}} + 0.5 \cdot 0.5\iota \in Z_1 = H$, hence $0.5 \cdot \mathbf{z}^{\mathbf{v}} + 0.5 \cdot 0.5\iota \in B$, therefore $\mathbf{z}^{\mathbf{v}} \notin c_{\{P^{\mathbf{v}}\}}(B)$, contradiction.

Step 2.2. Assume for a contradiction that $I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}}) > 0.5$. Then $(0.5 - 0.02 I_P(\mathbf{v})) + 0.02\mathbf{v} \in H$, and $I_P((0.5 - 0.02 I_P(\mathbf{v}))\iota + 0.02\mathbf{v}) = 0.5$, hence $(0.5 - 0.02 I_P(\mathbf{v}))\iota + 0.02\mathbf{v} \in F$. Next, since $\mathbf{z}^v = 0.5 \cdot ((0.5 - 0.02 I_P(\mathbf{v}))\iota + 0.02\mathbf{v}) + 0.5 \cdot 0.5\iota$, then $I_{P^{\mathbf{v}}}((0.5 - 0.02 I_P(\mathbf{v}))\iota + 0.02\mathbf{v}) > I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}})$. We conclude that $\mathbf{z}^{\mathbf{v}} \notin c_{\{P^{\mathbf{v}}\}}(B)$, contradiction.

Step 2.3. We conclude that $I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}}) = 0.5$. Notice that for arbitrary $\mathbf{v}' \in O$ we have $I_P(\mathbf{z}^{\mathbf{v}'}) = 0.5$, hence $\mathbf{z}^{\mathbf{v}'} \in B$. Therefore, $I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}'}) \leq I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}}) = 0.5$. For arbitrary $\mathbf{v} \in O$ consider

$$\mathcal{C}^{\mathbf{v}} = \left\{ P' \in \mathcal{A}\backslash\{P\} \;\middle|\; I_{P'}(\mathbf{z}^{\mathbf{v}}) = 0.5 \text{ and } I_{P'}(\mathbf{z}^{\mathbf{v}'}) \leq 0.5\ \forall \mathbf{v}' \in O \right\} \neq \varnothing$$

where $\mathcal{C}^{\mathbf{v}} \neq \varnothing$ for all $\mathbf{v} \in O$ by our calculations. Then for all $\mathbf{v} \in O$

$$I_P(\mathbf{z}^{\mathbf{v}}) \;=\; 0.5 \;=\; \max_{\mathbf{v}' \in O} \max_{P' \in \mathcal{C}^{\mathbf{v}'}} I_{P'}(\mathbf{z}^{\mathbf{v}}) \;=\; \max_{P' \in \mathcal{C}} I_{P'}(\mathbf{z}^{\mathbf{v}})$$

where

$$\mathcal{C} \;=\; \bigcup_{\mathbf{v}' \in O} \mathcal{C}^{\mathbf{v}'} \;\subseteq\; \mathcal{A} \backslash \{P\}$$

Step 2.4. Now consider arbitrary $\mathbf{z} \in R^{|S|}$ and recall its decomposition $\mathbf{z} = \overline{z} \cdot \iota + \mathbf{z}_\perp$ given by eq. (B.9). If $\mathbf{z}_\perp = \mathbf{0}$, then $I_P(\mathbf{z}) = \overline{z} = \max_{P' \in \mathcal{C}} I_{P'}(\mathbf{z})$. Otherwise $\dfrac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|} \in O$, and

$$I_P(\mathbf{z}) \;\;=\;\; \overline{z} + |\mathbf{z}_\perp| \cdot I_P\left(\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}\right) \;\;=\;\; \overline{z} + |\mathbf{z}_\perp| \cdot \max_{P' \in \mathcal{C}} I_{P'}\left(\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}\right) \;\;=\;\; \max_{P' \in \mathcal{C}} I_{P'}(\mathbf{z})$$

We conclude that P is a non-trivial coherent intersection of $\mathcal{C}$.

Step 3. Let $(U, \mathcal{B})$ be the maximum framed ambiguity representation of $c(\cdot)$ existing by Theorem 2.2.3, and $(U', \mathcal{A}')$ be any other framed ambiguity representation of $c(\cdot)$. By Theorem 2.2.3, it is WLOG to consider $U' = U$. Assume $P \in \mathcal{B} \backslash \mathcal{A}'$. Then for any $B \in \mathcal{K}$

$$c_{\mathcal{A}\backslash\{P\}}(B) \;\;=\;\; c_{\mathcal{A}}(B) \;\;\subseteq\;\; c_{\mathcal{A}\cup\{P\}}(B) \;\;\subseteq\;\; c_{\mathcal{B}}(B) \;\;=\;\; c_{\mathcal{A}}(B) \;\;=\;\; c_{\mathcal{A}\backslash\{P\}}(B)$$

Therefore, by Step 2 P is a non-trivial coherent intersection of a subcollection $\mathcal{C} \subseteq \mathcal{A}$. From the other hand, if $P \in \mathcal{B} \cap \mathcal{A}'$, then P is a trivial coherent intersection of $\mathcal{C} = \{P\} \subseteq \mathcal{A}'$. We conclude that $\mathcal{B}$ consists of coherent intersections of subcollection of sets of beliefs in $\mathcal{A}'$. Proposition 2.3.2 is proven. ■

Proof of Corollary 2.3.3. If all $P = \{\mu\} \in \mathcal{A}$ are singletons, then $P_1 \cap P_2 = \varnothing$ for any two different frames. Thus by Proposition 2.3.1 a non-empty set of beliefs

is a coherent intersection of $\mathcal{C}$ only if $\mathcal{C} = \{P\} = \{\{\mu\}\}$ which is a trivial coherent intersection. Thus by . Finally, if all frames are singletons, then $\{\mu_1\} \cap \{\mu_2\} = \varnothing$ for any two different frames. ■

B.1.5 Proof of Proposition 2.4.1

Lemma B.1.16 *Let $c(\cdot)$ satisfies axioms 1-7. Then the following conditions are equivalent for $c(\cdot)$: (i) every representation $(U, \mathcal{A})$ of $c(\cdot)$ has a unique frame: $|\mathcal{A}| = 1$ (ii) β (iii) WARP (iv) Normality (v) Direct Ambiguity Aversion (vi) Pairwise No-C-Hedging.*

Proof. Firstly note that *WARP* and β are equivalent under axioms 1-7, since one of the axioms is α. We then are going to show the following implications: (1) $|\mathcal{A}| = 1 \implies$ *Direct Ambiguity Aversion, Pairwise No-C-Hedging, WARP* $\implies$ *Normality*, and (2) $|\mathcal{A}| > 1 \implies$ violation of *WARP*, violation of *Direct Ambiguity Aversion*, violation of *No-C-Hedging* $\implies$ violation of *Normality*. Taken together, these implications are equivalent to the statement of the Lemma.

Step 1. Let $\mathcal{A} = \{P\}$. Notice that $U(\cdot)$ is non-degenerate, otherwise $c(A) = A$ for all $A \in \mathcal{K}$, and *C-Non-Degeneracy* fails. Then

$$W_P(\lambda f+(1-\lambda)g) \geq \min\{W_P(f), W_P(g)\}, \quad W_P(\lambda f+(1-\lambda)p) = \lambda W_P(f)+(1-\lambda)U(p)$$

for any $f, g \in H$, $p \in H_0$ and $\lambda \in (0,1)$. Take any $A \in \mathcal{K}$ and suppose $f \in c(A \cup \{f\})$, $g \in c(A \cup \{g\})$. Then $W_P(f), W_P(g) \geq W_P(h)$ for all $h \in A$. Therefore, $W_P(\lambda f+(1-\lambda)g) \geq W_P(\lambda f+(1-\lambda)g) \geq \min\{W_P(f), W_P(g)\} \geq W_P(h)$ for all $h \in A$. Hence, $\lambda f + (1-\lambda)g \in c(A \cup \{\lambda f + (1-\lambda)g\})$, and *Direct Ambiguity Aversion* holds. Suppose that $h \in c(\{h, f\})$ and $h \in c(\{h, p\})$. Then $W_P(h) \geq W_P(f), U(p)$. There-

fore, $W_P(h) \geq \lambda W_P(f) + (1-\lambda)U(p) = W_P(\lambda f + (1-\lambda)p)$, $h \in c(\{h, \lambda f + (1-\lambda)p\})$, and *Pariwise No-C-Hedging* holds. Notice that with a single frame P choice correspondence $c(\cdot)$ is a result of the maximization of utility function W_P, hence it satisfies *WARP*. *Normality* follows from $f \in c(A) \iff W_P(f) \geq W_P(g)\ \forall g \in A \iff f \in c(\{f, g\}) \forall g \in A$.

Step 2. Assume $|\mathcal{A}| > 1$. Then there are $P_1, P_2 \in \mathcal{A}$ and act $f \in H$ such that $W_{P_1}(f) > W_{P_2}(f)$, otherwise $P_1 = P_2$ for all $P_1, P_2 \in \mathcal{A}$. It is easy to see that there are constant acts q_1, q_2 such that $W_{P_1}(f) = U(q_1)$ and $W_{P_2}(f) = U(q_2)$. Indeed, take $\underline{q} = f\left(\arg\min_{s \in S} U(f(s))\right)$ and $\overline{q} = f\left(\arg\max_{s \in S} U(f(s))\right)$, then $U(\underline{q}) \leq W_{P_2}(f) < W_{P_1}(f) \leq U(\overline{q})$, and we can take q_1 and q_2 as convex combinations of $\underline{q}$ and $\overline{q}$ (choice of $s \in$ argmin..., if it is not a singleton does not matter).

Step 2.1. Notice that $c(\{q_2, f\}) = \{q_2, f\}$, but $c(\{q_1, q_2, f\}) = \{q_1, f\}$, since $W_P(q_1) = U(q_1) > U(q_2) = W_P(q_2)$ for any $P \in \mathcal{A} \subseteq \Pi$. Thus, condition β is violated: $\{q_2, f\} \cap c(\{q_1, q_2, f\}) = \{f\} \neq \varnothing$, but $q_2 \in c(\{q_2, f\}) \not\subseteq c(\{q_1, q_2, f\})$. Therefore, *WARP* is violated.

Step 2.2. Now take $g = 0.25q_1 + 0.75q_2$ and consider $h' = 0.5f + 0.25q_1 + 0.25q_2$, $h = 0.5f + 0.5g = 0.5f + 0.125q_1 + 0.375q_2$. Let menu $A = \{h'\}$. Then $W_{P_1}(f) = U(q_1) \geq 0.75U(q_1) + 0.25U(q_2) = W_{P_1}(h')$, hence $f \in c(\{A \cup \{f\}\})$. Next, $W_{P_2}(h') = 0.25U(q_1) + 0.75U(q_2) = W_{P_2}(g)$, hence $g \in c(A \cup \{g\})$. However, h' strictly state-wise dominates h, therefore by *Strict Monotonicity* $h \notin c(\{h', h\}) = c(\{A \cup \{h\}\})$. Therefore, *Direct Ambiguity Aversion* is violated.

Step 2.3. Consider $p = 0.5q_1 + 0.5q_2 \in H_0$. Notice that $h' = 0.5f + 0.5p$. We have: $W_{P_2}(h) = W_{P_2}(0.5f + 0.5(0.25q_1 + 0.75q_2)) = 0.125U(q_1) + 0.875U(q_2) \geq U(q_2) =$

$W_{P_2}(f)$, hence $h \in c(\{h, f\})$. Next, $W_{P_1}(h) = 0.625U(q_1) + 0.375U(q_2) \geq 0.5U(q_1) + 0.5U(q_2) = U(p)$, hence $h \in c(\{h, p\})$. However, $h \notin c(\{h, h'\}) = c(\{h, 0.5f + 0.5p\})$. Thus, *No-C-Hedging** is violated.

Step 2.4. Finally, suppose that *Normality* holds. Recall that *No-C-Hedging* and condition α hold by the assumption of Proposition 2.4.1. Consider arbitrary $h, f \in H$ and $p \in H_0$ and $\lambda \in (0.1)$. Suppose that $h \in c(\{f, h\})$ and $h \in c(\{h, p\})$. Then $h \succeq_c f$ and $h \succeq_c p$, hence by *Normality* we have $h \in c(\{h, f, p\})$. Therefore, by *No-C-Hedging* we get $h \in c(\{h, f, p, \lambda f + (1 - \lambda)p\})$. Hence, by condition α we have $h \in c(\{h, \lambda f + (1 - \lambda)p\})$. We conclude that *No-C-hedging** holds, in a contradiction to what we have proven before. ■

Proposition 2.4.1 follows from lemma B.1.16. ■

B.1.6 Proof of Proposition 2.5.1

Lemma B.1.17 *Let c_1 and c_2 have framed ambiguity representations $(U, \mathcal{A}_1)$ amd $(U, \mathcal{A}_2)$. Assume that for all $f, g \in H$ we have $c_2(\{f, g\}) = \{f\} \implies c_1(\{f, g\}) = \{f\}$. Then U_1 and U_2 are positive affine transformations of each other.*

Proof Let x maximizes U_2 and y minimizes U_2 over $p \in \triangle X$ (exist by continuity of $U(\cdot)$ and compactness of $\triangle X$), then $U_2(x) > U_2(y)$ by non-degeneracy. Assume that $U_1(\overline{x}) > U_1(x)$. Then for small enough $\epsilon > 0$ we have $c_1(\{\epsilon y + (1 - \epsilon)\overline{x}, x\}) = \{\epsilon y + (1 - \epsilon)\overline{x}\}$, but $c_2(\{\epsilon y + (1 - \epsilon)\overline{x}, x\}) = \{x\}$, contradiction. Similarly, if $U_1(\overline{y}) < U_1(y)$, then for small enough $\epsilon > 0$ we have $c_1(\{\epsilon x + (1-\epsilon)\overline{y}, y\}) = \{y\}$, but $c_2(\{\epsilon x + (1-\epsilon)\overline{y}, y\}) = \{\epsilon x + (1-\epsilon)\overline{y}\}$, contradiction. Using continuity of the vNM expected utility function, we conclude that for any $p \in \triangle X$ there is $\lambda(p) \in [0, 1]$ such that $U_2(p) = \lambda(p)U_2(x) + (1 - \lambda(p))U_2(y)$,

and it is easy to see that $U_1(p) = \lambda(p)U_1(x) + (1 - \lambda(p))U_1(y)$ (apply $c_2(\{f,g\}) = \{f\} \implies c_1(\{f,g\}) = \{f\}$ for choices from menus $\{p, \lambda x + (1-\lambda)y\}$ with $\lambda \in [0,1]$). Normalizing $U_1(x) = U_2(x) = 1$, $U_1(y) = U_2(y) = 0$ we get $U_1(p) = U_2(p)$ for all $p \in \triangle X$. ■

If U_1 is a positive affine transformation of U_2 and $\mathcal{A}_1 \subseteq (\mathcal{A}_2)_{coh}$, then for all $B \in \mathcal{K}$ we have

$$c_1(B) = \bigcup_{P \in \mathcal{A}_1} \arg\max_{f \in B} W_P(f) \subseteq \bigcup_{P \in (\mathcal{A}_2)_{coh}} \arg\max_{f \in B} W_P(f) = c_2(B)$$

Hence, DM 1 is more decisive than DM 2. Conversely, assume that DM 1 is more decisive than DM 2. Then by Lemma B.1.17 U_1 is a positive affine transformation of U_2. In particular, $(U_2, \mathcal{A}_1)$ represents $c_1(\cdot)$ as well, and we can assume that W_P is evaluated using vNM expected utility U_2 and set of beliefs $P \in \Pi$. Since

$$\bigcup_{P \in \mathcal{A}_1} \arg\max_{f \in B} W_P(f) = c_1(B) \subseteq c_2(B) = \bigcup_{P \in (\mathcal{A}_2)_{coh}} \arg\max_{f \in B} W_P(f)$$

then $(U_2, \mathcal{A}_1 \cup (\mathcal{A}_2)_{coh})$ represents $c_2(\cdot)$. Therefore, by Proposition 2.3.2 we have $\mathcal{A}_1 \subseteq \mathcal{A}_1 \cup (\mathcal{A}_2)_{coh} \subseteq (\mathcal{A}_2)_{coh}$. ■

B.1.7 Proof of Theorem 2.5.2

Throughout the proof $I_P(\cdot)$ is a support functional of set of beliefs P defined by eq. (B.11). We'll firstly prove the supporting lemmas, and then will prove implications $(i) \implies (iii) \implies (ii) \implies (i)$ for the statements of Theorem 2.5.2.

Lemma B.1.18 *If $c(\cdot)$ has a framed ambiguity representation with finite number of frames, then any framed ambiguity representation of $c(\cdot)$ has a finite number of frames.*

Proof. Let $(U, \mathcal{A})$ be a representation of $c(\cdot)$ with finite number of frames $|\mathcal{A}| < \infty$ and non-degenerate $U(\cdot)$. Then $|\,(\mathcal{A})_{coh}\,| \leq 2^{|\mathcal{A}|} < \infty$ from eq. (2.3). Then we can use Proposition 2.3.2 to conclude the proof. ■

Recall Definition 10: DM 1 is more consistent than DM 2 if for all $A \in \mathcal{K}$ $|c_2(A)| = 1 \implies |c_1(A)| = 1$.

Lemma B.1.19 *Let $c_1(\cdot)$ and $c_2(\cdot)$ have framed ambiguity representations. Then the following statements are equivalent:*

(i) For all $A \in \mathcal{K}$ $|c_2(A)| = 1 \implies |c_1(A)| = 1$.

(ii) For all $A \in \mathcal{K}$ and $f \in H$ $c_2(A) = \{f\} \implies c_1(A) = \{f\}$.

(iii) For all $f, g \in H$ $c_2(\{f, g\} = \{f\} \implies c_1(\{f, g\}) = \{f\}$.

Proof. Implications (ii) $\implies$ (i), (iii) are obvious. Let $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$ be framed ambiguity representations $c_1(\cdot)$ and $c_2(\cdot)$. Given a choice correspondence $c(\cdot)$, denote $\succeq_c$ to be its induced binary relation given by $f \succeq_c g$ if $f \in c(\{f, g\})$; let $\succ_c$ be the strict part of $\succeq_c$.

Step 1. (iii) $\implies$ (ii). By Lemma B.1.17 U_1 and U_2 are positive affine transformations of each other. Let $W_P(\cdot)$ be a utility function of the maxmin model with founded above vNM utility function $U_1 = U_2$ and set of beliefs $P \in \Pi$. If $c_2(A) = \{f\}$, then by α we have $f \succ_{c_2} g$ for all $g \in A \backslash \{f\}$. Therefore by (iii) we have $f \succ_{c_1} g$ for all $g \in A \backslash \{f\}$. It follows that $W_P(f) > W_P(g)$ for all $g \in A \backslash \{f\}$ for all $P \in \mathcal{A}_1$, hence $c_1(A) = \{f\}$, and (ii) holds.

Step 2. (i) $\Longrightarrow$ (ii). Assume that (i) holds. Let $x \in \triangle X$ be a maximizer and $y \in \triangle X$ be a minimizer of U_1 over $\triangle X$. Let $p = 0.5x + 0.5y$. Assume for a contradiction that (ii) fails for DM 1 and DM 2, then there is a menu A such that $c_2(A) = \{f\}$ and $c_1(A) = \{g\}$ with $g \neq f$. By *C-Independence* it is equivalent to $c_2(A_{0.5}) = \{f_{0.5}\}$ and $c_1(A_{0.5}) = \{g_{0.5}\}$, where $A_{0.5} = 0.5A + 0.5p$, $f_{0.5} = 0.5f + 0.5p$, etc. Let $W_P(\cdot)$ be the maxmin utility function with vNM expected utility $U_1(\cdot)$ and set of beliefs $P \in \Pi$, and similarly $\overline{W}_P(\cdot)$ for $U_2(\cdot)$ and $P \in \Pi$. Then $\overline{W}_P(f_{0.5}) > \overline{W}_P(g_{0.5})$ for all $P \in \mathcal{A}_2$. Since $\mathcal{A}_2$ is closed and $W_P(g_{0.5})$ is continuous in both P and $g_{0.5}$, then $\exists \epsilon > 0$ such that $\overline{W}_P(f_{0.5}) > \overline{W}_P(h)$ for all $h \in H : \ d(h, g_{0.5}) < \epsilon$. Thus, $c_2(A_{0.5} \cup \{h\}) = \{f_{0.5}\}$ for any $h \in V_\epsilon(g_{0.5}) = \{h \in H \mid d(h, g_{0.5}) < \epsilon\}$. Consider $\zeta : \ V_\epsilon(g_{0.5}) \to R$ given by $\zeta(h) = \min\limits_{P \in \mathcal{A}_1} (W_P(g_{0.5}) - W_P(h))$. Clearly, ζ is a continuous function. Next, for big enough $D > 0$ act $(1 - \epsilon/D)g_{0.5} + (\epsilon/D)x$ strictly state-wise dominates $g_{0.5}$ and belongs to $V_\epsilon(g_{0.5})$. Similarly, $(1 - \epsilon/D)g_{0.5} + (\epsilon/D)y$ is strictly state-wise dominated by $g_{0.5}$ and also belongs to $V_\epsilon(g_{0.5})$. Thus, ζ takes positive and negative values on $V_\epsilon(g_{0.5})$, hence $\zeta(h) = 0$ for some $h \in V_\epsilon(g_{0.5})$, because $V_\epsilon(g_{0.5})$ is a connected set. Since $\mathcal{A}_1$ is closed, it follows that there exists $P \in \mathcal{A}_1$ such that $W_P(h) = W_P(g_{0.5}) \geq W_P(h')$ for any $h' \in A$. Therefore, $\{h, g_{0.5}\} \in c_1(A_{0.5} \cup \{h\})$, contradicting $|c_2(A_{0.5} \cup \{h\})| = 1 \implies |c_1(A_{0.5} \cup \{h\})| = 1$. ■

Lemma B.1.20 *Let $\mathcal{C} \subseteq \Pi$ be a non-empty closed collection of sets of beliefs. Then:*

(i) If P is a convex union of $\mathcal{C}$ given by eq. (2.4), then $P \in \Pi$, and $I_P(\mathbf{z}) = \min\limits_{P' \in \mathcal{C}} I_{P'}(\mathbf{z})$.

(ii) There is unique $P \in \Pi$ such that $I_P(\mathbf{z}) = \min\limits_{P' \in \mathcal{C}} I_{P'}(\mathbf{z})$. Moreover, P is a convex union of $\mathcal{C}$.

Proof of (i). Obviously, P is non-empty and convex. Next, let $\mu^n \in \bigcup\limits_{P' \in \mathcal{C}} P'$ be such that $\mu^n \longrightarrow \mu \in \Pi$. Then for every $n = 1, 2, ...$ there is $P^n \in \mathcal{C}$ such that $\mu^n \in P^n$. Since $\mathcal{C}$ is compact (as a closed subset of a compact set Π), there is a convergent

subsequence $P^{n_k} \longrightarrow P' \in \mathcal{C}$. Thus, there is $\hat{\mu}^{n_k} \in P'$ for $k = 1, 2, ...$ such that $|\hat{\mu}^{n_k} - \mu^{n_k}| \longrightarrow 0$. Therefore, $|\hat{\mu}^{n_k} - \mu| \leq |\hat{\mu}^{n_k} - \mu^{n_k}| + |\mu^{n_k} - \mu| \to 0$. Since P' is closed, $\hat{\mu}^{n_k} \longrightarrow \mu \in P'$. Thus, $\bigcup_{P' \in \mathcal{C}} P'$ is closed. Therefore, its convex hull P is closed (by Corollary 5.33 of Aliprantis and Border (2005)), and $P \in \Pi$. Notice that by the Theorem of Maximum the support functional $I_{P'}(\mathbf{z})$ is continuous in P', and recall that $\mathcal{C}$ is compact. Then

$$I_P(\mathbf{z}) \;=\; \min_{\mu \in P} \sum_{s \in S} \mu(s)\mathbf{z}_s \;=\; \min_{P' \in \mathcal{C}} \min_{\mu \in P'} \sum_{s \in S} \mu(s)\mathbf{z}_s \;=\; \min_{P' \in \mathcal{C}} \; I_{P'}(\mathbf{z})$$

Proof of (ii). Denote $I(\mathbf{z}) = \min_{P' \in \mathcal{C}} I_{P'}(\mathbf{z})$ for $\mathbf{z} \in R^{|S|}$. Notice that $I(\cdot)$ is monotone, positively homogeneous, C-additive and normalized. Also, $I(\lambda \mathbf{z} + (1-\lambda)\mathbf{z}') = \min_{P' \in \mathcal{C}} I_{P'}(\lambda \mathbf{z} + (1-\lambda)\mathbf{z}') \geq \min_{P' \in \mathcal{C}} \lambda I_{P'}(\mathbf{z}) + (1-\lambda) I_{P'}(\mathbf{z}') \geq \lambda \min_{P' \in \mathcal{C}} I_{P'}(\mathbf{z}) + (1-\lambda) \min_{P' \in \mathcal{C}} I_{P'}(\mathbf{z}') = \lambda I(\mathbf{z}) + (1-\lambda) I(\mathbf{z}')$ for $\lambda \in (0,1)$. Thus, $I(\cdot)$ is concave. By lemma B.1.10 there is a unique $P \in \Pi$ given by eq. (B.12) such that $I_P(\cdot) = I(\cdot)$. From the other hand, by (i) we have $I_{\widehat{P}}(\cdot) = I(\cdot)$, where $\widehat{P} \in \Pi$ is a convex union of $\mathcal{C}$. Therefore, by lemma B.1.10 $P = \sigma(I) = \widehat{P}$. ■

Lemma B.1.21 *Let $\mathcal{C} = \{P_1, ..., P_N\} \subseteq \Pi$ be a non-empty finite collection of sets of beliefs, and $\lambda \in \triangle(\{1, ..., N\})$. Then:*

(i) If P is a convex combination of $\mathcal{C}$ with respect to weights λ given by eq. (2.5), then $P \in \Pi$, and $I_P(\mathbf{z}) = \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z})$

(ii) Function $\zeta : \triangle(\{1, ..., N\}) \to \Pi$ given by $\zeta(\lambda) = \sum_{i=1}^{N} \lambda_i P_i$ is continuous.

(iii) There is a unique $P \in \Pi$ such that $I_P(\mathbf{z}) = \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z})$. Moreover, P is a convex combination of $\mathcal{C}$ with respect to weights λ.

Proof of (i). Obviously, P is non-empty. Next, let $\mu^n \in P$ for $n = 1, 2, ...$, and $\mu^n \longrightarrow \mu$. Then there exists $\mu_i^n \in P_i$ for all $n = 1, 2, ...$, $i = 1, ..., N$ such that $\mu^n = \sum_{i=1}^{N} \lambda_i \mu_i^n$ for all $n = 1, 2, ...$ Notice that $\underset{i=1}{\overset{N}{\times}} P_i$ is compact with respect to the

product topology, thus there is a subsequence μ^{n_k} such that $\mu_i^{n_k} \longrightarrow \mu_i \in P_i$ for all $i = 1, ..., N$. Then $\mu = \lim_{k\to\infty} \mu^{n_k} = \lim_{k\to\infty} \sum_{i=1}^{N} \lambda_i \mu_i^{n_k} = \sum_{i=1}^{N} \lambda_i \mu_i$, hence $\mu \in P$, and P is closed. If $\mu, \mu' \in P$, then there are $\mu_i, \mu_i' \in P_i$ for $i = 1, ..., N$ such that $\mu = \sum_{i=1}^{N} \lambda_i \mu_i$ and $\mu' = \sum_{i=1}^{N} \lambda_i \mu_i'$. Since each P_i is convex, $\rho\mu_i + (1-\rho)\mu_i' \in P_i$ for all $\rho \in (0,1)$, hence $\rho\mu + (1-\rho)\mu' = \sum_{i=1}^{N} \lambda_i(\rho\mu_i + (1-\rho)\mu_i')$, and P is convex. Finally,

$$I_P(\mathbf{z}) = \min_{\mu\in P} \sum_{s\in S} \mu(s) z_s = \min_{\mu_i \in P_i \ \forall i=1,...,N} \sum_{s\in S} \sum_{i=1}^{N} \lambda_i \mu_i(s) z_s =$$

$$= \sum_{i=1}^{N} \lambda_i \left(\min_{\mu_i \in P_i} \sum_{s\in S} \mu_i(s) z_s \right) = \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z})$$

Proof of (ii). Let $\mu \in \sum_{i=1}^{N} \lambda_i P_i$, then there is $\mu_i \in P_i$ for $i = 1, ..., N$ such that $\mu = \sum_{i=1}^{N} \lambda_i \mu_i$. Consider $\mu' = \sum_{i=1}^{N} \lambda_i' \mu_i \in \sum_{i=1}^{N} \lambda_i' P_i$. Then $\mu - \mu' = \sum_{i=1}^{N} (\lambda_i - \lambda_i')\mu_i \to 0$ when $\lambda' \to \lambda$. A symmetric argument shows that for any $\mu' \in \sum_{i=1}^{N} \lambda_i' P_i$ there is $\mu \in \sum_{i=1}^{N} \lambda_i P_i$ such that $\mu' - \mu \to 0$ when $\lambda' \to \lambda$. Thus, $\sum_{i=1}^{N} \lambda_i' P_i \to \sum_{i=1}^{N} \lambda_i P_i$ with respect to the Hausdorff metric, when $\lambda' \to \lambda$. We conclude that $\zeta(\lambda) = \sum_{i=1}^{N} \lambda_i P_i$ is continuous w.r.t. $\lambda \in \triangle(\{1, ..., N\})$.

Proof of (iii). Let $I(\mathbf{z}) = \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z})$ for $\mathbf{z} \in R^{|S|}$. It is easy to see that $I(\cdot)$ is monotone, positively homogeneous, concave, C-additive and normalized. By lemma B.1.10 there is a unique $P \in \Pi$ such that $I_P(\cdot) = I(\cdot)$. From the other hand, by (i) we have $I_{\widehat{P}}(\cdot) = I(\cdot)$, where $\widehat{P}$ is a convex combination of $\mathcal{C}$ with weights λ. Therefore, by lemma B.1.10 $P = \sigma(I) = \widehat{P}$ ■

When vNM expected utility function is fixed (up to the positive affine transformation), denote $c_{\mathcal{D}}(\cdot)$ to be the choice correspondence represented by the framed ambiguity model $(U, \mathcal{D})$, and $\succeq_{\mathcal{D}}$ to be the corresponding binary relation given by $f \succeq_{\mathcal{D}} g$

if $f \in c_{\mathcal{D}}(\{f,g\})$, with $\succ_{\mathcal{D}}$ being its strict part. If $\mathcal{D} = \{P\}$ is a singleton, we'll use $\succeq_P$ instead of $\succeq_{\{P\}}$ for brevity.

Lemma B.1.22 *Let $c_1(\cdot)$ and $c_2(\cdot)$ have framed ambiguity representations $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$. Then the following statements are equivalent:*

(i) DM 1 is more consistent that DM 2.

(ii) U_1 is a positive affine transformation of U_2, and

$[f \succeq_{P_2} g \ \ \forall P_2 \in \mathcal{A}_2]$ implies $[f \succeq_{P_1} g \ \ \forall P_1 \in \mathcal{A}_1]$.

(iii) U_1 is a positive affine transformation of U_2, and

$[f \succ_{P_2} g \ \ \forall P_2 \in \mathcal{A}_2]$ implies $[f \succ_{P_1} g \ \ \forall P_1 \in \mathcal{A}_1]$.

Proof. Step 1. (i) $\Longrightarrow$ (iii). Let (i) holds. By Step 1 of the proof of lemma B.1.19 we know that U_1 is a positive affine transformation of U_2. Next, $[f \succ_{P_2} g \ \ \forall P_2 \in \mathcal{A}_2]$ if and only if $\{f\} = c_2(\{f,g\})$, which by lemma B.1.19 implies $\{f\} = c_1(\{f,g\})$, that is equivalent to $[f \succ_{P_1} g \ \ \forall P \in \mathcal{A}_1]$.

Step 2. (iii) $\Longrightarrow$ (i). Let (iii) holds. Normalize $U(\cdot) = U_1(\cdot) = U_2(\cdot)$ such that $1 = U(x) \geq U(p) \geq U(y) = 0$ for all $p \in \triangle X$. Since $[f \succ_P g \ \ \forall P \in \mathcal{A}_i]$ if and only if $\{f\} = c_i(\{f,g\})$, then $\{f\} = c_2(\{f,g\})$ implies $\{f\} = c_1(\{f,g\})$. Now assume $f \in c_2(\{f,g\})$. Consider $f_{0.5} = 0.5f + 0.5(0.5x + 0.5y)$, similarly $g_{0.5}$, and $f^n_{0.5} = (1-1/n)f_{0.5} + (1/n)x$ for n=1,2,... . Then $\{f^n_{0.5}\} = c_2(\{f^n_{0.5}, g_{0.5}\})$, hence $\{f^n_{0.5}\} = c_1(\{f^n_{0.5}, g_{0.5}\})$. By *Act Continuity* we have $\{f_{0.5}\} \in c_1(\{f_{0.5}, g_{0.5}\})$, hence $f \in c_1(\{f,g\})$. Using lemma B.1.19, we conclude that DM 1 is more consistent than DM 2.

Step 3. (iii) $\Longrightarrow$ (ii). Let (iii) holds. Assume $[f \succeq_{P_2} g \ \ \forall P_2 \in \mathcal{A}_2]$. Using $f_{0.5}, f^n_{0.5}, g_{0.5}$ defined in Part 2, we get $[f^n_{0.5} \succ_{P_2} g_{0.5} \ \ \forall P_2 \in \mathcal{A}_2]$, hence $[f^n_{0.5} \succ_{P_1} g_{0.5} \ \ \forall P_1 \in \mathcal{A}_1]$. By continuity of $\succeq_{P_1}$ we get $[f_{0.5} \succeq_{P_1} g_{0.5} \ \ \forall P_1 \in \mathcal{A}_1]$, which implies $[f \succeq_{P_1} g \ \ \forall P_1 \in \mathcal{A}_1]$.

Step 4. (ii) $\Longrightarrow$ (iii). Let (ii) holds. Assume $[f \succ_{P_2} g \ \ \forall P_2 \in \mathcal{A}_2]$. Notice that $\zeta : \ \Pi \times H \times H$ given by $\zeta(P, f, g) = W_P(f) - W_P(g)$ is continuous on $\mathcal{A}_2 \times H \times H$. Since $\mathcal{A}_2$ is compact (as a closed subset of a compact set Π), $\mathcal{A}_2 \times H \times H$ is compact, therefore ζ is uniformly continuous on it. Then there is $\epsilon \in (0, 1)$ such that $W_P(f) > W_P(g) + 2\epsilon$ for all $P \in \mathcal{A}_2$. Recall the definition of $f_{0.5}, g_{0.5}$ from Part 2, then $W_P(f_{0.5}) > W_P(g_{0.5}) + \epsilon$ for all $P \in \mathcal{A}_2$. Consider $f^{\epsilon}_{0.5} = (1 - \epsilon)f_{0.5} + \epsilon y$, then $W_P(f^{\epsilon}_{0.5}) > W_P(g_{0.5})$ for all $P \in \mathcal{A}_2$, hence $[f^{\epsilon}_{0.5} \succeq_{P_2} g_{0.5} \ \ \forall P_2 \in \mathcal{A}_2]$ and $[f^{\epsilon}_{0.5} \succeq_{P_1} g_{0.5} \ \ \forall P_1 \in \mathcal{A}_1]$, and in particular $W_{P_1}(f_{0.5}) > W_{P_1}(f^{\epsilon}_{0.5}) \geq W_{P_1}(g_{0.5})$ for all $P_1 \in \mathcal{A}_1$. It follows that $[f_{0.5} \succ_{P_1} g_{0.5} \ \ \forall P_1 \in \mathcal{A}_1]$ and $[f \succ_{P_1} g \ \ \forall P_1 \in \mathcal{A}_1]$. ■

The following lemma is closely related to lemmas 1-7 in Crès et al. (2011). The difference is that $\succeq_P$ does not satisfy EUA axiom from Crès et al. (2011) with respect to $\succeq_{P_i}$, but only the Unanimity axiom. We also borrowed some of the proofs and ideas of proofs from their lemmas 1-7.

Lemma B.1.23 *Fix a non-degenerate vNM function U. Let $\mathcal{D} = \{P_1, ..., P_N\} \subseteq \Pi$ be a non-empty finite collection of sets of beliefs, and $\varnothing \neq P \in \Pi$. If $\forall f, g \in H$ we have $[f \succeq_{P_i} g \ \forall i = 1, ..., N] \implies f \succeq_P g$, then there is a function $\psi : \ R^N \to R$ such that $I_P(\mathbf{z}) = \psi(I_{P_1}(\mathbf{z}), ..., I_{P_N}(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$. Moreover, we can choose ψ to be simultaneously:*

(i) Positively homogeneous: $\psi(\lambda t) = \lambda\psi(t)$ for all $\lambda \geq 0$.

(ii) Monotone: $t \geq t' \implies \psi(t) \geq \psi(t')$.

(iii) C-additive: $\psi(t + \beta \cdot (1, ..., 1)) = \psi(t) + \psi(\beta \cdot (1, ..., 1))$ for all $\beta \in R$.

(iv) Normalized: $\psi(1, ..., 1) = 1$.

Proof. Denote $e = (1, ..., 1) \in R^N$. Normalize vNM expected utility function $1 \geq U(x) \geq U(p) \geq U(y) = -1$ for all $p \in \triangle X$ and consider acts in the utility space $(U(f(s)))_{s \in S} = \mathbf{z} \in H = [-1, 1]^{|S|}$. This way $W_{\widetilde{P}}(\mathbf{z}) = I_{\widetilde{P}}(\mathbf{z})$ for $\mathbf{z} \in [-1, 1]^{|S|}$,

where $I_{\widetilde{P}}(\cdot)$ is a support functional of set of beliefs $\widetilde{P}$ given by eq. (B.11). Define $\mathbf{I}^N = (I_1, ..., I_N)$ for brevity. Denote also $\sim_{\widetilde{P}}$ to be the symmetric part of $\succeq_{\widetilde{P}}$.

Step 1. Take arbitrary $\mathbf{z}', \mathbf{z} \in H$. We argue that if $\mathbf{I}^N(\mathbf{z}') \geq \mathbf{I}^N(\mathbf{z})$, then $I_P(\mathbf{z}') \geq I_P(\mathbf{z})$, and if $\mathbf{I}^N(\mathbf{z}') = \mathbf{I}^N(\mathbf{z})$, then $I_P(\mathbf{z}') = I_P(\mathbf{z})$. Indeed, if $\mathbf{I}^N(\mathbf{z}') \geq \mathbf{I}^N(\mathbf{z})$, then $\mathbf{z}' \succeq_{P_i} \mathbf{z}$ for all $i = 1, ..., N$, hence $\mathbf{z}' \succeq_P \mathbf{z}$, and $I_P(\mathbf{z}') \geq I_P(\mathbf{z})$.

Step 2. Let $Q = \mathbf{I}^N([-1,1]^{|S|})$ be the image of the set of acts $H = [-1,1]^{|S|}$ with respect to the vector of utility functions $\mathbf{I}^N = (I_{P_i})_{i=1,...,N}$, representing $\succeq_{P_i}$, $i = 1, ..., N$, and let $cone(Q)$ be a minimum cone that contains Q, which is given by $cone(Q) = \{t \in R^N \mid \exists \widetilde{t} \in Q \text{ and } \gamma \geq 0 : \ t = \gamma \widetilde{t}\}$. Define function $\phi : \ cone(Q) \to R$ by

$$\phi(t) \quad = \quad \gamma I_P(\mathbf{z}) \ \text{ for some } \gamma > 0, \mathbf{z} \in H \text{ such that } \ \mathbf{I}^N(\mathbf{z}) = \gamma^{-1} t \tag{B.17}$$

Notice that ϕ is well-defined. Indeed, firstly, let $t \in cone(Q)$, then there is $\gamma > 0$ and $\hat{t} \in Q$ such that $t = \gamma \hat{t}$ and there is $\mathbf{z} \in H$ such that $\mathbf{I}^N(\mathbf{z}) = \hat{t} = \gamma^{-1} t$. Secondly, let $\mathbf{z}, \mathbf{z}' \in H$ be such that $\mathbf{I}^N(\mathbf{z}) = \gamma^{-1} t$ and $\mathbf{I}^N(\mathbf{z}') = (\gamma')^{-1} t$ for some $\gamma, \gamma' > 0$. WLOG, $\gamma \geq \gamma'$. Then $(\gamma'/\gamma)\, \mathbf{z}' \in H$, $\mathbf{I}^N((\gamma'/\gamma)\, \mathbf{z}') = (\gamma'/\gamma)\, \mathbf{I}^N(\mathbf{z}') = \gamma^{-1} t = \mathbf{I}^N(\mathbf{z})$. Therefore, by Step 1 we have $(\gamma'/\gamma)\, I_P(\mathbf{z}') = I_P((\gamma'/\gamma)\, \mathbf{z}') = I_P(\mathbf{z})$. Thus, $\phi(t) = \gamma' I_P(\mathbf{z}') = \gamma I_P(\mathbf{z})$ does not depend on the choice of γ and $\mathbf{z}$ such that $\mathbf{I}^N(\mathbf{z}) = \gamma^{-1} t$.

Step 3. Notice that, since $\mathbf{I}^N(0) = 0$, and $I_P(0) = 0$, then $\phi(0) = 0$. Let us show that ϕ is positively homogeneous, monotone, C-additive and normalized. We consider arbitrary $t, t' \in cone(Q)$. Then there are $\mathbf{z}', \mathbf{z} \in H$ and $\gamma, \gamma' > 0$ such that

$\mathbf{I}^N(\mathbf{z}) = \gamma^{-1}t$, $\mathbf{I}^N(\mathbf{z}') = (\gamma')^{-1}t'$, and we have $\phi(t) = \gamma I_P(\mathbf{z})$, $\phi(t') = \gamma' I_P(\mathbf{z}')$.

Step 3.1. Let $t' = \beta t$ for $\beta \geq 0$. If $t = 0$, then $t' = 0$ and $\phi(t') = 0 = \beta \cdot 0 = \beta \cdot \phi(t)$. If $\beta = 0$, then $t' = 0$, and $\phi(t') = 0 = 0 \cdot \phi(t)$. In other cases, $\beta > 0$ and $t, t' \neq 0$. Suppose that $(\gamma\beta)/\gamma' \leq 1$. Then $(\gamma\beta)/\gamma' \cdot \mathbf{z} \in H$. Next, $\mathbf{I}^N((\gamma\beta)/\gamma' \cdot \mathbf{z}) = ((\gamma\beta)/\gamma') \cdot \gamma^{-1}t = (\gamma')^{-1}\beta \cdot t = (\gamma')^{-1}t' = \mathbf{I}^N(\mathbf{z}')$. Thus, by Step 1 $(\gamma\beta)/\gamma' \cdot I_P(\mathbf{z}) = I_P((\gamma\beta)/\gamma' \cdot \mathbf{z}) = I_P(\mathbf{z}')$. Finally, $\phi(t') = \gamma' I_P(\mathbf{z}') = \gamma' \cdot (\gamma\beta)/\gamma' \cdot I_P(\mathbf{z}) = \beta\gamma I_P(\mathbf{z}) = \beta\phi(t)$. The case when $(\gamma\beta)/\gamma' > 1$ is proven by a symmetric argument, considering $(\gamma'/(\gamma\beta)) \cdot \mathbf{z}' \in H$. Thus, ϕ is positively homogeneous.

Step 3.2. Let $t \geq t'$. Assume $\gamma/\gamma' \leq 1$. Then $(\gamma/\gamma') \cdot \mathbf{z} \in H$, and $\mathbf{I}^N((\gamma/\gamma') \cdot \mathbf{z}) = (\gamma/\gamma') \cdot \mathbf{I}^N(\mathbf{z}) = (\gamma')^{-1}t \geq (\gamma')^{-1}t' = \mathbf{I}^N(\mathbf{z}')$. Thus by Step 1 $\phi(t) = \gamma I_P(\mathbf{z}) = \gamma' I_P((\gamma/\gamma') \cdot \mathbf{z}) \geq \gamma' I_P(\mathbf{z}') = \phi(t')$. We conclude that ϕ is monotone.

Step 3.3. Consider $\iota = (1, ..., 1) \in H$. Then $\mathbf{I}^N(\iota) = e$, hence $\phi(e) = I_P(\iota) = 1$. Therefore, ϕ is normalized.

Step 3.4. Let $t = t' + \beta \cdot e$ for $\beta \in R$. If $\beta = 0$, then $\phi(t') = \phi(t) = \phi(t) + \phi(0 \cdot e)$, thus C-additivity holds. WLOG, $\beta > 0$ (if $\beta < 0$, then consider $t' = t + (-\beta) \cdot e$). Denote $\delta = \min\{\gamma'/(3\beta), 1/3\}$. Then $\delta \cdot \mathbf{z}' + \delta\beta(\gamma')^{-1} \cdot \iota \in H$. We have $\mathbf{I}^N(\delta \cdot \mathbf{z}' + \delta\beta(\gamma')^{-1} \cdot \iota) = \delta \cdot \mathbf{I}^N(\mathbf{z}') + \delta\beta(\gamma')^{-1} \cdot e = \delta(\gamma')^{-1} \cdot (t' + \beta \cdot e) = \delta(\gamma')^{-1} \cdot t$. Therefore, $\phi(t) = (\gamma'/\delta) \cdot I_P(\delta\mathbf{z}' + \delta\beta(\gamma')^{-1} \cdot \iota) = (\gamma'/\delta) \cdot (\delta I_P(\mathbf{z}') + \delta\beta(\gamma')^{-1}) = \gamma' I_P(\mathbf{z}') + \beta = \phi(t') + \beta = \phi(t') + \beta\phi(e) = \phi(t') + \phi(\beta \cdot e)$ by Steps 3.3. and 3.2. Thus, ϕ is C-additive.

Step 4. Take $\gamma = 1$, then by the definition of ϕ we have $I_P(\mathbf{z}) = \phi(\mathbf{I}^N(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$ whenever $t = \mathbf{I}^N(\mathbf{z}) \in cone(Q)$. To see that this always holds, $\mathbf{I}^N(\mathbf{z}) = |\mathbf{z}| \cdot \mathbf{I}^N(\mathbf{z}/(|\mathbf{z}|)) \in cone(Q)$ for $|\mathbf{z}| \neq 0$, and $\mathbf{I}^N(0) \in Q \subset cone(Q)$. Thus, $I_P(\mathbf{z}) = \phi(\mathbf{I}^N(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$.

Step 5. Define $\psi: \ R^N \to R$ by

$$\psi(t) \quad = \quad \inf \ \{\phi(t') \mid t' \in cone(Q) \text{ and } t' \geq t\} \tag{B.18}$$

Notice that, since $\iota, (-\iota) \in H$, then $e, (-e) \in Q$, and $\beta \cdot e \in cone(Q)$ for all $\beta \in R$. Since $\left(\min_{i=1,\ldots,N} t_i\right) \cdot \ e \leq t \leq \left(\max_{i=1,\ldots,N} t_i\right) \cdot \ e$, then inf in eq. (B.18) exists, and $\psi(\cdot)$ is well-defined.

Step 5.1. Let us show that $I_P(\mathbf{z}) = \psi(\mathbf{I}^N(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$. We argue that $\psi(\mathbf{I}^N(\mathbf{z})) = \phi(\mathbf{I}^N(\mathbf{z}))$. Indeed, $\mathbf{I}^N(\mathbf{z}) = |\mathbf{z}| \cdot \mathbf{I}^N(\mathbf{z}/(|\mathbf{z}|)) \in cone(Q)$ for $|\mathbf{z}| \neq 0$, and $\mathbf{I}^N(0) \in Q \subset cone(Q)$, thus $\phi(\mathbf{I}^N(\mathbf{z}))$ is well-defined; if $t' \geq \mathbf{I}^N(\mathbf{z})$, then $\phi(t') \geq \phi(\mathbf{I}^N(\mathbf{z}))$, since $\phi(\cdot)$ is monotone by Step 3.2. Step 4 finishes the proof.

Step 5.2. Let $\gamma > 0$ and $t \in R^{|S|}$. Then

$$\begin{aligned}
\psi(\gamma t) \quad &= \quad \inf \{\phi(t') \mid t' \in cone(Q) \text{ and } t' \geq \gamma t\} \quad = \\
&= \quad \inf \{\phi(t') \mid t' \in cone(Q) \text{ and } \gamma^{-1} t' \geq t\} \quad = \\
&= \quad \inf \{\phi(t') \mid \gamma^{-1} t' \in cone(Q) \text{ and } \gamma^{-1} t' \geq t\} \quad = \\
&= \quad \inf \{\phi(\gamma t'') \mid t'' \in cone(Q) \text{ and } t'' \geq t\} \quad = \\
&= \quad \inf \{\gamma \phi(t'') \mid t'' \in cone(Q) \text{ and } t'' \geq t\} \quad = \\
&= \quad \gamma \cdot \inf \{\phi(t'') \mid t'' \in cone(Q) \text{ and } t'' \geq t\} \quad = \quad \gamma \psi(t)
\end{aligned}$$

If $\gamma = 0$, then $\psi(\gamma t) = \psi(0) = \gamma\psi(t)$. Thus, $\psi(\cdot)$ is positively homogeneous.

Step 5.3. Let $t \geq t'$ for some $t, t' \in R^{|S|}$. Then

$$\{\ t'' \in cone(Q) \mid t'' \geq t\ \} \quad \subseteq \quad \{\ t'' \in cone(Q) \mid t'' \geq t'\ \}$$

Therefore, $\psi(t) \geq \psi(t')$. We conclude that $\psi(\cdot)$ is monotone.

Step 5.4. Since $e \in cone(Q)$, and $\phi(\cdot)$ is monotone and normalized, then $\psi(e) = \phi(e) = 1$. Hence, $\psi(\cdot)$ is normalized.

Step 5.5. Let's firstly show that $t \in cone(Q)$ implies $t + \beta e \in cone(Q)$ for all $\beta \in R$. We already know this for $t = 0$. Case $\beta = 0$ is also trivial, so consider $t \neq 0$ and $\beta \neq 0$. By the definition of $cone(Q)$ there is $\mathbf{z} \in H$ and $\gamma > 0$ such that $t = \gamma \mathbf{I}^N(\mathbf{z})$ (since $t \neq 0$, $\gamma \neq 0$). Take $\delta = \min\{\gamma/(3|\beta|), 1/3\} > 0$. Then $\delta\mathbf{z} + \delta\gamma^{-1}\beta \cdot \iota \in H$. Therefore, $\delta\mathbf{I}^N(\mathbf{z}) + \delta\gamma^{-1}\beta \cdot e = \mathbf{I}^N(\delta\mathbf{z} + \delta\gamma^{-1}\beta \cdot \iota) \in Q$. Hence, $t + \beta e = \gamma\mathbf{I}^N(\mathbf{z}) + \beta e \in cone(Q)$. To see that $\psi(\cdot)$ is C-additive,

$$\begin{aligned}
\psi(t + \beta e) &= \inf\{\phi(t') \mid t' \in cone(Q) \text{ and } t' \geq t + \beta e\} \ = \\
&= \inf\{\phi(t') \mid t' \in cone(Q) \text{ and } t' - \beta e \geq t\} \ = \\
&= \inf\{\phi(t') \mid t' - \beta e \in cone(Q) \text{ and } t' - \beta e \geq t\} \ = \\
&= \inf\{\phi(t'' + \beta e) \mid t'' \in cone(Q) \text{ and } t'' \geq t\} \ = \\
&= \inf\{\phi(t'') + \beta \mid t' - \beta e \in cone(Q) \text{ and } t' - \beta e \geq t\} \ = \\
&= \inf\{\phi(t'') \mid t'' \in cone(Q) \text{ and } t'' \geq t\} + \beta \ = \\
&= \psi(t) + \beta \ = \ \psi(t) + \beta\psi(e) \ = \ \psi(t) + \psi(\beta e)
\end{aligned}$$

To summarise, function $\psi : R^N \to R$ given by eq. (B.18) is positively homogeneous, monotone, C-additive, normalized, and satisfies $I_P(\mathbf{z}) = \psi(\mathbf{I}^N(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$. ■

We next want to make use results from Frick et al. (2019a).

Lemma B.1.24 *Let function* $\psi : [0,1]^N \to R$ *be positively homogeneous, monotone, C-additive and normalized, then there is a non-empty compact (w.r.t. Hausdorff metric) collection* Θ *of non-empty compact and convex sets of weights* $\Lambda \subseteq \triangle(\{1,...,N\})$ *such that for all* $t \in [0,1]^N$

$$\psi(t) \quad = \quad \max_{\Lambda \in \Theta} \min_{\lambda \in \Lambda} \sum_{i=1}^{N} \lambda_i t_i \tag{B.19}$$

Proof. This formula is eq. (2) in Frick et al. (2019a), and later is discussed in their Remark (iv). Frick et al. (2019a) give the formal proof in their Appendix B1. ■

The next lemma is Proposition 1[1] in Crès et al. (2011) formulated in the utility space:

Lemma B.1.25 *Let* $\Lambda \subseteq \triangle(1,...,N)$ *be a non-empty closed and convex set of weights, then*

$$I_P(\mathbf{z}) \;=\; \min_{\lambda \in \Lambda} \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z}) \quad \forall \mathbf{z} \in R^{|S|}$$

if and only if

$$P \;=\; \left\{ \mu \in \triangle S \;\middle|\; \exists \lambda \in \Lambda \text{ and } \mu_i \in P_i : \; \mu = \sum_{i=1}^{N} \lambda_i \mu_i \right\}$$

Proof. Normalize vNM expected utility function $U(\cdot)$ such that $1 = U(x) \geq U(p) \geq U(y) = -1$ for all $p \in \triangle X$. Then use Proposition 1 from Crès et al. (2011) to argue that the statement is true if we change the domain of $I_P(\mathbf{z})$ and $I_{P_i}(\mathbf{z})$ from $R^{|S|}$ to $[-1,1]^{|S|}$. Then use positive homogeneity of I_P, I_{P_i} to argue that the statement remains true if we expand the domain from $[-1,1]^{|S|}$ to $R^{|S|}$. ■

[1] We demand Λ to be non-empty, so that J_0 in eq. (5) of Crès et al. (2011) is well-defined.

Fix a vNM expected utility function $U(\cdot)$. For an arbitrary non-empty collection of frames $\mathcal{D}$ denote $(\mathcal{D})_{agg}$ to be its closure with respect to the operations of convex union, coherent intersection and convex combination, given by Definition 11.

Lemma B.1.26 *For any non-empty collection of sets of beliefs $\mathcal{D} \subseteq \Pi$ its closure with respect to operations of convex union, coherent intersection and convex combination exists and is unique.*

Proof. Let $\mathbb{F}$ be the set of collections of sets of beliefs $\mathcal{F}$ such that $\mathcal{D} \subseteq \mathcal{F} \subseteq \Pi$ and $\mathcal{F}$ is closed with respect to the operations of convex union, coherent intersection and convex combination for all $\mathcal{F} \in \mathbb{F}$. Notice that $\Pi \in \mathbb{F} \neq \varnothing$. Then the unique $(\mathcal{D})_{agg}$ is given by $(\mathcal{D})_{agg} = \bigcap_{\mathcal{F} \in \mathbb{F}} \mathcal{F}$ ■

Lemma B.1.27 *Let vNM expected utility function $U(\cdot)$ be fixed, and $\varnothing \neq \mathcal{D} \subseteq \Pi$. If $P \in (\mathcal{D})_{agg}$, then $[f \succ_{P'} g \quad \forall P' \in \mathcal{D}]$ implies $f \succ_P g$.*

Proof. Let $\mathcal{E} \subseteq (\mathcal{D})_{agg}$ be the collection of all sets of beliefs $P \in (\mathcal{D})_{agg}$ such that $[f \succ_{P'} g \quad \forall P' \in \mathcal{D}]$ implies $f \succ_P g$. Notice that $\varnothing \neq \mathcal{D} \subseteq \mathcal{E}$. Take an arbitrary non-empty closed subcollection $\mathcal{C} \subseteq \mathcal{E}$. Normalize $1 = U(x) \leq U(p) \leq U(y) = 0$ for all $p \in \triangle X$.

Case 1. Assume that P is a convex union of $\mathcal{C}$ given by eq. (2.4). Using lemma B.1.20, we get $W_P(f) = \min_{P' \in \mathcal{C}} W_{P'}(f)$ for all $f \in H$.

Case 2. Assume that P is a coherent intersection of $\mathcal{C}$ given by eq. (B.15). Then $W_P(f) = \max_{P' \in \mathcal{C}} W_{P'}(f)$ for all $f \in H$.

Case 3. Assume P is a convex combination of $\mathcal{C} = \{P_1, ..., P_N\}$ with weights λ, given by eq. (2.5). Using lemma B.1.21, we get $W_P(f) = \sum_{i=1}^{N} \lambda_i W_{P_i}(f)$ for all $f \in H$. Now let $[f \succ_{P'} g \quad \forall P' \in \mathcal{D}]$. Since $\mathcal{C} \subseteq \mathcal{E}$, then $f \succ_{P'} g$ for all $P' \in \mathcal{C}$, thus $W_{P'}(f) > W_{P'}(g)$ for all $P' \in \mathcal{C}$, which in all three cases implies $W_P(f) > W_P(g)$ and $f \succ_P g$. Notice that since $\mathcal{C} \subseteq \mathcal{E} \subseteq (\mathcal{D})_{agg}$, and $(\mathcal{D})_{agg}$ is closed with respect to

all three operations, then $P \in (\mathcal{D})_{agg}$. Therefore, $P \in \mathcal{E}$. Hence, $\mathcal{E}$ is closed with respect to all three operations. Since $\mathcal{D} \subseteq \mathcal{E} \subseteq (\mathcal{D})_{agg}$, we conclude that $\mathcal{E} = (\mathcal{D})_{agg}$, which proves the lemma. ■

Lemma B.1.28 *Let $c_1(\cdot)$ and $c_2(\cdot)$ have framed ambiguity representations $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$ with $|\mathcal{A}_2| < \infty$. Then the following statements are equivalent:*

(i) DM 1 is more consistent than DM 2.

(ii) U_1 is a positive affine transformation of U_2, and $\mathcal{A}_1$ belongs to the closure of $\mathcal{A}_2$ with respect to operations of convex union, coherent intersection and convex combination.

(iii) U_1 is a positive affine transformation of U_2, and any $P \in \mathcal{A}_1$ is a coherent intersection of convex unions of convex combinations of sets of beliefs in $\mathcal{A}_2$

Proof. Step 1. (i) $\Longrightarrow$ (iii). Suppose DM 1 is more consistent than DM 2. Let $P \in \mathcal{A}_1$. By lemma B.1.22 U_1 is a positive affine transformation of U_2, thus we normalize $U_1(\cdot) = U_2(\cdot) = U(\cdot)$ without loss of generality. Lemma B.1.22 also implies that for any $f, g \in H$ if $[f \succeq_{P_i} g \; \forall i = 1, ..., N]$, then $f \succeq_P g$. Therefore, by lemma B.1.23 there is a positively homogeneous, monotone, C-additive and normalized function $\psi : R^N \to R$ such that $I_P(\mathbf{z}) = \psi(I_{P_1}(\mathbf{z}), ..., I_{P_N}(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$. By lemma B.1.24 there is a non-empty compact collection $\Theta \subseteq 2^{\triangle(\{1,...,N\})}$ of non-empty compact convex sets of weights $\Lambda \subseteq \triangle(\{1, ..., N\})$ such that

$$I_P(\mathbf{z}) \quad = \quad \max_{\Lambda \in \Theta} \min_{\lambda \in \Lambda} \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z}) \qquad \forall \mathbf{z} \in R^{|S|} \tag{B.20}$$

Notice that by lemma B.1.21 for arbitrary $\lambda \in \Lambda \in \Theta$ there is a unique $P^\lambda \in \Pi$ given by $P^\lambda = \sum_{i=1}^{N} \lambda_i P_i$ (eq. 2.5) such that $I_{P^\lambda}(\mathbf{z}) = \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z})$ for all $\mathbf{z} \in R^{|S|}$. Thus, P^λ is a convex combination of P_i, $i = 1, ..., N$ with weights λ. Therefore, for arbitrary

$\Lambda \in \Theta$ we have $\min_{\lambda\in\Lambda} \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z}) = \min_{\lambda\in\Lambda} I_{P^\lambda}(\mathbf{z})\ \forall \mathbf{z} \in R^{|S|}$. For arbitrary $\Lambda \in \Theta$ consider $\mathbb{D}_\Lambda \equiv \left\{ \sum_{i=1}^{N} \lambda_i P_i \right\}_{\lambda\in\Lambda} = \left\{ P^\lambda \right\}_{\lambda\in\Lambda} \subseteq \Pi$. Since Λ is compact, and since by lemma B.1.21 $\sum_{i=1}^{N} \lambda_i P_i$ is continuous with respect to $\lambda \in \triangle(\{1, ..., N\})$, then $\mathbb{D}_\Lambda$ is compact. Therefore, by lemma B.1.20 there is a unique P^Λ given by a convex union of P^λ, $\lambda \in \Lambda$ (eq. 2.4) such that

$$I_{P^\Lambda}(\mathbf{z}) = \min_{\lambda\in\Lambda} I_{P^\lambda}(\mathbf{z}) = \min_{\lambda\in\Lambda} \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z}) \qquad \forall \mathbf{z} \in R^{|S|} \tag{B.21}$$

Therefore,

$$I_P(\mathbf{z}) = \max_{\Lambda\in\Theta} I_{P^\Lambda}(\mathbf{z}) \qquad \forall \mathbf{z} \in R^{|S|} \tag{B.22}$$

We conclude that any $P \in \mathcal{A}_1$ is a coherent intersection of sets of beliefs P^Λ, $\Lambda \in \Theta$, where each P^Λ is a convex union of sets of beliefs P^λ, $\lambda \in \Lambda$, and, in turn, each P^λ is a convex combination of $P_i \in \{P_1, ..., P_N\} = \mathcal{A}_2$ with weights λ.

Proof. Step 2. (iii) $\Longrightarrow$ (ii). The statement is obvious.

Step 3. (ii) $\Longrightarrow$ (i). By lemma B.1.27 for any $P \in \mathcal{A}_1$ for any $f, g \in H$ if $[f \succ_{P'} g\ \forall P' \in \mathcal{A}_2]$, then $f \succ_P g$. Therefore, by lemma B.1.22 DM 1 is more consistent than DM 2. ■

Theorem 2.5.2 follows from lemma B.1.28. ■

B.1.8 Proof of Proposition 2.5.3, Corollary 2.5.4 and Corollary 2.5.5

Proposition 2.5.3 follows from lemma B.1.19. ■

Proof of Corollary 2.5.4, Part 1. Assume firstly that $c(\{f,g\}) = c'(\{f,g\})$, then by Proposition 2.5.3 DM with choice correspondence $c(\cdot)$ is more consistent than DM' with choice correspondence $c'(\cdot)$, and vice versa DM' is more consistent than DM. Therefore, by Theorem 2.5.2 $U(\cdot)$ and $U'(\cdot)$ are positive affine transformations of each other, and $\mathcal{A} \subseteq (\mathcal{A}')_{agg}$ and $\mathcal{A}' \subseteq (\mathcal{A})_{agg}$, where $(\cdot)_{agg}$ denotes the closure with respect to the operations of convex union, convex combination and coherent intersection. Observe that $(\cdot)_{agg}$ is monotone with respect to set inclusion, thus $(\mathcal{A})_{agg} \subseteq ((\mathcal{A}')_{agg})_{agg} = (\mathcal{A}')_{agg}$ and $(\mathcal{A}')_{agg} \subseteq ((\mathcal{A})_{agg})_{agg} = (\mathcal{A})_{agg}$. Therefore, $(\mathcal{A})_{agg} = (\mathcal{A}')_{agg}$.

Part 2. Assume now that U and U' are positive affine transformations of each other, and $(\mathcal{A})_{agg} = (\mathcal{A}')_{agg}$. Then $\mathcal{A} \subseteq (\mathcal{A}')_{agg}$ and $\mathcal{A}' \subseteq (\mathcal{A})_{agg}$, and DM is more consistent than DM', and DM' is more consistent than DM by Theorem 2.5.2. Therefore, by Proposition 2.5.3 for all $f, g \in H$ we have $c(\{f,g\}) \subseteq c'(\{f,g\})$ and $c'(\{f,g\}) \subseteq c(\{f,g\})$. We conclude that $c(\{f,g\}) = c'(\{f,g\})$ for all binary menus $A = \{f,g\}$. ■

Proof of Corollary 2.5.5. Consider DM 2 with a with choice correspondence $c_2(\cdot)$ represented by a framed ambiguity model $(U, \{P_i\}_{i=1,...,N})$, where $U(\cdot)$ is a common vNM expected utility function that is a part of representation of $\succeq_i$, $i = 1, ..., N$ and $\succeq$, and DM 1 with with choice correspondence $c_1(\cdot)$, represented by $(U, \{P\})$. These models are indeed well-defined, since $U(\cdot)$ is non-degenerate because of non-degeneracy of $\succeq_i$, $\succeq$, and families of frames are finite. Therefore, by lemma B.1.22 $\succeq$

satisfies *Unanimity* with respect to $(U, \{P_i\}_{i=1,...,N})$ if and only if DM 1 is more consistent than DM 2, which is equivalent by Theorem 2.5.2 to $\{P\} \subseteq (\{P_1, ..., P_N\})_{agg}$ ■

B.1.9 Proof of Proposition 2.5.6

Lemma B.1.29 *Let Condition 1 or 2 hold. Then*

$$I_{\bigcap\limits_{i=1,...,N} P_i}(\mathbf{z}) \quad = \quad sup\left\{ \sum_{i=1}^{N} I_{P_i}(\mathbf{z}_i) \;\middle|\; \sum_{i=1}^{N} \mathbf{z}_i = \mathbf{z} \right\} \tag{B.23}$$

where for each $\mathbf{z} \in R^{|S|}$ *the suprenum is attained, and* $I_P(\cdot)$ *is a support functional of set* P *given by eq. B.11*[2]*.*

Proof. If Condition 2 holds, the statement follows from Corollary 16.4.1 in Rockafellar (1970). If Condition 1 holds, the statement follows from Theorem 20.1 in Rockafellar (1970) for the indicator functions $f_i(\cdot) = \delta(\cdot|P_i)$, where $f_i(\cdot)$ are polyhedral because P_i are polyhedral (Corollary 19.2.1 in Rockafellar (1970)). ■

Now let's prove Proposition 2.5.6. Rename $\mathcal{A}_1 = \{P_1\}$ to $\mathcal{A}_1 = \{P\}$ for convenience and denote $\mathcal{A}_2 = \{P_1, ..., P_N\}$.

Part 1. Step 1. Assume that U_1 is a positive affine transformation of U_2, and $P = \bigcap\limits_{i=1}^{N} P_i$. WLOG, $U_1(\cdot) = U_2(\cdot) = U(\cdot)$ normalized such that $U(x) = 1$ and $U(y) = 0$, where x is the best, and y is the worst prize, and we use this vNM utility function in our calculations. Take $q = 0.5x + 0.5y$.

[2]Notice that we define support functional as a minimum of a linear function over a set, while Rockafellar (1970) defines it as a maximum. Hence we get *sup* instead of his inf in lemma B.23.

Step 2. Consider arbitrary $f \in H$. By lemma B.1.29 there exist $z_1, ..., z_N \in R^{|S|}$ such that $W_P(U(f)) = \sum_{i=1}^{N} I_{P_i}(z_i)$ and $U(f) = \sum_{i=1}^{N}(z_i)$, where $U(f)_s = U(f(s))$. Consider $\tilde{z}_i = z_i + \left(\sum_{j=1}^{N} I_{P_j}(z_j) - I_i(z_i)\right)\iota$. Since $I_{P_i}(\cdot)$ is C-additive and normalized, then $z = \sum_{i=1}^{N} \tilde{z}_i$, $W_P(U(f)) = \sum_{i=1}^{N} I_{P_i}(\tilde{z}_i)$, and $I_{P_i}(\tilde{z}_i) = W_P(U(f))$ for all $i = 1, ..., N$. Take $\lambda = 0.1\left(\max\left\{1, \max_{i=1,...,N}|\tilde{z}_i|\right\}\right)^{-1} > 0$. Then $\hat{z}_i \equiv \lambda\tilde{z}_i + (1-\lambda)0.5\iota \in Z_{0.1}$ and $\lambda U(f) + (1-\lambda)0.5\iota = \sum_{i=1}^{N} \hat{z}_i$, and by positive homogeneity and C-additivity of the support functional we have $I_P(\lambda U(f) + (1-\lambda)0.5\iota) = \sum_{i=1}^{N} I_{P_i}(\hat{z}_i)$. Consider $f_i \in H$, $i = 1, ..., N$ given by

$$f_i(s) \;=\; \frac{\lambda}{N} f(s) + \left((\hat{z}_i)_s - \frac{\lambda}{N} U(f(s))\right) x + \left(1 - (\hat{z}_i(s))_s + \frac{\lambda}{N} U(f(s))\right) y$$

Notice that because of our choice of λ we have $(\hat{z}_i)_s - \frac{\lambda}{N} U(f(s)) \in [0.3, 0.7]$, thus f_i are valid acts. It is easy to see that $\lambda f + (1-\lambda)q = \sum_{i=1}^{k} \sigma_i f_i$ for $\sigma_i = 1/N$. Next, $U(f_i(s)) = U((\hat{z}_i)_s)$, thus $W_{P_i}(f_i) = I_{P_i}(U(f_i)) = I_P(\lambda U(f) + (1-\lambda)0.5\iota) = W_P(\lambda f + (1-\lambda)q)$ for all $i = 1, ..., N$. Since $\lambda f + (1-\lambda)q \in c_1(\{\lambda f + (1-\lambda)q, p\}) \iff W_P(\lambda f + (1-\lambda)q) \geq U(p)$, then $W_{P_i}(f_i) \geq U(p)$, thus $f_i \in c_2(\{f_i, p\})$.

Step 3. Take σ_i, f_i and λ from the previous step. Assume $f_i \in c_2(\{f_i, p\})$ for all $i = 1, ..., k$. Then $W_{\bigcap_{i=1}^{N} P_i}(f_j) \geq \max_{i=1,...,N} W_{P_i}(f_j) \geq U(p)$. Therefore, by quasi-concavity of $W(\cdot)$ we have $W_{\bigcap_{i=1}^{N} P_i}(\lambda f + (1-\lambda)q) \geq U(p)$, hence $\lambda f + (1-\lambda)q \in c_1(\{\lambda f + (1-\lambda)q, p\})$. We conclude that DM 1 with choice correspondence represented by (U, P) optimistically learns from DM 2.

Part 2. For the other direction, suppose DM 1 optimistically learns from DM 2. Notice that DM 1 satisfies *WARP*, and hence her choice is represented by the utility

function of the maxmin model $(U_1, \{P\})$. This utility function is equal to

$$W(f) \;=\; \sup_{\lambda\in(0,1]} \Big(\frac{1}{\lambda} \left(Q_\lambda(f)\right) - (1-\lambda)U(q) \Big)$$

where

$$Q_\lambda(f) \;=\; \sup\Big\{ U_2(p) \;\Big|\; p \in H_0 : \exists f_i, ..., f_N \in H, \sigma \in \triangle(\{1, ..., N\}) : \tag{B.24}$$

$$\sum_{i=1}^{N} \sigma_i f_i = \lambda f + (1-\lambda) q \;,\; f_i \in c_2(\{f_i, p\}) \Big\}$$

Notice that by our calculations in Part 1, maxmin model $(U_2, \bigcap\limits_{i=1,..,N} P_i)$ is represented by the same utility function. Therefore, these two model represent the same choice correspondence, hence U_2 is a positive affine transformation of U_1, and $P = \bigcap\limits_{i=1,..,N} P_i$.

■

B.2 Proofs of other statements

B.2.1 Proof of Proposition 2.2.2

Part 1. Independence of *Framed Uncertainty*. Let $X = \{x, y, z\}$ and S be arbitrary. Consider

$$c(A) \;=\; \left\{ f \in A \;\middle|\; \sum_{s\in S} f(s)(x) \geq \sum_{s\in S} g(s)(x) \;\forall g \in A \right\} \;\bigcup$$

$$\bigcup \; \left\{ f \in A \;\middle|\; \sum_{s\in S} f(s)(y) \geq \sum_{s\in S} g(s)(y) \;\forall g \in A \right\}$$

Notice that $c(\{x, 0.5x + 0.5y\}) = \{x, 0.5x + 0.5y\}$ and $c(\{x, 0.5x + 0.5y, y\}) = \{x, y\}$, thus C-β, and, hence, *Framed Uncertainty* is violated. It is easy to see

that *C-Independence, No-C-Hedging* and *Strict Monotonicity* are satisfied. Next, let the premise of the *Indirect Ambiguity Aversion* holds for acts h, f, g and menu A. Suppose that for some $i \in \{x, y\}$ we have $\sum\limits_{s\in S} h(s)(i) \geq \sum\limits_{s\in S} h'(s)(i)$ for all $h' \in A$. Then $\sum\limits_{s\in S} h(s)(i) < \sum\limits_{s\in S} f(s)(i)$ and $\sum\limits_{s\in S} h(s)(i) < \sum\limits_{s\in S} g(s)(i)$, therefore $\sum\limits_{s\in S} h(s)(i) < \sum\limits_{s\in S} (\lambda f(s)(i) + (1-\lambda)g(s)(i))$, hence $h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$, and the *Indirect Ambiguity Aversion* holds. Next, let $f^k \in c(A^k)$ for $k = 1, 2, ...,$ and $(A^k, f^k) \longrightarrow (A, f)$. Then at least for one $i \in \{x, y\}$ there exists a subsequence (A^{k_l}, f^{k_l}) such that $\sum\limits_{s\in S} f^{k_l}(s)(i) \geq \sum\limits_{s\in S} g(s)(i)$ for all $g \in A^{k_l}$. Since $(A^{k_l}, f^{k_l}) \longrightarrow (A, f)$, it follows that $\sum\limits_{s\in S} f(s)(i) \geq \sum\limits_{s\in S} g(s)(i)$ for all $g \in A$, hence $f \in c(A)$, and *Continuity* holds. Finally, $z \notin c(\{0.5x + 0.5y, z\})$, thus *C-Non-Degeneracy* holds.

Part 2. Independence of *C-Independence*. Let $X = \{x, y\}$, $S = \{1, 2\}$. Consider

$$c(A) = \arg\max_{f\in A} W(f), \qquad \text{where} \quad W(f) = \frac{f(1)(x) + f(2)(x)}{1 + f(2)(x)}$$

Let $f = (0.4x + 0.6y, y)$ and $g = (y, x)$. Then $W(f) = 0.4 < 0.5 = W(g)$, hence $c(\{f, g\}) = \{g\}$. Next, let $p = (x, x)$, and consider $f' = 0.5f + 0.5p = (0.7x + 0.3y, 0.5x + 0.5y)$ and $g' = 0.5g + 0.5p = (0.5x + 0.5y, x)$. Then $W(f') = 1.2/1.5 = 4/5 > 3/4 = 1.5/2.0 = W(g')$, hence $c(0.5\{f, g\} + 0.5\{p\}) = \{f'\} = \{0.5f + 0.5p\}$, contradicting *C-Independence*. Next, $c(\cdot)$ has a continuous utility representation, thus *WARP*, and, hence, *Framed Ambiguity* holds, and also *Continuity* holds. Clearly, *C-Non-Degeneracy* holds: $W(y, y) = 0 < 1 = W(x, x)$. Next, since $W(p) \geq W(q)$ iff $p(x) \geq q(x)$, and $W(f) = 1 - \dfrac{1 - f(1)(x)}{1 + f(2)(x)}$, *Strict Monotonicity* holds as well. Finally, since $\dfrac{a}{b} \leq \dfrac{c}{d}$ implies $\dfrac{a}{b} \leq \dfrac{\lambda a + (1-\lambda)c}{\lambda b + (1-\lambda)d} \leq \dfrac{c}{d}$ for positive a, b, c, d and $\lambda \in (0, 1)$, it follows

that $\min\{W(f), W(g)\} \leq W(\lambda f + (1-\lambda)g) \leq \max\{W(f), W(g)\}$, which imply *No-C-Hedging* and *Indirect Ambiguity Aversion*. Indeed, the premise of the former axiom implies $W(h) \geq \max\{W(f), W(p)\}$ for $h \in c(A)$, and the premise of the latter axiom implies $W(h) < \min\{W(f), W(g)\}$ whenever $h \in c(A)$.

Part 3. Independence of *No-C-Hedging*. Let $X = \{x, y\}$ and $S = \{1, 2\}$. Then $H \simeq [0,1]^2$. Take some $0 < \underline{\mu} \leq \overline{\mu} \leq 1$ and consider

$$c(B) = \bigcup_{k \in [0,\infty)} \arg\max_{f \in B} W_k(f)$$

where

$$W_k(f) = \begin{cases} (1-\overline{\mu})(f(1) - f(2)) + f(2) & \text{if } f(1) \geq f(2) \\ f(1) + \underline{\mu} \cdot \min\{(f(2) - f(1)), k\} & \text{if } f(1) < f(2) \end{cases}$$

for any $B \in \mathcal{K}$. Take menu $A = \{(0.2, 0.3), (0.2, 1.0), p\}$, where $p = (0.2 + 0.1\underline{\mu}, 0.2 + 0.1\underline{\mu})$ is a constant act. Then $(0.2, 0.3) \in \arg\max_{f \in A} W_k(f)$ if and only if $k = 0.1$. Notice that $W_{0.1}(0.5 \cdot (0.2, 1.0) + 0.5 \cdot p) = W_{0.1}(0.2 + 0.05\underline{\mu}, 0.6 + 0.05\underline{\mu}) > W_{0.1}(0.2, 0.3)$. Therefore, $(0.2, 0.3) \in c(A)$, but $(0.2, 0.3) \notin c(A \cup \{0.5(0.2, 1.0) + 0.5p)\}\})$, violating *No-C-Hedging*.

It is easy to check that α, *Aizerman's Property*, *C-*β, *Strict Monotonicity* and *C-Non-Degeneracy* are satisfied for $c(\cdot)$. Notice that for arbitrary $k \in [0, \infty)$, $\lambda \in (0,1)$, act $f \in H = [0,1]^2$ and constant act $p = (a, a)$ we have $W_k(\lambda f + (1-\lambda)p) = \lambda W_{k/\lambda}(f) + (1-\lambda)a$, hence $W_k(\lambda f + (1-\lambda)p) \geq W_k(\lambda g + (1-\lambda)p) \iff W_{k/\lambda}(f) \geq W_{k/\lambda}(g)$. Since $\lambda : [0, \infty) \to [0, \infty)$ given by $\lambda(k) = k/\lambda$ is a bijection for $\lambda \in (0,1)$, we conclude that $c(\cdot)$ satisfies *C-Independence*. Next, $W_k(f) = \min\{ (1-\overline{\mu})(f(1) - f(2)) + f(2),$

$f(1) + \underline{\mu}(f(2) - f(1)),\ f(1) + \underline{\mu}k \Big\}$ is quazi-concave, thus $W_k(h) < W_k(f), W_k(g)$ implies $W_k(h) < W_k(\lambda f + (1-\lambda)g)$, hence *Indirect Ambiguity Aversion* holds for $c(\cdot)$. Finally, consider $(A^n, f^n) \longrightarrow (A, f)$ such that $f^n \in A^n$ for all $n = 1, 2, ...$ Notice that $W_k(h) = W_1(h)$ for all $k > 1$ and $h \in H$. Therefore, for all $n = 1, 2, ...$ there is $k(n) \in [0, 1]$ such that $W_{k(n)}(f^n) \geq W_{k(n)}(h)$ for all $h \in A^n$. Since $[0, 1]$ is compact, there is a convergent subsequence $k(n_m) \longrightarrow k \in [0, 1]$. Next, since $A^{n_m} \longrightarrow A$, for any $g \in A$ there is $g^{n_m} \in A^{n_m}$ such that $g^{n_m} \longrightarrow g$. Hence, $W_{k(n_m)}(f^{n_m}) \geq W_{k(n_m)}(g^{n_m})$. Obviously, $W_k(h)$ is continuous w.r.t. $(k, h) \in [0, 1] \times H$. Therefore, $W_k(f) \geq W_k(g)$. Since this holds for arbitrary $g \in A$, we conclude that $c(\cdot)$ satisfies *Continuity*.

Part 4. Independence of *Strict Monotonocity*. Let $X = \{x, y\}$, $S = \{1, 2\}$, and

$$c(A) \ = \ \arg\max_{f \in A} W(f), \qquad \text{where} \quad W(f) \ = \ 2f(1)(x) - f(2)(x)$$

Consider $f = (0.3x + 0.7y, 0.3x + 0.7y)$ and $g = (0.4x + 0.6y, 0.7x + 0.3y)$. Then $f(1) \notin c(\{f(1), g(1)\})$ and $f(2) \notin c(\{f(2), g(2)\})$, but $W(f) = 0.3 > 0.1 = W(g)$, hence $f \in c(\{f, g\})$, and *Strict Monotonicity* is violated. Since $c(\cdot)$ has a continuous utility representation, it satisfies *WARP*, and thus *Framed Uncertainty*, and also it satisfies *Continuity*. Next, $W(\lambda f + (1-\lambda)g) = \lambda W(f) + (1-\lambda)W(g)$, thus *C-Independence, No-C-Hedging, Indirect Ambiguity Aversion* hold. Finally, $W(x) = 1, W(y) = 0$, hence *C-Non-Degeneracy* holds as well.

Part 5. Independence of *Indirect Ambiguity Aversion*. Let $X = \{x, y\}$, $S = \{1, 2\}$, and

$$c(A) \ = \ \arg\max_{f \in A} W(f), \qquad \text{where} \quad W(f) \ = \ \max\{f(1)(x), f(2)(x)\}$$

Let $h = (0.7x + 0.3y, 0.7x + 0.3y)$, and $A = \{h\}$. Notice that $h \notin c(\{h, (x, y)\})$ and $h \notin c(\{h, (y, x)\})$, but $h \in c(\{h, (0.5x + 0.5y, 0.5x + 0.5y)\})$, hence *Indirect Ambiguity Aversion* fails. Since $c(\cdot)$ has a continuous utility representation, it satisfies *Continuity* and *Framed Uncertainty*. Next, since $W(\lambda f + (1 - \lambda)p) = \lambda W(f) + (1 - \lambda)p(x)$, thus *C-Independence* and *No-C-Hedging* hold. Finally, $W(x) = 1 > 0 = W(y)$, thus *C-Non-Degeneracy* holds.

Part 6. Independence of *Continuity*. A choice corresondence that maximizes a lexicographic preference relation works as a standard example. Another example is a model that uses representation given by eq. (2.1) and eq.(2.2) with $X = \{x, y\}$, $S = \{1, 2\}$, $U(p) = p(x)$, $\mathcal{A} = \{\{\mu\} \subseteq \triangle S \mid 0 \leq \mu(1) < 1\}$, then $(x, y) \in c(\{(x, y), (\lambda x + (1 - \lambda)y, \lambda x + (1 - \lambda)y)\})$ for $\lambda \in [0, 1)$, but not for $\lambda = 1$, so *Continuity* fails. One can see that other axioms are satisfied.

Part 7. Independence of *C-Non-Degeneracy*. A trivial choice correspondence $c(A) = A$ for all $A \in \mathcal{K}$ satisfies all axioms except *C-Non-Degeneracy*. ■

B.2.2 Proof of Lemma B.1.10

Step 1. We firstly argue that $\tau(\cdot)$ and $\sigma(\cdot)$ are well defined. One can see that if $P \in \Pi$, then indeed $\tau(P) \in \mathcal{I}$. Now consider $I \in \mathcal{I}$. Notice that $I(\mathbf{0}) = 0$ by positive homogeneity. Next, notice that $I(\cdot)$ is continuous. Indeed, since $\boldsymbol{z} + \max_{s \in S}\{z_s - z'_s\} \cdot \boldsymbol{\iota} \geq \boldsymbol{z'}$, we get $|I(\boldsymbol{z}) - I(\boldsymbol{z'})| \leq |\boldsymbol{z} - \boldsymbol{z'}|$. Thus for arbitrary $\boldsymbol{z} \in R^{|S|}$ set $D(\boldsymbol{z}) = \{\tilde{\boldsymbol{z}} \in R^{|S|} \mid I(\tilde{\boldsymbol{z}}) \geq I(\boldsymbol{z})\}$ is closed. Next, $D(\boldsymbol{z})$ is convex, because $I(\cdot)$ is concave. Moreover, since $I(\cdot)$ is monotone, $\boldsymbol{z} - (1/n)\boldsymbol{\iota} \notin D(\boldsymbol{z})$ for any $n = 1, 2, ...,$ and $\boldsymbol{z} - (1/n)\boldsymbol{\iota} \to \boldsymbol{z}$, hence $\boldsymbol{z}$ is boundary point of $D(\boldsymbol{z})$, and by the Supporting Hyperplane Theorem there exist $\mu \in R^{|S|}$ and $c \in R$ such that

$\sum_{s\in S}\mu(s)z_s + c = I(\boldsymbol{z})$ and $\sum_{s\in S}\mu(s)\tilde{z}_s + c \geq I(\tilde{\boldsymbol{z}})$ for all $\tilde{\boldsymbol{z}} \in R^{|S|}$. Positive homogeneity of $I(\cdot)$ implies that $c \geq \alpha\left(\sum_{s\in S}\mu(s)z_s - I(\boldsymbol{z})\right) = \alpha c$ for all $\alpha \geq 0$, hence $c = 0$. Next, assume $\mu(\overline{s}) < 0$ for some $\overline{s} \in S$. Consider $e_{\overline{s}} \in R^{|S|}$ such that $(e_{\overline{s}})_s = \mathbb{1}\{s = \overline{s}\}$. Then, since $e_{\overline{s}} \geq \mathbf{0}$, we get $0 > \sum_{s\in S}\mu(s)\,(e_{\overline{s}})_s \geq I(e_{\overline{s}}) \geq I(\mathbf{0}) = 0$, contradiction. Finally, $\sum_{s\in S}\mu(s)\cdot 1 \geq I(\boldsymbol{\iota}) = 1$ and $\sum_{s\in S}\mu(s)\cdot(-1) \geq I(-\boldsymbol{\iota}) = -1$ imply $\sum_{s\in S}\mu(s) = 1$. Therefore, $\mu \in \triangle S$, and $\sigma(I) \neq \varnothing$.

Next, function $\xi : \triangle S \times [-1,1]^{|S|} \to R$ given by $\xi(\mu, \boldsymbol{z}) = \min_{s\in S}\mu(s)z_s - I(\boldsymbol{z})$ is continuous, therefore by the Theorem of Maximum function $\eta : \triangle S \to R$ given by $\eta(\mu) = \min_{\boldsymbol{z}\in[-1,1]^{|S|}} \xi(\mu, \boldsymbol{z})$ is continuous. Take a sequence $\mu^n \in \sigma(I)$ that converges to some $\mu \in \triangle S$. Then $\eta(\mu^n) \geq 0$ for all n, and we conclude that $\eta(\mu) \geq 0$. By positive homogeneity of $I(\cdot)$ we get $\sum_{s\in S}\mu(s)z_s - I(\boldsymbol{z}) \geq 0\ \forall \boldsymbol{z} \in [-1,1]^{|S|} \Leftrightarrow \sum_{s\in S}\mu(s)z_s - I(\boldsymbol{z}) \geq 0\ \forall \boldsymbol{z} \in R^{|S|}$, thus $\mu \in \sigma(I)$, hence $\sigma(I)$ is closed. Finally, $\sigma(I)$ is obviously convex, thus $\sigma(I) \in \Pi$.

Step 2. Let us now show that $\tau(\sigma(I)) = I$ for all $I \in \mathcal{I}$. Obviously, $\tau(\sigma(I))(\boldsymbol{z}) \geq I(\boldsymbol{z})$ for all $\boldsymbol{z} \in R^{|S|}$, since $\sum_{s\in S}\mu(s)z_s \geq I(\boldsymbol{z})$ for all $\mu \in \sigma(I)$ for all $\boldsymbol{z} \in R^{|S|}$. From the other hand, by Step 1 for all $\boldsymbol{z} \in R^{|S|}$ there is $\mu \in \sigma(I)$ such that $\sum_{s\in S}\mu(s)z_s = I(\boldsymbol{z})$. Thus $\tau(\sigma(I))(\boldsymbol{z}) \leq I(\boldsymbol{z})$ for all $\boldsymbol{z} \in R^{|S|}$. We conclude that $\tau(\sigma(I)) = I$.

Step 3. Let us show that $\sigma(\tau(P)) = P$ for all $P \in \Pi$. Take any $\mu \in P$, then $\sum_{s\in S}\mu(s)z_s \geq \min_{\mu'\in P}\sum_{s\in S}\mu'(s)z_s = P(\boldsymbol{z})$ for any $\boldsymbol{z} \in R^{|S|}$, thus $P \subseteq \sigma(\tau(P))$. Assume for a contradiction that there exists $\mu \in \sigma(\tau(P))\backslash P \neq \varnothing$. Since $\{\mu\}$ and P are disjoint compact subsets of $R^{|S|}$, by the Separating Hyperplane Theorem there exists $\boldsymbol{z} \in R^{|S|}$ such that $\sum_{s\in S}\mu(s)z_s < c < \min_{\mu'\in P}\sum_{s\in S}\mu'(s)z_s = P(\boldsymbol{z})$, contradiction. Thus, $\sigma(\tau(P)) \subseteq P$.

Step 4. Since $\tau(\sigma(I)) = I$ for all $I \in \mathcal{I}$ and $\sigma(\tau(P)) = P$ for all $P \in \Pi$,

then $\tau : \Pi \to \mathcal{I}$ is a bijection with the inverse $\tau^{-1} = \sigma$. Indeed, $\tau(P) = \tau(P')$ implies $P = \sigma(\tau(P)) = \sigma(\tau(P')) = P'$, thus τ is an injection. And for any $P \in \Pi$ we have $\tau(\sigma(P)) = P$, thus τ is a surjection. ■

B.2.3 Other Proofs

Lemma B.2.1 *There is a choice correspondence $c(\cdot)$ that satisfies axioms 1-6 and violates axiom 7 (C-Non-Degeneracy) such that $c(A) \neq A$ for some $A \in \mathcal{K}$; moreover, $c(\{f, g\}) \neq \{f, g\}$ for some $f, g \in H$ and $c(A) = A$ for all $A \in \mathcal{K}_0$.*

Proof. Let $X = \{x, y\}$, $S = \{1, 2\}$, then $H \simeq [0, 1]^2$. Consider

$$c(A) \;=\; \{\; f \in A \mid f(1) \geq f(2) \;\} \;\cup\; \{\; f \in A \mid f(2) \geq g(2)\ \forall g \in A \;\}$$

Obviously, $c(A) \subseteq A$. Since $A \in \mathcal{K}$ is a compact set and function $u : H \to R$ given by $u(f) \equiv f(2)$ is continuous, there exists $f \in A$ such that $f(2) \geq g(2)$ for all $g \in A$. Thus, $c(\cdot)$ is a valid choice correspondence. Consider $f = (0, 0.5),\ g = (0.1, 0.7)$ and $A = \{f, g\}$, then $c(A) = \{g\} \neq \{f, g\} = A$. Notice that $p(1) = p(2)$ for all $p \in H_0$, therefore $c(A) = A$ for all $A \in \mathcal{K}_0$, and in particular *C-Non-Degeneracy* is violated. It remains to check that other axioms are satisfied:

(α) : Let $f \in c(A \cup B) \cap A$. If $f(1) \geq f(2)$, then $f \in c(A)$. If $f(1) < f(2)$, then $f(2) \geq g(2)$ for all $g \in A \cup B$, thus $f \in c(A)$.

C-β : Follows from $c(A) = A$ for all $A \in \mathcal{K}_0$

***Aizerman's Property*:** Let $h \in c(A)$. If $h(1) \geq h(2)$, then $h \in c(A \cup \{f\})$. Otherwise $h(2) \geq g(2)$ for all $g \in A$. Then if $h(2) \geq f(2)$, then $h \in c(A \cup \{f\})$. Otherwise $f(2) > h(2) \geq g(2)$ for all $g \in A$, hence $f \in c(A \cup \{f\})$.

***C-Independence*:** For $p \in H_0$ denote $f_\lambda = \lambda f + (1 - \lambda)p$, etc. Notice

that $p(1) = p(2)$. C-independence holds since for any $\lambda \in (0.1)$ we have: $f_\lambda(1) \geq f_\lambda(2) \iff f(1) \geq f(2)$ and $f_\lambda(2) \geq g_\lambda(2)\ \forall g_\lambda \in \lambda A + (1-\lambda)p \iff f_\lambda(2) \geq g_\lambda(2)\ \forall g \in A$.

***No-C-Hedging*:** Let $h \in c(A)$ with $f, p \in A$. If $h(1) \geq h(2)$, then $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$. Otherwise, $h(2) \geq g(2)$ for all $g \in A$, and in particular $h(2) \geq f(2), p(2)$. Thus, $h(2) \geq \lambda f(2) + (1-\lambda)p(2)$. Therefore, $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$.

***Strict Monotonicity*:** Since $c(\{p, q\}) = \{p, q\}$ for all $p, q \in H_0$, the premise of the axiom is never satisfied.

Indirect Ambiguity Aversion Consider $h \in A$. If $h \notin c(A)$, then, by proven condition α we have $h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$. Suppose $h \in c(A)$. If $h(1) \geq h(2)$, then $h \in c(A \cup \{\lambda f + (1-\lambda)g\})$. Otherwise, $h(2) \geq h'(2)$ for all $h' \in A$. Let now $h \notin c(A \cup \{f\})$ and $h \notin A \cup \{g\}$. Then $h(2) < f(2), g(2)$. Therefore, $h(2) < \lambda f(2) + (1-\lambda)g(2)$. Given that $h(1) < h(2)$, it follows that $h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$.

***Continuity*:** Let $(A^n, f^n) \to (A, f)$ be such that $f^n \in c(A^n)$ for all $n = 1, 2, ...$. If $f(1) \geq f(2)$, then $f \in c(A)$. Otherwise $f(1) < f(2)$ and, since $f^n \to f$, then $f^n(1) < f^n(2)$ for large enough n. Therefore, $f^n(2) \geq g(2)$ for all $g \in A^n$. Notice that function $v : \mathcal{K} \to R$ given by $v(A) = \max_{g \in A} g(2)$ is continuous, thus function $u : \mathcal{K} \times H \to R$ given by $u(h, A) = h(2) - \max_{g \in A} g(2)$ is continuous. Hence, $f^n(2) - \max_{g \in A^n} g(2) = 0$ for all large enough n implies $f(2) - \max_{g \in A} g(2) = 0$ in the limit. It follows that $f(2) \geq g(2)$ for all $g \in A$, hence $f \in c(A)$. ■

Lemma B.2.2 *Let $\mathcal{A} \subseteq \Pi$ be a non-empty family of sets of beliefs, then $(\mathcal{A})_{coh}$ given by eq. (2.3) is such that: (i) $\mathcal{A} \subseteq (\mathcal{A})_{coh}$, (ii) if a non-empty subcollection $\mathcal{C} \subseteq (\mathcal{A})_{coh}$ has a coherent intersection P, then $P \in (\mathcal{A})_{coh}$, and (iii) if $\mathcal{A}'$ is such that $\mathcal{A} \subseteq \mathcal{A}'$,*

and if a non-empty subcollection $\mathcal{C} \subseteq \mathcal{A}'$ has a non-empty intersection P, then $P \in \mathcal{A}'$, then $(\mathcal{A})_{coh} \subseteq \mathcal{A}'$.

Proof. Statement (i) follows from $P \in \mathcal{A}$ being a trivial coherent intersection of $\{P\} \subseteq \mathcal{A}$. Statement (iii) follows from the definition of $(\mathcal{A})_{coh}$ and the fact that $\mathcal{C} \subseteq \mathcal{A}'$ for any non-empty subcollection $\mathcal{C} \subseteq \mathcal{A}$. To prove (ii), consider an arbitrary non-empty subcollection $\mathcal{C} \subseteq (\mathcal{A})_{coh}$ that has a coherent intersection P. Then for each $P' \in \mathcal{C}$ there exists $\mathcal{C}_{P'} \subseteq \mathcal{A}$ such that P' is a coherent intersection of $\mathcal{C}_{P'}$. Therefore, for any $z \in R^{|S|}$

$$I_P(z) \quad = \quad \max_{P' \in \mathcal{C}} I_{P'}(z) \quad = \quad \max_{P' \in \mathcal{C}} \max_{P'' \in \mathcal{C}_{P'}} I_{P''}(z) \quad = \quad \max_{P'' \in \left(\bigcup\limits_{P' \in \mathcal{C}} \mathcal{C}_{P'} \right)} I_{P''}(z)$$

We conclude that P is a coherent intersection of $\bigcup\limits_{P' \in \mathcal{C}} \mathcal{C}_{P'} \subseteq \mathcal{A}$, hence $P \in (\mathcal{A})_{coh}$. ■

Appendix C

Appendices for Chapter 1

C.1 Proofs

C.1.1 Proof of Proposition 3.2.1

Let BAUP model (S, π, σ, μ) represents an extended random choice ρ. For each frame f, consider the distribution r over utility functions $v: X \to \mathbb{R}$ given by the following formula:

$$r_f(v) \;=\; \sum_u \mu(u) \cdot Pr\Big(u^{M(f)} = v\Big)$$

Then

$$\rho_f^A(x \;=\; \sum_u \mu(u) \cdot Pr\Big(\Big\{u^{M(f)}(x) \geq u^{M(f)}(y)\ \forall y \in A\Big\}\Big) \;=$$
$$\sum_u \mu(u) \cdot Pr\Big(\Big\{u^{M(f)} = v \;\Big|\; v(x) \geq v(y)\ \forall y \in A\Big\}\Big) \;=\; r_f\Big(\Big\{v \;\Big|\; v(x) \geq v(y)\ \forall y \in A\Big\}\Big)$$

Thus, for each frame $f \in F$, a random utility model r_f represents the random choice ρ_f within this frame. ■

C.1.2 Proof of Corollary 3.2.2

When $|S| = 1$, the state space is irrelevant, and the expected utility is given by function u_1. Hence,

$$\rho_f^A(x) \;=\; \mu\Big(\Big\{u \;\Big|\; u_1(x) \geq u_1(y)\ \forall y \in A\Big\}\Big)$$

The Corollary 3.2.2 follows immediately ■

C.1.3 Proof of Theorem 3.2.3

We start with the second statement. Without loss of generality, assume that any combination of frame details is a possible frame; that is, $F = 2^D$.

Let O be the set of all linear orders (complete, transitive and antisymmetric binary relations) on the set of alternatives X, and let $\mathcal{O} = O^F$ be the set of vectors of linear orders indexed by the frames $f \in F$. For $o \in \mathcal{O}$, we write xo_fy when $x \in X$ is ordered higher than $y \in X$ according to order o_f under frame $f \in F$. Let $p \in \triangle\mathcal{O}$ be a probability distribution over $\mathcal{O}$. We say that a BAUP model (S, π, σ, μ) represents (or induces) probability distribution p over vectors of linear orders $o \in \mathcal{O}$ if for all $f \in F$, and $x, y \in A$,

$$p(o) \;=\; \sum_u \mu(u) \cdot Pr\Big(\Big\{u^{M(f)}(x) \geq u^{M(f)}(y) \iff xo_fy \;\;\forall f \in F\Big\}\Big) \tag{C.1}$$

where the summation is performed over all state-dependent utility functions $u : S \times X \to \mathbb{R}$ such that $\mu(u) > 0$, and $u^{M(f)}(x)$ is a function given in eq. (3.2) of a random signal $M(f)$ associated with frame f and distributed according to eq. (3.5).

Lemma C.1.1 *There is a state space S and a framing function $\sigma : D \to \Sigma(S)$ such that for any $\epsilon > 0$, there is a prior probability $\pi \in \triangle(S)$ such that for any $o \in \mathcal{O}$, there is a state-dependent utility function $u : S \times X \to \mathbb{R}$ such that BAUP model $(S, \pi, \sigma, \delta_u)$*

induces a distribution over vectors of orders $p \in \triangle(\mathcal{O})$ such that $p(o) \geq 1 - \epsilon$, where δ_u denotes a degenerate distribution that places probability 1 on u.

Proof of Lemma C.1.1. Consider a state space $S = \{0,1\}^D$; thus, $s = (s_1, ..., s_{|D|}) \in S$, where $s_d \in \{0,1\}$ for $d \in D$. Consider binary message spaces $\mathcal{M}_d = \{d\} \times \{0,1\}$ for frame details $d \in D$. Denote by $\mathcal{M}$ the space of joint messages (all combinations of messages/absence of messages for different frame details) and define the signal structures $\sigma(d)$ as follows:

$$\sigma(d)(s,(d,i)) \;=\; \mathbb{1}\{s_d = i\} \qquad \text{for } i \in \{0,1\}$$

In other words, signal (d,i) reveals the d-th component of the state: $s_d = i$.

Denote by ι the state $\iota \in S$ such that $\iota_d = 1$ for all $d \in D$. Define $\pi \in \triangle(S)$ as follows:

$$\pi(\iota) \;=\; 1 - \epsilon, \qquad \pi(s) \;=\; \frac{\epsilon}{2^{|D|} - 1} \quad \text{if } s \neq \iota$$

For all $f \in F$, let $V_f : X \to \mathbb{R}$ be the following utility function:

$$V_f(x) \;=\; |\{y \in X | x o_f y\}| \tag{C.2}$$

Next, for $s \in S$, denote by

$$L(s) = \{d \in D | s_d = 1\}$$

and define the following state-dependent utility function:

$$u_s(x) \;=\; (\pi(s))^{-1} \cdot \sum_{J:\; L(s) \subseteq J \subseteq D} (-1)^{|J| - |L(s)|} \cdot 2^{|D| - |J|} \cdot V_J(x) + v(x)$$

where function $v : X \to (0,1)$ will be specified later.

Consider an arbitrary frame $f \in F$. Note that state s is a vertex of a binary D-dimensional cube. For convenience, denote by s_f the projection of s on $f \subseteq D$. Consider signal realization $\iota_f \equiv \{(d,1)\}_{d \in f} \in \mathcal{M}$. The corresponding expected utility of item x is

$$u^{\iota_f}(x) \;=\; \frac{\sum_{s:\ s_f=\iota_f} \pi(s) u_s(x)}{\sum_{s:\ s_f=\iota_f} \pi(s)} \;=$$

$$\frac{1}{\pi(\{s \in S | s_f = \iota_f\})} \sum_{s:\ s_f=\iota_f} \quad \sum_{J:\ L(s) \subseteq J \subseteq D} (-1)^{|J|-|L(s)|} 2^{|D|-|J|} V_J(x) + v(x) \;=$$

$$\frac{1}{1 - \frac{2^{|D|}-2^{|f|}}{2^{|D|}-1}\epsilon} \sum_{L,J:\ f \subseteq L \subseteq J \subseteq D} (-1)^{|J|-|L|} 2^{|D|-|J|} V_J(x) + v(x) \;=$$

$$\frac{1}{1 - \frac{2^{|D|}-2^{|f|}}{2^{|D|}-1}\epsilon} \Big[2^{|D|-|f|} V_f(x) + \sum_{L,J:\ f \subseteq L \subseteq J \subseteq D,\ J \neq f} (-1)^{|J|-|L|} 2^{|D|-|J|} V_J(x) \Big] + v(x) \;=$$

$$\frac{1}{1 - \frac{2^{|D|}-2^{|f|}}{2^{|D|}-1}\epsilon} \Big[2^{|D|-|f|} V_f(x) + \sum_{J:\ f \subseteq J \subseteq D,\ J \neq f} (-1)^{|J|} 2^{|D|-|J|} V_J(x) \sum_{L:\ f \subseteq L \subseteq J} (-1)^{-|L|} \Big] + v(x) \;=$$

$$\frac{1}{1 - \frac{2^{|D|}-2^{|f|}}{2^{|D|}-1}\epsilon} V_f(x) + v(x)$$

where we used the Binomial theorem to get

$$\sum_{L:\ f \subseteq L \subseteq J} (-1)^{-|L|} \;=\; \sum_{k=|f|}^{|J|} \quad \sum_{L:\ f \subseteq L \subseteq J,\ |L|=k} (-1)^{-|L|} \;=$$

$$(-1)^{-|f|} \sum_{k=0}^{|J|-|f|} \binom{|J|-|f|}{k} (-1)^{-k} \;=\; (-1)^{-|f|} \cdot (1-1)^{|J|-|f|} \;=\; 0$$

for $J \neq f$.

Claim C.1.2 *There is a function $v : X \to (0,1)$ such that for any frame $f \in F$, function $u^{\iota_f}(\cdot)$ represents linear order o_f, and for any signal realization $m \in \{0,1\}^f$, the expected utility $u^m(\cdot)$ represents some linear order $\tilde{o} \in O$.*

Proof. Note that $[xo_f y \text{ and } x \neq y]$ implies

$$\frac{1}{1-\frac{2^{|D|}-2^{|f|}}{2^{|D|}-1}\cdot\epsilon}\cdot V_f(x) - \frac{1}{1-\frac{2^{|D|}-2^{|f|}}{2^{|D|}-1}\cdot\epsilon}\cdot V_f(y) \;>\; \frac{1}{1-\frac{2^{|D|}-2^{|f|}}{2^{|D|}-1}\cdot\epsilon} \;\geq\; 1$$

Hence, $u^{\iota_f}(x) \geq u^{\iota_f}(y)$ if and only if $xo_f y$, proving that the first statement of the Claim holds for any function $v : X \to (0,1)$.

Next, note that for any signal realization $m \in \mathcal{M}$, for any $x, y \in X$,

$$u^m(x) - u^m(y) \;=\; C_m(x) - C_m(y) + v(x) - v(y)$$

where $\{C_m(x)\}_{m\in\mathcal{M},x\in X}$ is a finite set of quantities that does not depend on $v(\cdot)$. Let

$$\widetilde{C} \;=\; \min_{m\in M; x,y\in X}\Big\{|C_m(x) - C_m(y)| \text{ such that } |C_m(x) - C_m(y)| > 0\Big\}$$

hence, $\widetilde{C} > 0$. Enumerate alternatives $x \in X$ arbitrary by $k = 1, 2, 3, ..., |X|$ and let $k(x)$ to be the number of alternative x. Define

$$v(x) \;=\; \min\{\widetilde{C}, 1\}\cdot e^{-k(x)}$$

It is straightforward to see that $u^m(x) \neq u^m(y)$ for all $m \in \mathcal{M}$, $x, y \in X$, hence the second statement of the Claim holds as well. □

By Claim C.1.2, BAUP model $(S, \pi, \sigma, \delta_u)$ induces some distribution of linear orders p. Next, note that for state $\iota = (1, ..., 1)$,

$$\sigma(d)(\iota, (d, 1)) \;=\; 1$$

Hence, using Claim C.1.2, we get

$$p(o) \;\geq\; Pr\Big(\Big\{M_f = \iota_f \;\forall f \in F\Big\}\Big) \;=\; \pi(\iota) \;=\; 1-\epsilon$$

proving the Lemma. ■

Lemma C.1.3 *Let $Y = \{1, ..., n, n+1\}$, and $y^j = (y^j_1, ..., y^j_{n+1})$ for $j = 1, 2, ...$ be a sequence such that $y^j_i \in \triangle(Y)$ for all i, j, and $y^j_i \longrightarrow \delta_i$ for all $i = 1, ..., n, n+1$. Then for any $p \in \triangle(Y)$ such that $p(i) > 0$ for all $i = 1, ..., n, n+1$, there is j such that $p \in conv(\{y^j_1, ..., y^j_{n+1}\})$.*

Proof of Lemma C.1.3. Since $\sum_{i=1}^{n+1} y'_i = 1$ for all $y' \in \triangle(Y)$, it is enough to consider $Y_0 = \{x \in \mathbb{R}^n | x_i \geq 0 \;\forall i \in \{1, ..., n\},\; \sum_{i=1}^n x_i \leq 1\}$ and a sequence $z^j = (z^j_1, ..., z^j_n, z^j_{n+1})$, $j = 1, 2, ..$ such that $z^j \in Y_0$, $z^j_i \longrightarrow e_i$ for $i \in \{1, ..., n\}$, where $e_{ii} = 1$, e_{ik}=0, for $k \neq i$, and $z^j_{n+1} \longrightarrow e_0$, where $e_0 = (0, ..., 0)$. Consider now $q \in Y_0$ with $q_i = p_i$ for $i \in \{1, ..., n\}$. Note that half-spaces $x_i \geq 0$, $\sum_i^n x_i \leq 1$ include n = 1-dimensional faces $F_i = \text{conv}\Big(\{e_0, e_1, ..., e_n\} \backslash \{e_i\}\Big)$, $i = 0, 1, ..., n$ of the polytope S_0, and their intersection is equal to Y_0. Clearly, for large enough j, $F^j_i = \text{conv}\Big(\{y^j_0, y^j_1, ..., y^j_n\} \backslash \{y^j_i\}\Big)$ for $i = 0, 1, ..., n$ comprise the set of $n-1$ dimensional faces of $Z^j = \text{conv}\{y^j_0, y^j_1, ..., y^j_n\}$. Since $z^j_i \longrightarrow e_i$, it follows that Z^j is equal to the intersection of half-spaces H^j_i, $i \in \{0, 1, ..., n\}$, where $H^j_i = \{x \in \mathbb{R}^n | \sum_{k=1}^n \epsilon^j_{ik} x_k + x_i \geq \epsilon^j_{i0}\}$ for $i \in \{1, ..., n\}$, $H^j_0 = \{x \in \mathbb{R}^n | \sum_{k=1}^n (1 + \epsilon^j_{0k}) x_k \leq 1 + \epsilon^j_{00}\}$ for some $\epsilon^j_{ik} \longrightarrow 0$. Since $q_i > 0$, and $\sum_{i=1}^n q_i < 1$, for large enough j, $\sum_{k=1}^n \epsilon^j_{ik} q_k + q_i \geq \epsilon^j_{i0}$, and $\sum_{k=1}^n (1 + \epsilon^j_{0k}) q_k \leq 1 + \epsilon^j_{00}$. Thus, $q \in H^j_i$ for all $i \in \{0, 1, ..., n\}$, hence $q \in Z^j$, and $p \in \text{conv}(\{y^j_1, ..., y^j_{n+1}\})$. ■

Lemma C.1.4 *There is a state space S and a framing function $\sigma : D \to \Sigma(S)$ such that if the distribution $p \in \triangle\mathcal{O}$ of vectors of linear orders satisfies $p(o) > 0$ for all*

$o \in \mathcal{O}$, then there is a prior $\pi \in \triangle(S)$ and a distribution over state-dependent utility functions μ such that the BAUP model (S, π, σ, μ) induces p.

Proof of Lemma C.1.4. Let S be a state space, σ be a framing function, and, for given $\epsilon > 0$, let $\pi \in \triangle(S)$ be a prior probability and for each $o \in \mathcal{O}$, let $u^\epsilon(o) : \ S \times X \to \mathbb{R}$ be a state-dependent utility function such that $(S, \pi, \sigma, \delta_{u_o^\epsilon})$ induces a distribution of vectors of linear orders p_o^ϵ with the property $p_o^\epsilon(o) \geq 1 - \epsilon$. By Lemma C.1.1, we can find such objects. Next, a space $\triangle\mathcal{O}$ of a probability distributions over vectors of linear orders is a $|X!|^{|F|}$-dimensional probability simplex, and $p_o^\epsilon \longrightarrow \delta_o$ for $\epsilon \longrightarrow 0$. Therefore, by Lemma C.1.3, for small enough ϵ, $p \in \text{conv}\Big(\{p_o^\epsilon\}_{o \in \mathcal{O}}\Big)$, hence $p = \sum_{o \in \mathcal{O}} \alpha_o p_o^\epsilon$ for some convex weights α. Consider a distribution over state-dependent utility functions μ such that $\mu(u_o^\epsilon) = \alpha_o$, then a BAUP model (S, π, σ, μ) represents p. ■

Now, let ρ be an extended random choice such that $q_f(A, x) > 0$ for all $f \in F$, $A \in \mathcal{X}$, $x \in A$, where quantities q are defined in eq. (3.6). Consider $\widetilde{\rho} : F \times \mathcal{X} \times X \to \mathbb{R}$ defined by

$$\widetilde{\rho}_f^A(x) \ = \ \frac{1}{1-\epsilon}\rho_f^A(x) - \frac{\epsilon}{1-\epsilon} \cdot \frac{\mathbb{1}\{x \in A\}}{|A|}$$

where $\epsilon > 0$ is a small parameter. Note that $\widetilde{\rho}_f^A(x) = 0$ for $x \notin A$, and $\sum_{x \in A} \widetilde{\rho}_f^A(x) = 1$. Moreover, since $\rho_f^A(x) = q_f(x, X \backslash A) + \sum_{B \subsetneq A} q_f(x, B) > 0$ for $x \in A$, it follows that for small enough ϵ, $\widetilde{\rho}_f^A(x) > 0$ for all $f \in F$, $A \in \mathcal{X}$, $x \in A$, hence $\widetilde{\rho}$ is an extended random choice. Let $\widetilde{q}$ be an associated with $\widetilde{\rho}$ set of linear polynomials according to eq. (3.6). Then for small enough ϵ, $\widetilde{q}_f^A(x) \geq 0$ for all $f \in F$, $A \in \mathcal{X}$, $x \in A$. For each frame f, by the results of Falmagne (1978), $\widetilde{\rho}_f$ admits a random utility representation; following Barberá and Pattanaik (1986), we consider the associated distribution over linear orders $\widetilde{\tau}_f \in \triangle(O)$. Let $\tau^{uniform}$ be a uniform distribution over linear orders given by $\tau^{uniform}(d) = 1/|O|$ for all $d \in O$. Notice that $\tau^{uniform}$ represents random

choice given by $Pr(\text{choose } x \text{ from } A) = \dfrac{\mathbb{1}\{x \in A\}}{|A|}$. Since

$$\rho_f^A(x) \;=\; (1-\epsilon)\cdot \widetilde{\rho}_f^A(x) + \epsilon \cdot \frac{\mathbb{1}\{x \in A\}}{|A|}$$

it follows that ρ^f has a random utility (random order) representation given by

$$\tau_f \;=\; (1-\epsilon)\cdot \widetilde{\tau}_f + \epsilon \cdot \tau^{uniform}$$

Construct a distribution of linear orders $p \in \mathcal{O}$ given by

$$p(o) \;=\; \prod_{f \in F} \tau_f(o_f)$$

Note that, since $\tau_f(o_f) \geq \epsilon/|O|$, then for any $o \in \mathcal{O}$, $p(o) \geq \left(\dfrac{\epsilon}{|O|}\right)^{|F|} > 0$. By Lemma C.1.4, p has some BAUP representation (S, π, σ, μ). Since marginal distributions of p are given by τ_f for each frame f, and they comprise random utility representations of random choice ρ_f for each f, then (S, π, σ, μ) is a BAUP representaion of the extended random choice ρ. Thus, we have proven the second statement of the Theorem.[1]

Suppose now that $q_f(x, A) \geq 0$ for all $f \in F$, $A \in \mathcal{X}$, $x \in A$, and let $\epsilon > 0$ is given. Consider an extended random choice τ given by

$$\tau_f^A(x) \;=\; (1-\epsilon/2)\cdot \rho_f^A(x) + (\epsilon/2)\cdot \frac{\mathbb{1}\{x \in A\}}{|A|}$$

Then

$$|\rho_f^A(x) - \tau_f^A(x)| \;\leq\; (\epsilon/2)\cdot \left|\frac{\mathbb{1}\{x \in A\}}{|A|} - \rho_f^A(x)\right| \;<\; \epsilon$$

[1]To see that our assumption $F = 2^D$ is without loss of generality, notice that we can always augment the set of frames to 2^D be creating fictitious random choice frequencies that replicate those in one of the existing frames. We then can apply the proof for the case $F = 2^D$

Next, let $q^{uniform}(x, A)$ for $x \in X$, $A \in \mathcal{X}$ be the Falmagne polynomials associated with the uniform random choice $Pr(\text{choose } x \text{ from } A) = \dfrac{\mathbb{1}\{x \in A\}}{|A|}$. According to Barberá and Pattanaik (1986), $q(x, A)$ is the probability that A is an upper counter set of x according to a random utility model that rationalizes the random choice. Since uniform distribution over linear orders rationalizes uniform random choice, it follows that $q^{uniform}(x, A) > 0$ for all $A \in \mathcal{X}$, $a \notin A$. Let $\tilde{q}_f(x, A)$ for $f \in F$, $A \in \mathcal{X}$, $x \in A$ be Falmagne polynomials constructed according to eq. 3.6 from the extended random choice τ, then

$$\tilde{q}_f(x, A) \;=\; (1 - \epsilon/2) \cdot q_f(x, A) + (\epsilon/2) \cdot q^{uniform}(x, A)$$

Hence, $\tilde{q}_f(x, A) > 0$ for all $f \in F$, $A \in \mathcal{X}$, $x \in A$. By the proven second statement of the Theorem, the extended random choice τ admits a BAUP representation. Since this holds for arbitrary $\epsilon > 0$, the if part of the first statement of the Theorem is proven.

Finally, assume that an extended random choice ρ has a BAUP approximation given by a sequence of BAUP models $\left\{\left(S^j, \pi^j, \sigma^j, \mu^j\right)\right\}_{j=1,2,\ldots}$. For any j, a BAUP model $\left(S^j, \pi^j, \sigma^j, \mu^j\right)$ induces a random utility model within each frame $f \in F$ with the induced random choice ρ_f^j. By the results of Falmagne (1978), it must be that $q_f^j(x, A) \geq 0$ for the associated Falmagne polynomials. Since $\rho_f^j \longrightarrow \rho_f$ for every $f \in F$, it follows that $q_f^j(x, A) \to q_f(x, A)$, implying $q_f(x, A) \geq 0$ for all $f \in F$, $A \in \mathcal{X}$, $x \in A$. The only if part of the first statement of the Theorem is proven. ■

C.1.4 Proof of Corollary 3.2.4

The Corollary follows immediately from Theorem 3.2.3 and the results of Falmagne (1978), Barberá and Pattanaik (1986) that the random choice ρ_f has a random utility representation if and only if $q_f(x, A) \geq 0$ for all $x \in X$, $A \in \mathcal{X}$. ■

C.1.5 Proof of Proposition 3.3.1

The major part of the proof is given in our analysis in Section 3.3.1. We complement it by calculating the set Θ. Note also that for our purposes, we do not need to calculate the set Θ precisely; it suffice to calculate a set $\Theta_0 \subseteq \Theta$ such that for each $\theta \in \Theta$, there is $\theta_0 \in \Theta_0$ such that $\theta \subseteq \theta_0$. It follows from the fact that $\psi \not\subseteq \theta_0$ for all $\theta_0 \in \Theta_0$ if and only if $\psi \not\subseteq \theta$ for all $\theta \in \Theta$.

Thus, it is without loss of generality to assume that both $u_1(\cdot)$ and $u_2(\cdot)$ do not exhibit ties. There is always a permutation of x, y, z such that $u_1(x) > u_1(y) > u_1(z)$. If $u_2(z) > u_2(y) > u_2(x)$, then either $O(u) = \theta_1 = \{xyz, xzy, zxy, zyx\}$, or $O(u) = \theta_2 = \{xyz, yxz, yzx, zyx\}$; see Figure 3.1. If $u_2(y) > u_2(z) > u_2(x)$, then $O(u) = \{xyz, yxz, yzx\} \subset \theta_2$. If $u_2(z) > u_2(x) > u_2(y)$, then $O(u) = \{xyz, xzy, zxy\} \subset \theta_1$. If $u_2(y) > u_2(x) > u_2(z)$, then $O(u) = \{xyz, yxz\} \subset \theta_2$. If $u_2(x) > u_2(z) > u_2(y)$, then $O(u) = \{xyz, xzy\} \subset \theta_1$. Finally, if $u_2(x) > u_2(y) > u_2(z)$, then $O(u) = \{xyz\} \subset \theta_1, \theta_2$. It follows that the set Θ_0 consists of the sets θ_1, θ_2 and the sets induced from θ_1 and θ_2 by the permutation of items x, y, z, supporting the analysis provided in Section 3.3.1.

Next, consider $\psi = \{xyz, yzx, zxy\}$. Note that $\psi \not\subseteq \theta_1, \theta_2$. Next, the orbit of ψ with respect to permutations of x, y, z consists of ψ and $\psi' = \{yxz, xzy, zyx\}$. We can check that $yxz \notin \theta_1$, and $xzy \notin \theta_2$, proving that ψ is not a subset of any $\theta \in \Theta_0$, and hence, ψ is not a subset of any $\theta \in \Theta$. The rest of the analysis for the necessity of $W_{fgh}^{xyz} \leq 2$ is provided in Section 3.3.1.

Consider now $\psi = \{xzy, yxz, yzx, zxy\}$. Since $|\psi| = 4 = |\theta_1| = |\theta_2|$, it suffice to see that ψ is not a result of applying of some permutation of x, y, z to θ_1, since the latter always features only 2 and not 3 different items on the top of the preference order, and, similarly, it is not a result of applying of some permutation of x, y, z to θ_2, since the latter always features only 2 and not 3 different items at the bottom of the

preference order. The rest of the analysis for the necessity of $Q^{xyz}_{fghj} \leq 3$ is provided in Section 3.3.1. ■

C.1.6 Proof of Lemma 3.3.2

Let $|X| = n$, and assume $O(u) = O$ for a state-dependent utility function u. Consider a probability simplex $\triangle S$ as a subset of $R^{|S|-1}$ consisting of points $p \in R^{|S|-1}$ such that $p_i \geq 0$, $\sum_{i=1}^{|S|-1} p_i \leq 1$. Define (expected) utility of item $x \in X$ depending on $p \in R^{|S|-1}$ to be

$$V(p,x) \;=\; \sum_{i=1}^{|S|-1} p_i \cdot u_i(x) + \left(1 - \sum_{i=1}^{|S|-1} p_i\right) u_{|S|}(x)$$

Since all linear orders in O are induced by an expected utility for some $\pi \in \triangle S$, it follows that for each pair of items $x, y \in X$, there is an open half-space of $\mathbb{R}^{|S|-1}$ such that $V(p,x) > V(p,y)$ and an open half-space of $\mathbb{R}^{|S|-1}$ such that $V(p,y) > V(p,x)$, divided by an $|S|-2$ dimensional hyperplane consisting of points p such that $V(p,x) = V(p,y)$. Thus, $\frac{n(n-1)}{2}$ hyperplanes given by equations $V(p,x) = V(p,y)$ for various pairs $x, y \in X$ divide $R^{|S|-1}$ on multiple cells. Within each cell, the expected utility $V(p,x)$ induces the same linear order over X. The total number of linear orders induced by $V(p,\cdot)$ for various p may not exceed the maximum number of potential cells Y. By Orlik and Terao (2013) (page 1),

$$Y \;=\; \sum_{k=0}^{|S|-1} \left(\begin{array}{c} n(n-1)/2 \\ k \end{array} \right)$$

where $\begin{pmatrix} m \\ l \end{pmatrix} = \frac{m!}{l!(m-l)!}$ is a Binomial coefficient. Since $|O| = |X|! = n!$, we have

$$\sum_{k=0}^{|S|-1} \begin{pmatrix} n(n-1)/2 \\ k \end{pmatrix} \geq n!$$

For $n > 2|S|$, we have

$$n! \leq \sum_{k=0}^{|S|-1} \begin{pmatrix} n(n-1)/2 \\ k \end{pmatrix} \leq |S| \cdot \begin{pmatrix} n(n-1)/2 \\ |S| \end{pmatrix} \leq |S| \cdot \left(n^2/2\right)^{|S|}$$

Since any fixed k, $n! > \text{const} \cdot n^k$ for large enough n, the above inequality does not hold for large enough n. Hence, for large enough $n = |X|$, $O(u) \neq O$, proving the Lemma. ■

C.1.7 Proof of Theorem 3.3.3

Suppose an extended random choice ρ has a BAUP representation (S, π, σ, μ). For each $u \in \text{supp}(\mu)$, consider an induced family of random utility representations $\{p_f^u\}_{f \in F}$ given by

$$p_f^u(\succeq) = Pr\Big(\Big\{x \succ y \implies U^{M(f)}(x) > U^{M(f)}(y) \ \forall x, y \in X\Big\}\Big)$$

and a family of random utility representations $\{p_f\}_{f \in F}$ given by

$$p_f(\succeq) = \sum_u \mu(u) p_f^u(\succeq_i)$$

Consider arbitrary $\psi = \{\succeq_1, ..., \succeq_n\} \in \Psi^k$, then for any state-dependent utility function u, there are linear orders $\succeq_{i_1}, ..., \succeq_{i_k} \in \psi \backslash O(u)$. It follows from the definition of

$O(u)$ that

$$p_f^u(\succeq_{i_j}) \;=\; 0 \qquad \forall j = 1, ..., k$$

Hence,

$$\sum_{i=1}^{n} p_f^u(\succeq_i) \;=\; \sum_{i=1}^{n} \mathbb{1}\{i \neq i_1, ..., i_k\} \cdot p_f^u(\succeq_i) \;\leq\; \sum_{i=1}^{n} \mathbb{1}\{i \neq i_1, ..., i_k\} \;=\; n - k$$

It follows that

$$\sum_{i=1}^{n} p_f(\succeq_i) \;=\; \sum_{i=1}^{n} \sum_{u} \mu(u) p_f^u(\succeq_i) \;=\; \sum_{u} \mu(u) \sum_{i=1}^{n} p_f^u(\succeq_i) \;\leq\; n - k$$

Finally,

$$p_f\Big(\Big\{ \succeq \in O | x \succ y \; \forall y \in A \backslash x \Big\}\Big) \;=\; \sum_{u} \mu(u) Pr\Big(\Big\{ U^{M(f)}(x) > U^{M(f)}(y) \; \forall y \in A \backslash x \Big\}\Big) \;=\; \rho_f^A(x)$$

Thus, a family of random utility models $\{p_f\}_{f \in F}$ satisfies all desired properties. Since non-strict inequalities are preserved in the limit, the statement also holds for a BAUP approximation with state space $|S|$. ■

C.1.8 Proof of Corollary 3.3.4

Let $|X| = n$, $|F| = m = n!$, and let $O = \{\succeq_1, ..., \succeq_m\}$ be the set of linear orders on X; enumerate frames by $f_1, ..., f_m \in F$. Consider an extended random choice ρ induced by the family of distributions of linear $\{p_f\}_{f \in F}$ such that $p_{f_i} = \delta_{\succeq_i}$. That is,

$$\rho_{f_i}^A(x) \;=\; \mathbb{1}\{x \succ_i y \; \forall y \in A \backslash x\}$$

Towards a contradiction, assume that ρ has a BAUP approximation $(S, \pi^j, \sigma^j, \mu^j)$, $j = 1, 2, ...$ with state space S. That is, for any $\epsilon > 0$, if j is large enough, a

BAUP model $(S, \pi^j, \sigma^j, \mu^j)$ induces an extended random choice τ^j such that $\left|\rho_f^A(x) - \left(\tau^j\right)_f^A(x)\right| < \epsilon$ for all x, A, f.

Claim C.1.5 *The family $\{p_f\}_{f\in F}$ is the unique family of random utility representations associated with the extended random choice ρ that it induces.*

Proof. Assume that some other family $\{t_f\}_{f\in F}$ is associated with the same extended random choice. Then there is a frame $f_i \in F$ and a pair of items x, y such that $x \succ_i y$ and $t_f(\{\succ \in O | x \succ y\}) < 1$, but this contradicts τ being associated with ρ, since $\rho_f^{\{x,y\}}(x) = 1$. □

By Lemma 3.3.2, if n is large enough, $O \notin \Theta$. By Theorem 3.3.3, for any BAUP model $(S, \pi^j, \sigma^j, \mu^j)$, there is a family of random utility representations p^j (in fact, induced by the corresponding BAUP model) with the property that

$$\sum_{i=1}^{|O|} p_{f_i}^j(\succeq_i) \;\leq\; |O| - 1$$

Consider $\epsilon = (2|O|)^{-2}$. Then the above inequality implies that there is $i \in \{1, ..., |O|\}$ and a linear order $\succ \neq \succeq_i$ such that $p_{f_i}^j(\succeq) > \epsilon$. Since $\succeq \neq \succeq_i$, then there are items $x, y \in X$ such that $x \succ_i y$ and $y \succ x$. Hence, $\rho_{f_i}^{\{x,y\}}(x) = 1$, $(\tau^j)_{f_i}^{\{x,y\}}(x) < 1 - \epsilon$. Since such frame and pair of items could be found for any j, this contradicts our assumption that $(S, \pi^j, \sigma^j, \mu^j)$ approximates ρ. ■

C.1.9 Proof of Theorem 3.3.5

The proof of Theorem 3.3.5 partially repeats the proof of Theorem 3.3.3.

Assume that an extended random choice ρ has a BAUP representation (S, π, σ, μ) with state space S. For each $u \in \text{supp}(\mu)$, consider an induced family of random

utility representations $\{p_f^u\}_{f\in F}$ given by

$$p_f^u(\succeq) \;=\; Pr\Big(\Big\{x \succ y \implies U^{M(f)}(x) > U^{M(f)}(y)\ \forall x, y \in X\Big\}\Big)$$

and a family of random utility representations $\{p_f\}_{f\in F}$ given by

$$p_f(\succeq) \;=\; \sum_u \mu(u) p_f^u(\succeq_i)$$

Note that p_f represents ρ_f for each frame $f \in F$. Therefore,[2] for $x \notin A$,

$$q_f(x, A) \;=\; p_f\Big(\Big\{\ \succeq \in O \;\Big|\; y \succ x \ \Leftrightarrow y \in A\Big\}\Big)$$

Then

$$\begin{aligned}\sum_{i=1}^n q_{f_i}(x_i, A_i) \;&=\; \sum_{i=1}^n \sum_u \mu(u) p_{f_i}^u\Big(\Big\{\ \succeq \in O \;\Big|\; y \succ x_i \ \Leftrightarrow y \in A_i\Big\}\Big) \;= \\ &\sum_u \mu(u) \sum_{i=1}^n p_{f_i}^u\Big(\Big\{\ \succeq \in O \;\Big|\; y \succ x_i \ \Leftrightarrow y \in A_i\Big\}\Big) \;\leq\; n - k\end{aligned}$$

where we used the definition of Ξ to find that for any state-dependent utility function u, there are at least k pairs (x, A) such that for any f, $p_f^u\Big(\Big\{\ \succeq \in O \;\Big|\; y \succ x \ \Leftrightarrow y \in A\Big\}\Big) = 0$ and, therefore,

$$\sum_{i=1}^n p_{f_i}^u\Big(\Big\{\ \succeq \in O \;\Big|\; y \succ x_i \ \Leftrightarrow y \in A_i\Big\}\Big) \;\leq\; n - k$$

for any state-dependent utility function u. Thus, we have proven the necessity of constraints for a BAUP representation with a fixed state space S. Since the constraints are linear polynomials in choice frequencies, they continue to hold for a BAUP approximation with state space S as well. ■

[2]See the argument in Barberá and Pattanaik (1986).

References

Abreu, D. and A. Rubinstein (1988): "The structure of Nash equilibrium in repeated games with finite automata," *Econometrica: Journal of the Econometric Society*, 1259–1281.

Ahn, D. S. and H. Ergin (2010): "Framing contingencies," *Econometrica*, 78, 655–695.

Ahn, D. S. and T. Sarver (2013): "Preference for flexibility and random choice," *Econometrica*, 81, 341–361.

Aliprantis, C. D. and K. C. Border (2005): *Infinite Dimensional Analysis, A Hitchhiker's Guide, 3rd edition*, Springer.

Banks, J. S. and R. K. Sundaram (1990): "Repeated games, finite automata, and complexity," *Games and Economic Behavior*, 2, 97–117.

Barberá, S. and P. K. Pattanaik (1986): "Falmagne and the rationalizability of stochastic choices in terms of random orderings," *Econometrica: Journal of the Econometric Society*, 707–715.

Block, H. D. and J. Marschak (1960): "Random orderings and stochastic theories of responses," *Contributions to probability and statistics*, 97–132.

Börgers, T. and A. Morales (2004): "Complexity constraints in two-armed bandit problems: an example," Tech. rep., Citeseer.

Bourgeois-Gironde, S. and R. Giraud (2009): "Framing effects as violations of extensionality," *Theory and Decision*, 67, 385–404.

Caplin, A. and D. J. Martin (2020): "Framing, Information, and Welfare," Tech. rep., National Bureau of Economic Research.

Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and A. Rustichini (2020): "Multinomial logit processes and preference discovery: inside and outside the black box," *arXiv preprint arXiv:2004.13376*.

Chandrasekher, M. (2019): "Source-dependent uncertainty aversion," Tech. rep., Working paper.

Cover, T. and M. Hellman (1970): "The two-armed-bandit problem with time-invariant finite memory," *IEEE Transactions on Information Theory*, 16, 185–195.

Cover, T. M. (1969): "Hypothesis testing with finite statistics," *The Annals of Mathematical Statistics*, 40, 828–835.

Crès, H., I. Gilboa, and N. Vieille (2011): "Aggregation of multiple prior opinions," *Journal of Economic Theory*, 146, 2563–2582.

Danan, E., T. Gajdos, B. Hill, and J.-M. Tallon (2016): "Robust social decisions," *American Economic Review*, 106, 2407–25.

Deb, R. and L. Renou (2021): "Dynamic Choice and Common Learning," *CEPR Discussion Paper No. DP16160*.

Dekel, E., B. L. Lipman, and A. Rustichini (2001): "Representing preferences with a unique subjective state space," *Econometrica*, 69, 891–934.

Dillenberger, D., J. S. Lleras, P. Sadowski, and N. Takeoka (2014): "A theory of subjective learning," *Journal of Economic Theory*, 153, 287–312.

Dow, J. (1991): "Search decisions with limited memory," *The Review of Economic Studies*, 58, 1–14.

Falmagne, J.-C. (1978): "A representation theorem for finite random scale systems," *Journal of Mathematical Psychology*, 18, 52–72.

Frick, M., R. Iijima, and Y. Le Yaouanq (2019a): "Boolean expected utility," .

Frick, M., R. Iijima, and T. Strzalecki (2019b): "Dynamic random utility," *Econometrica*, 87, 1941–2002.

Galaabaatar, T. and E. Karni (2013): "Subjective expected utility with incomplete preferences," *Econometrica*, 81, 255–284.

Ghirardato, P., F. Maccheroni, and M. Marinacci (2004): "Differentiating ambiguity and ambiguity attitude," *Journal of Economic Theory*, 118, 133–173.

Gilboa, I., F. Maccheroni, M. Marinacci, and D. Schmeidler (2010): "Objective and subjective rationality in a multiple prior model," *Econometrica*, 78, 755–770.

Gilboa, I. and D. Schmeidler (1989): "Maxmin Expected Utility with Non-Unique Prior," *Journal of Mathematical Economics*, 18, 141–153.

Goldin, J. and D. Reck (2020): "Revealed-Preference Analysis with Framing Effects," *Journal of Political Economy*, 128, 2759–2795.

Gul, F. and W. Pesendorfer (2006): "Random expected utility," *Econometrica*, 74, 121–146.

Hara, K., E. A. Ok, and G. Riella (2019): "Coalitional Expected Multi-Utility Theory," *Econometrica*, 87, 933–980.

Heller, Y. (2012): "Justifiable choice," *Games and Economic Behavior*, 76, 375–390.

Hellman, M. E. and T. M. Cover (1970): "Learning with Finite Memory," *The Annals of Mathematical Statistics*, 41, 765–782.

——— (1971): "On memory saved by randomization," *The Annals of Mathematical Statistics*, 1075–1078.

Hill, B. (2011): *Unanimity and the aggregation of multiple prior opinions*, Groupe HEC.

Huber, J., J. W. Payne, and C. Puto (1982): "Adding asymmetrically dominated alternatives: Violations of regularity and the similarity hypothesis," *Journal of consumer research*, 9, 90–98.

Kalai, E. and E. Solan (2003): "Randomization and simplification in dynamic decision-making," *Journal of Economic Theory*, 111, 251–264.

Kalai, E. and W. Stanford (1988): "Finite rationality and interpersonal complexity in repeated games," *Econometrica: Journal of the Econometric Society*, 397–410.

Kandori, M., G. J. Mailath, and R. Rob (1993): "Learning, mutation, and long run equilibria in games," *Econometrica: Journal of the Econometric Society*, 29–56.

Karni, E. and Z. Safra (2016): "A theory of stochastic choice under uncertainty," *Journal of Mathematical Economics*, 63, 164–173.

Kifer, Y. (1988): "Random perturbations of dynamical systems," *Nonlinear Problems in Future Particle Accelerators*, 189.

Klabjan, D., W. Olszewski, and A. Wolinsky (2014): "Attributes," *Games and Economic Behavior*, 88, 190–206.

Kocer, Y. (2010): "Endogenous learning with bounded memory," *Economic Theory Center Working Paper*.

Kohn, M. G. and S. Shavell (1974): "The theory of search," *Journal of Economic Theory*, 9, 93–123.

Kopylov, I. (2019): "Multiple Priors and Comparative Ignorance," *Mimeo*.

Kreps, D. M. (1979): "A representation theorem for" preference for flexibility"," *Econometrica: Journal of the Econometric Society*, 565–577.

Lehrer, E. and R. Teper (2011): "Justifiable preferences," *Journal of Economic Theory*, 146, 762–774.

Lu, J. (2014): "Random Ambiguity," *Unpublished working paper, UCLA*.

——— (2016): "Random choice and private information," *Econometrica*, 84, 1983–2027.

Luce, R. D. (1959): *Individual choice behavior*, John Wiley.

Mandler, M., P. Manzini, and M. Mariotti (2012): "A million answers to twenty questions: Choosing by checklist," *Journal of Economic Theory*, 147, 71–92.

Manzini, P. and M. Mariotti (2014): "Stochastic choice and consideration sets," *Econometrica*, 82, 1153–1176.

Morgan, P. and R. Manning (1985): "Optimal search," *Econometrica: Journal of the Econometric Society*, 923–944.

Nascimento, L. (2012): "The ex ante aggregation of opinions under uncertainty," *Theoretical Economics*, 7, 535–570.

Natenzon, P. (2019): "Random choice and learning," *Journal of Political Economy*, 127, 419–457.

Ok, E. A., P. Ortoleva, and G. Riella (2012): "Incomplete preferences under uncertainty: Indecisiveness in beliefs versus tastes," *Econometrica*, 80, 1791–1808.

Ok, E. A. and G. Tserenjigmid (2019): "Deterministic Rationality of Stochastic Choice Behavior," *en. In*, 39.

Oprea, R. (2020): "What makes a rule complex?" *American Economic Review*, 110, 3913–51.

Orlik, P. and H. Terao (2013): *Arrangements of hyperplanes*, vol. 300, Springer Science & Business Media.

Piermont, E. (2017): "Context dependent beliefs," *Journal of Mathematical Economics*, 71, 63–73.

Rockafellar, R. T. (1970): *Convex analysis*, 28, Princeton university press.

Rubinstein, A. (1986): "Finite automata play the repeated prisoner's dilemma," *Journal of Economic Theory*, 39, 83–96.

Rustichini, A. (2020): "Neural and normative theories of stochastic choice," Tech. rep., Working paper, University of Minnesota.

Salant, Y. and A. Rubinstein (2008): "(A, f): Choice with Frames," *The Review of Economic Studies*, 75, 1287–1296.

Sanjurjo, A. (2014): "The role of memory in search and choice," *Available at SSRN 2479561*.

——— (2015): "Search, memory, and choice error: An experiment," *PloS one*, 10, e0126508.

——— (2017): "Search with multiple attributes: Theory and empirics," *Games and Economic Behavior*, 104, 535–562.

Shmaya, E. and L. Yariv (2016): "Experiments on Decisions under Uncertainty: A Theoretical Framework," *American Economic Review*, 106, 1775–1801.

Tversky, A. and D. Kahneman (1981): "The framing of decisions and the psychology of choice," *science*, 211, 453–458.

Tversky, A. and I. Simonson (1993): "Context-dependent preferences," *Management Science*, 39, 1179–1189.

Weitzman, M. L. (1979): "Optimal search for the best alternative," *Econometrica: Journal of the Econometric Society*, 641–654.

Wilson, A. (2014): "Bounded memory and biases in information processing," *Econometrica*, 82, 2257–2294.